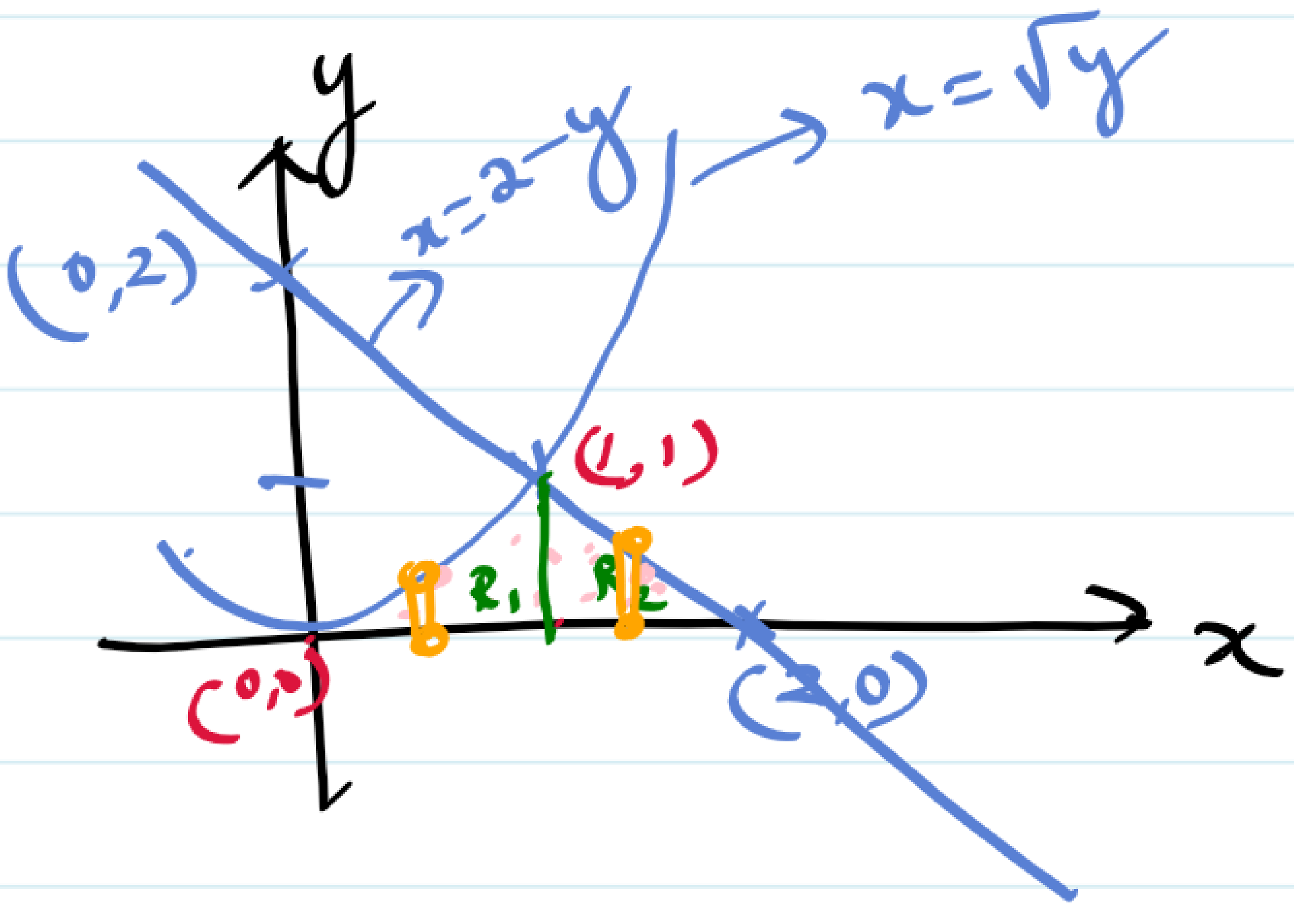
① Change the order of integration in the integral

$$\int_0^1 \int_{\sqrt{y}}^{2-y} xy \, dx \, dy$$



The given region is bounded by

$x=\sqrt{y}$, $x=2-y$, $y=0$, $y=1$

$y=x^2$

$y=x^2$, $x+y=2$

$$I = \iint_{R_1} xy \, dy \, dx + \iint_{R_2} xy \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} xy \, dy \, dx + \int_{x=1}^{2} \int_{y=0}^{2-x} xy \, dy \, dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{x^2} dx + \int_{x=1}^{2} x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \int_0^1 \frac{x^5}{2} \, dx + \int_{x=1}^{2} \frac{1}{2} x (2-x)^2 \, dx$$

