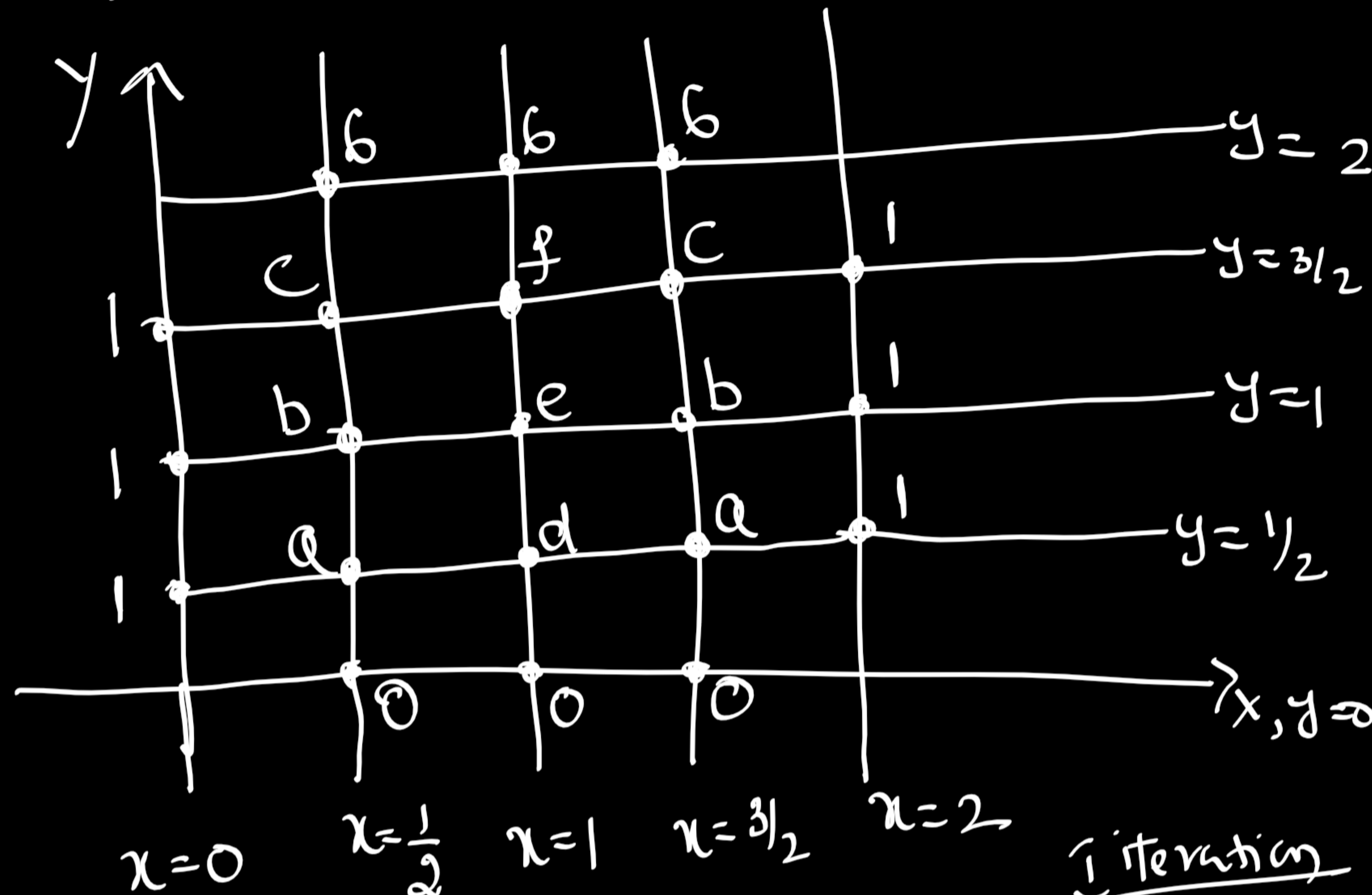
2. Solve: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 2, \quad h = \frac{1}{2}$

