Four Different Fourier Representation

Time Domain	Periodic (t, n)	Non periodic (t, n)	
Continuous (t)	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_o t}\, dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$	Nonperiodic (k, ω)
Discrete (n)	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n}\, d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	Periodic (k, Ω)
	Discrete (k)	Continuous (ω, Ω)	Frequency Domain

DFT to the rescue!

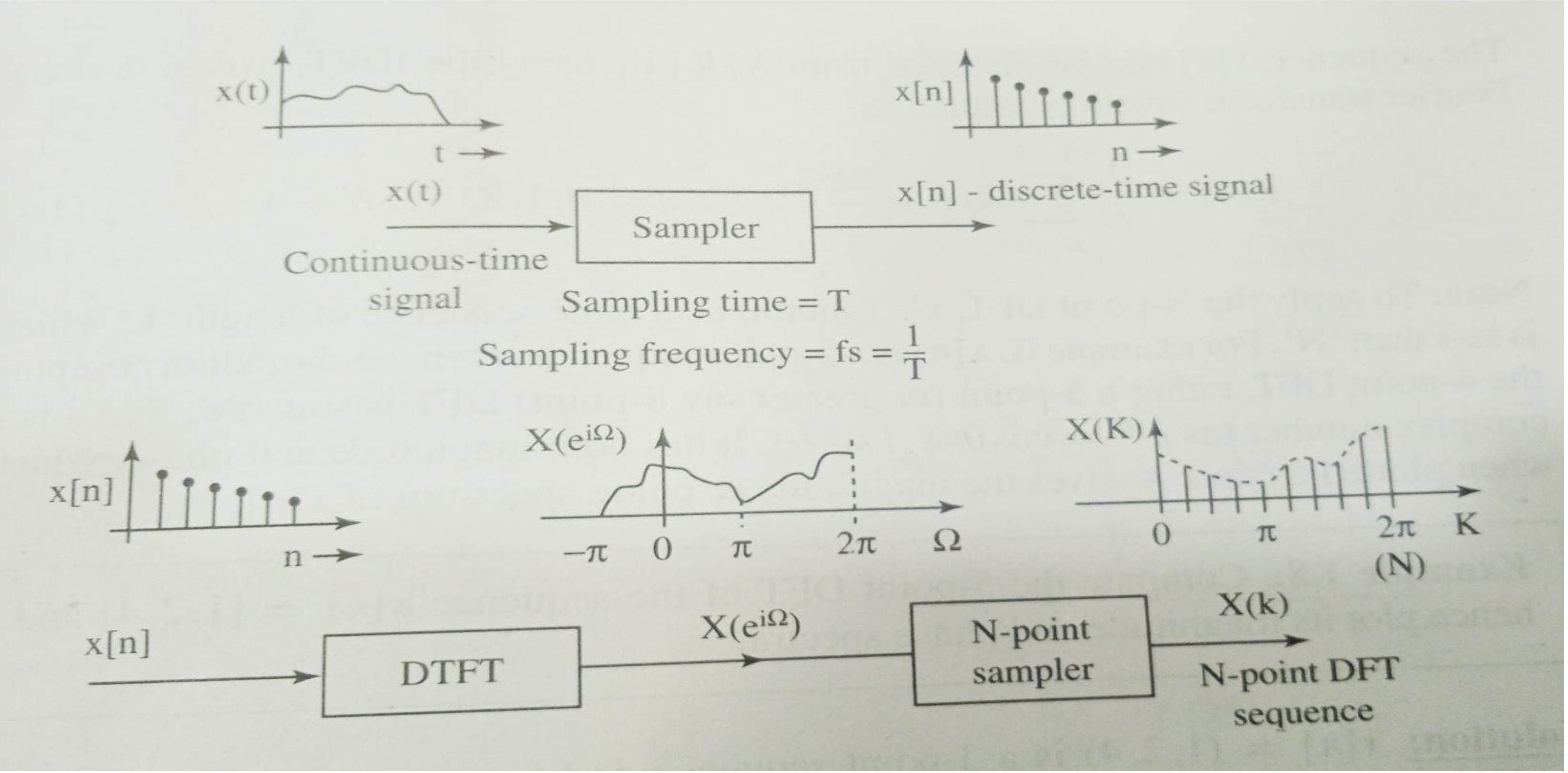
Could we calculate the **frequency spectrum** of a signal using a **digital computer** with **CTFT/DTFT**?

- Both CTFT and DTFT produce continuous function of frequency → can't calculate an infinite continuum of frequencies using a computer

- Most real-world data is not in the form like $a^n u(n)$

DFT can be used as a FT approximation that can calculate a **finite set of discrete-frequency spectrum values** from a finite set of discrete-time samples of an analog signal

Discrete Fourier Transform (DFT) Realization



Building the DFT formula (cont)

Continuous time signal **x(t)**

↓ sample

Discrete time signal **x(n)**

↓ window

Discrete time signal $\mathbf{x_0(n)}$

Finite length

⟷ DTFT

Discrete Time Fourier Transform (DTFT), $X_o(\Omega)$

(periodic over $[0,2\pi)$)

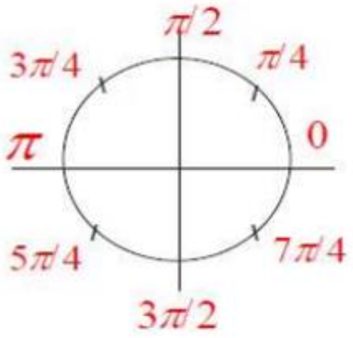
$$X_0(\Omega) = \sum_{n=-\infty} x_0[n]e^{-j\Omega n} = \sum_{n=0} x_0[n]e^{-j\Omega n}$$

Discrete Fourier Transform DFT **X(k)**

Discrete + periodic with period **N**

$$\mathbf{X(k) = \sum_{n=0}^{N-1} x_0(n) e^{-j2\pi kn/N}}$$

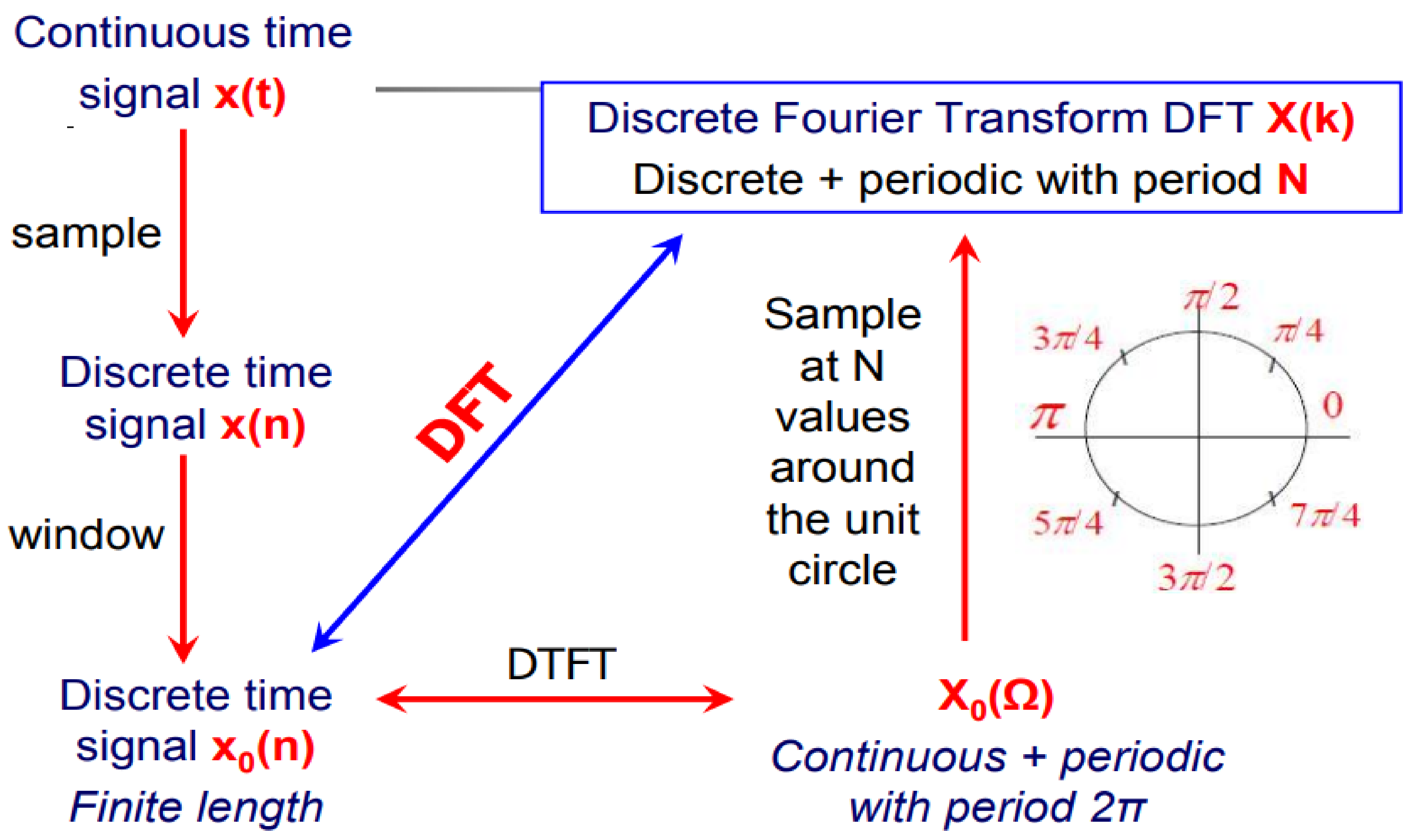
↑ Sample at N values around the unit circle



$$\mathbf{X(\Omega)|_{\Omega=\frac{2\pi k}{N}} = X(k)}$$

DFT

Building the DFT formula (cont)



Computation of DFT by extracting one period of DFS

To a finite-length sequence $x(n)$:

$$\underset{DFT}{X(k)} \Leftrightarrow DFS[x((n))_N]$$

Periodical copies

DFS of periodic sequence

Get DFT by extracting one period of DFS

Attention: DFT is acquired by extracting one period of DFS, it is not a new kind of Fourier Transform.

6.4 RELATION BETWEEN DFT AND Z-TRANSFORM

The Z-transform of N-point sequence $x(n)$ is given by

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

Let us evaluate $X(z)$ at N equally spaced points on the unit circle, i.e., at $z = e^{j(2\pi/N)k}$.

$$\left|e^{j(2\pi/N)k}\right| = 1 \text{ and } \angle e^{j(2\pi/N)k} = \frac{2\pi}{N}k$$

Hence, when k is varied from 0 to $N - 1$, we get N equally spaced points on the unit circle in the z-plane.

$$\therefore \qquad X(z)\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n)\, z^{-n}\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)kn}$$

By the definition of N-point DFT, we get

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)kn}$$

From the above equations, we get

$$X(k) = X(z)\Big|_{z = e^{j(2\pi/N)k}}$$

Discrete Fourier Transform DFT

- The formulas for DFT and IDFT may be expressed as (in terms of twiddle factor)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad 0 \le n \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\,\sin\left(\frac{2\pi}{N}\right)$$

- The relationship between $x(n)$ and $X(k)$ is denoted as

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

DFT and IDFT examples

Ex.4. Given $x(n) = \delta(n)$
$N = 4$. Find $X(k)$.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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DFT and IDFT examples

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$$\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn} X(k)}$$

DFT and IDFT examples

Ex.4. Given x(n) = δ(n)
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$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn} X(k) = W_4^0 = 1} \quad \textcolor{red}{\mathbf{where\ k = 0 \to N-1}}$$

DFT and IDFT examples

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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$$X[0] = \mathbf{1}$$
$$X[1] = \mathbf{1}$$
$$X[2] = \mathbf{1}$$
$$X[3] = \mathbf{1}$$

$$\mathbf{x(n) = \delta[n - n_0]}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\boxed{\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n - n_0) W_N^{kn}}}$$

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$$\boxed{\mathbf{X(k) = W_N^{kn_0}}}$$

$$\boxed{\mathbf{X(k) = e^{-j\frac{2\pi k n_0}{N}}}}$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

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$$= \frac{1}{2}\left[N\delta(k-k_0) + N\delta(k+k_0)\right]$$

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DFT and IDFT examples

Ex.4. Given $x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + \delta(n-3)$ and $N = 4$. Find $X(k)$.

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$$= 1 + 2(-j)^k + 3(-1)^k + (j)^k$$

$$X[0] = 7$$

$$X[1] = 1 - 2j - 3 + j = -2 - j$$

$$X[2] = 1 - 2 + 3 - 1 = 1$$

$$X[3] = 1 + 2j - 3 - j = -2 + j$$

DFT and IDFT examples

Ex.5. Given X(k) = 2δ(k) + 2δ(k-2) and N = 4. Find x(n).

