Half range Series:

In some applications of Fourier Series especially in solving partial differential equations, it becomes necessary to expand $f(x)$ in $(0, c)$. This range is called half range.

Sine Series:

If it is required to expand $f(x)$ as a sine Series in $(0, c)$, we extend the function in $(-c, 0)$ reflecting it in the origin so that $f(-x) = -f(-x)$. Then the extended function is odd in $(-c, c)$. The extended function is given by

$$g(x) = \begin{cases} -f(-x), & -c < x < 0 \\ f(x), & 0 < x < c. \end{cases}$$

and the expansion will give the desired Fourier Sine series. Hence the half range fourier Sine series is given by $f(x) = \sum b_n \sin \frac{n\pi x}{c}$ where

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx$$

Cosine Series:

If it is required to expand $f(x)$ as a cosine series in $(0, c)$. We extend the function in $(-c, 0)$ reflecting it in y-axis so that $f(-x) = +f(-x)$. Then the extended function is even in $(-c, c)$ and is given by

$$g(x) = \begin{cases} f(-x), & -c < x < 0 \\ f(x), & 0 < x < c \end{cases}$$

Hence the half range cosine Series is given by $f(x) = a_0/2 + \sum a_n \cos \frac{n\pi x}{c}$

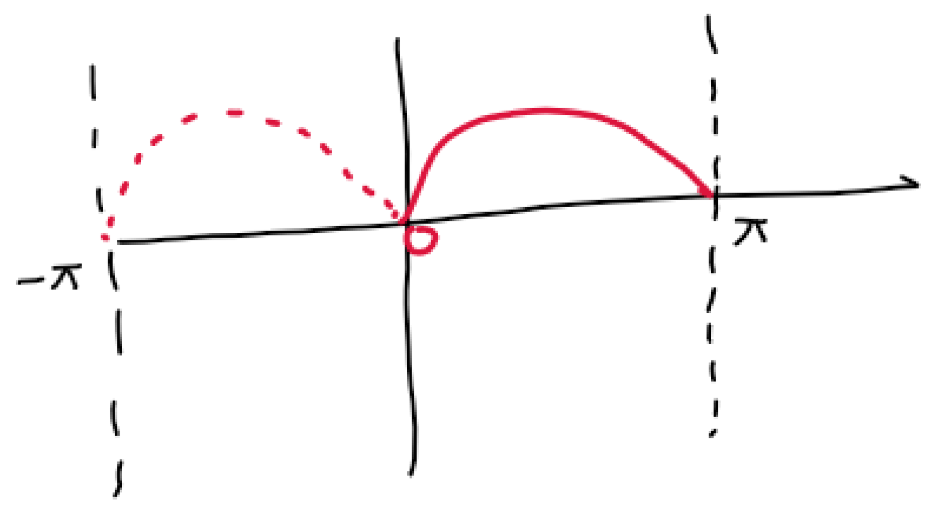
where $a_0 = \frac{2}{c} \int_0^c f(x)\, dx$

$$a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c}\, dx.$$

Exercise:

1) Obtain the half range sine and cosine series expansion of $f(x) = x\sin x$, $0 \le x \le \pi$.

Soln:- Cosine series:



$f(-x) = +f(-x) = (-x)\sin(-x) = x\sin x, \quad -\pi \le x \le 0.$

$f(x) = a_0/2 + \sum a_n \cos\frac{n\pi x}{c}$ where $c =$ U.L. in the given question.