$u(x,0) = 0, \quad u(x,2) = 6, \quad u(0,y) = u(2,y) = 1.$



$4a - b - d = 1$

$4b - a - c - e = 1$

$4c - b - f = 7$

$4d - 2a - e = 0$

$4e - 2b - f - d = 0$

$4f - 2c - e = 6$

$a = \frac{1}{4}\left[b + d + 1\right]$

$b = \frac{1}{4}\left[a + c + e + 1\right]$

$c = \frac{1}{4}\left[b + f + 7\right]$

$d = \frac{1}{4}\left[2a + e\right]$

$e = \frac{1}{4}\left[2b + f + d\right]$

$f = \frac{1}{4}\left[2c + e\right]$

I iteration

$a = 0.25$

$b = 0.3125$

$c = 1.828$

$d = 0.125$

$e = 0.1875$

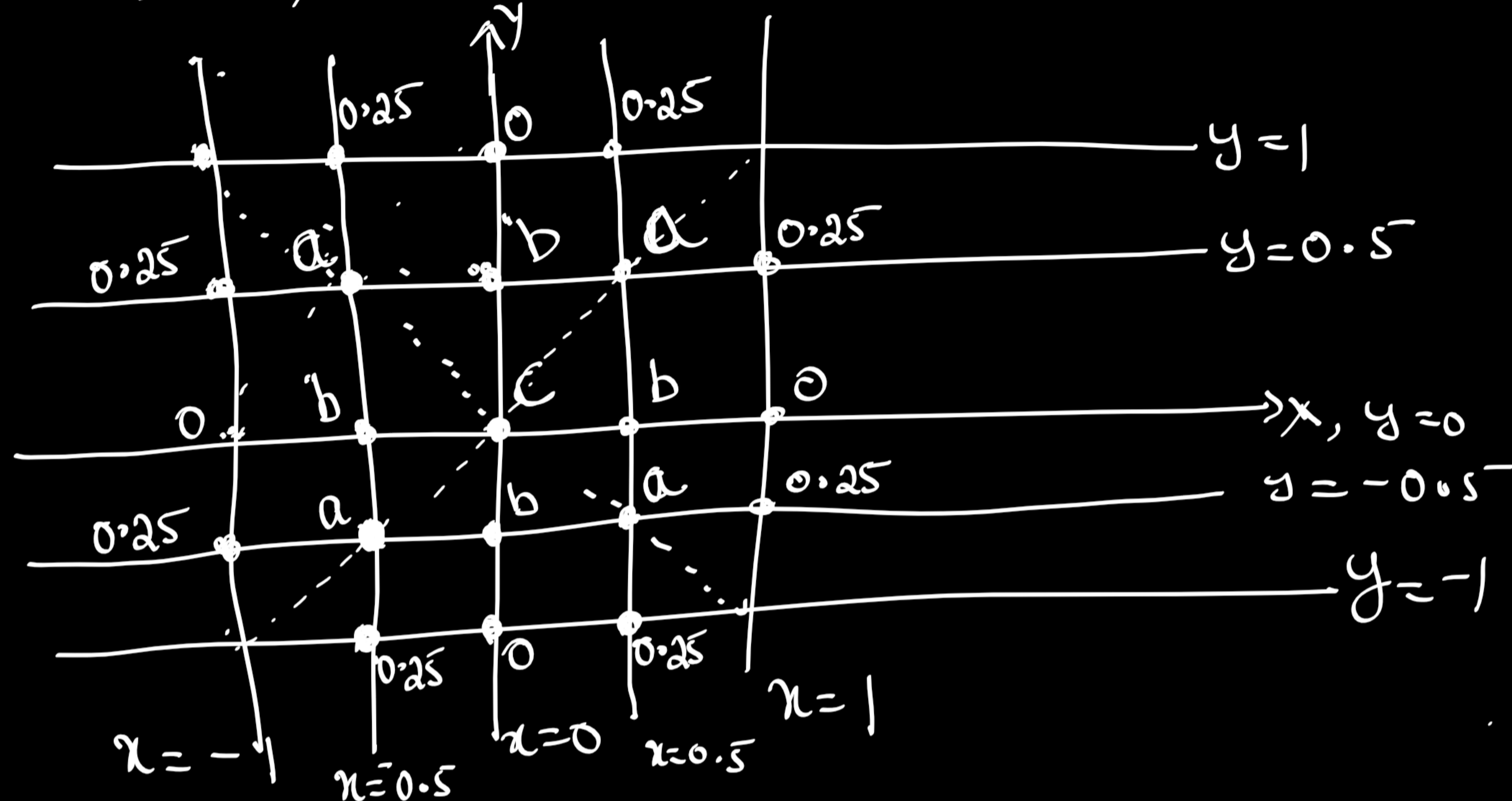
$f = 0.9608$

Repeat the iterations till you get the desired accuracy.

