⑥

⑤ A solid of revolution is formed by rotating about x-axis, the area bounded by x-axis, the lines x=0 and x=1 and a curve through the points

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 0.0 | 0.25 | 0.50 | 0.75 | 1.00 |
| $y$ : | 1.00 | 0.9896 | 0.9589 | 0.9089 | 0.8415 |

Estimate the volume of the solid by numerical integration.

Note that Simpson's $\frac{1}{3}$rd Rule & $\frac{3}{8}$th Rule are not applicable.

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 0.0 | 0.25 | 0.50 | 0.75 | 1.00 |
| $y^2$ : | 1.0000 | 0.9793 | 0.9195 | 0.8261 | 0.7081 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

$$\text{Volume} = \int_0^1 \pi y^2 dx = \pi \times \frac{h}{2}\left[(y_0+y_4) + 2\,(y_1+y_2+y_3)\right]$$

$$= 2.8109 \text{ cubic units}$$

⑥ Using Simpson's $\frac{1}{3}$rd Rule evaluate $\int_0^1 \frac{1}{1+x}\,dx$ with $h=\frac{1}{6}$. Hence find an approximate value of $\log 2$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 0 | $\frac{1}{6}$ | $2/6$ | $3/6$ | $4/6$ | $5/6$ | 1 |
| $y=\frac{1}{1+x}$ : | 1 | $6/7$ | $6/8$ | $6/9$ | $6/10$ | $6/11$ | $1/2$ |

$$I = \int_0^1 \frac{1}{1+x}\,dx = \frac{1/6}{3}\left[\left(1+\tfrac{1}{2}\right) + 2\left(6/8 + 6/10\right) + 4\left(6/7 + 6/9 + 6/11\right)\right]$$

$$= 0.69317$$

$$\int_0^1 \frac{1}{1+x}\,dx = \log(1+x)\Big|_0^1 = \log 2 - \log 1 = \log 2$$

$\therefore$ Approximate value of $\log 2 = 0.69317$

Actual value
$\log 2 = 0.69314$

5

1) Trapezoidal Rule: $\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$

2) Simpson's $\frac{1}{3}$rd Rule: $\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots+y_{n-2}) + 4(y_1+y_3+\cdots+y_{n-1})\right]$

applicable only when number of subintervals = even.
number of points = odd.

3) Simpson's $\frac{3}{8}$th Rule: $\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\cdots+y_{n-3}) + 3(y_1+y_2+y_4+\cdots+y_{n-2}+y_{n-1})\right]$

applicable only when number of subintervals is a multiple of 3.
number of points is of the form $3k+1$, for some integer $k$.

4 A curve passes through the points $(1,2), (1.5, 2.4), (2, 2.7), (2.5, 2.8), (3,3), (3.5, 2.6), (4, 2.1)$.
Obtain the area bounded by the curve, $x$-axis and the lines $x=1$ & $x=4$.
Also find the volume of solid of revolution obtained by revolving this area about $x$-axis.

Solution:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $y$ : | 2 | 2.4 | 2.7 | 2.8 | 3 | 2.6 | 2.1 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