$= \pi.$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\,dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - 1\cdot(\cancel{-\sin x})^{0}\right]_0^\pi$$

$$= 2$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[-x\left(\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right) + 1\cdot\left[\cancel{\frac{\sin(1+n)x}{(1+n)^2} + \frac{\sin(1-n)x}{(1-n)^2}}^{\,0}\right]\right]_0^\pi$$

$$= -\left[\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n}\right] = \frac{2(-1)^n}{1-n^2}, \quad n \ne 1.$$

$$a_1 = \frac{2}{c}\int_0^c f(x)\cos\frac{\pi x}{c}\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

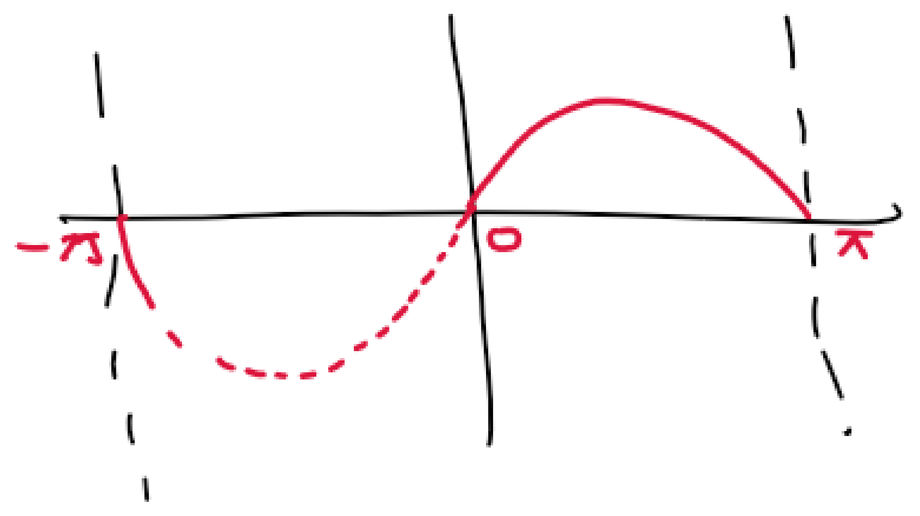
$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\cancel{\left(\frac{-\sin 2x}{4}\right)}^{0}\right]_0^\pi$$

$$= -1/2$$

$$\therefore x\sin x = \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^n}{1-n^2}\cos nx$$

Sine Series:

$f(-x) = -f(-x) = -\left[(-x)(\sin(-x))\right] = -x\sin x.$



$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$ where $c =$ U.L

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left[\cancel{\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}}^{\,0}\right] - 1\cdot\left[\frac{-\cos(1-n)x}{(1-n)^2} + \frac{\cos(1+n)x}{(1+n)^2}\right]\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{1-n} - 1}{(1-n)^2} - \frac{(-1)^{1+n} - 1}{(1+n)^2}\right] = \frac{(-1)^{1+n} - 1}{\pi}\left[\frac{1+n^2+2n-1+2n-n^2}{(1+n)^2(1-n)^2}\right]$$

$$= \frac{4n}{\pi}\frac{\left[(-1)^{1+n} - 1\right]}{(1+n)^2(1-n)^2}, \quad n \ne 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[1-\cos 2x]\, dx$$

$$= \frac{1}{\pi}\left\{ x\left[x - \cancel{\frac{\sin 2x}{2}}\right] - 1\cdot\left[\frac{x^2}{2} + \cancel{\frac{\cos 2x}{4}}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \pi/2$$

$$\therefore x\sin x = \pi/2 \sin x + \sum_{n=2}^{\infty} \frac{4n((-1)^{1+n}-1)}{\pi(1+n)^2(1-n)^2} \sin nx \quad //$$

