7. Find the deflection $u(x,t)$ of the vibrating string of length π with zero initial velocity and initial deflection $f(x) = k\sin 3x$. Take $c^2 = 1$

8. $u(x,0) = k(\sin x - \sin 2x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$. Find $u(x,t)$ Take $c^2 = 1$.

9. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Given $l = \pi$, $u(0,t) = 0$
$u(\pi, t) = 0$

$$u(x,0) = f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$= \sum \left(\frac{40}{n\pi} \cos\frac{n\pi}{2} + \frac{200}{n^2\pi^2} \sin\frac{n\pi}{2} \right) \sin\frac{n\pi x}{10} \, e^{-\frac{n^2\pi^2 t}{100}}$$

Exercise

- Solve:

(1) $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

(2) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$

(3) $u_{yy} = u_x$

(4) $u_x + u_y = 2(x+y)u$

(5) $u_{xx} + u_{xy} - 2u_{yy} = 0$. Given $v = x+y$, $z = 2x-y$

(6) $u_{xx} - 2u_{xy} + u_{yy} = 0$, $v = x$, $z = x+y$.

$u(x,0) = f(x) \Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$ which is a half range sine series

$$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

Example

1. Find the temperature $u(x,t)$ satisfying $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in a silver rod of length 10 cm.

$$u(x,0) = \begin{cases} x, & 0 \le x \le 5 \\ 10-x, & 5 \le x \le 10 \end{cases}$$

$c^2 = 1 \quad l = 10 \quad f(x) = \begin{cases} x, & 0 \le x \le 5 \\ 10-x, & 5 \le x \le 10 \end{cases}$

$$a_n = \frac{2}{10}\int_0^{10} f(x) \sin \frac{n\pi x}{10}\, dx$$

$$= \frac{1}{5}\left\{ \int_0^5 x \sin \frac{n\pi x}{10}\, dx + (10-x) \sin \frac{n\pi x}{10}\, dx \right\}$$

$$= \frac{1}{5}\left\{ \left[x\left(-\cos \frac{n\pi x}{10} \times \frac{10}{n\pi}\right) - \left(-\sin \frac{n\pi x}{10} \times \frac{100}{n^2\pi^2}\right) \right]_0^5 + \left[(10-x)\left(-\cos \frac{n\pi x}{10} \times \frac{10}{n\pi}\right) - (-1)\left(-\sin \frac{n\pi x}{10} \frac{100}{n^2\pi^2}\right) \right]_5^{10} \right.$$

$$a_n = \frac{-10}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{50}{n\pi} \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \times \frac{100}{n^2\pi^2}$$

One dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ $u(x,t) \to$ temperature

Using variable Seperable method we obtain 3 types of solutions

(1) $u(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right) c_3 e^{c^2p^2t}$

(2) $u(x,t) = \left(c_1 \cos px + c_2 \sin px\right) c_3 e^{-c^2p^2t}$

(3) $u(x,t) = \left(c_1 x + c_2\right) c_3$

$u(x,t)$ decreases as time t increases

$\therefore$ $u(x,t) = \left(c_1 \cos px + c_2 \sin px\right) c_3 e^{-c^2p^2t}$

Boundary conditions

Suppose a bar in which heat flows is of finite length l. End points are $x=0$ and $x=l$.

$$u(0,t)=0, \quad u(l,t)=0$$

Let $u(x,0) = f(x)$ be the initial temp

By applying the initial condition and the boundary conditions we get

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}$$

$$= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\, e^{-\lambda_n^2 t}$$

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos nct+b_n\sin nct\right)\sin nx$$

Given $\quad u(x,0)=f(x)=kx(\pi-x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)=0 \Rightarrow b_n=0$$

$$\therefore\ a_n=\frac{2k}{\pi}\int_0^{\pi}x(\pi-x)\sin nx\,dx$$

$$=\frac{2k}{\pi}\left[x(\pi-x)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$=-\frac{4k}{\pi}\left[\frac{\cos n\pi-1}{n^3}\right]=\frac{4k\left(1-(-1)^n\right)}{\pi n^3}$$

$$u(x,t)=\frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n^3}\cos nct\sin nx$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-a_n \frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + b_n \frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\right] \times \sin\frac{n\pi x}{l}$$

$$u(x,0) = f(x) \Rightarrow \quad f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad \rightarrow (1)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \rightarrow \quad g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l}\sin\frac{n\pi x}{l} \quad \rightarrow (2)$$

