① Solve $(D^3+2D^2+D)y = e^{-x} + \sin 2x$

To find CF:

Auxiliary equation is

$$m^3+2m^2+m=0$$

$$m(m^2+2m+1)=0$$

$$m(m+1)^2=0$$

$$m=0,-1,-1.$$

$$CF = C_1 + (C_2x+C_3)e^{-x}$$

$R(x) = e^{ax}$

Replace D by a

$R(x) = \sin ax$

Replace D^2 by $-a^2$

To find PI:

$$PI = \frac{1}{D^3+2D^2+D}\left(e^{-x} + \sin 2x\right)$$

$$= \frac{1}{D^3+2D^2+D}e^{-x} + \frac{1}{D^3+2D^2+D}\sin 2x$$

$a^2=-4$

$$= x\frac{1}{3D^2+4D+1}e^{-x} + \frac{1}{D(-4)+2(-4)+D}\sin 2x$$

$$= x \quad x\frac{1}{6D+4}e^{-x} + \frac{1}{-3D-8}\sin 2x$$

$$= x^2\frac{1}{6(-1)+4}e^{-x} - \frac{(3D-8)}{(3D+8)(3D-8)}\sin 2x$$

$$= \frac{x^2e^{-x}}{-2} - \frac{(3D-8)}{9D^2-64}\sin 2x$$

$$= -\frac{x^2e^{-x}}{2} - \frac{(3D-8)}{9(-4)-64}\sin 2x$$

$$= -\frac{x^2e^{-x}}{2} + \frac{1}{100}(3D-8)\sin 2x$$

$$= \frac{-x^2e^{-x}}{2} + \frac{1}{100}\left[3\ D\sin 2x - 8\sin 2x\right]$$

$$= \frac{-x^2e^{-x}}{2} + \frac{1}{100}\left[3\cdot 2\cos 2x - 8\sin 2x\right]$$

$$= \frac{-x^2e^{-x}}{2} + \frac{1}{50}\left[3\cos 2x - 4\sin 2x\right]$$

Complete solution is

$$y = CF + PI = C_0 + (C_1x+C_2)e^{-x} - \frac{x^2e^{-x}}{2} + \frac{1}{50}\left[3\cos 2x - 4\sin 2x\right]$$

② Solve: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$

$CF = (c_1 x + c_2)\, e^{-x}$

$PI = \frac{1}{D^2+2D+1} e^{2x} - \frac{1}{D^2+2D+1}\cos^2 x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{1}{2^2+2(2)+1} e^{2x} - \frac{1}{D^2+2D+1}\, \frac{1+\cos 2x}{2}$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[\frac{1}{D^2+2D+1} e^{0x} + \frac{1}{D^2+2D+1}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[\frac{1}{0+0+1}\cdot e^{0x} + \frac{1}{-4+2D+1}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{1}{2D-3}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{2D+3}{(2D-3)(2D+3)}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{2D+3}{4D^2-9}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{2D+3}{4(-4)-9}\cos 2x\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 - \frac{1}{25}\left(2\, D\cos 2x + 3\cos 2x\right)\right]$

$= \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{2}\,\frac{1}{25}\left(2(-2\sin 2x) + 3\cos 2x\right)$

$PI = \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{50}\left[-4\sin 2x + 3\cos 2x\right]$

$y = CF + PI = (c_1 x + c_2)e^{-x} + \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{50}\left[-4\sin 2x + 3\cos 2x\right]$

Case 3: Let $R(x) = x^m$, $m \in N$.

If $f(D)\{h(x)\} = x^m$, then $PI = h(x)$.

Polynomial: decreasing order of degree

$f(D)$: increasing order of degree of D.

① Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$.

$$(D^2+D)y = x^2+2x+4$$

$$CF = c_1 + c_2 e^{-x}$$

$$D+D^2 \overline{\big)\, x^2+2x+4\,}\ \ \frac{x^3}{3}+4x$$

$$\begin{array}{l} x^2+2x+4 \\ \underset{(-)}{\cancel{x^2}}+\underset{(-)}{\cancel{2x}} \\ \hline 4 \\ \underset{(-)}{4}+0 \\ \hline 0. \end{array}$$

