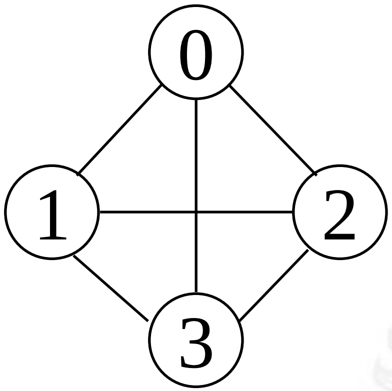
GRAPHS

Definitions

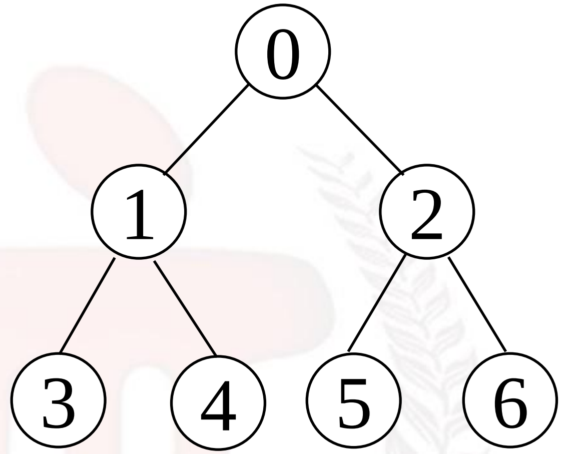
- A graph, *G=(V, E)*, consists of two sets:
  - a finite set of ***vertices(V)***, and
  - a finite, possibly empty set of edges***(E)***
  - *V(G)* and *E(G)* represent the sets of vertices and edges of *G*, respectively
- Undirected graph
  - The pairs of vertices representing any edges is ***unordered***
  - e.g., (v_0, v_1) and (v_1, v_0) represent the same edge (v0, v1) = (v1,v0)
- Directed graph
  - Each edge as a directed pair of vertices <v0, v1> != <v1,v0>
  - e.g. $<v_0, v_1>$ represents an edge, v_0 is the tail and v_1 is the head

Examples for Graph



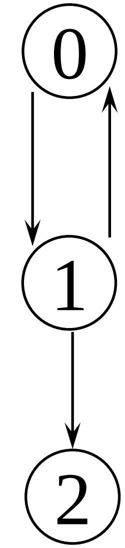
G_1

complete graph



G_2

incomplete graph



G_3

$V(G_1)=\{0,1,2,3\}$ $E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

$V(G_2)=\{0,1,2,3,4,5,6\}$ $E(G_2)=\{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3)=\{0,1,2\}$ $E(G_3)=\{<0,1>,<1,0>,<1,2>\}$

complete undirected graph: $n(n-1)/2$ edges

complete directed graph: $n(n-1)$ edges

Complete Graph

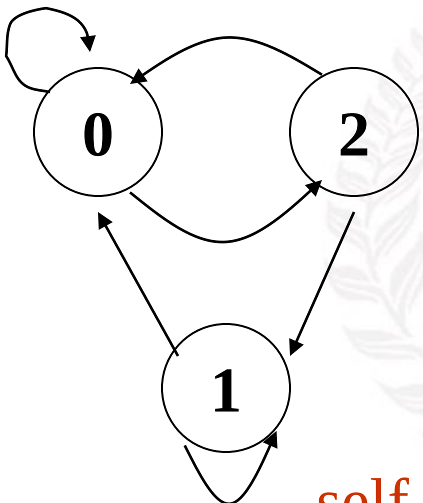
A complete graph is a graph that has the maximum number of edges

- for undirected graph with n vertices, the maximum number of edges is $n(n-1)/2$
- for directed graph with n vertices, the maximum number of edges is $n(n-1)$
- example: G1 (previous slide) is a complete graph

Adjacent and Incident

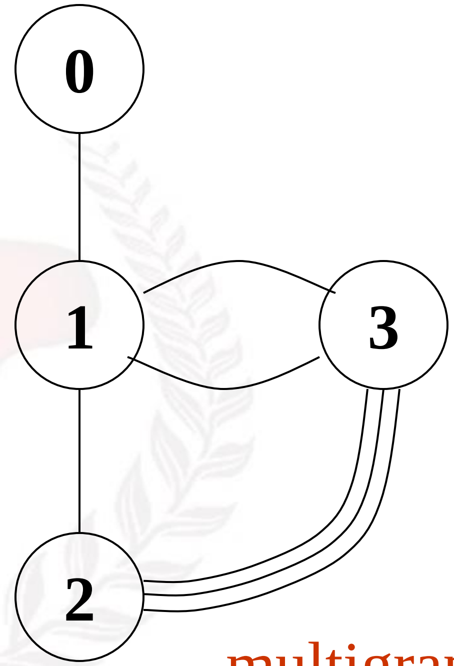
- If (v_0, v_1) is an edge in an undirected graph,
  - v_0 and v_1 are adjacent
  - The edge (v_0, v_1) is incident on vertices v_0 and v_1
- If $<v_0, v_1>$ is an edge in a directed graph
  - v_0 is adjacent to v_1, and v_1 is adjacent from v_0
  - The edge $<v_0, v_1>$ is incident on v_0 and v_1

Example of a graph with feedback loops and a multigraph



self edge

(a)