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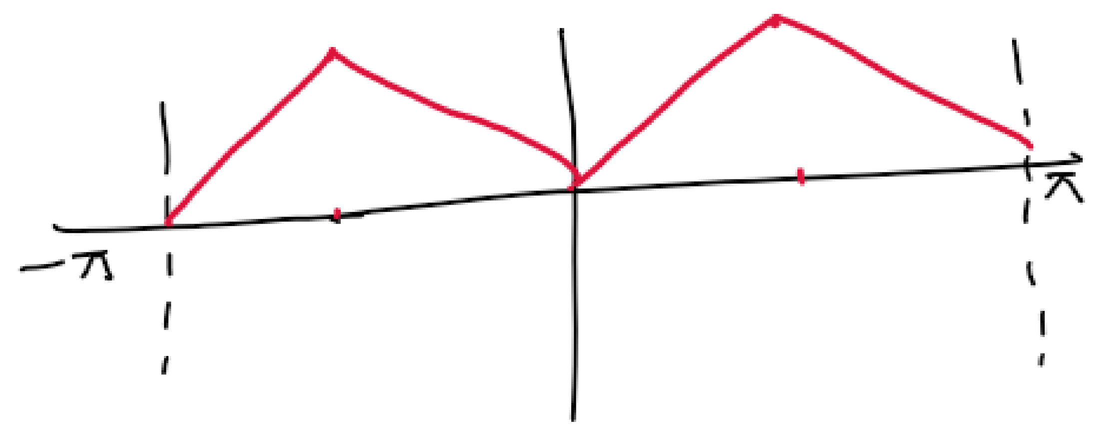
2) $f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$

(1) Cosine Series:

$$f(-x) = +f(-x) = \begin{cases} -x, & -\pi/2 \le x \le 0 \\ \pi + x, & -\pi \le x \le -\pi/2 \end{cases}$$



$f(x) = a_0/2 + \sum a_n \cos \frac{n\pi x}{c}$ where $c = U.L = \pi$.

$$a_0 = \frac{2}{c}\int_0^c f(x)\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi}\cdot\frac{\pi^2}{8}\left[1+8-4-4+1\right] = \pi/2 \, //.$$

$$a_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\pi/2} - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\Big|_0^{\pi/2} + (\pi - x)\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi} - (-1)\left(\frac{-\cos nx}{n^2}\right)\Big|_{\pi/2}^{\pi}\right]$$

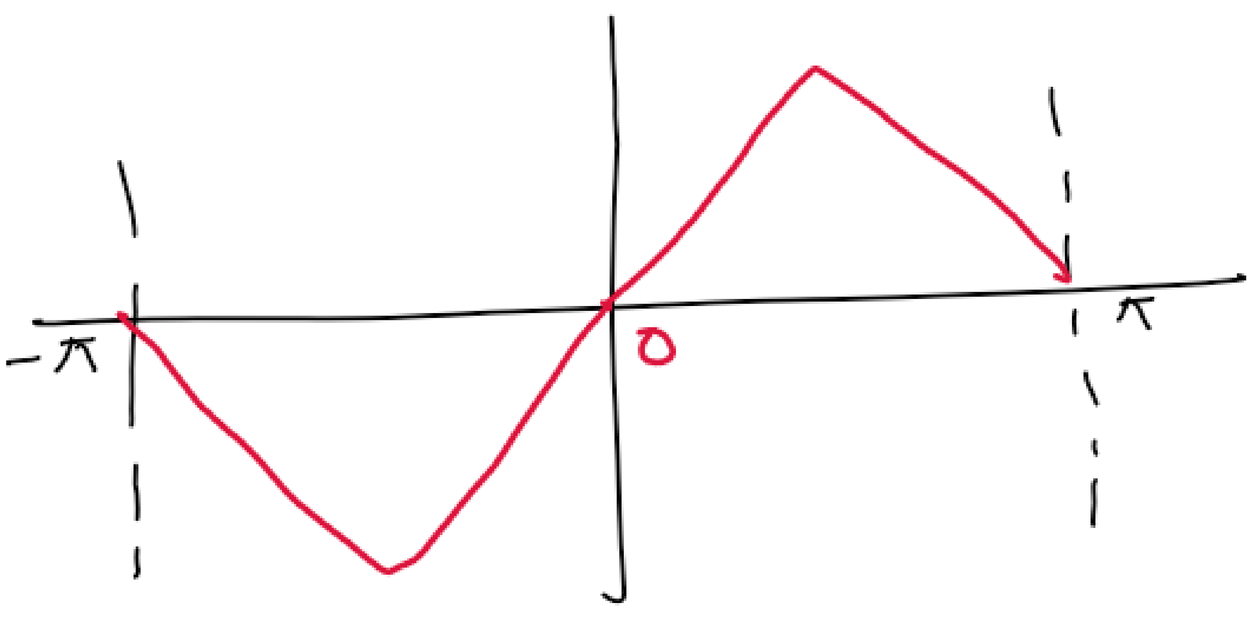
$$= \frac{2}{\pi}\left[\frac{\pi/2 \sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{\pi/2 \sin n\pi/2}{n} - \frac{(-1)^n}{n^2} + \frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos n\pi/2 - 1 - (-1)^n\right]$$

