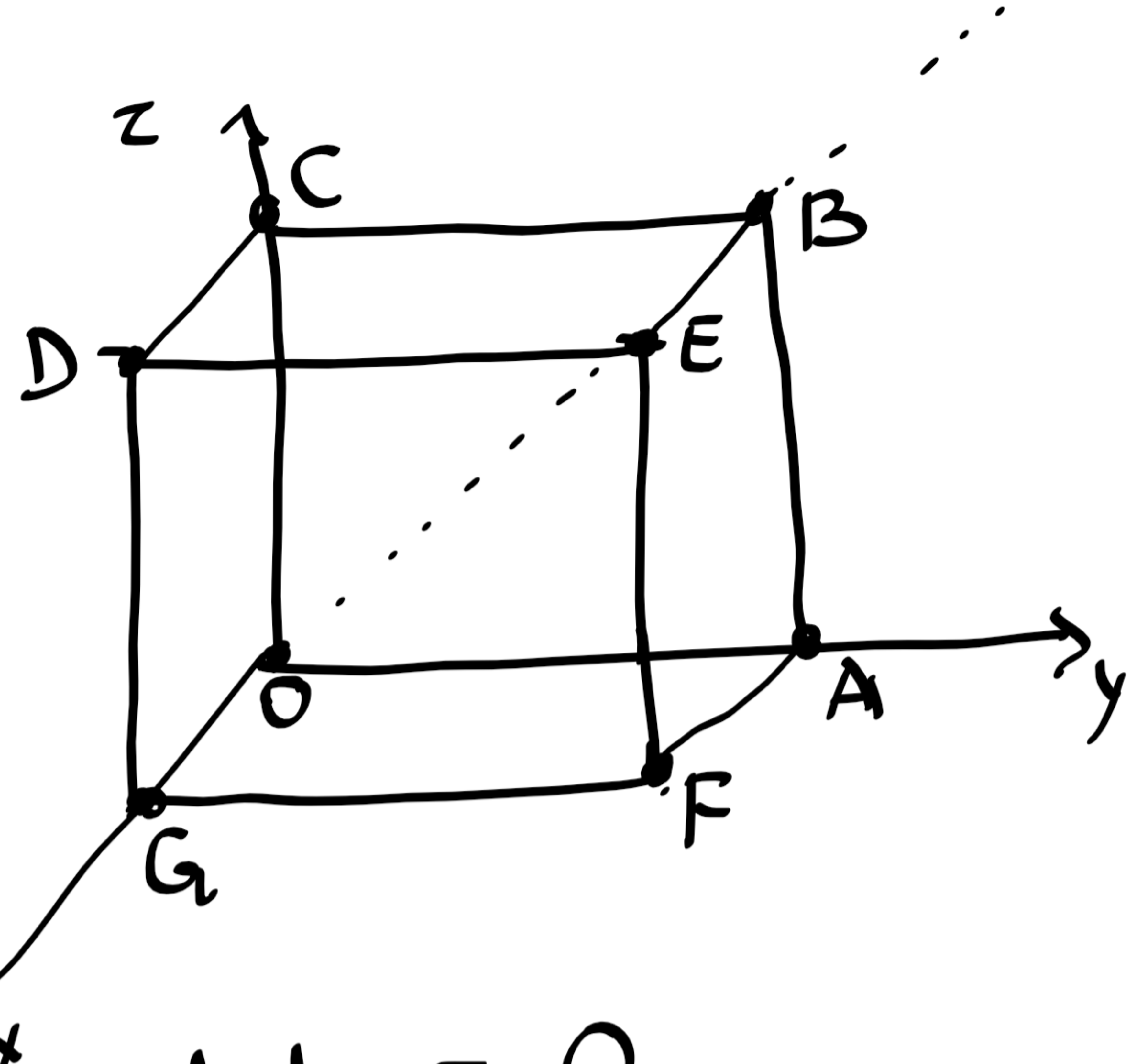
$$\iint_S \vec{F} \cdot \hat{n} \, ds = 2 - 1 + \frac{1}{2} = 3/2.$$

Face DEFG: $n = i$, $x = 1$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1} 4z\,\frac{dy\,dz}{|n\cdot i|}$$

$$= \underline{\underline{2}}$$



Face ABCO: $n = -i$, $x = 0$

$$\iint_{ABCO} \vec{F}\cdot\hat{n}\,ds = \int_{0}^{1}\int_{0}^{1} (-y^2 j + yz k)\cdot(-i)\,dy\,dz = 0$$

Face ABEF: $n = j$, $y = 1$.

