If $n = 1$ then $z = a$ is a simple pole.

if $n = 2$ then $z = a$ is a double pole.

If the no: of $-$ve powers of $(z-a)$ is infinite then $z = a$ is called an <u>essential singularity</u>.

If all the $-$ve powers of $(z-a)$ are zero then $z = a$ is called a removable singularity.

Find <u>the nature of</u> Singularities

1) $f(z) = \dfrac{z - \sin z}{z^2}$, $z=0$ is a singular point

$$\frac{z-\sin z}{z^2} = \frac{1}{z^2}\left\{ z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right\}$$

$$= \frac{1}{z^2}\left\{ \frac{z^3}{3!} - z^5/5! + z^7/7! - \cdots\right\}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots$$

No $-$ve powers. $\therefore$ $z=0$ is a removable singularity.

Ex:-

$$f(z) = \frac{1}{z-2}, \quad z = 2 \text{ is a singular point.}$$

$$f(z) = \frac{1}{(z-1)(z-3)}, \quad z = 1, 3 \text{ are singular points}$$

Isolated singular point

A singular point z_0 is said to be an isolated singular point if there exists a neighbourhood of z_0 which does not contain any other singular points.

Ex:- $f(z) = \frac{1}{z}$ has an isolated singularity at $z = 0$

If $z = a$ is an isolated singularity of $f(z)$, then $f(z)$ can be expanded in a Laurent's series about $z = a$

$f(z) = \dfrac{1}{\sin \pi/z}$ has no isolated singularities

singular points $z = \pm 1/n$.

Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \longrightarrow (1)$$

If the no: of $-ve$ powers of $(z-a)$ in (1) is finite, say n then a is called a pole of order n.

A function $f(z)$ is said to be analytic at ∞ if $f(1/z)$ is analytic at $z=0$.

<u>Ex:-</u> (1) $f(z) = e^{1/z}$ is analytic at $z=\infty$

since $f(1/z) = e^{z}$ is analytic at $z=0$

(2) Since z^2 is analytic at $z=0$, $\frac{1}{z^2}$ is analytic at $z=\infty$

<u>Definition</u>

A point $z=a$ at which a function $f(z)$ fails to be analytic is called a singular point of $f(z)$.

$f(z)$ is said to have a singular point at ∞ if $f(1/z)$ has a singular point at $z=0$.

Zeroes and Singularities

A point $z=a$ is called a <u>Zero</u> of an analytic function $f(z)$ if $f(a)=0$

If all the derivatives $f', f'', f''', \ldots, f^{n-1}$ are also zero at $z=a$ and $f^{(n)}(a) \neq 0$ then $f(z)$ is said to have a zero of order n.

Zero of order 1 is called simple zero.

ie $f(a)=0$, $f'(a) \neq 0$ then $z=a$ is simple zero.

Ex:-

$$f(z) = (z-5)^3$$

$$f'(z) = 3(z-5)^2$$

$$f''(z) = 6(z-5)$$

$$f'''(z) = 6$$

At $z=5$

$$f(z) = f'(z) = f'(z) = 0$$

$$f'''(z) \neq 0$$

$\therefore z=5$ is a zero of order 3.

$f(z)$ is said to have a zero of order n at ∞ if $f(1/z)$ has a zero at $z=0$

Ex:- $f(z) = \dfrac{1}{1-z}$ has a simple zero at ∞

$$f(1/z) = \frac{z}{z-1}$$

$$f'(1/z) = \frac{(z-1)-z}{(z-1)^2} \neq 0 \text{ at } z=0$$

$$\frac{4z-1}{z^4-1} = \left(\frac{1-4i}{4i}\right)\frac{1}{z-i} - \left(\frac{1+4i}{4i}\right)\frac{1}{z+i} + \frac{5}{4}\cdot\frac{1}{z+1} + \frac{3}{4}\cdot\frac{1}{z-1}$$

