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Modern Control Theory (ICE 3153)

State Feedback Controller – Pole Placement Method & Observers

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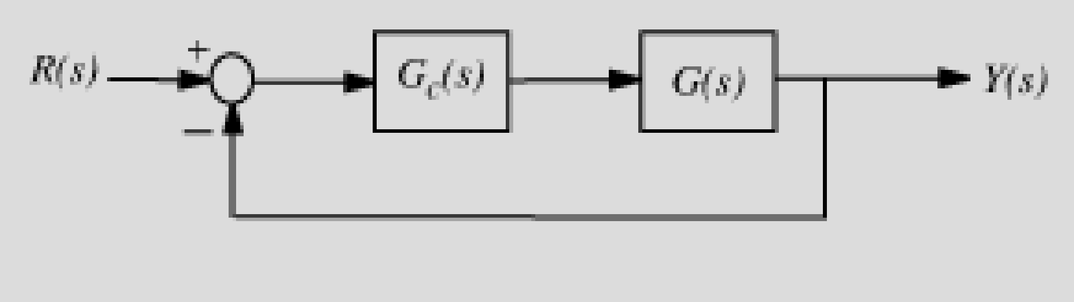
State Feedback controller

The classical control loop is given as,



Why do we need a controller for the plant G(s)?

We use classical controllers like P, PI, PID, Lead, Lag..

Conventional controller fails in meeting the desired response for higher order systems…

Here comes the concepts of State feedback controller….

- Advantage of state feedback…
- You can place pole anywhere in the S plane. Poles governs the performance. As you get the freedom to place the pole anywhere, you can obtain any desired performance.

- There are some practical issues….

- As we place the pole away from the imaginary axis, time constant reduces. And we need high quality sensors to measure this state..

- If some states are not measurable…

POLE PL.ACEMENT

- We assume that all state variables are measurable and are available for feedback.
- If the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

- The design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

- Let us assume that we decide that the desired closed-loop poles are to be at $s = \mu_1, s = \mu_2, \dots \dots .. s = \mu_n$

- *By* choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.

Design by Pole Placement:

- Different from specifying only dominant closed-loop poles (the conventional design approach), the pole-placement approach specifies all closed-loop poles.
- Consider the system described by

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du \qquad (1)$$

x = state vector (n-vector), **u** = control vector (r-vector), y = output vector (m-vector), **A** = *n* X *n* matrix, **B** = *n* X r matrix, **C** = m X *n* matrix, D = constant (scalar).

We shall choose the control signal to be,

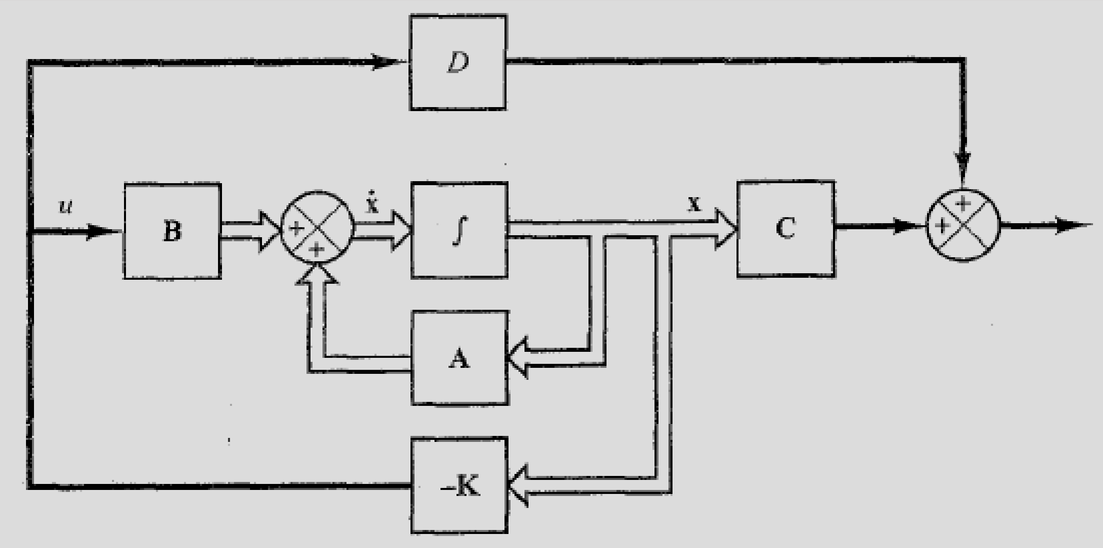
$$u = -Kx \qquad (2)$$

- This means that the control signal *u* is determined by an instantaneous state. Such a scheme is called state feedback.
- The 1 X n matrix K is called the state feedback gain matrix.

