II Method of Variation of Parameters:

Consider the differential equation.

$$(aD^2+bD+c)y = R(x),$$ a, b, c are constants, $a \neq 0$.

Let y_1 and y_2 be the two linearly independent solutions of

$$(aD^2+bD+c)y=0.$$

Then the Particular Integral is given by

$$\boxed{PI = A(x)y_1 + B(x)y_2}$$

where

$$A(x) = -\int \frac{R(x)\,y_2}{W}\,dx, \qquad B(x) = \int \frac{R(x)\,y_1}{W}\,dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$ is called the Wronskian of the functions y_1 & y_2.

n functions $y_1, y_2, \ldots, y_n$ are linearly independent

$\Leftrightarrow$ Their wronskian $\neq 0$.

Solve the following differential equations:

① $(D^2+1)y = \operatorname{cosec} x$

To find CF:

$m^2+1=0$

$m = \pm i \qquad (\alpha=0, \beta=1)$

$CF = c_1\cos x + c_2 \sin x$

To find PI:

$y_1 = \cos x, \quad y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Let $PI = A(x)y_1 + B(x)y_2$

$$A(x) = -\int \frac{R(x)\,y_2}{W}\,dx = -\int \frac{\operatorname{cosec} x \cdot \sin x}{1}\,dx = -\int 1\,dx = -x$$

$$B(x) = \int \frac{R(x)\,y_1}{W}\,dx = \int \operatorname{cosec} x \cdot \cos x\,dx = \int \cot x\,dx = \log \sin x$$

$\therefore PI = -x\cos x + (\log \sin x)\sin x. \qquad \therefore y = CF + PI =$

② $(D^2+4)y = \tan 2x$

$$CF = c_1 \cos 2x + c_2 \sin 2x$$

To find PI:

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

Let $\quad PI = A(x)\,y_1 + B(x)\,y_2$

$$A(x) = -\int \frac{R(x)\,y_2}{W}\,dx = -\int \frac{\tan 2x \,\sin 2x}{2}\,dx = -\int \frac{\sin^2 2x}{2\cos 2x}\,dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = -\frac{1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{\sin 2x}{4}$$

$$B(x) = \int \frac{R(x)\,y_1}{W}\,dx = \int \frac{\tan 2x \cdot \cos 2x}{2}\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{\cos 2x}{4}$$

$$\therefore PI = \left[-\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{\sin 2x}{4}\right]\cos 2x + \left[-\frac{\cos 2x}{4}\right]\sin 2x$$

$$= -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

$$\therefore y = CF + PI = c_1 \cos 2x + c_2 \sin 2x - \frac{\cos 2x}{4}\log(\sec 2x + \tan 2x).$$

(3) $y'' - 2y' + y = \dfrac{e^x}{x^5}$

$m^2 - 2m + 1 = 0$

$m = 1, 1.$

$CF = (c_1 x + c_2)e^x = c_1 x e^x + c_2 e^x$

To find PI:

$y_1 = xe^x, \quad y_2 = e^x$

$W = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x & e^x \end{vmatrix} = xe^x e^x - e^x(xe^x + e^x) = -e^{2x}$ $\qquad \int x^n dx = \dfrac{x^{n+1}}{n+1}$

At $PI = A(x)y_1 + B(x)y_2$

$A(x) = -\int \dfrac{R(x)\,y_2}{W} dx = -\int \dfrac{e^x}{x^5} \times \dfrac{e^x}{-e^{2x}} dx = \int \dfrac{1}{x^5} dx = \dfrac{-1}{4x^4}$

$B(x) = \int \dfrac{R(x)\,y_1}{W} dx = \int \dfrac{e^x}{x^5} \times \dfrac{xe^x}{-e^{2x}} = -\int \dfrac{1}{x^4} dx = \dfrac{1}{3x^3}$

$\therefore PI = -\dfrac{1}{4x^4} \cdot xe^x + \dfrac{1}{3x^3} e^x = -\dfrac{e^x}{4x^3} + \dfrac{e^x}{3x^3} = \dfrac{e^x}{12x^3}$

$\therefore y = CF + PI = (c_1 x + c_2)e^x + \dfrac{e^x}{12x^3}$

(4) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

$m^2 - 1 = 0$

$m = \pm 1$

$CF = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x, \quad y_2 = e^{-x},$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$= -e^x e^{-x} - e^x e^{-x}$

$= -2$

$PI = A(x)y_1 + B(x)y_2$

$A(x) = -\int \dfrac{R(x)\,y_2}{W} dx = -\int \dfrac{2}{1+e^x} \cdot \dfrac{e^{-x}}{-2} dx$

