Cauchy's and Legendre's Differential Equations:

The linear differential equation of the form

$$(ax+b)^n \frac{d^ny}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + k_2 (ax+b)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_n y = R(x) \quad \text{——} (1)$$

is called Legendre's differential equation, where $a, b, k_1, k_2, \ldots, k_n$ are constants and $R(x)$ is a function of x alone.

In particular, if $a=1$, $b=0$, (1) reduces to

$$x^n \frac{d^ny}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_n y = R(x) \quad \text{——} (2)$$

This equation is called Cauchy's differential equation.

To solve (1) we proceed as follows:

Denote $\frac{d}{dt} = D$, put $ax+b = e^t$, i.e. $t = \log(ax+b)$.

$$\frac{dt}{dx} = \frac{1}{ax+b} \cdot a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = Dy \cdot \frac{a}{ax+b} = a \frac{1}{ax+b} Dy \quad \text{——} (3) \qquad \frac{a}{ax+b} \cdot \frac{dy}{dt}$$

$$(ax+b) \frac{dy}{dx} = aDy \quad \text{-----} (i)$$

Differentiating (3) again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{a}{ax+b} \frac{d}{dx}\left(\frac{dy}{dt}\right) + Dy \frac{d}{dx}\left(\frac{a}{ax+b}\right)$$

$$= \frac{a}{ax+b} \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} + Dy \cdot \frac{-a}{(ax+b)^2} \times a$$

$$= \frac{a}{ax+b} \cdot D^2y \cdot \frac{a}{ax+b} - \frac{a^2}{(ax+b)^2} Dy$$

$$= \frac{a^2}{(ax+b)^2} \left[D^2y - Dy\right]$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y \quad \text{----} (ii)$$

Illy

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y \quad \text{-----} (iii)$$

Substituting (i), (ii), (iii), ... in equation (1), we get a linear differential equation with constant coefficients and hence can be solved.

① Solve: $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2$

Cauchy's homogeneous differential equation.

Put $x = e^t$, i.e. $t = \log x$, denote $\frac{d}{dt} = D$.

Then

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Substituting, we get,

$$D(D-1)y - 4Dy + 6y = (e^t)^2$$

$$(D^2 - D - 4D + 6)y = e^{2t}$$

$$(D^2 - 5D + 6)y = e^{2t}$$

To find CF:

Auxiliary equation is

$$m^2 - 5m + 6 = 0$$

roots $= 2, 3$

$$CF = c_1 e^{2t} + c_1 e^{3t} = c_1(e^t)^2 + c_2(e^t)^3 = c_1 x^2 + c_2 x^3$$

To find PI:

$$PI = \frac{1}{D^2 - 5D + 6} e^{2t}$$

$$= t\frac{1}{2D-5} e^{2t}$$

$$= t\frac{1}{2\times 2 - 5} e^{2t}$$

$$= -te^{2t} = -\log x \cdot x^2$$

∴ complete solution is

$$y = CF + PI$$

$$= c_1 x^2 + c_2 x^3 - x^2 \log x.$$

② Solve: $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3\log x$

$x = e^t, \quad t = \log x, \quad \frac{d}{dt} = D$

$x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y + 2Dy - 12y = (e^t)^3\, t$

$(D^2 - D + 2D - 12)y = te^{3t}$

$(D^2 + D - 12)y = te^{3t}$

To find CF:

Auxiliary equation is

$m^2 + m - 12 = 0$

Roots: $3, -4$

$e^{3t} = (e^t)^3 = x^3$

$e^{-4t} = (e^t)^{-4} = x^{-4}$

$CF = c_1 e^{3t} + c_2 e^{-4t} = c_1 x^3 + c_2 (x)^{-4} = c_1 x^3 + \frac{c_2}{x^4}$

To find PI:

$PI = \frac{1}{D^2 + D - 12}\, te^{3t}$

$= e^{3t}\frac{1}{(D+3)^2 + (D+3) - 12}\, t$

$= e^{3t}\frac{1}{D^2 + 7D}\, t$

$= e^{3t}\left(\frac{t^2}{14} - \frac{t}{49}\right)$

$$\begin{array}{r|l} & \frac{t^2}{14} - \frac{t}{49} \\ \hline 7D + D^2 & t \\ & \underset{(-)}{t} + \underset{(-)}{\frac{1}{7}} \\ \hline & -\frac{1}{7} \\ & -\frac{1}{7} \\ \hline & 0 \end{array}$$

$7D\left(\frac{t^2}{14}\right) = 7\cdot\frac{2t}{14} = t$

$D^2\left(\frac{t^2}{14}\right) = \frac{1}{7}$

$PI = \frac{x^3}{7}\left[\frac{(\log x)^2}{2} - \frac{\log x}{7}\right]$

$y = CF + PI$

Note: $\frac{1}{D^2 + 7D}\, t = \frac{1}{7D\left(1 + \frac{D}{7}\right)}\, t$

$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \cdots$

$= \frac{1}{7D}\left[1 - \frac{D}{7} + \frac{D^2}{49} - \cdots\right] t$

$= \frac{1}{7D}\left[t - \frac{1}{7}\right]$

$= \frac{1}{7}\left[\int t\,dt - \int \frac{1}{7}\,dt\right] = \frac{1}{7}\left[\frac{t^2}{2} - \frac{t}{7}\right]$

(3) Solve: $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Legendre's differential equation.

