Exercise

1. Let $\vec{A} = (2y+3)i + xzj + (yz-x)k$. Evaluate $\int_C \vec{A}\cdot dr$ along the following paths C:

(i) $x = 2t^2,\ y = t,\ z = t^3$ from $t=0$ to 1.

(ii) the straight lines from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(2,1,1)$

(iii) The straight line joining $(0,0,0)$ and $(2,1,1)$.

2. Suppose $\vec{A} = (4xy - 3x^2z^2)i + 2x^2j - 2x^3zk$.

(i) S.T. $\vec{A}$ is conservative.

(ii) find its scalar potential.

3. Let $\vec{F} = 4xzi - y^2j + yzk$. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2) - 2a^2}{\sqrt{a^2-x^2-y^2}} \, dy\, dx$$

$x = r\cos\theta, \quad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2-r^2}} \, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{\left[3(r^2-a^2) + a^2\right]}{\sqrt{a^2-r^2}} \, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left[-3r\sqrt{a^2-r^2} + \frac{a^2 r}{\sqrt{a^2-r^2}}\right] dr\,d\theta$$

$a^2 - r^2 = t$

$-2r\,dr = dt$

$r\,dr = -\frac{dt}{2}$

$$\int -3r\sqrt{a^2-r^2}\, dr = \frac{3}{2}\int \sqrt{t}\, dt = t^{3/2} = (a^2-r^2)^{3/2}$$

$$\int \frac{a^2 r\, dr}{\sqrt{a^2-r^2}} = -\frac{a^2}{2}\int \frac{1}{\sqrt{t}}\, dt = -a^2\sqrt{a^2-r^2}$$

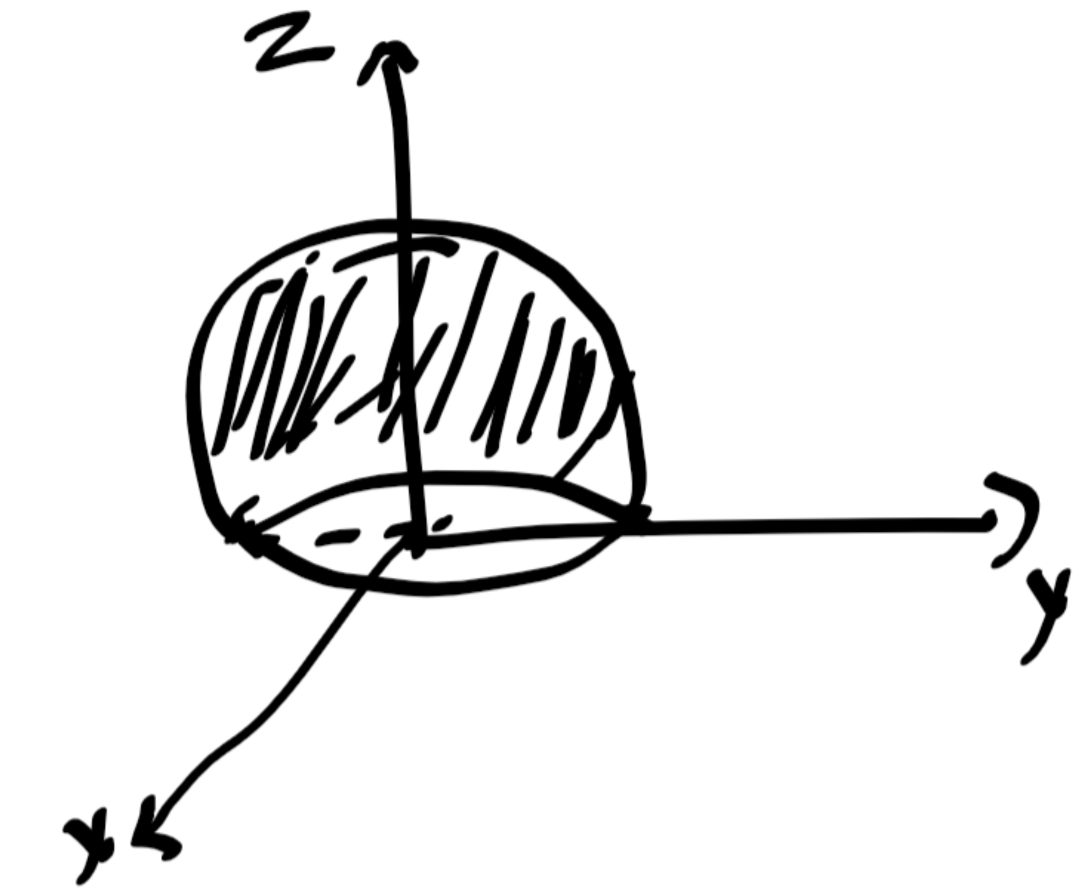
$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^{2\pi} \int_0^{a} \left[(a^2-r^2)^{3/2} - a^2\sqrt{a^2-r^2}\right]_0^a d\theta$$