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]\, W_4^{-kn}$$

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$$\boxed{x[n] = [1 \quad 0 \quad 1 \quad 0]}$$

$$\mathbf{n = 0, 1, 2, 3}$$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$\mathbf{X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}}$$

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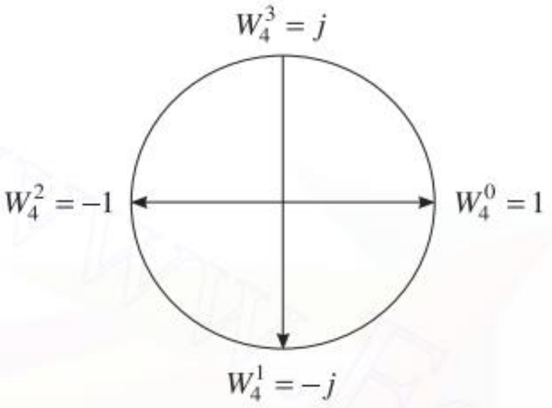

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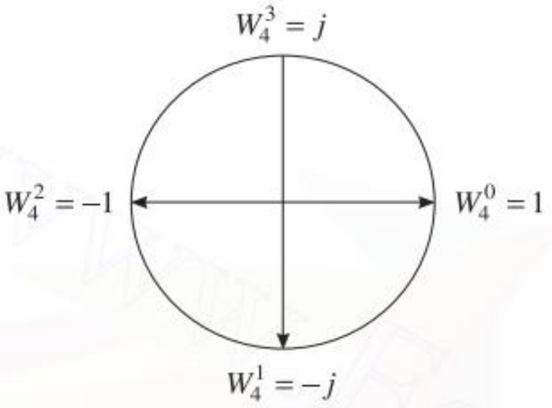

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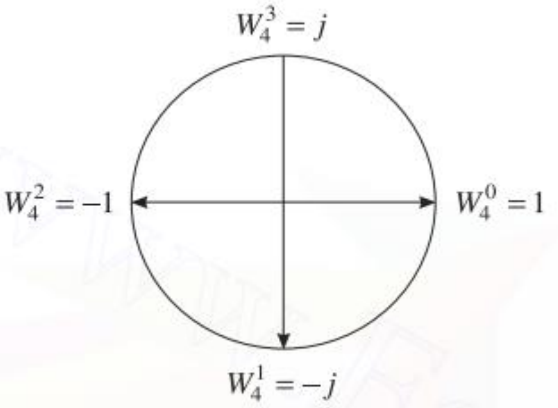

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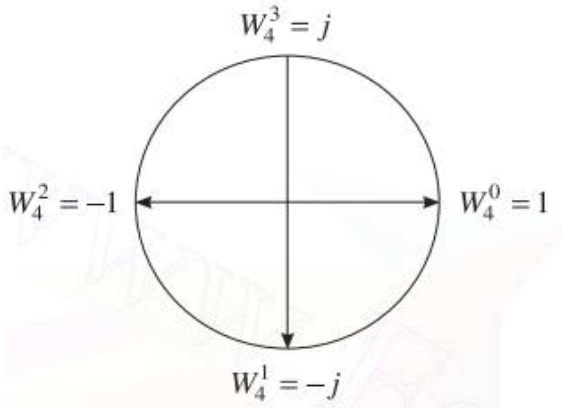

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where k=0, 1, 2, 3

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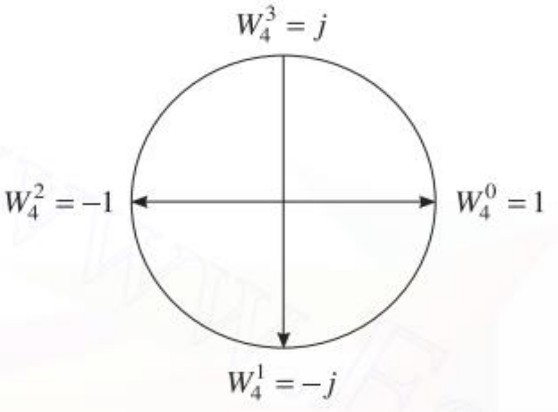


(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

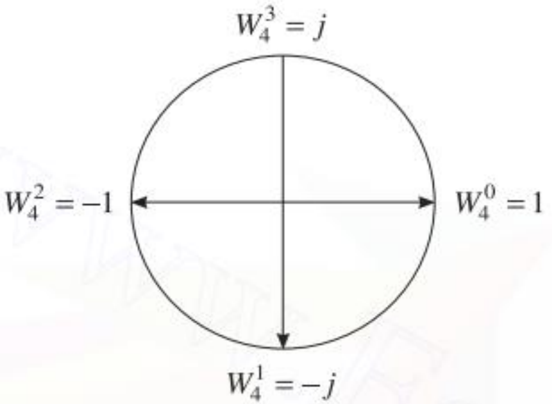
$$\mathbf{X(k) = \sum_{n=0}^{3} x[n]W_4^{nk}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

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$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

$$\mathbf{X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6}$$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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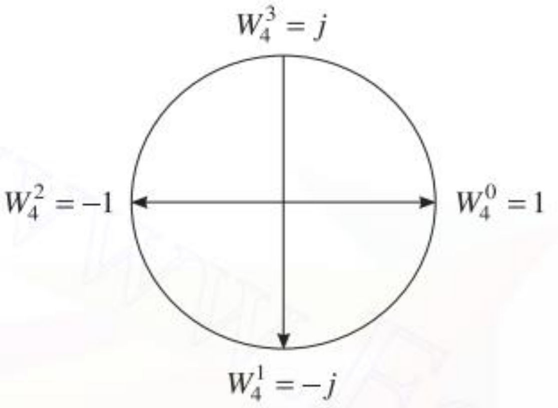
$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0$$

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$$X(2) = 1 + 1 + 2 + 2 = 6$$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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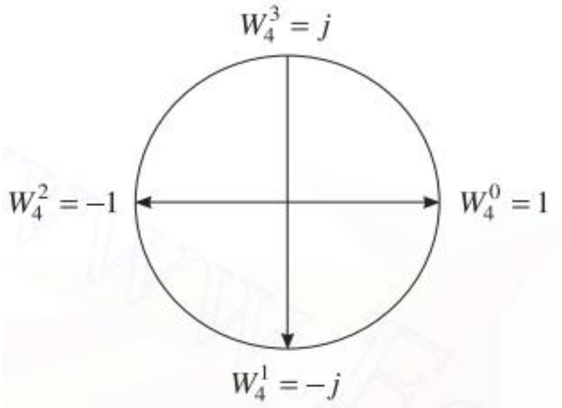
where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

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$$X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9$$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}$$

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where k=0, 1, 2, 3

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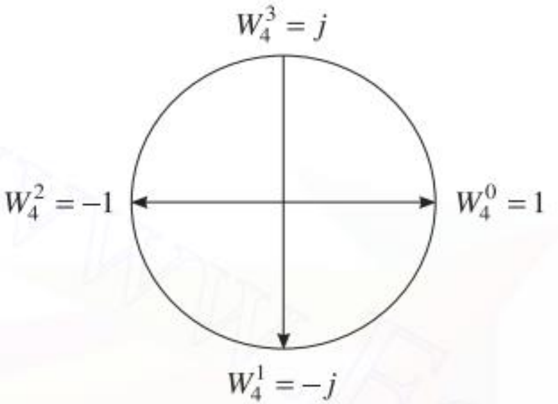

(a) 4-point DFT

$$X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0$$

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$$X(2) = 1 + 1 + 2 + 2 = 6$$

$$X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9$$

$$X(3) = 1 - j - 2 + 2j = -1 + j$$

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N} \mathbf{x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}, \quad \mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

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MATRIX FORMULATION OF THE DFT AND IDFT

$$W_N = e^{-j\frac{2\pi}{N}}$$

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$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ & & & \\ & & & \end{bmatrix} \end{array}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4,} \quad \mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ & & & \end{bmatrix} \end{array}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4, \quad k = 0, 1, 2, 3 \ \& \ n = 0, 1, 2, 3}$

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MATRIX FORMULATION OF THE DFT AND IDFT

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$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

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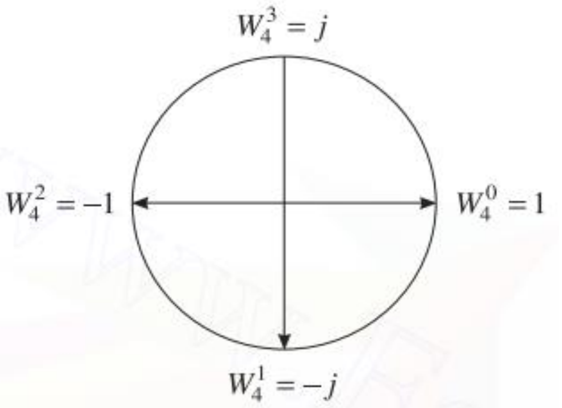

(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $N = 4$, $k = 0, 1, 2, 3$ & $n = 0, 1, 2, 3$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

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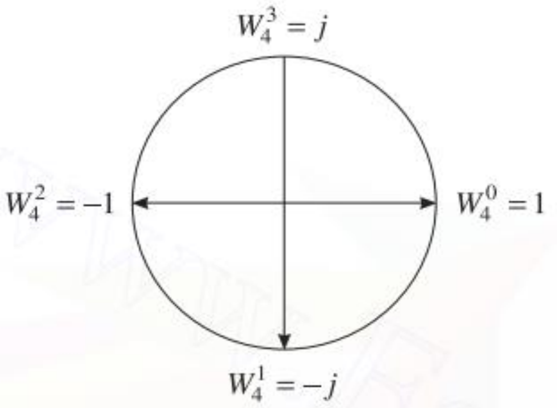
$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & \boxed{w_4^0} & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

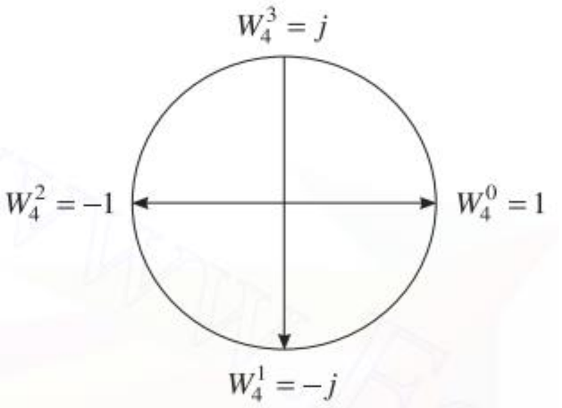
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

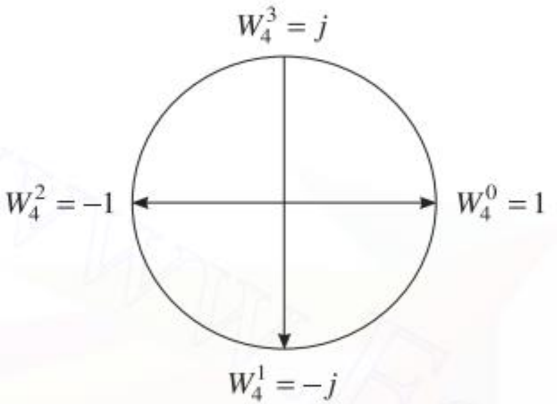
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

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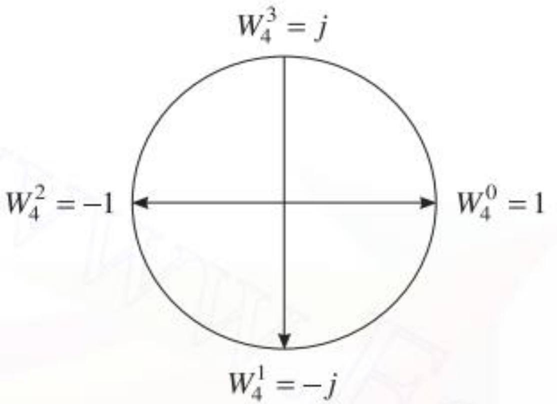
$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

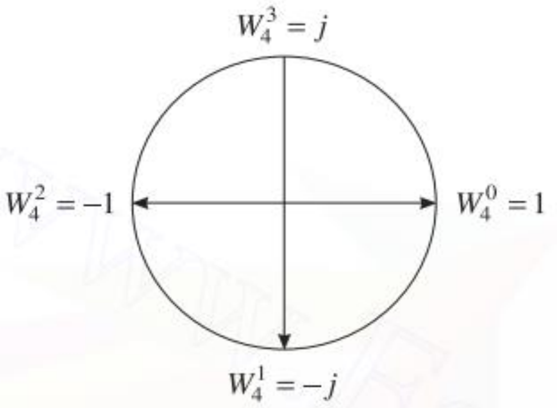
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $N = 4$, $k = 0, 1, 2, 3$ & $n = 0, 1, 2, 3$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

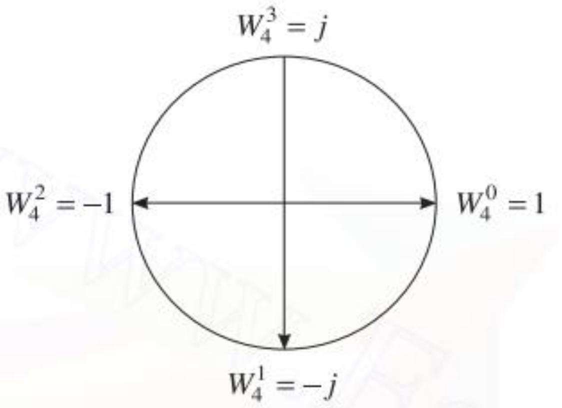
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

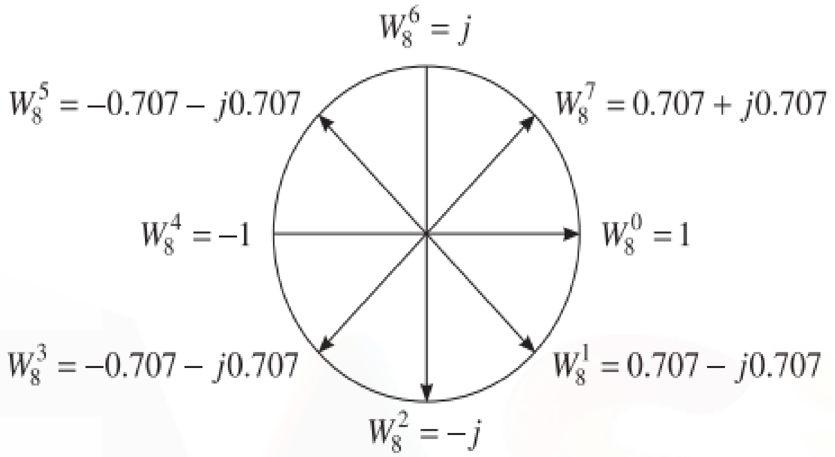
$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

$$\mathbf{W_4^{kn}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$



(b) 8-point DFT

EXAMPLE 6.8 Find the DFT of the sequence **and also draw the phase and magnitude spectrum**

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

EXAMPLE 6.8 Find the DFT of the sequence **and also draw the phase and magnitude spectrum**

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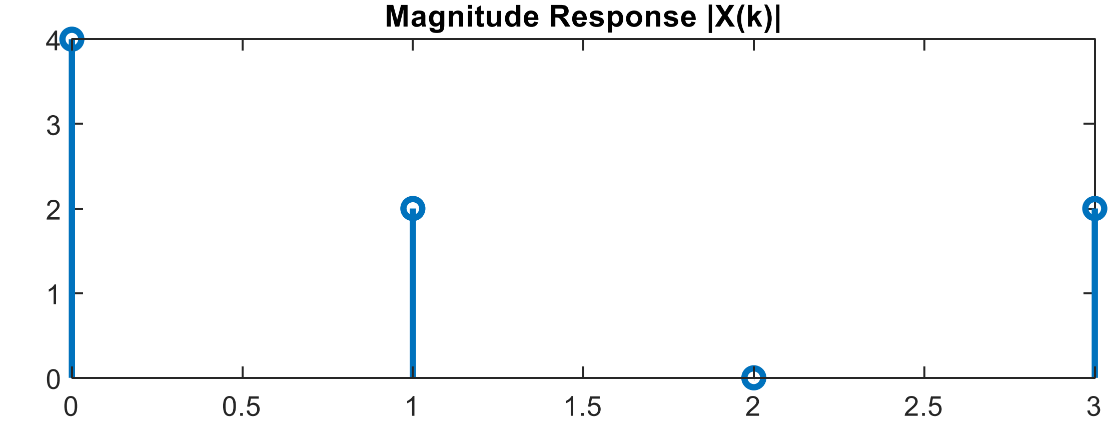

