Impulse invariant methods

$$\frac{1}{s-p_i} \xrightarrow{\text{(is tranformed to)}} \frac{1}{1-e^{p_i T}z^{-1}}$$

Refer Class Note for derivation

1. $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left(\frac{1}{1-e^{-sT}z^{-1}}\right);\quad s=p_i$$

2. $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

3. $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

So the relationship between analog frequency Ω and digital frequency ω is $\omega = \Omega T$ or $\Omega = \dfrac{\omega}{T}$.

Refer Class Note for derivation

Impulse invariant methods

EXAMPLE 8.4 For the analog transfer function

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

determine $H(z)$ if (a) $T = 1$ s and (b) $T = 0.5$ s using impulse invariant method.

Solution: Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)\,H_a(s)\big|_{s=-1} = \frac{2}{s+3}\bigg|_{s=-1} = 1$$

$$B = (s+3)\,H_a(s)\big|_{s=-3} = \frac{2}{s+1}\bigg|_{s=-3} = -1$$

$$\therefore \qquad H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{s-(-3)}$$

By impulse invariant transformation, we know that

$$\frac{A_i}{s-p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A_i}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = -1$ and $p_2 = -3$.

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

(a) When $T = 1$ s

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-3}z^{-1}}$$

$$= \frac{1}{1-0.3678\,z^{-1}} - \frac{1}{1-0.0497\,z^{-1}}$$

$$= \frac{(1-0.0497\,z^{-1})-(1-0.3678\,z^{-1})}{(1-0.3678\,z^{-1})(1-0.0497\,z^{-1})}$$

$$= \frac{0.3181\,z^{-1}}{1-0.4175\,z^{-1}+0.0182\,z^{-2}}$$

Impulse invariant methods

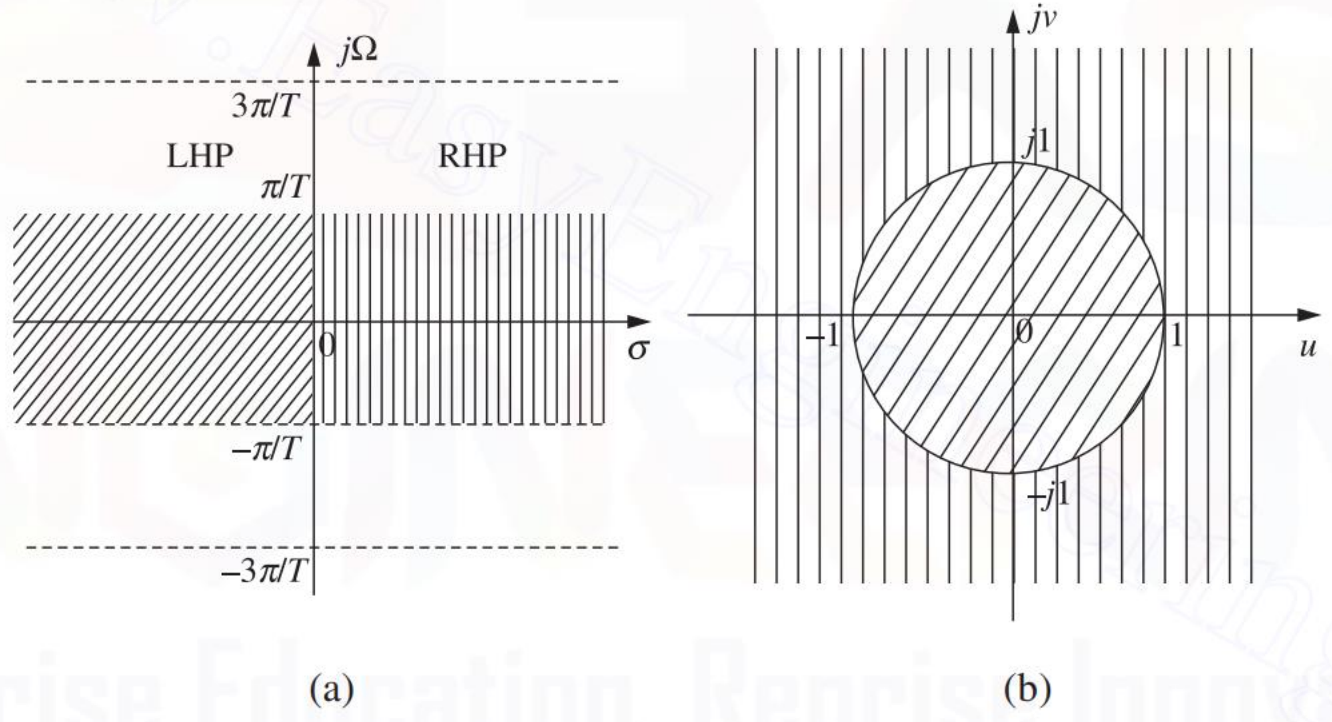


Figure 8.3 Mapping of (a) s-plane into (b) z-plane by impulse invariant transformation.

Bilinear Transformation

This is the relation between analog and digital poles in bilinear transformation. So to convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{in } H_a(s)$$

Refer Class Note for derivation

$\therefore$ The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently, we have $\omega = 2\tan^{-1}\frac{\Omega T}{2}$.

Refer Class Note for derivation

EXAMPLE 8.12 Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

with $T = 0.5$ s and find $H(z)$.

$$\therefore \qquad H(z) = \left.\frac{4}{(s+3)(s+4)}\right|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \left.\frac{4}{(s+3)(s+4)}\right|_{s=4\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{4}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+3\right]\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+4\right]}$$

$$= \frac{4}{\left[\frac{4-4z^{-1}+3+3z^{-1}}{1+z^{-1}}\right]\left[\frac{4-4z^{-1}+4+4z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4(1+z^{-1})^2}{(7-z^{-1})\,8}$$

