MANIPAL INSTITUTE OF TECHNOLOGY
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Modern Control Theory (ICE 3153)

Linearization of Nonlinear Systems

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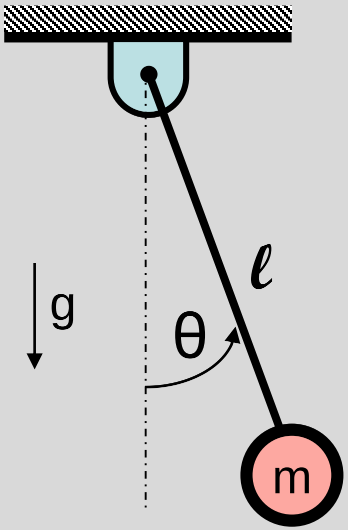
Introduction

- In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around the equilibrium.

- If the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system.

- Such a linear system is equivalent to the nonlinear system considered within a limited operating range.

Example: Consider the undamped simple pendulum



$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

$$x_1 = \theta\,,\ x_2 = \dot{\theta} = \omega$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\frac{g}{\ell}\sin\theta$$

Equilibrium points: (0, 0) and (π, 0)

Stable Unstable

$$\left.\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g}{\ell}\sin\theta\end{aligned}\right\}$$

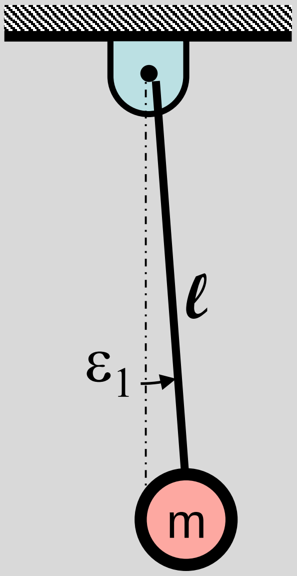
For ℓ=1 m

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -9.81\sin\theta$$

At the equilibrium, all derivatives are zero

$$\left.\begin{aligned}0 &= \omega_d \\ 0 &= -9.81\sin\theta_d\end{aligned}\right\}$$

$$\omega_d = 0$$
$$\theta_d = \begin{cases}0 \text{ rad} \\ \pi \text{ rad}\end{cases}$$



Consider the small perturbations around the equilibrium point θ_d=0

$$\theta = \theta_d + \varepsilon_1 = \varepsilon_1$$
$$\omega = \omega_d + \varepsilon_2 = \varepsilon_2$$

Nonlinear terms can be linearized using the Maclaurin series.

$$\dot{\varepsilon}_1 = \varepsilon_2$$
$$\dot{\varepsilon}_2 = -9.81\ \sin\varepsilon_1$$

$$\boxed{f(\varepsilon) = f(0) + \frac{f'(0)}{1!}\varepsilon + \frac{f''(0)}{2!}\varepsilon^2 + \ldots}$$

For θ_d=0

$$\sin(\varepsilon_1) = \sin(0) + \frac{\cos(0)}{1}\varepsilon_1 - \frac{\sin(0)}{2}\varepsilon^2 + \ldots$$

$$\sin(\varepsilon_1) \approx \varepsilon_1$$

$$\boxed{\begin{aligned}\cos(\varepsilon_1) &= \cos(0) - \frac{\sin(0)}{1}\varepsilon_1 - \frac{\cos(0)}{2}\varepsilon^2 + \ldots \\ \cos(\varepsilon_1) &\approx 1\end{aligned}}$$

Higher order term

$$\dot\varepsilon_1 = \varepsilon_2$$
$$\dot\varepsilon_2 = -9.81\ \varepsilon_1$$

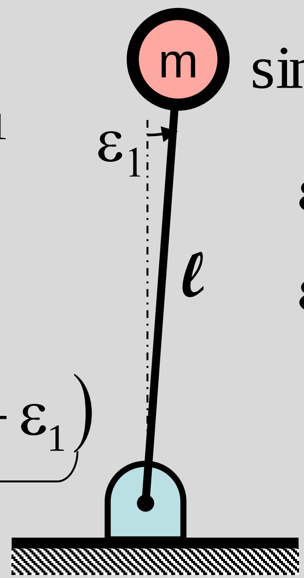
$$\begin{bmatrix}\dot\varepsilon_1\\ \dot\varepsilon_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -9.81 & 0\end{bmatrix}}_{A}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}$$

```
clc;clear;
A=[0 1;-9.81 0];
eig(A)

ans =

   0 + 3.1321i
   0 - 3.1321i
```

