2) Suppose $\varphi(x, y, z) = 3x^2y - y^2z^2$. Find $\nabla\varphi$ at $(1, -2, -1)$

$$\nabla\varphi = 6xy\,\hat{i} + (3x^2 - 2yz^2)\,\hat{j} - 2y^2z\,k$$

At $(1, -2, -1)$

$$\nabla\varphi = -12\hat{i} + 7\hat{j} + 2k$$

Directional derivative

Consider the scalar function $\varphi(x, y, z)$. Then the directional derivative of ϕ in the direction of a vector $\vec{A}$ is given by $\nabla\varphi \cdot \hat{a}$ where $\hat{a}$ is the unit vector in the direction of $\vec{A}$.

Ex:- Let $\varphi(x, y, z) = x^2 + y^2 + xz$

Find the directional derivative of φ at the point $(2, -1, 3)$ in the direction of $\vec{A} = i + 2j + k$.

$$\nabla\varphi = (2x + z)i + 2yj + xk$$

At $(2, -1, 3)$ $\quad \nabla\varphi = 7i - 2j + 2k$

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{i + 2j + k}{\sqrt{1+4+1}} = \frac{i + 2j + k}{\sqrt{6}}$$

Directional derivative $= \nabla\varphi \cdot \hat{a} = (7i - 2j + 2k) \cdot \left(\frac{i + 2j + k}{\sqrt{6}}\right)$

$$= \frac{7}{\sqrt{6}} - \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} = 5/\sqrt{6}$$

Gradient

The vector differential operator del, written ∇, is defined as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

Let $\varphi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space. (ie, φ defined a differentiable scalar field)

Then the gradient of φ, written $\nabla\phi$ or grad ϕ defined as,

$$\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \varphi(x, y, z)$$

$$= \frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k$$

Note :- $\nabla\varphi$ is a vector field.

Example

1) Suppose $\varphi(x, y, z) = 3xy^3 - y^2 z^2$. Find $\nabla\varphi$ at $(1, 1, 2)$.

$$\nabla\varphi = 3y^3 i + (9xy^2 - 2yz^2) j - 2y^2 z k$$

At $(1, 1, 2)$,

$$\nabla\varphi = 3i + j - 4k.$$

$$\vec{r} = x(s)i + y(s)j + z(s)k$$

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k$$

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

$$= \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$$

$\Rightarrow \frac{d\vec{r}}{ds}$ is a unit vector.

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(dz/dt\right)^2}$$

$\rightarrow$ Derivative of arclength.

Ex:- Suppose a particle P moves along a curve whose parametric equations, where t is time, follows:

$$x = 40t^2 + 8t, \quad y = 2\cos 3t, \quad z = 2\sin 3t$$

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitude of velocity and acceleration at $t=0$.

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (40t^2+8t)\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

(a) $$\vec{V} = \frac{d\vec{r}}{dt} = (80t+8)\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$$

$$\vec{a} = \frac{d^2r}{dt^2} = 80\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}.$$

(b) At $t=0$, $\vec{v} = 8\hat{i} + 6\hat{k}$, $\vec{a} = 80\hat{i} - 18\hat{j}$

$$|\vec{v}| = \sqrt{64+36} = 10, \quad |\vec{a}| = \sqrt{(80)^2 + (-18)^2}$$

$$= \underline{\underline{82}}$$

2) A curve C is defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$ where s is the arc length of C measured from a fixed point on C. If $\vec{r}$ is the position vector at any point on C, show that $\frac{d\vec{r}}{ds}$ is a unit vector tangent to C.

Vector Calculus

Gradient
Divergence
Curl
Vector integration
Both single and double integrals
Triple integrals

$f(x)$ is a real function of one variable

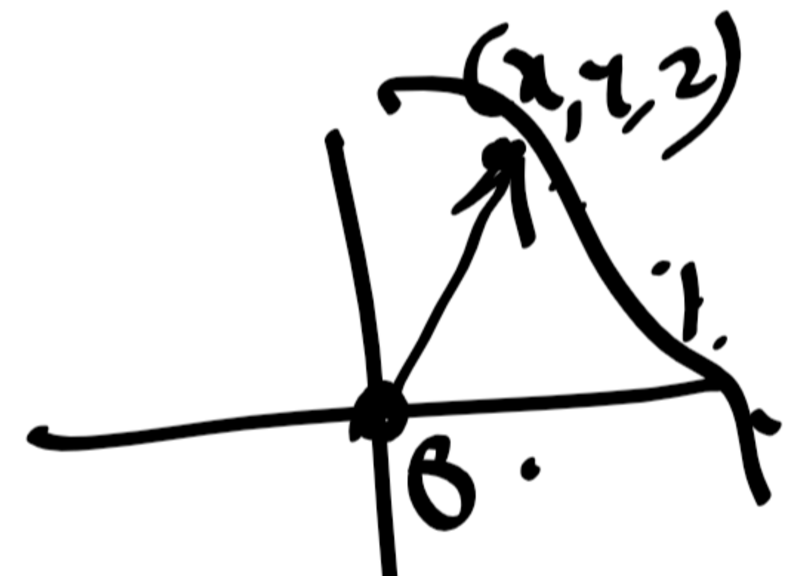
then $f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

If $\vec{R}(u)$ is a vector depending on a scalar u then

$$\frac{d\vec{R}}{du} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

Δu is increment in u.

