Introduction

Let $Z = f(x, y)$ be a two dimensional function. Then x and y are called independent variables and z is called dependent variable. An equation which involves partial derivatives of an independent varaible is called a PDE. We employ the following notations. $\frac{\delta z}{\delta x} = p$, $\frac{\delta z}{\delta y} = q$, $\frac{\delta^2 z}{\delta x^2} = r$, $\frac{\delta^2 z}{\delta x \delta y} = s$, $\frac{\delta^2 z}{\delta y^2} = t$.

Then $2\frac{\delta^2 z}{\delta x^2} + 3xy\frac{\delta^2 z}{\delta x \delta y} = s + 7\frac{\delta z}{\delta x} = 5$, is an example of a PDE which can be written as $2r + 3xys + 7p = 5$

Formation of PDE

Unlike the case of ordinary differential equation (ODE) which arise from the elimination of arbitrary constants, the PDE can be formed either by the elimination of arbitrary constants or by elimination of arbitrary functions from a relation involving three or more variables.

Example 1

Derive the PDE by eleminating the constants from the equation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{1}$$

Solution: Differentiating w.r.t x , we get $2\frac{\delta z}{\delta x} = \frac{2x}{a^2}$
OR $\frac{1}{a^2} = \frac{1}{x}\frac{\delta z}{\delta x} = \frac{p}{x}$
Differentiating w.r.t y , we get $2\frac{\delta z}{\delta y} = \frac{2y}{b^2}$
OR $\frac{1}{b^2} = \frac{1}{y}\frac{\delta z}{\delta y} = \frac{q}{y}$
Substituting values of $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in given equation (1)we get
$2z = xp + yq$
as the desired PDE of first order.

Example 2

Form the PDE by eliminating arbitrary functions from
$Z = f(x^2 + y^2)$
Solution: Then $p = \frac{\delta z}{\delta x} = f^1(x^2 + y^2)2x$
$q = \frac{\delta z}{\delta y} = f^1(x^2 + y^2)2y$
Dividing these two equations we get
$p/q = x/y$ or $yp - xq = 0$ is the required PDE

Equations solvable by direct integration

Example 3

Solve $\frac{\delta^3 z}{\delta^2 x \delta y} + 18xy^2 + sin(2x - y) = 0$

Solution: Integrating w.r.t x (keeping y as constant), we get

$\frac{\delta^2 z}{\delta x \delta y} + 9x^2y^2 - \frac{1}{2}cos(2x - y) = f(y)$

Again Integrating w.r.t x , $\frac{\delta z}{\delta y} + 3x^3y^2 - \frac{1}{4}sin(2x - y) = xf(y) + g(y)$

Now, integrating w.r.t y keeping y fixed, we get

$z + x^3y^3 - \frac{1}{4}cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$

Thus $z = \frac{1}{4}cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$

where $\int f(y)dy = u(y)$ and $\int g(y)dy = v(y)$

Example 4

Solve $\frac{\delta^2 z}{\delta x^2} + z = 0$ given that when $x = 0, z = e^y$ and $\frac{\delta z}{\delta x} = 1$

Solution: If z were functions of x alone then the above equation may be written as $(D^2 + 1)z = 0$ whose solution is known to be $z = Asinx + Bcosx$ where A and B are constants. Since z is a function of x and y we must have A and B as functions of y. Hence the solution of the given PDE is of the form $z = f(y)sinx + \phi(y)cosx$.

Then $\frac{\delta z}{\delta x} = f(y)cosx - \phi(y)sinx$

When $x = 0, z = e^y$ implies $e^y = \phi(y)$

When $x = 0, \frac{\delta z}{\delta x} = 1$ which implies $f(y) = 1$

Hence the desired solution is $z = sinx + e^y cosx$

Method of separation of variables

It involves a solution which breaksup into a product of functions each of which contains only one of the variables.

