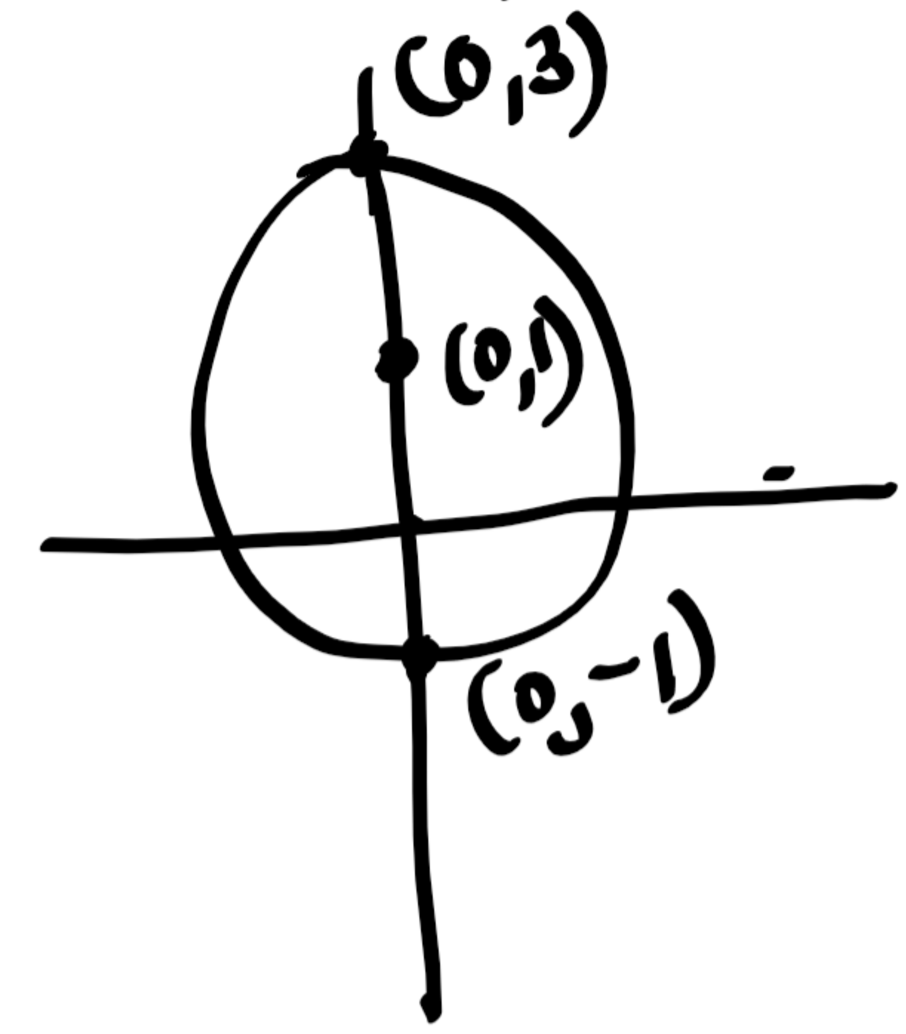
4) Evaluate: $\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ where $C: |z-i| = 2$



Poles are $z = -1 \rightarrow$ double pole

$z = 2 \rightarrow$ simple pole.

$(0,1),\ (-1,0)$

$\sqrt{1+1} = \sqrt{2} = 1.414 < 2$

$(0,1),\ (2,0)$

$\sqrt{4+1} = \sqrt{5} = 2.23 > 2$

$z = -1$ lies inside the circle and $z = 2$ lies outside the circle.

