1) Fourier Series ✓

2) Complex Analysis ✓ → Schaum Series

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3) Vectors →

4) Partial differential Equations ✓

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Preliminaries:

Periodic function:

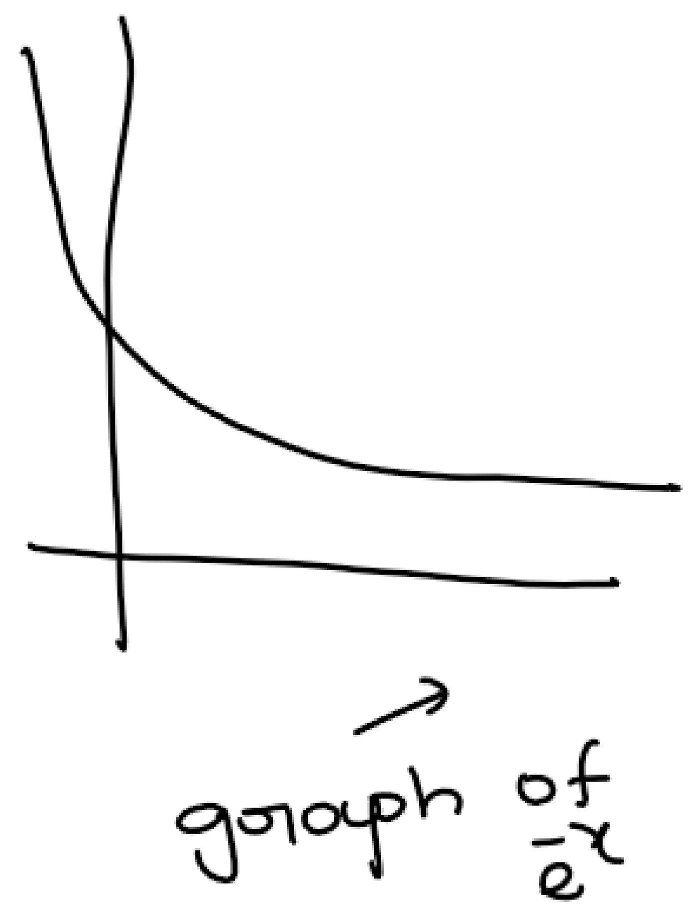
A function $f(t)$ is said to be periodic with period T if $f(t+T) = f(t)$ for all t.

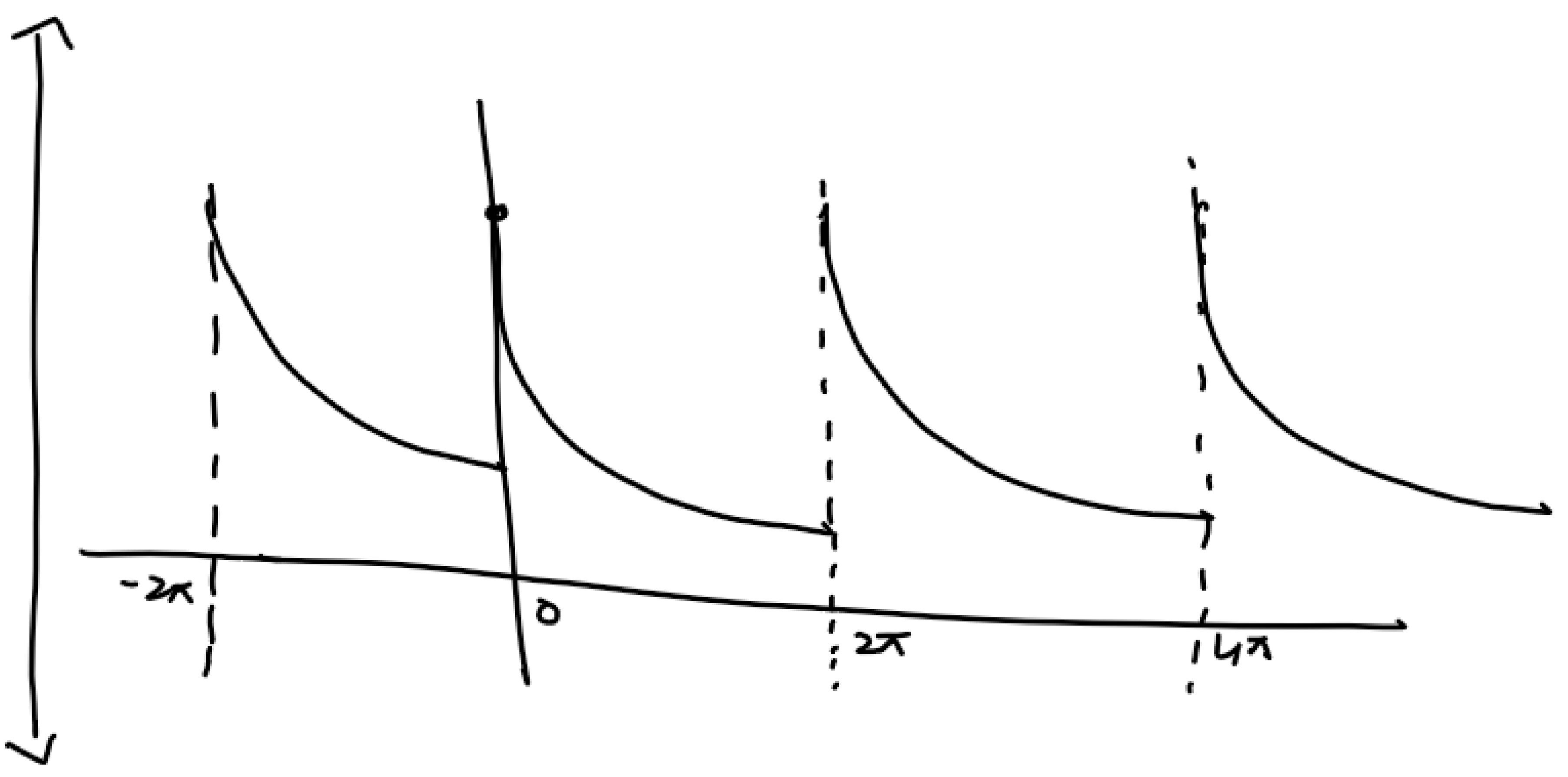
Note: If $f(t)$ is periodic with period T, then $f(t+nT) = f(t)$ for all integers n.