(b)

multigraph:
multiple occurrences
of the same edge

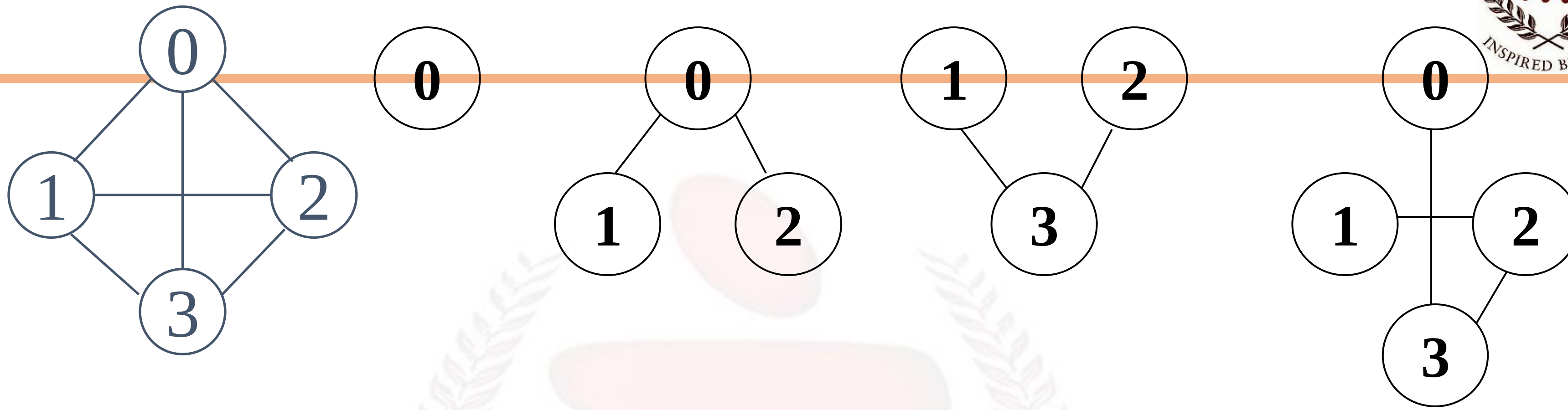
Subgraph and Path

- A subgraph of G is a graph G' such that V(G') is a subset of V(G) and E(G') is a subset of E(G).

- A path from vertex v_p to vertex v_q in a graph G, is a sequence of vertices, v_p, v_{i1}, v_{i2}, ..., v_{in}, v_q, such that (v_p, v_{i1}), (v_{i1}, v_{i2}), ..., (v_{in}, v_q) are edges in an undirected graph.

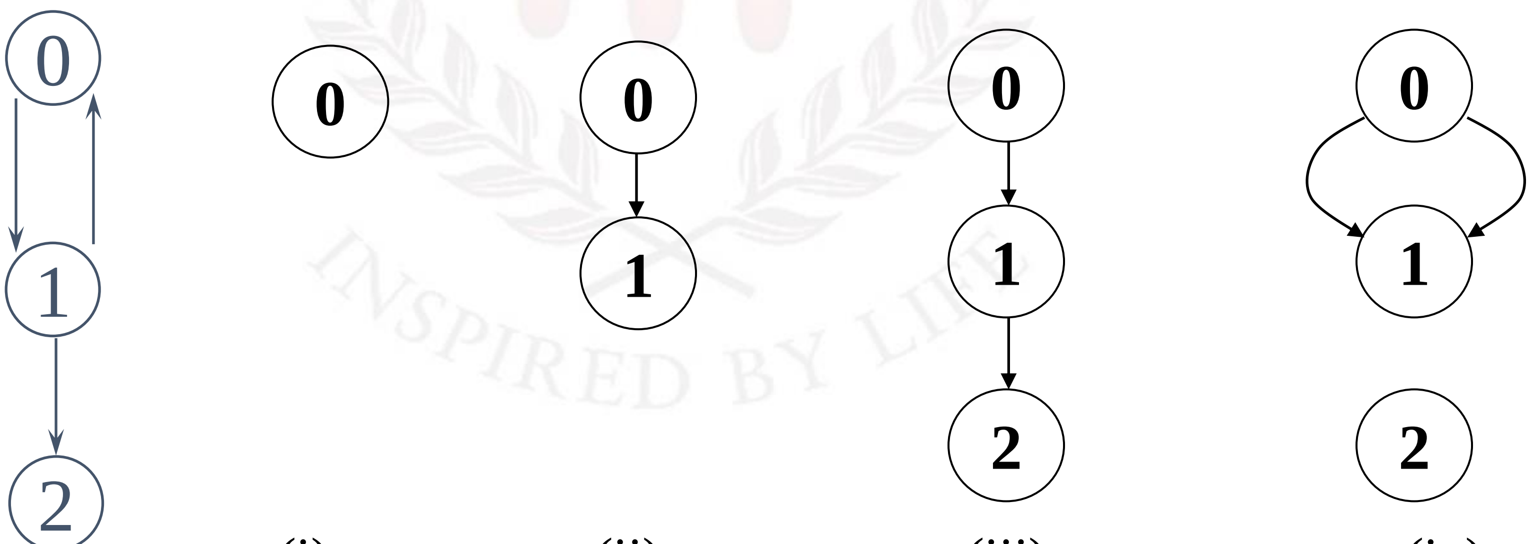
- The length of a path is the number of edges on it.

Subgraphs of G_1 and G_3



G_1 (i) (ii) (iii) (iv)

(a) Some of the subgraph of G_1



(i) (ii) (iii) (iv)

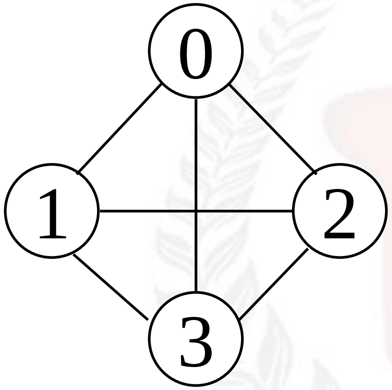
(b) Some of the subgraph of G_3

G_3

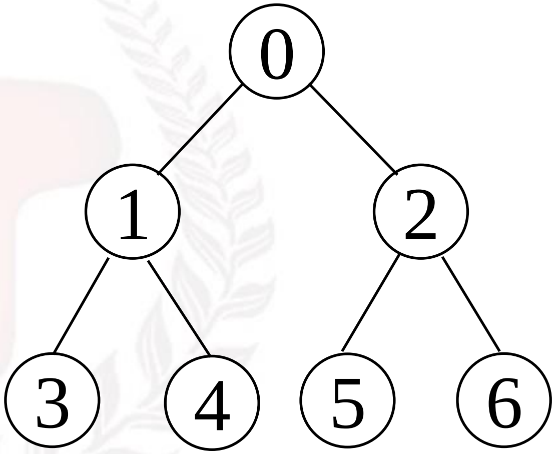
Simple Path and Style

- A simple path is a path in which all vertices, except possibly the first and the last, are distinct
- A cycle is a simple path in which the first and the last vertices are the same
- In an undirected graph G, ***two vertices***, v_0 and v_1, are connected if there is a path in G from v_0 to v_1
- An undirected ***graph*** is connected if, for every pair of distinct vertices v_i, vj, there is a path from vi to v_j

connected



G_1



G_2

tree (acyclic graph)

