Fourier Transforms

Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Integral transforms

$$\int_a^b \underbrace{K(s,t)}_{\text{Kernel.}} f(t)\,dt$$

Fourier transform

Definition

F.T of a function $f(t)$ is defined as

$$F(s) = F\left(f(t)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

and $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-ist}\, ds$ is called inverse Fourier transform.

Properties

1. $F\left(af(t) + bg(t)\right) = aF(s) + bG(s)$

   where $F(s) = F\left(f(t)\right)$

   $G(s) = F\left(g(t)\right)$

2. $F\left(f(t-a)\right) = e^{ias} F(s)$

3. $F\left(e^{iat} f(t)\right) = F(s+a)$

4. $F\left(f(at)\right) = \frac{1}{|a|} F(s/a), \quad a \neq 0$

5. $F\left(t^n f(t)\right) = (-i)^n \frac{d^n}{ds^n} F(s)$

6. $F\left(\frac{d^n}{dt^n} f(x)\right) = (-is)^n F(s)$

Parseval's identity

A function $f(t)$ and its transform satisfy the identity

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(s)|^2 \, ds .$$

Examples

1. Find the F.T. of $f(x)$ defined by

$$f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$$

$-\infty \ldots \quad a \quad b \quad \infty$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{a} 0 \cdot e^{isx} dx + \int_{a}^{b} 1 \cdot e^{isx} dx + \int_{b}^{\infty} 0 \cdot e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{a}^{b}$$

$$= \frac{-i}{s\sqrt{2\pi}} \left(e^{isb} - e^{isa} \right)$$

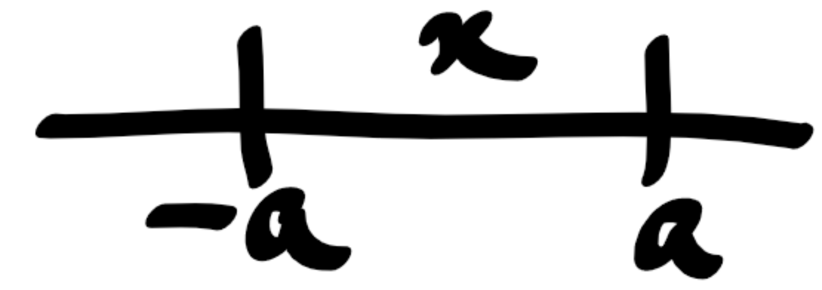
$$\frac{1}{i} = \frac{i}{i^2} = -i$$

2. Find the F.T. of $f(x)$ if

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$|x| < a$

$\Rightarrow -a < x < a$



$|x| > a$

$\Rightarrow x \notin (-a, a)$

Deduce that

(i) $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

(ii) $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

(iii) $$\int_0^\infty \frac{\sin as \cos sx}{s}\,ds = \begin{cases} \frac{\pi}{2}, & |x| < a \\ \pi/4, & |x| = a \\ 0, & |x| > a \end{cases}$$

solution

$$\hat{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} 0 \cdot e^{isx}\, dx + \int_{-a}^{a} 1 \cdot e^{isx}\, dx + \int_{a}^{\infty} 0 \cdot e^{isx}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\underbrace{\cos sx}_{\text{even}} + i \underbrace{\sin sx}_{\text{odd}})\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{a} \cos sx\, dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin sx}{s}\Big|_{0}^{a}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin as}{s}$$

(iii) Using inverse F.T

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin as}{s} (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi}\, 2 \int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds$$

$$\int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} \times 1, \quad |x| < a$$

$$\int_0^\infty \frac{\sin as \cos sx}{s}\, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2}\times 1 = \frac{\pi}{2}, \quad |x|<a$$

$$\frac{\pi}{2}\times 0 = 0, \quad |x|>a$$

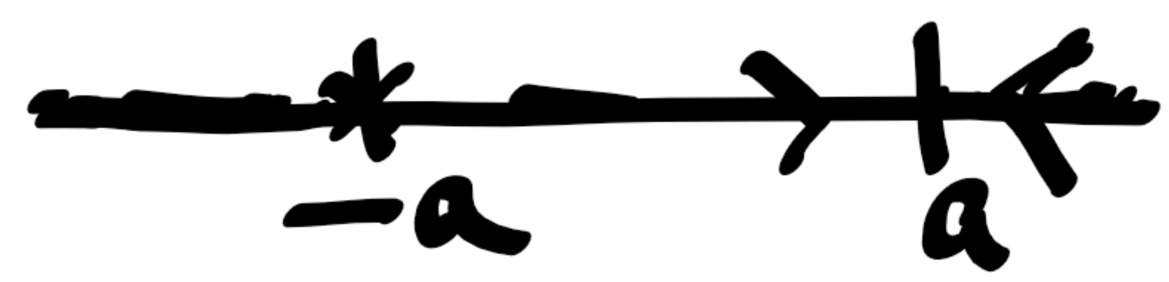
$$\frac{\pi}{2}\times \frac{1}{2} = \frac{\pi}{4}, \quad |x|=a$$