Marginally stable

For θ_d=$\boldsymbol{\pi}$

$$\theta = \theta_d + \varepsilon_1 = \pi + \varepsilon_1$$
$$\omega = \omega_d + \varepsilon_2 = \varepsilon_2$$
$$\dot\varepsilon_1 = \varepsilon_2$$
$$\dot\varepsilon_2 = -9.81\ \underbrace{\sin(\pi + \varepsilon_1)}_{-\varepsilon_1}$$



$$\sin(\pi + \varepsilon_1) = \sin(\pi)\cos(\varepsilon_1) + \cos(\pi)\sin(\varepsilon_1) \approx -\varepsilon_1$$

$$\dot\varepsilon_1 = \varepsilon_2$$
$$\dot\varepsilon_2 = 9.81\ \varepsilon_1$$

$$\begin{bmatrix}\dot\varepsilon_1\\ \dot\varepsilon_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 9.81 & 0\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}$$

```
clc;clear;
A=[0 1;9.81 0];
eig(A)

ans =

    3.1321
   -3.1321
```

Unstable

Example:

Mathematical model of a nonlinear system is given by the equation

$$2\ddot{x} + 18\dot{x} + 128000\frac{x^2}{(x+2)} = 0.03f$$

Where f(t) is the input and x(t) is the output of the system.

Find the equilibrium points for f=80 and linearize the system for small deviations from the equilibrium points. Find the response of the system

The state variables are chosen as x_1=x and x_2=dx/dt=dx_1/dt

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -9x_2 - 64000\frac{x_1^2}{(x_1+2)} + 0.015f$$

>>solve('64000*x1^2/(x1+2)=1.2')

x_{1d}=0.00613, x_{2d}=0

For the equilibrium condition

$$0 = x_{2d}$$

$$0 = -9x_{2d} - 64000\frac{x_{1d}^2}{(x_{1d}+2)} + 0.015*80$$

$x_1 = x_{1d} + \varepsilon_1 = 0.00613 + \varepsilon_1$

$x_2 = x_{2d} + \varepsilon_2 = \varepsilon_2$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -9x_2 - 64000\frac{x_1^2}{(x_1+2)} + 0.015\,f$$

$x_1 = x_{1d} + \varepsilon_1 = 0.00613 + \varepsilon_1$

$x_2 = x_{2d} + \varepsilon_2 = \varepsilon_2$

$f = f_d + u$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 - 64000\frac{(x_{1d}+\varepsilon_1)^2}{(x_{1d}+2+\varepsilon_1)} + 0.015(f_d + u)$$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 - 64000\left(\frac{1}{2+x_{1d}} - \frac{1}{(2+x_{1d})^2}\varepsilon_1\right)\left(x_{1d}^2 + 2x_{1d}\varepsilon_1 \cancel{+\varepsilon_1^2}\right) + 0.015\,f_d + 0.015\,u$$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 - \left(\frac{64000}{2+x_{1d}} - \frac{64000}{(2+x_{1d})^2}\varepsilon_1\right)\left(x_{1d}^2 + 2x_{1d}\varepsilon_1 + \varepsilon_1^2\right) + 0.015\,f_d + 0.015\,u$$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 - \left(\frac{64000\,x_{1d}^2}{2+x_{1d}} + \frac{128000\,x_{1d}\varepsilon_1}{2+x_{1d}} - \frac{64000}{(2+x_{1d})^2}\varepsilon_1 x_{1d}^2 - \cancel{\frac{128000\,x_{1d}\varepsilon_1^2}{(2+x_{1d})^2}}\right) + 0.015\,f_d + 0.015\,u$$

$$0 = x_{2d}$$

$$0 = -9\underbrace{x_{2d}}_{=0} - 64000\frac{x_{1d}^2}{(x_{1d}+2)} + 0.015 * fd$$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 \underbrace{- \frac{64000\,x_{1d}^2}{2+x_{1d}} + 0.015\,f_d}_{=0} \underbrace{- \frac{128000\,x_{1d}\varepsilon_1}{2+x_{1d}} + \frac{64000}{(2+x_{1d})^2}\varepsilon_1 x_{1d}^2}_{-390.52\,\varepsilon_1} + 0.015\,u$$

$$\dot{\varepsilon}_1 = \varepsilon_2$$

$$\dot{\varepsilon}_2 = -9\varepsilon_2 - 390.52\,\varepsilon_1 + 0.015\,u$$

$$\begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -390.52 & -9 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.015 \end{bmatrix} u$$

```
clc;clear;
A=[0 1;-390.52 -9];
eig(A)
```

```
ans =

  -4.5000 +19.2424i
  -4.5000 -19.2424i
```

Stable system