Degree

- The degree of a vertex is the number of edges incident to that vertex

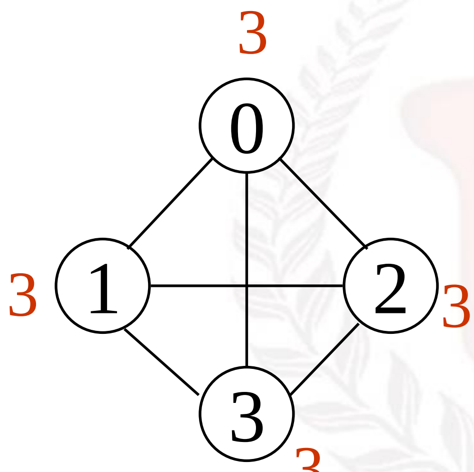
- For directed graph,
  - the in-degree of a vertex v is the number of edges that have v as the head
  - the out-degree of a vertex v is the number of edges that have v as the tail

  - if d_i is the degree of a vertex i in a graph G with n vertices and e edges, the number of edges is

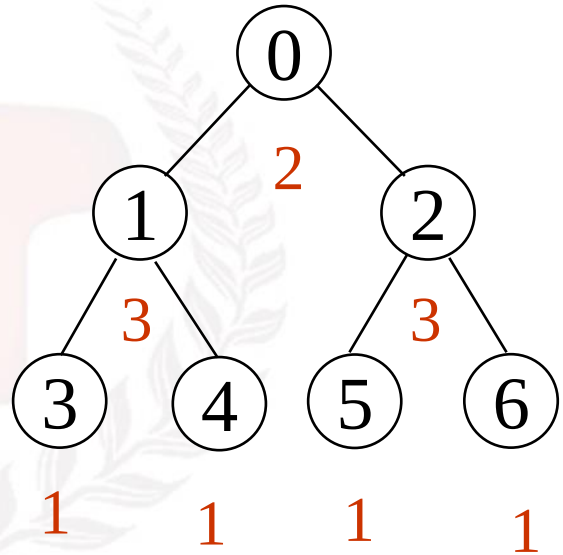
$$e = \left(\sum_{0}^{n-1} d_i\right) / 2$$

