MANIPAL INSTITUTE OF TECHNOLOGY
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Modern Control Theory (ICE 3153)

Transfer Function to State Space
State space representation using phase variable & Canonical Forms

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The phase variables are defined as those **particular state variables which are obtained from one of the system variables and its derivatives**.

Usually the **variable used is the system output** and the remaining state variables are then **derivatives of output**.

The state model using phase variable can be easily determined if the system model is already known in the differential equation or transfer function form.

Advantages of Phase Variables:

The state space model can be directly formed by inspection from the differential equations governing the system.

Disadvantages of Phase Variables:

The phase variables are not physical variables of the system and are therefore not available for measurement and control purposes.

<u>Case 1</u>:

Consider the differential equation

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \ldots + a_{n-1}\dot{y} + a_n y = u$$

Choosing the output, *y(t),* and its (*n* - 1) derivatives as the state variables. This choice is called the *phase-variable choice*

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\vdots \\ x_n &= \overset{(n-1)}{y} \end{aligned}$$

The state equations are evaluated as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= a_n x_1 - \cdots - a_1 x_n + u \end{aligned}$$

The phase-variable form of the state equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

The solution to the differential equation is y(t), or x_1 and the output equation is:

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Question 1:
MCS, 12th Edition, Dorf and Bishop

Consider a fourth-order transfer function G(s) given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Obtain the state space representation. Draw the signal flow graph and block diagram representation.

From Mason's signal-flow gain formula

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k P_k \Delta_k}{\Delta}.$$

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q} = \frac{\text{Sum of the forward-path factors}}{1 - \text{sum of the feedback loop factors}}.$$

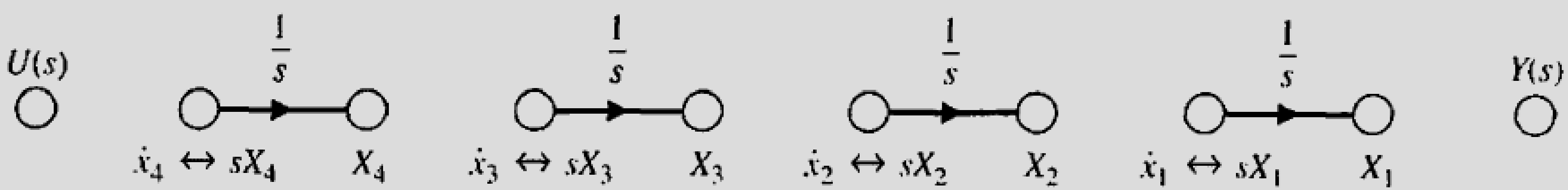
Where P_K = path gain of K^{th} forward path.

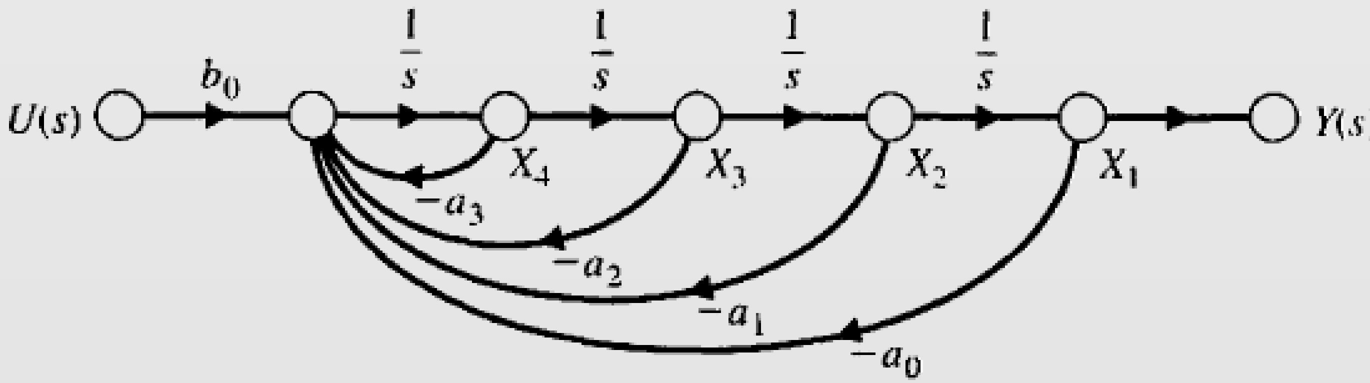
$\Delta = 1 -$ (sum of loop gain of all individual loops)