- Substituting Eqn (2) into eqn (1) gives,

$$\dot{x} = (A - BK)x$$
$$y = (C - DK)x \qquad (3)$$

The block diagram representation of the above eqns can be drawn as,



This closed-loop system has no input.

Its objective is to maintain the zero output.

- Such a system where the reference input is always zero is called a regulator system. (Note that if the reference input to the system is always a nonzero constant, the system is also called a regulator system.)

- Solution of eqn (3) can be given as,

$$x(t) = e^{(A-BK)t}x(0)$$

- where x(0) is the initial state caused by external disturbances.

- The stability and transient response characteristics are determined by the eigenvalues of matrix **A** - **BK.**

- If the matrix **K** is chosen properly, the matrix **A - BK** can be made an asymptotically stable matrix, and for all $x(0) \neq 0$ it is possible to make x(t) approach 0 as ***t*** approaches infinity.
- The eigenvalues of matrix **A** - **BK** are called the regulator poles (closed loop poles).
- If these regulator poles are placed in the left-half s plane, then **x(t)** approaches 0 as t approaches infinity.

- The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

Necessary and Sufficient Condition for Arbitrary Pole Placement

- A necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable.

- if the system is not completely state controllable, then there are eigenvalues of matrix **A** that cannot be arbitrarily placed.
- Therefore, to place the eigenvalues of matrix **A** - **BK** arbitrarily, the system must be completely state controllable (necessary condition).

- Sufficient condition: that is, if the system is completely state controllable, then all eigenvalues of matrix **A** can be arbitrarily placed.

Determination of Matrix K Using Transformation Matrix T

- Suppose that the system is defined by $\dot{x} = Ax + Bu$
- and the control signal is given by $u = -Kx$

- The feedback gain matrix **K** that forces the eigenvalues of **A - BK** to be, $\mu_1, \mu_2, \ldots \ldots .. \mu_n$ (desired values) can be determined by the following steps. (if μ_i is a complex eigenvalue, then its conjugate must also be an eigenvalue of **A** - BK):

- *Step 1:* Check the controllability condition for the system. If the system is completely state controllable, then use the following steps:

- *Step* 2: From the characteristic polynomial for matrix **A** determine the values of $a_1, a_2, \ldots \ldots .. a_n$

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

- *Step 3:* Determine the transformation matrix **T** that transforms the system state equation into the controllable canonical form. (If the given system equation is already in the controllable canonical form, then ***T* = I**.)
- The transformation matrix T is given by,

$$\mathbf{T} = \mathbf{MW}$$

- where M is the controllability matrix, $\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- *Step 4:* Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial: and determine the values of $\alpha_1, \alpha_2, \dots \dots . . \alpha_n$

$$(s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

- *Step 5:* The required state feedback gain matrix K can be determined from Equation

$$\mathbf{K} = \left[\alpha_n - a_n \;\vdots\; \alpha_{n-1} - a_{n-1} \;\vdots\; \cdots \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1\right]\mathbf{T}^{-1}$$

Determination of Matrix K Using Direct Substitution Method

- If the system is of low order ($n \leq 3$), direct substitution of matrix K into the desired characteristic polynomial may be simpler.
- For example, if *n* = 3, then we can write the state feedback gain matrix K as,

$$K = [K_1 \quad K_2 \quad K_3]$$

Substitute this K matrix into the desired characteristic polynomial,

$|SI - A + BK|$ and equate it to $(s - \mu_1)\,(s - \mu_2)\,(s - \mu_2)$ or

$$|SI - A + BK| = (s - \mu_1)\,(s - \mu_2)\,(s - \mu_2)$$

- Since both sides of this characteristic equation are polynomials in **s,** by equating the coefficients of the like powers of **s** on both sides, it is possible to determine the values of $K_1, K_2\,, K_3$

Determination of Matrix K Using Ackermann's Formula

- Suppose that the system is defined by $\dot{x} = Ax + Bu$
- and the control signal is given by $u = -Kx$

- We assume that the system is completely state controllable and the desired closed-loop poles are at $s = \mu_1, s = \mu_2, \dots \dots .. s = \mu_n$