undirected graph

degree



G1

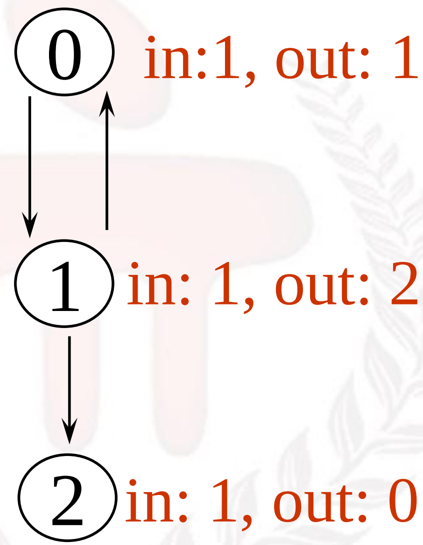


G2

directed graph

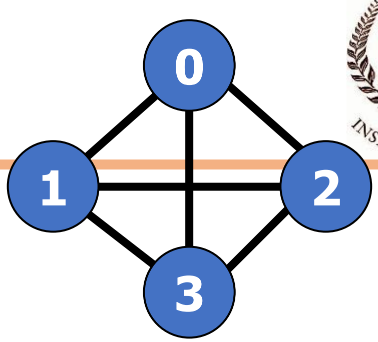
in-degree

out-degree

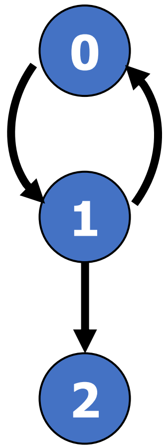


G_3

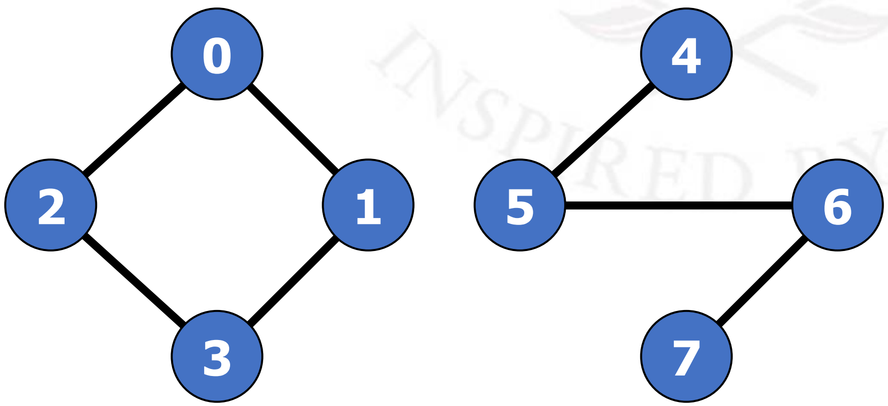
- Adjacency matrices
- Adjacency lists



G1

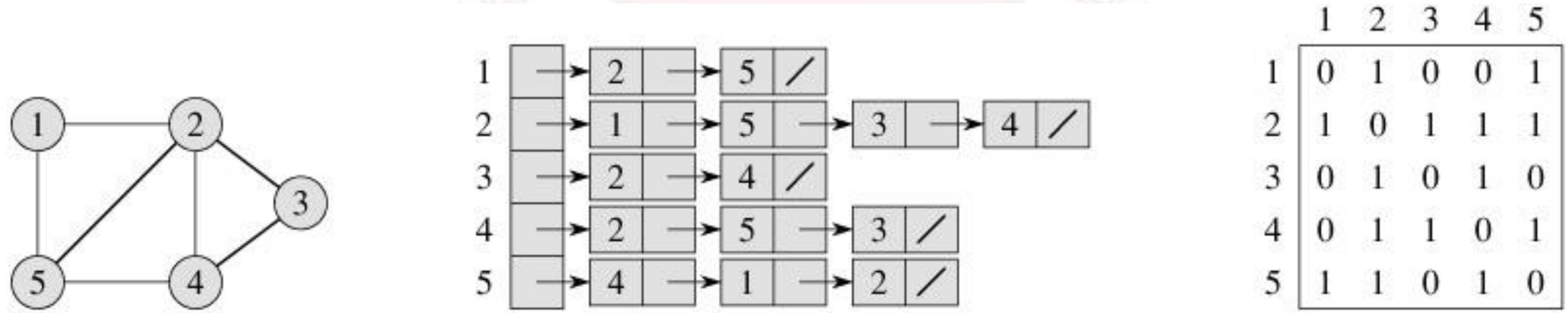


G3



G4

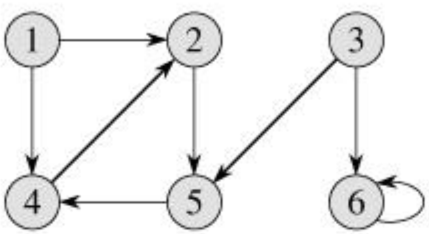
Graph representation – undirected



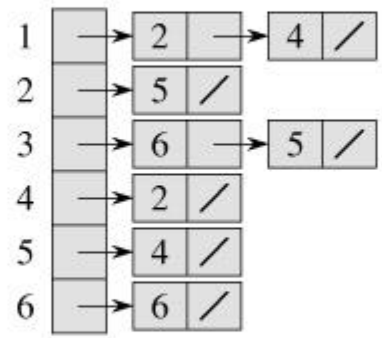
(a) (b) (c)

graph Adjacency list Adjacency matrix

Graph representation – directed



(a)



(b)

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

graph

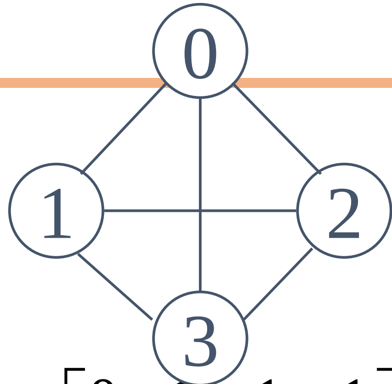
Adjacency list

Adjacency matrix

Adjacency Matrix

- Let G=(V,E) be a graph with n vertices.
- The adjacency matrix of G is a two-dimensional n by n array, say adj_mat
- If the edge (vi, vj) is in E(G), adj_mat[i][j]=1
- If there is no such edge in E(G), adj_mat[i][j]=0
- The adjacency matrix for an undirected graph is symmetric; the adjacency matrix for a digraph need not be symmetric

Examples for Adjacency Matrix



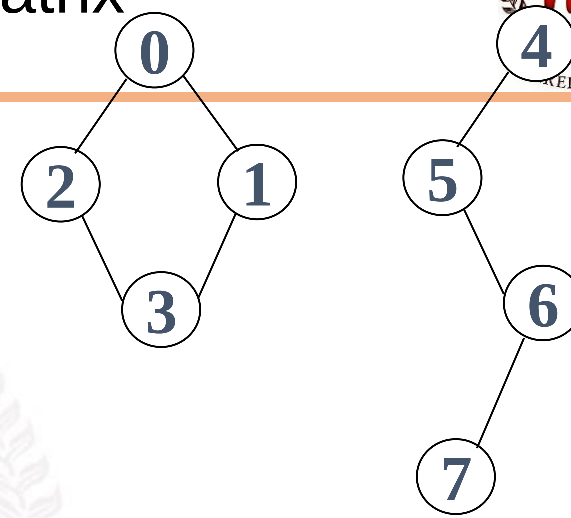
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

G_2

symmetric



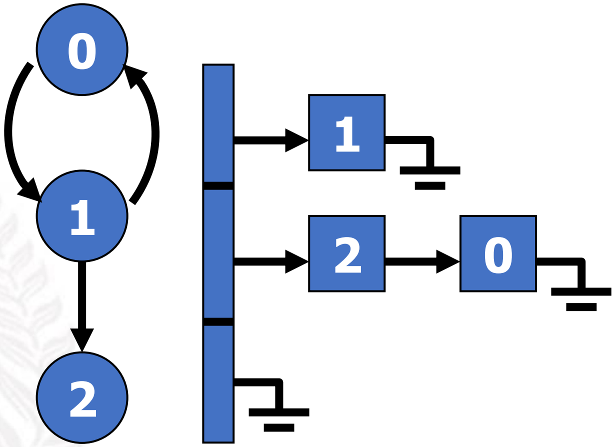
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

