3. If $\vec{v} = \vec{\omega} \times \vec{r}$ then P.T $\omega = \frac{1}{2} \text{curl} \vec{v}$ where $\vec{\omega}$ is a constant vector.

Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$

$\vec{r} = x i + y j + z k$.

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y) i - (\omega_1 z - \omega_3 x) j + (\omega_1 y - \omega_2 x) k$$

$$\text{curl} \vec{v} = \nabla \times \vec{v} = \nabla \times (\vec{\omega} \times \vec{r})$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z)\right] i$$

$$- \left[\frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right] k$$

$$= (\omega_1 + \omega_1) i - (-\omega_2 - \omega_2) j + (\omega_3 + \omega_3) k$$

$$= 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\vec{\omega}$$

$$\therefore \vec{\omega} = \frac{1}{2} \text{curl} \vec{v}$$

2. If $\nabla \times \vec{A} = 0$ then evaluate $\nabla \cdot (\vec{A} \times \vec{r})$

Let $\vec{A} = A_1 i + A_2 j + A_3 k$.

$\vec{r} = x i + y j + z k$.

$$\vec{A} \times \vec{r} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (A_2 z - A_3 y) i - (A_1 z - A_3 x) j + (A_1 y - A_2 x) k$$

$$\nabla \cdot (\vec{A} \times \vec{r}) = z \frac{\partial A_2}{\partial x} - y \frac{\partial A_3}{\partial y} + x \frac{\partial A_3}{\partial y} - z \frac{\partial A_1}{\partial y} + y \frac{\partial A_1}{\partial z} - x \frac{\partial A_2}{\partial z}$$

$$= x \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + y \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial y} \right) + z \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= (x i + y j + z k) \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial y} \right) j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) k \right]$$

$$= \vec{r} \cdot (\nabla \times \vec{A}) = 0 \quad \because \nabla \times \vec{A} = 0.$$

$$= \left(\frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(-2y^2z^2)\right) i$$

$$- \left(\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial z}(x^2z^2)\right) j$$

$$+ \left(\frac{\partial}{\partial x}(-2y^2z^2) - \frac{\partial}{\partial y}(x^2z^2)\right) k$$

$$= (2xyz + 4yz^2) i - (y^2z - 2x^2z) j +$$

At $(1, -1, 1)$

$$\nabla \times \vec{A} = \underline{\underline{2i + j}}$$

The curl

Suppose $\vec{V}(x, y, z) = V_1 i + V_2 j + V_3 k$ is a differentiable vector field. Then the curl of $\vec{V}$ or rotation of $\vec{V}$, written $\nabla \times \vec{V}$ or curl $\vec{V}$ is given by

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Note:- Curl gives the measure of angular velocity of an object. If curl is zero, it means the object is not rotating.

If curl is not zero, its magnitude represents the speed of the object and its direction denotes the axis of rotation.

Examples

1. Let $\vec{A} = x^2 z^2 i - 2y^2 z^2 j + x y^2 z k$.

Find $\nabla \times \vec{A}$ at $(1, -1, 1)$

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ x^2 z^2 & -2y^2 z^2 & x y^2 z \end{vmatrix}$$

$\vec{V}$ is solenoidal if $\nabla\cdot\vec{V}=0$.

$$\nabla\cdot\vec{V} = -4+1+a = 0 \Rightarrow -3+a=0 \Rightarrow \underline{\underline{a=3}}$$

6. Prove that $\nabla^2|\vec{r}|^n = n(n+1)|\vec{r}|^{n-2}$

$$\nabla^2|\vec{r}|^n = \nabla\cdot\nabla|\vec{r}|^n$$

$$= \nabla\cdot\left(n|\vec{r}|^{n-2}\vec{r}\right)$$

$$= n\left[\nabla|\vec{r}|^{n-2}\cdot\vec{r} + |\vec{r}|^{n-2}\,\nabla\cdot\vec{r}\right]$$

$$= n\left[(n-2)|\vec{r}|^{n-4}\,\vec{r}\cdot\vec{r} + |\vec{r}|^{n-2}\,3\right]$$

$$= n\left[(n-2)|\vec{r}|^{n-4}|\vec{r}|^2 + 3|\vec{r}|^{n-2}\right]$$

$$= \underline{\underline{n(n+1)|\vec{r}|^{n-2}}}$$

7. If $\vec{A} = (2x^2+8xy^2z)i + (3x^3y-3xy)j - (4y^2z^2+2x^3z)k$

Iden S.T. $xyz^2\vec{A}$ is solenoidal.