$$\therefore \operatorname{Res}_{z=-1} f(z) = \frac{1}{(2-1)!}\left\{\frac{d}{dz}\,\cancel{(z+1)^2}\,\frac{(z-1)}{\cancel{(z+1)^2}(z-2)}\right\}_{z=-1}$$

$$= \left\{\frac{(z-2)-(z-1)}{(z-2)^2}\right\}_{z=-1} = \frac{(-3)-(-2)}{(-1-2)^2}$$

$$= -1/9.$$

$$\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz = 2\pi i\left(-1/9\right) = -\frac{2\pi i}{9}$$

$$\operatorname*{Res}_{z=3i} f(z) = \operatorname*{Lt}_{z\to 3i} \frac{\cancel{(z-3i)}(3z^3+2)}{(z-1)(z+3i)\cancel{(z-3i)}}$$

$$= \frac{-81i+2}{(3i-1)6i} = \frac{2-81i}{6i(3i-1)}$$

$$\operatorname*{Res}_{z=-3i} f(z) = \operatorname*{Lt}_{z\to -3i} \frac{\cancel{(z+3i)}(3z^3+2)}{(z-1)\cancel{(z+3i)}(z-3i)}$$

$$= \frac{81i+2}{(-3i-1)(-6i)} = \frac{81i+2}{6i(3i+1)}$$

$$\therefore \oint_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\left\{\frac{1}{2} + \frac{2-81i}{6i(3i-1)} + \frac{81i+2}{6i(3i+1)}\right\}$$

$$= \pi i\left\{1 + \frac{2-81i}{3i(3i-1)} + \frac{81i+2}{3i(3i+1)}\right\}$$

$$= \pi i\left\{1 + \frac{(2-81i)(3i+1)+(2+81i)(3i-1)}{-30i}\right\}$$

$$= \pi i\left\{1 + \frac{12i-162i}{-30i}\right\} = \pi i\{1+5\}$$

$$= \underline{\underline{6\pi i}}$$

$$\therefore \oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}\,dz = 2\pi i\,(2\pi+1+1)$$

$$= 2\pi i\,(2\pi+2)$$

$$= \underline{4\pi i\,(\pi+1)}$$

3) Evaluate $\displaystyle\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$ where (i) $C: |z-2|=2$