RHS of (1) and (2) are half range Fourier Sine Series for $f(x)$ and $g(x)$ respectively.

$$\therefore \quad a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\left(\frac{n\pi c}{l}\right) b_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Examples

1. Find the deflection $u(x,t)$ of the vibrating string of length π units fixed at both ends and vibrating zero initial velocity and initial deflection is given by $u(x,0) = kx(\pi - x)$

Soln is $\quad u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{\pi} + b_n \sin\frac{n\pi ct}{\pi}\right)\sin\frac{n\pi x}{\pi}$

Boundary conditions

Since the string is fixed at the ends at $x=0$ and $x=l$ we have two boundary conditions

$$u(0,t)=0, \quad u(l,t)=0$$

Let the initial deflection $u(x,0) = f(x)$ and the initial velocity $\left.\frac{\partial y}{\partial t}\right|_{t=0} = g(x)$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$u(0,t)=0 \Rightarrow c_1 (c_3 \cos cpt + c_4 \sin cpt) = 0 \Rightarrow c_1 = 0$$

$$u(x,t) = c_2 \sin px \,(c_3 \cos cpt + c_4 \sin cpt)$$

$$u(l,t)=0 \Rightarrow c_2 \sin pl \,(c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\Rightarrow \sin pl = 0 \Rightarrow pl = n\pi, \quad n=1,2,\dots$$

$$\therefore p = \frac{n\pi}{l}$$

$$u_n(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi}{l} ct + c_4 \sin \frac{n\pi}{l} ct\right)$$

$\therefore$ General solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$G'' - p^2c^2G = 0$$

$$(D^2 - p^2c^2)G = 0$$

$$\therefore G(t) = c_3 e^{cpt} + c_4 e^{-cpt}$$

$$\therefore u(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{cpt} + c_4 e^{-cpt}\right)$$

(2) $k < 0$, $k = -p^2$

$$F'' + p^2F = 0$$

$$F(x) = c_1 \cos px + c_2 \sin px$$

$$G'' + p^2c^2G = 0$$

$$G(t) = c_3 \cos cpt + c_4 \sin cpt$$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

(3) $k = 0$

$$F'' = 0, \quad G'' = 0$$

$$F(x) = c_1 + c_2 x, \quad G(t) = c_3 + c_4 t$$

$$u(x,t) = (c_1 + c_2 x)(c_3 + c_4 t)$$

As we are dealing with problems on vibrations $u(x,t)$ must be a periodic function of x and t. Hence it must contain trigonometric terms.

$$\therefore u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

One dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution by Variable Separable method

Consider an elastic string which is stretched to length l and then fixed at the end points. The problem is to determine the vibrations of the string at any point x and at any time $t > 0$.

Let $u(x,t) = F(x)\,G(t)$ be the solution.

$$u_x = G\frac{dF}{dx} \qquad u_{xx} = G\frac{d^2F}{dx^2} = GF''$$

$$u_{tt} = F\frac{d^2G}{dt^2} = FG''$$

$$FG'' = c^2 G F''$$

$$\frac{F''}{F} = \frac{G''}{c^2 G} = k$$

$$\therefore\ F'' - kF = 0\ ;\quad G'' - kc^2 G = 0$$

(1) $k > 0$. Let $k = p^2$

$$F'' - p^2 F = 0$$

if $\frac{d}{dx} = D$ then $(D^2 - p^2)F = 0$

$$F(x) = c_1 e^{px} + c_2 e^{-px}$$

$$u_y = u_v v_y + u_z z_y = x u_z \qquad u_z$$

$$u_{yy} = (v u_z)_y = v \quad z + v\,(u_z)_y$$

$$= v\left[(u_z)_v v_y + u_{zz} z_y\right]$$

$$= v\, u_{zz} x = v^2 u_{zz}$$

$v = x$
$z = xy$
$y = \frac{z}{v}$

$\therefore$ The Given P.D.E becomes

$$v\left[\, v u_{vz} + u_z + z u_{zz}\right] - \frac{z}{v} v^2 u_{zz} - v u_z = 0$$