G_4

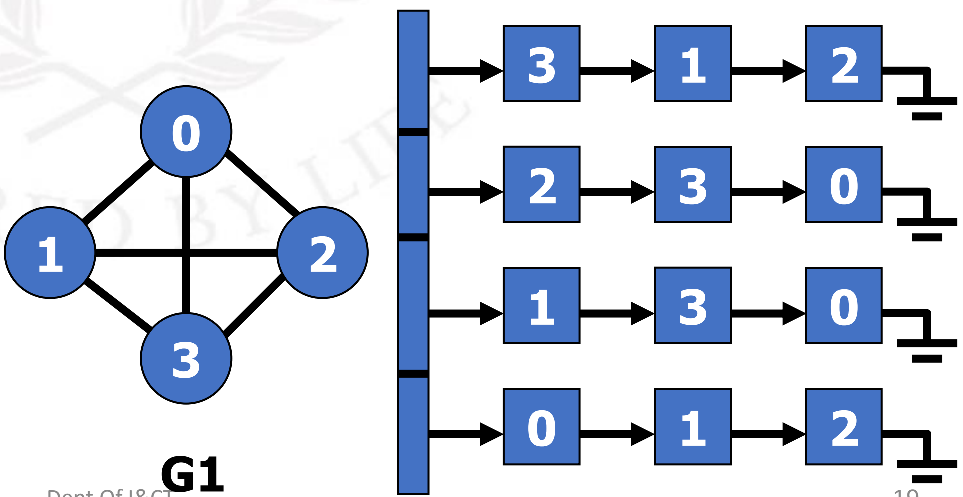
undirected: $n^2/2$

directed: n^2

Adjacency lists

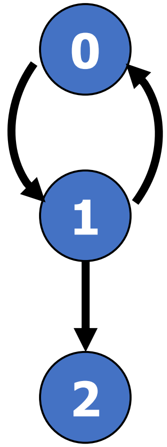


G3



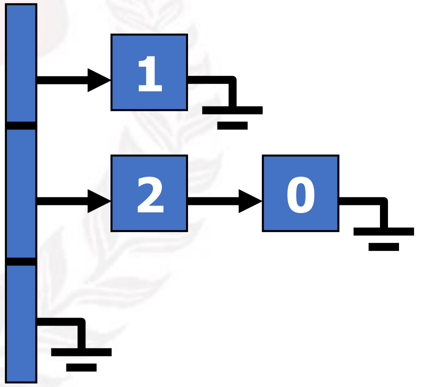
G1

Adjacency lists



G3

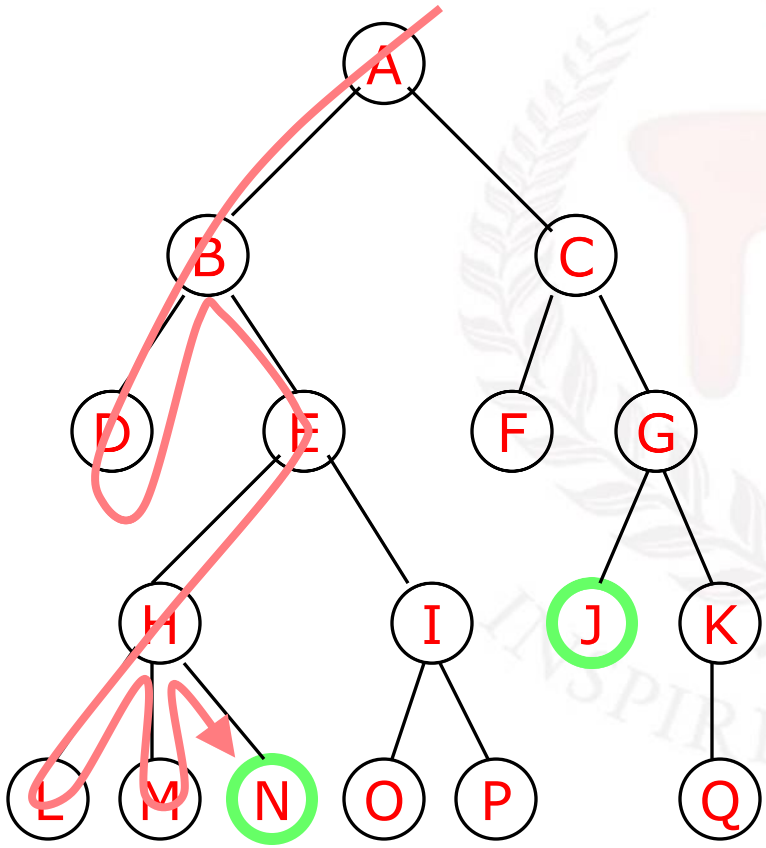
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