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1} (4xz i - j + z k)\cdot j\,\frac{dx\,dz}{|n\cdot j|} = \underline{\underline{-1}}$$

Face: OGDC: $n = -j$, $y = 0$

$$\iint_{OGDC} \vec{F}\cdot\hat{n}\,ds = 0$$

Face BCDE, $n = k$, $z = 1$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = 1/2$$

Face: AFGO: $n = -k$, $z = 0$

$$\iint_{AFGO} \vec{F}\cdot n\,ds = 0$$

Divergence Theorem

Suppose V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function with continuous derivatives. Then

$$\iiint_V \nabla \cdot \vec{A} \, dV = \iint_S \vec{A} \cdot \hat{n} \, ds$$

verify Divergence theorem for $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = 4xz\,i - y^2 j + yz\,k$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

$$\iiint_V \nabla \cdot \vec{F} \, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} \left[\frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)\right] dz\,dy\,dx$$

$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + y)\, dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left(2z^2 - yz\right)_0^1 = \int_0^1\int_0^1 (2-y)\,dy\,dx$$

$$= \int_0^1 \left(2y - y^2/2\right)_0^1 dx = \int_0^1 \left(2 - \tfrac{1}{2}\right) dx$$

$$= 3/2$$

$$= \left[\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right]_{\pi/2}^{0}$$

$$= 1 - \frac{\pi^2}{4\pi} - \frac{2}{\pi} = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$\therefore \oint_C (y - \sin x)\,dx + \cos x\,dy = -1 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = \underline{\underline{-\frac{\pi}{4} - \frac{2}{\pi}}}$$

$M = y - \sin x \qquad N = \cos x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -\sin x.$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\,dx$$

$$= \int_0^{\pi/2} -y\sin x - y \Big|_0^{2x/\pi} dx$$

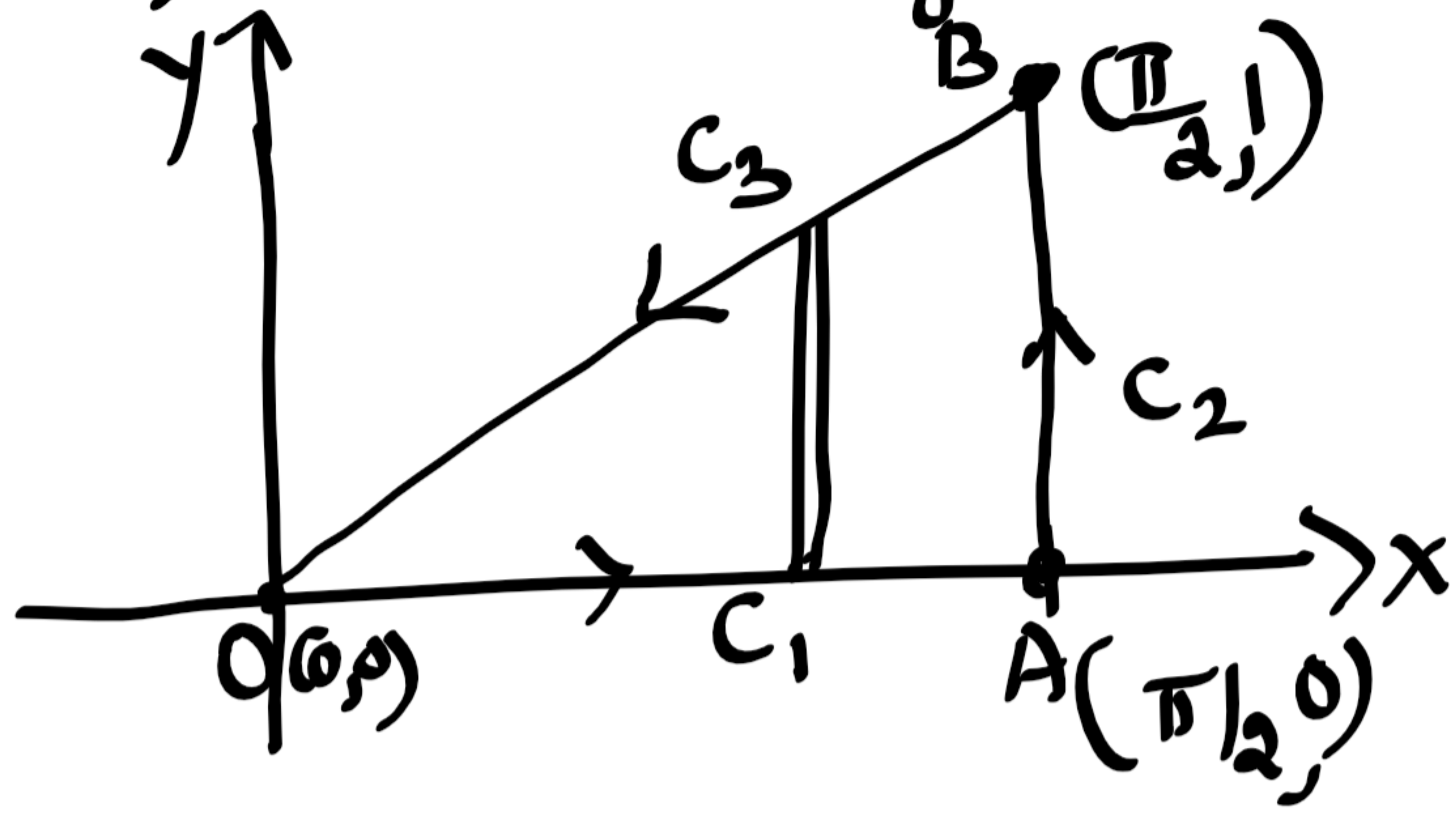
$$= -\frac{2}{\pi} \int_0^{\pi/2} (x\sin x + x)\,dx$$

