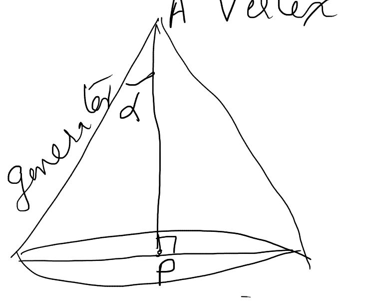
Lecture - 5 , RCC & Right circular cylinder

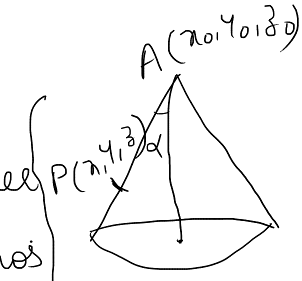


Right Circular Cone (RCC)

A RCC is a surface generated by a straight line which passes through a fixed point (vertex) and makes a constant angle with a fixed line. The constant angle 'α' is called the semi-vertical angle and the fixed line AP is called the axis.

The section of a RCC by a plane $\perp^r$ to the axis is a circle

Equation of a RCC



Let (x_0, y_0, z_0) be the co-ordinates of the vertex A and (a, b, c) be the direction ratios of the axis. Consider any point $P(x, y, z)$ on the cone. Then the direction ratios of the generator AP are $(x - x_0, y - y_0, z - z_0)$, then

$$\cos\alpha = \frac{a(x - x_0) + b(y - y_0) + c(z - z_0)}{\sqrt{a^2+b^2+c^2}\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \quad \text{or}$$

$$\left[a(x - x_0) + b(y - y_0) + c(z - z_0)\right]^2 = (a^2+b^2+c^2)\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)\cos^2\alpha$$

The equation holds for any point P on the cone

Problems

1. Find the equation of the RCC whose vertex is the origin, axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which has semi vertical angle of 30°.

Ans $\rightarrow$ $(x_0, y_0, z_0) = (0, 0, 0)$.

D.R's of the axis $(a, b, c) = (1, 2, 3)$, $\alpha = 30^\circ$.

$\therefore$ the equation of RCC

$$(1(x) + 2y + 3z)^2 = (1^2 + 2^2 + 3^2)(x^2 + y^2 + z^2)\cos^2 30^\circ$$

$$x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6xz = \overset{7}{\cancel{14}}(x^2+y^2+z^2)\frac{3}{\cancel{4}_2}$$

$$2x^2 + 8y^2 + 18z^2 + 8xy + 24yz + 12xz = 21x^2 + 21y^2 + 21z^2$$

$$19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12xz = 0$$

2) Find the eqn of the RCC whose vertical angle is $\pi/2$ which has its vertex at the origin and axis along z-axis.

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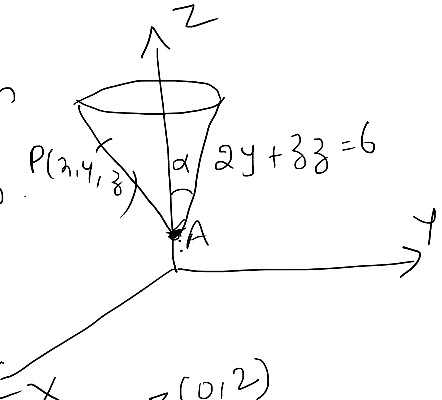
3) Find the equation of the RCC generated when the straight line $2y+3z=6$, $x=0$ revolves about the z-axis.



Ans $\rightarrow$ The vertex is the point of intersection of the line $2y+3z=6$, $x=0$ and z-axis.

$\therefore$ Vertex A $(0,0,2)$

D.R's of the axis is $(0,0,1) \rightarrow (a,b,c)$

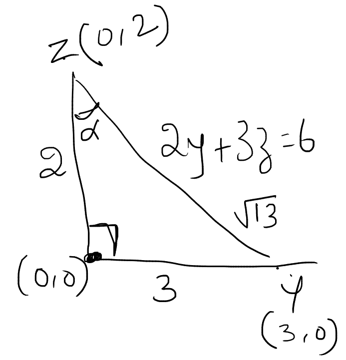
Any point on the generator $(0,3,0)$

Vertex A $(0,0,2)$

DR's of the generator $(0,3,-2)$

$\rightarrow (x-x_0, y-y_0, z-z_0)$

$\cos\alpha = \dfrac{2}{\sqrt{13}}$



$$\cos\alpha = \frac{0\cdot 0 + 3(0) + -2(1)}{\sqrt{0+9+4}\ \sqrt{0+0+1}} = \frac{-2}{\sqrt{13}}$$

