2) The temperature T at any point (x,y,z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$.

$$T = 400\, xyz^2 \qquad \phi = x^2+y^2+z^2-1 = 0$$

$$F = T + \lambda\phi$$

$$F = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$

$$F_x = 0 \Rightarrow 400yz^2 + \lambda(2x) = 0 \Rightarrow -\lambda = \frac{200yz^2}{x}$$

$$F_y = 0 \Rightarrow 400xz^2 + \lambda(2y) = 0 \Rightarrow -\lambda = \frac{200xz^2}{y}$$

$$F_z = 0 \Rightarrow 800xyz + \lambda(2z) = 0 \Rightarrow -\lambda = 400xy$$

$$\Rightarrow \frac{\cancel{200}\, y\cancel{z^2}}{x} = \frac{\cancel{200}\, x\cancel{z^2}}{y}$$

$$\underline{\underline{x = \pm y}}$$

$$\frac{200\cancel{x}z^2}{y} = 400\cancel{x}y$$

$$z^2 = 2y^2$$

$$\underline{\underline{z = \pm\sqrt{2}\, y}}$$

Now $\phi(x,y,z) = 0$

$$x^2+y^2+z^2-1 = 0$$

$$x^2 + x^2 + 2x^2 = 1$$

$$4x^2 = 1$$

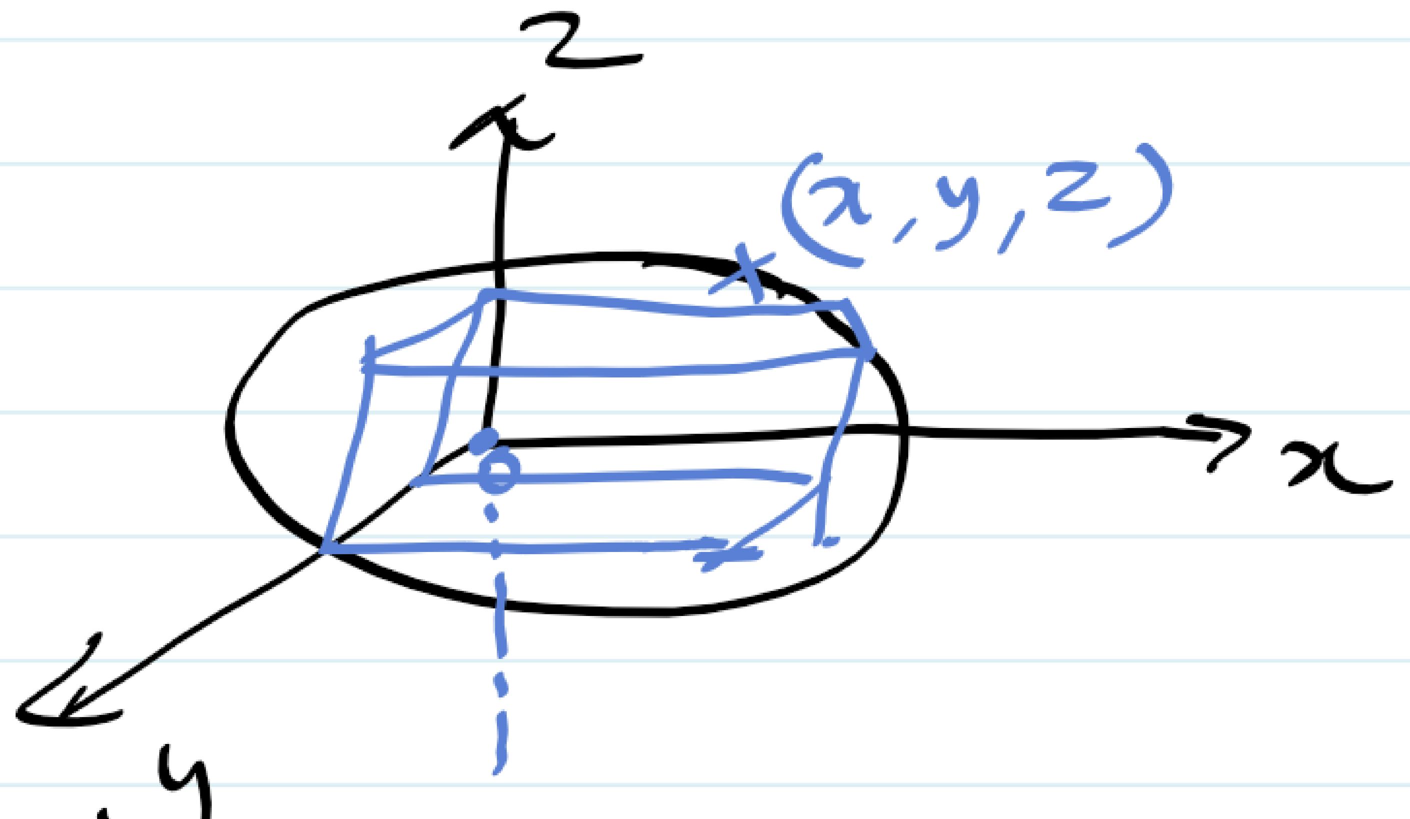
$$x^2 = 1/4 \Rightarrow \boxed{x = \pm 1/2}$$

$$\therefore y = \pm 1/2, \quad z = \pm 1/\sqrt{2}$$

$\therefore$ Highest temparature on the surface is,

$$T = 400\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)^2 = \underline{\underline{50}} \text{ unit}$$

③ Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



<u>Ans:</u> Let the edges of the parallelopiped be $2x$, $2y$, $2z$ which are parallel to the axes.