The smallest positive number T satisfying the above property is called primitive period or simply the period of the function $f(t)$.

The graph of periodic function $f(t)$ with period T, periodically repeats in an interval of width T.

Eg:- If the given function $f(x) = e^{-x}$ in $(0, 2\pi)$ $f(x+2\pi) = f(x)$ then the graph of $f(x)$ is





Since the function has period 2π, the same shape of curve repeats

Note:

If $f(x)$ and $g(x)$ are periodic functions of x with period T_1 and T_2 respectively, then $c_1 f(x) + c_2 g(x)$ is also periodic with period $T = \text{lcm}(T_1, T_2)$.

Example:

1) The primitive period of

$$f(x) = 2 + 3\cos 3x + 2\sin 6x \text{ is } \_\_\_\_ .$$

period of const function = 0.

" " $\cos x$ = 2π.

" " $\sin x$ = 2π.

$\therefore 3x = 2\pi \Rightarrow x = 2\pi/3$.

$6x = 2\pi \Rightarrow x = \frac{2\pi}{6} = \pi/3$

multiples of $\frac{2\pi}{3} \Rightarrow \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \ldots\ldots$

" of $\pi/3 \Rightarrow \pi/3, 2\pi/3, 3\pi/3, \ldots\ldots$

$\therefore \text{lcm}\left(\frac{2\pi}{3}, \pi/3\right) = \frac{2\pi}{3}.$

$\therefore$ primitive period $= \frac{2\pi}{3}.$

Note: The functions $\sin kt$ and $\cos kt$ have period $T = \frac{2\pi}{k}$. since

$$f\left(t + \frac{2\pi}{k}\right) = \sin k\left(t + \frac{2\pi}{k}\right) = \sin(kt + 2\pi)$$
$$= \underline{\sin kt}.$$

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2) The primitive period of

$f(x) = 3 + \cos 2x + 3\sin 3x$ is ____.

Soln:- $2x = 2\pi$, $x = \pi$ | $3x = 2\pi$, $x = \frac{2\pi}{3}$.

$\pi \rightarrow \pi, 2\pi, 3\pi, \ldots\ldots$

$\frac{2\pi}{3} \rightarrow \frac{2\pi}{3}, 2\pi, \frac{4\pi}{3}, \ldots\ldots$

$\text{lcm}(\pi, 2\pi/3) = \underline{2\pi}$

$\therefore$ Primitive Period $= 2\pi$.