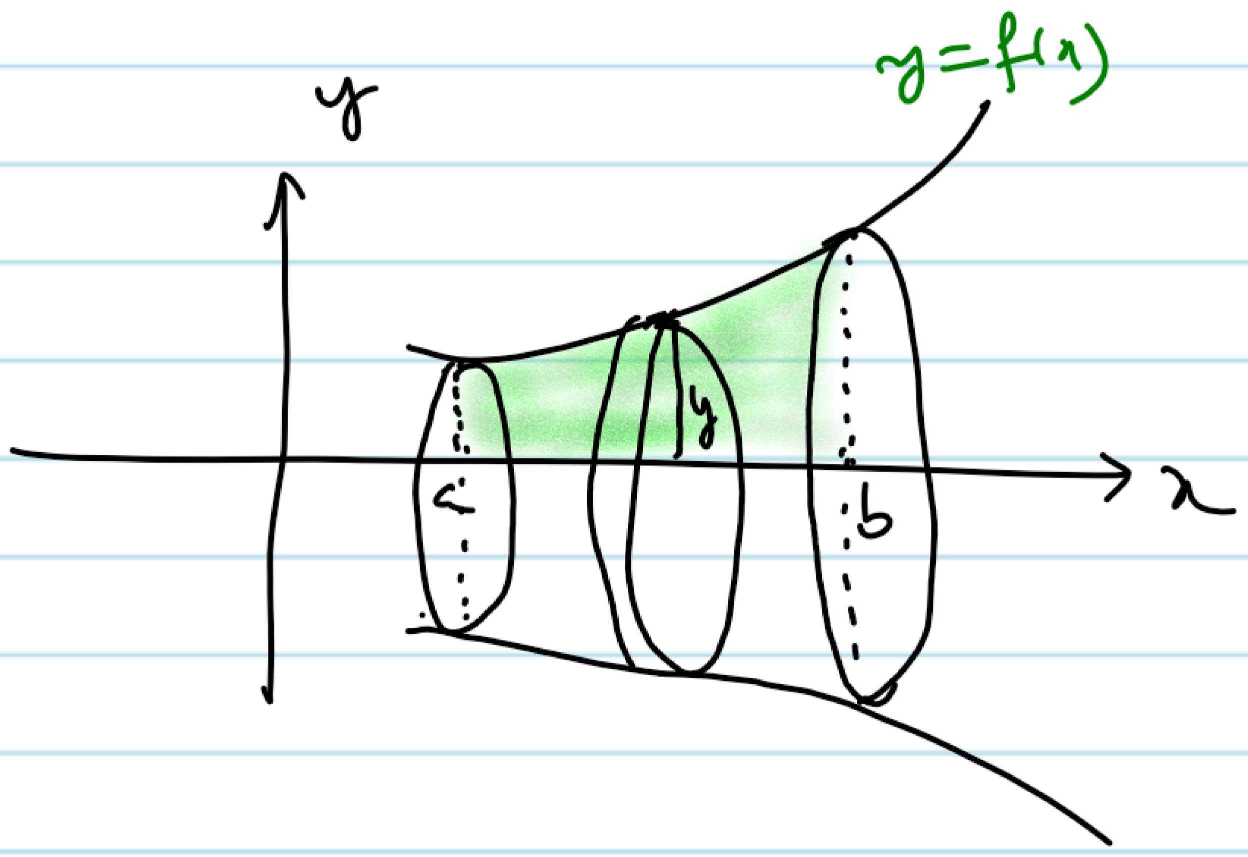
(All three formulae are applicable).

$$\text{Area} = \int_{x=1}^{4} y\,dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{0.5}{3}\left[(2+2.1) + 2(2.7+3) + 4(2.4+2.8+2.6)\right]$$

$$= 7.7833 \text{ sq. units}$$

Volume of Solid of Revolution:




Circular disk of thickness $= dx$
radius $= y$
$\therefore$ Area $= \pi y^2$
volume of the disk $= \pi y^2 dx$

$\therefore$ Volume of solid of revolution is

$$V = \int_{x=a}^{b} \pi y^2 dx$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $y^2$ : | 4 | 5.76 | 7.29 | 7.84 | 9 | 6.76 | 4.41 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

$$\therefore \text{Volume} = \int_{x=1}^{4} \pi y^2 dx = 3.1416 \times \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$= 64.1043$ cubic units

$64.1042$ cubic units.

(ii) Simpson's $\frac{1}{3}^{rd}$ Rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.0270)+4(0.5+0.1+0.0384)+2(0.2+0.0588)\right]$$

$$= 1.3660$$

(iii) Simpson's $\frac{3}{8}^{th}$ Rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= 1.35697$$

$$= 1.357$$

Actual value $= \int_0^6 \frac{1}{1+x^2}dx = \tan^{-1}x\Big|_0^6 = \tan^{-1}6 = 1.4056$

(2) Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Trapezoidal rule. Take $h=0.2$. Find an approximate value of $\pi$.

| $x$: | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $y=\frac{1}{1+x^2}$: | 1 | 0.96154 | 0.86207 | 0.73529 | 0.60976 | 0.5 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |

Note That Simpson's $1/3^{rd}$ & $\frac{3}{8}^{th}$ rules are not applicable.

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= 0.7837$$

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} = 0.7837 \Rightarrow \pi = 3.1348$$

actual value $= 3.14159$.

(3) Compute $\int_{0.2}^{1.4}(\sin x - \log x + e^x)\,dx$ using Simpson's $\frac{3}{8}^{th}$ rule.