$$= -\frac{2}{\pi} \left\{ -x\cos x + \sin x + \frac{x^2}{2} \right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left\{ 1 + \frac{\pi^2}{8} \right\} = \underline{\underline{-\frac{2}{\pi} - \frac{\pi}{4}}}$$

Hence the verification

4. Verify Green's theorem in the plane for $\oint_C (y-\sin x)\,dx + \cos x\,dy$ where C is the triangle



Along C_1: $y=0$ $dy=0$

$$\int_{C_1} (y-\sin x)\,dx+\cos x\,dy = \int_{x=0}^{\pi/2} -\sin x\,dx = \underline{\underline{-1}}$$

Along C_2: $x=\frac{\pi}{2}$, $dx=0$

$$\int_{C_2} (y-\sin x)\,dx+\cos x\,dy = 0.$$

Eqn. of the line joining $(0,0)$ and $(\pi/2,1)$

$$\frac{x}{\pi/2} = \frac{y}{1} \Rightarrow y=\frac{2x}{\pi}$$

$$dy=\frac{2}{\pi}dx$$

Along C_3: $y=\frac{2x}{\pi}$, $dy=\frac{2}{\pi}dx$

$$\int_{C_3} (y-\sin x)\,dx+\cos x\,dy = \int_{x=\pi/2}^{0}\left(\frac{2x}{\pi}-\sin x+\frac{2}{\pi}\cos x\right)dx$$

2. S.T. the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

Area by double integrals
$A = \iint_R dx\,dy$

$M = -y, \quad N = x$

$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$

$$\oint_C x\,dy - y\,dx = \iint_R 1-(-1)\,dx\,dy$$

$$= 2\iint_R dx\,dy = 2\text{Area}$$

$$\therefore \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

3. Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos\theta\, b\cos\theta\,d\theta - b\sin\theta(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta$$

$$= \pi ab$$

Along C_2: $y = x$

x varies from 1 to 0

$dy = dx$

$$\int_{C_2} (xy + y^2)\,dx + x^2\,dy = \int_{x=1}^{0} 3x^2\,dx = -1.$$

$$\therefore \oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = \underline{\underline{-\frac{1}{20}}}$$

$M = xy + y^2$; $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$; $\frac{\partial N}{\partial x} = 2x$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\,dy\,dx$$

$$= \int_0^1 \left(xy - y^2\right)\Big|_{x^2}^{x} dx = \int_0^1 0 - (x^3 - x^4)\,dx$$

$$= -\frac{x^4}{4} + \frac{x^5}{5}\Big|_0^1 = -\frac{1}{4} + \frac{1}{5}$$

$$= \underline{\underline{-\frac{1}{20}}}$$

Hence the verification.

Green's theorem in the plane.