+ (sum of gain products of all possible combinations of two non-touching loops) – ………

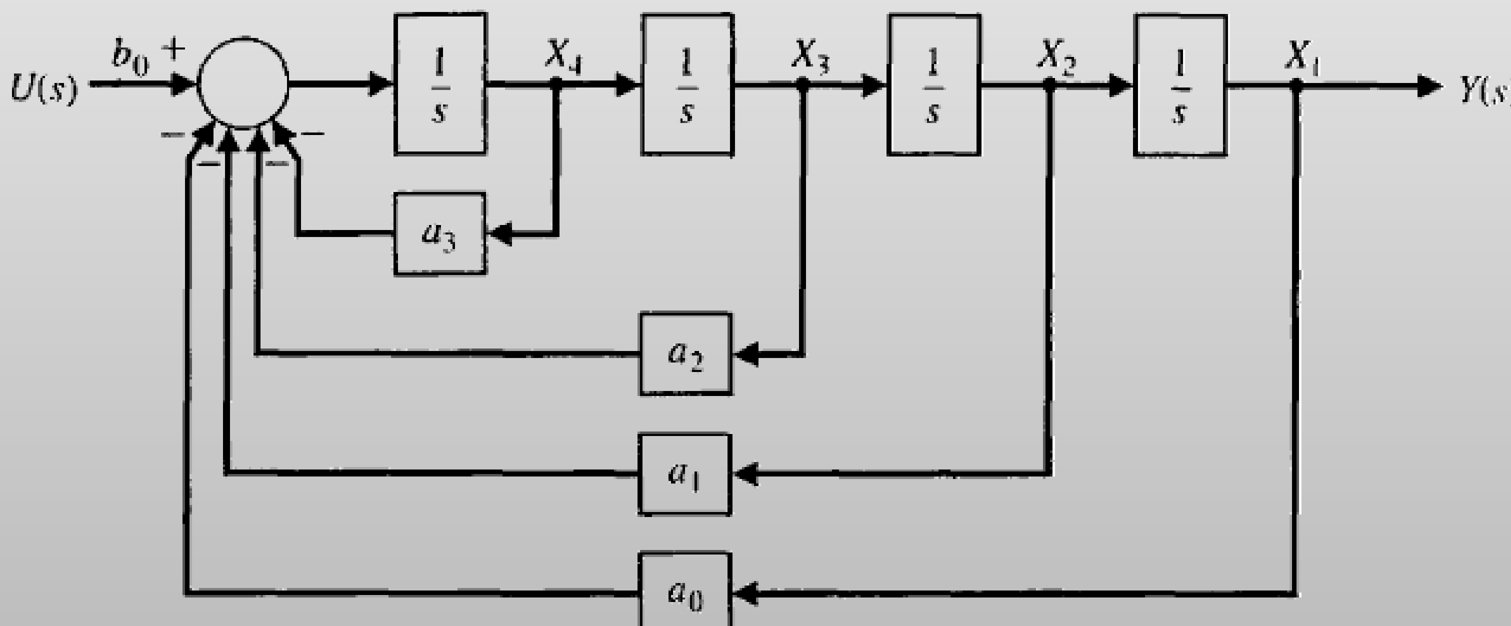
$\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path.