$nh = 1.4-0.2 = 1.2$, $\quad n=6$, $\quad h = \frac{1.2}{6} = 0.2$

| $x$ : | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|
| $y=\sin x-\log x+e^x$ : | 3.0295 | 2.7975 | 2.8976 | 3.1660 | 3.5597 | 4.0698 | 4.7042 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{8}\left[(y_0+y_6)+2y_3+3(y_1+y_2+y_4+y_5)\right]$$

$$= 4.0529$$

## Simpson's $\frac{3}{8}$th Rule:

| | | | | |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| $x_1$ | $y_1$ | $\Delta y_0$ | | |
| $x_2$ | $y_2$ | $\Delta y_1$ | $\Delta^2 y_0$ | $\Delta^3 y_0$. |
| $x_3$ | $y_3$ | $\Delta y_2$ | $\Delta^2 y_1$ | |

Put $n=3$ in (*)

$$\int_{x_0}^{x_3} y\,dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \frac{3^2(6-3)}{12}\Delta^2 y_0 + \frac{3^2(3-2)^2}{24}\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{27}{12}(y_2-2y_1+y_0) + \frac{9}{24}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}\left[y_0+3y_1+3y_2+y_3\right]$$

Similarly

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3+3y_4+3y_5+y_6\right], \dots \quad \int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$$

Adding all, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+\dots+y_{n-3})+3(y_1+y_2+y_4+y_5+y_7+\dots)\right]$$

This is the Simpson's $\frac{3}{8}$th formula.

Error $= -\frac{3}{80}(x_n-x_0)h^4 y^{iv}(\alpha)$ $\quad y^{iv}(\alpha)$ is the maximum value of the fourth order derivative of $y$.

Simpson's $\frac{3}{8}$th Rule can be applied when the range $[x_0, x_n]$ is divided into a number of subintervals, which must be a multiple of 3. (or functions values must be $3k+1$).

① Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using (i) Trapezoidal Rule (ii) Simpson's $\frac{1}{3}$rd Rule (iii) Simpson's $\frac{3}{8}$th Rule by dividing the interval into 6 equal parts. Compare the result with the actual value.

$n$ = number of subintervals, $h$ is width. Then $nh = 6-0$

$h = \frac{6}{6} = 1 \quad \because n=6$

| $x$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y=\frac{1}{1+x^2}$: | 1 | 0.5 | 0.2 | 0.1 | 0.0588 | 0.0384 | 0.0270 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

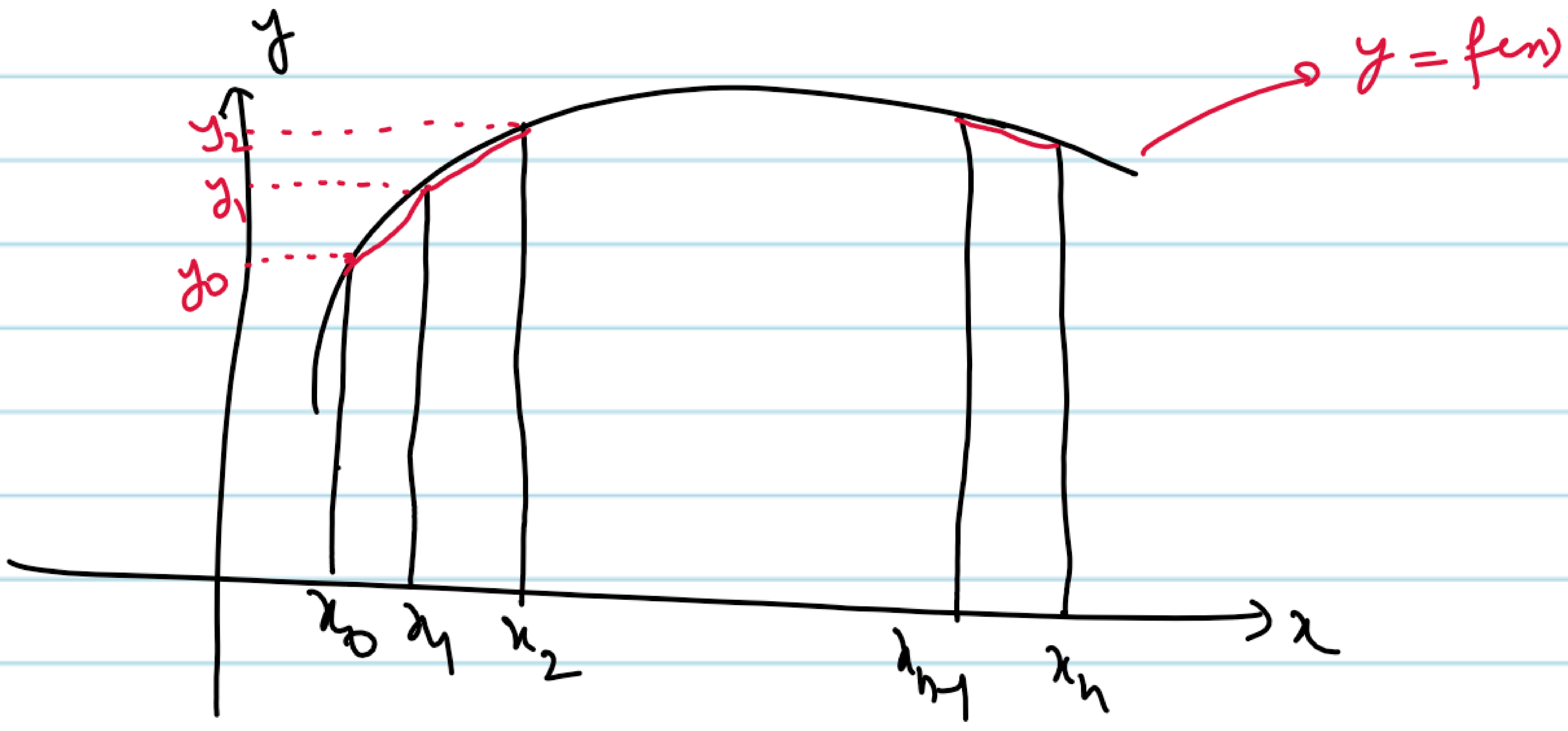
(i) Trapezoidal Rule:

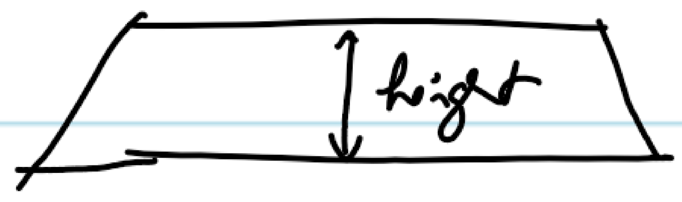
$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+\dots+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.0270)+2(0.5+0.2+0.1+0.0588+0.0384)\right]$$

$$= 1.407$$

## Geometric Significance:




Area of Trapezium

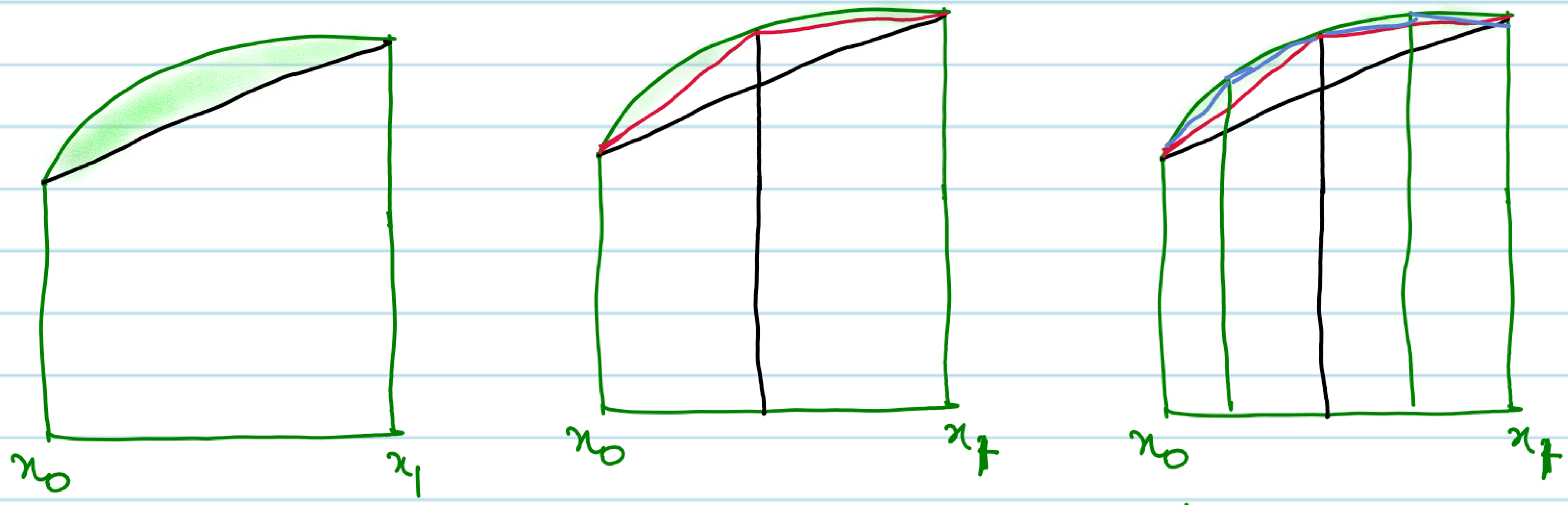
$= \frac{1}{2} \times \text{height} \times (\text{sum of } \|^{\text{el}} \text{ sides})$

1st trapezium, Area $= \frac{1}{2} \times h \times (y_0 + y_1)$

2nd trapezium Area $= \frac{1}{2} \times h \times (y_1 + y_2)$

Here the curve is replaced by n straight line segments joining the points $(x_0, y_0)$ & $(x_1, y_1)$ ; $(x_1, y_1)$ & $(x_2, y_n)$ ; ... $(x_{n-1}, y_{n-1})$ & $(x_n, y_n)$.