Let $P(x,y,z)$ be any point on the cone. So that the dr's of AP is $(x,y,z-2)$

$$\text{Eqn of RCC, } \cos\alpha = \frac{-2}{\sqrt{13}} = \frac{x(0)+y(0)+(z-2)1}{\sqrt{x^2+y^2+(z-2)^2}\ \sqrt{1}}$$

Squaring $\quad 4(x^2+y^2+(z-2)^2) = 13(z-2)^2$

$$4x^2+4y^2-9z^2+36z-36=0$$

4) Find the equation of the RCC passing through the coordinate axes having vertex at the origin. Find the semi-vertical angle and equation of the axis.

Ans → Let the cone intersect the co-ordinate axes at

(x-axis) (y-axis) (z-axis)

$A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

Let (l,m,n) be the D. Cosines of the axis

$\cos\alpha$ = angle between the axis and OA

$$\cos\alpha = \frac{l.a + m.0 + n.0}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+0+0}} = \frac{al}{a} = l \qquad \underline{l^2+m^2+n^2=1}$$

|||ly $\cos\alpha = m$ (OB) (y-axis) $\cos\alpha = n$ (OC) z-axis

$$l^2+m^2+n^2 = 1 \qquad \rightarrow l^2+l^2+l^2=1$$

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \qquad l^2 = \frac{1}{3}$$

$$3\cos^2\alpha = 1 \qquad l = \pm\frac{1}{\sqrt{3}}$$

$$\cos^2\alpha = \frac{1}{3}$$

$$\cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Let $P(x,y,z)$ be any point on the cone.

DR's of OP → (x,y,z) Vertex is the origin

$$\cos\alpha = \frac{lx+my+nz}{\sqrt{l^2+m^2+n^2}\sqrt{x^2+y^2+z^2}} = \frac{lx+ly+lz}{1\sqrt{x^2+y^2+z^2}}$$

$$\pm\frac{1}{\sqrt{3}} = \frac{l(x+y+z)}{\sqrt{x^2+y^2+z^2}} \quad \text{or} \quad \frac{1}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}}\frac{(x+y+z)}{\sqrt{x^2+y^2+z^2}}$$

Squaring $x^2+y^2+z^2 = (x+y+z)^2$

$$\underline{\underline{xy+yz+zx=0}}$$

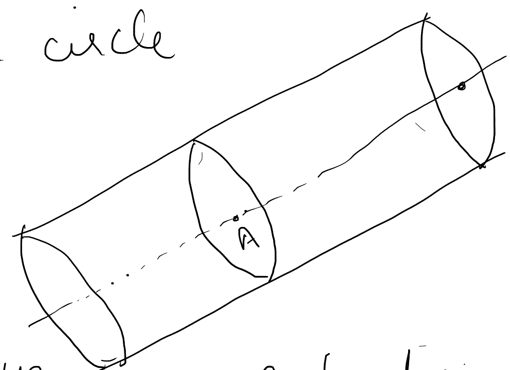
Semi. vertical angle $\alpha = \cos^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$

Equation of the axis $\dfrac{x}{1/\sqrt{3}} = \dfrac{y}{1/\sqrt{3}} = \dfrac{z}{1/\sqrt{3}}$ or $\underline{\underline{x=y=z}}$

Right Circular cylinder

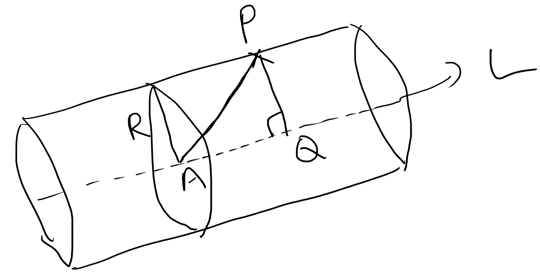
A RCC is a surface generated by a straight line, which is parallel to a straight line and is at a constant distance from it. The constant distance is called the radius of the cylinder

<u>Note</u> A plane $\perp^r$ to the axis of a RCC cuts the cylinder along a circle whose centre lies on the axis and whose radius is equal to the radius of the cylinder. This circle is called base circle or guiding circle of the cylinder.