$$= \frac{1}{2}\,\frac{(1+z^{-1})^2}{(7-z^{-1})}$$

Frequency Warping

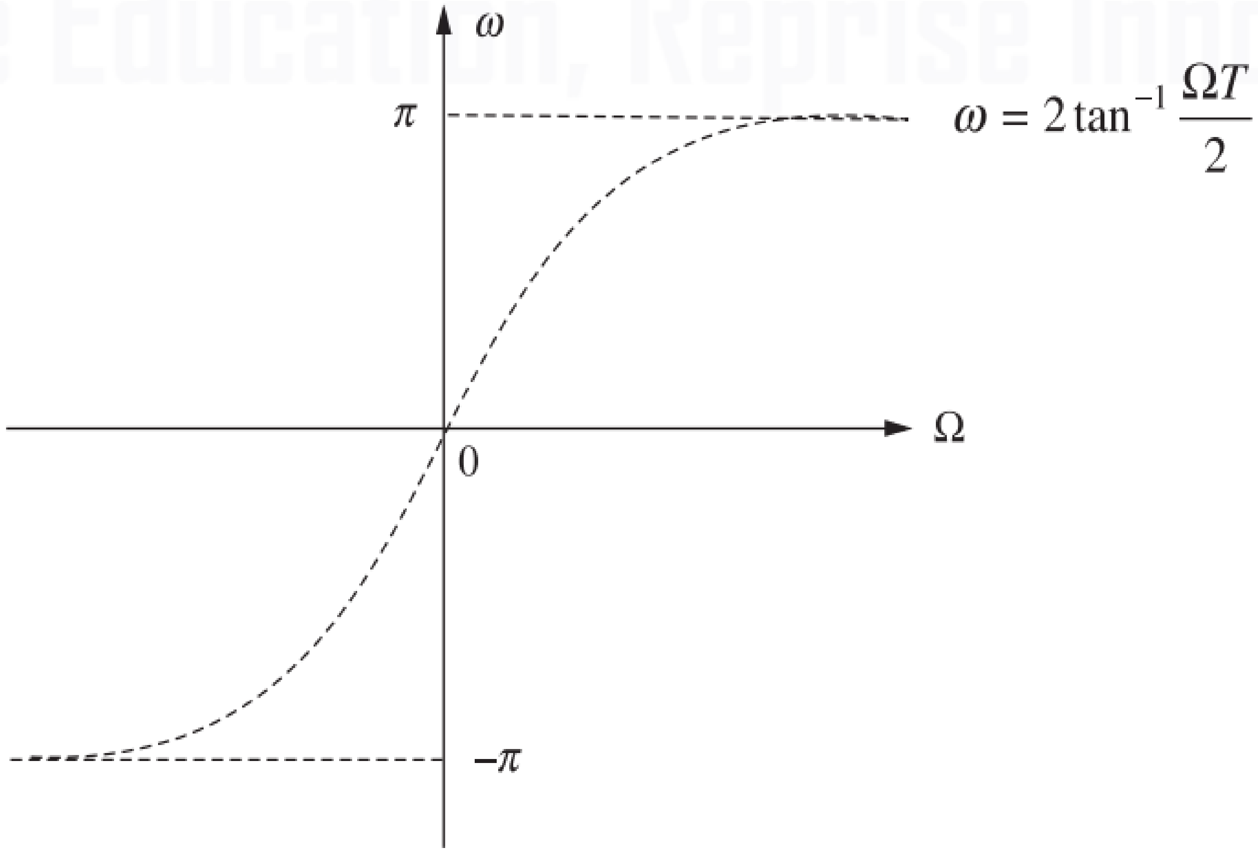
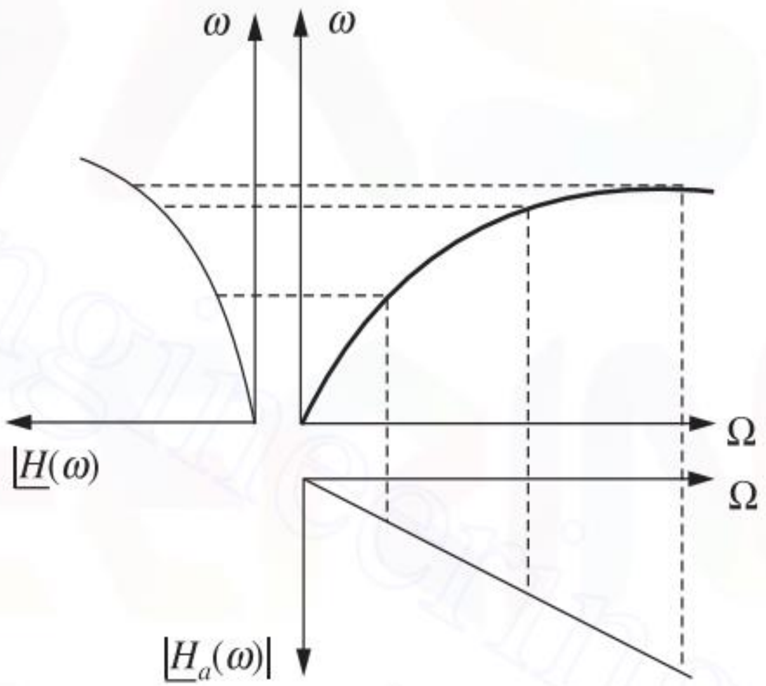


Figure 8.4 Mapping between Ω and ω in bilinear transformation.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 8.5(b). The phase response of corresponding digital filter will be nonlinear.



(b)

Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

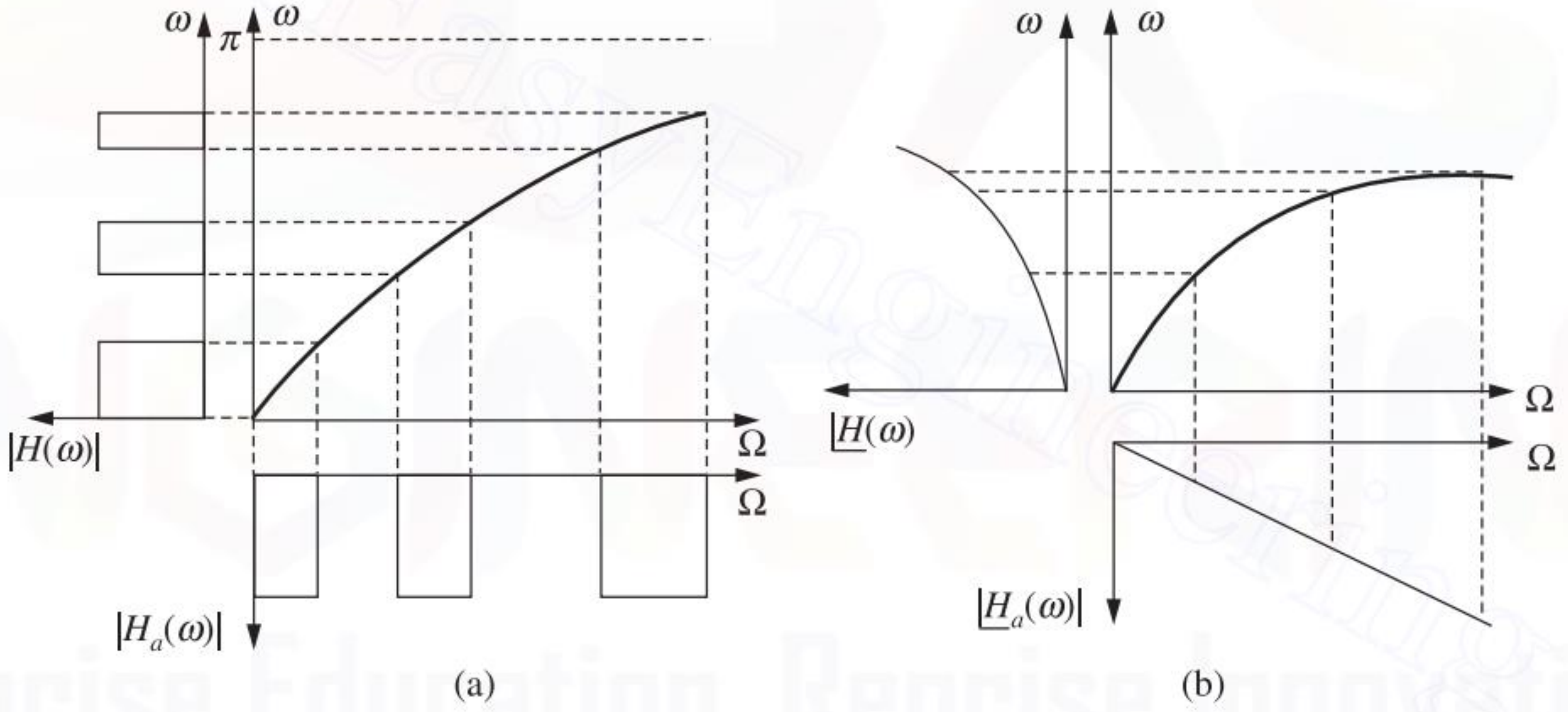


Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

From the earlier discussions, it can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore, the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed into a linear phase digital filter using the bilinear transformation.