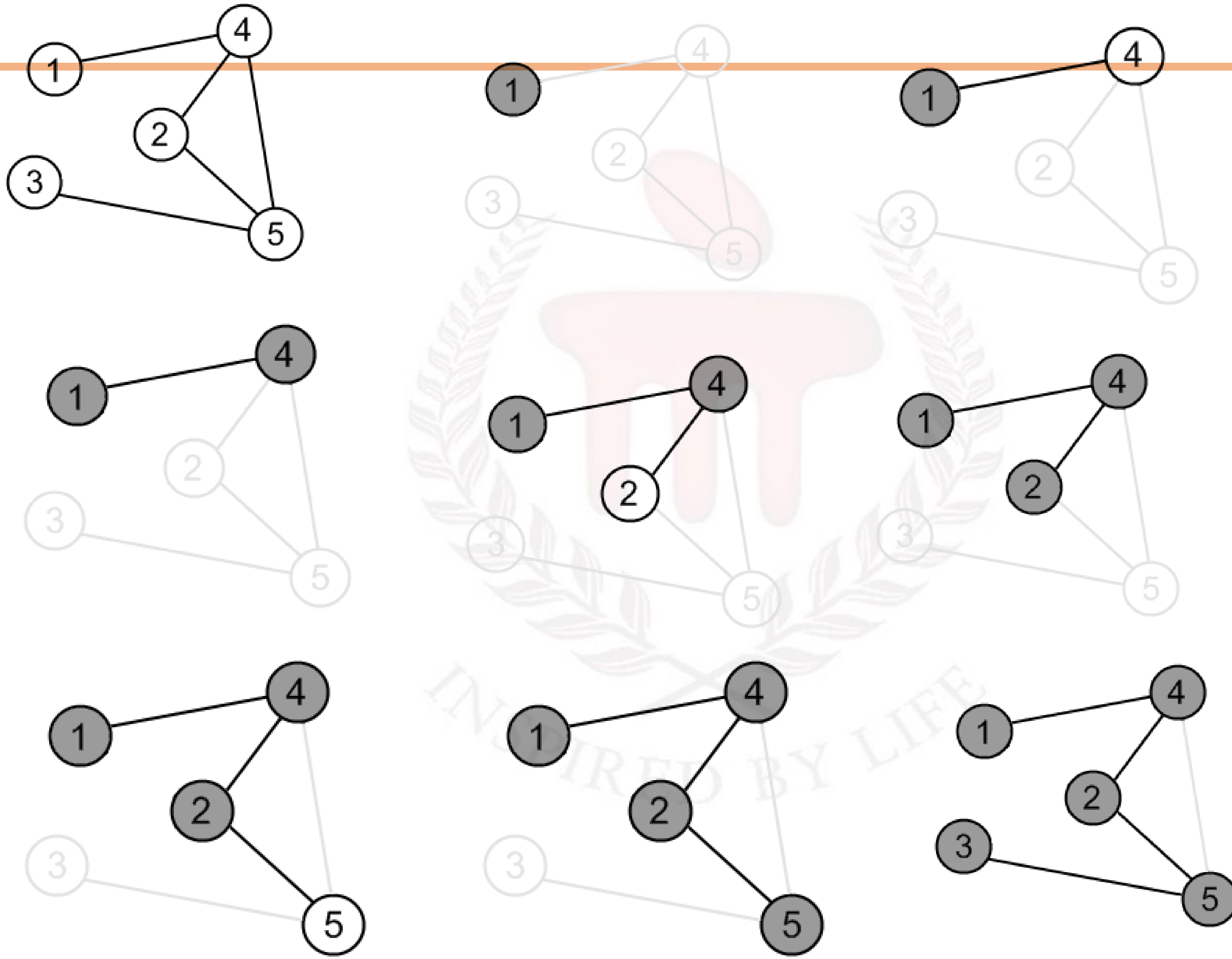
Some Graph Operations

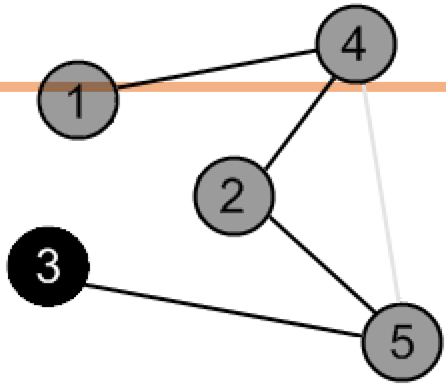
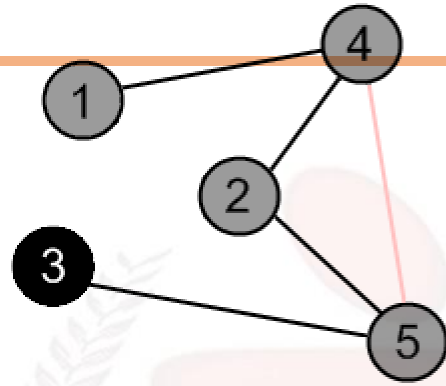
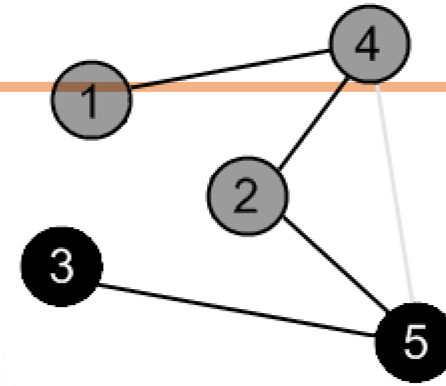
- Traversal
  Given G=(V,E) and vertex v, find all $w \in V$, such that w connects v.
  - Depth First Search (DFS)
    preorder tree traversal
  - Breadth First Search (BFS)
    level order tree traversal

Depth-first search



- A depth-first search (DFS) explores a path all the way to a leaf before backtracking and exploring another path
- For example, after searching A, then B, then D, the search backtracks and tries another path from B
- Node are explored in the order A B D E H L M N I O P C F G J K Q




4
1
2
3
5

4
1
2
3
5

4
1
2
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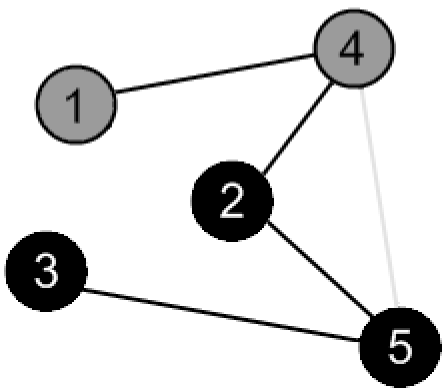
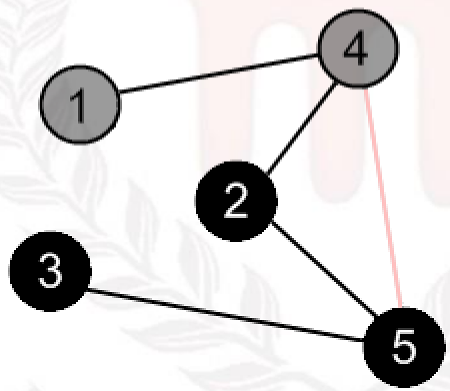

4
1
2
3
5

4
1
2
3
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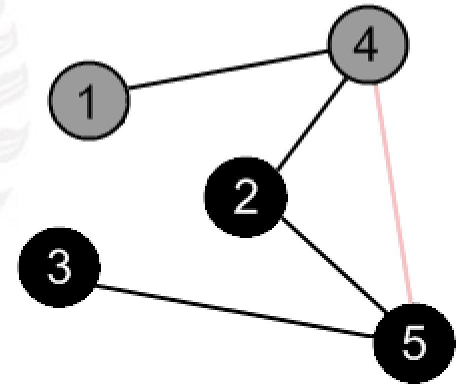
4
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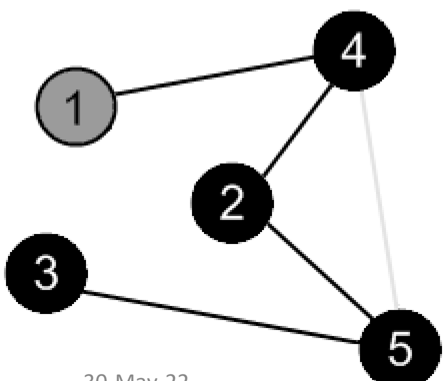
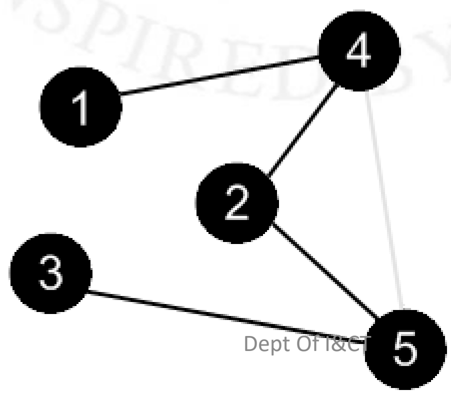