$$f(x) = \frac{1}{2}(\pi/2) + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[2\cos n\pi/2 - 1 - (-1)^n\right] \cos nx.$$

Sine Series:

$$f(-x) = -f(-x) = \begin{cases} x, & -\pi/2 \le x \le 0 \\ -(\pi + x), & -\pi \le x \le -\pi/2 \end{cases}$$



$$f(x) = \sum b_n \sin \frac{n\pi x}{c}$$

$$b_n = \frac{2}{c}\int_0^c f(x) \sin\frac{n\pi x}{c}\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(x\frac{(-\cos nx)}{n} - 1\cdot\frac{(-\sin nx)}{n^2}\right)_0^{\pi/2} + \left(\frac{(\pi-x)(-\cos nx)}{n} - \frac{(-1)(-\sin nx)}{n^2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} + \frac{\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2}\right]$$

$$= \frac{4}{n^2\pi} \sin n\pi/2 .$$

$$\therefore f(x) = \sum \frac{4}{n^2\pi} \sin n\pi/2 \, \sin nx \;//.$$

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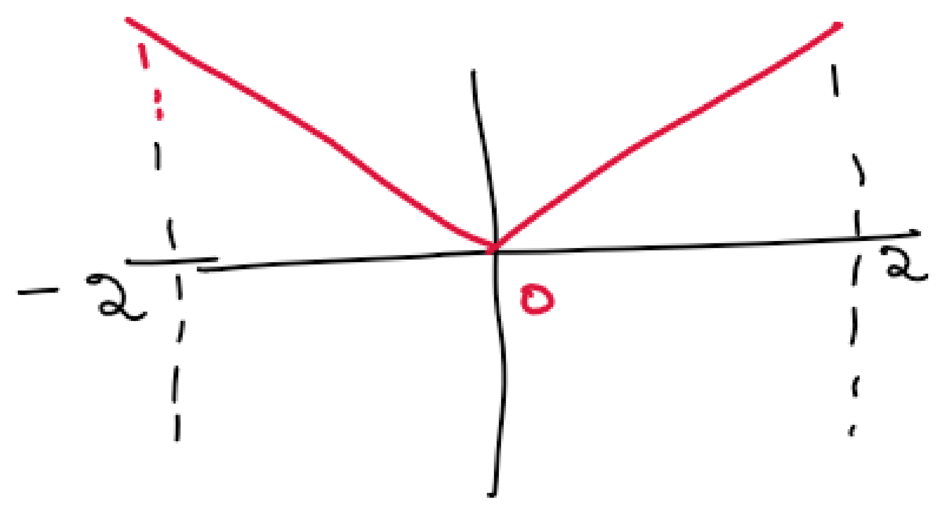
3) $f(x) = x, \quad 0 < x < 2$.

