Odd and Even Functions:

A function $f(x)$ is said to be even in $(-c, c)$ if $f(-x) = f(x)$.

Eg: $\cos x$, $x \in (-\pi, \pi)$

A function $f(x)$ is said to be odd in $(-c, c)$ if $f(-x) = -f(x)$

Eg:- $\sin x$, $x \in (-\pi, \pi)$

w.k.t

$$\int_{-c}^{c} f(x)\,dx = \begin{cases} 2\int_0^c f(x)\,dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd.} \end{cases}$$

$\therefore$ The F.S. expansion are given by

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{c} + \sum b_n \sin \frac{n\pi x}{c}$$

If $f(x)$ and $g(x)$ are even, then the product $f(x) \cdot g(x)$ is even. When both $f(x)$ and $g(x)$ are odd. Then the product is even. When $f(x)$ is even and $g(x)$ is odd, then the product is odd.

Case(I): When $f(x)$ is even

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{2}{c}\int_0^c f(x)\,dx.$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = 0.$$

Case(II): When $f(x)$ is odd,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = 0.$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = 0.$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

Even and odd nature of $f(x)$ in $(0, 2c)$

$f(x)$ is said to be even like if

$$f(2c-x) = f(x)$$

$f(x)$ is said to be oddlike if

$$f(2c-x) = -f(x)$$

w.k.t

$$\int_0^{2c} f(x)\,dx = \begin{cases} 0, & f(2c-x) = -f(x) \\ 2\int_0^c f(x)\,dx, & f(2c-x) = f(x). \end{cases}$$

Case (I): When $f(x)$ is even like, we have

$b_n = 0$.

$$a_0 = 2\int_0^c f(x)\,dx$$

$$a_n = 2\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Case (II) : when $f(x)$ is odd like,

$$a_0 = 0 = a_n$$

$$b_n = 2\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

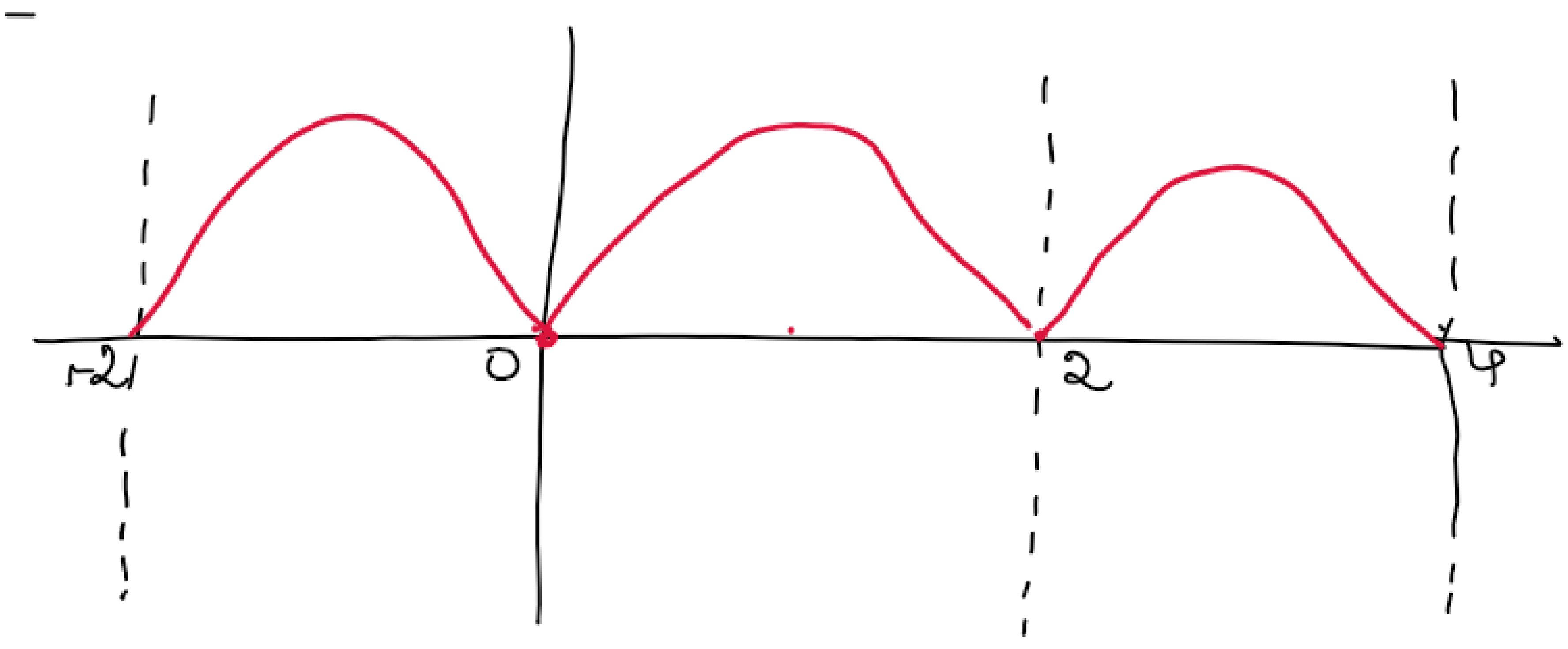
The graph of even (or even like) function is symmetric about y-axis (1^{st} & 2^{nd} quadrant same shape of the curve present).