Example

$$\textit{Solve}: \qquad u_{xx} + u_{yy} = 0 \tag{2}$$

Solution: Let $u(x, y) = F(x)G(y)$

Then $u_x = G\frac{dF}{dx}$ and $u_y = F\frac{dG}{dy}$

$u_{xx} = G\frac{d^2F}{dx^2}$ and $u_{yy} = F\frac{d^2G}{dy^2}$

Substituting in equation (2)

$$G\frac{d^2F}{dx^2} + F\frac{d^2G}{dy^2} = 0$$

ie,

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} \tag{3}$$

Since x and y are independent variables, equation (3) can hold good if each side of equation (3) is equal to a constant k (say).
Then equation (3) gives

$$\frac{d^2F}{dx^2} = kx, \qquad \frac{d^2g}{dy^2} = -ky$$

case I: Let k be positive. Let $k = p^2$

Then $\frac{d^2F}{dx^2} - p^2x = 0, \quad \frac{d^2G}{dy^2} + p^2y = 0$

On solving the above equations we get,

$F(x) = c_1e^{px} + c_2e^{-px}$ and $G(y) = c_3cospy + c_4sinpy$

$\therefore \quad u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3cospy + c_4sinpy)$

case II: Let k be negative. Let $k = -p^2$

Then $\frac{d^2F}{dx^2} + p^2x = 0, \quad \frac{d^2G}{dy^2} - p^2y = 0$

On solving the above equations we get,

$F(x) = c_5 cospx + c_6 sinpx$ and $G(y) = c_7 e^{py} + c_8 e^{-py}$

$\therefore \quad u(x,y) = (c_5 cospx + c_6 sinpx)(c_7 e^{py} + c_8 e^{-py})$

case III: Let $k = 0$

Then $\frac{d^2F}{dx^2} = 0, \quad \frac{d^2G}{dy^2} = 0$

On solving the above equations we get,

$F(x) = c_9 x + c_{10}$ and $G(y) = c_{11} y + c_{12}$

$\therefore \quad u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Exercise Problems

1. Solve: $xu_x + yu_y = 0$

2. Solve: $u_{yy} = u_x$

Indicated transformations

We can transform a given partial differential equation in a suitable way by introducing new independent variables and hence we can obtain the solution.

Example 1

Solve: $u_{xy} - u_{yy} = 0$ using the transformations $v = x,\, z = x + y$

Solution: $u_y = u_v v_y + u_z z_y = u_v.0 + u_z.1 = u_z$ ‘

$$u_{xy} = (u_z)_x = u_{vz} v_x + u_{zz} z_x = u_{vz}.1 + u_{zz}.1 = u_{vz} + u_{zz}$$‘

$$u_{yy} = (u_z)_y = u_{vz} v_y + u_{zz} z_y = u_{vz}.0 + u_{zz}.1 = u_{zz}$$

$$u_{xy} - u_{yy} = u_{vz} + u_{zz} - u_{zz} = 0$$

$$\therefore \quad u_{vz} = 0$$

Integrating with respect to v, keeping z fixed, we get

$$u_z = h(z)$$

Again integrating with respect to z, we get

$$u = \int h(z)dz + g(v)$$

$\therefore$ the solution is

$$u(x, y) = f(z) + g(v) = f(x + y) + g(x).$$

Example 2.

Solve: $xu_{xy} = yu_{yy} + u_y$ using the transformations $v = x, z = xy$

Solution: $u_y = u_v v_y + u_z z_y = u_v.0 + u_z.x = vu_z$ ‘

$$u_{xy} = (vu_z)_x = (vu_z)_v v_x + (vu_z)_z z_x = u_z + vu_{vz} + vu_{zz}y$$

$$= u_z + vu_{vz} + vu_{zz}\frac{z}{v} = u_z + vu_{vz} + zu_{zz}$$

$$u_{yy} = (vu_z)_y = (vu_z)_v v_y + (vu_z)_z z_y = (vu_z)_v.0 + vu_{zz}x = v^2u_{zz}$$

$$xu_{xy} - yu_{yy} - u_y = v(u_z + vu_{vz} + zu_{zz}) - \frac{z}{v}(v^2u_{zz}) - vu_z$$

$$\therefore \quad v^2u_{vz} = 0$$

$$u_{vz} = 0$$

Integrating with respect to v, keeping z fixed, we get

$$u_z = h(z)$$

Again integrating with respect to z, we get

$$u = \int h(z)dz + g(v)$$

$\therefore$ the solution is $u(x, y) = f(z) + g(v) = f(xy) + g(x)$.

