Homogeneous D.E.

$y = vx$ or $x = vy$

reduce to

Variable separable.

Variable separable.

$f(x)dx + g(y)dy = 0$.

Simply integrate.

Equations with non-homogeneous coeffts.

straight lines

parallel

↓

variable separable

intersecting

(h, k)

$x = X + h$

$y = Y + k$

reduces the equation VS.

First order
First degree DE

$\frac{dy}{dx} = f(x, y)$, $Mdx + Ndy = 0$

Exact DE.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\int\limits_{y \text{ const}} M dx + \int (\text{terms of } N \text{ ind of } x) dy = C$

$\int\limits_{x \text{ const}} N dy + \int (\text{terms of } M \text{ ind of } y) dx = C$.

Reducible to Exact Differential Equations

1. IF found by inspection

2. $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \Rightarrow IF = e^{\int f(x)\, dx}$$

3. $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = g(y) \Rightarrow IF = e^{\int g(y)\, dy}$$

4. Linear DE in y: $\frac{dy}{dx} + Py = Q$, $IF = e^{\int P dx}$

$$y(IF) = \int Q\,(IF)\, dx + c.$$

5. Bernoulli's DE: $\frac{dy}{dx} + Py = Qy^n$,

$\div y^n$, put $\frac{1}{y^{n-1}} = t$.

Equation reduces to Linear DE.

1. $\sin y (x + \sin y) dx + 2x^2 \cos y\, dy = 0$

$\sin y = t$, $\cos y\, dy = dt$

$t(x + t)\, dx + 2x^2\, dt = 0$

$2x^2 \frac{dt}{dx} + xt + t^2 = 0$.

$2x^2 \frac{dt}{dx} + xt = -t^2$

Bernoulli's in t.

$\div t^2$ & proceed.

② $(y\log x - 2)y\,dx - x\,dy = 0.$

$(y\log x - 2)y - x\frac{dy}{dx} = 0.$

$-x\frac{dy}{dx} + \log x\; y^2 - 2y = 0.$ Bernoulli's in y.

③ $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

$\underbrace{(\sin x + x\cos y + x)}_{N}\,dy + \underbrace{(y\cos x + \sin y + y)}_{M}\,dx = 0.$

$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$ $\quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1$ Exact.

④ $y^2\,dx + (3xy + y^2 - 1)\,dy = 0.$

$y^2\frac{dx}{dy} + 3xy + y^2 - 1 = 0.$ Linear in x

⑤ $\underbrace{(xy^3 + y)}_{M}\,dx + \underbrace{2(x^2y^2 + x + y^4)}_{N}\,dy = 0.$

$\frac{\partial M}{\partial y} = x3y^2 + 1$ $\quad \frac{\partial N}{\partial x} = 2(2xy^2 + 1)$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -xy^2 - 1 = -(xy^2 + 1)$ $\quad y(xy^2+1)$

$\div -M$ & proceed.

⑥ $(x^2 + y^2)\,dx - xy\,dy = 0$

Homogeneous. DE.

⑦ $x(x-y)\,dy + y^2\,dx = 0.$

Homogeneous. DE.

⑧ $\frac{dy}{dx} = 1 + 6xe^{x-y}$ $\quad x - y = t$

$1 - \frac{dt}{dx} = 1 + 6xe^t$ $\quad 1 - \frac{dy}{dx} = \frac{dt}{dx}$

⑨ $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$y - ay^2 = (a + x)\frac{dy}{dx}$

Variable separable.

(10) $2x\left[3x + y - y e^{-x^2}\right] dx + \left(x^2 + 3y^2 + e^{-x^2}\right) dy = 0.$

$$\underbrace{2x\left[3x + y - y e^{-x^2}\right]}_{M} \qquad \underbrace{\left(x^2 + 3y^2 + e^{-x^2}\right)}_{N}$$

$$\frac{\partial M}{\partial y} = 2x - 2x e^{-x^2} \qquad \frac{\partial N}{\partial x} = 2x - 2x e^{-x^2}$$

Exact.

observe: $\int\limits_{x \text{ const}} N\,dy + \int \ldots$ is easier in this case.