The graph of odd (or odd like) function is symmetric about origin (1^{st} & 3^{rd} quadrant same shape of the curve present).

Exercise:

1) Find the Fourier Series representation of $f(x) = (2-x)x$, $0 \le x \le 2$, $f(x+2) = f(x)$. Deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$.

Soln:-



curve is symmetric about y-axis $\therefore$ fn is even

$b_n = 0$.

OR Since $f(x)$ is given in 0 to 2, Replace x by $2-x$, we get.

$$f(2-x) = (2-(2-x))(2-x)$$
$$= x(2-x)$$
$$= f(x)$$

– Even like $\therefore b_n = 0$

F.S representation of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{c} \quad \text{where } c = \frac{2-0}{2} = 1$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 (2-x)x\,dx$$

$$= 2\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 (2-x)x\cos n\pi x\,dx$$

$$= 2\left[\cancel{(2x-x^2)\frac{\sin n\pi x}{n\pi}}^{\,0} - (2-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\cancel{\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)}^{\,0}\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[0 - (2-0)\cos 0\right]$$

$$= \frac{-4}{n^2\pi^2}$$

$\therefore$ F.S rep of $f(x)$ is

$$(2-x)x = \frac{1}{2}\left(\frac{4}{3}\right) - \frac{4}{\pi^2}\sum \frac{\cos n\pi x}{n^2}$$

$$= \frac{2}{3} - \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \dots\right]$$

Put $x = 1$

$$1 = \frac{2}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \dots\dots = \frac{\pi^2}{4}\left[1 - \frac{2}{3}\right] = \frac{\pi^2}{12}$$