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$





U(s)
b0
1/s
X4
X3
X2
X1
Y(s)
−a3
−a2
−a1
−a0

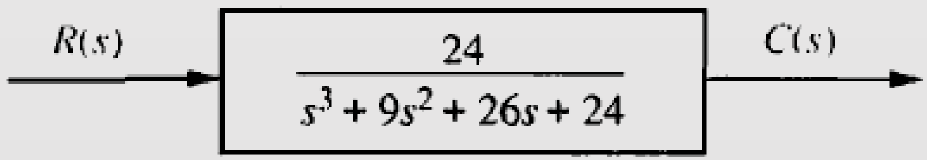


U(s)
b0
+
1/s
X4
X3
X2
X1
Y(s)
a3
a2
a1
a0

Question 2:

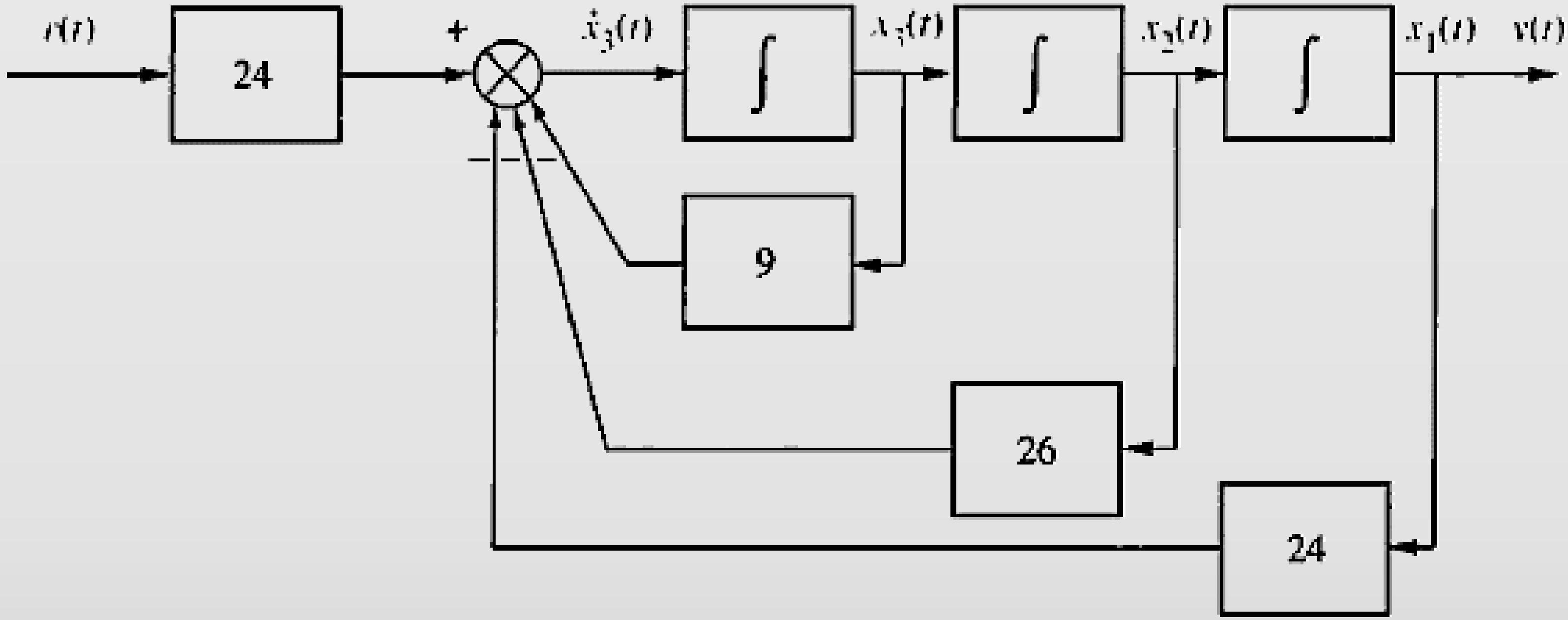
Ex 3.4, CSE – 6th Edition, Norman. S. Nise

Find the state-space representation in phase-variable form for the transfer function shown in figure.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



r(t)
24
+
$\dot{x}_3(t)$
$\int$
$x_3(t)$
$\int$
$x_2(t)$
$\int$
$x_1(t)$
y(t)
9
26
24