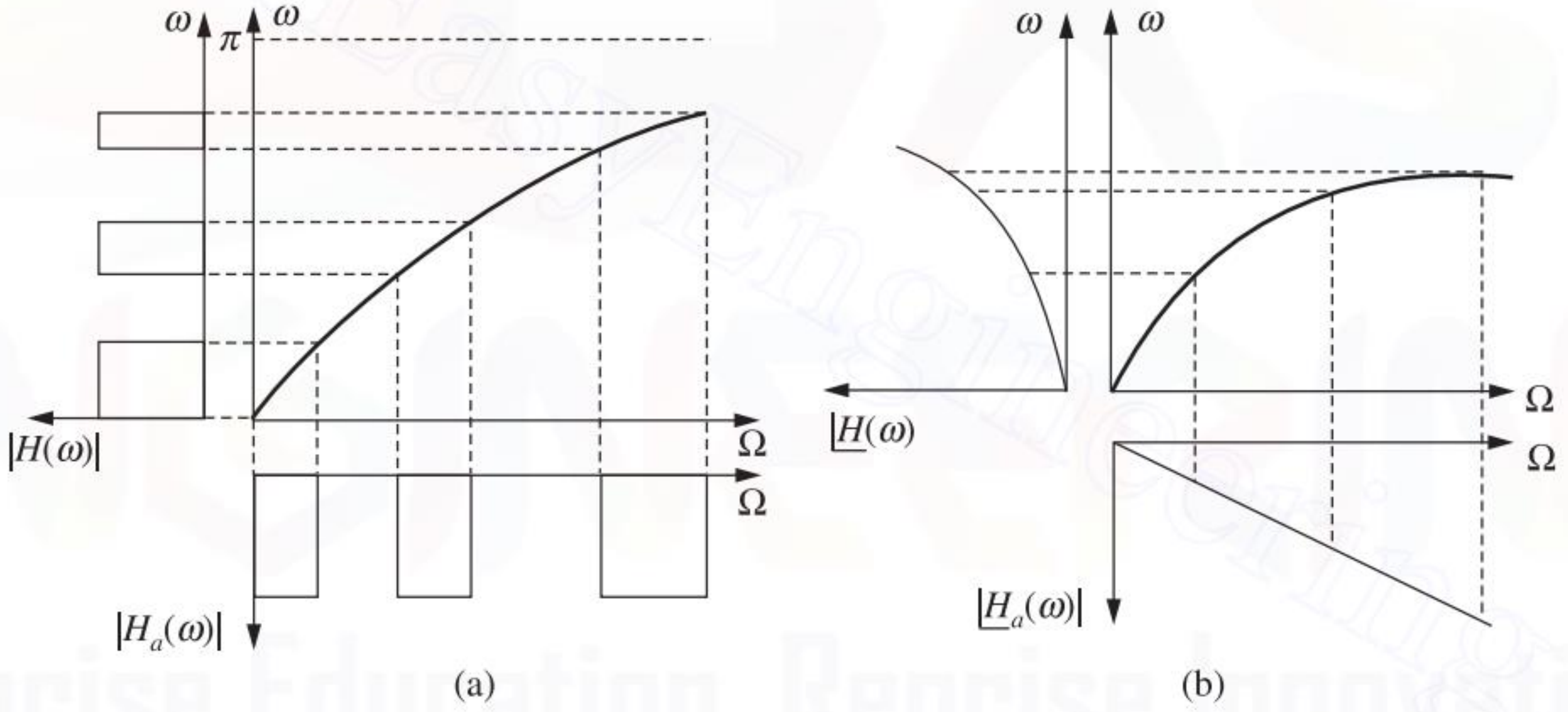


Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

Due to Frequency Warping, phase response of analog filter is not preserved but magnitude response can be preserved by pre-warping analog frequencies

Sr No	Impulse Invariance	Bilinear Transformation
1	In this method IIR filters are designed having a unit sample response h(n) that is sampled version of the impulse response of the analog filter.	This method of IIR filters design is based on the trapezoidal formula for numerical integration.
2	In this method small value of T is selected to minimize the effect of aliasing.	The bilinear transformation is a conformal mapping that transforms the j Ω axis into the unit circle in the z plane only once, thus avoiding aliasing of frequency components.
3	They are generally used for low frequencies like design of IIR LPF and a limited class of bandpass filter	For designing of LPF, HPF and almost all types of Band pass and band stop filters this method is used.
4	Frequency relationship is linear.	Frequency relationship is non-linear. Frequency warping or frequency compression is due to non-linearity.
5	All poles are mapped from the s plane to the z plane by the relationship $Z^k = e^{p_k T}$. But the zeros in two domain does not satisfy the same relationship.	All poles and zeros are mapped.