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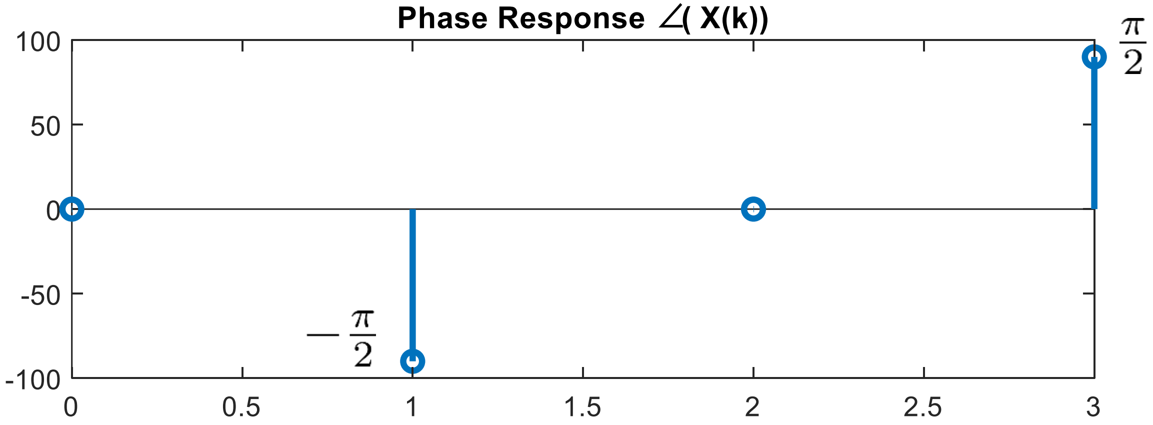

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Find IDFT of $X(k) = \{4, -2j, 0, 2j\}$

$$\mathbf{IDFT[X(k)] = \frac{1}{N}(W_N^{nk})^{*}X(k) = x(n)}$$

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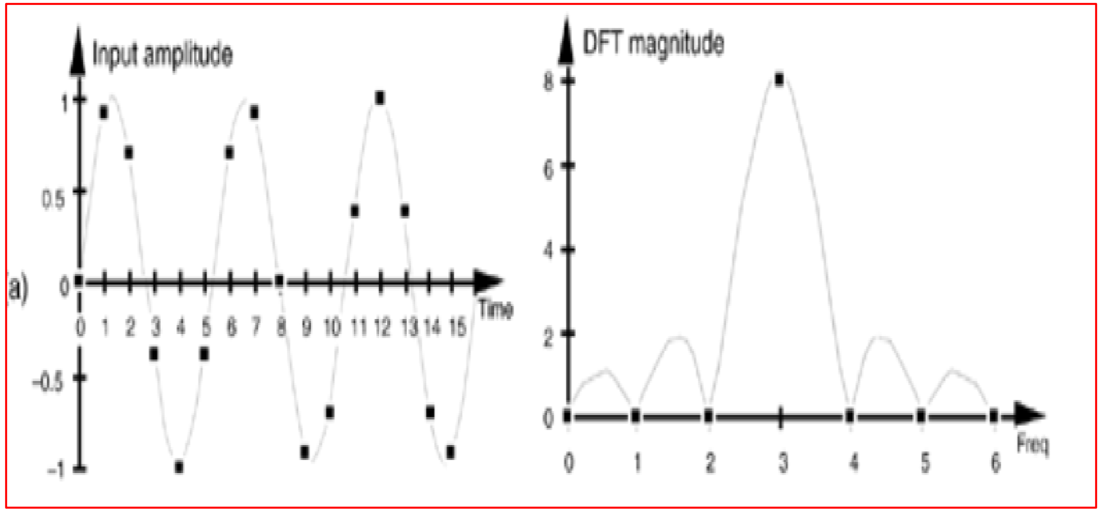
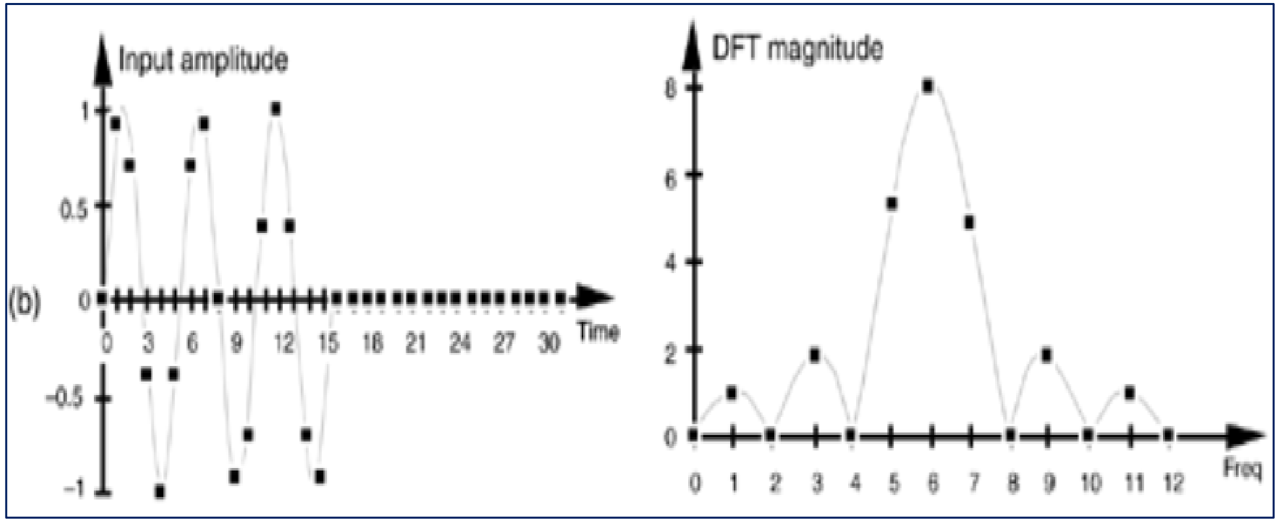
Zero Padding

One popular method used to improve DFT spectral estimation is known as zero padding. This process involves the addition of zero-valued data samples to an original DFT input sequence to increase the total number of input data samples

The more points in our DFT, the better our DFT output approximates the CFT.



(a) 16 input data samples and N = 16



(b) 16 input data samples, 16 padded zeros, and N = 32

Properties of DFT

Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Periodicity

If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

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Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT}\ \{x_1(n)\} = X_1(k)$$

and

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Then for any real valued or complex valued constants a and b,

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

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$$= aX_1(k) + bX_2(k)$$

Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

$$x[(-n), \bmod N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\begin{aligned} \text{DFT}\,\{x(-n), \bmod N\} &= \text{DFT}\,\{x(N-n)\} \\ &= X[(-k), \bmod N] = X(N-k) \end{aligned}$$

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Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

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$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N} = X(N-k)$$

Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

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$$\mathbf{= \left[\sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}}\right] e^{\frac{-j2\pi m}{N}}}$$

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Circular Frequency

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then, $\quad \text{DFT}\{\mathbf{x(n)e^{j2\pi ln/N}}\} = X\{(k - l), (\text{mod } N)]$

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Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then $\quad \text{DFT}\ \{x^*(n)\} = X^*(N-k) = X^*[(-k), \bmod N]$

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$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof: $\quad \text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$

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Proof:

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Proof: $$\text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$$

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Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)\,x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

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Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Multiplication

If $\quad DFT\,[x_1(n)] = X_1(k)$

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Then $\quad DFT[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

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Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

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Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]}$$

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$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

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Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]}$$

Rearranging the exponential terms

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]}$$

Multiplication

If $\quad \mathrm{DFT}\,[x_1(n)] = X_1(k)$

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Then $\quad \mathrm{DFT}[x_1(n)x_2(n)] = \dfrac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]}$$

Rearranging the exponential terms

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]}$$

Using defination of DFT

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)X_2((k-m))_N}$$

Multiplication

If $\quad \mathrm{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \mathrm{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \mathrm{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathrm{DFT}\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}$$

Replacing $x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}$$

Rearranging the order of the summation.

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]$$

Rearranging the exponential terms

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]$$

Using defination of DFT

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)X_2((k-m))_N$$

By defination of Circular Convalution

$$\mathrm{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$$

Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Numerical Problem Based on Properties of DFT

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(a)Circular Convalution (b) Linear Convalution using DFT.

Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

Step3: $\mathbf{Y(k)} = \mathbf{X(k)} \times \mathbf{H(k)}$.

Step4: $y(n) = \mathbf{IDFT(Y(k))}$.

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}}$$

$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

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$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$$\mathbf{X(0) = 1w_2^0 + 2w_2^0} = 1 + 2 = 3$$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}}$$

$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0} = 1 + 2 = 3$

$\mathbf{X(1) = 1w_2^0 + 2w_2^1} = 1 - 2 = -1$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n)w_2^{kn}}$$

$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0}$ = 1+ 2 =3

$\mathbf{X(1) = 1w_2^0 + 2w_2^1}$ = 1- 2=-1

$\mathbf{X(k) = \{3, -1\}}$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n)w_2^{kn}}$$

$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$$\mathbf{X(0) = 1w_2^0 + 2w_2^0} = 1+2=3$$

$$\mathbf{X(1) = 1w_2^0 + 2w_2^1} = 1-2=-1$$

$$\mathbf{X(k) = \{3, -1\}}$$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}}$$

$$\mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

If $x(n)=\{1,2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N=2}$$

$$\mathbf{w_N^{kn}=e^{-j2\pi kn/N}=w_2^1=e^{-j\pi}=-1}$$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n])=X(k)=\sum_{n=0}^{1}x(n)w_2^{kn}}$$

$$\mathbf{X(k)=x(0)w_2^0+x(1)w_2^k}$$

where $\mathbf{k=0,1}$

$$\mathbf{X(0)=1w_2^0+2w_2^0}=1+2=3$$

$$\mathbf{X(1)=1w_2^0+2w_2^1}=1-2=-1$$

$$\mathbf{X(k)=\{3,-1\}}$$

$$\mathbf{DFT(h[n])=H(k)=\sum_{n=0}^{1}h(n)w_2^{kn}}$$

$$\mathbf{H(k)=h(0)w_2^0+h(1)w_2^k}$$

where $\mathbf{k=0,1}$

$$\mathbf{H(0)=2w_2^0+w_2^0}=2+1=3$$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}}$$

$$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$$\mathbf{X(0) = 1w_2^0 + 2w_2^0} = 1+2=3$$

$$\mathbf{X(1) = 1w_2^0 + 2w_2^1} = 1-2=-1$$

$$\mathbf{X(k) = \{3, -1\}}$$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}}$$

$$\mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k}$$

where $\mathbf{k = 0, 1}$

$$\mathbf{H(0) = 2w_2^0 + w_2^0} = 2+1=3$$

$$\mathbf{H(1) = 2w_2^0 + 1w_2^1} = 2-1=1$$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n)w_2^{kn}} \qquad \mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$$\mathbf{X(0) = 1w_2^0 + 2w_2^0} = 1+ 2 =3 \qquad \mathbf{X(1) = 1w_2^0 + 2w_2^1} = 1- 2=-1 \qquad \mathbf{X(k) = \{3, -1\}}$$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$$\mathbf{H(0) = 2w_2^0 + w_2^0} = 2+ 1 =3 \qquad \mathbf{H(1) = 2w_2^0 + 1w_2^1} = 2- 1=1 \qquad \mathbf{H(k) = \{3, 1\}}$$

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k)w_2^{-kn}}$$

$$\mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)}$$ where $\mathbf{n = 0, 1}$

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

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$$\mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)}$$

where $\mathbf{n = 0, 1}$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9 - 1)/2 = 4}$$

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

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$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k) w_2^{-kn}}$$

$$\mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)}$$ where $\mathbf{n = 0, 1}$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9-1)/2 = 4}$$

$$\mathbf{y(1) = \left(Y(0)w_2^0 + Y(1)w_2^{-1}\right) = (9+1)/2 = 5}$$

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 2}$$

$$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

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$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k)w_2^{-kn}}$$

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where $\mathbf{n = 0, 1}$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9-1)/2 = 4}$$

$$\mathbf{y(1) = \left(Y(0)w_2^0 + Y(1)w_2^{-1}\right) = (9+1)/2 = 5}$$

$$\mathbf{y(n) = \{4, 5\}}$$

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

Step2: By Zero padding make lenght of of $\mathbf{x(n)}$ and $\mathbf{h(n)}$ same as $\mathbf{y(n)}$.

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

Step2: By Zero padding make lenght of of $\mathbf{x(n)}$ and $\mathbf{h(n)}$ same as $\mathbf{y(n)}$.

After zero padding $x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

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After zero padding $x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

Step3: $\mathbf{Y(k) = X(k) \times H(k)}$.

Step4: $y(n) = \mathbf{IDFT(Y(k))}$.

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}}$$

$$\mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$\mathbf{N = 3}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^{1} = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}}$$

$$\mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

(b) Linear Convalution using DFT.

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^{1} = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt(3)/2) = j\sqrt{3}}$

(b) Linear Convalution using DFT.

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$$\mathbf{N = 3} \qquad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k} \qquad \text{where } \mathbf{k = 0, 1, 2}$$

$$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$$

$$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt(3)/2) = j\sqrt{3}}$$

$$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt(3)) = j\sqrt{3}}$$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

(b) Linear Convalution using DFT.

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k} \qquad \text{where } \mathbf{k = 0, 1, 2}$$

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$$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt(3)) = j\sqrt{3}}$$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k} \qquad \text{where } \mathbf{k = 0, 1, 2}$$

Circular convolution $x_1(n) \circledast x_2(n) \longleftrightarrow X_1(k)X_2(k)$

(b) Linear Convalution using DFT.

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}}$$

$\mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$ where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1+ 2 =3$

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$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}}$$

$\mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k}$ where $\mathbf{k = 0, 1, 2}$

$\mathbf{H(0) = 2w_3^0 + 1w_3^0} = 1+ 2 =3$

(b) Linear Convalution using DFT.

Circular convolution $x_1(n) \circledast x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1+ 2 =3$

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$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt(3)) = j\sqrt{3}}$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{H(0) = 2w_3^0 + 1w_3^0} = 1+ 2 =3$

$\mathbf{H(1) = 2w_3^0 + w_3^1 = 2 - (1/2 - j\sqrt(3)/2) = 3/2 + j\sqrt(3)/2)}$

(b) Linear Convalution using DFT.

Circular convolution	$x_1(n) \circledast x_2(n)$	$\longleftrightarrow$	$X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

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$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k}$$

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$\mathbf{H(0) = 2w_3^0 + 1w_3^0} = 1 + 2 = 3$

$\mathbf{H(1) = 2w_3^0 + w_3^1 = 2 - (1/2 - j\sqrt(3)/2) = 3/2 + j\sqrt(3)/2)}$

$$\mathbf{H(2) = w_3^0 + 2w_3^2 = 2 - (1/2 - j\sqrt(3)/2) = 3/2 + j\sqrt{3}/2}$$

$$X(k) = \{3, j\sqrt{3}, j\sqrt(3)\}.$$

$$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt(3)/2\}.$$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$$

$$X(k) = \{3, j\sqrt{3}, j\sqrt(3)\}.$$

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Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$$

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1.5 - j2.598, -1.5 + j2.598\}}.$

$$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$$

$$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$$

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1.5 - j2.598, -1.5 + j2.598\}}.$

Step4: $\mathbf{y(n) = IDFT(Y(k))}.$

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k) w_3^{-kn}}$$

$$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$$

$$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

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$$\mathbf{y(0) = \frac{1}{3}\left(Y(0)w_3^0 + Y(1)w_3^{-n} + Y(3)w_3^{-2n}\right) = 6/3 = 2}$$

$$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$$

$$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$$

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1.5 - j2.598, -1.5 + j2.598\}}.$

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$$\mathbf{y(n) = \{2, 5, 2\}}$$

Find Linear and Circular Convalution using DFT and IDFT

$$x_1(n) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(n) = \{4, 3, 2, 1\}$$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \circledast x(n)$
(f) Signal energy

Q2. Find Linear and Circular Convalution using DFT and IDFT

$$x_1(n) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(n) = \{4, 3, 2, 1\}$$

EXAMPLE 6.17 Let $x(n) = \{A, 2, 3, 4, 5, 6, 7, B\}$. If $X(0) = 20$ and $X(4) = 0$, find A and B.