4
1
2
3
5

4
1
2
3
5

DFS Algorithm

Mark all the n vertices as not visited.

insert source into stack and mark it visited

while(Stack is not empty)

{

delete Stack element into variable u

place all the adjacent (not visited) vertices of u into Stack and also mark them visited

print u

}

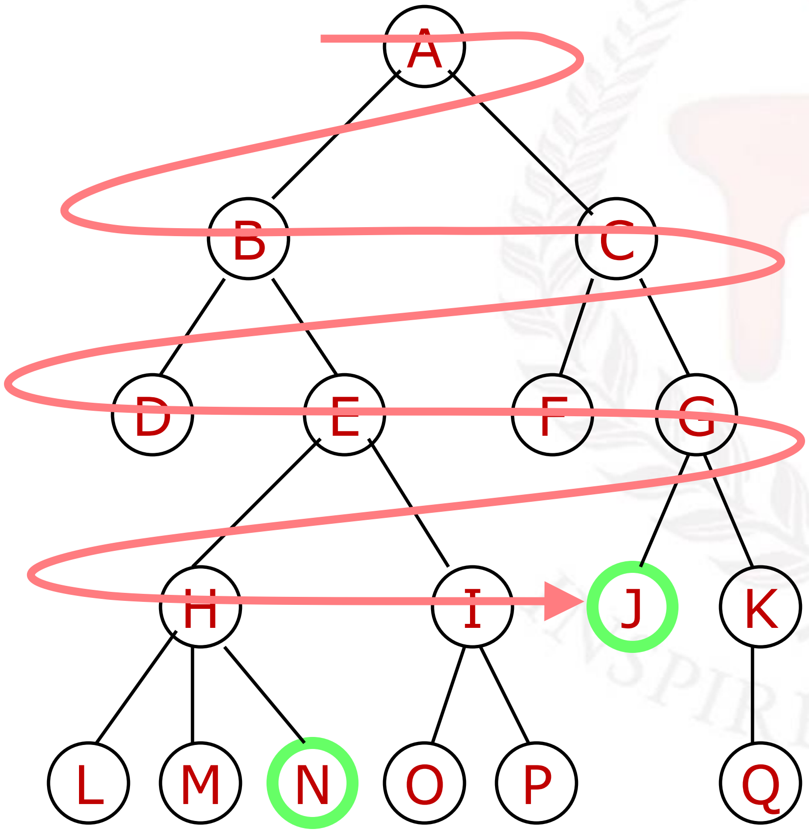
```
void dfs(int a[20][20],int n,int source)
{  int visited[10],u,v,i;
   for(i=1;i<=n;i++)        visited[i]=0;
   int S[20],top=-1;
   S[++top]=source;
   visited[source]=1;
   while(top>=0)
   {  u=S[top--];
      for(v=1;v<=n;v++)
      {   if(a[u][v]==1 && visited[v]==0)
          {
             visited[v]=1;          S[++top]=v;
          }
      }
      cout<<u<<" ";
   }
```

Breadth first search

It is so named because

It discovers all vertices at distance k from s before discovering vertices at distance k+1.

Breadth-first search



- A breadth-first search (BFS) explores nodes nearest the root before exploring nodes further away
- For example, after searching A, then B, then C, the search proceeds with D, E, F, G
- Node are explored in the order A B C D E F G H I J K L M N O P Q

Algorithm BFS

```
Mark all the n vertices as not visited.
insert source into Q and mark it visited
while(Q is not empty)
{
        delete Q element into variable u
        place all the adjacent (not visited) vertices of u into Q and also
mark them visited
        print u
}
```

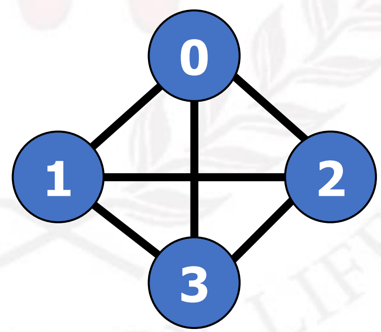
```
void bfs(int a[20][20],int n,int source)
{
int visited[10],u,v,i;

  for(i=1;i<=n;i++)        visited[i]=0;
    int Q[20],f=-1,r=-1;
   Q[++r]=source;    visited[source]=1;
   while(f<r)
   {
       u=Q[++f];
       for(v=1;v<=n;v++)
       {   if(a[u][v]==1 && visited[v]==0)
           {
               visited[v]=1;
               Q[++r]=v;
           }
       }
       cout<<u<<" ";
   }
}
```

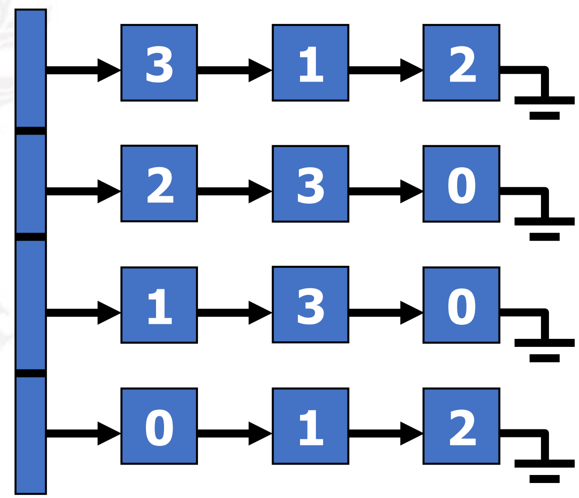
```
#include<iostream.h>
void bfs(int a[20][20],int n,int source);
void dfs(int a[20][20],int n,int source);
int main()
{
    int a[20][20],source, n,i,j;
    cout<<"Enter the no of vertices: ";    cin>>n;
    cout<<"Enter the adjacency matrix: ";
    for(i=1;i<=n;i++)        for(j=1;j<=n;j++)            cin>>a[i][j];
    cout<<"Enter the source: ";
    cin>>source;
    cout<<"\n BFS: ";    bfs(a,n,source);
    cout<<"\n DFS: ";    dfs(a,n,source);
    return 1;
}
```

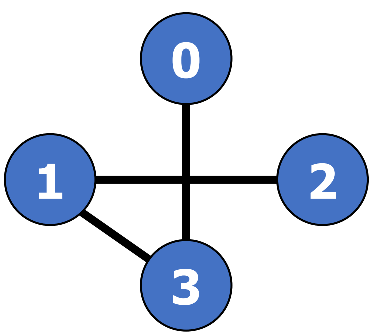
Spanning Tree (ST)

- A spanning tree is a minimal subgraph G’, such that V(G’)=V(G) and G’ is connected. Spanning Tree is always acyclic.