- The state feedback gain matrix **K** can be computed as,

$$K = [0\ 0\ \ \dots ..0\ \ 1][B \vdots\ AB \vdots\ \ \dots . \vdots A^{n-1}B]^{-1}\phi(A)$$

Where,

$$\phi(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_{n-1}A + \alpha_n I$$

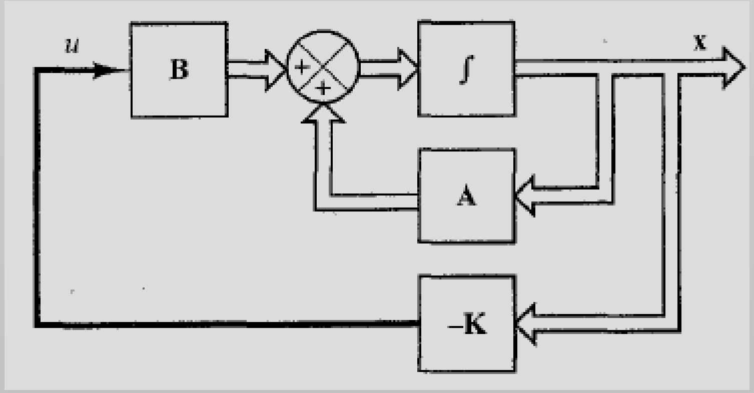
Question:1

Ex 12-1, Modern Control Engineering , 4th Edition Ogata

- Consider the regulator system shown in Figure below. The plant is given by $\dot{x} = Ax + Bu$. Where,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The system uses the state feedback control $u = -Kx$. Let us choose the desired closed-loop poles at $s = -2 + j4, s = -2 - j4, s = -10$. Determine the state feedback gain matrix K.



Method 1:

- Step-1
- First we need to check the controllability of the system.
- $M = [B \;\vdots\; AB \;\vdots\; A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$ $|M| = -1$ and the system is completely state controllable and arbitrary pole placement is possible.

- Step-2 : From the characteristic polynomial for matrix **A** determine the values of $a_1, a_2, \ldots \ldots .. a_n$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix} = s^3 + 6s^2 + 5s + 1 = s^3 + a_1s^2 + a_2s + a_3 = 0$$

$$a_1 = 6, \qquad a_2 = 5, \qquad a_3 = 1$$

- Step-3 : Find T? **T=I**
- Step-4 : The desired characteristic equation is,

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200 = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

$$\alpha_1 = 14, \qquad \alpha_2 = 60, \qquad \alpha_3 = 200$$

- Step-5 : Find K

$$\mathbf{K} = [\alpha_n - a_n \mid \alpha_{n-1} - a_{n-1} \mid \cdots \mid \alpha_2 - a_2 \mid \alpha_1 - a_1]\mathbf{T}^{-1}$$

$$\mathbf{K} = [200 - 1 \mid 60 - 5 \mid 14 - 6]$$
$$= [199 \quad 55 \quad 8]$$

Method 2:

- By defining the desired state feedback gain matrix K as, $K = [K_1 \quad K_2 \quad K_3]$

and equating $|SI - A + BK|$ with the desired characteristic equation, we obtain

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix} = s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

- Thus we get, $6 + k_3 = 14$, $5 + k_2 = 60$ and $1 + k_1 = 200$
- Or $\mathrm{K} = [199 \quad 55 \quad 8]$

Method 3:

- Using Ackerman's formula,

$$K = [0\ 0\ \ \ldots\ldots 0\ \ 1][B \vdots\ AB \vdots\ \ \ldots . \vdots A^{n-1}B]^{-1}\phi(A)$$

Where,

$$\phi(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_{n-1}A + \alpha_n I$$

- $\phi(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

And $[B \vdots\ AB \vdots A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$

we obtain, $K = [0\ 0\ 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix} = [199\ \ 55\ \ \ 8]$

When there is an input r?

- Suppose that the system is defined by $\dot{x} = Ax + Bu$
- and the control signal is given by $u = -Kx + r$

$$\dot{x} = Ax + B(-Kx + r)$$
$$= Ax - BKx + Br)$$
$$\dot{x} = (A - BK)x + Br$$

STATE OBSERVERS

- Estimation of unmeasurable state variables is commonly called *observation.*
- A device (or a computer program) that estimates or observes the state variables is called a *state observer,* or simply an *observer.*
- If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called *a full-order state observer.*
- An observer that estimates fewer than *n* state variables, where n is the dimension of the state vector, is called a *reduced-order state observer* or, simply, a *reduced-order observer.*
- If the order of the reduced-order state observer is the minimum possible, the

observer is called a *minimum-order state observer* or *minimum-order observer.*

- State observers can be designed if and only if the observability condition is satisfied.