$$v^2 u_{vz} + \cancel{v u_z} + \cancel{v z u_{zz}} - \cancel{v z u_{zz}} - \cancel{v u_z} = 0$$

$$v^2 u_{vz} = 0 \Rightarrow u_{vz} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial v \partial z} = 0$$

Integrate w.r.to v

$$\frac{\partial u}{\partial z} = f_1(z)$$

Integrate w.r.to z

$$u = \int f_1(z)\, dz + f_2(v)$$

$$u(v,z) = F_1(z) + f_2(v)$$

$$\therefore u(x,y) = \underline{F_1(xy) + f_2(x)}$$

$$\therefore\ u_{vv} + 2u_{vz} + u_{zz} - 2u_{vz} - 2u_{zz} + u_{zz} = 0$$

$$\Rightarrow u_{vv} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial v^2} = 0$$

$$\frac{\partial u}{\partial v} = f(z)$$

$$u = \int f(z)\,dz + g(z) = F(z) + g(z)$$

$$u(x,y) = F(\underline{\underline{x-y}}) + g(x-y)$$

3. Solve: $x u_{xy} = y u_{yy} + u_y$

Given $v = x, \quad z = xy.$

$$u_x = u_v v_x + u_z z_x = u_v + u_z y$$

$$u_{xy} = (u_v + y u_z)_y = (u_v)_y + (y u_z)_y$$

$$= (u_v)_v \cdot v_y + (u_v)_z \cdot z_y$$

$$+ u_z + y (u_z)_y$$

$$= x u_{vz} + u_z + y\left[(u_z)_v v_y + u_{zz} x\right]$$

$$= v u_{vz} + u_z + z u_{zz}$$

Integration w.r.to v

$$\frac{\partial u}{\partial z} = f(z)$$

Integrate w.r.to z

$$u = \int f(z)dz + g(v) = F(z) + g(v)$$

$$\underline{u(x,y) = F(x+y) + g(x)}$$

2. solve: $u_{xx} + 2u_{xy} + u_{yy} = 0$

Given $v = x$, $z = x - y$.

$$u_x = u_v v_x + u_z z_x = u_v + u_z$$

$$u_{xx} = (u_v + u_z)_x = (u_v + u_z)_v v_x + (u_v + u_z)_z \cdot z_x$$

$$= u_{vv} + u_{zv} + u_{vz} + u_{zz}$$

$$\therefore u_{xx} = u_{vv} + 2u_{vz} + u_{zz}.$$

$$u_{xy} = (u_v + u_z)_y = (u_v + u_z)_v \cdot v_y + (u_v + u_z)_z \cdot z_y$$

$$= -u_{vz} - u_{zz}$$

$$u_y = u_v v_y + u_z z_y = -u_z$$

$$u_{yy} = (-u_z)_y = (-u_z)_v v_y + (-u_z)_z z_y$$

$$= u_{zz}$$

4. Method of Indicated Transformations.

We can transform a given PDE in a suitable way by introducing new independent variables.

