4. S.T. $f(z) = \log z$ is analytic everywhere except at $z = 0$.

$$f(z) = \log z = \log re^{i\theta} = \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1} y/x$$

$\therefore u(x,y) = \frac{1}{2}\log(x^2+y^2)$, $v(x,y) = \tan^{-1} y/x$.

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2} \times 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+y^2/x^2}\left(-y/x^2\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore \log z$ is analytic except at $z=0$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= -e^{-x}\cos y + i\, e^{-x}\sin y$$

$$= -e^{-x}(\cos y - i\sin y)$$

$$= -e^{-x}\cdot e^{-iy} = -e^{-(x+iy)} = \underline{\underline{-e^{-z}}}$$

2. S.T. $f(z) = z^2$ is analytic.

$$f(z) = z^2 = (x+iy)^2 = x^2 + i2xy - y^2$$

$$\therefore u(x,y) = x^2 - y^2, \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(C-R eqns)}$$

$\therefore f(z) = z^2$ is analytic

3. $f(z) = \bar{z} = x - iy$

$$u = x, \quad v = -y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\therefore \bar{z}$ is not analytic

$\therefore f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \longrightarrow (3)$

Since $f(z)$ is analytic, both the values of $f'(z)$ given by (2) and (3) must be the same.

i.e $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Examples

1. S.T. $f(z) = e^{-z}$ is analytic everywhere. Find $f'(z)$

$$f(z) = e^{-z} = e^{-(x+iy)} = e^{-x-iy}$$

$$= e^{-x} \cdot e^{-iy}$$

$$= e^{-x}(\cos y - i\sin y)$$

$$= e^{-x}\cos y - i e^{-x}\sin y$$

$\therefore u(x,y) = e^{-x}\cos y \qquad v(x,y) = -e^{-x}\sin y$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial v}{\partial x} = e^{-x}\sin y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y$

CR eqns satisfied $\therefore e^{-z}$ is analytic.

Proof:- Let $f(z) = u(x,y) + iv(x,y)$ be analytic.

Then $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \longrightarrow (1)$$

exists. $\Delta z = \Delta x + i\Delta y$.

Let $\Delta z \to 0$ along real axis. Then $\Delta x \to 0$, $\Delta y = 0$

$$\therefore f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (2)$$

Let $\Delta z \to 0$ along imaginary axis. Then $\Delta y \to 0$ and $\Delta x = 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u - iv}{i\Delta y}$$

$$= -i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

Analytical functions (Regular functions)

A function $f(z)$ is said to be analytic in a region R if $f(z)$ is defined and differentiable at all points of R.

Ex:- 1. $z, z^2 \ldots$ are Analytic in the entire complex plane.

2. $f(z) = \frac{1}{1-z}$ is analytic everywhere in the complex plane except at $z=1$.

3 $|z|, \bar{z}, |z|^2, |\bar{z}|^2$ are not analytic anywhere in the complex plane.

The necessary and sufficient condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic for all z in a region R are

(1) The partial derivatives of u and v exist

(2) they satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex:-2 Let $f(z) = \bar{z}$

$z = x+iy$
$\bar{z} = x-iy$
$\Delta z = \Delta x + i\Delta y$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{(x+iy+\Delta x+i\Delta y)} - (x-iy)}{\Delta x + i\Delta y}$$

$$= \frac{\overline{(x+\Delta x)+i(y+\Delta y)} - (x-iy)}{\Delta x+i\Delta y}$$

$$= \frac{(x+\Delta x) - i(y+\Delta y) - (x-iy)}{\Delta x + i\Delta y}$$

$$= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$\Delta z \to 0$
$\Delta x + i\Delta y \to 0$

$$\underset{\substack{\Delta x \to 0 \\ \Delta y = 0}}{Lt} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \underset{\Delta x \to 0}{Lt} \frac{\Delta x}{\Delta x} = 1$$

$$\underset{\substack{\Delta y \to 0 \\ \Delta x = 0}}{Lt} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = -1$$

Since both limits are not equal,
$f(z) = \bar{z}$ is not differentiable

Note An open connected set is called a domain

Complex function

A complex function is denoted by

$z = x+iy = (x,y)$

$$W = f(z)$$ where z is a complex variable.

We write $f(z) = u(x,y) + iv(x,y)$

$$u = Re(f) \qquad V = Im(f)$$

A function $f(z)$ is said to be differentiable at a point $z = z_0$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ exists and is denoted by $f'(z_0)$

Example 1 Find the derivative of $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}$$

$$= \underline{\underline{2z}}$$

Open set

A set S is said to open if every point of S has a neighbourhood consisting only of points of S.

Ex:- Open circular disk

$|z-a|=r$

$|z|=1$

$|z-a|<r$

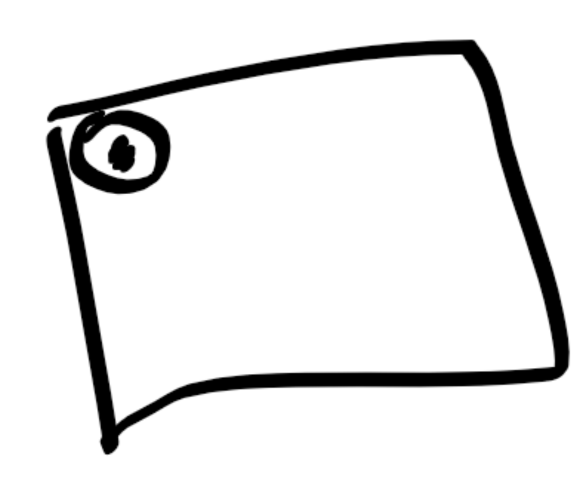
$|z-a|\le r$

$r_1 \le |z-a| \le r_2$

Closed set :- A set S is called closed if its complement is open.

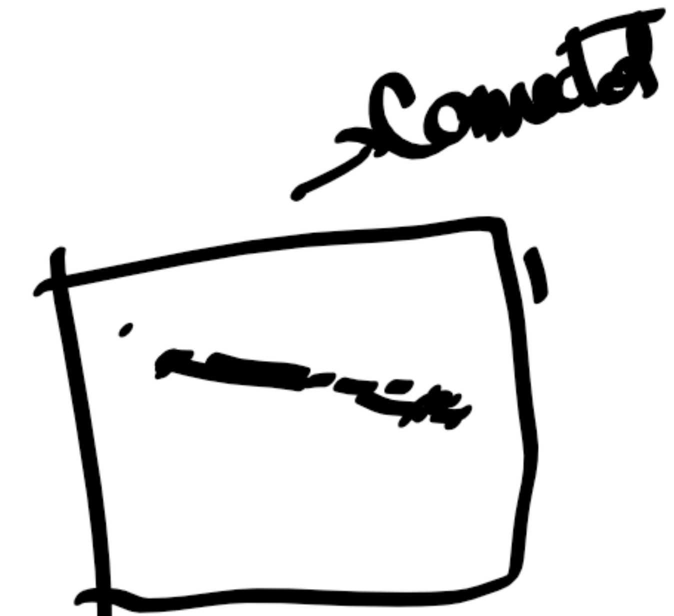
Ex:- closed circular disk.

Bounded set :- A set S is called bounded if all of its points lie within a circle of sufficiently large radius.



Ex.- The points inside a rectangle form a bounded set.

The points on a straight line do not form a bounded set.



Connected set :- An open set S is said to be connected if any two of its points can be joined by a polygonal line of finitely many line segments whose points belong to S