Lecture 7

20-04-21

HOMOGENEOUS FUNCTIONS

Defn: A function $z = f(x,y)$ is said to be homogeneous if for real λ, $f(\lambda x, \lambda y) = \lambda^k f(x,y)$. Here k is the degree of the function

Eg: 1) $f(x,y) = x^2+y^2$

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2(x^2+y^2) = \lambda^2 f(x,y)$$

$\Rightarrow f$ is homogeneous fn with degree 2

2) $f(x,y) = x^2+y^2+2$
is not homogeneous fn

Note: If $f(x,y) = x^n \phi(y/x)$ or $y^n \phi(x/y)$ Then we can say that $f(x,y)$ is homogeneous.

consider $u = \dfrac{x^2+y^2}{x-y} = \dfrac{x^2\left(1+(y/x)^2\right)}{x(1-y/x)} = x^1 \phi(y/x) \rightarrow$ homogen with deg $n=1$

Euler's Theorem :- If u is a homogeneous function of degree n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: Since u is a homogeneous fn of degree n in x & y we can write $u(x,y) = x^n \phi(y/x)$

diff u partially w.r.to x

$$\frac{\partial u}{\partial x} = x^n \phi'(y/x)\cdot(-y/x^2) + \phi(y/x)\, n x^{n-1}$$

$$x\frac{\partial u}{\partial x} = -x^{n-2+1} y\, \phi'(y/x) + n \underbrace{x^n \phi(y/x)} \quad \because \text{multiply by } x$$

$$x\frac{\partial u}{\partial x} = -x^{n-1} y\, \phi'(y/x) + nu \quad \text{--- (1)}$$

diff u partially w.r.to y

$$\frac{\partial u}{\partial y} = x^n \phi'(y/x) \times \frac{1}{x} = x^{n-1}\phi'(y/x)$$

$$\therefore y\frac{\partial u}{\partial y} = y\, x^{n-1}\phi'(y/x) \quad \text{--- (2)}$$

Adding (1) & (2) we get $\boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu}$

Eg: $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(y/x)$ Then $xu_x + yu_y =$ ____

$$= \sin^{-1}(x/y) + \tan^{-1}\left(\frac{1}{(x/y)}\right)$$

$= y^0 \phi(x/y)$, $n=0 \Rightarrow u$ is homo of deg $=0$

$\therefore xu_x + yu_y = nu = 0$

2. If $u = \ln\left[\frac{x^4+y^4}{x+y}\right]$ then find the value of

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Soln:

$$u = \ln_e\left(\frac{x^4+y^4}{x+y}\right)$$

$$\Rightarrow e^u = \frac{x^4+y^4}{x+y} = \frac{x^{\cancel{4}\,3}\left(1+\left(y/x\right)^4\right)}{\cancel{x}\left(1+y/x\right)} = x^3\phi\left(y/x\right)$$

$\Rightarrow z = e^u$ is a homo fn of deg 3.

$\therefore$ By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$\therefore x\frac{\partial(e^u)}{\partial x} + y\frac{\partial(e^u)}{\partial y} = 3z$$

$$\cancel{e^u}\, x\frac{\partial u}{\partial x} + \cancel{e^u}\cdot y\frac{\partial u}{\partial y} = 3\cancel{e^u} \qquad \frac{\partial(e^u)}{\partial x} = e^u\cdot\frac{\partial u}{\partial x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \underline{\underline{3}}$$

3. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, P.T $x u_x + y u_y = \tan u$

Note that $\sin u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+y^2/x^2\right)}{x\left(1+y/x\right)} = x^1\phi\left(y/x\right)$

$\Rightarrow$ $z = \sin u$ is homogeneous eqn with deg 1

By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = n\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = n\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\sin u}{\cos u} = \underline{\underline{1\cdot\tan u}}$$

Corollary :- If u is a homogeneous fn of deg n in x & y then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof : We have $u = x^n \phi(y/x)$

By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ — (1)

Diff (1) w.r.t x partially.

$$x\frac{\partial^2 u}{\partial x^2} + 1.\frac{\partial u}{\partial x} + y.\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

Multiply by x

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = nx\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial x} = (n-1)x\frac{\partial u}{\partial x} \quad — (2)$$

Diff (1) partially w.r.to y

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = n.\frac{\partial u}{\partial y}$$

Multiply by y

$$xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = ny\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial y} = (n-1)y\frac{\partial u}{\partial y} \quad (3)$$

Adding (2) & (3) we obtain

$$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = [n-1]\{xu_x + yu_y\}$$

$$= \underline{\underline{(n-1)nu}} \quad \text{from (1)}$$

Eg : If $u = x^4y^6\cos^{-1}(y/x) - x^3y^7\cot^{-1}(x/y)$

Then the value of $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$

$$u(tx, ty) = (tx)^4(ty)^6\cos^{-1}\left(\frac{\cancel{t}y}{\cancel{t}x}\right) - (tx)^3(ty)^7\cot^{-1}\left(\frac{\cancel{t}x}{\cancel{t}y}\right)$$

$$= t^{10}u(x,y)$$

$n = 10$, u is homo.

$\therefore x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = n(n-1)u = 10 \times 9u = \underline{90u}$

1. If $u = \sin^{-1}\left(\frac{x^2+y^2}{2x+3y}\right)$ then P.T $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \tan^3 u$

Observe that $\sin u = \frac{x^2+y^2}{2x+3y} = x\,\phi(y/x)$ is homo fn of deg 1.