Examples

1. Solve: $u_{xy} - u_{yy} = 0$, $v = x$, $z = x + y$.

$$u_x = u_v \cdot v_x + u_z \cdot z_x = u_v + u_z$$

$$u_{xy} = (u_v + u_z)_y$$

$$= (u_v + u_z)_v \cdot v_y + (u_v + u_z)_z \cdot z_y$$

$$= (u_v + u_z)_v \cdot 0 + (u_{vz} + u_{zz}) \cdot 1$$

$$u_{xy} = u_{vz} + u_{zz}$$

$$u_y = u_v v_y + u_z z_y = u_z$$

$$u_{yy} = (u_z)_v \cdot v_y + (u_z)_z z_y$$

$$= u_{zz}$$

$$\therefore u_{vz} + u_{zz} - u_{zz} = 0$$

$$u_{vz} = 0$$

$$\frac{\partial^2 u}{\partial v \partial z} = 0$$

Case 2 $k < 0$. Let $k = -p^2$

$$F = c_1 x^{-p^2}, \quad G = c_2 y^{p^2}$$

$$u(x,y) = c_1 c_2 x^{-p^2} y^{p^2}$$

Case 3 $k = 0$

$$\frac{x}{F}\frac{dF}{dx} = 0 \Rightarrow \frac{dF}{F} = 0$$

$$\log F = \log c_1 \Rightarrow F(x) = c_1$$

|||ly $G(y) = c_2$

$$u(x,y) = c_1 c_2 = C$$

2. solve: $xu_x + yu_y = 0$

Let $u(x,y) = F(x)G(y)$

$u_x = G\frac{dF}{dx}$; $u_y = F\frac{dG}{dy}$

$\therefore xG\frac{dF}{dx} + yF\frac{dG}{dy} = 0$

$\frac{x}{F}\frac{dF}{dx} + \frac{y}{G}\frac{dG}{dy} = 0$

$\frac{x}{F}\frac{dF}{dx} = k$, $-\frac{y}{G}\frac{dG}{dy} = k$

Case 1 $k > 0$, Let $k = p^2$

$\frac{x}{F}\frac{dF}{dx} = p^2$

$\frac{dF}{dx} = \frac{p^2 F}{x}$

$\frac{dF}{F} = p^2\frac{dx}{x}$

$\log F(x) = p^2\log x + \log c_1$

$= \log c_1 x^{p^2}$

$F(x) = c_1 x^{p^2}$

$\frac{dG}{dy} = -\frac{p^2 G}{y}$

$\frac{dG}{G} = -p^2\frac{dy}{y}$

$\log G(y) = -p^2\log y + \log c_2$

$\log G(y) = \log c_2 y^{-p^2}$

$G(y) = c_2 y^{-p^2}$

$\therefore u(x,y) = c_1c_2 x^{p^2} y^{-p^2} = \underline{c\, x^{p^2} y^{-p^2}}$

Case 2 $\quad k<0 \quad$ Let $k=-p^2$

$$\frac{d^2F}{dx^2} = -p^2F \quad ; \quad \frac{d^2G}{dy^2} = p^2G.$$

$$(D^2+p^2)F=0 \qquad\qquad (D^2-p^2)G=0$$

$$F(x)=c_1\cos px+c_2\sin px \quad ; \quad G(y)=c_3e^{py}+c_4e^{-py}$$

$$\therefore\ u(x,y)=(c_1\cos px+c_2\sin px)(c_3e^{py}+c_4e^{-py})$$

Case 3 $\quad k=0$

$$\frac{1}{F}\frac{d^2F}{dx^2}=0, \qquad \frac{1}{G}\frac{d^2G}{dy^2}=0$$

$$\frac{d^2F}{dx^2}=0 \Rightarrow D^2F=0$$

$$m^2=0,\ m=0,0$$

$$F(x)=c_1x+c_2 \qquad G(y)=c_3y+c_4$$

$$\therefore\ u(x,y)=(c_1x+c_2)(c_3y+c_4)$$

Case 1 Let $k > 0$ say $k = p^2$, $p \neq 0$

$$\frac{d^2F}{dx^2} = p^2 F \quad ; \quad \frac{d^2G}{dy^2} + p^2 G = 0$$

$$\frac{d^2F}{dx^2} - p^2 F = 0 \qquad (D^2 + p^2) G = 0$$

$$D = \pm p i$$

$$(D^2 - p^2) F = 0$$

$$D = \pm p \qquad G(y) = c_3 \cos py + c_4 \sin py.$$

$$\therefore F(x) = c_1 e^{px} + c_2 e^{-px}$$

$$\therefore u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right)$$

3. Method of Separation of Variables.

It involves a solution which break up into a product of functions each of which contains only one of the variables.