$x(2-x)^2$

$x(4+x^2-4x)$

$4x+x^3-4x^2$

$$= \frac{1}{2}\left(\frac{x^6}{6}\right)_0^1 + \frac{1}{2}\left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3}\right]_1^2$$

$$= 39/24 \quad \frac{27}{24}$$

2) Evaluate $\displaystyle\int_0^a \int_x^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$.

Changing the order of integration –

The given region is bounded by $x=0$, $x = a-\sqrt{a^2-y^2}$

$y=0$, $y=a$

$x-a = -\sqrt{a^2-y^2}$

squaring $(x-a)^2 = a^2-y^2$

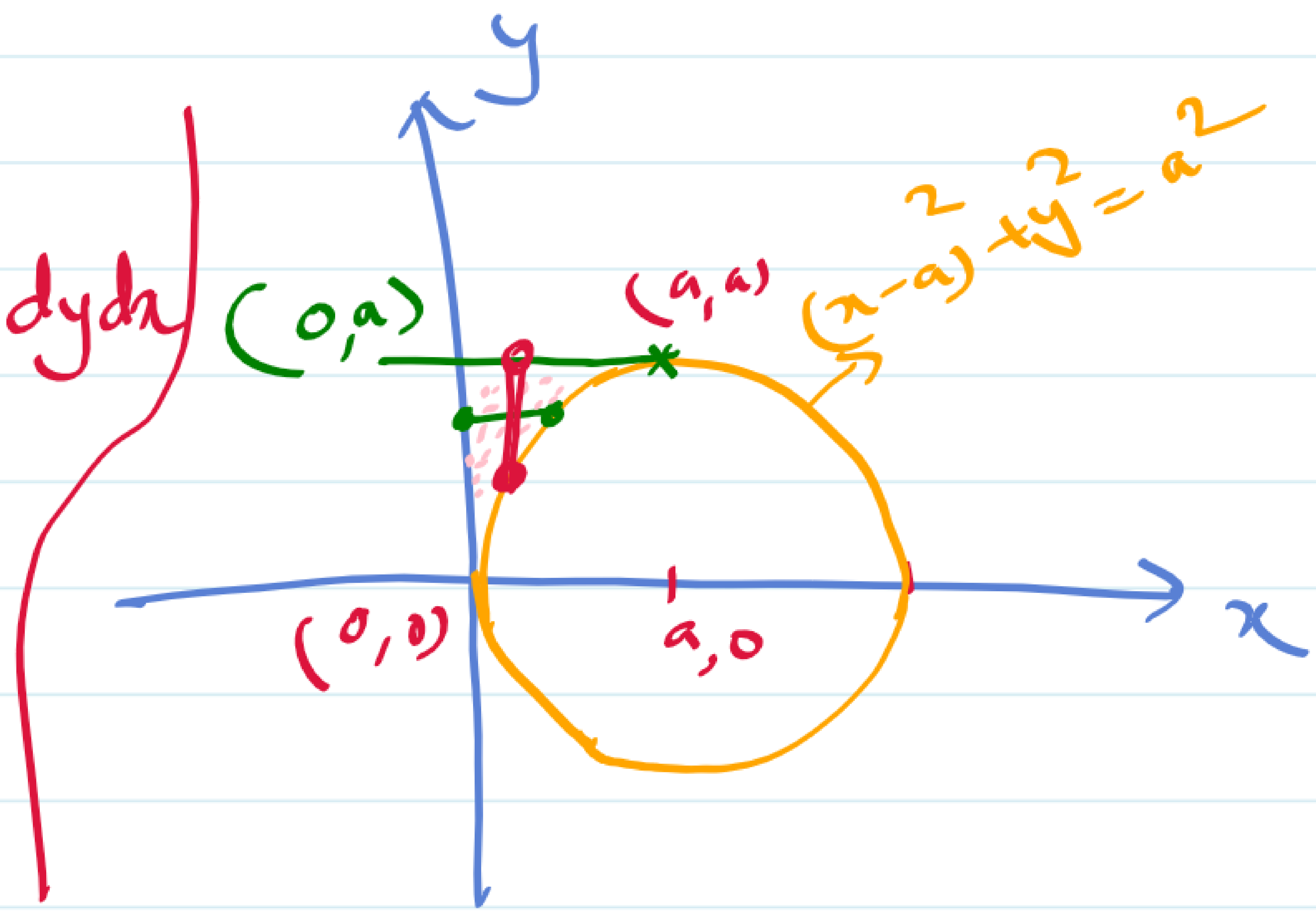
$(x-a)^2+y^2 = a^2$

$\downarrow$

circle, centre $(0,a)$

$r=a$

$$I = \int_{x=0}^{a}\int_{y=\sqrt{2ax-x^2}}^{a} \frac{x\log(x+a)}{(x-a)^2}\, y\, dy\, dx$$



$(x-a)^2+y^2=a^2$

$y^2 = a^2-(x-a)^2$

$y^2 = a^2-(x^2+a^2-2ax)$

$y^2 = -x^2+2ax$

$y = \sqrt{2ax-x^2}$

$$I = \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{y^2}{2}\right)_{\sqrt{2ax-x^2}}^{a} dx$$

$$= \frac{1}{2}\int_0^a \frac{x\log(x+a)}{\cancel{(x-a)^2}}\left(\cancel{a^2-2ax+x^2}\right) dx$$

$$= \frac{1}{2}\int_0^a \frac{x\log(x+a)}{2} dx$$

$$= \frac{1}{2}\left\{\left[\log(x+a)\,\frac{x^2}{2}\right]_0^a - \int_0^a \frac{x^2}{2}\times\frac{1}{x+a}\,dx\right\}$$

$$= \frac{1}{2}\log(x+a)\frac{x^2}{2}\Big)_0^a - \frac{1}{2\times2}\int_0^a \frac{x^2-a^2+a^2}{x+a}\,dx$$

$$= \frac{1}{4} a^2 \log 2a - \frac{1}{4} \int_0^a \left(x - a + \frac{a^2}{x+a} \right) dx$$

$$= \frac{1}{4} a^2 \log(2a) - \left[\frac{1}{4} \left(\frac{x^2}{2} - ax + a^2 \log(x+a) \right) \right]_0^a$$

$$= \frac{1}{4} a^2 \log(2a) - \frac{1}{4} \left[\frac{a^2}{2} - a^2 + a^2 \log 2a - a^2 \log a \right]$$

$$= \frac{a^2 \log(2a)}{4} - \frac{1}{4} \left(-\frac{a^2}{2} + a^2 \log 2a - a^2 \log a \right)$$