$$= \int_0^{2\pi} (a^3 - a^3)\, d\theta = \underline{\underline{0}}$$

3. Suppose $\vec{F} = y\,i + (x - 2xz)\,j - xy\,k$.

Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

$$\nabla \times \vec{F} = x\,i + y\,j - 2z\,k$$



$$\hat{n} = \frac{\nabla(x^2+y^2+z^2)}{|\nabla(x^2+y^2+z^2)|}$$

$$= \frac{2x\,i + 2y\,j + 2z\,k}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\,i + y\,j + z\,k}{a}$$

$$\hat{n} \cdot \hat{k} = z/a$$

$$(\nabla \times \vec{F}) \cdot \hat{n} = (x\,i + y\,j - 2z\,k) \cdot \left(\frac{x\,i + y\,j + z\,k}{a}\right)$$

$$= \frac{x^2+y^2-2z^2}{a}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_{R_{xy}} \frac{x^2+y^2-2z^2}{a} \, \frac{dx\,dy}{z/a}$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2-2(a^2-x^2-y^2)}{\sqrt{a^2-x^2-y^2}} \, dy\, dx.$$

$$\hat{n} = \frac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$|\hat{n}\cdot\hat{j}| = y/4.$$

$$\vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2 z k)\cdot\left(\frac{x\hat{i}+y\hat{j}}{4}\right)$$

$$= \frac{xz+xy}{4}$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{R_{xz}} \left(\frac{xz+xy}{4}\right)\frac{dx\,dz}{y/4} = \iint_{R_{xz}} \frac{xz+xy}{y}\,dx\,dz$$

$$x^2+y^2 = 16 \Rightarrow y = \sqrt{16-x^2}$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}} + x\right)dx\,dz$$