(ii) $C: |z|=4$

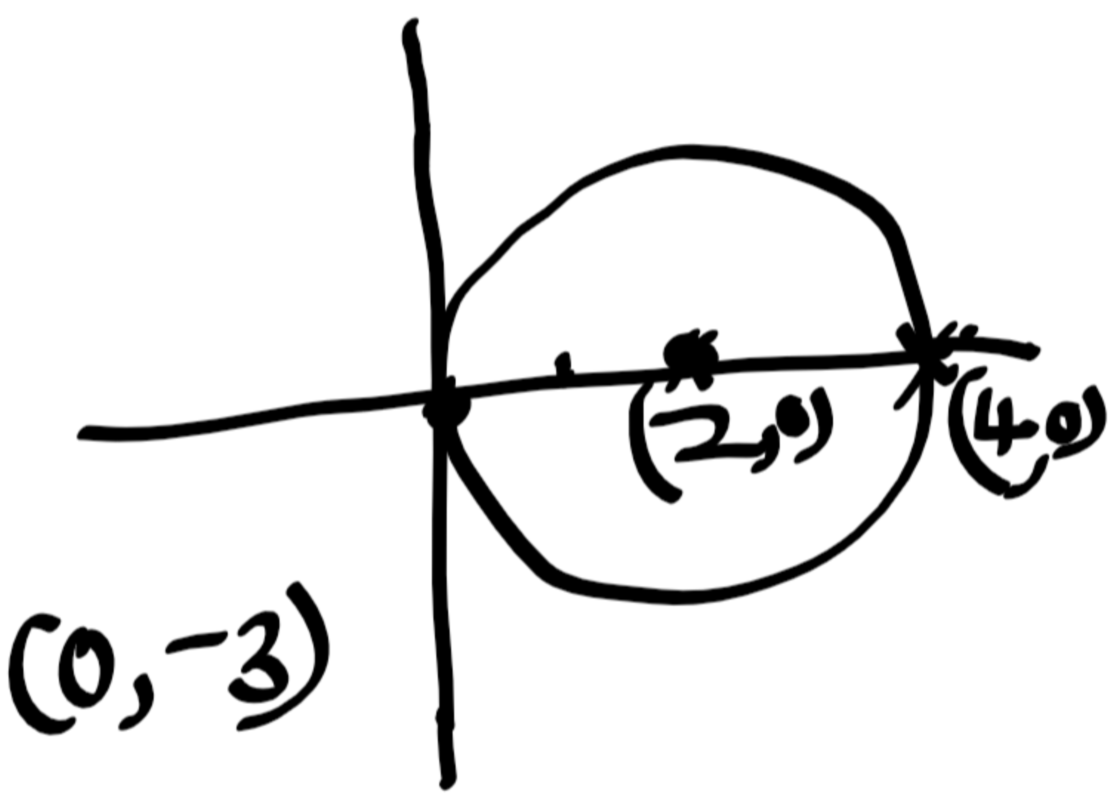
Poles are $z=1, \pm 3i$

z^2+9

$=(z+3i)(z-3i)$

(i) $C: |z-2|=2$

$z=1$ lies inside C.



$$\therefore \underset{z=1}{\operatorname{Res}} f(z) = \lim_{z\to 1} \cancel{(z-1)}\,\frac{3z^3+2}{\cancel{(z-1)}(z^2+9)}$$

$$= \frac{5}{10} = \underline{\underline{1/2}}$$

$$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i \times \frac{1}{2} = \underline{\underline{\pi i}}$$

(2,0), (0,−3)

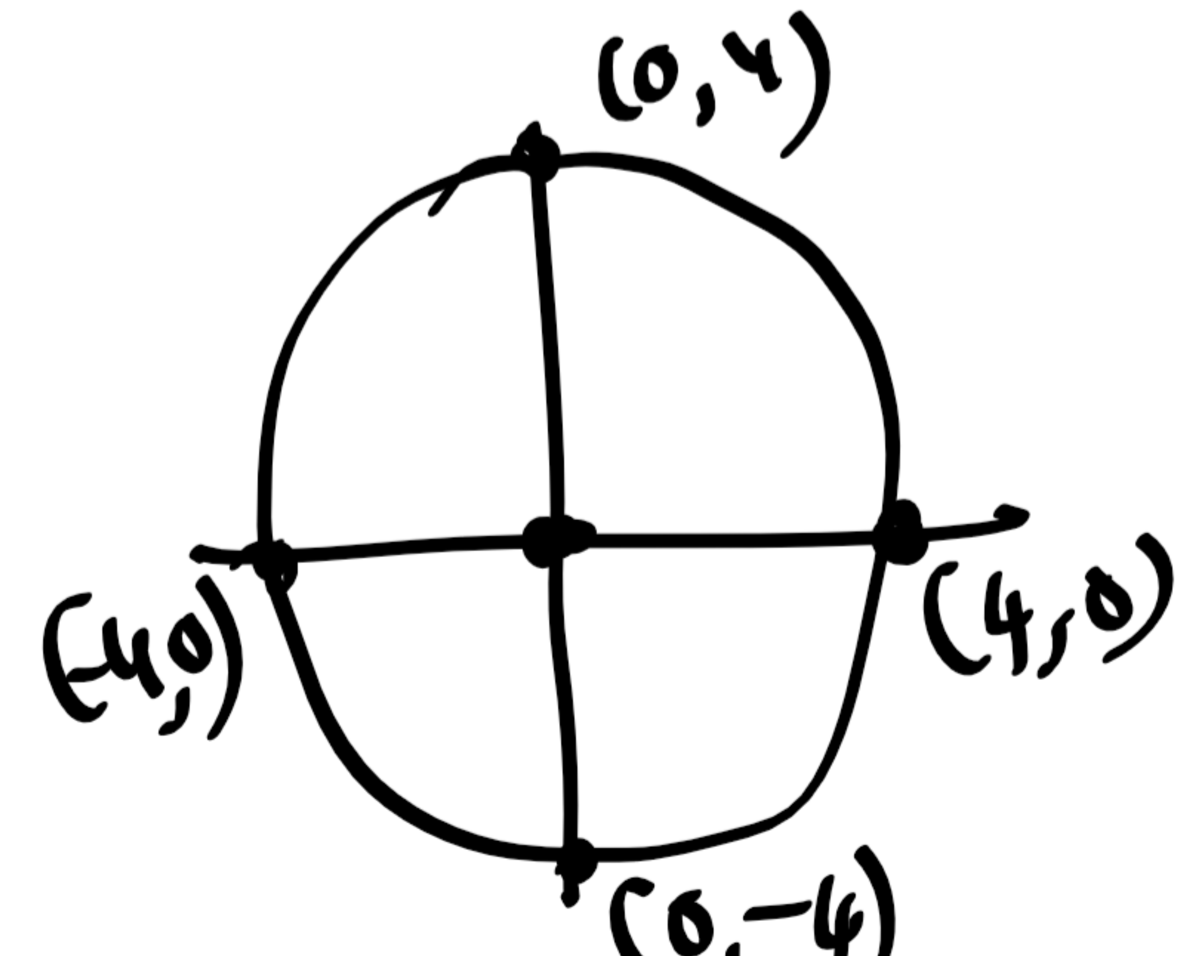
(2,0) (0,3)