The circular area bounded by this circle is called a normal section of the cylinder, its radius is the radius of the cylinder

Equation of RCC



Let (l, m, n) be the direction cosines (d.c) of the axis L. Let $A(x_0, y_0, z_0)$ be a point on L. Consider an arbitrary point $P(x, y, z)$ on the cylinder. If Q is the foot of the $\perp^r$ from P onto L then $l(PQ) = R$, the radius of the cylinder.

Also AQ is the projection of AP on L

$$\therefore AQ = l(x-x_0) + m(y-y_0) + n(z-z_0)$$

$$AP^2 = AQ^2 + PQ^2 \rightarrow AP^2 - AQ^2 = PQ^2$$

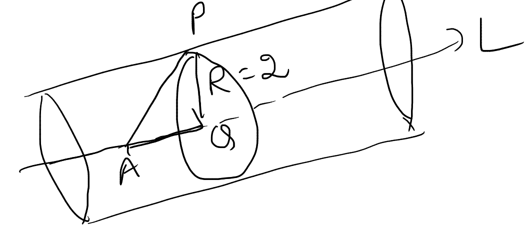
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - \left[l(x-x_0) + m(y-y_0) + n(z-z_0) \right]^2 = R^2$$

is the eqn of the RCC. This eqn holds for an arbitrary point $P(x, y, z)$ on the cylinder.

Problems

1. The radius of a normal section of a RCC is 2 units. The axis lies along the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$. Find its equation.

Solution



Let $P(x,y,z)$ be any point on the cylinder. Draw $PQ \perp$ axis L.

Then PQ = radius $R = 2$.

Any point on the axis $A = (1,-3,2)$

AQ is the projection of AP onto L

DR's of $AQ = (2,-1,5)$

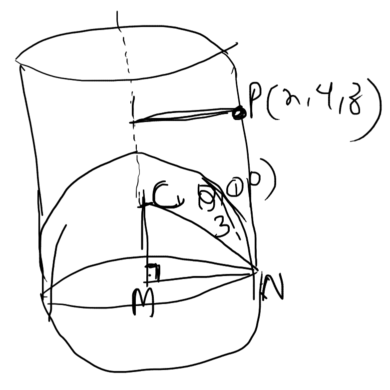
DC's of $AQ = \left(\frac{2}{\sqrt{2^2+(-1)^2+5^2}}_{(l)}, \frac{-1}{\sqrt{30}}_{(m)}, \frac{5}{\sqrt{30}}_{(n)} \right)$

Required equation is

$$(x-1)^2 + (y+3)^2 + (z-2)^2 - \left[\frac{2}{\sqrt{30}}(x-1) - \frac{1}{\sqrt{30}}(y+3) + \frac{5}{\sqrt{30}}(z-2) \right]^2 = 2^2$$

2) Find the equation of the RCC having the circle $x^2+y^2+z^2 = 9$, $x-y+z = 3$ as a base circle.

Solution



Here base circle of the required RCC is the circle of intersection of the sphere S $(x^2+y^2+z^2 = 9)$ and the plane q $(x-y+z = 3)$.

Centre of the sphere $C(0,0,0)$

D R's of normal to the plane q $(1,-1,1)$

D.C of the axis $(l,m,n) = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

$\perp^r$ distance from O onto the plane q is

$$CM = \left| \frac{-3}{\sqrt{1^2+(-1)^2+1^2}} \right| = \left| \frac{-3}{\sqrt{3}} \right| = \sqrt{3}$$

$$CN = 3$$

$$MN = \sqrt{CN^2 - CM^2} = \sqrt{3^2 - (\sqrt{3})^2} = \sqrt{6}.$$

Equation of the RCC

$$(x-0)^2 + (y-0)^2 + (z-0)^2 - \left\{ \frac{1}{\sqrt{3}} \left((x-0) - (y-0) + (z-0) \right) \right\}^2 = (\sqrt{6})^2$$

$$3(x^2+y^2+z^2) - [x - y + z]^2 = 18$$

$$3x^2 + 3y^2 + 3z^2 - x^2 - y^2 - z^2 + 2xy + 2yz - 2xz = 18$$

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2yz - 2xz = 18$$

$$x^2 + y^2 + z^2 + xy + yz - xz = 9$$

3) Find the equation of the RCC of radius 3 units and axis $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$

4) Find the equation of RCC of radius 2 units, axis passes through (1,2,3) and has d.c's proportional to (2,-3,6).