$$= \int_{z=0}^{5} -z\sqrt{16-x^2} + \frac{x^2}{2}\Bigg|_0^4\,dz$$

$$= \int_0^5 (0+8+4z)\,dz = 8z+2z^2\Big|_0^5$$

$$= 40+50 = \underline{\underline{90}}$$

$$\int \frac{x}{\sqrt{16-x^2}}\,dx$$

Put $t = 16-x^2$

$dt = -2x\,dx$

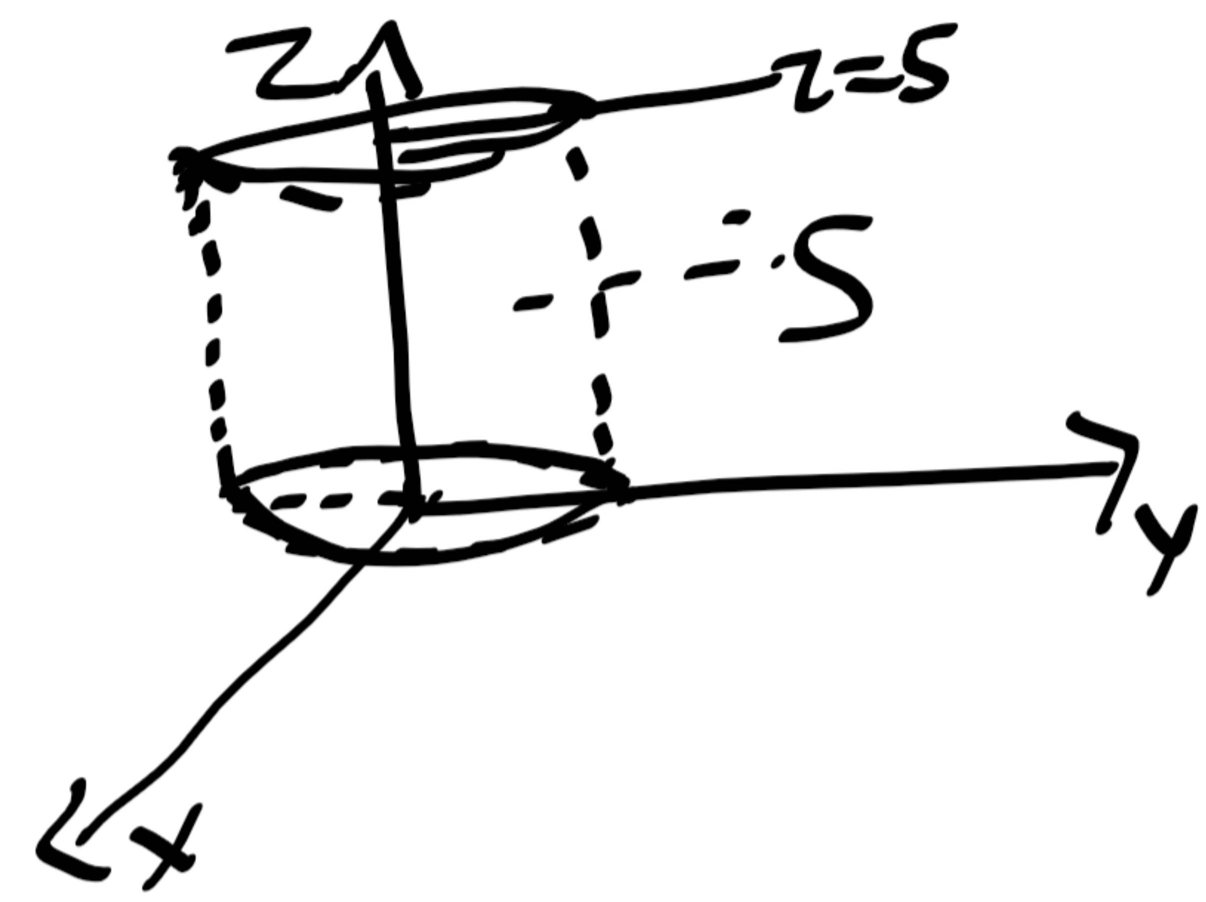
$$-\frac{1}{2}\int\frac{1}{\sqrt{t}}\,dt$$

$$= -\frac{1}{\cancel{2}}\,\cancel{2}\sqrt{t}$$

$$= -\sqrt{16-x^2}$$

2. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = z\,i + x\,j - 3y^2 z\,k$

and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.



Taking projection of S on the xz-plane

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{R_{xz}} \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot j|}$$

$$\hat{n} = \frac{2\hat{i} + 3j + 6k}{\sqrt{4+9+36}} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}k$$

$$|\hat{n}\cdot\hat{k}| = 6/7.$$

$$\vec{A}\cdot\hat{n} = (18z\hat{i} - 12\hat{j} + 3yk)\cdot\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}k\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

$$z = \frac{12 - 2x - 3y}{6}$$

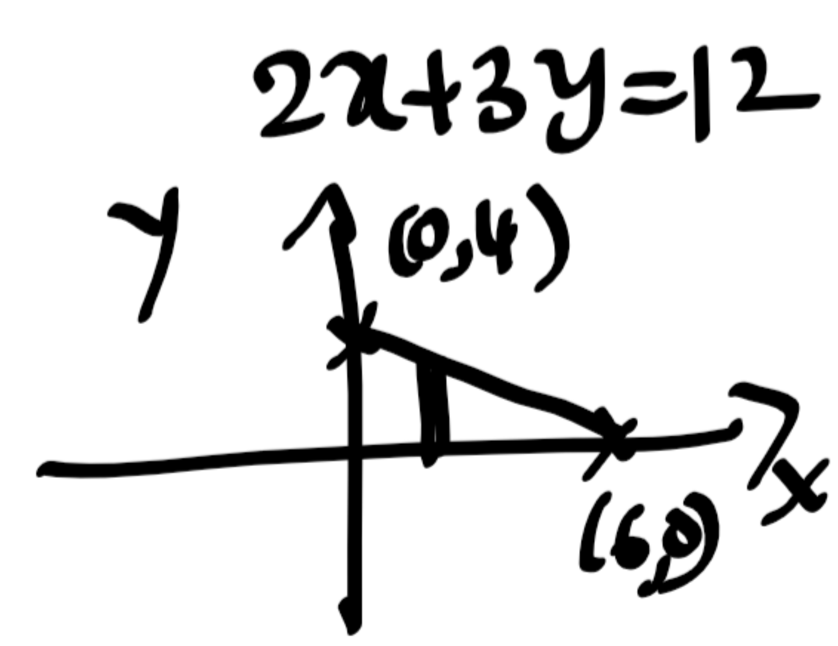
$$\therefore \vec{A}\cdot\hat{n} = \frac{72 - 12x - 18y - 36 + 18y}{7} = \frac{36-12x}{7}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{R_{xy}} \frac{36-12x}{7}\cdot\frac{dx\,dy}{6/7} = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