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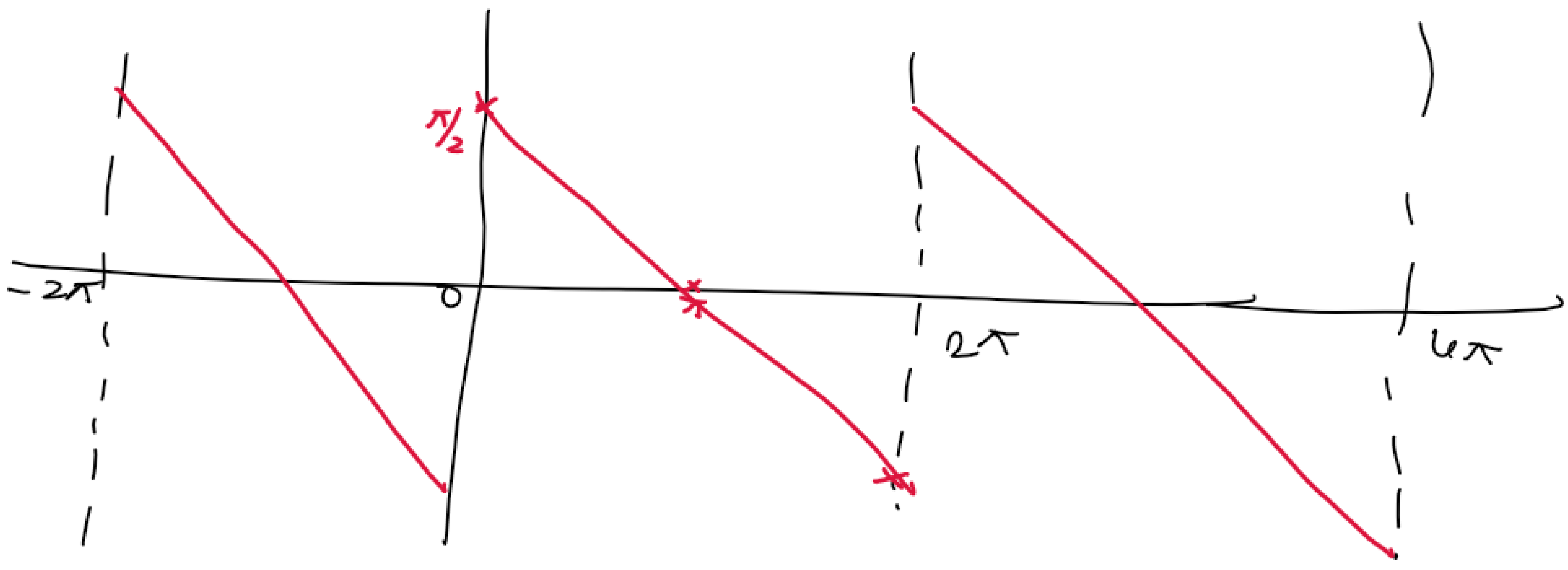
2) $f(x) = \frac{\pi - x}{2}$ in $(0, 2\pi)$ $\quad f(x+2\pi) = f(x)$.



The curve is symmetric about origin.

$\therefore$ Function is odd

$$a_0 = 0 = a_n.$$

Also $f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{-\pi + x}{2} = -\frac{(\pi - x)}{2}$

$$= -f(x).$$

— odd like.

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx \qquad c = \frac{2\pi - 0}{2} = \pi$$

$$= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi - x}{2}\right)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi - x}{2}\right)\left(\frac{-\cos nx}{n}\right) - \cancel{\left(-\frac{1}{2}\right)\left(\frac{-\sin nx}{n^2}\right)}\right]_0^\pi$$

$$= -\frac{1}{\pi n}\left[0 - (\pi\cos 0)\right] = \frac{1}{n}.$$

$$\therefore \left(\frac{\pi - x}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