Low Pass Filter Design

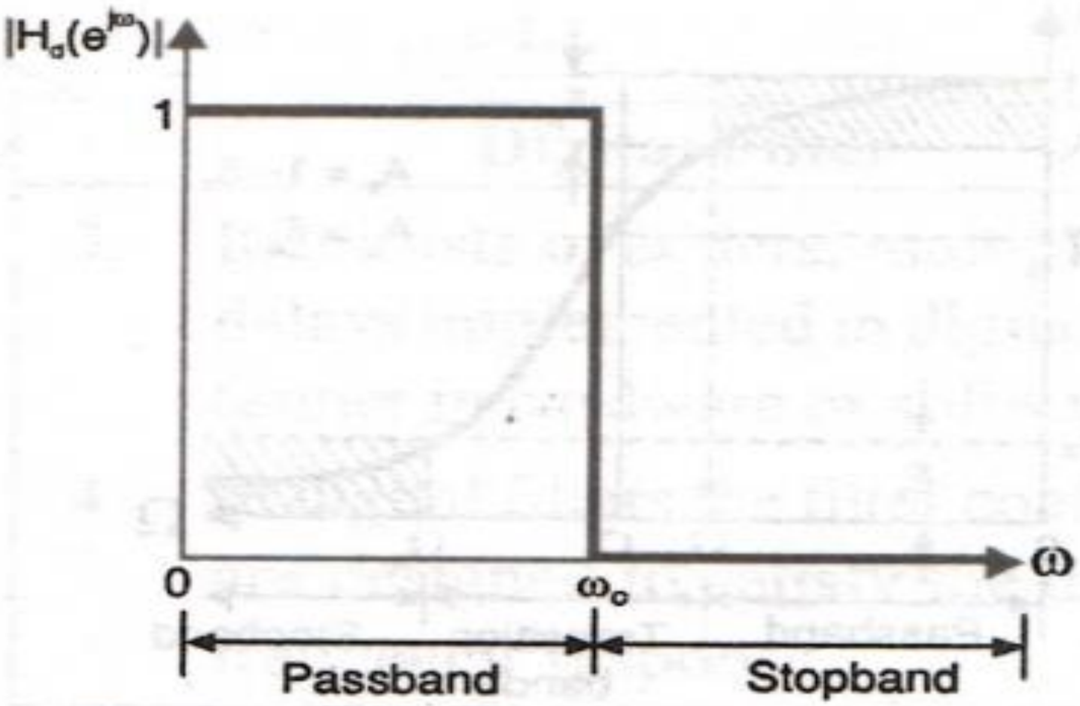


Fig a : *Normalized magnitude response of ideal digital IIR lowpass filter.*

Low Pass Filter Design

- Specification:

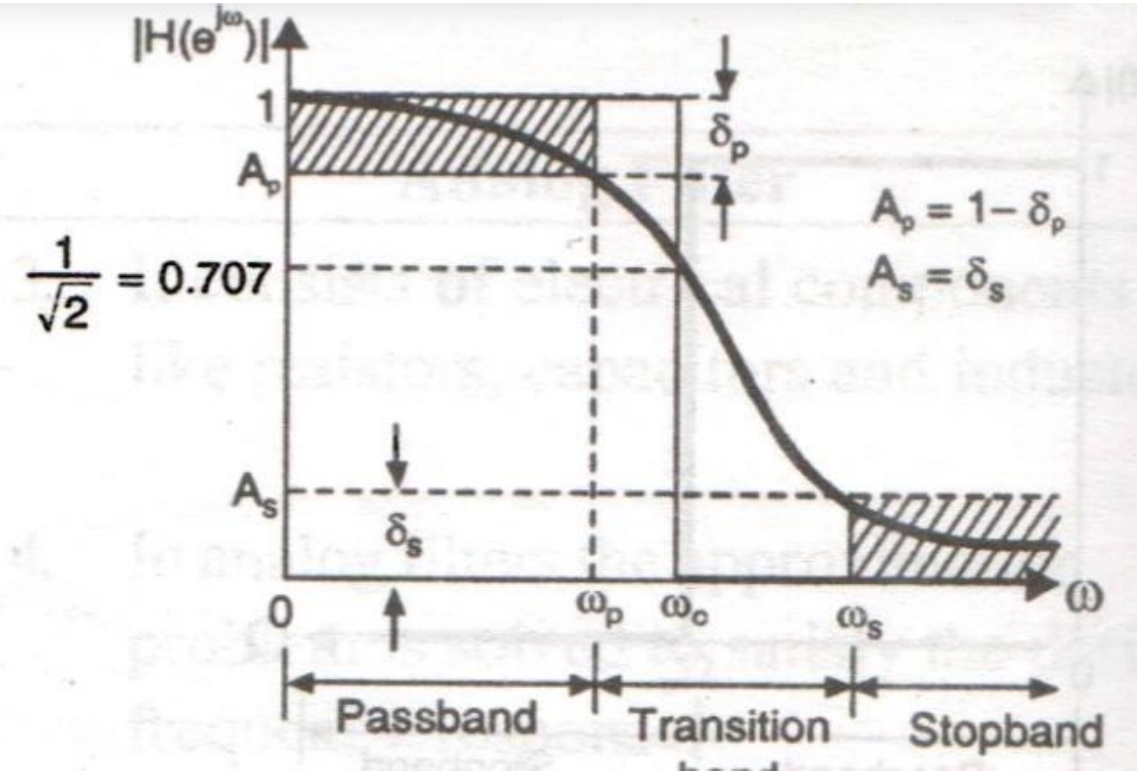
Gain at passband edge frequency $(\mathbf{A_p})$

Gain at stopband edge frequency $(\mathbf{A_s})$

Passband edge digital frequency $(\omega_\mathbf{p})$

Stopband edge digital frequency $(\omega_\mathbf{s})$

Sampling time $(\mathbf{T})$



Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at ∞).
2. The filter order N completely specifies the filter.
3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.

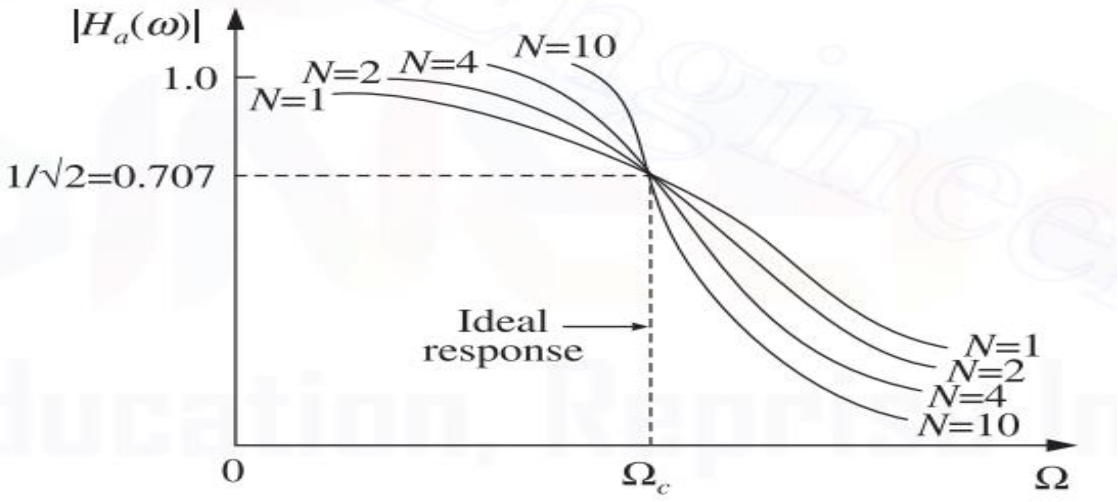
Analog Butterworth Filter

The magnitude response of low-pass filter obtained by this approximation is given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the 3 dB cutoff frequency and N is the order of the filter.