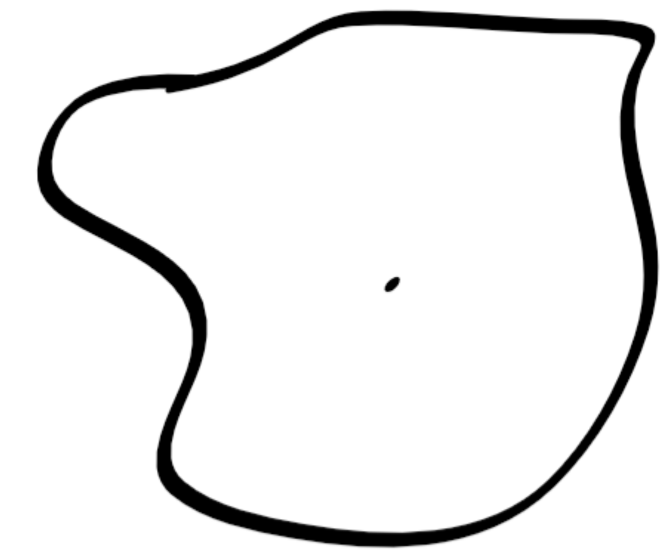
$$= \int_{x=0}^{6} (6-2x)\,y\Big|_0^{\frac{12-2x}{3}} dx = \int_0^6 \left(24 - 12x + \frac{4x^2}{3}\right)dx$$

$$= 24x - 6x^2 + \frac{4x^3}{9}\Big|_0^6$$

$$= 144 - 216 + 96 = \underline{\underline{24}}$$



Surface integrals



S has projection on xy-plane

$$\iint_S \vec{A}\cdot\hat{n}\, dS = \iint_{R_{xy}} \vec{A}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