Exercise Problems

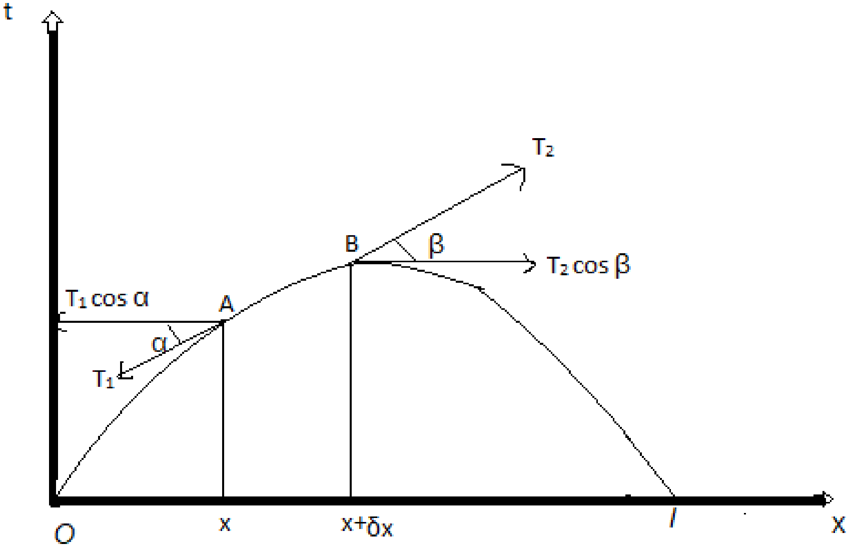
1. solve: $u_{xx} + u_{xy} - 2u_{yy} = 0$ using the transformations $v = x + y, \quad z = 2x - y.$

2. solve: $u_{xx} - 2u_{xy} + u_{yy} = 0$ using the transformations $v = x, \quad z = x + y.$

One dimensional wave equation

Consider a flexible string tightly stretched between two fixed points at a distance 'l' apart. Let 'ρ be the mass per unit length of the string. We assume that

- The string is perfectly elastic and the mass per unit length of the string is constant i.e, it is homogeneous.
- The tension on the string is large enough to neglect the gravitational force on the string.
- The horizontal components are constants on the string i.e., no horizontal motion.


t
T2
B
β
T2 cos β
T1 cos α
A
α
T1
O
x
x+δx
l
X

Let T_1 and T_2 be the tensions acting on the string at points A and B.

Since there is no horizontal motion, the horizontal components $T_1 \cos \alpha = T_2 \cos \beta = T$, a constant.

The vertical components $-T_1 \sin \alpha$ and $T_2 \sin \beta$, are the net forces.

Let δx be the length of the small portion AB of the string and $u(x, t)$ be the vertical displacement of the string at position x and time t.

The resultant force is given by $T_2 \sin \beta - T_1 \sin \alpha$.

By applying Newton's law $F = ma$

$$T_2 sin\beta - T_1 sin\alpha = \rho\delta x \frac{\partial^2 U}{\partial t^2}$$

Dividing throughout by T, $\frac{T_2 sin\beta}{T} - \frac{T_1 sin\alpha}{T} = \frac{\rho}{T}\delta x \frac{\partial^2 U}{\partial t^2}$

But $\frac{T_1}{T} = \frac{1}{cos\alpha}$; $\frac{T_2}{T} = \frac{1}{cos\beta}$

Thus $\frac{sin\beta}{cos\beta} - \frac{sin\alpha}{cos\alpha} = \frac{\rho}{T}\delta x \frac{\partial^2 U}{\partial t^2}$

$$\tan\beta - \tan\alpha = \frac{\rho}{T}\delta x \frac{\partial^2 U}{\partial t^2}$$

But $\tan\alpha$ and $\tan\beta$ represent the slopes.