$|x|=a$

$\Rightarrow x = \pm a$

$f(x)$ at $x=a$ is

$$\frac{f(a_+)+f(a_-)}{2} = \frac{0+1}{2}$$



(ii) Using Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

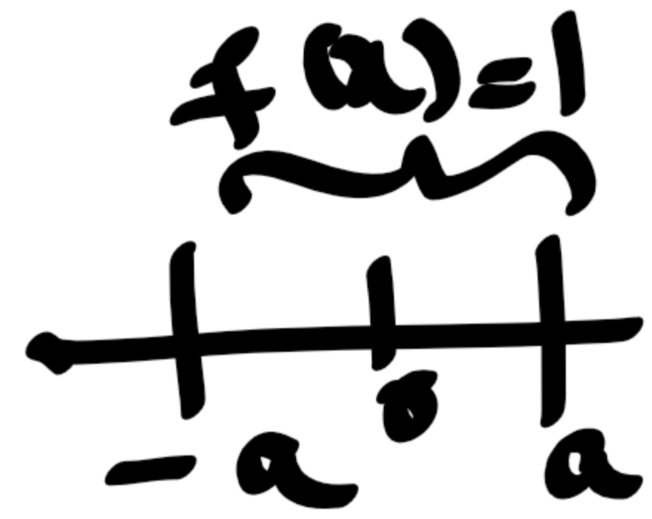
$$\int_{-a}^{a} 1\cdot dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin as}{s}\right)^2 ds$$

$$\cancel{2}a = \frac{\cancel{2}}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$a\pi = 2\int_0^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi}{2}a$, Put $as = t$, $\cancel{a}\int_0^\infty \left(\frac{\sin t}{t}\right)^2 \frac{dt}{\cancel{a}} = \frac{\pi}{2}a$

(i) $\displaystyle\int_0^\infty \frac{\sin as \cos sx}{s}\, ds = \frac{\pi}{2} f(x)$

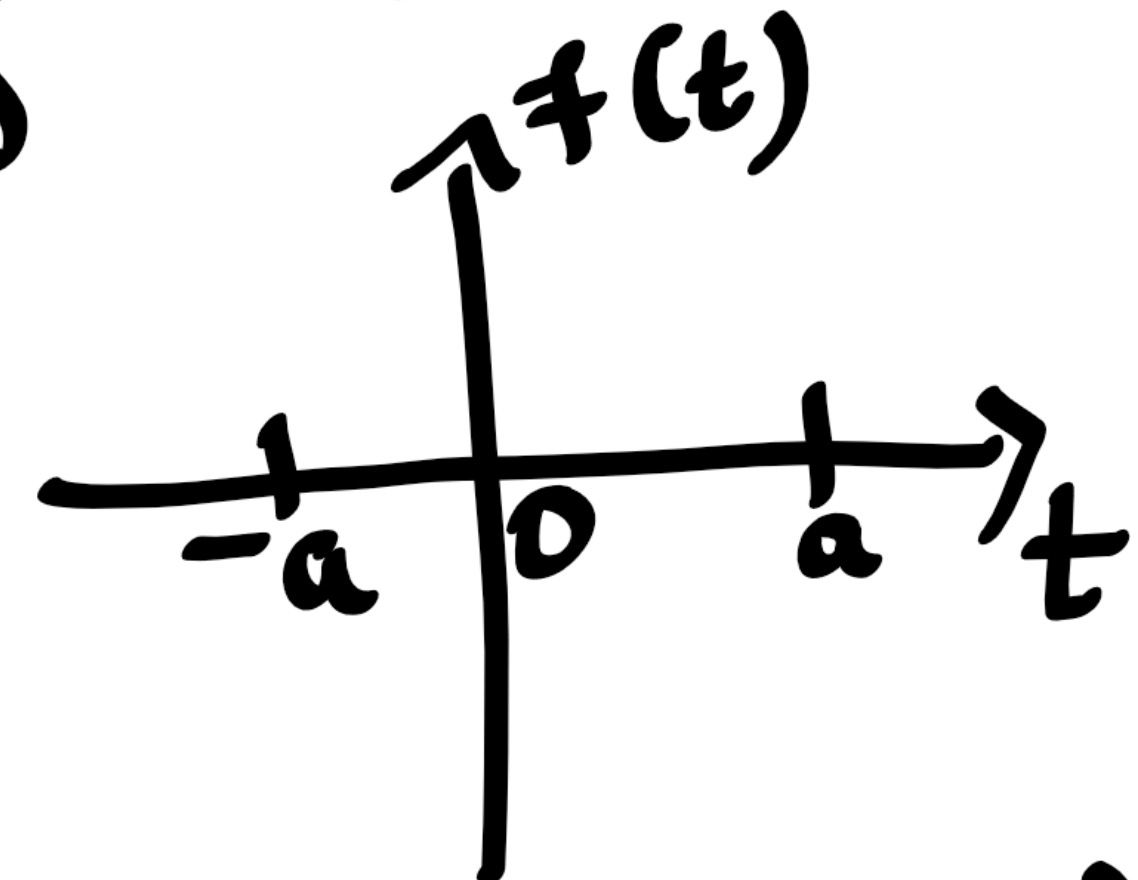


Put $x = 0$

$$\int_0^\infty \frac{\sin as}{s}\, ds = \frac{\pi}{2} f(0) = \frac{\pi}{2} \times 1 = \frac{\pi}{2}$$

Put $as = t$ $\quad ds = dt/a$

$$\int_0^\infty \frac{\sin t}{t/a}\, dt/a = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$



$x \in (-a, a)$

3. Find the F.T. of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right) \right\}_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{\cos s}{s^2} + \frac{1}{s^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right)$$