EXAMPLE 6.10 Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.
(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

EXAMPLE 3.23 Determine the inverse Z-transform of

(a) $X(z) = \dfrac{1}{z-a}$; ROC; $|z| > a$ (b) $X(z) = \dfrac{1}{1-az^{-1}}$; ROC; $|z| > a$

(c) $X(z) = \dfrac{1}{1-z^{-4}}$; ROC; $|z| > 1$

EXAMPLE 3.17 Find the convolution of the sequences

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

using (a) Convolution property of Z-transforms and (b) Time domain method.

EXAMPLE 3.4 Find the Z-transform and ROC of

$$x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n)$$

Sketch the ROC and pole-zero location.

Find the Z.T. of the following sequence & determine the ROC.

i) $x(n) = 3^{n+1}u(n) - 2\left(\frac{1}{2}\right)^n u(-n-1)$

(ii) $3e^{-2n}u(n) + 2\left[4^n u(-n-1)\right] + 5\delta(n)$

Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
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Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n - 2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\,\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\,X(k) = e^{-j\pi k}\,X(k)$$

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$$= \{X(0)\,e^{0}, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

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$$= \{4(1), -j2(-1), 0(1), j2(-1)\}$$

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$$= \{X(0)\,e^{0}, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

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Solution:

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$$\text{DFT}\,\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\,X(k) = e^{-j\pi k}\,X(k)$$

$$= \{X(0)\,e^{0}, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

(b) Using the flipping (time reversal) property of DFT, we have

$$\text{DFT}\ \{x(-n)\} = X(-k) = X^*(k) =$$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n - 2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\,\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\,X(k) = e^{-j\pi k}\,X(k)$$

$$= \{X(0)\,e^{0}, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1), \,-j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, \,-j2\}$$

(b) Using the flipping (time reversal) property of DFT, we have

$$\text{DFT}\;\{x(-n)\} = X(-k) = X^*(k) = \{4, \,-j2, 0, j2\}^* = \{4, j2, 0, \,-j2\}$$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

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(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT } \{x^*(n)\} = X^*(-k)$$

Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
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Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

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Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT }\{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT}\ \{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

Since $\text{DFT}\ \{x^*(n)\} = \text{DFT}\ \{x(n)\}$, we can say that $x(n)$ is real valued.

(d) Using the property of convolution of product of two signals, we have

$$\text{DFT}\ \{x(n)\,x(n)\} = \frac{1}{N}[X(k) \oplus X(k)]$$

Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \text{mod } N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT }\{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

Since $\text{DFT }\{x^*(n)\} = \text{DFT }\{x(n)\}$, we can say that $x(n)$ is real valued.

(d) Using the property of convolution of product of two signals, we have

$$\text{DFT }\{x(n)\,x(n)\} = \frac{1}{N}[X(k) \oplus X(k)] = \frac{1}{4}[(4, -j2, 0, j2) \oplus (4, -j2, 0, j2)]$$

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$$\boxed{= \{6, -j4, 0, j4\}}$$

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Solution:

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$$\text{DFT}\,\{x(n) \oplus x(n)\} = [X(k)\,X(k)]$$

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Solution:

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$$\text{Signal energy} = \frac{1}{4}\sum |X(k)|^2$$

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(f) Using Parseval's theorem, we have

$$\text{Signal energy} = \frac{1}{4}\sum |X(k)|^2 = \frac{1}{4}\sum |\{4, \ -j2, 0, j2\}|^2$$
$$= \frac{1}{4}[16 + 4 + 0 + 4] = 6$$

EXAMPLE 6.23 If IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$, using properties of DFT, find

(a) IDFT $\{X(k-1)\}$
(b) IDFT $\{X(k) \oplus X(k)\}$
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(d) Signal energy

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Solution: Given IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$

(a) Using modulation property, we have

$$\text{IDFT } \{X(k-1)\} = x(n)\, e^{j2\pi n/4} = x(n)\, e^{j\pi\frac{n}{2}}$$

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(b) Using periodic convolution property, we have

$$\text{IDFT } \{X(k) \oplus X(k)\} = Nx^2(n)$$

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(b) Using periodic convolution property, we have

$$\text{IDFT } \{X(k) \oplus X(k)\} = Nx^2(n) = 4\{1, 2, 1, 0\}^2 = \{4, 16, 4, 0\}$$

EXAMPLE 6.23 If IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$, using properties of DFT, find

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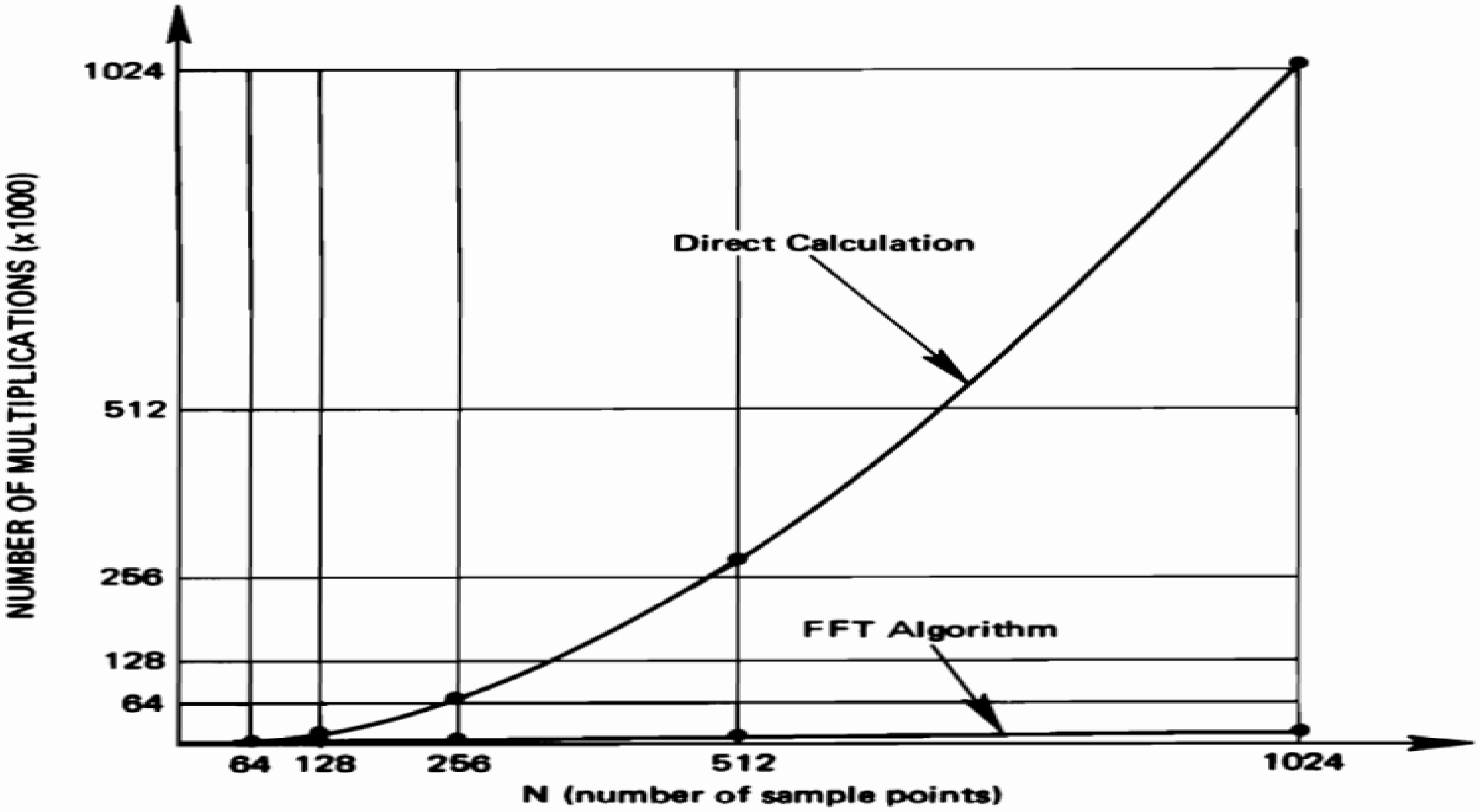
$$= (1)^2 + (2)^2 + (1)^2 + (0)^2 = 6$$

Fast Furier Transform

- Fast Fourier transform (FFT) is an efficient algorithm to compute the DFT with reduced computations.
- Due to the efficiency offered by FFT, the DFT is widely used for the spectrum analysis, convolutions, correlations, and for linear filtering.
- FFT is only a computational algorithm and not another transform.
- FFT algorithm is developed by Cooley and Tukey in 1965.
- Two FFT algorithms are known as decimation-in-time (DIT) and decimation-in-frequency (DIF) algorithms.

Fast Furier Transform



Fast Furier Transform

- **The simplest and perhaps best-known method for computing the FFT is the Radix-2 Decimation in Time algorithm.**

- **The main limitation of the radix-2 method is that it only works if $N = 2^m$, where m is an integer. If $N = 37$ (for example), this method cannot be used.**

- **In decimation in time (DIT) algorithm, time sequence $x(n)$ is decimated and smaller point DFTs are combined to get the result of N point DFT.**

- **In general, we can say that N point DFT can be realized from $N/2$ points DFT. Similarly, $N/2$ points DFT can be calculated by $N/4$ points DFT and so on.**

- **The decimation can be performed up to m times, where $\mathrm{N} = 2^m$ and $m = \log_2(N)$**

Fast Furier Transform

- **In general, we can say that N point DFT can be realized from $N/2$ points DFT. Similarly, $N/2$ points DFT can be calculated by $N/4$ points DFT and so on.**

- **If $N = 8$, the decimation can be performed up to $m = \log_2(N) = \log_2(8) = 3$**

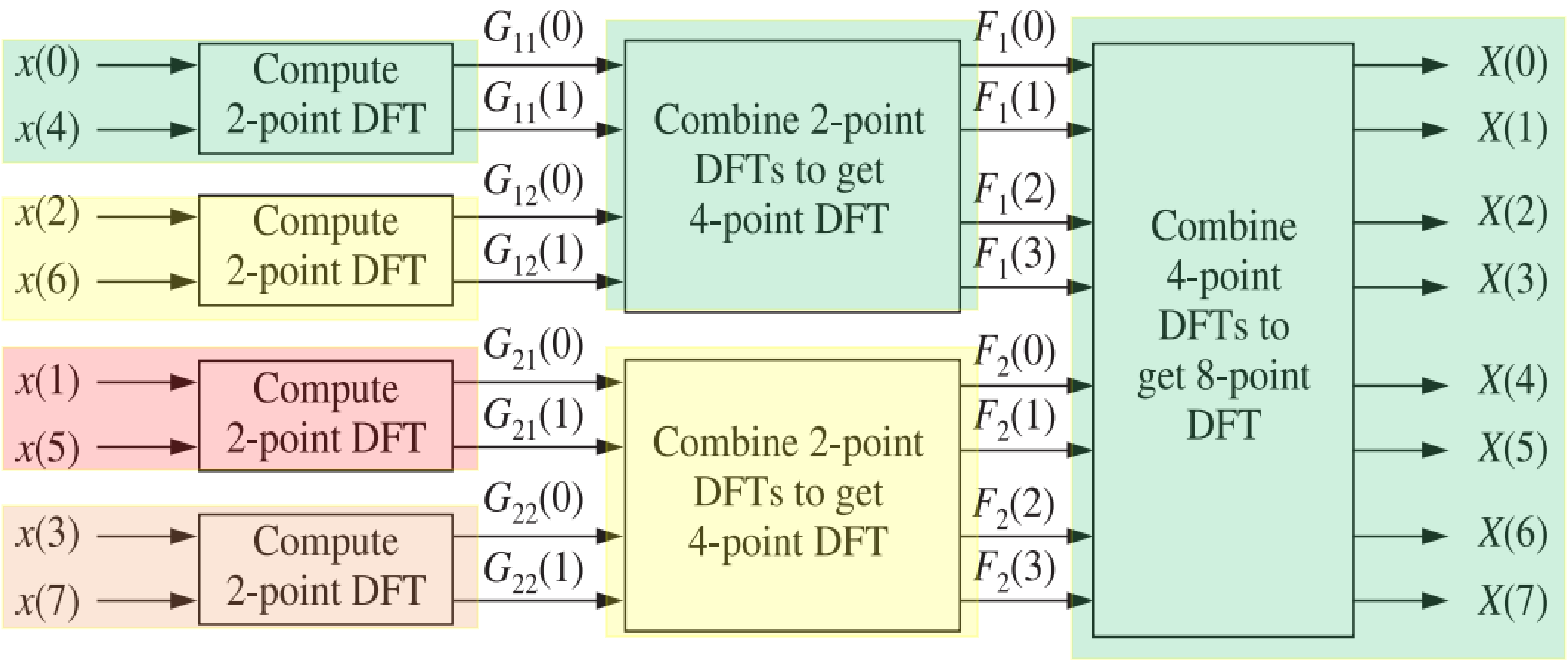


Figure 7.5 Three stages of computation in 8-point DFT.