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3) The primitive period of

$f(x) = 3 + \sin\left(\frac{2\pi}{3}x\right) + \cos\left(\pi/3\, x\right)$ is ____

$\frac{2\pi}{3}x = 2\pi$, $x = 3$, $\pi/3\, x = 2\pi$, $x = 6$

$\text{lcm}(3, 6) = 6.$

$\therefore$ primitive period $= 6$.

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4) The primitive period of

$f(x) = \cos\left(\pi/6\, x\right) + \sin(\pi x)$ is ____

$\pi/6\, x = 2\pi$, $x = 12$ | $\pi x = 2\pi$, $x = 2$

primitive period $= \underline{12}$.

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5) The primitive period of

$f(x) = 3\cos 3x + \sin\left(2\pi/3\, x\right)$ is ____

$3x = 2\pi$, $x = 2\pi/3$ | $2\pi/3\, x = 2\pi$, $x = 3$

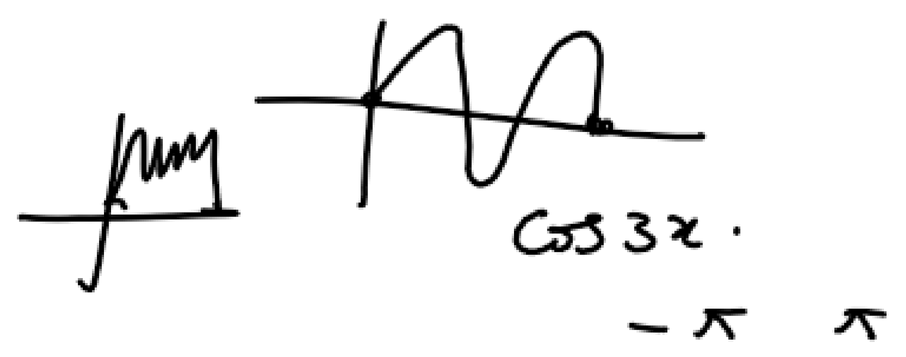
Primitive period doesnot exist,

Dirichlet conditions:

1) $f(x)$ is periodic, single valued and finite

2) $f(x)$ has finite number of discontinuities in any one period

3) $f(x)$ has atmost a finite number of maxima and minima.



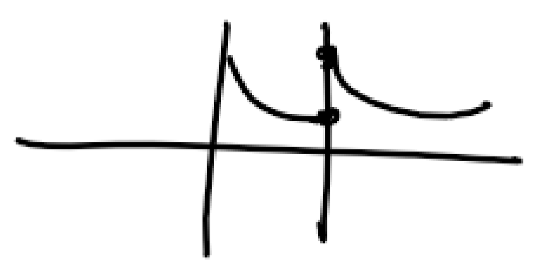
Fourier Series:

Let $f(x)$ be a periodic function with period c. and is defined in an interval $[\alpha, \alpha+2c]$ Then the expansion of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

if exists is called the Fourier Series expansion of $f(x)$. Here a_0, a_n, b_n are called fourier co-effecients. and $f(x)$ should satisfy derichlet conditions.

Eg:- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots = \frac{1}{6}\left[\frac{2\pi^2}{3} - \pi\right]$



The fourier co-effecients a_0, a_n, b_n are given by Euler's formulae.

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Some standard formulae:

1) $\cos n\pi = (-1)^n$; $\sin n\pi = 0$

2) $\cos(2n-1)\frac{\pi}{2} = 0$, ; $\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$, $n = 1, 2, 3, \ldots$

3) $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$

4) To find c, $c = \frac{U.L - L.L}{2} = \frac{(\alpha+2c)-\alpha}{2}$

5) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

$$f(x) = a_0/2 + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c} \quad \text{(1)}$$

To find a_0

Integrate (1) from α to $\alpha+2c$

$$\int_\alpha^{\alpha+2c} f(x)\,dx = \frac{1}{2}\int_\alpha^{\alpha+2c} a_0\,dx + \int_\alpha^{\alpha+2c}\sum a_n \cos\frac{n\pi x}{c}\,dx + \int_\alpha^{\alpha+2c}\sum b_n \sin\frac{n\pi x}{c}\,dx.$$

$$= \frac{a_0}{2}(\cancel{\alpha}+2c-\cancel{\alpha}) + \left(\sum a_n \sin\frac{n\pi x}{c}\right)\Big|_\alpha^{\alpha+2c} - \left(\sum b_n \cos\frac{n\pi x}{c}\right)\Big|_\alpha^{\alpha+2c}$$