Practice questions -

① Evaluate $\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{(2-x^2)(1-x^2y^2)}$ by changing the order of integration

[Ans: $\left(1-\frac{\pi}{4}\right)$]

② Evaluate $\int_0^a \int_{y^2/a}^{2a-y} xy\,dx\,dy$ by changing the order of integration

[Ans: $\frac{5a^4}{24}$]

③ Evaluate $\int_0^\infty \int_0^x x\,e^{-x^2/2}\,dy\,dx$

[Ans: $1/2$]

4) Show that $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \frac{a\pi}{4}$

⑤ Evaluate $\int_0^1 dx \int_1^\infty e^{-y} y^x \log y\,dy$

[Ans: $1/e$]

Change of variables in double integrals -

Certain double integrals may be impossible to evaluate by ordinary methods, specially when the region of integration R happens to be complecated. In such cases, it is necessary to substitute the variables x, y in terms of new variables u, v by the relation $x = g_1(u,v)$, $y = g_2(u,v)$.

Jacobian

If x and y are functions of two variables u and v,

then the determinant $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the

Jocobian of x and y w.r.to u, v.

$$(x,y) \longrightarrow (u,v)$$

Similarly, the Jacobian of x, y, z w.r.to u, v, w

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = J = \begin{vmatrix} \frac{\partial x}{\partial u} & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \\ \partial x/\partial w & \partial y/\partial w & \partial z/\partial w \end{vmatrix}$$

$$\begin{tikzcd} u,v \arrow[r] \arrow[rr, bend left] & r,s \arrow[r] & x,y \end{tikzcd}$$

Properties:

① If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

② Chain rule: If u, v are functions of r, s and r, s are functions of x, y then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$