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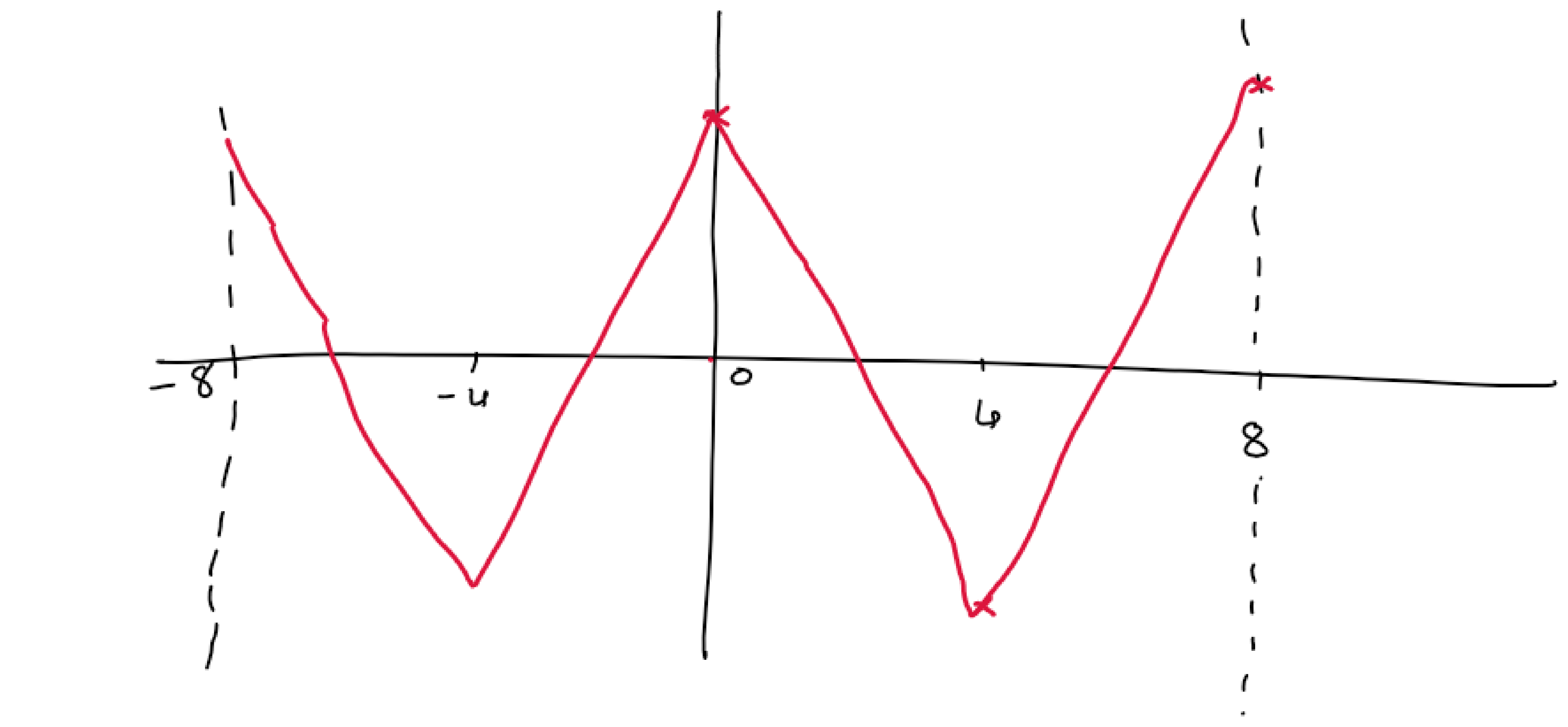
3) $f(x) = \begin{cases} 2 - x, & 0 < x < 4 \\ x - 6, & 4 < x < 8 \end{cases} \qquad f(x+8) = f(x).$



Curve is sym about y axis $\therefore f(x)$ is even

$b_n = 0$

OR $\quad f(8-x) = \begin{cases} 2-(8-x), & 0 < 8-x < 4 \\ (8-x)-6, & 4 < 8-x < 8 \end{cases}$

$= \begin{cases} -6+x, & -8 < -x < -4 \\ 2-x, & -4 < -x < 0 \end{cases}$

$= \begin{cases} x-6 & 4 < x < 8 \\ 2-x, & 0 < x < 4. \end{cases}$

$= f(x)$, even like.

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx \quad \text{where } c = \frac{8-0}{2} = 4$$

$$= \frac{2}{4}\int_0^4 (2-x)\,dx = \frac{2}{4}\int_4^8 (x-6)\,dx$$

$$= \frac{1}{2}\left[2x - \frac{x^2}{2}\right]_0^4$$

$$= 0$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{4}\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{2}\left[\cancel{(2-x)\sin\frac{n\pi x}{4}\left(\frac{4}{n\pi}\right)}^{\,0} - (-1)\left(\frac{4}{n\pi}\right)^2\left(-\cos\frac{n\pi x}{4}\right)\right]_0^4$$

$$= -\frac{1}{2}\,\frac{16}{n^2\pi^2}(\cos n\pi - 1) = \frac{8}{n^2\pi^2}\left(1-(-1)^n\right).$$

$$= \begin{cases} \dfrac{16}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even}. \end{cases}$$

$$f(x) = 0 + \sum_{n=1,3,5,\ldots} \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}$$