$$= a_0 c + \left[\sum a_n\left(\sin\frac{n\pi}{c}[\alpha+2c] - \sin\frac{n\pi\alpha}{c}\right)\right] - \left[\sum b_n\left(\cos\frac{n\pi}{c}(\alpha+2c) - \cos\frac{n\pi\alpha}{c}\right)\right]$$

$$= a_0 c + \sum a_n\left(\sin\frac{n\pi\alpha}{c} - \sin\frac{n\pi\alpha}{c}\right) - \sum b_n\left(\cos\frac{n\pi\alpha}{c} - \cos\frac{n\pi\alpha}{c}\right)$$

$$= a_0 c.$$

$$\boxed{\begin{gathered} f(t+nT) = f(t) \\ \sin\left(\frac{n\pi\alpha}{c} + 2n\pi\right) = \sin\frac{n\pi\alpha}{c}\end{gathered}}$$

$$\int_\alpha^{\alpha+2c} \cos\frac{m\pi x}{c}\cos\frac{n\pi x}{c}\,dx = \begin{cases} 0, & m \neq n \\ c, & m = n.\end{cases}$$

$$\int_\alpha^{\alpha+2c} \sin\frac{m\pi x}{c}\sin\frac{n\pi x}{c}\,dx = \begin{cases} 0, & m \neq n \\ c, & m = n\end{cases}$$

$$\int_\alpha^{\alpha+2c} \sin\frac{m\pi x}{c}\cos\frac{n\pi x}{c}\,dx = 0, \quad \forall\, m,n.$$

x'ly ① by $\cos\frac{n\pi x}{c}$ & Integrate from α to $\alpha+2c$

$$\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx = \int_{\alpha}^{\alpha+2c} \frac{a_0}{2}\cos\frac{n\pi x}{c}\,dx +$$

$$\int_{\alpha}^{\alpha+2c}\left[a_1\cos\frac{\pi x}{c}\cos\frac{n\pi x}{c} + \cdots + a_n\cos^2\frac{n\pi x}{c} + \cdots\right]dx$$

$$+ \int_{\alpha}^{\alpha+2c}\left[\sum b_n \sin\frac{n\pi x}{c}\cos\frac{n\pi x}{c}\right]dx$$

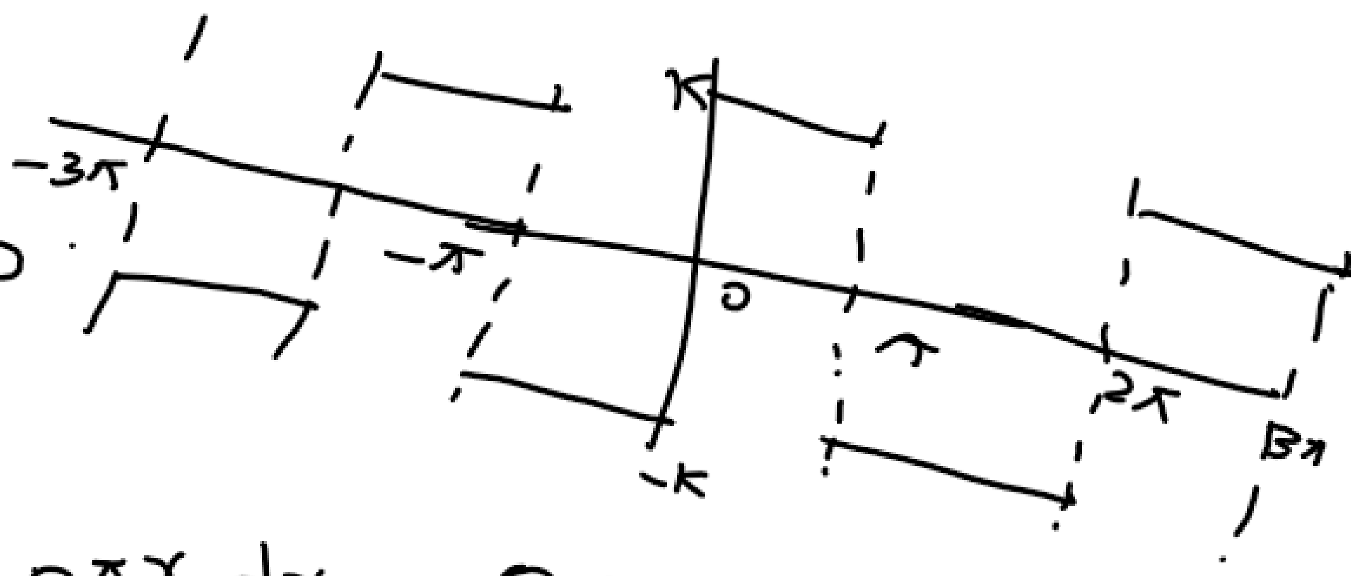
$$= \frac{a_0}{2}\cdot 0 + a_1\cdot 0 + a_2\cdot 0 + \cdots + a_n c + 0 + \cdots + \sum b_n\cdot 0 .$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx .$$