<u>Case 2</u>: **<u>Controllable Canonical Form</u>**

Consider the differential equation

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

So the transfer function of system defined by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u
$$

$$
y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u
$$

Observable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

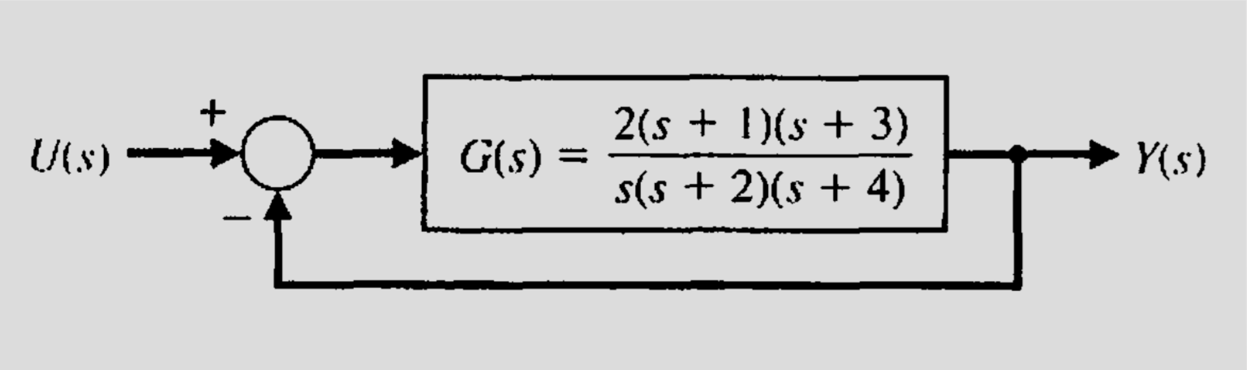
$$y = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$\begin{aligned} A_{obs} &= A_{cont}^T \\ B_{obs} &= C_{cont}^T \\ C_{obs} &= B_{cont}^T \\ D_{obs} &= D_{cont} \end{aligned}$$

Tutorial -2

<u>Question 1:</u>
<u>Ex 3.2, MCS – 12th Edition, Dorf and Bishop</u>

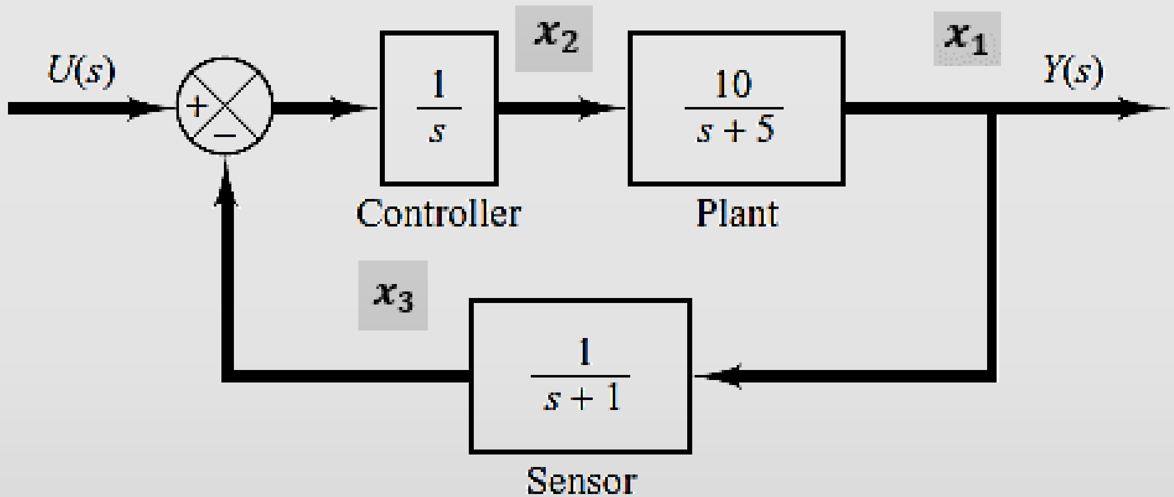
For the system shown in figure, derive the controllable canonical form and observable canonical form.



$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

Question 2:
A-3-8, MCE – 5th Edition, K. Ogata

Obtain a state-space model of the system shown in figure.