Examples

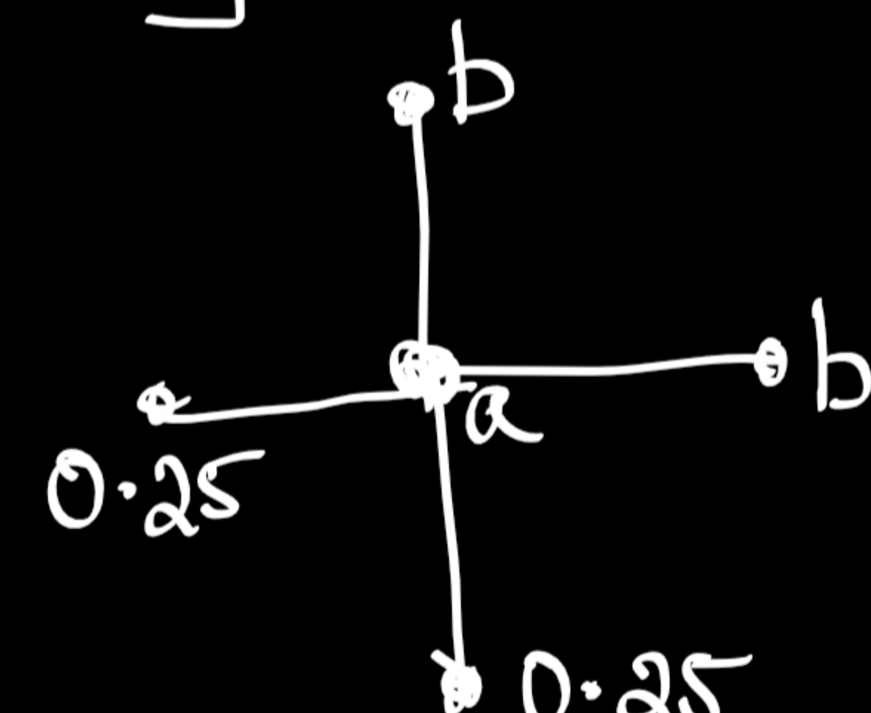
1. Solve: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $|x|<1$, $|y|<1$, $h = \frac{1}{2}$

$u(x, \pm 1) = x^2$, $u(\pm 1, y) = y^2$



[Gauss-Seidel used to solve a PDE using finite difference method is known as Liebmann's method]

$$a = \frac{1}{4}\left[2b + 0.5\right]$$



$$4a - 2b = 0.5 \longrightarrow (1)$$

$$b = \frac{1}{4}\left[2a + c\right]$$

$$-2a + 4b - c = 0 \longrightarrow (2)$$

$$c = \frac{1}{4}\left[4b\right] \Rightarrow b = c \rightarrow (3)$$

$$a = 3/16, \quad b = 1/8 = c$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$$

$$\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{\partial^2 u_{ij}}{\partial y^2} = 0$$

ie $$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} = 0$$

$$u_{i+1,j} - 4u_{ij} + u_{i-1,j} + u_{ij+1} + u_{ij-1} = 0$$

$$\therefore u_{ij} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{ij+1} + u_{ij-1}\right]$$