Examples

1. Solve: $u_{xx} + u_{yy} = 0 \rightarrow (1)$ (Laplace's eqn)

Let $u(x,y) = F(x)G(y)$ be the solution of (1)

$$u_x = G\frac{dF}{dx}$$

$$u_{xx} = G\frac{d^2F}{dx^2}$$

$$u_{yy} = F\frac{d^2G}{dy^2}$$

$$\therefore G\frac{d^2F}{dx^2} + F\frac{d^2G}{dy^2} = 0$$

$\div$ by FG

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} \rightarrow (2)$$

Since x and y are independent (2) can hold good only if each side of (2) is equal to a constant k.

ie $\frac{1}{F}\frac{d^2F}{dx^2} = k$, $-\frac{1}{G}\frac{d^2G}{dy^2} = k$.

$$Z = c_1\cos x + c_2\sin x$$

When $x=0$, $Z = e^y$

$$\therefore e^y = c_1$$

$$\frac{\partial z}{\partial x} = -c_1\sin x + c_2\cos x$$

$x=0$, $\frac{\partial z}{\partial x} = 1$

$$\therefore 1 = c_2$$

$$\therefore Z = \underline{e^y\cos x} + \sin x$$

2. Solve $\frac{\partial u}{\partial y} + 2yu = 0$

$$\frac{du}{dy} + 2yu = 0$$

$$\frac{du}{u} + 2y\,dy = 0$$

soln: is $\log u + y^2 = c$

$$\log u = c - y^2 \Rightarrow \underline{u = e^{c-y^2}}$$

3. Solve: $u_x = 2xyu.$

$$\frac{du}{dx} = 2xyu$$

$$\frac{du}{u} = 2y\,x\,dx$$

$$\log u = 2y\frac{x^2}{2} = x^2y + c$$

$$u = e^{x^2y+c} \quad //$$

2. solve : $\dfrac{\partial^2 z}{\partial x \partial y} = 4x \sin(3xy)$

Integration w.r.to y

$$\frac{\partial z}{\partial x} = -4\cancel{x}\,\frac{\cos(3xy)}{3\cancel{x}} + f(x)$$

$$\frac{\partial z}{\partial x} = -\frac{4}{3}\cos(3xy) + f(x)$$

integrate w.r.to x

$$z = -\frac{4}{3}\,\frac{\sin(3xy)}{3y} + \int f(x)\,dx + g(y)$$

$$z = -\frac{4}{9}\,\frac{\sin(3xy)}{y} + F(x) + g(y)$$

2. If a PDE involves derivatives w.r.to one of the independent variables only, we may solve it like an ODE treating the other independent variable as parameter.

Ex:

1. solve : $\dfrac{\partial^2 z}{\partial x^2} + z = 0$. Given when $x = 0$, $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$.

$\dfrac{\partial}{\partial x} = D$

$$(D^2+1)z = 0$$

Auxilliary eqn: $m^2+1=0 \Rightarrow m = \pm i$

Partial differential Equations

An equation involving partial differential Coefficients of a function of two or more variables is known as a Partial differential equation (P.D.E)

Solution of PDE

1. Direct integration

solve:

1. $$\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0.$$

Integrate w.r.to x

$$\frac{\partial^2 z}{\partial x \partial y} + 18 \cdot \frac{x^2}{2} \cdot y^2 - \frac{\cos(2x-y)}{2} = f(y)$$

Integrate w.r.to x

$$\frac{\partial z}{\partial y} + 9 \frac{x^3}{3} y^2 - \frac{\sin(2x-y)}{4} = xf(y) + g(y)$$

Integrate w.r.to y

$$z + x^3y^3 - \frac{\cos(2x-y)}{4} = x\int f(y)\,dy + \int g(y)\,dy + h(x)$$

$$z + x^3y^3 - \frac{\cos(2x-y)}{4} = xF(y) + G(y) + h(x)$$