(1) Cosine series:

$$f(-x) = +f(-x) = -x, \quad -2 < x < 0$$



$$f(x) = a_0/2 + \sum a_n \cos\frac{n\pi x}{c} \quad \text{where} \quad c = U.L = 2$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx = 2.$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{2}\int_0^2 x \cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\frac{\cancel{\sin n\pi x/2}}{n\pi/2}^{\,\to 0} - 1\cdot\frac{(-\cos n\pi x/2)}{(n\pi/2)^2}\right]_0^2$$

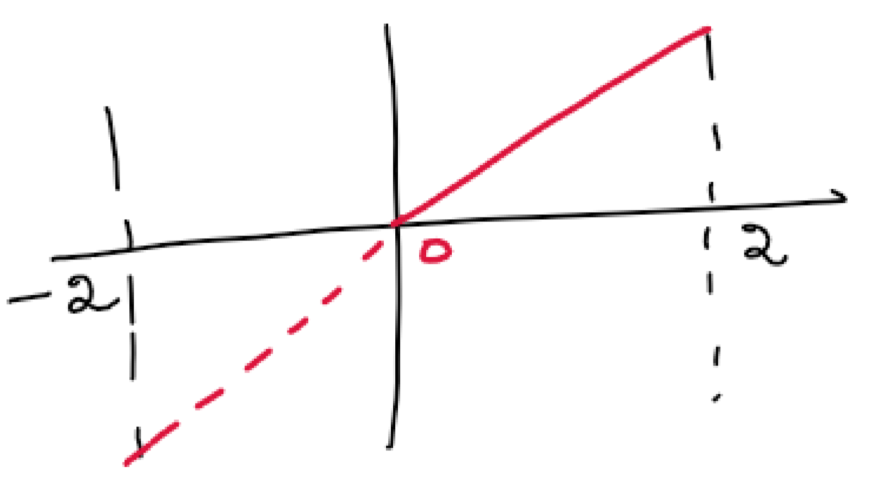
$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$\therefore x = \frac{2}{2} + \sum{}' \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \cdots\cdots\right]$$

Sine Series:

$$f(-x) = -f(-x) = +x, \quad -2 < x < 0$$



$$f(x) = \sum b_n \sin\frac{n\pi x}{c} \quad \text{where } b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{2}\int_0^2 x \sin\frac{n\pi x}{2}\,dx = \left[x\frac{(-\cos\frac{n\pi x}{2})}{n\pi/2} - 1\cdot\frac{\cancel{(-\sin\frac{n\pi x}{2})}}{(n\pi/2)^2}^{\,0}\right]_0^2$$

$$= \frac{2}{n\pi}\left[2(-1)^{1+n}\right]$$

$$\therefore x = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{1+n} \sin\frac{n\pi x}{2} \;//$$