(2,0), (1,0) $\sqrt{4+9} = \sqrt{13}$

$= 3.6 > 2$

(ii) $C: |z|=4$

$z=1, \pm 3i$ lie inside C.



$$\underset{z=1}{\operatorname{Res}} f(z) = 1/2$$

Examples

1) Evaluate $\oint_C \frac{z^2}{(z-1)^2(z+2)} dz$ where $C: |z| = 2.5$

using Cauchy's residue theorem.

$$\operatorname{Res}_{z=-2} f(z) = 4/9, \quad \operatorname{Res}_{z=1} f(z) = 5/9.$$

$$\oint_C \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i \left(4/9 + 5/9\right) = \underline{\underline{2\pi i}}$$

2) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz, \quad C: |z| = 3.$

$z = 1 \rightarrow$ Double pole, $z = 2 \rightarrow$ Simple pole.

$$\operatorname{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \left\{ \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{\sin \pi z^2 + \cos \pi z^2}{\cancel{(z-1)^2}(z-2)} \right] \right\}_{z=1}$$

$$= \left\{ \frac{(z-2)\left[\cos \pi z^2 \cdot 2\pi z - \sin \pi z^2 \cdot 2\pi z\right] - \left(\sin \pi z^2 + \cos \pi z^2\right)}{(z-2)^2} \right\}_{z=1}$$