Hence $\tan\alpha = (\frac{\partial U}{\partial x})_{(x,t)}$ and $\tan\beta = (\frac{\partial U}{\partial x})_{(x+\delta x,t)}$

$$(\frac{\partial U}{\partial x})_{(x+\delta x,t)} - (\frac{\partial U}{\partial x})_{(x,t)} = \frac{\rho}{T}\delta x \frac{\partial^2 U}{\partial t^2}$$

$$\frac{(\frac{\partial U}{\partial x})_{(x+\delta x,t)} - (\frac{\partial U}{\partial x})_{(x,t)}}{\delta x} = \frac{\rho}{T}\frac{\partial^2 U}{\partial t^2}$$

As $\delta x \to 0$, LHS becomes $\frac{\partial^2 U}{\partial x^2}$

Hence $\frac{\partial^2 U}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 U}{\partial t^2}$

$$\frac{T}{\rho}\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$$

$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$

-One dimensional wave equation

Derivation of one dimensional heat equation using Gauss divergence theorem

Heat flows in the direction of decreasing temparature . Physical aspects show that the velocity of flow of heat is directly proportional to negative gradient of temparature U.

$\overrightarrow{V} \propto -\nabla U$

$\overrightarrow{V} = -k.\nabla U$ where k is called the thermal conductivity, a physical quantity which is constant.

If V is the region bounded by the surface S, the amount of heat leaving the region V per unit time is given by,

$H = \iint_S \overrightarrow{V}.\hat{n}\, dS$, where $\hat{n}$ is unit outward normal vector to the element dS.

According to Gauss Divergence theorem,

$$\iint_S \overrightarrow{f}.\hat{n}\,dS = \iiint_V div\,\overrightarrow{f}\;dV$$

Thus

$$\begin{aligned}
H &= \iint_S \overrightarrow{V}.\hat{n}\,dS \\
&= \iiint_V div\,\overrightarrow{V}\,dV = \iiint_V \nabla.\overrightarrow{V}\,dV \\
&= \iiint_V \nabla.(-k\nabla U)\,dV \\
&= -k\iiint_V \nabla^2 UdV \\
H &= -k\iiint_V \nabla^2 Udxdydz \qquad (4)
\end{aligned}$$

On the other hand, the total amount of heat T in V is
$T = \iiint_V \sigma\rho U dxdydz$

where σ is the specific heat of the material ρ is the density of the material. Hence rate of decrease of heat is

$$-\frac{\partial T}{\partial t} = -\iiint_V \sigma\rho\frac{\partial U}{\partial t} dxdydz \tag{5}$$

Observe that this rate of decrease of heat must be equal to the amount of heat leaving V per unit time. i.e.
$-\frac{\partial T}{\partial t} = H$

From Equations(4) and (5) we obtain

$k \iiint_V \nabla^2 U dxdydz = \iiint_V \sigma\rho\frac{\partial U}{\partial t} dxdydz$

$$\iiint_v \left(\sigma\rho\frac{\partial T}{\partial t} - k\nabla^2 U\right) dxdydz = 0$$

Since this holds for any region V in the body, the integrand must be zero everywhere.

Hence, $\sigma\rho\frac{\partial U}{\partial t} - k\nabla^2 U$=0

$$\frac{\partial U}{\partial t} = \frac{k}{\sigma\rho}\nabla^2 U$$

$\frac{\partial U}{\partial t} = c^2\nabla^2 U$, where $c^2 = \frac{k}{\sigma\rho}$, thermal diffusivity constant.

If we consider a thin isolated rod of uniform cross section in which heat flows only in x-direction. Then U is a function of x and t only. Thus it reduces to, $\frac{\partial U}{\partial t} = c^2\frac{\partial^2 U}{\partial x^2}$, One dimensional heat equation.

Consider the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Let $u = XT$, where $X = X(x)$ and $T = T(x)$ be the solution of PDE.

Therefore, the PDE becomes $\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$

$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$

Dividing by XT, separate x and t and equate it to a constant k we get
$\frac{1}{X} \frac{\partial^2 T}{\partial t^2} = \frac{1}{c^2 T} \frac{\partial^2 X}{\partial x^2}$

$\frac{\partial^2 X}{\partial x^2} - kX = 0$; $\frac{\partial^2 T}{\partial t^2} - c^2 kT = 0$

$(D^2 - k)X = 0$; $(D^2 - c^2k)T = 0$ where, $D^2 = \frac{d^2}{dx^2}$ and $\frac{d^2}{dt^2}$

<u>*Case*(*i*) :</u> Let $k = 0$

Then $D^2X = 0$; $D^2T = 0$

Therefore the solutions are $X = c_1 + c_2x$; $T = c_3 + c_4t$

Hence $u(x,t) = (c_1 + c_2x)(c_3 + c_4t)$

Applying boundary conditions $u(0,t) = 0, u(l,t) = 0$;

The solution is $u = 0$.