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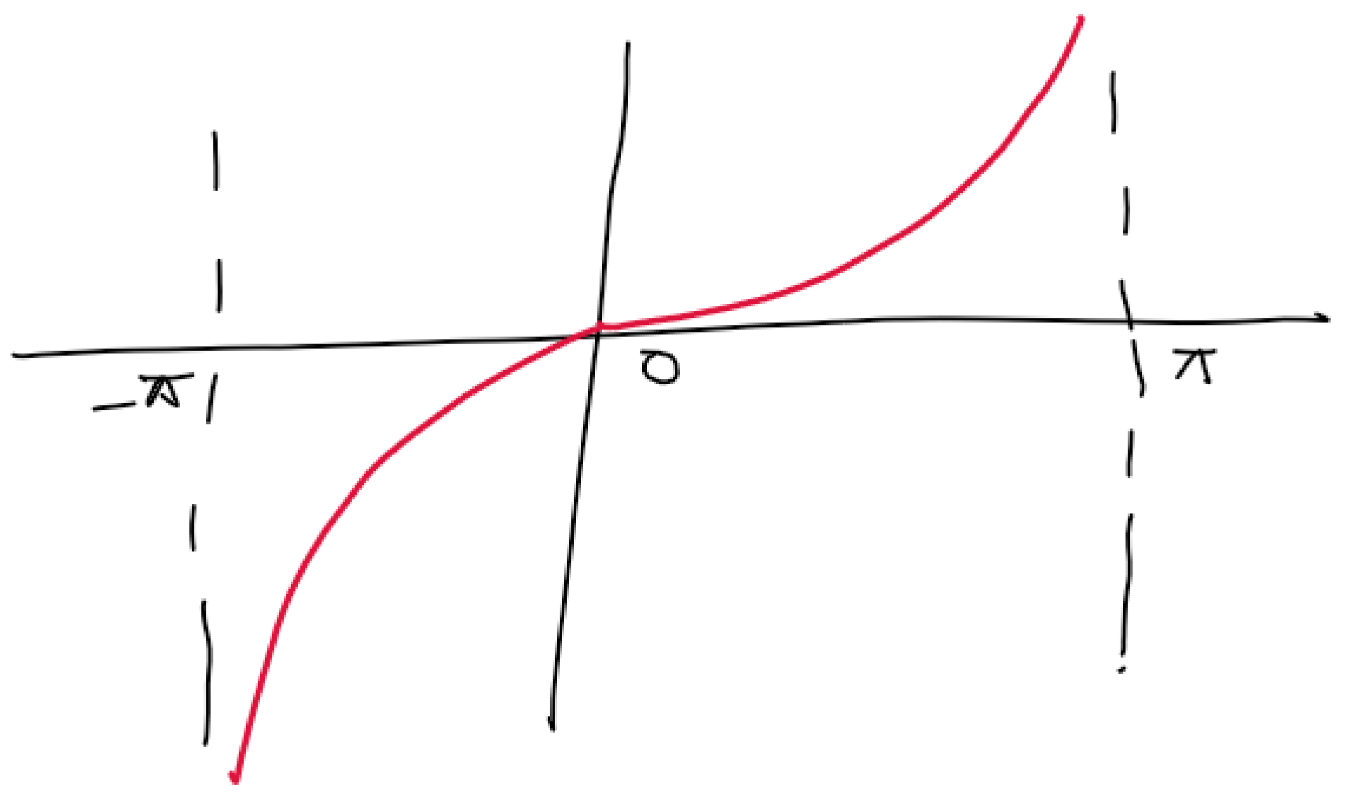
4) $f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$ $\qquad f(x+2\pi) = f(x).$



$f(x)$ is odd, $a_0 = a_n = 0$.

$$f(-x) = \begin{cases} (-x)^2, & 0 \le -x \le \pi \\ -(-x)^2, & -\pi \le -x \le 0 \end{cases}$$

$$= \begin{cases} x^2, & 0 \ge x \ge -\pi \\ -x^2, & \pi \ge x \ge 0 \end{cases}$$

$$= -\begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases} = -f(x).$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx = \frac{2}{\pi}\int_{-\pi}^{0}(-x^2)\sin nx\,dx$$

$$= -\frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\cancelto{0}{\left(\frac{-\sin nx}{n^2}\right)} + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{0}$$

$$= -\frac{2}{\pi}\left[0 + \frac{\pi^2\cos n\pi}{n} + \frac{2}{n^3}\left[1 - \cos n\pi\right]\right]$$

$$= -\frac{2\pi\cos n\pi}{n} - \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