Frequency Response of the Butterworth Filter



Chebyshev filters

Chebyshev filters are analog or digital filters that have a steeper roll-off than Butterworth filters, and have either passband ripple (type I) or stopband ripple (type II). Chebyshev filters have the property that they minimize the error between the idealized and the actual filter characteristic over the range of the filter

type I Chebyshev filter

The magnitude response of type-1 Chebyshev low-pass filter is given by

$$|H_a(\Omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)}}$$

where ε = attenuation constant Or ripple factor

$C_N\left(\frac{\Omega}{\Omega_c}\right)$ = Chebyshev polynomial of the first kind of degree N.

type I Chebyshev filter

The filter parameter ϵ is related to the ripple in the passband

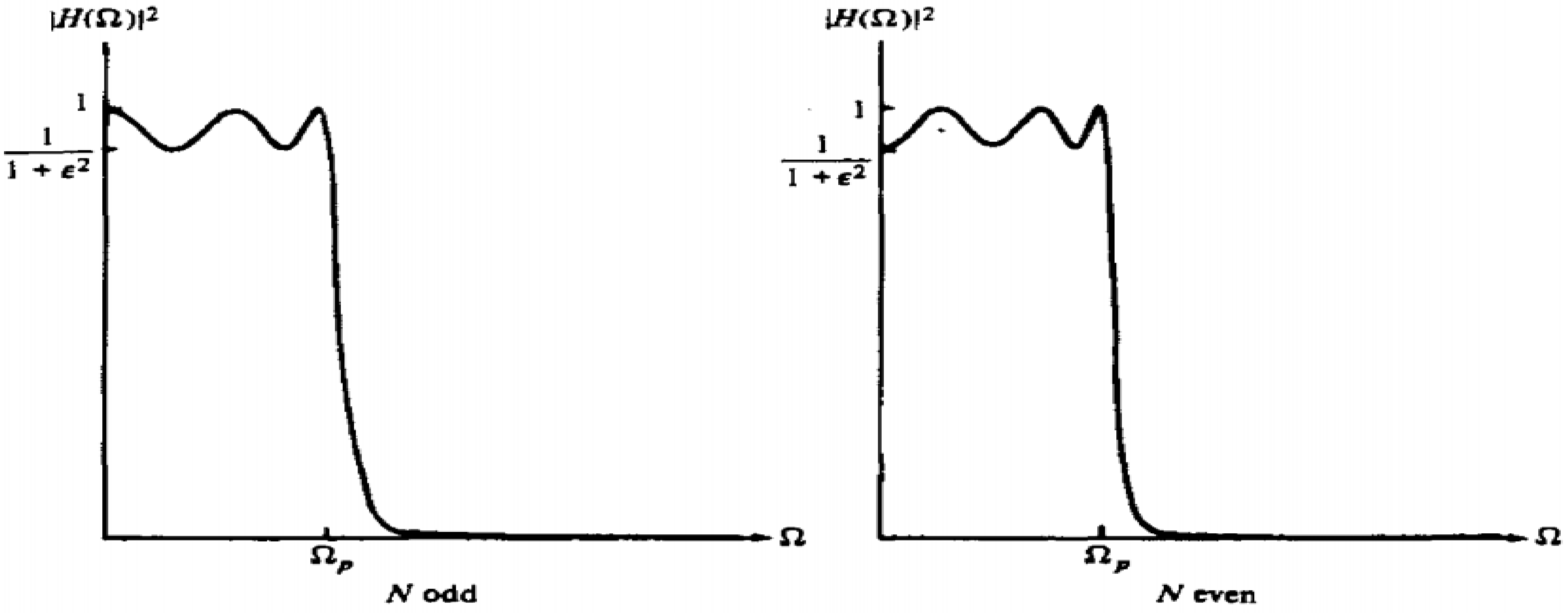


Figure 8.39 Type I Chebyshev filter characteristic.

Butterworth	*Chebyshev type*-1
(i) All pole design.	(i) All pole design.
(ii) The poles lie on a circle in s-plane.	(ii) The poles lie on an ellipse in s-plane.
(iii) The magnitude response is maximally flat at the origin and monotonically decreasing function of Ω.	(iii) The magnitude response is equiripple in passband and monotonically decreasing in the stopband.
(iv) The normalized magnitude response has a value of $1/\sqrt{2}$ at the cutoff frequency Ω_c.	(iv) The normalized magnitude response has a value of $1/\sqrt{1+\varepsilon^2}$ at the cutoff frequency Ω_c.
(v) Only a few parameters have to be calculated to determine the transfer function.	(v) A large number of parameters have to be calculated to determine the transfer function.

Butterworth analog low pass filter response	$\lvert H(\Omega)\rvert^2 = \dfrac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \dfrac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$ $N = \dfrac{log\left(\frac{1}{\delta_2{}^2}-1\right)}{2log\left(\frac{\Omega_s}{\Omega_c}\right)}$ $s_k = \Omega_c e^{j\phi_k}$	N: order of the filter $\lvert H(\Omega)\rvert^2$: Squared magnitude response Ω_p: Pass band edge frequency Ω_p: Stop band edge frequency Ω_c: 3-dB cut-off frequency $\frac{1}{1+\epsilon^2}$: Pass band edge value of $\lvert H(\Omega)\rvert^2$ (Ap) $\delta_2{}^2$: Stop band edge value of $\lvert H(\Omega)\rvert^2$ = (Hs) $\delta_1 = 1 - A_p$ s_k: Poles of H(s) $\phi_k = pole\ angle = \dfrac{\pi}{2} + \dfrac{(2k+1)\pi}{2N}$, $k = 0, 1, 2, \ldots\ldots N\text{-}1$
Chebyshev analog low pass filter response	$\lvert H(\Omega)\rvert^2 = \dfrac{1}{1+\epsilon^2 T_N{}^2\left(\frac{\Omega}{\Omega_p}\right)}$ $T_N(x) = \begin{cases} cos(N cos^{-1}x), & \lvert x\rvert \le 1 \\ cosh(N cosh^{-1}x), & \lvert x\rvert > 1 \end{cases}$	

20 P

Low Pass Filter Design

- Specification:

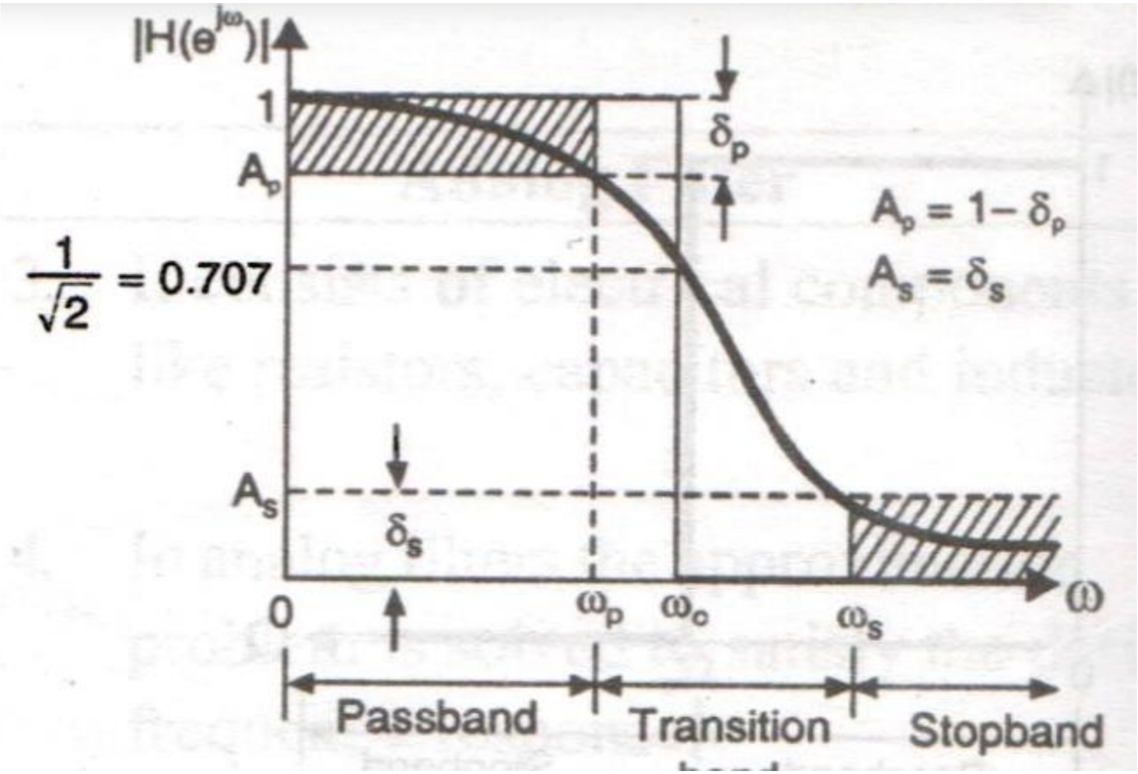
Gain at passband edge frequency $(\mathbf{A_p})$