<u>*Case*(*ii*) :</u> Let k be positive integer. i.e., $k = +p^2$(say)

Then $(D^2 - p^2)X = 0$; $(D^2 - c^2p^2)T = 0$

Hence $u(x,t) = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt})$

Applying boundary conditions $u(0,t) = 0, u(l,t) = 0$;

The solution is $u = 0$.

$\underline{Case(iii)}$: Let k be negative integer. i.e., $k = -p^2$(say)
Then $(D^2 + p^2)X = 0$; $(D^2 + c^2p^2)T = 0$
Hence the standard solution of wave equation is
$u(x,t) = (Acos(px) + Bsin(px))(Ccos(cpt) + Dsin(cpt))$
Applying boundary conditions $u(0,t) = 0, u(l,t) = 0$;
$X(0) = 0 \Rightarrow c_1 = 0$
$X(l) = 0 \Rightarrow c_2 sin(pl) = 0$
$sin(pl) = 0$ if $c_2 \neq 0$
$p = \frac{n\pi}{l}; n = 0, 1, ...$
Therefore $X(x) = c_2 sin(\frac{n\pi}{l})x$

Now, $T = c_3 cos(\lambda_n)t + c_4 sin(\lambda_n)t$, where $\lambda_n = \frac{n\pi}{l}$
Thus $u(x,t) = \sum_{n=0}^{\infty}[c_5 cos(\lambda_n)t + c_6 sin(\lambda_n)t] sin(\frac{n\pi}{l})x$
where, $c_5 = c_3 c_2$ and $c_6 = c_4 c_2$
Applying initial condition $u(x,0) = f(x)$;
$f(x) = \sum_{n=0}^{\infty} c_5 sin(\frac{n\pi}{l})x$
By Fourier half range sine series expansion, we get
$c_5 = \frac{2}{l}\int_0^l f(x) sin(\frac{n\pi}{l})x dx$
Again apply the second initial condition $\frac{\partial u}{\partial t}(x,0) = g(x)$,
$g(x) = \sum_{n=0}^{\infty} c_6 sin(\frac{n\pi}{l})x$
By Fourier half range sine series expansion, we get
$c_6 = \frac{2}{\lambda_n l}\int_0^l g(x) sin(\frac{n\pi}{l})x dx$

Consider the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
Put $v = x + ct, w = x - ct$ [u is a function of v and w]
We use chain rule;
$u_x = u_v.v_x + u_w.w_x$
Since $v_x = w_x = 1$
$u_x = u_v + u_w$
Now $u_{xx} = (u_{vv}.v_x + u_{vw}.w_x) + (u_{wv}.v_x + u_{ww}.w_x)$
$u_{xx} = u_{vv} + 2u_{vw} + u_{ww}$
Similarly, $u_t = u_v.v_t + u_w.w_t$
Since $v_t = c, w_t = -c$
Then $u_{tt} = c^2[u_{vv} - 2u_{vw} + u_{ww}]$

Substitute u_{xx} and u_{tt} in wave equation,
$c^2[u_{vv} - 2u_{vw} + u_{ww}] = c^2[u_{vv} + 2u_{vw} + u_{ww}]$
$\Rightarrow -4u_{vw} = 0$
$\Rightarrow u_{vw} = 0$
Integrate,
$u_v = f(v)$
$u = F(v) + G(w)$, where $F(v) = \int f(v)dv$
Hence $u = F(x + ct) + G(x - ct)$
Applying initial conditions;
$u(x, 0) = f(x)$; $\frac{\partial u}{\partial t}(x, 0) = g(x)$ for $x \leq 0$.
Therefore,

$$f(x) = F(x) + G(x) \tag{6}$$

Now $g(x) = c[G'(x) - F'(x)]$
Integrate with respect to x;

$$\frac{1}{c}\int g(x)dx + k = G(x) - F(x) \tag{7}$$

Adding Equation(6) and Equation(7);
$G(x) = \frac{1}{2}\{f(x) + \frac{1}{c}\int g(x)dx + k\}$
Subtracting Equation(7) from Equation(6) ;
$F(x) = \frac{1}{2}\{f(x) - \frac{1}{c}\int g(x)dx - k\}$
But we require to replace x by $x + ct$, then
$u(x,t) = \frac{1}{2}f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(s)ds$
When $g(x) = 0$, then
$u(x,t) = \frac{1}{2}f(x+ct) + f(x-ct)$