$D(?) = x^2$

$D\left(\frac{x^3}{3}\right) = x^2$

$D^2\left(\frac{x^3}{3}\right) = D(x^2) = 2x$

$D(?) = 4$

$D(4x) = 4$

$$\therefore PI = \frac{x^3}{3}+4x$$

Verification:

$y = \frac{x^3}{3}+4x$

$\frac{dy}{dx} = x^2+4$, $\frac{d^2y}{dx^2} = 2x$

$\therefore$ LHS $= \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + x^2+4 =$ RHS

② Solve: $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2+1$

$$(D^2-D+1)y = x^3-3x^2+1$$

CF: $m^2-m+1 = 0$

$m = \frac{1\pm i\sqrt{3}}{2}$ $\alpha = \frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$

$$CF = e^{\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

PI:

$$PI = \frac{1}{D^2-D+1}\, x^3-3x^2+1$$

$$= x^3-6x-5$$

$$1-D+D^2 \overline{\big)\, x^3-3x^2+1\,}\ \ x^3-6x-5$$

$$\begin{array}{l} \cancel{x^3}-\cancel{3x^2}+1 \\ \underset{(-)}{\cancel{x^3}}\ \underset{(+)}{-\cancel{3x^2}}\ \underset{(-)}{+6x} \\ \hline -\cancel{6x}+1 \\ \underset{(+)}{-\cancel{6x}}\ \underset{(-)}{+6}+0 \\ \hline -5 \\ \underset{(+)}{-5} \\ \hline 0 \end{array}$$

$1(?) = x^3$

$D(x^3) = 3x^2$

$D^2(x^3) = 6x$

$D(-6x) = -6$

$D^2(-6x) = 0$

$\therefore$ Complete solution is

$$y = CF + PI$$

$$= e^{\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}x}{2}\right)\right] + x^3-6x-5$$

Verification: $y = x^3-6x-5$

$Dy = 3x^2-6$

$D^2y = 6x$

LHS $= D^2y - Dy + y = \cancel{6x} - 3x^2+6+x^3-\cancel{6x}-5$

$= x^3-3x^2+1 =$ RHS

③ $(D^3+8)y = x^4+2x+1$

To find CF:

Auxiliary equation is

$m^3+8=0$

$(m+2)(m^2-2m+4)=0$

$m=-2$ $\quad m = \frac{2\pm\sqrt{4-16}}{2}$

$= 1\pm i\sqrt{3}$ $\quad \alpha=1, \beta=\sqrt{3}$

$m=-2$ is a root.

$(-2)^3+8 = -8+8=0$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & +4 & -8 \\ \hline & 1 & -2 & 4 & 0 \\ & \downarrow & \downarrow & \downarrow & \\ & m^2 & m & \text{constant} & \end{array}$$

$CF = c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x).$

To find PI:

$PI = \frac{1}{D^3+8} x^4+2x+1$

$$\begin{array}{r|l} & \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8} \\ \hline 8+D^3 & \cancel{x^4} + 2x + 1 \\ & \underline{(-)\cancel{x^4} \;\; (+)3x} \\ & \quad -x+1 \\ & \quad \underline{-x+0} \\ & \qquad\quad 1 \\ & \qquad\quad \underline{1} \\ & \qquad\quad 0 \end{array}$$

$D^3\left(\frac{x^4}{8}\right) = \frac{24x}{8}$

$= 3x$

$= \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}$

$y = CF + PI$

$= c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x). + \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}$

Problems for Practice:

Solve: ① $(D^3 - D^2 - 6D)y = x^2+1$

② $(D^2+2D+1)y = 2x+x^2$

③ $(D^2+2D+1)y = 3x+2$

Case 4: $R(x) = e^{ax} V$, V is a function of x.

Property 2

$f(D)\, e^{ax} y = e^{ax} f(D+a)\, y$

$$PI = \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

① Solve: $(D^2 - 2D + 4)\, y = e^{x} \cos x$

CF: exercise.