Fast Furier Transform

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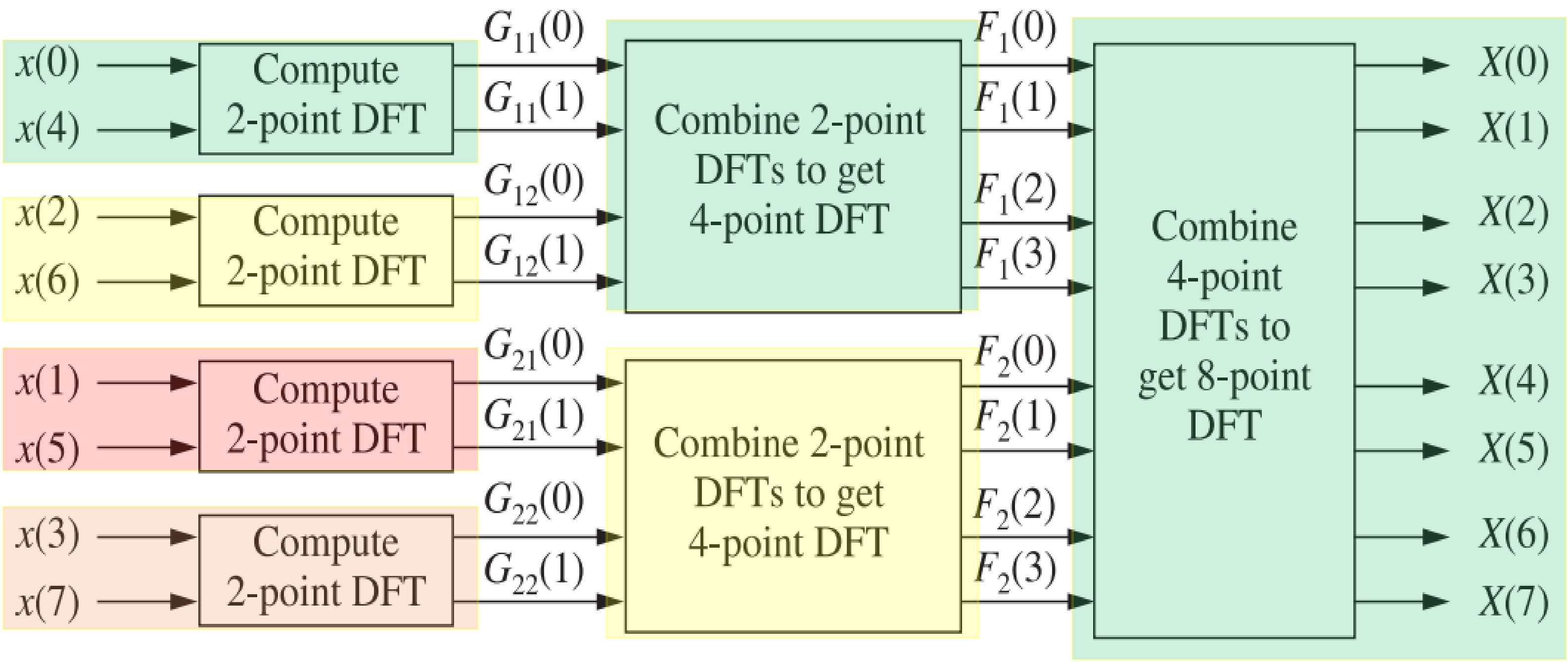
How to arrange inputs?



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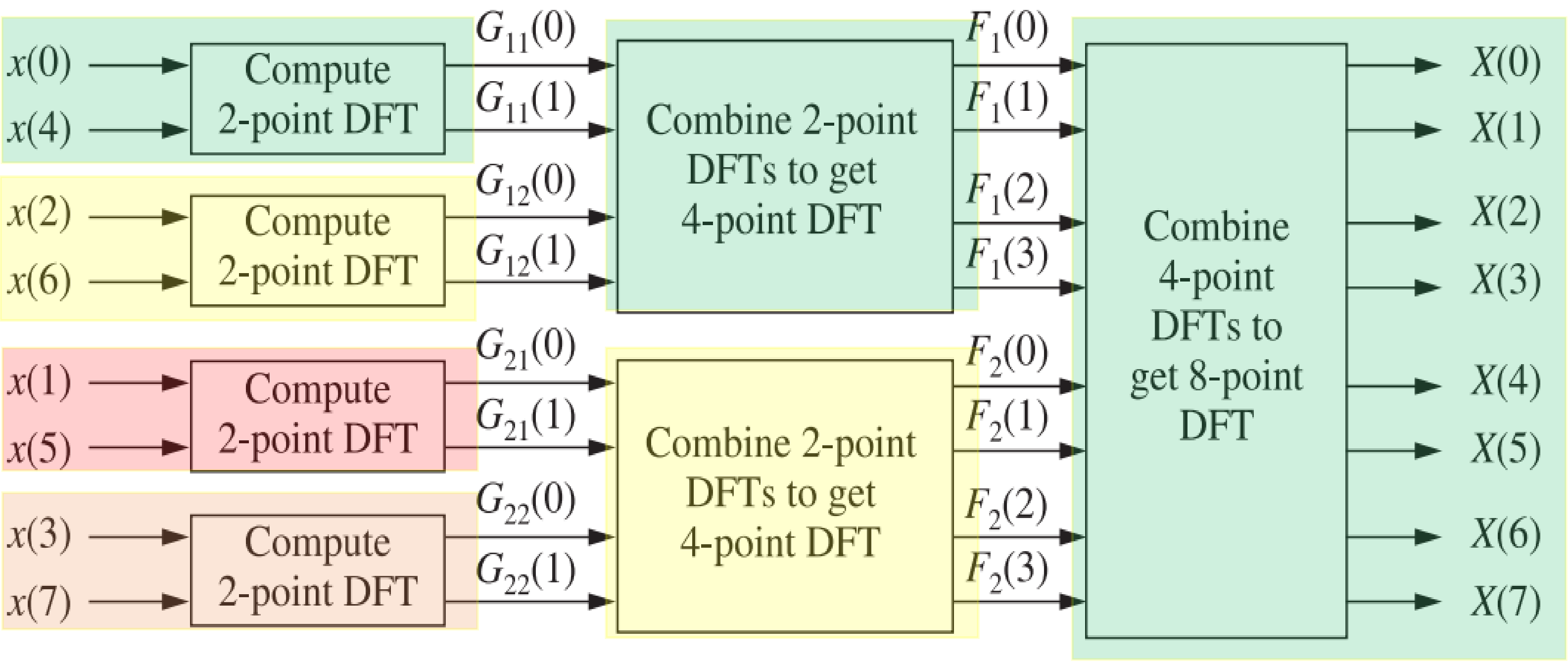
Original	Binary Form	Reversed Form	Final
0	000		
1	001		
2	010		
3	011		
4	100		
5	101		
6	110		
7	111		



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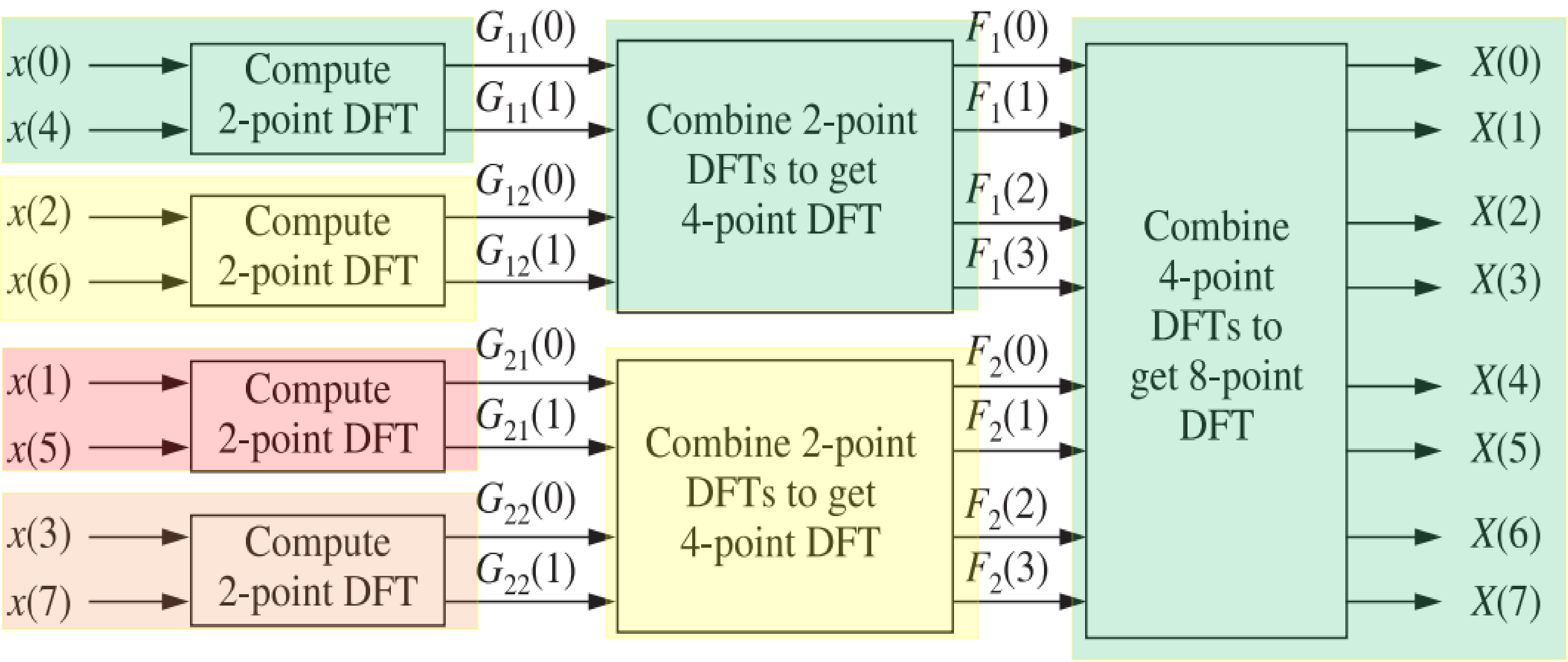

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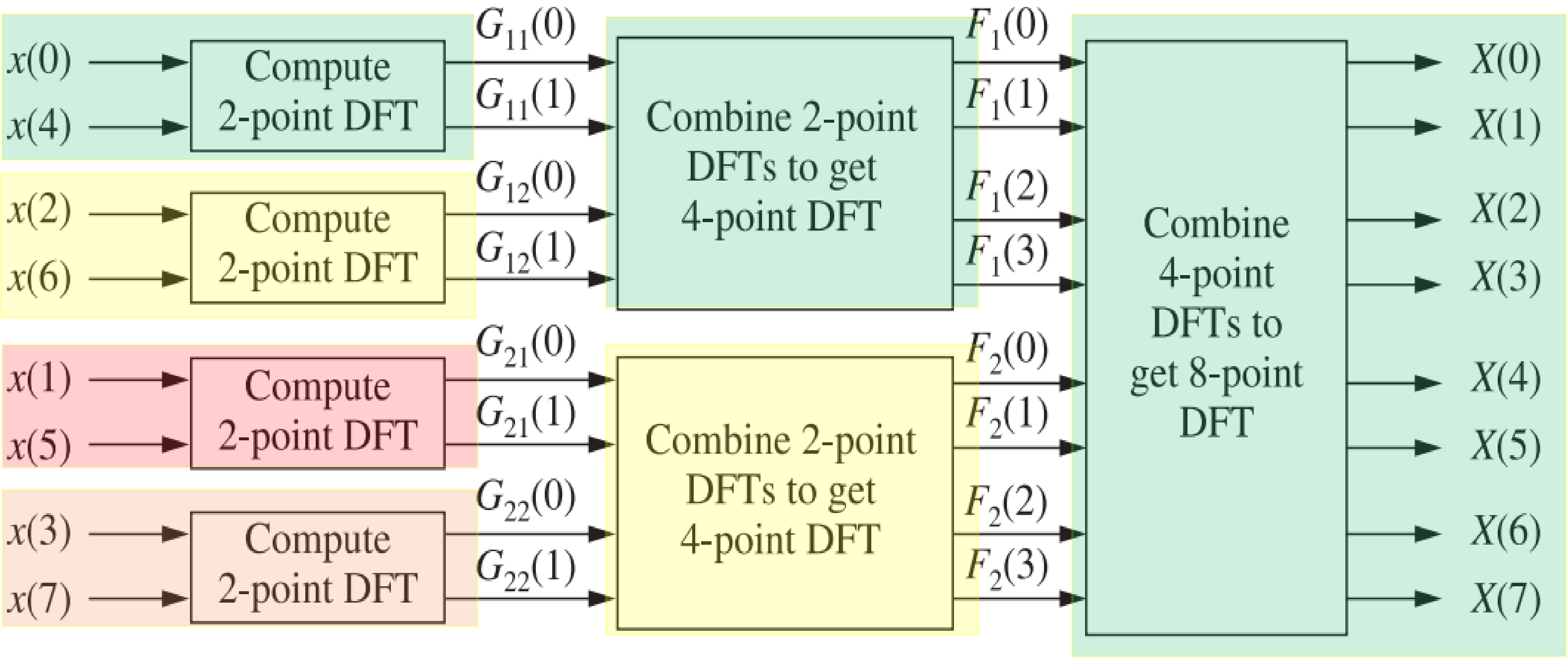

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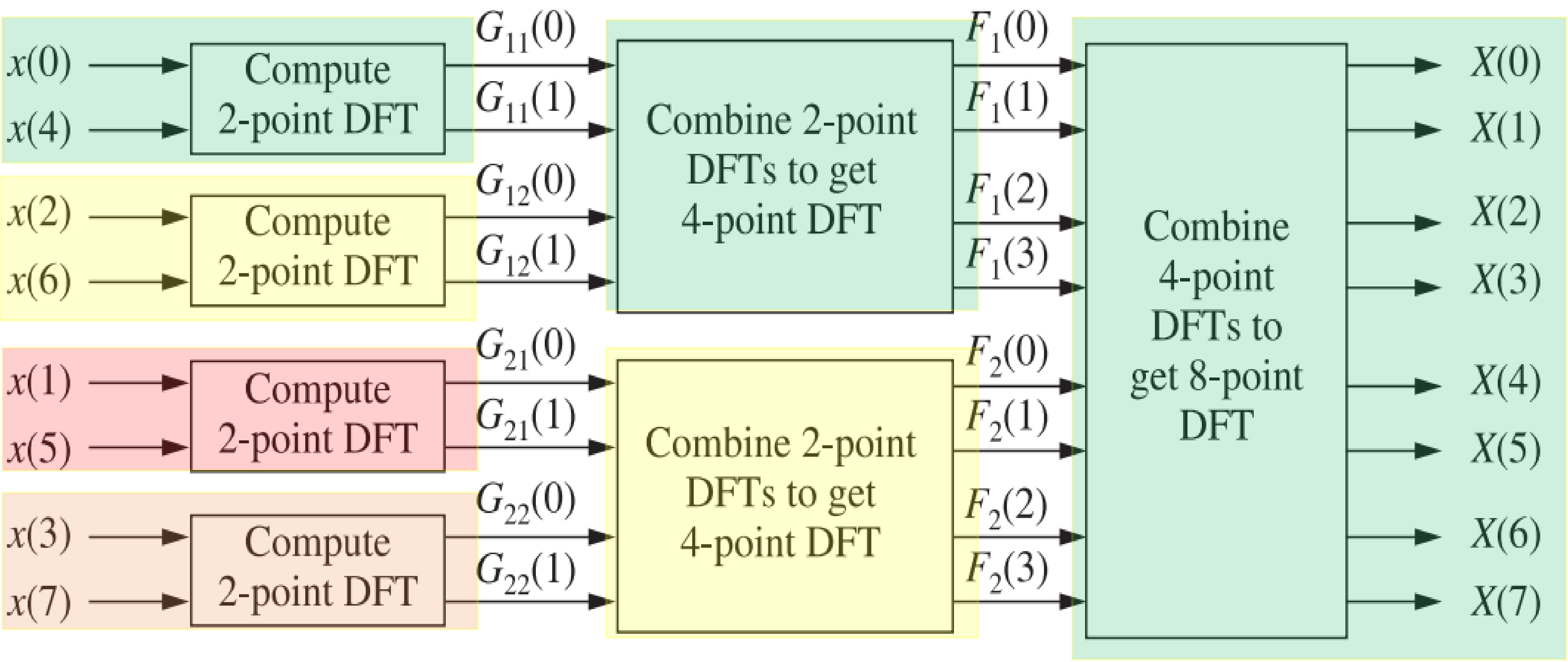