$\rightarrow$ Standard five point formula

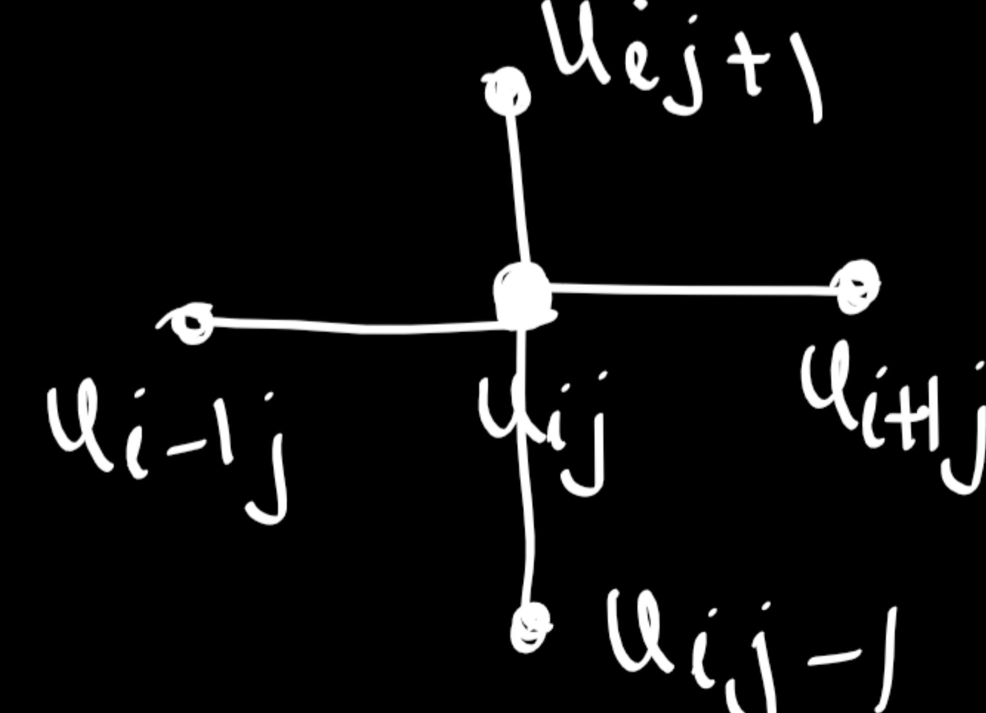
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

$$\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{\partial^2 u_{ij}}{\partial y^2} = f_{ij}$$

ie $$\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} = f_{ij}$$

$$u_{i+1j} - 4u_{ij} + u_{i-1j} + u_{ij+1} + u_{ij-1} = h^2 f_{ij}$$

$$\therefore u_{ij} = \frac{1}{4}\left[u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1} - h^2 f_{ij}\right]$$



To solve Laplace's and Poisson's equations in a rectangular region, we divide the region into small squares. Let h be the step size for x and y values. The points of intersection of the lines parallel to the co-ordinate axes are called mesh points or grid points.

Let $x_i = x_0 + ih$, $y_j = y_0 + jh$.

$$u_{ij} = u(x_i, y_j)$$

$$u_{i+1,j} = u(x_i + h, y_j) = u(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \cdots$$

$$\therefore u_{i+1,j} = u_{ij} + h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2}$$

$$u_{i-1,j} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2}$$

$$u_{i+1,j} + u_{i-1,j} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2}$$

$$\frac{\partial^2 u_{ij}}{\partial x^2} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}$$

Similarly, $$\frac{\partial^2 u_{ij}}{\partial y^2} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2}$$