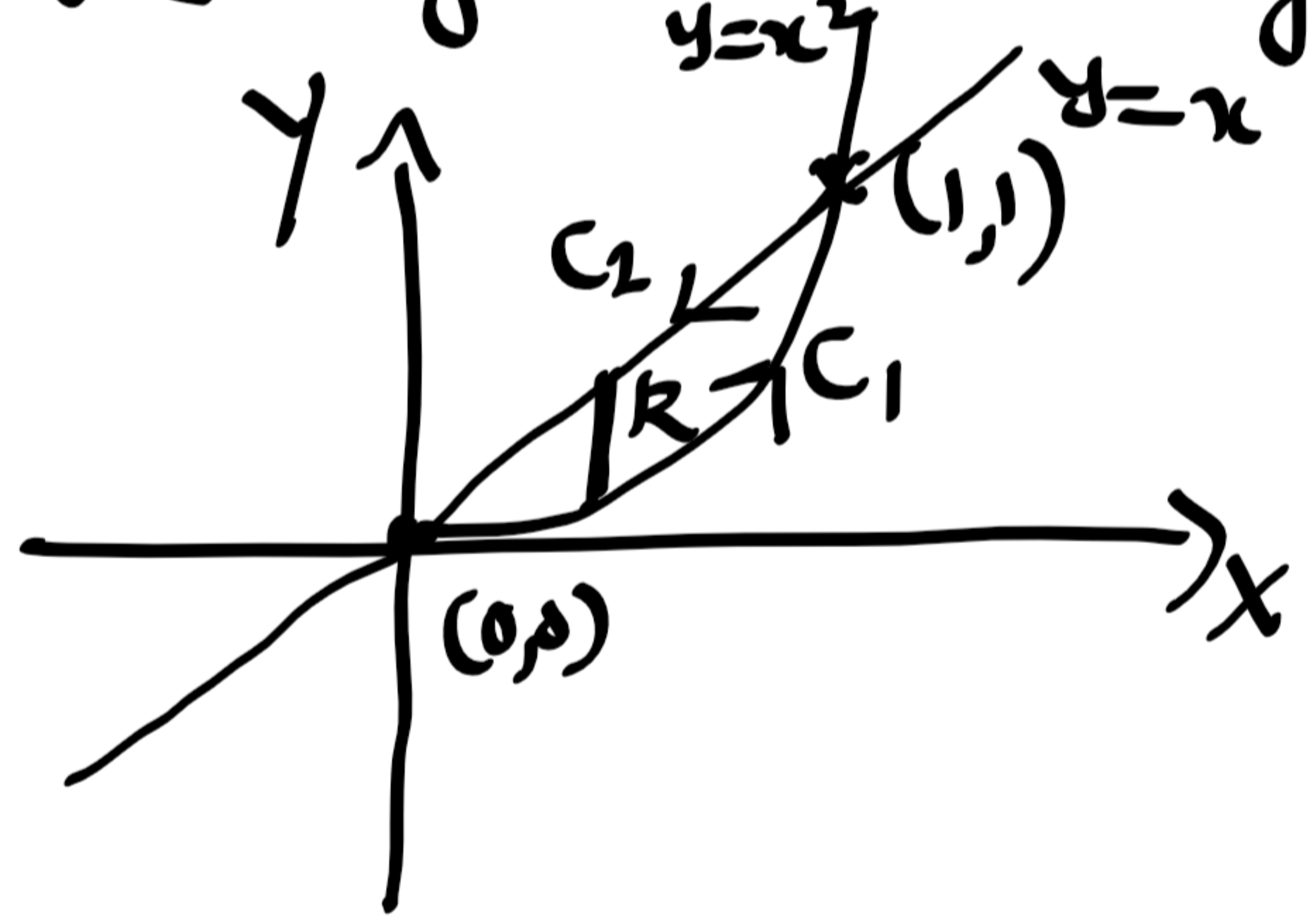
Suppose R is a closed region in the xy-plane bounded by a simple closed curve C and suppose M and N are continuous functions of x and y having continuous derivatives in R.

Then
$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

Examples

1. Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2\,dy$

where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.



$C: C_1 \cup C_2$

Along $C_1: y=x^2$

x varies from 0 to 1.

$$y=x^2 \Rightarrow dy = 2x\,dx.$$

$$\int_{C_1} (xy+y^2)\,dx + x^2\,dy = \int_{x=0}^{1} \left[x\cdot x^2 + (x^2)^2\right]dx + x^2\cdot 2x\,dx$$

$$= \int_{x=0}^{1} (3x^3+x^4)\,dx = \frac{3x^4}{4} + \frac{x^5}{5}\Bigg|_0^1 = \frac{19}{20}$$