(11) $y' = x^3 - 2xy$

Linear in y

(12) $(y - x + xy\cot x)\, dx + x\,dy = 0$

$y\,dx + x\,dy - x\,dx + xy\cot x\, dx.$

$$\frac{d(xy)}{xy} - \frac{1}{y}dx + \cot x\, dx = 0.$$

$$x\frac{dy}{dx} + \underbrace{y + xy\cot x}_{(1 + x\cot x)y} - x = 0.$$

Linear in y.

Higher Order Linear Differential Equations:

A differential equation is said to be a linear differential equation if all the derivatives and the unknown variable occur in degree one and are not multiplied.

A linear differential equation of order n with constant coefficients is of the form,

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = R(x)$$

where $k_1, k_2, \ldots, k_n$ are real numbers and $R(x)$ is a function of x alone.

If we denote $\frac{d}{dx}$ by D, then the above equation can be written as

$$D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \cdots + k_n y = R(x)$$

$$\left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n\right) y = R(x)$$

i.e. $$f(D)\, y = R(x) \quad \text{———} (1)$$

where $f(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n$

Equation (1) is said to be homogeneous if $R(x) = 0$ otherwise the equation is non-homogeneous.

Thus a homogeneous equation is of the form

$$f(D)\, y = 0 \quad \text{———} (2)$$

D is called the differential operator (it has properties of an algebraic operator)

Note: $(D-\alpha)(D-\beta) = (D-\beta)(D-\alpha)$.

Equation (2) has n linearly independent solutions. Let $y_1, y_2, \ldots, y_n$ be the n linearly independent solutions of (2). Then the general solution of (2) is given by

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Let y_p be a particular solution of (1). Then, the complete solution of (1) is given by

$$y = y_c + y_p.$$

y_c is called the complementary function (CF)

y_p is called the particular integral (PI).

Thus, the complete solution of (1) is

$$\boxed{y = CF + PI}$$

Rules for Finding CF:

Consider the differential equation

$$f(D)\,y = 0$$

The equation $f(m) = 0$ is called auxiliary equation. → polynomial equation, degree n.

Let $m_1, m_2, \dots, m_n$ be the roots of this equation.

Case 1: Suppose that all the roots are real and distinct.

Then general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

[$(D - m_n)(D - m_{n-1}) \cdots (D - m_1)\,y = 0$.

Consider $(D - m_1)\,y = 0$

$$\frac{dy}{dx} - m_1 y = 0 \Rightarrow \frac{dy}{y} = m_1\,dx$$

$$\text{ie } \log y = m_1 x + \log c_1$$

$$\log\left(\frac{y}{c_1}\right) = m_1 x$$

$$\frac{y}{c_1} = e^{m_1 x} \qquad \therefore y = c_1 e^{m_1 x}$$

$|||^{ly}$ the remaining solutions are

$$c_2 e^{m_2 x}, \dots, c_n e^{m_n x} \ ]$$

Case 2: All roots real and distinct, except for two roots, say m_1 & m_2, which are equal. In this case, the solution is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

If $m_1 = m_2 = m_3$, then

$$y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

[m_1 is a repeated root. $\therefore$ factor $(D - m_1)^2$

$$(D - m_1)^2 y = 0$$

$$(D - m_1)\underbrace{(D - m_1)\,y}_{z} = 0$$

$$(D - m_1)\,z = 0 \qquad \text{where } (D - m_1)\,y = z$$

$$z = c_1 e^{m_1 x}$$

$$(D - m_1)\,y = c_1 e^{m_1 x}$$

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}, \quad \text{linear in } y.$$

$$\text{IF} = e^{-\int m_1\,dx} = e^{-m_1 x}, \qquad y\left(e^{-m_1 x}\right) = \int c_1 e^{m_1 x} e^{-m_1 x}\,dx + c_2$$

$$y e^{-m_1 x} = c_1 x + c_2$$

$$y = (c_1 x + c_2)\, e^{m_1 x}$$

2 independent solutions, $e^{m_1 x}$ & $x e^{m_1 x}$]

Case 3: Suppose that 2 of the roots are complex, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, all other roots real and distinct.

The general solution is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 i - c_2 i)\sin\beta x]$$

$$= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

if $\alpha \pm i\beta$ are roots, 2 linearly independent solutions are

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x.$$

Suppose that $m_1 = \alpha + i\beta = m_2$, $m_3 = \alpha - i\beta = m_4$, then the solution is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$