Example:

A tightly stretched string has its ends fixed at x = 0 and x = l. At time t = 0, the string is given a shape defined by $f(x) = \mu x(l-x)$, where μ is a constant and then released. Find the displacement of any point x, of the string at any time $t > 0$.

<u>*Solution*</u> : $u(0,t) = 0, u(l,t) = 0, u(x,0) = f(x) = \mu x(l-x)$

$u_t(x,0) = g(x) = 0$

We know that $u(x,t) = \sum_{n=0}^{\infty}[A_n cos\frac{n\pi ct}{l} + B_n sin\frac{n\pi ct}{l}]sin(\frac{n\pi}{l})x$

Here $B_n = 0$

Therefore $A_n = \frac{2}{l}\int_0^l \mu x(l-x)sin(\frac{n\pi}{l})xdx$

Solving we get,

$A_n = \frac{4\mu l^2}{n^3\pi^3}[(-1)^n - 1]$

Hence the solution is $u(x,t) = \sum_{n=0}^{\infty}[\frac{4\mu l^2}{n^3\pi^3}[(-1)^n - 1]]cos\frac{n\pi ct}{l}sin(\frac{n\pi}{l})x$

One Dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u = XT$, where $X = X(x)$, $T = T(t)$

$$\frac{\partial (XT)}{\partial t} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X.\frac{\partial T}{\partial t} = c^2 T.\frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{c^2 T}.\frac{\partial T}{\partial t} = \frac{1}{X}.\frac{\partial^2 X}{\partial x^2}$$

Equating to a constant k,
$\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = k$ and $\frac{1}{c^2 T} \cdot \frac{\partial^2 T}{\partial t^2} = k$

$$(D^2 - k)X = 0 \qquad \text{and} \qquad (D^2 - c^2 k)T = 0.$$

Case (i): Let $k = 0$, Auxiliary equation becomes $m^2 = 0$ and $m = 0$ then the solutions are

$$X = c_1 + c_2 x \qquad \text{and} \qquad T = c_1 e^{0t} = c_3$$

Therefore

$$u = (c_1 + c_2 x)c_3$$

Case (ii): $k = p^2$, Auxiliary equation becomes, $m^2 - p^2 = 0$ and $m - c^2p^2 = 0$.

$$X = c_1 e^{px} + c_2 e^{-px} \qquad \text{and} \qquad T = c_3 e^{c^2p^2t}$$

Therefore, $u = (c_1 e^{px} + c_2 e^{-px})c_3 e^{c^2p^2t}$.
Case (iii): $k = -p^2$, then Auxiliary equation becomes, $m^2 + p^2 = 0$ and $m + c^2p^2 = 0$

$$X = (c_1 cospx + c_2 sinpx) \qquad \text{and} \qquad T = c_3 e^{-c^2p^2t}$$

Therefore, $u = (c_1 cospx + c_2 sinpx)c_3 e^{-c^2p^2t}$

Boundary Conditions

Boundary conditions are, $u(0,t) = 0$ and $u(l,t) = 0$
Case (i): $X(0) = 0, X(l) = 0$

$$X(0) = c_1 + c_2(0) \qquad \text{and} \qquad X(l) = c_1 + c_2 l$$

implies, $c_1 = 0$ and $c_2 = 0$ therefore
$X(x) = c_1 + c_2 x = 0$ and $u = 0$.
Case (ii): $X(0) = c_1 + c_2$ implies $c_1 = -c_2$
$X(l) = c_1 e^{pl} + c_2 e^{-pl}$ implies $c_1 = c_2 = 0$.
therefore $u = 0$.