$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s+5}$$
$$\frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s}$$
$$\frac{X_3(s)}{X_1(s)} = \frac{1}{s+1}$$
$$Y(s) = X_1(s)$$

$$sX_1(s) = -5X_1(s) + 10X_2(s)$$
$$sX_2(s) = -X_3(s) + U(s)$$
$$sX_3(s) = X_1(s) - X_3(s)$$
$$Y(s) = X_1(s)$$

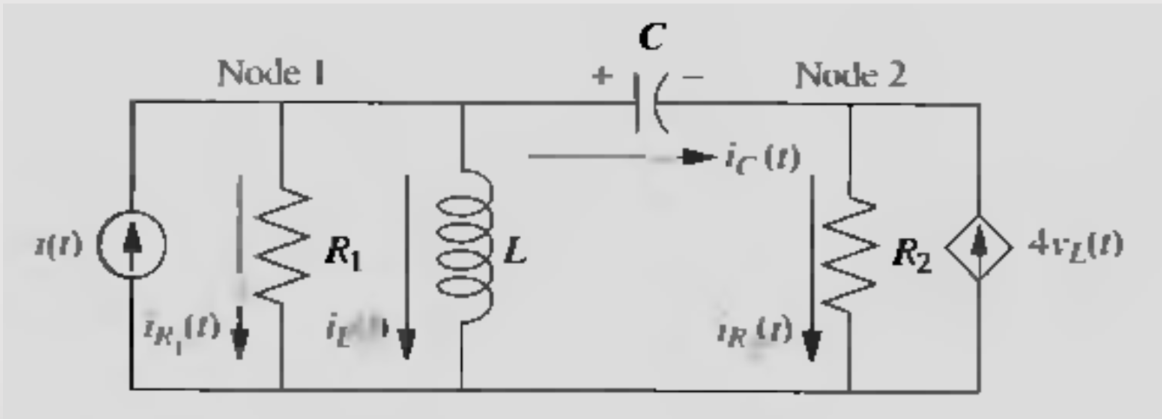
$$\dot{x}_1 = -5x_1 + 10x_2$$
$$\dot{x}_2 = -x_3 + u$$
$$\dot{x}_3 = x_1 - x_3$$
$$y = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Question 3:
Ex 3.2, CSE – Norman S Nice

Find the state and output equations for the electrical network shown in the figure if the output variables are v_{R_2} i_{R_2}



$$L\frac{di_L}{dt} = v_L$$

$$C\frac{dv_C}{dt} = i_C$$

$x_1 = i_L;$ $x_2 = v_C$

$i_{R_2} = i_C + 4v_L.$

$v_L = v_C + v_{R_2} = v_C + i_{R_2}R_2$

$v_L = v_C + (i_C + 4v_L)R_2$

$$v_L = \frac{1}{1 - 4R_2}(v_C + i_C R_2)$$

$$v_L = \frac{1}{\Delta}[R_2 i_L - v_C - R_2 i(t)]$$

$$\begin{aligned} i_C &= i(t) - i_{R_1} - i_L \\ &= i(t) - \frac{v_{R_1}}{R_1} - i_L \\ &= i(t) - \frac{v_L}{R_1} - i_L \end{aligned}$$

$$i_C = \frac{1}{\Delta}\left[(1 - 4R_2)i_L + \frac{1}{R_1}v_C - (1 - 4R_2)i(t)\right]$$

$$\Delta = -\left[(1 - 4R_2) + \frac{R_2}{R_1}\right]$$

$$(1 - 4R_2)v_L - R_2 i_C = v_C$$

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t)$$

$$
\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} R_2/(L\Delta) & -1/(L\Delta) \\ (1-4R_2)/(C\Delta) & 1/(R_1 C\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/(L\Delta) \\ -(1-4R_2)/(C\Delta) \end{bmatrix} i(t)
$$

$$
\begin{aligned}
v_{R_2} &= -v_C + v_L \\
i_{R_2} &= i_C + 4v_L
\end{aligned}
$$

$$
\begin{bmatrix} v_{R_2} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1+1/\Delta) \\ 1/\Delta & (1-4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t)
$$

Direct, cascade, parallel realizations

Decomposition of Transfer Function

- Process of going from TF to State Model is called decomposition

- Direct -> Phase variable form
- Parallel -> Canonical form
- Cascade