Gain at stopband edge frequency $(\mathbf{A_s})$

Passband edge digital frequency $(\omega_\mathbf{p})$

Stopband edge digital frequency $(\omega_\mathbf{s})$

Sampling time $(\mathbf{T})$



EXAMPLE 8.24 Determine the order and the poles of a low-pass Butterworth filter that has a –3dB bandwidth of 500 Hz and an attenuation of 40 dB at 1000 Hz. Take sampling frequency(f_s) 2000 Hz

Solution:

$$\omega_c = 500\text{Hz} \qquad \omega_c = \frac{500}{2000}2\pi = 0.5\pi \qquad \omega_s = 1000\text{Hz} \qquad \omega_s = \frac{1000}{2000}2\pi = \pi$$

Applying Impulse Invariant Method

$$\Omega_c = \frac{\omega_c}{T} = 0.5\pi f_s = 1000\pi \qquad \Omega_s = \frac{\omega_s}{T} = \pi f_s = 2000\pi$$

Gain at stopband edge $k_2 = -40$ dB, $\therefore \quad A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

$$N = \frac{\log\left(\frac{1}{\delta_2^2} - 1\right)}{2\log\left(\frac{\Omega_s}{\Omega_c}\right)} = \frac{\log\left(\frac{1}{(0.01)^2} - 1\right)}{2\log\left(\frac{2000\pi}{1000\pi}\right)} = 6.6438 = 7$$

The pole positions are:

$$s_k = \Omega_c e^{j\left[\frac{\pi}{2}+(2k+1)\pi/2N\right]}$$

$$= 1000\pi e^{j\left[\frac{\pi}{2}+(2k+1)\pi/14\right]}, \quad k = 0, 1, 2, 3, 4, 5, 6$$

where Ω_c is 3 dB cutoff frequency.

Unnormalized Transferfunction

$$H[s] = \frac{1}{\prod_{i=1}^{7}(s - P_i)}$$

EXAMPLE 8.24 Determine the order and the poles of a low-pass Butterworth filter that has a –3dB bandwidth of 500 Hz and an attenuation of 40 dB at 1000 Hz. Take sampling frequency(f_s) 2000 Hz

Solution:

$$\omega_c = 500\text{Hz} \qquad \omega_c = \frac{500}{2000}2\pi = 0.5\pi \qquad \omega_s = 1000\text{Hz} \qquad \omega_s = \frac{1000}{2000}2\pi = \pi$$

Applying Impulse Invariant Method

$$\Omega_c = \frac{\omega_c}{T} = 0.5\pi f_s = 1000\pi \qquad \Omega_s = \frac{\omega_s}{T} = \pi f_s = 2000\pi$$

Gain at stopband edge $k_2 = -40$ dB, $\therefore \quad A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

$$N = \frac{\log\left(\frac{1}{\delta_2^2} - 1\right)}{2\log\left(\frac{\Omega_s}{\Omega_c}\right)} = \frac{\log\left(\frac{1}{(0.01)^2} - 1\right)}{2\log\left(\frac{2000\pi}{1000\pi}\right)} = 6.6438 = 7$$

When $\Omega_c = 1$ we get normalized poles

The pole positions are:

$$s_k = e^{j\left[\frac{\pi}{2}+(2k+1)\pi/2N\right]}$$

$$= \pi e^{j\left[\frac{\pi}{2}+(2k+1)\pi/14\right]}, \quad k = 0, 1, 2, 3, 4, 5, 6$$

where Ω_c is 3 dB cutoff frequency.

Normalized Transferfunction is

$$H[s_n] = \frac{1}{\prod_{i=1}^{7}(s_n - P_i)}$$

For LP put $s_n = \frac{s}{\Omega_c}$

EXAMPLE 8.24 Determine the order and the poles of a HP Butterworth filter that has cutoff frequency of 1000 Hz and an attenuation of 40 dB at 500 Hz. Take sampling frequency(f_s) 2000 Hz

High Pass Filter Specification:

$$\omega_c = 1000\text{Hz} \qquad \omega_s = 500\text{Hz}$$

Low Pass Filter Specification:

$$\omega_c = 500\text{Hz} \qquad \omega_c = \frac{500}{2000}2\pi = 0.5\pi \qquad \omega_s = 1000\text{Hz} \qquad \omega_s = \frac{1000}{2000}2\pi = \pi$$

Applying Impulse Invariant Method

$$\Omega_c = \frac{\omega_c}{T} = 0.5\pi f_s = 1000\pi \qquad \Omega_s = \frac{\omega_s}{T} = \pi f_s = 2000\pi$$

Gain at stopband edge $k_2 = -40$ dB, $\therefore \quad A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

$$N = \frac{\log\left(\frac{1}{\delta_2^2} - 1\right)}{2\log\left(\frac{\Omega_s}{\Omega_c}\right)} = \frac{\log\left(\frac{1}{(0.01)^2} - 1\right)}{2\log\left(\frac{2000\pi}{1000\pi}\right)} = 6.6438 = 7$$

When $\Omega_c = 1$ we get normalized poles

The pole positions are:

$$s_k = e^{j\left[\frac{\pi}{2}+(2k+1)\pi/2N\right]}$$

$$= \pi e^{j\left[\frac{\pi}{2}+(2k+1)\pi/14\right]}, \quad k = 0, 1, 2, 3, 4, 5, 6$$

where Ω_c is 3 dB cutoff frequency.

Normalized Transferfunction is

$$H[s_n] = \frac{1}{\prod_{i=1}^{7}(s_n - P_i)}$$

For HP put $s_n = \frac{\Omega_c}{s}$

All pole IIR system	$y(n) = \sum_{k=1}^{N} a_k y(n-k) + b_0 x(n)$ $H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 - \sum_{k=1}^{N} a_k z^{-k}}$
All zero FIR system	$y(n) = \sum_{k=0}^{M} b_k x(n-k)$ $H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$

Determine the poles of low pass Butterworth filter for N=3.