Then, its volume, $V = 8xyz$.

Now, we have to find the max value of V subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$F_x = 0 \Rightarrow 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \quad \text{— ①}$$

$$F_y = 0 \Rightarrow 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad \text{— ②}$$

$$F_z = 0 \Rightarrow 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad \text{— ③}$$

From ① $\quad 8yz = -\lambda\frac{2x}{a^2} \Rightarrow 4yz = -\frac{\lambda x}{a^2}$

$$\Rightarrow \underline{\underline{\frac{-\lambda}{4} = \frac{yz\,a^2}{x}}}$$

$|||^{ly}$ from ② $-\frac{\lambda}{4} = \frac{xzb^2}{y}$

from ③ $-\frac{\lambda}{4} = \frac{xyc^2}{z}$

Equating the values of λ,

$$\frac{yza^2}{x} = \frac{xzb^2}{y} = \frac{xyc^2}{z}$$

$$\frac{y\cancel{z}a^2}{x} = \frac{x\cancel{z}b^2}{y} \qquad\qquad \frac{\cancel{x}zb^2}{y} = \frac{\cancel{x}yc^2}{z}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \qquad\qquad \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\phi(x,y,z) = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$\frac{x^2}{a^2} = 1/3$$

$$x^2 = a^2/3 \Rightarrow x = \pm a/\sqrt{3}$$

$|||^{ly}$ $y = \pm b/\sqrt{3}$, $z = \pm c/\sqrt{3}$

$\therefore$ Greatest volume, $V = 8xyz$

$$= \frac{8abc}{3\sqrt{3}}$$

4) Find the maximum and minimum distance from the point $(1,2,3)$ to the sphere $x^2+y^2+z^2=56$ using Lagrange's method.

Ans:

Distance from $(1,2,3)$ to (x,y,z) is,

$$f = D^2 = (x-1)^2+(y-2)^2+(z-3)^2.$$

$$F = (x-1)^2+(y-2)^2+(z-3)^2 + \lambda(x^2+y^2+z^2-56)$$

$F_x=0 \Rightarrow 2(x-1)+2x\lambda=0 \Rightarrow -\lambda = \frac{x-1}{x}$

$F_y=0 \Rightarrow 2(y-2)+2y\lambda=0 \Rightarrow -\lambda = \frac{y-2}{y}$

$F_z=0 \Rightarrow 2(z-3)+2z\lambda=0 \Rightarrow -\lambda = \frac{z-3}{z}$

$$\frac{x-1}{x} = \frac{y-2}{y} \Rightarrow \cancel{xy}-y = \cancel{xy}-2x$$

$$y=2x$$

$$\frac{x-1}{x} = \frac{z-3}{z} \Rightarrow \cancel{xz}-z = \cancel{xz}-3x$$

$$\Rightarrow z=3x$$

$\phi(x,y,z)=0$

$$x^2+y^2+z^2=56$$

$$x^2+4x^2+9x^2=56$$

$$14x^2=56 \Rightarrow x^2=\frac{56}{14}=4$$

$$x=\pm 2$$

$\therefore\ y=\pm 4 \quad , \quad z=\pm 6$

Max. Distance $D^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$.

$= (-2-1)^2 + (-4-2)^2 + (-6-3)^2$.

$= 9 + 36 + 81 = 126$

$D = \underline{\underline{\sqrt{126}}}$.

Min Distance $D^2 = (2-1)^2 + (4-2)^2 + (6-3)^2$

$= 1 + 4 + 9 = 14$

Min distance, $D = \underline{\underline{\sqrt{14}}}$

Evaluate the following questions using the Lagrange's Method of Undetermined Multipliers.

① It is given that the sum of 3 numbers x, y, z are constant. Find the numbers if xy^2z^3 is maximum.

Ans: $f = xy^2z^3 = \frac{n^6}{432}$

② Show that the rectangular solid of maximum volume can be inscribed in a sphere with centre at origin is a cube.

Hint: $F = f + \lambda\phi$

$F = 8xyz + \lambda(x^2+y^2+z^2-a^2)$

Ans: $x = y = z$

③ Find the maximum and minimum distance from the origin to the curve $5x^2+6xy+5y^2-8=0$

Ans: Max $D = 2$

Min $D = 1$

4) Given an aluminium sheet of area $2a$, find the maximum volume of parallelopiped that can be formed.

Ans: $V = \left(\sqrt{a/3}\right)^3$

5) Divide 24 as a sum of three numbers such that the continued product of the first, square of the second and cube of the third is maximum.

Ans: 4, 8, 12