$= \int \dfrac{e^{-x}}{1+e^x} dx.$

$= \int \dfrac{1}{e^x(1+e^x)} dx$

$= \int \left(\dfrac{1}{e^x} - \dfrac{1}{1+e^x}\right) dx$

$= \int \left[e^{-x} - \dfrac{1+e^x-e^x}{1+e^x}\right] dx$

$= \int \left[e^{-x} - 1 + \dfrac{e^x}{1+e^x}\right] dx$

$= -e^{-x} - x + \log(1+e^x)$

$\int \dfrac{e^{-x}\,dx}{1+\frac{1}{e^{-x}}}$

$= \int \dfrac{e^{-x}\, e^{-x} dx}{e^{-x}+1} = \int \dfrac{(t-1)(-dt)}{t}$

$e^{-x}+1 = t$

$$B(x) = \int \frac{R(x)\, y_1}{W}\, dx = \int \frac{2}{1+e^{x}} \cdot \frac{e^{x}}{-2}\, dx = -\int \frac{e^{x}}{1+e^{x}}\, dx = -\log(1+e^{x})$$

$$\therefore\ PI = \left[-e^{-x} - x + \log(1+e^{x})\right] e^{x} + \left[-\log(1+e^{x})\right] e^{-x}$$

$$= -1 - xe^{x} + e^{x}\log(1+e^{x}) - e^{-x}\log(1+e^{x}).$$

$$y = CF + PI =$$

5. $(D^2+2D+1)y = (e^{x}-1)^{-2}$

$$CF = (C_1 x + C_2)e^{-x} = C_1 x e^{-x} + C_2 e^{-x}$$

$$y_1 = x e^{-x}, \quad y_2 = e^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} xe^{-x} & e^{-x} \\ -xe^{-x}+e^{-x} & -e^{-x} \end{vmatrix} = -xe^{-x}e^{-x} + xe^{-x}e^{-x} - e^{-x}e^{-x}$$

$$= -e^{-2x}$$

$$PI = A(x)y_1 + B(x)y_2$$

$e^{x}-1=t$

$e^{x}dx = dt$

$$A(x) = -\int \frac{R(x)\, y_2}{W}\, dx = -\int \frac{(e^{x}-1)^{-2} \cdot e^{-x}}{-e^{-2x}}\, dx = \int (e^{x}-1)^{-2} e^{-x} e^{2x}\, dx = \int \frac{1}{(e^{x}-1)^2} e^{x}\, dx$$

$$= \int \frac{dt}{t^2} = -\frac{1}{t} = \frac{-1}{e^{x}-1}$$

$$B(x) = \int \frac{R(x)\, y_1}{W}\, dx = \int \frac{(e^{x}-1)^{-2}\, xe^{-x}}{-e^{-2x}}\, dx$$

$$= -\int \frac{1}{(e^{x}-1)^2}\, x\, e^{x}\, dx$$

$$= -\int \underset{\mathrm{I}}{x}\ \underset{\mathrm{II}}{\frac{e^{x}}{(e^{x}-1)^2}}\, dx$$

$$\int \frac{e^{x}}{(e^{x}-1)^2}\, dx = \frac{-1}{e^{x}-1}$$

$$= -\left[x\left(\frac{-1}{e^{x}-1}\right) - \int \left(\frac{-1}{e^{x}-1}\right) dx\right]$$

$$= \frac{x}{e^{x}-1} + \int \frac{-1}{e^{x}-1}\, dx$$

$$= \frac{x}{e^{x}-1} + \int \frac{e^{x}-1-e^{x}}{e^{x}-1}\, dx$$

$$= \frac{x}{e^{x}-1} + \left[x - \log(e^{x}-1)\right]$$

$$\int \frac{e^{x}-1-e^{x}}{e^{x}-1}\, dx = \int \left[\frac{e^{x}-1}{e^{x}-1} - \frac{e^{x}}{e^{x}-1}\right] dx$$

$$= \int 1\, dx - \int \frac{e^{x}dx}{e^{x}-1}$$

$$= x - \log(e^{x}-1)$$

$e^{x}-1=t$

$$PI = \frac{-1}{e^{x}+1}\, xe^{-x} + \left[\frac{x}{e^{x}-1} + x - \log(e^{x}-1)\right] e^{-x}$$

$$= \left[x - \log(e^{x}-1)\right] e^{-x}.$$

Problems for Practice

Solve

1. $y'' + 2y' + 2y = e^{-x} \sec^3 x$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \log x$

3. $(D^2+1)y = \sec^3 x$

4. $(D^2 - 3D + 2)y = \cos(e^{-x})$

5. $(D^2-1)y = e^{-2x} \sin(e^{-x})$