$$PI = \frac{1}{D^2 - 2D + 4} e^{x} \cos x$$

$$= e^{x} \frac{1}{(D+1)^2 - 2(D+1)\, 4} \cos x$$

$$= e^{x} \frac{1}{D^2 + 3} \cos x$$

$$= e^{x} \frac{1}{-1+3} \cos x$$

$$= \frac{e^{x} \cos x}{2}.$$

② $(D^2 + 5D + 6)\, y = e^{-2x} \sin 2x$

CF: exercise

$$PI = \frac{1}{D^2 + 5D + 6} e^{-2x} \sin 2x$$

$$= e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sin 2x$$

$$= e^{-2x} \frac{1}{D^2 + D} \sin 2x$$

$$= e^{-2x} \frac{1}{-4 + D} \sin 2x$$

$$= e^{-2x} \frac{D+4}{(D-4)(D+4)} \sin 2x$$

$$= e^{-2x} \frac{D+4}{D^2 - 16} \sin 2x$$

$$= \frac{e^{-2x}}{-20} \left[D \sin 2x + 4 \sin 2x \right]$$

$$= -\frac{e^{-2x}}{20} \left[2\cos 2x + 4 \sin 2x \right] = -\frac{e^{-2x}}{10} \left[\cos 2x + 2 \sin 2x \right]$$

③ $(D^2-2D+1)y = xe^x \sin x$

CF: exercise

$$PI = \frac{1}{D^2-2D+1} xe^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x x\sin x$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D}\left[\int x \sin x\, dx\right]$$

$$= e^x \int \left[-\underset{\downarrow}{x\cos x} + \sin x\right] dx$$

$$= e^x\left[(-x\sin x - \cos x) + (-\cos x)\right]$$

$$= -e^x\left[x\sin x + 2\cos x\right]$$

Bernoulli's Generalized Rule of Integration by parts:

$$\int uv\,dx = \boxed{uv_1} - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$u', u'', \ldots$ represent derivatives

$v_1, v_2, \ldots$ represent integrals

i) $\int x\sin x\,dx = x(-\cos x) - (1)(-\sin x) + (0) = -x\cos x + \sin x$

ii) $\int x\cos x\,dx = x(\sin x) - (1)(-\cos x) + 0 = x\sin x + \cos x$

iii) $\int \underset{I}{x^2}\, \underset{II}{e^{2x}}\,dx = x^2\left(\frac{e^{2x}}{2}\right) - (2x)\left(\frac{e^{2x}}{4}\right) + (2)\left(\frac{e^{2x}}{8}\right) - (0)$

ILATE.

Solve the following differential equations:

① $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

To find CF:

$$CF = c_1 e^{-x} + c_2 e^{-3x}$$

To find PI:

$$PI = \frac{1}{D^2+4D+3}\left\{e^{-x}\sin x + x\,e^{3x}\right\}$$

$$= \frac{1}{D^2+4D+3}e^{-x}\sin x + \frac{1}{D^2+4D+3}xe^{3x}$$

$$PI_1 = \frac{1}{D^2+4D+3}e^{-x}\sin x$$

$$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$= e^{-x}\frac{1}{D^2+2D}\sin x$$

$$= e^{-x}\frac{1}{-1+2D}\sin x$$

$$= e^{-x}\frac{2D+1}{(2D-1)(2D+1)}\sin x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}\sin x = e^{-x}\frac{(2D+1)}{-4-1}\sin x$$

$$= -\frac{e^{-x}}{5}(2D\sin x + \sin x)$$

$$= -\frac{e^{-x}}{5}\left[2\cos x + \sin x\right]$$

$$PI_2 = \frac{1}{D^2+4D+3}xe^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$= e^{3x}\frac{1}{D^2+10D+24}x$$

$$= e^{3x}\left(\frac{x}{24} - \frac{5}{288}\right)$$

$$24+10D+D^2 \,\overline{)\, x \,}\quad \frac{x}{24} - \frac{5}{288}$$

$$x + \frac{5}{12}$$

$$- \quad -$$

$$-\frac{5}{12}$$

$$-\frac{5}{12}$$

$$0$$

$$10D\left(\frac{x}{24}\right) = \frac{10}{24} = \frac{5}{12}$$

$$\left[\frac{x}{24} - \frac{5}{288}\right]$$

$$\therefore PI = PI_1 + PI_2 = -\frac{e^{-x}}{5}\left[2\cos x + \sin x\right] + e^{3x}\left[\frac{x}{24} - \frac{5}{288}\right]$$

$$y = CF + PI =$$

Note:

$$PI_2 = e^{3x} \frac{1}{24+10D+D^2} x$$

$$\frac{1}{1-r} = 1+r+r^2+r^3+\dots$$

$$= \frac{e^{3x}}{24} \frac{1}{1+\frac{5}{12}D+\frac{D^2}{24}} x$$

$$r = -\left(\frac{5}{12}D + \frac{D^2}{24}\right)$$

$$= \frac{e^{3x}}{24}\left(1-\left(\frac{5D}{12}+\frac{D^2}{24}\right)+\left(\quad\right)^2 \dots\right)x$$

$$= \frac{e^{3x}}{24}\left[1-\frac{5D}{12}\right]x$$

$$= \frac{e^{3x}}{24}\left[x-\frac{5}{12}Dx\right]$$

$$= \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

② $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

To find CF:

$$m^4+m^2+1=0.$$

$$(m^2+m+1)(m^2-m+1)=0.$$

roots: $\frac{-1\pm i\sqrt{3}}{2}$, $\frac{1\pm i\sqrt{3}}{2}$

$(\alpha=-1/2, \beta=\sqrt{3}/2)$ $(\alpha=1/2, \beta=\sqrt{3}/2)$

$$CF = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + e^{\frac{x}{2}}\left[c_3\cos\frac{\sqrt{3}x}{2} + c_4\sin\frac{\sqrt{3}x}{2}\right]$$

To find PI:

$$PI = \frac{1}{D^4+D^2+1} e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) = e^{-\frac{x}{2}} \frac{1}{(D-1/2)^4+(D-1/2)^2+1}\cos\frac{\sqrt{3}}{2}x$$

continue....

Another method:

$$PI = \frac{1}{D^4+D^2+1} e^{-\frac{x}{2}} \text{ Real Part of } e^{i\frac{\sqrt{3}}{2}x}$$

$$e^{i\frac{\sqrt{3}}{2}x} = \cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x$$

$$= \text{Real Part of } \frac{1}{D^4+D^2+1} e^{-\frac{x}{2}} e^{i\frac{\sqrt{3}}{2}x}$$

$$\boxed{\begin{aligned} e^{ix} &= \cos x + i\sin x \\ \cos x &= RP \text{ of } e^{ix} \\ \sin x &= IP \text{ of } e^{ix} \end{aligned}}$$

$$= RP \text{ of } \frac{1}{D^4+D^2+1} e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}$$

$$w = \frac{-1+i\sqrt{3}}{2}$$

$$= RP \text{ of } x \frac{1}{4D^3+2D} e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}$$

$$w^3 = 1$$

$$= RP \text{ of } x \frac{1}{4w^3+2w} e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}$$

$$4w^3+2w = 4+(-1+i\sqrt{3})$$
$$= 3+i\sqrt{3}$$

$$= RP\text{ of } x \frac{1}{3+i\sqrt{3}} e^{\frac{-x}{2}} e^{i\frac{\sqrt{3}}{2}x}$$

$$= RP\text{ of } x \frac{3-i\sqrt{3}}{(3+i\sqrt{3})(3-i\sqrt{3})} e^{\frac{-x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$$

$$3^2-(i\sqrt{3})^2$$
$$=9-(-1)(3)$$
$$=12$$

$$= RP\text{ of } x \frac{(3-i\sqrt{3})}{9+3} e^{\frac{-x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{x}{12} e^{-x/2}\left[3\cos\frac{\sqrt{3}}{2}x + \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right].$$

$y = CF + PI$

③ $(D^2-1)y = x\sin 3x$

$$CF = c_1e^{x} + c_2e^{-x}$$

$$PI = \frac{1}{D^2-1}x\sin 3x$$

one method

$$\frac{1}{D^2-1} = \frac{1}{(D+1)(D-1)}$$ Split into partial fraction.

Apply $\frac{1}{D-a}R(x)$ formula.

$$= \frac{1}{D^2-1}x\ IP\text{ of } \left(e^{i3x}\right)$$

$e^{i3x} = \cos 3x + i\sin 3x$

$\sin 3x =$ Imaginary part of e^{i3x}

$$= IP\text{ of } \frac{1}{D^2-1}x e^{i3x}$$

$$= IP\text{ of } e^{i3x}\frac{1}{(D+i3)^2-1}x.$$

$$(D+i3)^2 = D^2 + (i3)^2 + 6iD$$

$$= IP\text{ of } e^{i3x}\frac{1}{D^2+6iD-10}x$$

$$\frac{-x}{10} - \frac{3i}{50}$$

$$-10+6iD+D^2 \,\Big)\, x$$

$$x - \frac{3i}{5}$$

$$(-)\qquad(+)$$

$$\frac{3i}{5}$$

$$\frac{3i}{5}$$

$$-10\left(\frac{-x}{10}\right) = x$$

$$6iD\left(\frac{-x}{10}\right) = \frac{-3i}{5}$$

$$= IP\text{ of } e^{i3x}\left(\frac{-x}{10} - \frac{3i}{50}\right)$$

$$= IP\text{ of } (\cos 3x + i\sin 3x)\left(\frac{-x}{10} - \frac{3i}{50}\right)$$