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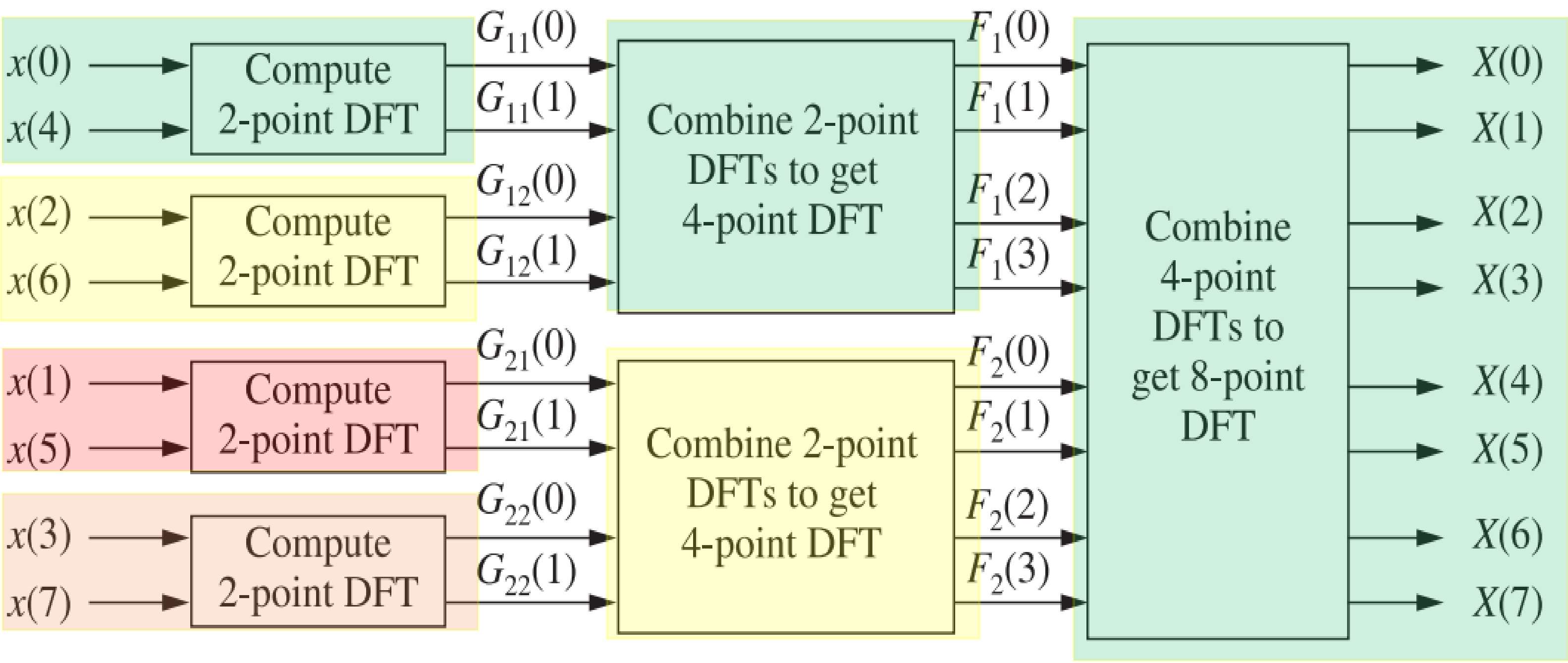

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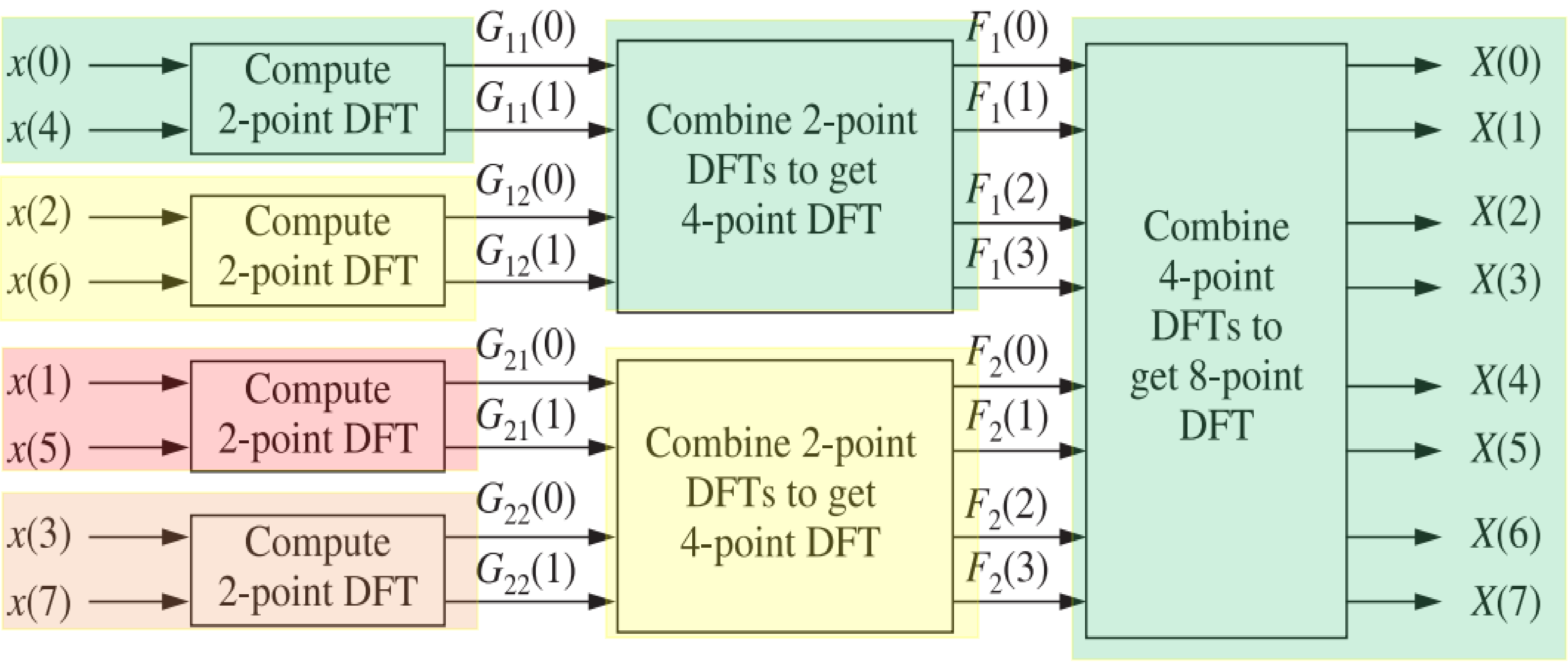

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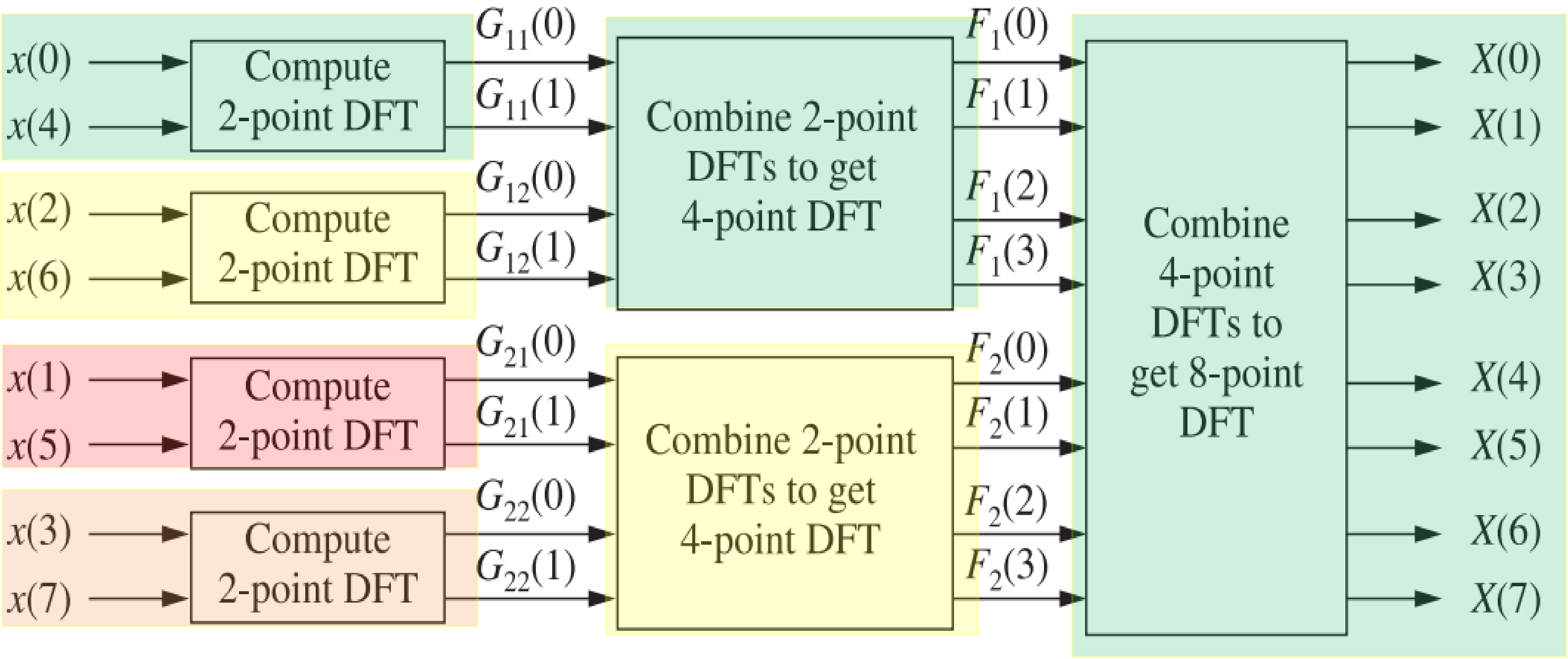

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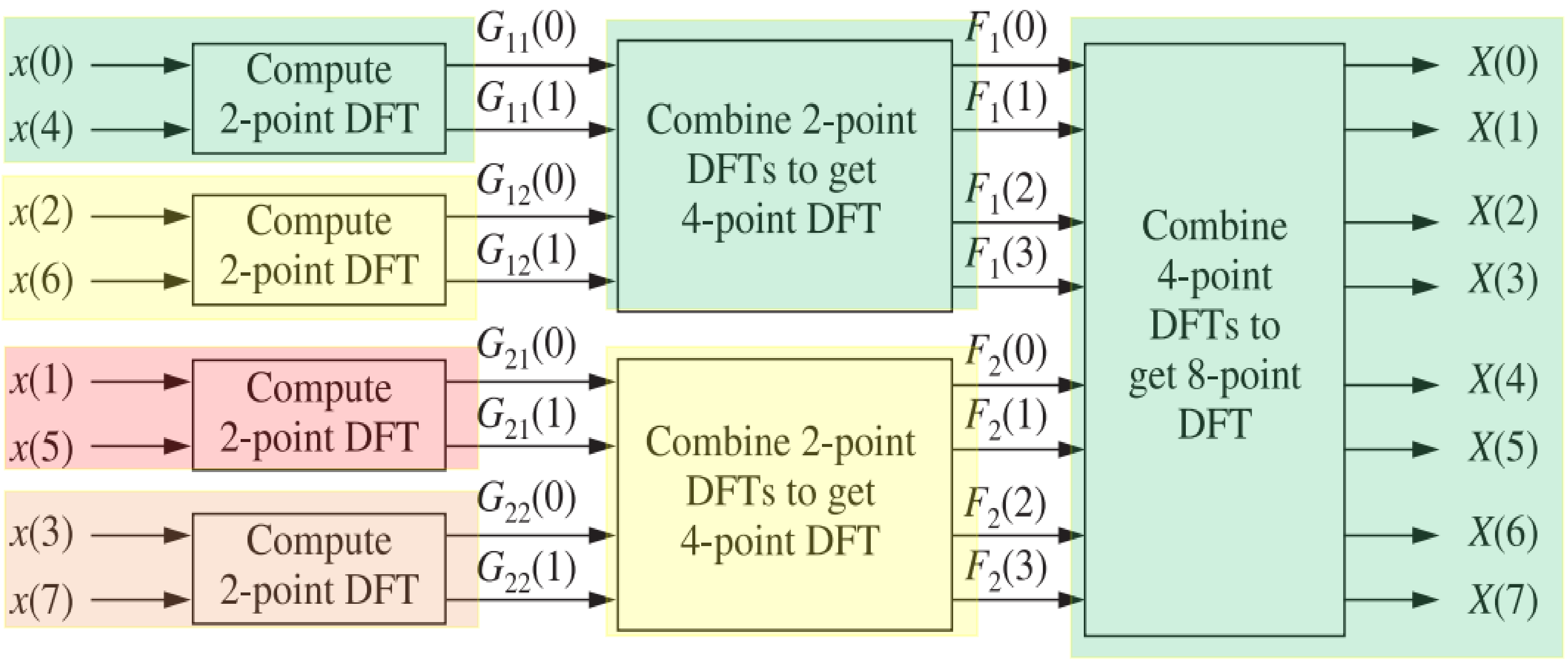
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7