Finite difference method to solve a second order PDE

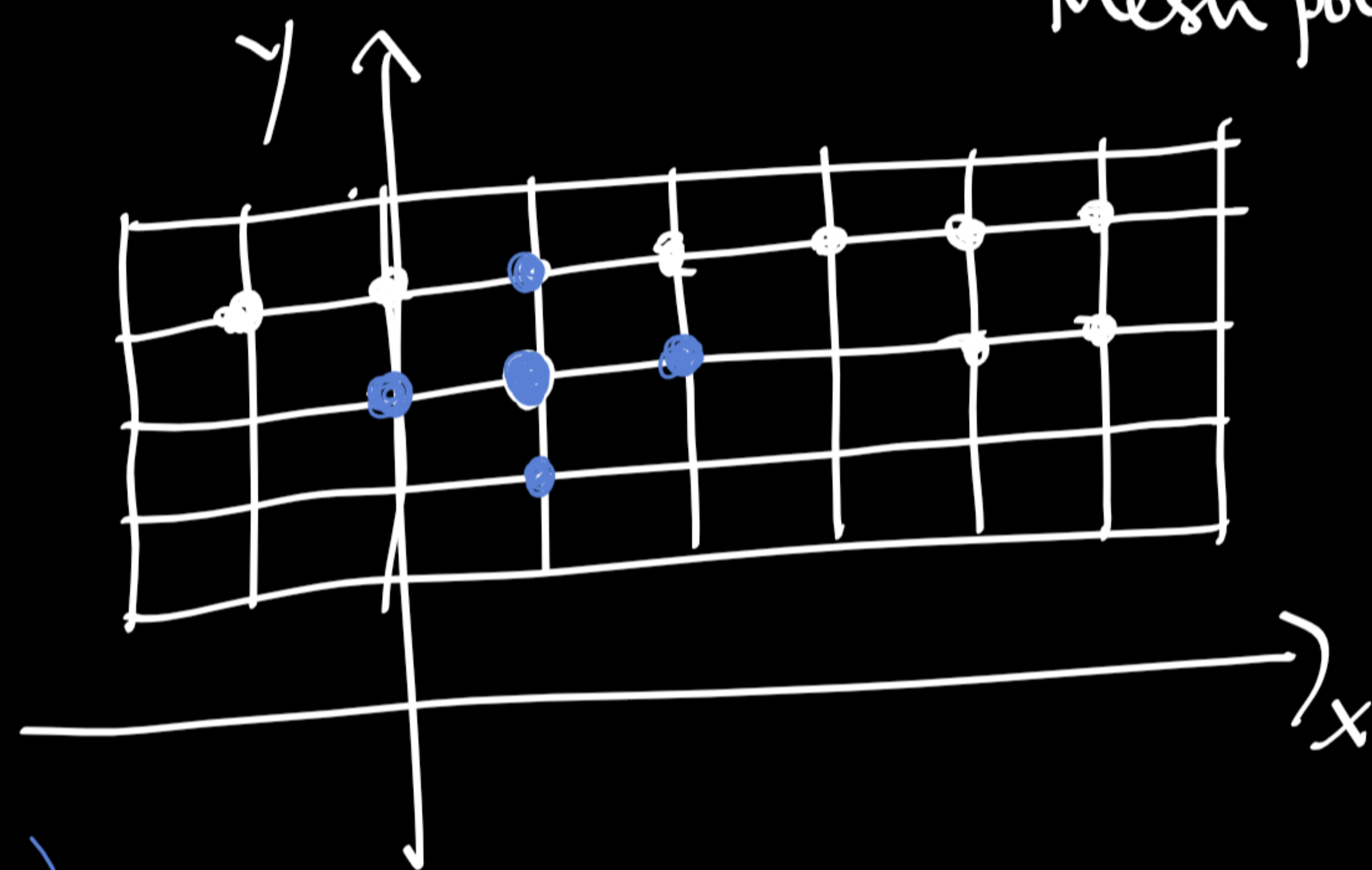
I.

(i) Laplace's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(ii) Poisson's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$

The above two equations are elliptic.

Grid points
Mesh points



$x = x_i$
$y = y_j$

$u(x_i, y_j) = u_{ij}$

$\bullet \rightarrow u_{ij} = u(x_i, y_j)$

$u_{i+1,j} = u(x_i + h, y_j)$

$u_{i-1,j} = u(x_i - h, y_j)$

$u_{i,j+1} = u(x_i, y_j + h)$

$u_{i,j-1} = u(x_i, y_j - h)$

$$f(a+h, b+k) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b) + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(a,b) + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}(a,b) + hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + \cdots$$

In Laplace's and Poisson's equations, the region is a closed region.

$$4x^2 + 16y^2 > 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/4)^2} > 1$$

$\therefore$ The eqn: is elliptic Outside the ellipse

$$\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/4)^2} = 1$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F(x, y, u, u_x, u_y) = 0$$

1) Parabolic $\rightarrow$ $B^2 - 4AC = 0$

2) Hyperbolic $\rightarrow$ $B^2 - 4AC > 0$

3) Elliptic $\rightarrow$ $B^2 - 4AC < 0$

Ex:- 1. $(1+x^2)u_{xx} + (5+2x^2)u_{xy} + (4+x^2)u_{yy} = 0$

$A = 1+x^2, \quad B = 5+2x^2, \quad C = 4+x^2$

$$\begin{aligned} B^2 - 4AC &= (5+2x^2)^2 - 4(1+x^2)(4+x^2) \\ &= 25 + 20x^2 + 4x^4 - 16 - 20x^2 - 4x^4 \\ &= 9 > 0 \end{aligned}$$

$\therefore$ Eqn: is Hyperbolic.

2. In which part of the xy-plane is the following equation elliptic?

$u_{xx} + u_{xy} + (x^2+4y^2)u_{yy} = 2\sin(xy)$

$A = 1, \quad B = 1, \quad C = x^2 + 4y^2$

$B^2 - 4AC = 1 - 4(x^2+4y^2)$

Eqn: is elliptic if $1 - 4(x^2+4y^2) < 0$

i.e $4(x^2+4y^2) > 1$