Find the Normalized Transfer Function, and comment on the nature of poles

-0.5000 + 0.8660i
-1.0000 + 0.0000i
-0.5000 - 0.8660i

Determine the poles of low pass Butterworth filter for N=5.

Find the Normalized Transfer Function, and comment on the nature of poles

-0.3090 + 0.9511i
-0.8090 + 0.5878i
-1.0000 + 0.0000i
-0.8090 - 0.5878i
-0.3090 - 0.9511i

7.3.1 Direct-Form Structures

The rational system function as given by (7.1.2) that characterizes an IIR system can be viewed as two systems in cascade, that is,

$$H(z) = H_1(z)H_2(z) \tag{7.3.1}$$

where $H_1(z)$ consists of the zeros of $H(z)$, and $H_2(z)$ consists of the poles of $H(z)$,

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k} \tag{7.3.2}$$

and

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{7.3.3}$$

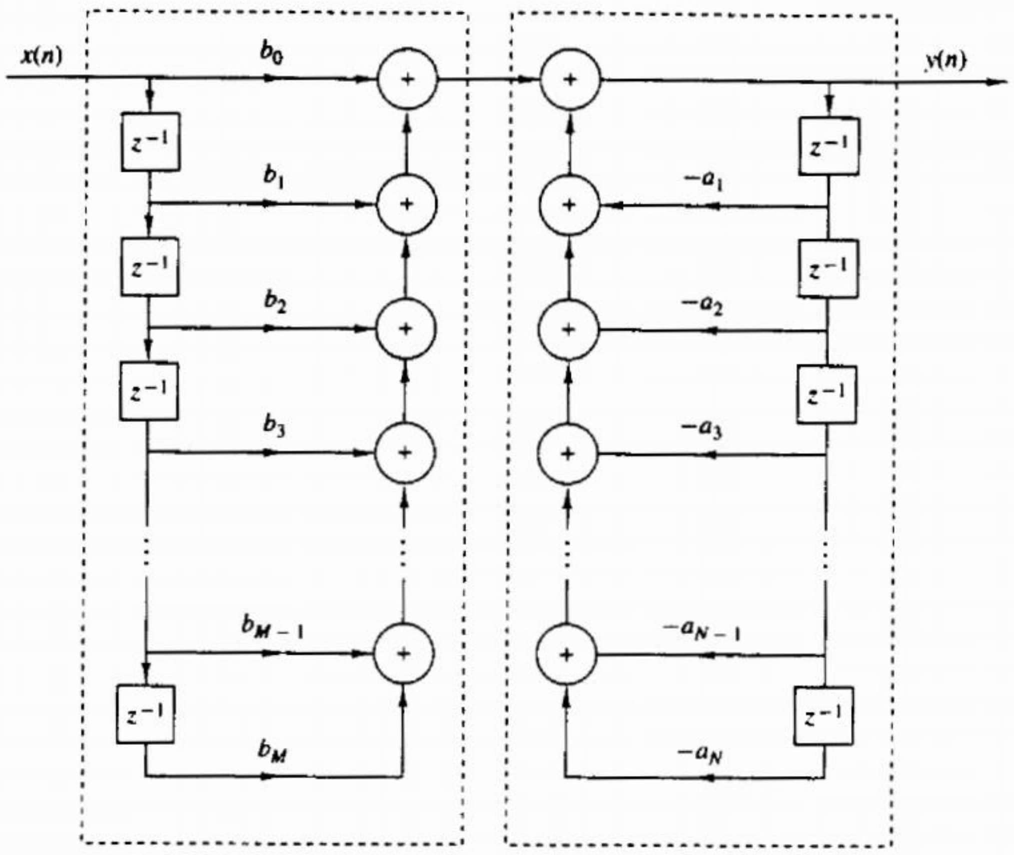


Figure 7.12 Direct form I realization.

Fig. 7.12. This realization requires $M + N + 1$ multiplications, $M + N$ additions, and $M + N$ memory locations. for the delay

If the all-pole filter $H_2(z)$ is placed before the all-zero filter $H_1(z)$, a more compact structure is obtained

and the maximum of $\{M, N\}$ memory locations. Since the direct form II realization minimizes the number of memory locations, it is said to be *canonic.*

The structures in Figs. 7.12 and 7.13 are both called "direct form" realizations because they are obtained directly from the system function $H(z)$ without any rearrangement of $H(z)$. Unfortunately, both structures are extremely sensitive to parameter quantization, in general, and are not recommended in practical applications.

when N is large, a small change in a filter coefficient due to parameter quantization, results in a large change in the location of the poles and zeros of the system.

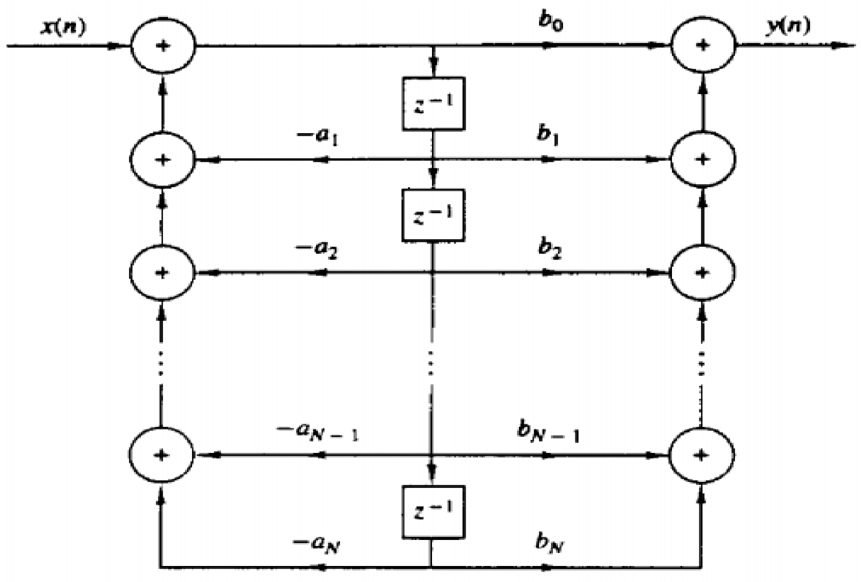


Figure 7.13 Direct form II realization ($N = M$).