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6	110	011	3
7	111	111	7

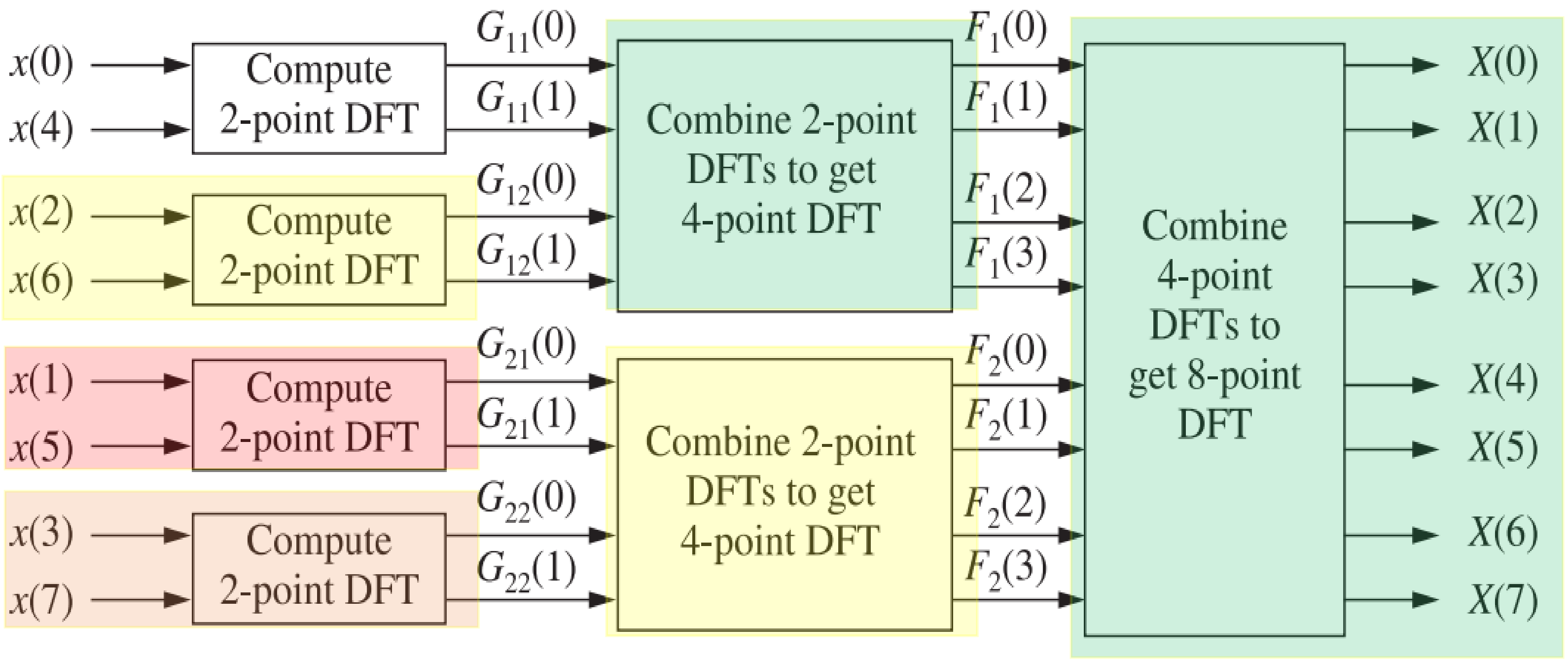
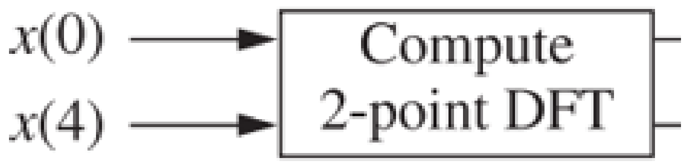
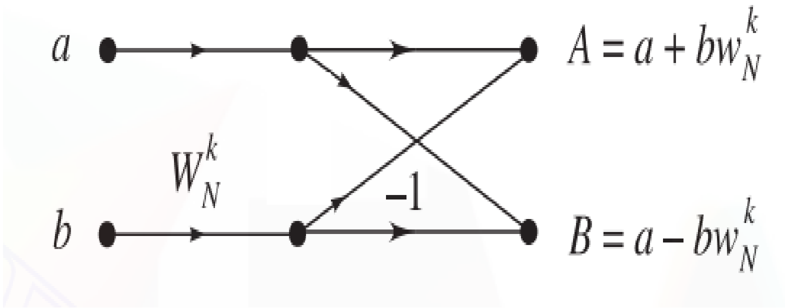
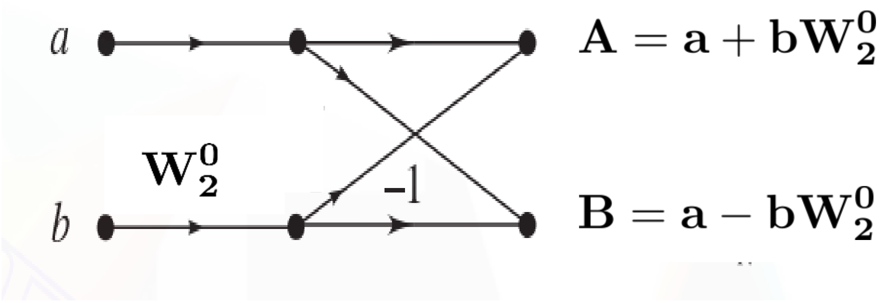


Figure 7.5 Three stages of computation in 8-point DFT.

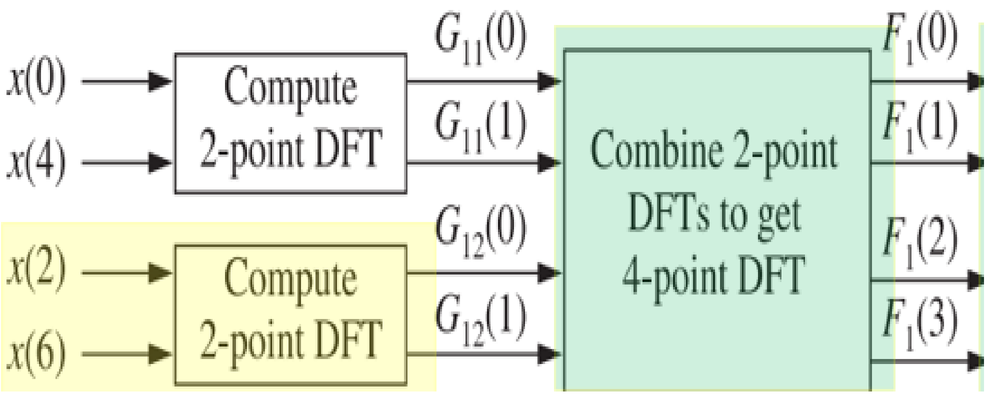
Fast Furier Transform

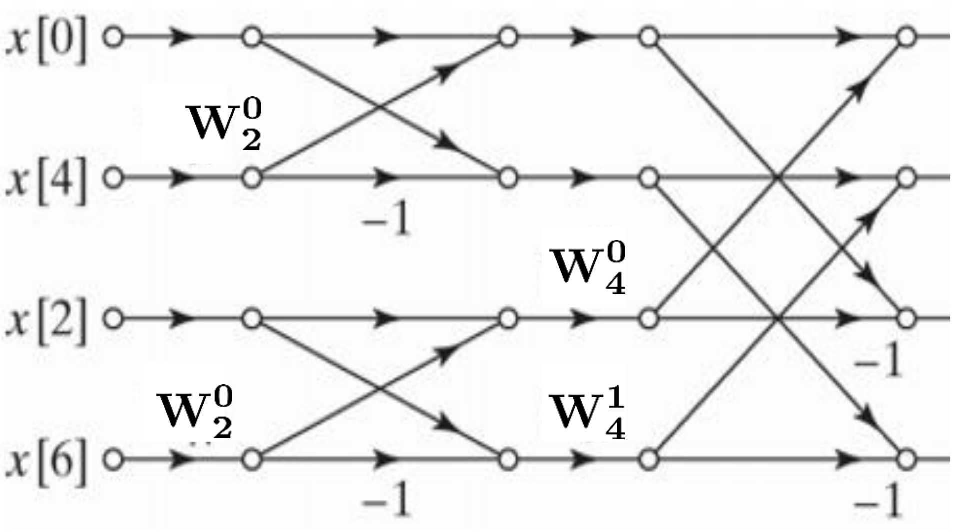






Fast Furier Transform





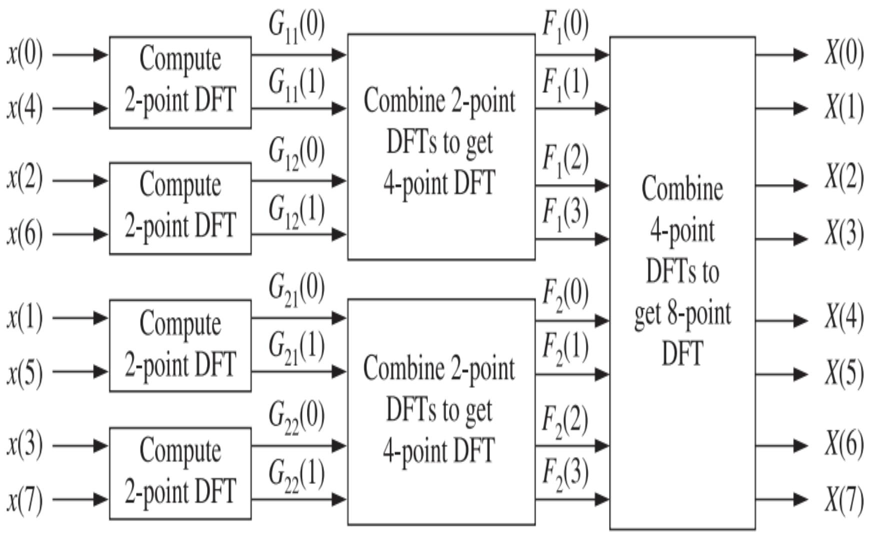
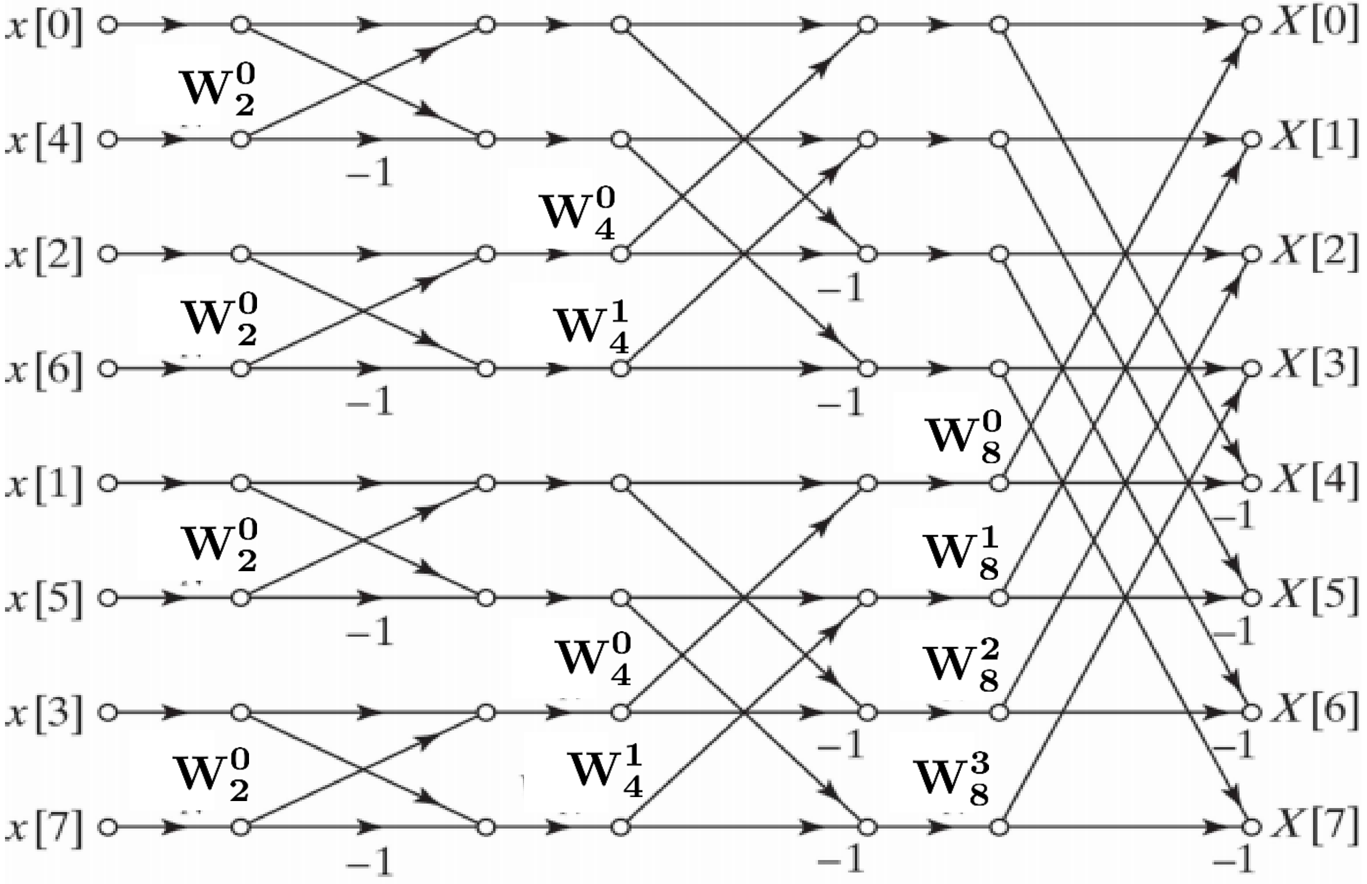
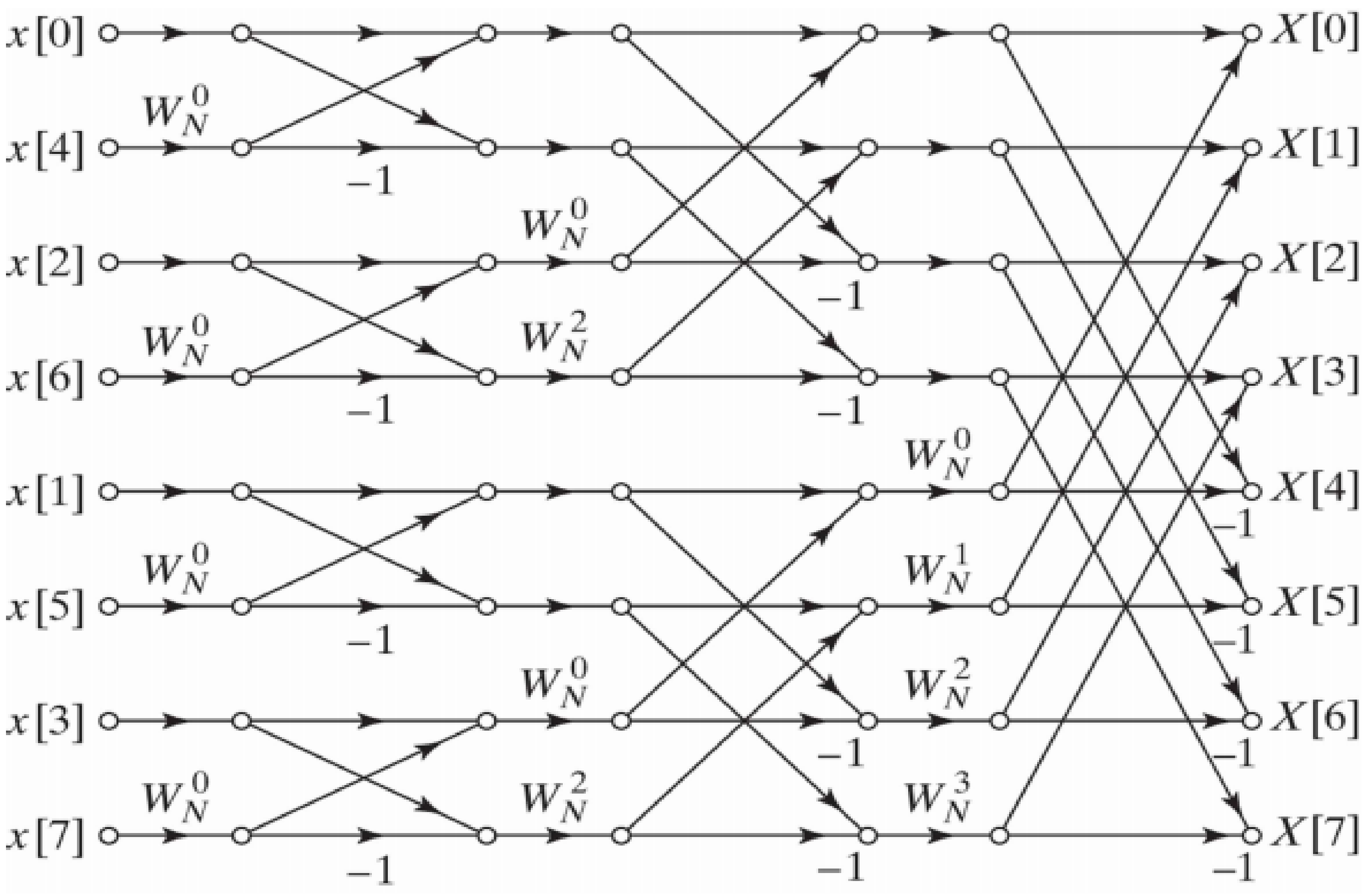


Figure 7.5 Three stages of computation in 8-point DFT.