(i) $$\frac{4z-1}{z^4-1} = \frac{1-4i}{4i}\left(-\frac{1}{i}\right)\frac{1}{(1-z/i)} - \left(\frac{1+4i}{4i}\right)\left(\frac{1}{i}\right)\frac{1}{(1+z/i)}$$

$$+\frac{5}{4}\left(\frac{1}{z+1}\right) - \frac{3}{4}\left(\frac{1}{1-z}\right)$$

$$= \left(\frac{1-4i}{4}\right)\left(1+\frac{z}{i}+\left(\frac{z}{i}\right)^2+\cdots\right) + \left(\frac{1+4i}{4}\right)\left(1-\frac{z}{i}+\left(\frac{z}{i}\right)^2-\cdots\right),$$

$$+\frac{5}{4}\left(1-z+z^2-\cdots\right) - \frac{3}{4}\left(1+z+z^2+\cdots\right),$$

$|z|<1$

(ii) $$f(z) = \left(\frac{1-4i}{4i}\right)\frac{1}{z(1-i/z)} - \left(\frac{1+4i}{4i}\right)\frac{1}{z(1+i/z)} + \frac{5}{4z}\frac{1}{1+1/z}$$

$$-\frac{3}{4z}\left(\frac{1}{1-1/z}\right)$$

$$= \left(\frac{1-4i}{4i}\right)\frac{1}{z}\left(1+i/z+(i/z)^2+\cdots\right) - \left(\frac{1+4i}{4}\right)\frac{1}{z}\left(1-i/z+(i/z)^2-\cdots\right)$$

$$+\frac{5}{4z}\left(1-1/z+1/z^2+\cdots\right)$$

$$-\frac{3}{4z}\left(1+1/z+1/z^2+\cdots\right)$$

$|z|>1.$

(ii) $\frac{1}{z^2+1} = \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{(z-i)\left[1+\frac{2i}{z-i}\right]}\right\}$

$= \frac{1}{2i(z-i)}\left\{1 - \left(1+\frac{2i}{z-i}\right)^{-1}\right\}$

$= \frac{1}{2i(z-i)}\left\{1-\left(1-\frac{2}{z-i}+\left(\frac{2i}{z-i}\right)^2 - \cdots\right)\right\},$

$\left|\frac{2i}{z-i}\right|<1$

$\Rightarrow |z-i| > |2i| = 2$

3) Find all possible expansions of $\frac{4z-1}{z^4-1}$ about the point $z=0$ and determine the precise region of convergence.

$\frac{4z-1}{z^4-1} = \frac{4z-1}{(z^2+1)(z^2-1)} = \frac{4z-1}{(z+i)(z-i)(z+1)(z-1)}$

$= \frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z-1}$

$A(z-i)(z^2-1) + B(z+i)(z^2-1) + C(z-1)(z^2+1) + D(z+1)(z^2+1) = 4z-1$

$z=i \Rightarrow -4iB = 4i-1 \Rightarrow B = \frac{1-4i}{4i}$

$z=-i \Rightarrow 4iA = -4i-1 \Rightarrow A = -\frac{(1+4i)}{4i}$

$z=1 \Rightarrow 4D = 3 \Rightarrow D = 3/4$

$z=-1 \Rightarrow -4C = -5 \Rightarrow C = 5/4$

Laurent's series

1) Expand $z^2 \sin 1/z$ about $z=0$.

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$$z^2 \sin 1/z = z^2\left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\right)$$

$$= z - \frac{1}{3!z} + \frac{1}{5!z^3} - \cdots$$

2) Find the Laurent's series of $f(z) = \frac{1}{z^2+1}$ about the point $z = i$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$= \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{z-i+2i}\right\}$$

(i) $$\frac{1}{z^2+1} = \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{2i\left[1+\frac{z-i}{2i}\right]}\right\}$$

$$= \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{2i}\left(1+\frac{z-i}{2i}\right)^{-1}\right\}$$

$$= \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{2i}\left(1 - \frac{z-i}{2i} + \left(\frac{z-i}{2i}\right)^2 - \cdots\right)\right\},$$

$$\left|\frac{z-i}{2i}\right| < 1$$

$$\Rightarrow |z-i| < |2i| = 2$$

$z = 2i = 0 + 2i$

$|z| = \sqrt{(0+2i)(0-2i)} = 2$

$= \sqrt{4}$.