Rectangular wave:

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

Soln:- $\quad a_0 = \int_{-\pi}^{\pi} f(x)\,dx = 0$



$$a_n = \int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx = 0 .$$

$$b_n = \int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx = \frac{2k}{n\pi}\left[1-\cos n\pi\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi x}{\pi}$$

$$= \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots$$

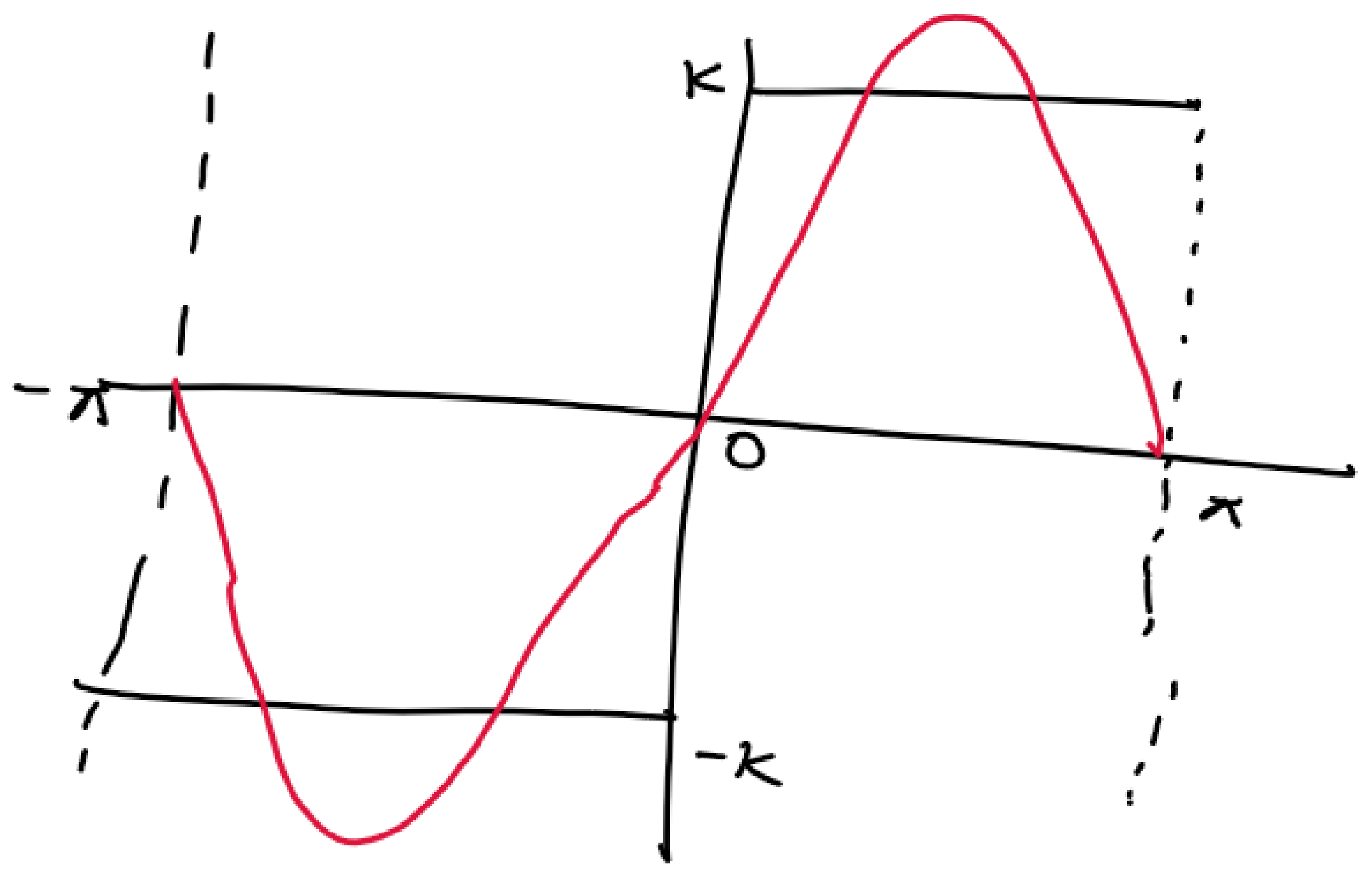
$$= \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\right]$$