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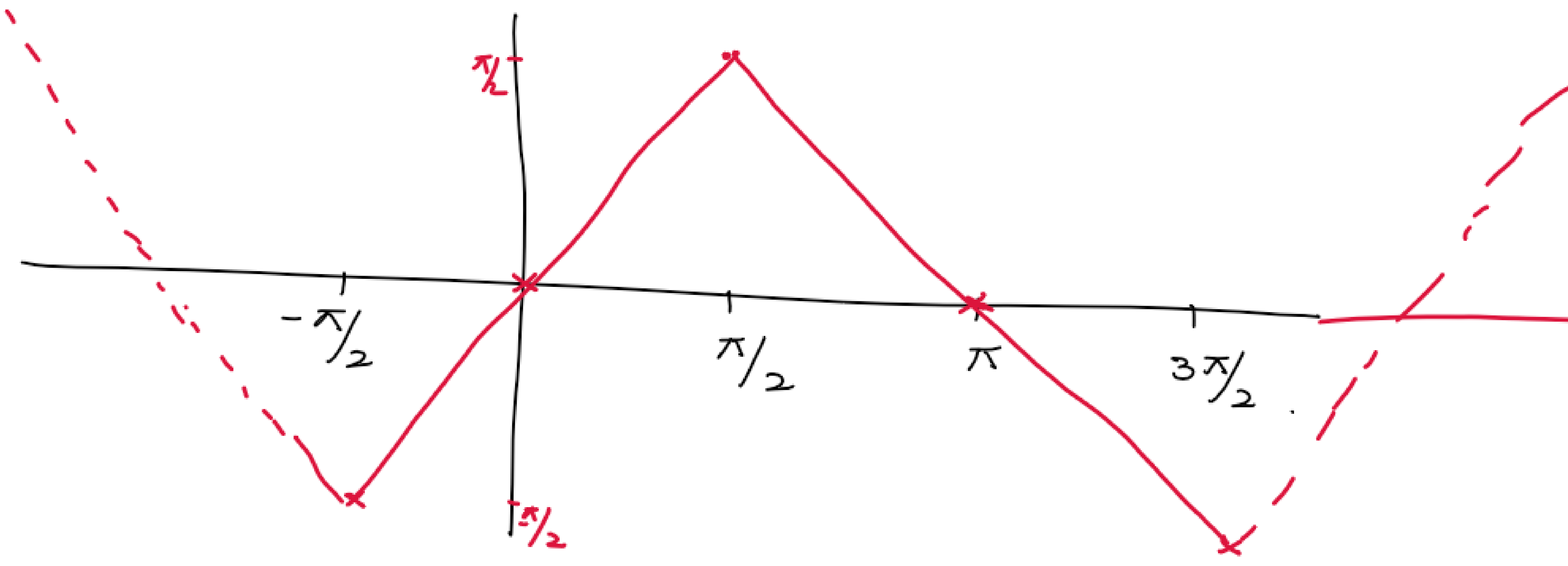
5) $f(x) = \begin{cases} x, & -\pi/2 < x < \pi/2 \\ \pi - x, & \pi/2 < x < 3\pi/2 \end{cases}$ $\quad f(x+2\pi) = f(x)$



$$c = \frac{3\pi/2 + \pi/2}{2} = \pi.$$

Curve is sym about origin. $\therefore f(x)$ is odd fn

$$a_0 = 0 = a_n$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right.$$

$$= \frac{2}{\pi}\left[-\frac{\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} + \frac{\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} \right]$$

$$= \frac{4}{n^2\pi} \sin n\pi/2$$

$$\therefore \; f(x) = \frac{4}{\pi} \sum_{1}^{\infty} \frac{\sin n\pi/2}{n^2} \sin nx$$