Take $z = \sin u$

By Euler's theorem. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{\sin u}{\cos u} = 1 \cdot \tan u \quad \text{--- (1)}$$

diff (1) w.r.t x, y partially

$$x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial y^2} = \sec^2 u \cdot \frac{\partial u}{\partial x}$$

x by by x

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = x \sec^2 u \frac{\partial u}{\partial x}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x \frac{\partial u}{\partial x} (\sec^2 u - 1) \quad \text{--- (2)}$$

My we obtain

$$y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = y \frac{\partial u}{\partial y} (\sec^2 u - 1) \quad \text{--- (3)}$$

Adding (2) & (3),

$$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = (\sec^2 u - 1)(x u_x + y u_y)$$
$$= (\sec^2 u - 1) \tan u$$
$$= \frac{(1 - \cos^2 u)}{\cos^2 u} \cdot \tan u$$
$$= \tan^2 u \cdot \tan u$$
$$= \tan^3 u$$

HW

2) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$ then

S.T (i) $x u_x + y u_y = \sin 2u$

ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = (1 - 4\sin^2 u) \sin 2u$

3. If $u = \cos^{-1}\left(\dfrac{x^2+y^2-z^2}{\sqrt{x^4+y^4-z^4}}\right)$ then

P.T $\quad x u_x + y u_y + z u_z = -\cot u$

$n = 0$

4. If $u = (x^2+y^2)^{1/3}$ P.T $\quad x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2u}{9}$

5. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$ P.T $\quad x^2 u_{xx} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\sin 2u$
$-\sin^2 u$

Total derivatives

Let $z = f(x, y)$ where $x = \phi(t)$ $\quad y = \psi(t)$
Now, we can express z as a function of t alone by putting x and y in $f(x, y)$.
Then z is a composite function of t.

Thus, The ordinary derivative $\dfrac{dz}{dt}$ is called the total derivative of z is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Total partial derivatives :-

If $z = f(u, v)$ where $u = \phi(x, y)$
composite fn $\qquad v = \psi(x, y)$

$$\begin{tikzcd} z \arrow[r] \arrow[rr, bend right, "\frac{\partial z}{\partial x}"'] \arrow[rr, bend right=60, "\frac{\partial z}{\partial y}"'] & (u, v) \arrow[r] & (x, y) \end{tikzcd}$$

then
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

1. Find $\frac{dz}{dt}$ if $z = xy^2 + x^2 y$, $x = at^2$, $y = 2at$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \text{——} (*)$$

$z = xy^2 + x^2 y$ $\qquad$ $x = at^2$, $y = 2at$

$\frac{\partial z}{\partial x} = y^2 + 2xy$, $\frac{\partial z}{\partial y} = 2xy + x^2$ $\qquad$ $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$\therefore$ $(*)$ becomes

$$\frac{dz}{dt} = (y^2 + 2xy)\, 2at + (2xy + x^2)\cdot 2a$$
$$= \left((2at)^2 + 2at^2 \times 2at\right) 2at + \left(2at^2 \times 2at + a^2t^4\right)\cdot 2a$$
$$= 16a^3t^3 + 10a^3t^4$$

2. $z = x^2 + y^2$, $x = \cos(uv)$, $y = \sin(u+v)$

find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. Also find $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

$$z \longrightarrow (x, y) \longrightarrow (u, v)$$

$z = x^2 + y^2$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2y$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$x = \cos(uv)$ $\qquad$ $y = \sin(u+v)$

$\frac{\partial x}{\partial u} = -\sin(uv)\cdot v$ $\qquad$ $\frac{\partial y}{\partial u} = \cos(u+v)$

$\frac{\partial x}{\partial v} = -\sin(uv)\cdot u$ $\qquad$ $\frac{\partial y}{\partial v} = \cos(u+v)$

$$\therefore \frac{\partial z}{\partial u} = 2x\cdot\left(-v\sin(uv)\right) + 2y\left(\cos(u+v)\right)$$
$$= 2\left\{-v\cos(uv)\sin(uv) + \sin(u+v)\cos(u+v)\right\}$$

III by $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$$= 2x\left\{-\sin(uv)\cdot u\right\} + 2y\left\{\cos(u+v)\right\}$$
$$= 2\left[-u\cos(uv)\sin(uv) + \sin(u+v)\cos(u+v)\right]$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \sin 2(uv)\,[u - v]$$

3. If $H = f(y-z,\ z-x,\ x-y)$ find

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z}$$

$$H = f(u,\ v,\ w)$$

where $u = y - z$, $u_x = 0$, $u_y = 1$, $u_z = -1$

$v = z - x$, $v_x = -1$, $v_y = 0$, $v_z = 1$

$w = x - y$, $w_x = 1$, $w_y = -1$, $w_z = 0$

$$H \longrightarrow (u, v, w) \longrightarrow (x, y, z)$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial H}{\partial x} = 0 - \frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} \longrightarrow (1)$$

|||ly
$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\,\frac{\partial w}{\partial y}$$

$$= \frac{\partial H}{\partial u} + 0 - \frac{\partial H}{\partial w} \longrightarrow (2)$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$= -\frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} + 0 \longrightarrow (3)$$

(1) + (2) + (3) gives

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$$

HW

1) $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$, find $\frac{du}{dt}$.

2) $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ find $\frac{du}{dt}$