$$= \underline{\underline{2\pi + 1}}$$

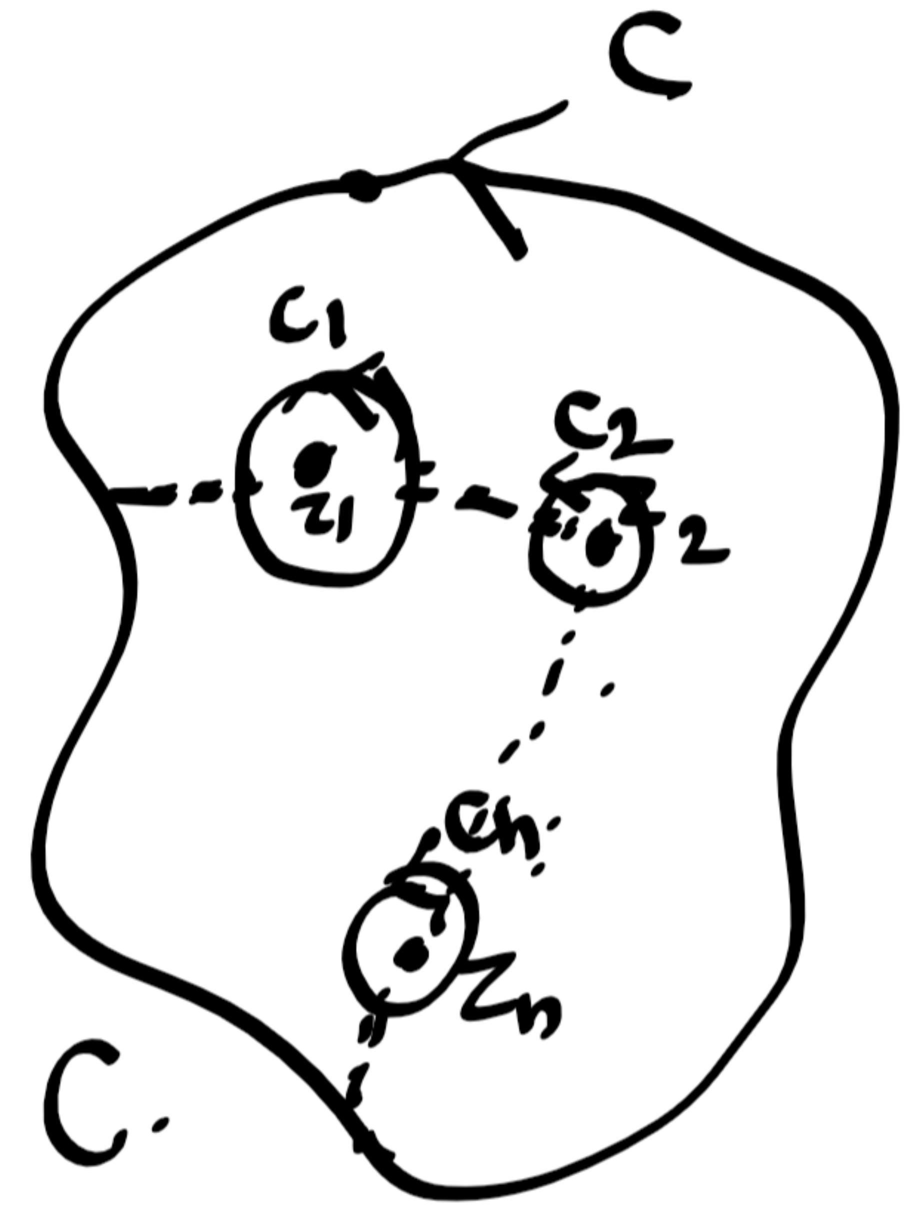
$$\operatorname{Res}_{z=2} f(z) = \lim_{z \to 2} \cancel{(z-2)} \frac{\left(\sin \pi z^2 + \cos \pi z^2\right)}{(z-1)^2 \cancel{(z-2)}}$$

$$= 1.$$

Cauchy's Residue Theorem

Let $f(z)$ be analytic within and on a simple closed curve C except at a finite no: of singular points $z_1, z_2, \ldots, z_n$ inside C.

Then $$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z=z_i} f(z)$$



Proof :-

Let $C_1, C_2, \ldots C_n$ be the circles with centres $z_1, z_1, \ldots z_n$ respectively and radius is so small that lies within C. Now $f(z)$ is analytic within C. By Cauchy's integral theorem

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$

$$= 2\pi i \left[\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z) + \cdots + \operatorname{Res}_{z=z_n} f(z) \right]$$

$$= 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z=z_i} f(z)$$ (By the definition of residues)

Residues

1) Residue of $\frac{\sinh z}{z^3}$ at $z=0$

$$\frac{\sinh z}{z^3} = \frac{e^z - e^{-z}}{2z^3}$$

$$= \frac{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^3}$$

$$= \frac{1}{z^2} + \frac{1}{3!} + \frac{z^2}{5!} + \cdots$$

Res of $\frac{\sinh z}{z^3}$ = coefft of $\frac{1}{z}$ in the above series

$= \underline{\underline{0}}$

$\cosh z = \frac{e^z + e^{-z}}{2}$

$\sinh z = \frac{e^z - e^{-z}}{2}$

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$

2) Residue of $\cot z$ at $z=0$

$$\cot z = \frac{\cos z}{\sin z} \quad \left(\frac{P(z)}{Q(z)}\right)$$

$P(a) \neq 0$
$Q(a) = 0$
$Q'(a) \neq 0$

$$\text{Residue of } \cot z = \frac{P(a)}{Q'(a)} = \frac{\cos 0}{\cos 0} = \underline{\underline{1}}$$