Draw DF I and DF II for given difference equation

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$$

Parallel form

Consider the equation $H(z) = \dfrac{(1+z^{-1})(1+3z^{-1})}{[1+(1/2)z^{-1}][1+(1/3)z^{-1}][1+(1/4)z^{-1}]}$

By partial fraction expansion, we have

$$H(z) = \frac{A}{1+(1/2)z^{-1}} + \frac{B}{1+(1/3)z^{-1}} + \frac{C}{1+(1/4)z^{-1}}$$

where, the coefficients A, B and C are

$$A = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-2} = \frac{(1-2)(1-6)}{\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)} = \frac{5}{\frac{1}{3}\cdot\frac{1}{2}} = 30$$

$$B = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-3} = \frac{(1-3)(1-9)}{\left(1-\frac{3}{2}\right)\left(1-\frac{3}{4}\right)} = \frac{16}{-\frac{1}{2}\cdot\frac{1}{4}} = -128$$

$$C = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}\Bigg|_{z^{-1}=-4} = \frac{(1-4)(1-12)}{(1-2)\left(1-\frac{4}{3}\right)} = \frac{33}{-1\cdot-\frac{1}{3}} = 99$$

$\therefore$
$$H(z) = \frac{30}{1+(1/2)z^{-1}} - \frac{128}{1+(1/3)z^{-1}} + \frac{99}{1+(1/4)z^{-1}}$$

Let
$$H(z) = \frac{Y(z)}{X(z)}$$

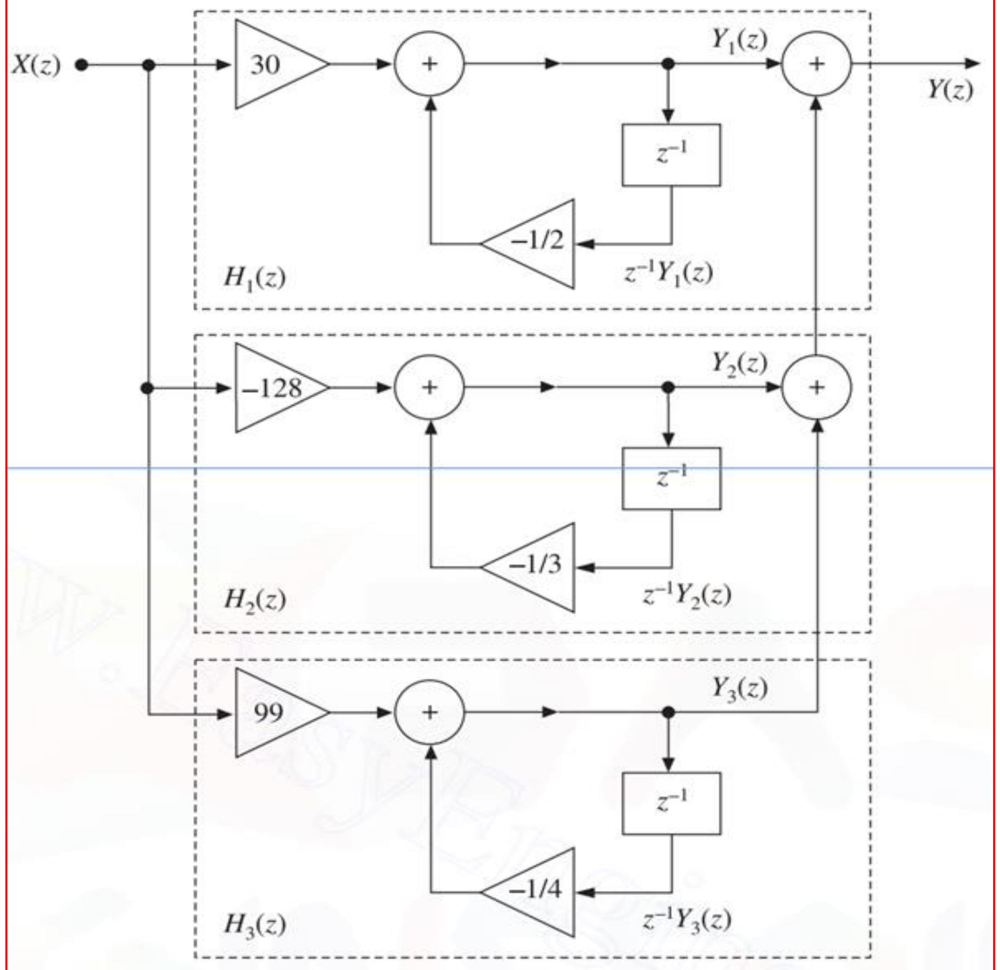


Figure 4.27 Parallel form realization (Example 4.6).

EXAMPLE 4.9 Realize the given system in cascade and parallel forms

$$H(z) = \frac{1 + (1/3)z^{-1}}{[1-(1/2)z^{-1} + (1/3)z^{-2}][1-(1/3)z^{-1} + (1/2)z^{-2}]}$$

Solution:

Cascade form

Let us realize the system as cascade of two second order sections.

$$H(z) = \frac{1 + (1/3)z^{-1}}{1-(1/2)z^{-1} + (1/3)z^{-2}} \; \frac{1}{1-(1/3)z^{-1} + (1/2)z^{-2}} = H_1(z)\; H_2(z)$$

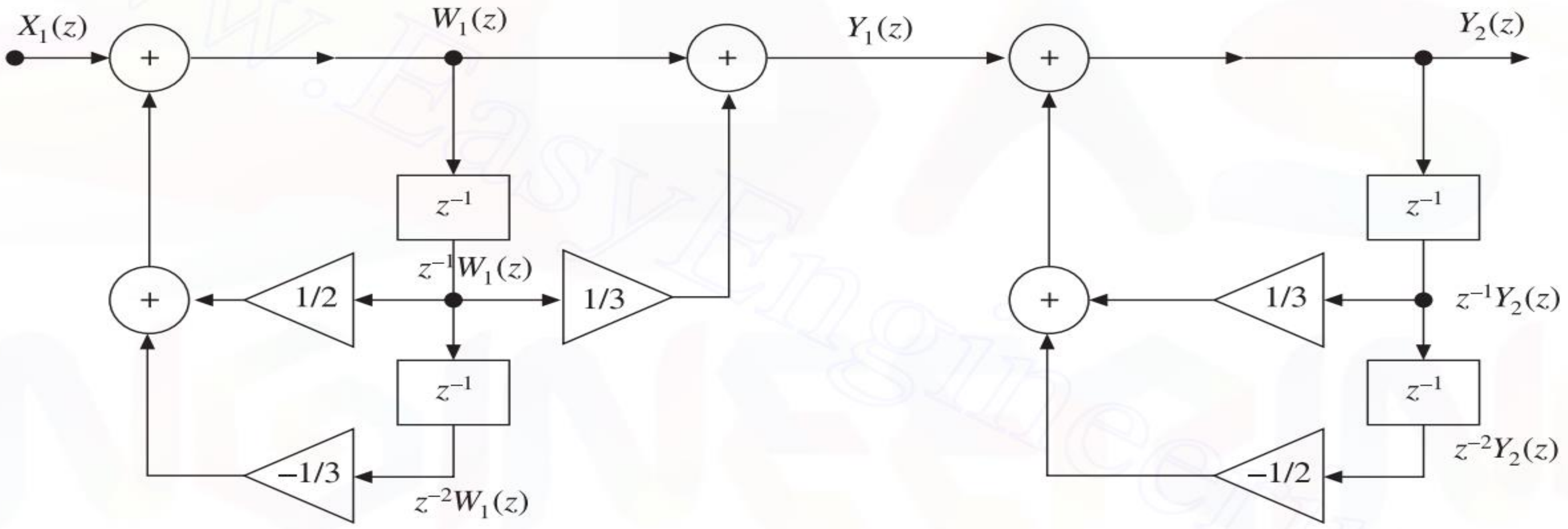


Figure 4.33 Cascade structure of $H(z)$ (Example 4.9).