Case (iii): $u = (c_1 cospx + c_2 sinpx)c_3 e^{-c^2 p^2 t}$
$X(0) = 0$ and $X(l) = 0$ implies $c_1 = 0$
and $X(l) = c_1 cospl + c_2 sinpl$ implies $0 = c_2 sinpl$ and if c_2 not equal to 0, $sinpl = 0$
implies $p = \frac{n\pi}{l}$ for $n = 0, 1, 2...$
therefore $u(x, t) = (c_2 sin\frac{n\pi}{l}x)c_3 e^{-c^2 n^2 \pi^2 t/t^2}$
The General solution is given by,

$$u = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l} e^{-\lambda^2 nt} \qquad \text{where} \qquad \lambda_n^2 = \frac{c^2 n^2 \pi^2}{l^2}.$$

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Initial Conditions

Now using $u(x,0) = f(x)$, $u(x,0) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l}$

$$f(x) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l} \qquad \text{where} \qquad B_n = \frac{2}{l}\int_0^l f(x) sin\frac{n\pi x}{l} dx.$$

Problems

1: Solve $u_t = 25u_{xx}$, $0 \leq x \leq 80$ subjected to $u(x,0) = sin\frac{3\pi x}{40}$ and $u(0,t) = u(l,t) = 0$.

Solution:

Given $u(x,0) = sin\frac{3\pi x}{40}$ and $l = 80$

$u(x,t) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l} e^{-\lambda^2 nt}$

$$f(x) = sin\frac{3\pi x}{40} = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l}$$

$$sin(\frac{3\pi x}{40} x\frac{2}{2}) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l}$$

$$sin(\frac{6\pi x}{80}) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l}$$

$$= B_1 sin\frac{\pi x}{l} + B_2 sin\frac{2\pi x}{l} + B_3 sin\frac{3\pi x}{l} + ...$$

$$= B_1 sin\frac{\pi x}{80} + B_2 sin\frac{2\pi x}{80} + B_3 sin\frac{3\pi x}{80} + ...$$

comparing and equating,

$$sin(\frac{6\pi x}{80}) = B_6 sin(\frac{6\pi x}{80})$$

we get, $B_6 = 1$ and $B_1 = B_2 = B_3 = B_4 = ... = 0$ implies $B_n = 0$; when n not equal to 6 therefore, $u(x,t) = B_6 sin(\frac{6\pi x}{80})e^{-\lambda_6^2 t}$, where $\lambda_6 = \frac{6n\pi}{80}$

2: A laterally insulated copper bar of length 80cm has its initial temperature $u(x,0) = sin^2(\frac{3\pi x}{40})$. If the ends of the bar are insulated, obtain the temperature distribution of the bar at any time.
Solution:
$l = 80cm$; $u(x,0) = sin^2(\frac{3\pi x}{40})$
since insulated, case (ii) the solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n cos \frac{n\pi x}{l} e^{-\lambda^2 nt}$$

$$u(x,0) = sin^2 \frac{3\pi x}{40} = \sum_{n=0}^{\infty} a_n cos \frac{n\pi x}{l}$$

$$\frac{1-cos\frac{6\pi x}{40}}{2}=\sum_{n=0}^{\infty}a_n cos\frac{n\pi x}{l}$$

$$\frac{1}{2}-\frac{1}{2}cos\frac{6\pi x}{40}=a_0+\sum_{n=0}^{\infty}a_n cos\frac{n\pi x}{l}$$

$$=a_0+a_1cos\frac{\pi x}{80}+a_2cos\frac{2\pi x}{80}+...$$

$$\frac{1}{2}-\frac{1}{2}cos(\frac{6\pi x}{40}.\frac{2}{2})=a_0+a_1cos\frac{\pi x}{80}+a_2cos\frac{2\pi x}{80}+...+a_{12}cos\frac{12\pi x}{80}+...$$

$a_0=\frac{1}{2}$ and $a_{12}=\frac{-1}{2}$

solution is,

$$= a_0 + a_{12} cos \frac{12\pi x}{80} e^{-\lambda_{12}^2 t}$$

$$u(x,t) = \frac{1}{2} - \frac{1}{2} cos \frac{12\pi x}{80} e^{-\lambda_{12}^2 t}$$

where $\lambda_{12} = \frac{12n\pi}{80}$.