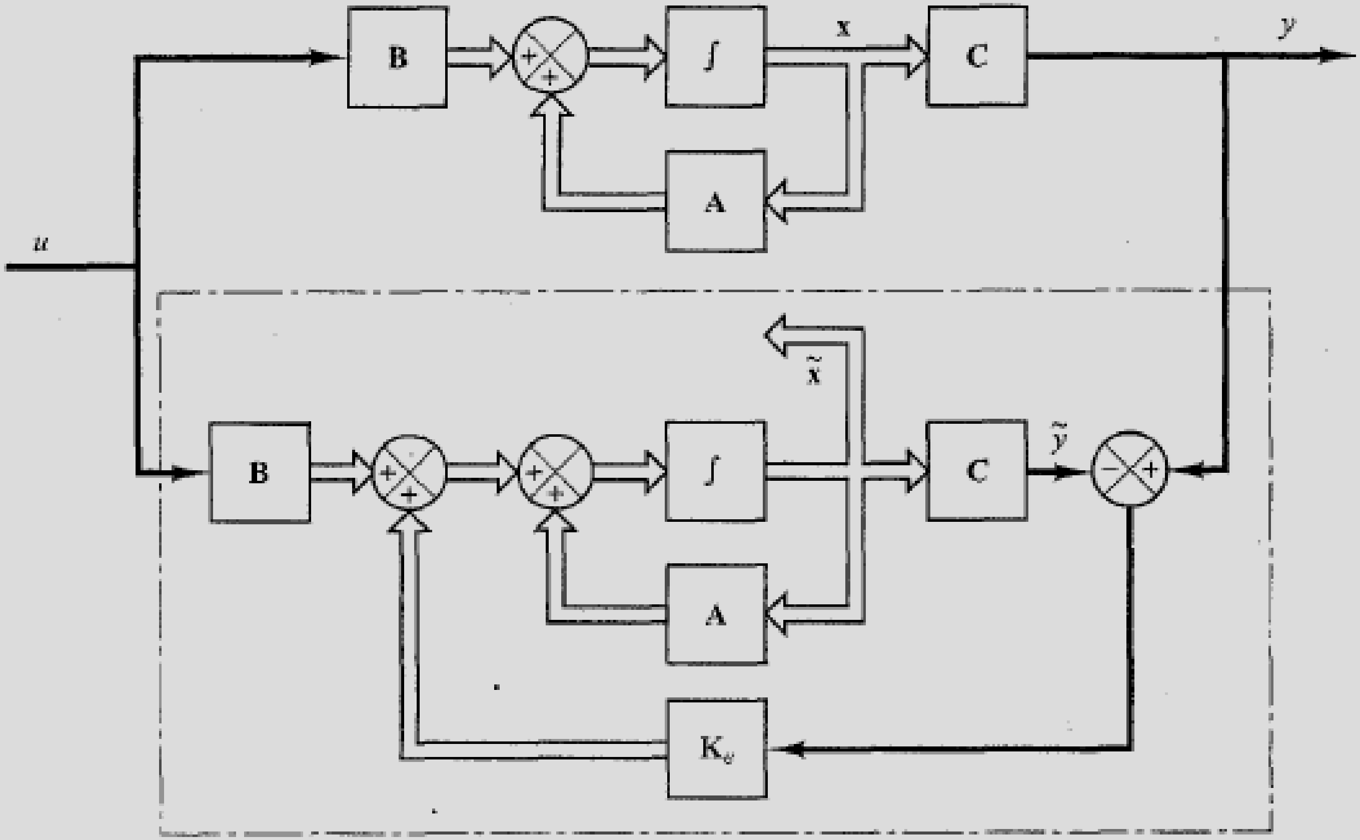
- Consider the system described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- The observer is a subsystem to reconstruct the state vector of the plant.
- The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices **A** and B and the lack of the initial error.
- we define the mathematical model of the observer to be

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \end{aligned}$$

- The inputs to the observer are the output y and the control input *u.*
- K_e is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output and the estimated output.



Full-order state observer