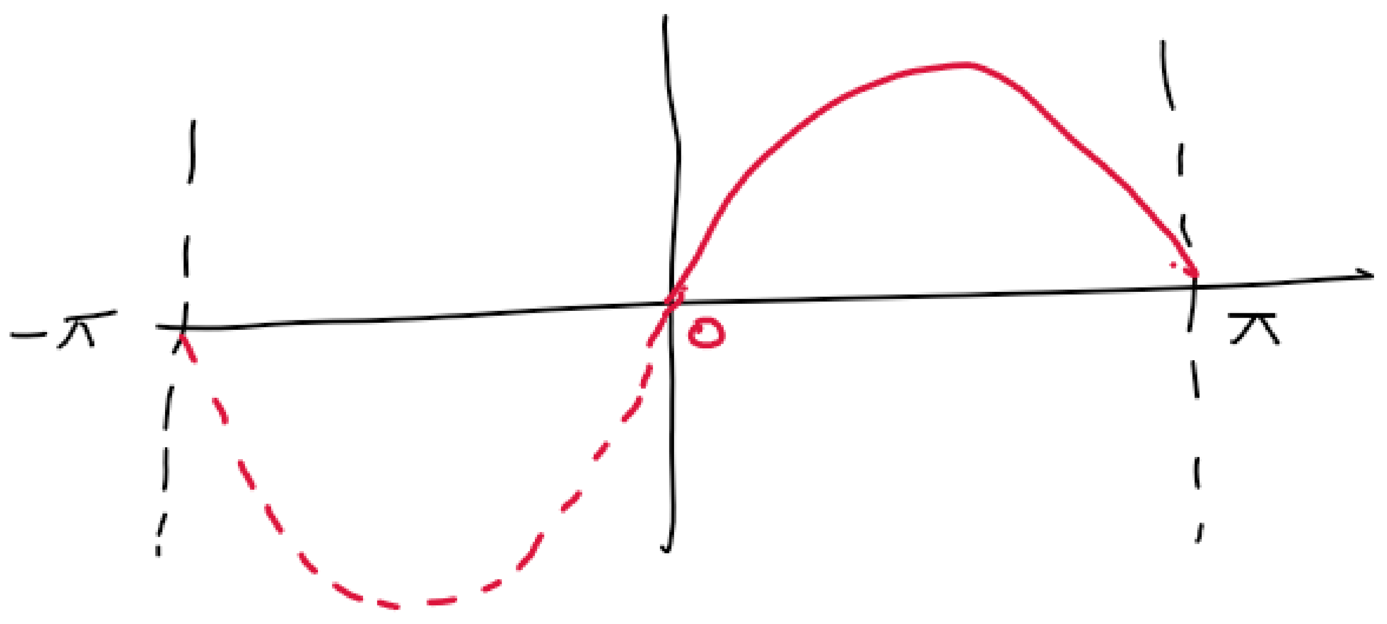
4) obtain the half range Sine Series for the function. $f(x) = |\sin x|$, $0 < x < \pi$. Also draw the periodic continuation of $f(x)$.

Soln:- $f(-x) = -f(-x)$.

$= -(\sin(-x))$

$= \sin x, \quad -\pi < x < 0$

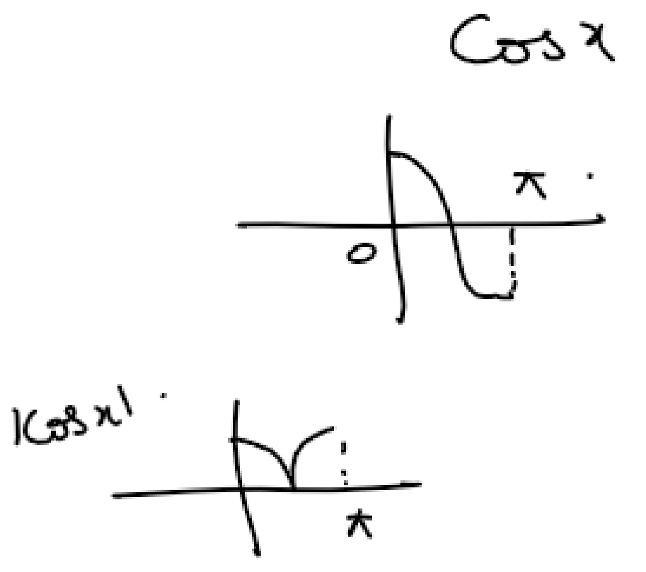
$f(x) = |\sin x|$
$= \sin x, 0 < x < \pi$



$|\cos x| \quad 0 < x < \pi$

$f(x) =$

Then $f(x) = \begin{cases} \cos x, & 0 < x < \pi/2 \\ -\cos x & \pi/2 < x < \pi \end{cases}$



$f(x) = \sum b_n \sin \frac{n\pi x}{c}$ where $c = U \cdot L$

$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx$

$= \frac{2}{\pi} \int_0^\pi \sin x \sin nx \, dx = \frac{2}{\pi} \begin{cases} 0 , & n \neq 1 . \\ \pi/2 , & n = 1 . \end{cases}$