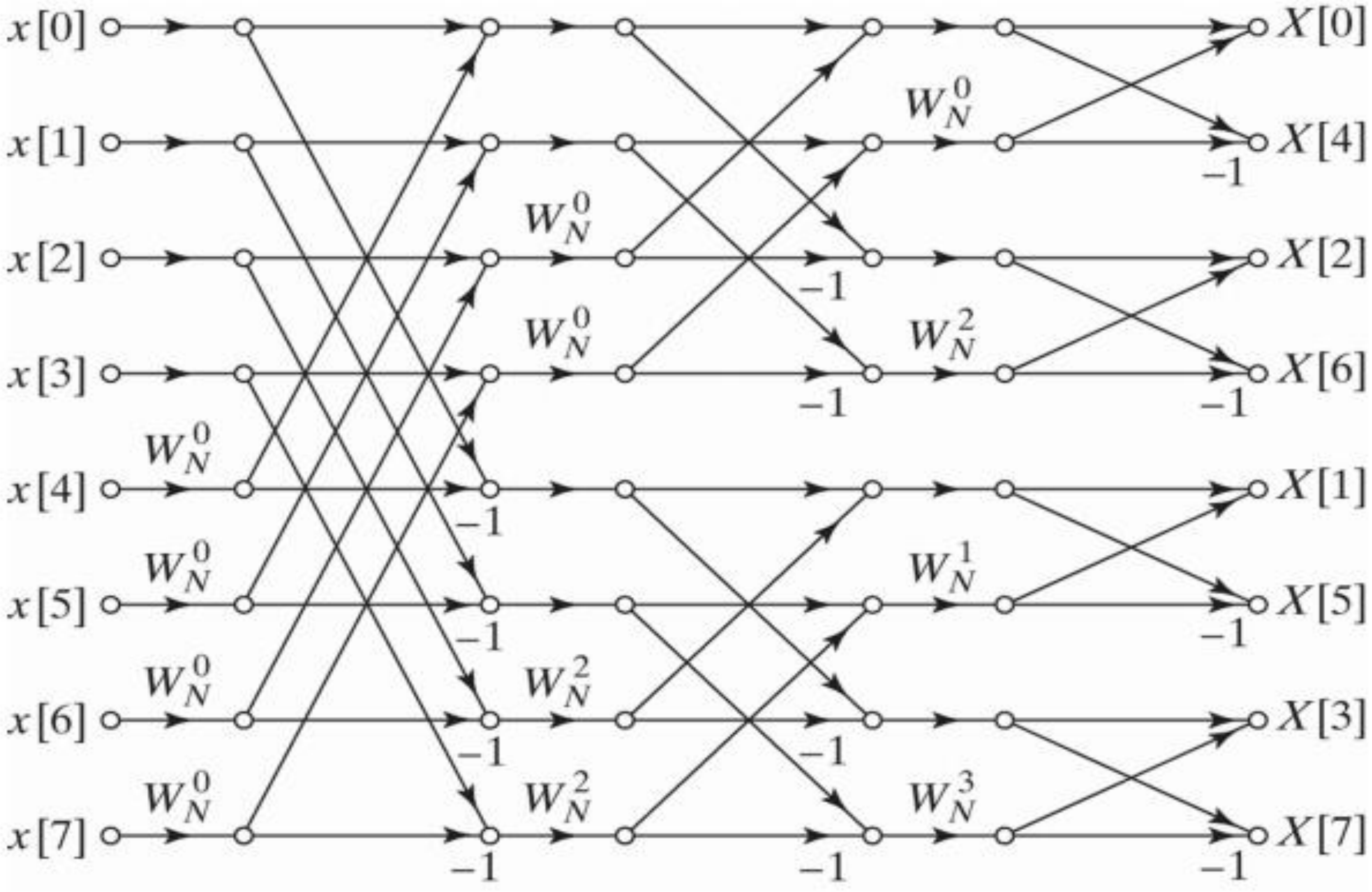


Fast Furier Transform

$\mathbf{N = 8}$



Fast Furier Transform



x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
W_N^0
W_N^0
W_N^0
W_N^0
W_N^0
W_N^0
W_N^2
W_N^2
W_N^0
W_N^2
W_N^1
W_N^3
–1
X[0]
X[4]
X[2]
X[6]
X[1]
X[5]
X[3]
X[7]

The End

Derivation

- The FT of DT aperiodic signal is represented by

$$X[\omega] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- Suppose that we sample X(ω) periodically in frequency domain at a spacing of $\delta\omega$ between two successive samples
- Since X(ω) is periodic with period 2π , only samples in period 0 to 2π are necessary.

- For convenience take N equidistance samples in the interval $0 \leq \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$. Replacing ω by ω_k.

$$\omega_k = \frac{2\pi k}{N} \qquad \mathbf{k = 0, 1, \ldots, N-1}$$

$$\therefore X[\omega_k] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega_k n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi k}{N} n}$$

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$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi k}{N}n}$$

- The summation can be divided in to infinite number of summations with N terms in each summation

$$\therefore X\left[\frac{2\pi k}{N}\right] = \ldots. + \sum_{n=-N}^{-1} x(n)e^{-j\frac{2\pi k}{N}n} + \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n} + \sum_{n=N}^{2N-1} x(n)e^{-j\frac{2\pi k}{N}n} + \ldots$$

$$\therefore X[\omega_k] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega_k n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi k}{N}n}$$

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi k}{N}n}$$

- The summation can be divided in to infinite number of summations with N terms in each summation

$$\therefore X\left[\frac{2\pi k}{N}\right] = \dots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \dots$$

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi k}{N}n}$$

- Replace n by n-lN

$$\therefore X\left[\frac{2\pi k}{N}\right]=\sum_{l=-\infty}^{\infty}\sum_{n=0}^{N-1}x(n-lN)e^{-j\frac{2\pi k}{N}n}\cancelto{1}{e^{j\frac{2\pi k}{N}lN}}$$

- Replace n by n-lN

$$\therefore X\left[\frac{2\pi k}{N}\right]=\sum_{l=-\infty}^{\infty}\sum_{n=0}^{N-1}x(n-lN)e^{-j\frac{2\pi k}{N}n}\cancelto{1}{e^{j\frac{2\pi k}{N}lN}}$$

- Interchanging the order of summation

$$\therefore X\left[\frac{2\pi k}{N}\right]=\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty}x(n-lN)\right]e^{-j\frac{2\pi k}{N}n}$$

- Replace n by n-lN

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-j\frac{2\pi k}{N}n} \cancelto{1}{e^{j\frac{2\pi k}{N}lN}}$$

- Interchanging the order of summation

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi k}{N}n}$$

- Let us define

$$\therefore x_p(n) = \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]$$

- Where x_p(n) is periodic extension of x(n) for every N samples.

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$

- Where x_p(n) is periodic extension of x(n) for every N samples.

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$

- x_p(n) can be expanded using FS as

$$\therefore x_p(n) = \sum_{k=0}^{N-1} C[k] e^{j\frac{2\pi k}{N}n}$$

(Discrete Fourier Series)

- Where $x_p(n)$ is periodic extension of $x(n)$ for every N samples.

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$

- $x_p(n)$ can be expanded using FS as

$$\therefore x_p(n) = \sum_{k=0}^{N-1} C[k] e^{j\frac{2\pi k}{N}n}$$

(Discrete Fourier Series)

- With FS coefficients

$$\therefore C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}$$

$$\therefore C[k] = \frac{1}{N} X\left[\frac{2\pi k}{N}\right] \quad \& \quad \therefore x_p(n) = \sum_{k=0}^{N-1} C[k] e^{j\frac{2\pi k}{N}n}$$

$$\therefore C[k] = \frac{1}{N} X\left[\frac{2\pi k}{N}\right]$$

&

$$\therefore x_p(n) = \sum_{k=0}^{N-1} C[k] e^{j\frac{2\pi k}{N} n}$$

Now substituting for C(k) in expression of x_p(n).

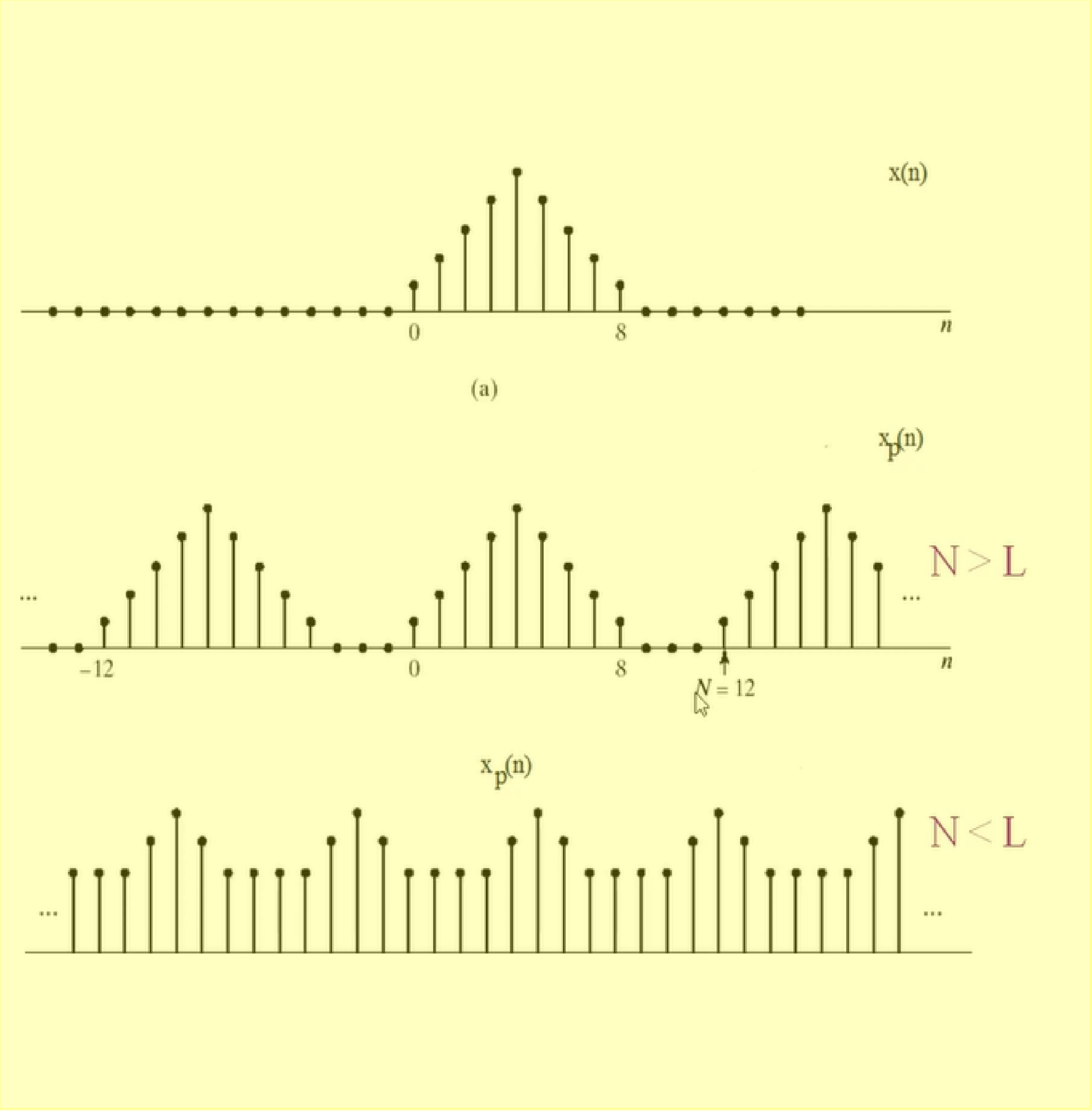
$$\therefore x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left[\frac{2\pi k}{N}\right] e^{j\frac{2\pi k}{N} n}$$

$$\therefore C[k] = \frac{1}{N} X\left[\frac{2\pi k}{N}\right] \quad \& \quad \therefore x_p(n) = \sum_{k=0}^{N-1} C[k] e^{j\frac{2\pi k}{N}n}$$

Now substituting for C(k) in expression of x_p(n).

$$\therefore x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left[\frac{2\pi k}{N}\right] e^{j\frac{2\pi k}{N}n}$$

- This relationship gives reconstruction of the periodic signal x_p(n) from X(ω).
- Since x_p(n) is periodic extension of x(n). x(n) can be recovered from x_p(n) obtained by above expression if there is no aliasing in the time domain.



- That is the length L of $x(n)$ should be smaller than N the period of $x_p(n)$.

$$\therefore X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$

Where $x(n)=x_p(n)$ for $0 \le n \le N-1$

$$\therefore x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left[\frac{2\pi k}{N}\right] e^{j\frac{2\pi k}{N}n}$$

- The expression obtained by sampling the $X(\omega)$ is called as discrete Fourier transform (DFT).

$$\therefore X[k] = X\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}$$

- And the reconstruction expression is referred as Inverse Discrete Fourier Transform(IDFT)

$$\therefore x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n}$$

- Important to note here that N the length of DFT should be larger than L the length of signal x(n), i.e. N≥L

Discrete Fourier Transform DFT

- The formulas for DFT and IDFT may be expressed as (in terms of twiddle factor)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad 0 \le n \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

- The relationship between $x(n)$ and $X(k)$ is denoted as

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$