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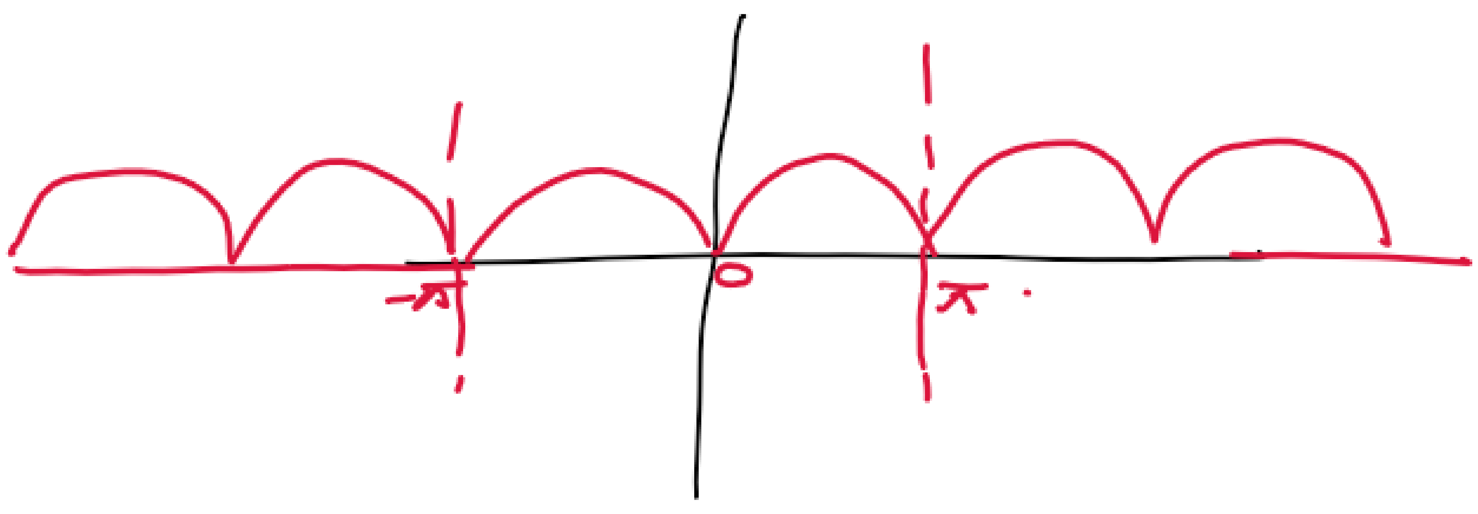
7) $f(x) = x\sin x$, $\quad -\pi \le x \le \pi$, $\quad f(x+2\pi) = f(x)$

Deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$

$= \frac{1}{4}(\pi - 2)$.



Since $x \in (-\pi, \pi)$ Replace x by $-x$.

$$f(-x) = (-x)\sin(-x) = +x\sin x = f(x)$$

$\therefore$ function is even $\quad b_n = 0, \; \forall n$.

[$f(x) = x\sin x$, $x \in (0, 2\pi)$, $f(2\pi - x) \ne f(x)$ neither even nor odd $\quad \therefore$ even though $b_n = 0$, b_1 was not equal to zero]

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\left[x(-\cos x) - 1\cdot\cancel{(-\sin x)}\right]_0^{\pi}$$

$$= -2\cos\pi = 2$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\left(\sin(1+n)x + \sin(1-n)x\right)dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right) + 1\cdot\cancel{\left(\frac{\sin(1+n)x}{(1+n)^2} + \frac{\sin(1-n)x}{(1-n)^2}\right)}\right]_0^{\pi}$$

$$= -\left[\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n}\right]$$

$$= (-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right] = \frac{2(-1)^n}{1-n^2}, \quad n \ne 1$$

Eg:- [$\frac{-1}{2} = \frac{1}{-2} = -\frac{1}{2}$ $\qquad (-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n$]

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{\pi} = -\frac{1}{2}$$

$$\therefore\; x\sin x = \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^n}{1-n^2}\cos nx$$

$$= 1 - \tfrac{1}{2}\cos x + \frac{2}{-3}\cos 2x - \frac{2}{8}\cos 3x + \cdots$$

Put $x = \pi/2$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 0 - \frac{2}{3}(-1) + 0 - \frac{2}{15}(1) + \cdots$$

$$\frac{1}{3} - \frac{1}{15} + \cdots = \left(1 - \frac{\pi}{2}\right)\left(-\frac{1}{2}\right)$$

$$= \frac{1}{4}(\pi - 2)$$