$2x+3 = e^t, \qquad t = \log(2x+3)$

$x = \frac{e^t-3}{2}, \qquad \frac{d}{dt} = D.$

Then

$(2x+3)\frac{dy}{dx} = 2Dy \qquad (2x+3)^2\frac{d^2y}{dx^2} = 2^2D(D-1)y$

$2^2D(D-1)y - 2Dy - 12y = 6\left(\frac{e^t-3}{2}\right)$

$(4D^2-6D-12)y = 3(e^t-3)$

To find CF:

$4m^2 - 6m - 12 = 0$

roots: $\frac{3\pm\sqrt{57}}{4}$

$CF = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t}$

To find PI:

$PI = \frac{1}{4D^2-6D-12}\,3(e^t-3)$

$= 3\left[\frac{1}{4D^2-6D-12}e^t - 3\frac{1}{4D^2-6D-12}e^{0t}\right]$ $\qquad 3 = 3\times1 = 3\times e^{0t}$

$= 3\left[\frac{1}{4(1)^2-6(1)-12}e^t - 3\frac{1}{0-0-12}\cdot1\right]$

$= 3\left[\frac{-e^t}{14} + \frac{1}{4}\right]$

$y = CF + PI = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t} - \frac{3e^t}{14} + \frac{3}{4}$

where $t = \log(2x+3)$.

④ Solve: $(x-1)^3 \frac{d^3y}{dx^3} + 2(x-1)^2 \frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1)$

$x-1 = e^t, \quad t = \log(x-1).$

$x = e^t + 1, \quad \frac{d}{dt} = D$

$(x-1)\frac{dy}{dx} = Dy$

$(x-1)^2 \frac{d^2y}{dx^2} = D(D-1)y, \quad (x-1)^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$

Substituting we get,

$$D(D-1)(D-2)y + 2D(D-1)y - 4Dy + 4y = 4t$$

$$(D^3 - D^2 - 4D + 4)y = 4t$$

To find CF:

$m^3 - m^2 - 4m + 4 = 0$

$m=1$ is a root

1	1	-1	-4	4
		1	0	-4
	1	0	-4	0

$m^2 - 4 = 0$

$m^2 = 4$

$\therefore 1, +2, -2$

$\therefore CF = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t} = c_1(x-1) + c_2(x-1)^2 + \frac{c_3}{(x-1)^2}$

To find PI:

$PI = 4 \frac{1}{D^3 - D^2 - 4D + 4} t$

$$4 - 4D - D^2 + D^3 \overline{\left)\begin{array}{l} \frac{t}{4} + \frac{1}{4} \\ t \\ \underline{(-)t \; (+)1} \\ 1 \\ \underline{(-)1} \\ 0 \end{array}\right.}$$

$4 \times \frac{t}{4} = t$

$-4D\left(\frac{t}{4}\right) = -1$

$= 4\left[\frac{t}{4} + \frac{1}{4}\right]$

$= t + 1. = \log(x-1) + 1$

$\therefore y = c_1(x-1) + c_2(x-1)^2 + \frac{c_3}{(x-1)^2} + \log(x-1) + 1.$

(5) Solve: $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = \sin(2\log(x+1))$

Legendre's DE.

$x+1 = e^t, \quad t = \log(x+1), \quad \frac{d}{dt} = D.$

$x = e^t - 1.$

$(x+1)\frac{dy}{dx} = Dy, \quad (x+1)^2 \frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y + Dy + y = \sin 2t$

$(D^2+1)y = \sin 2t$

To find CF:

$m^2+1=0$

$m = \pm i$

$CF = c_1 \cos t + c_2 \sin t = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$

To find PI:

$PI = \frac{1}{D^2+1} \sin 2t$

$= \frac{1}{-4+1} \sin 2t = \frac{-\sin 2t}{3} = -\frac{\sin(2\log(x+1))}{3}$

$\therefore$ Complete solution is

$y = CF + PI =$

Problems for Practice:

(1) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

(2) $x\frac{d^2y}{dx^2} - \frac{2}{x}y = x + \frac{1}{x^2}$ $\quad$ x^1y by x.

(3) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$

(4) $(3x+2)^2\frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2+x+1$

(5) $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1))$

(6) $(2x-1)^2\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$