Let $\vec{r}(u)$ be a vector joining the origin of a co-ordinate system and any point (x, y, z)



$$\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$$

$\frac{d\vec{r}}{du}$ is tangential to the curve with parametric eqns $x = x(u)$, $y = y(u)$, $z = z(u)$

If u denotes time t

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k \rightarrow \text{velocity.}$$

$$\frac{d^2 r}{dt^2} = \frac{d^2 x}{dt^2} i + \frac{d^2 y}{dt^2} j + \frac{d^2 z}{dt^2} k \rightarrow \text{acceleration.}$$

Exercise

Find the nature of the singularities

1) $\frac{\tan z}{z}$ (2) $\frac{1-e^{2z}}{z^4}$ (3) $\frac{1-\cos z}{z^4}$

(4) $\frac{1-\sin z}{z^5}$

Evaluate by Residue theorem

1) $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz$, $C: |z|=1$

2) $\int_C \frac{e^{2z}}{(z+1)^4} dz$, $C: |z|=3$

3) $\int_C \frac{1}{z^4-1} dz$, where C is

(i) $|z+1|=1$, (ii) $|z+3|=1$, (iii) $|z-i|=1$

(iv) $|z-1|=1$

4. $\int_C \frac{z^3-2z+1}{(z-i)^2} dz$, $C: |z|=2$

5. $\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz$, $C: |z|=1$

$$\underset{z=-1-2i}{\operatorname{Res}} f(z) = \lim_{z \to -1-2i} \cancel{(z+1+2i)} \frac{z-3}{(z+1-2i)\cancel{(z+1+2i)}}$$

$$= \frac{-4-2i}{-4i} = \frac{2+i}{2i}$$

$$\int_C f(z)\,dz = 2\pi i\left(\frac{2+i}{2i}\right) = \pi(2+i)$$

(i) $C : |z| = 1$

The points $(-1, \pm 2)$ lie outside the circle $|z| = 1$.

$\therefore$ By Cauchy's integral theorem

$$\int_C \frac{z-3}{z^2+2z+5}\,dz = 0.$$

(ii) $C : |z+1-i| = 2$

centre $= (-1, 1)$

The point $(-1, 2)$ lies inside and $(-1, -2)$ lies outside $|z+1-i| = 2$.

$$\underset{z=-1+2i}{\operatorname{Res} f(z)} = \lim_{z \to -1+2i} \cancel{(z+1-2i)} \frac{z-3}{\cancel{(z+1-2i)}(z+1+2i)}$$

$$= \frac{-4+2i}{4i} = -\frac{2+i}{2i}$$

$$\therefore \int_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i \left(\frac{i-2}{2i}\right) = (i-2)\pi.$$

(iii) $C : |z+1+i| = 2$.

Centre $= (-1, -1)$

$(-1, 2)$ lies outside & $(-1, -2)$ lies inside the circle

$(0,0), (-1, 2)$

$\sqrt{1+4} = \sqrt{5}$

$(0,0), (-1,-2)$

Distance $\sqrt{5}$

$(-1,1), (-1,2)$

Dist $= \sqrt{0+1}$

$= 1$

$(-1,1), (-1,-2)$

$D = \sqrt{0+9} = 3$

$(-1,-1), (-1,2)$

$D = \sqrt{0+9} = 3$

$(-1,-1), (-1,-2)$

$D = \sqrt{0+1} = 1$

Residue theorem

1. Evaluate $\int_C \sec z\,dz$ where $C: |z| = 2$.

$$f(z) = \sec z = \frac{1}{\cos z} \quad \left(\frac{P(z)}{Q(z)}\right)$$

$f(z)$ is not analytic at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$z = \pm \pi/2$ lie inside the circle $|z| = 2$.

$$\underset{z=\pi/2}{\operatorname{Res}} f(z) = \left.\frac{1}{-\sin z}\right|_{z=\pi/2} = -1.$$

$$\underset{z=-\pi/2}{\operatorname{Res}} f(z) = 1.$$

$$\therefore \int_C \sec z\,dz = 2\pi i\,(1-1) = \underline{\underline{0}}$$

2) Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$ where C is the circle

(i) $|z| = 1$, (ii) $|z+1-i| = 2$ (iii) $|z+1+i| = 2$

$f(z) = \frac{z-3}{z^2+2z+5}$ is not analytic

at $z = -1 \pm 2i$.

$-1+2i = (-1, 2)$

$-1-2i = (-1, -2)$

$$z^2+2z+5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$