The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ & $x = x_n$ and x-axis is then approximately equal to the sum of the areas of n trapeziums.




Accuracy the formula increases with decrease in h.

## Simpson's $\frac{1}{3}$rd Rule:

Put $n=2$ in (*). We get

n=2

| | | | |
|---|---|---|---|
| $x_0$ | $y_0$ | | |
| | | $\Delta y_0$ | |
| $x_1$ | $y_1$ | | $\Delta^2 y_0$ |
| | | $\Delta y_1$ | |
| $x_2$ | $y_2$ | | |

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{2^2(4-3)}{12}\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly $\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$, .... $\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$

Hence

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \cdots + \int_{x_{n-2}}^{x_n} y\,dx$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

This is the Simpson's $\frac{1}{3}$rd rule.

Error $= \frac{-(x_n - x_0)}{180} h^4 y^{iv}(\alpha)$, $y^{iv}(\alpha)$ is maximum value of fourth derivative.

Note: This formula requires the division of the interval into an even number of subintervals (or odd number of function values).

## Numerical Integration:

Consider $(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)$, satisfying $y = f(x)$, where $f$ is not known explicitly.

The process of finding the value of the integral

$$\int_{x_0}^{x_n} y\,dx.$$

is called numerical integration.

Suppose that $x_i = x_0 + ih$, $i = 1, 2, \ldots, n$, $h > 0$.

We replace $y$ by Newton's forward difference polynomial.

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdot \quad \cdot\right] dx$$

$$= \int_{x_0}^{x_n} \left[y_0 + \Delta y_0\, p + \frac{\Delta^2 y_0}{2!}(p^2 - p) + \frac{\Delta^3 y_0}{3!}(p^3 - 3p^2 + 2p) + \cdots\right] dx$$

$x = x_0 + ph$

$dx = h\,dp$

$x = x_0 \Rightarrow p = 0$

$x = x_n \Rightarrow p = n$

$$= \int_0^n \left[y_0 + p\Delta y_0 + \frac{\Delta^2 y_0}{2!}(p^2 - p) + \frac{\Delta^3 y_0}{3!}(p^3 - 3p^2 + 2p) + \cdots\right] h\,dp$$

$$= h\left[y_0 p + \Delta y_0 \frac{p^2}{2} + \frac{\Delta^2 y_0}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right) + \frac{\Delta^3 y_0}{3!}\left(\frac{p^4}{4} - p^3 + p^2\right) + \cdots\right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \frac{\Delta^3 y_0}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right) + \cdots\right]$$

$$\int_{x_0}^{x_n} y\,dx = h\left[n y_0 + n^2 \frac{\Delta y_0}{2} + \frac{n^2(2n-3)}{12}\Delta^2 y_0 + \frac{n^2(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad (*)$$

The above formula of integration is called "Newton-Cotes General Quadrature Formula".

For $n = 1, 2, 3, \ldots$ we obtain different integration formula using $(*)$.

## Trapezoidal Rule:

| | | |
|---|---|---|
| $x_0$ | $y_0$ | $\Delta y_0$ |
| $x_1$ | $y_1$ | |

Put $n = 1$, in $(*)$. We get,

$$\int_{x_0}^{x_1} y\,dx = h\left[1 \times y_0 + \frac{1^2}{2}(\Delta y_0)\right] = h\left[y_0 + \frac{y_1 - y_0}{2}\right] = \frac{h}{2}\left[y_0 + y_1\right]$$

||ly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}\left[y_1 + y_2\right], \quad \cdots \quad \int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Hence,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \cdots + \int_{x_{n-1}}^{x_n} y\,dx$$

$$= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n)\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

This is the Trapezoidal rule.

Error in Trapezoidal Rule $= -\frac{1}{12}h^3 n\, y''(\alpha) = -\frac{(x_n - x_0)}{12}h^2 y''(\alpha)$, where $y''(\alpha)$ is the maximum value of the $y''$.