$= \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 . \end{cases}$

D Expand $f(x) = |\cos x|$ as a Fourier Series in $(-\pi, \pi)$

$f(x+2\pi) = f(x)$

Soln:- $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

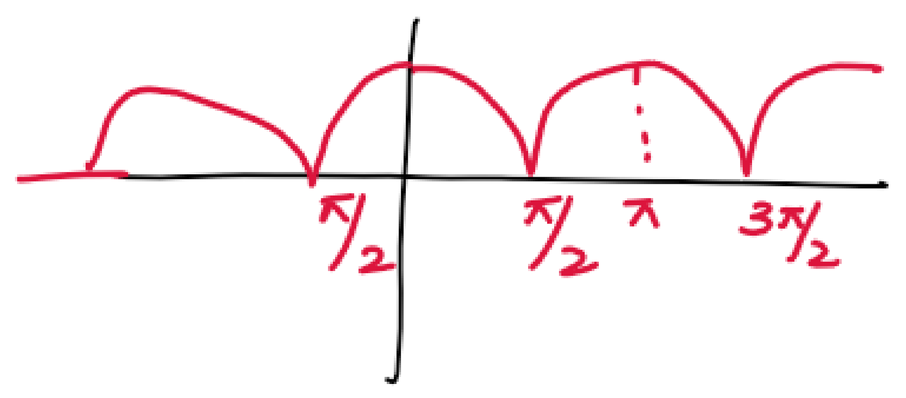
$\therefore f(x)$ is even $b_n = 0$.

But $f(x) > 0$ for all x.

In $(0, \pi)$ $|\cos x| = \begin{cases} \cos x, & 0 < x < \pi/2 \\ -\cos x, & \pi/2 < x < \pi \end{cases}$

$\therefore f(x) = a_0/2 + \sum a_n \cos \frac{n\pi x}{c}$ where $c = \frac{\pi - (-\pi)}{2} = \pi$

$a_0 = \frac{2}{c} \int_0^c f(x) dx =$ $\left(\cancel{} \ \frac{4}{\pi} \int_0^{\pi/2} \cos x \, dx \right)$.



$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx \right] = \frac{4}{\pi}$.

$a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx$

$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx \, dx - \int_{\pi/2}^{\pi} \cos x \cos nx \, dx \right]$

$= \frac{1}{\pi} \left[\int_0^{\pi/2} [\cos(1+n)x + \cos(1-n)x] \, dx \right.$

$\left. - \int_{\pi/2}^{\pi} [\cos(1+n)x + \cos(1-n)x] \, dx \right.$.

$$= \frac{1}{\pi}\left[\frac{\sin(1+n)x}{1+n}\Big|_0^{\pi/2} + \frac{\sin(1-n)x}{1-n}\Big|_0^{\pi/2} - \frac{\sin(1+n)x}{1+n}\Big|_{\pi/2}^{\pi} - \frac{\sin(1-n)x}{1-n}\Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(1+n)\pi/2}{1+n} + \frac{\sin(1-n)\pi/2}{1-n} \right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\pi/2 \cos n\pi/2 + \cancel{\cos\pi/2}\sin n\pi/2}{1+n} + \frac{\sin\pi/2\cos n\pi/2 - \cancel{\cos\pi/2}\sin n\pi/2}{1-n} \right]$$

$$= \frac{2}{\pi}\cos n\pi/2 \left[\frac{2}{1-n^2}\right] = \frac{4}{\pi[1-n^2]}\cos n\pi/2, \; n \neq 1$$

when $n=1$

$$a_1 = \frac{2}{c}\int_0^c f(x)\cos x\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x - \int_{\pi/2}^{\pi}\cos^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}(1+\cos 2x)dx - \int_{\pi/2}^{\pi}(1+\cos 2x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi/2 - \pi + \pi/2\right] = 0 \,//$$

$$\therefore |\cos x| = \frac{1}{2}\left(\frac{4}{\pi}\right) + \sum_{n=2}^{\infty} \frac{4}{\pi(1-n^2)}\cos n\pi/2 \cos nx \,//.$$