Put $x = \pi/2$

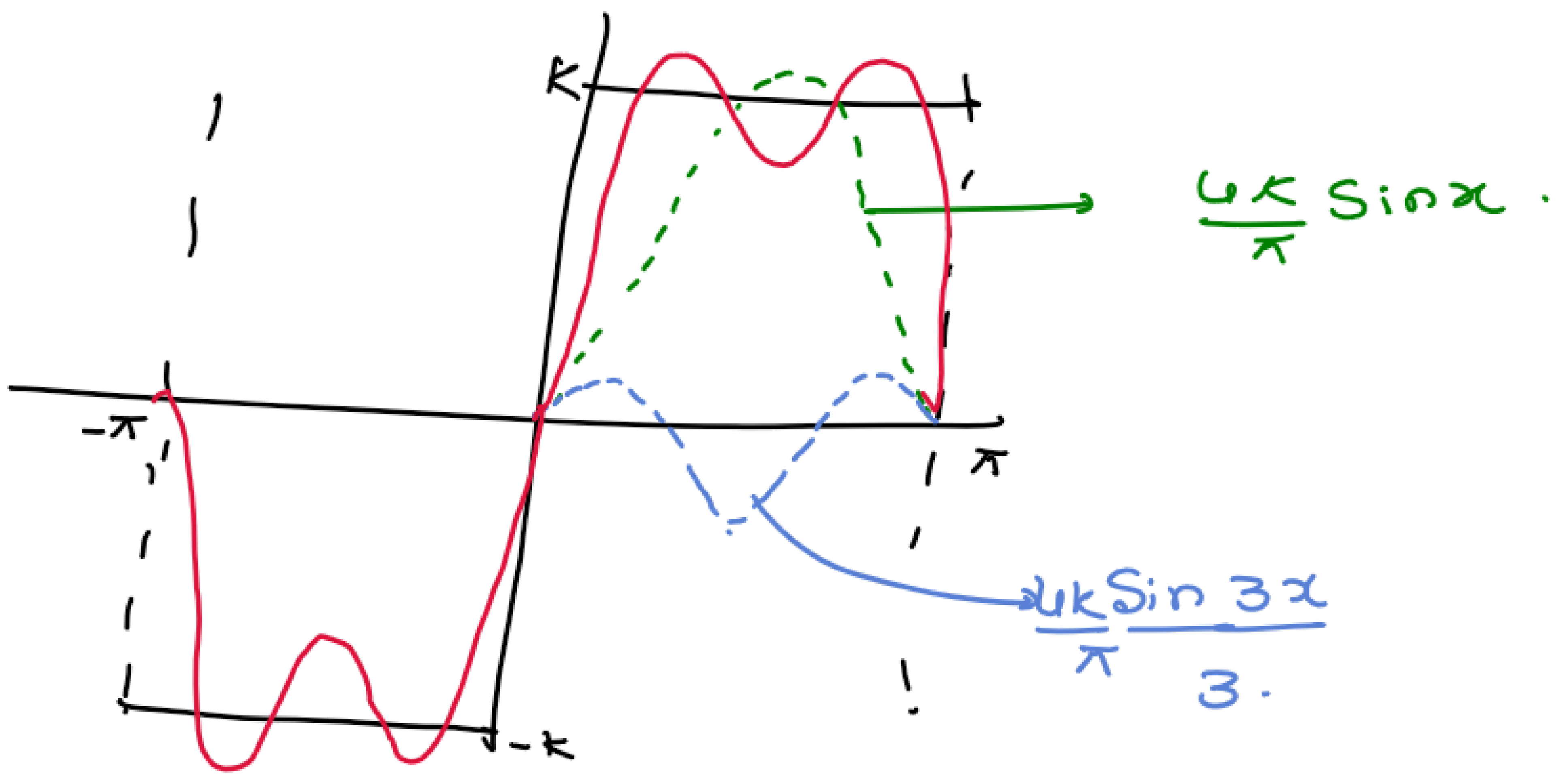
$$k = \frac{4k}{\pi}\left[\sin\pi/2 + \frac{\sin 3\pi/2}{3} + \frac{\sin 5\pi/2}{5} + \cdots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$$

$$S_1 = \frac{4K}{\pi}\sin x.$$



$$S_2 = \frac{4K}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right]$$



$$S_3 = \frac{4K}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right)$$