$$xyz^2\vec{A} = (2x^3yz^2+8x^2y^3z^3)i + (3x^4y^2z^2-3x^2y^2z^2)j - (4xy^3z^4+2x^4yz^3)k$$

$$\nabla\cdot(xyz^2\vec{A}) = 6x^2yz^2+16xy^3z^3+6x^4yz^2-6x^2yz^2-16xy^3z^3-6x^4yz^2$$

$= 0 \Rightarrow xyz^2\vec{A}$ is solenoidal

$$= 3|\vec{r}|^{-5}\,|\vec{r}|^{2} - 3|\vec{r}|^{-3}$$

$$= 3|\vec{r}|^{-3} - 3|\vec{r}|^{-3} = \underline{\underline{0}}$$

5. Determine the constant a so that the vector

$$\vec{V} = (-4x-6y+3z)\hat{i} + (-2x+y-5z)j + (5x+6y+az)k$$

is solenoidal.

2) Given $\varphi = 6x^3y^2z$. Find $\nabla\cdot\nabla\varphi$

$$\nabla\varphi = 18x^2y^2z\,i + 12x^3yz\,j + 6x^3y^2\,k.$$

$$\nabla\cdot\nabla\varphi = 36xy^2z + 12x^3z$$

3) Show that $\nabla\cdot\nabla\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ → Laplacian operator.

$$\nabla\cdot\nabla\varphi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left(\frac{\partial\varphi}{\partial x}i + \frac{\partial\varphi}{\partial y}j + \frac{\partial\varphi}{\partial z}k\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi$$

$$= \nabla^2\varphi.$$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ → Laplace's eqn.

4) P.T. $\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 0$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\nabla|\vec{r}|^n = n|\vec{r}|^{n-2}\vec{r}$

$$\nabla\cdot\nabla\varphi = \nabla^2\varphi$$

$$\nabla|\vec{r}|^{-1} = -|\vec{r}|^{-3}\vec{r}$$

$$\nabla^2|\vec{r}|^{-1} = \nabla\cdot\nabla|\vec{r}|^{-1} = \nabla\cdot\left(-|\vec{r}|^{-3}\vec{r}\right)$$

$\boxed{\nabla\cdot(xi+yj+zk)}$

$$\left[\ \nabla\cdot(\varphi\vec{A}) = \nabla\varphi\cdot\vec{A} + \varphi(\nabla\cdot\vec{A})\ \right]$$

$$\nabla^2|\vec{r}|^{-1} = -\nabla|\vec{r}|^{-3}\cdot\vec{r} + -|\vec{r}|^{-3}\,\nabla\cdot\vec{r}$$

$$= 3|\vec{r}|^{-5}\vec{r}\cdot\vec{r} - |\vec{r}|^{-3}3$$

Divergence

Suppose $\vec{V} = V_1 i + V_2 j + V_3 k$ is defined and differentiable at each point (x, y, z) in a region of space (i.e. $\vec{V}$ defines a differentiable vector field). Then divergence of $\vec{V}$, written $\nabla \cdot \vec{V}$ or div $\vec{V}$ is given by

$$\nabla \cdot \vec{V} = \nabla \cdot (V_1 i + V_2 j + V_3 k)$$
$$= \frac{\partial}{\partial x} V_1 + \frac{\partial}{\partial y} V_2 + \frac{\partial}{\partial z} V_3$$

$\nabla \cdot \vec{V}$ is a scalar.

Physical interpretation of divergence

A fluid moves so that its velocity at any point is $\vec{V}(x, y, z)$. Then the loss of fluid per unit volume per unit time in a small parallepiped is approximately given by $\nabla \cdot \vec{V}$.

If there is no loss of fluid anywhere then $\nabla \cdot \vec{V} = 0$ and $\vec{V}$ is called solenoidal.

Examples

1. Let $\vec{A} = x^2 z^2 i - 2y^2 z^2 j + xy^2 z k$. Find $\nabla \cdot \vec{A}$ at $(1, -1, 1)$.

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(x^2 z^2 i - 2y^2 z^2 j + xy^2 z k\right)$$
$$= 2xz^2 - 4yz^2 + xy^2 = 2 + 4 + 1 = 7$$

at $(1, -1, 1)$