$$= \cos 3x\left(\frac{-3}{50}\right) + \sin 3x\left(\frac{-x}{10}\right)$$

$$= \frac{-1}{10}\left[\frac{3}{5}\cos 3x + x\sin 3x\right]$$

④ $(D^2+2D+1)y = 2\cos x + 3x+2+3e^x$.

$CF = (c_1x+c_2)e^{-x}$

$PI = \frac{1}{D^2+2D+1}\left(2\cos x+3x+2+3e^x\right)$

$= 2\frac{1}{D^2+2D+1}\cos x + \frac{1}{D^2+2D+1}(3x+2) + 3\frac{1}{D^2+2D+1}e^x$

$PI_1 = 2\frac{1}{D^2+2D+1}\cos x$

$= 2\frac{1}{-1+2D+1}\cos x$

$= 2\frac{1}{2D}\cos x = \int \cos x\,dx = \sin x$

$PI_2 = \frac{1}{D^2+2D+1}(3x+2)$

$= 3x-4$

$$\begin{array}{r|l} & 3x-4 \\ \hline 1+2D+D^2 & 3x+2 \\ & 3x+6 \\ & (-)\quad(-) \\ \hline & -4 \\ & -4 \\ & (+) \\ & 0 \end{array}$$

$PI_3 = 3\frac{1}{D^2+2D+1}e^x$

$= 3\frac{1}{1^2+2\times1+1}e^x$

$= \frac{3e^x}{4}$

$\therefore PI = PI_1+PI_2+PI_3 = \sin x + 3x-4+\frac{3e^x}{4}$

$y = CF+PI = \dots$

⑤ $(D-2)^2y = e^{2x}+\sin 2x + x^2$

2, 2

$CF = (c_1x+c_2)e^{2x}$

$PI = \frac{1}{(D-2)^2}\left(e^{2x}+\sin 2x+x^2\right) = \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2$

$PI_1 = \frac{1}{(D-2)^2}e^{2x} = x\frac{1}{2(D-2)}e^{2x} = \frac{x}{2}\cdot x\frac{1}{1}e^{2x} = \frac{x^2e^{2x}}{2}$

$PI_2 = \frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-4-4D+4}\sin 2x = -\frac{1}{4D}\sin 2x$

$= \frac{-1}{4}\int \sin 2x\,dx$

$= -\frac{1}{4}\,\frac{-\cos 2x}{2}$

$= \frac{\cos 2x}{8}$

$$PI_3 = \frac{1}{(D-2)^2} x^2$$

$$= \frac{1}{D^2-4D+4} x^2$$

$$= \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

$$
\begin{array}{r|l}
 & \frac{x^2}{4} + \frac{x}{2} + 3/8 \\
\hline
4-4D+D^2 & x^2 \\
 & x^2 - 2x + 1/2 \\
 & \underline{(-) \quad (+) \quad (-)} \\
 & 2x - 1/2 \\
 & 2x - 2 \\
 & \underline{- \quad (+)} \\
 & 3/2 \\
 & \underline{(-)\,3/2} \\
 & 0
\end{array}
$$

$$4 \times \frac{x^2}{4} = 4$$

$$-4D\left(\frac{x^2}{4}\right) = -2x$$

$$D^2\left(\frac{x^2}{4}\right) = 1/2$$

$$\therefore PI = \frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}.$$

Problems for Practice:

① $(D^2+4)y = x \sin^2 x$

② $(D^4-1)y = \cos x \cosh x$

③ $y'' - 2y' + 2y = x + e^x \cos x$

④ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

⑤ $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$

⑥ $(D^2-4)y = x \sinh x$

$$\frac{1}{D^2} x^2 \sin 2x = \int \left[\int x^2 \sin 2x\, dx\right]$$