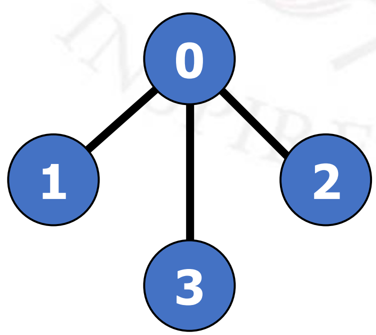


G1

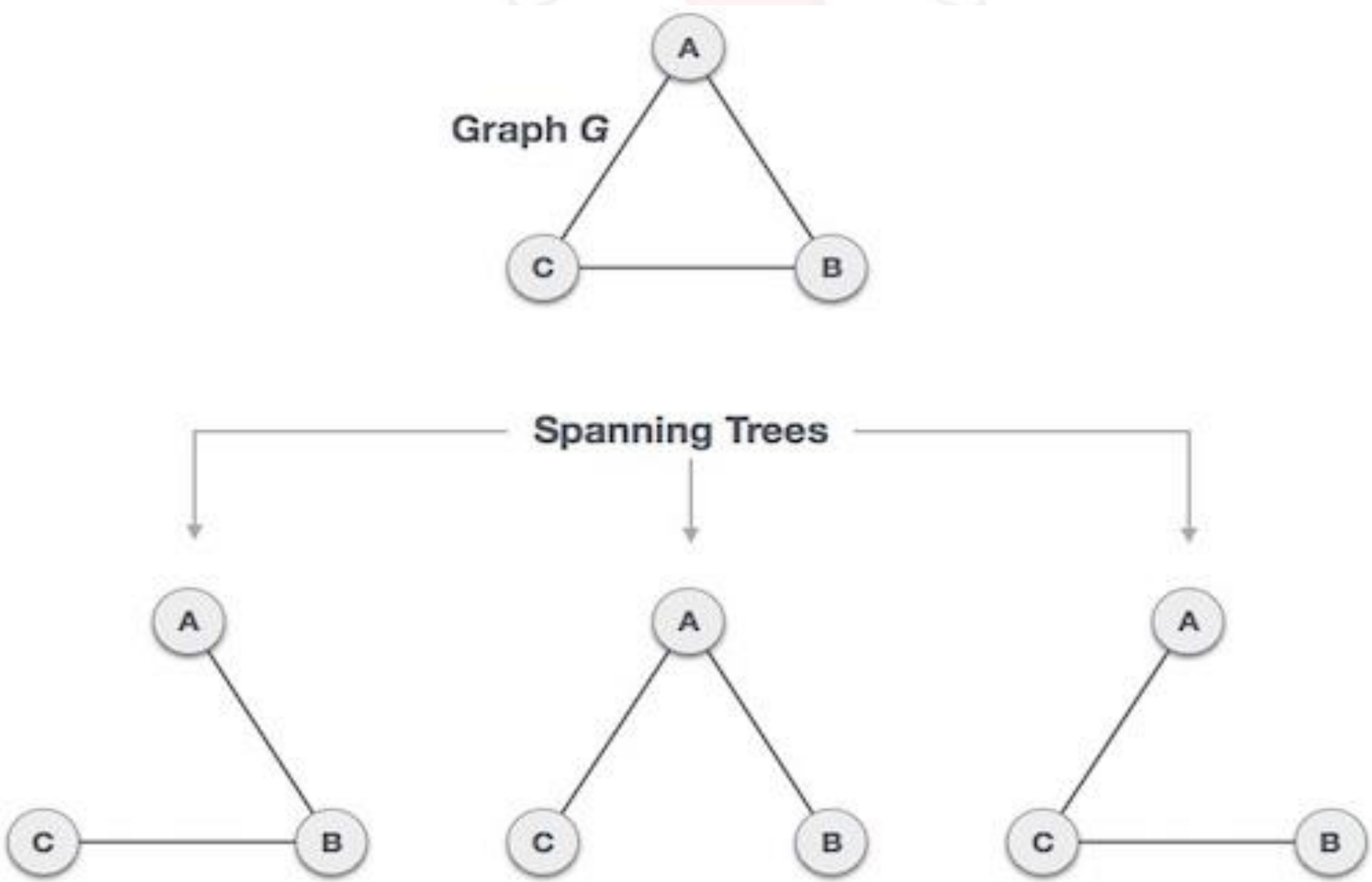


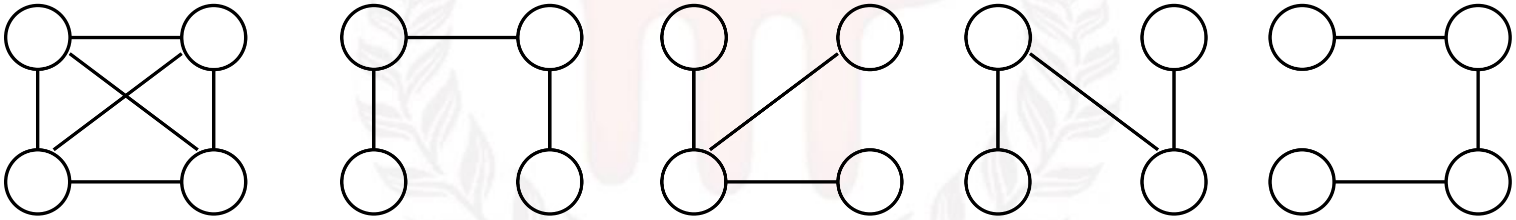


ST1(G)



ST2(G)


Graph G
A
C
B
Spanning Trees
A
C
B
A
C
B
A
C
B



A connected, undirected graph

Four of the spanning trees of the graph