S has projection on yz-plane

xy
yz
zx

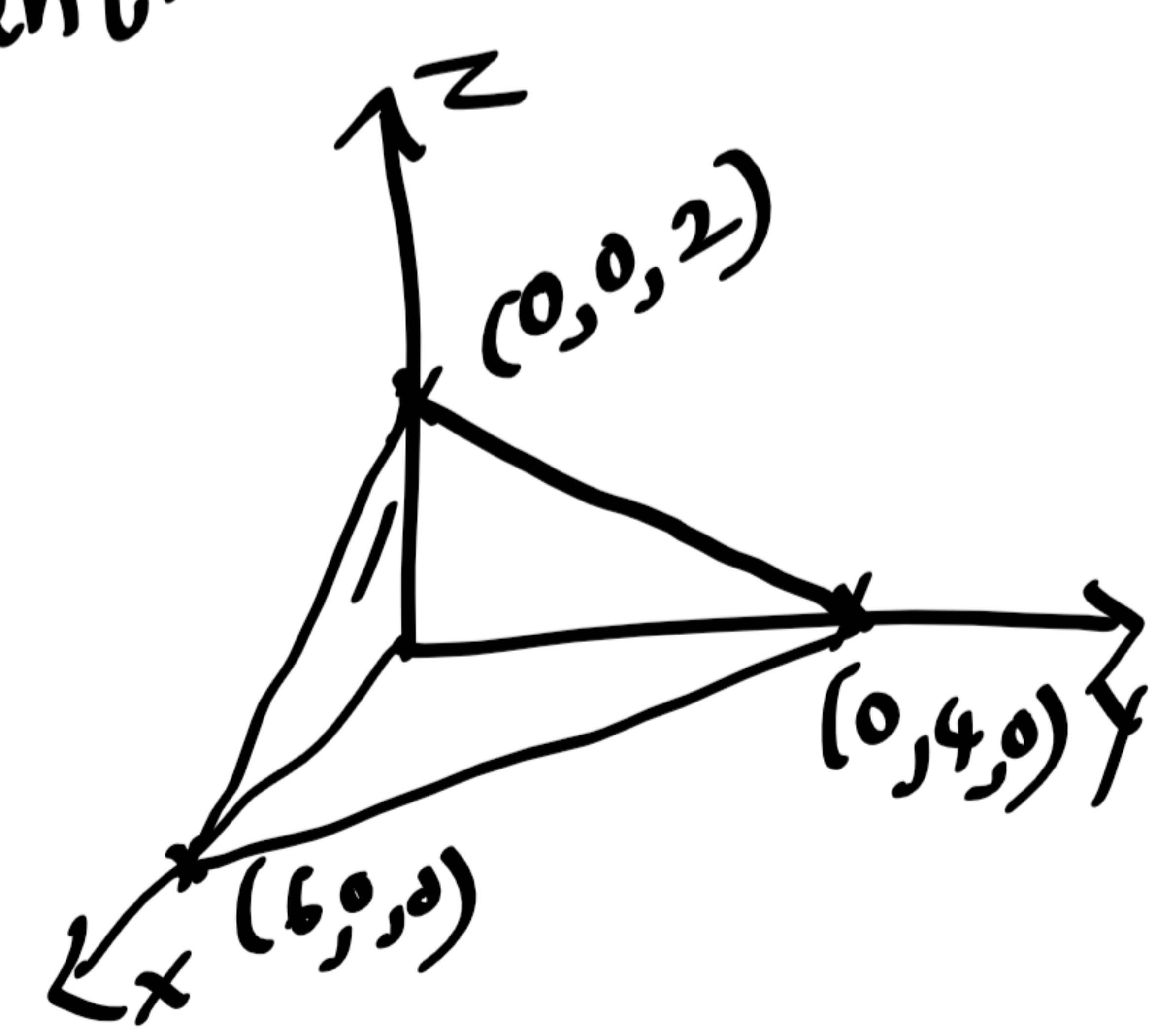
$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_{R_{yz}} \vec{A}\cdot\hat{n}\, \frac{dy\,dz}{|n\cdot\hat{i}|}$$

S has projection on xz-plane

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_{R_{xz}} \vec{A}\cdot\hat{n}\, \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

Examples

1. Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.



$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_{R_{xy}} \vec{A}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n} = \frac{\nabla(2x+3y+6z)}{|\nabla(2x+3y+6z)|}$$

$$\therefore d\varphi = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz$$

$$= (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz)$$

$$+ (-4dy + 2dz)$$

$$= d(y^2\sin x) + d(xz^3) + d(-4y+2z)$$

$$\therefore d\varphi = d(y^2\sin x + xz^3 - 4y + 2z)$$

$$\therefore \varphi(x,y,z) = \int d\varphi = y^2\sin x + xz^3 - 4y + 2z + C$$

(iii) $$\text{Workdone} = \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} \vec{F}\cdot d\vec{r} = \left[y^2\sin x + xz^3 - 4y + 2z\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$$

$$= (1 + 4\pi + 4 + 4) - (-4 - 2)$$

$$= 9 + 4\pi + 6 = \underline{\underline{15 + 4\pi}}$$

Conservative force field and scalar potential

1 (i) P.T $F = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2+2)k$ is a conservative force field.

(ii) Find its scalar potential

(iii) Find the workdone in moving an object in this field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$

(i)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]i - \left[\frac{\partial}{\partial x}(3xz^2+2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]j + \left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]k$$

$$= 0i - (3z^2 - 3z^2)j + (2y\cos x - 2y\cos x)k$$

$$= \underline{\underline{0}}$$

$\therefore \vec{F}$ is a conservative force field.

(ii) since $\vec{F}$ is conservative $\vec{F} = \nabla\phi$.

$$\vec{F}\cdot dr = \nabla\phi \cdot dr = d\phi$$