<u>Exercise</u>

1. Verify Green's theorem for $\oint_C (x^2 - 2xy)dx + (x^2y+3)dy$ where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

2. Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by
   (a) $y = \sqrt{x}$, $y = x^2$ (b) $x = 0$, $y = 0$, $x + y = 1$

3. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = 2xy\,i + yz^2 j + xz\,k$ and S is the surface of the parallelepiped bounded by $x=0, y=0, z=0, x=2, y=1, z=3$

4. Verify Stoke's theorem for $\vec{A} = (y - z + 2)\hat{i} + (yz+4)\hat{j} - xz\,k$ where S is the surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the xy-plane.

$$\hat{n} = \frac{\nabla(2x+y+2z)}{|\nabla(2x+y+2z)|} = \frac{2i+j+2k}{\sqrt{9}} = \frac{2}{3}i+\frac{1}{3}j+\frac{2}{3}k$$

$$(\nabla\times\vec{F})\cdot n = \frac{2}{3}x^2+\frac{1}{3}(x-2xy)$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_{R_{xz}} \frac{2}{3}x^2+\frac{1}{3}(x-2xy)\frac{dx\,dz}{|n\cdot j|}$$

$$= \int_{x=0}^{4}\int_{z=0}^{4-x}\left[2x^2+x-2x(8-2x-2z)\right]dz\,dx$$

$$= \int_0^4\int_0^{4-x}(6x^2-15x+4xz)\,dx\,dz$$

$$= \int_0^4\left[6x^2(4-x)-15x(4-x)+2x(4-x)^2\right]dx$$

$$= \int_0^4(-4x^3+23x^2-28x)\,dx = 32/3$$

Hence the verification

C_2 : $(0,8,0)$ to $(0,0,4)$

$x=0,\quad \frac{y-8}{-8} = \frac{z-0}{4} \Rightarrow 4y-32 = -8z$

$4y = 32-8z$

$y = 8-2z$

$$\int_{C_2} \vec{F}\cdot dr = \int -y\,dy = -\int_{z=0}^{4} (8-2z)(-2dz)$$

$$= 2\left[8z - z^2\right]_0^4 = 2\left[32-16\right] = \underline{\underline{32}}$$

C_3 : $(0,0,4)$ to $(4,0,0)$

$y=0,\quad \frac{x}{4} = \frac{z-4}{-4}$

$x = 4-z$ or $z = 4-x$.

$$\int_{C_3} \vec{F}\cdot dr = \int xz\,dx = \int_0^4 x(4-x)\,dx = 4\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^4$$

$$= 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$$

$$\therefore \oint_C \vec{F}\cdot dr = -32+32+\frac{32}{3} = \underline{\underline{\frac{32}{3}}}$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y & x^2y \end{vmatrix} = x^2\hat{i} - (2xy-x)\hat{j}$$

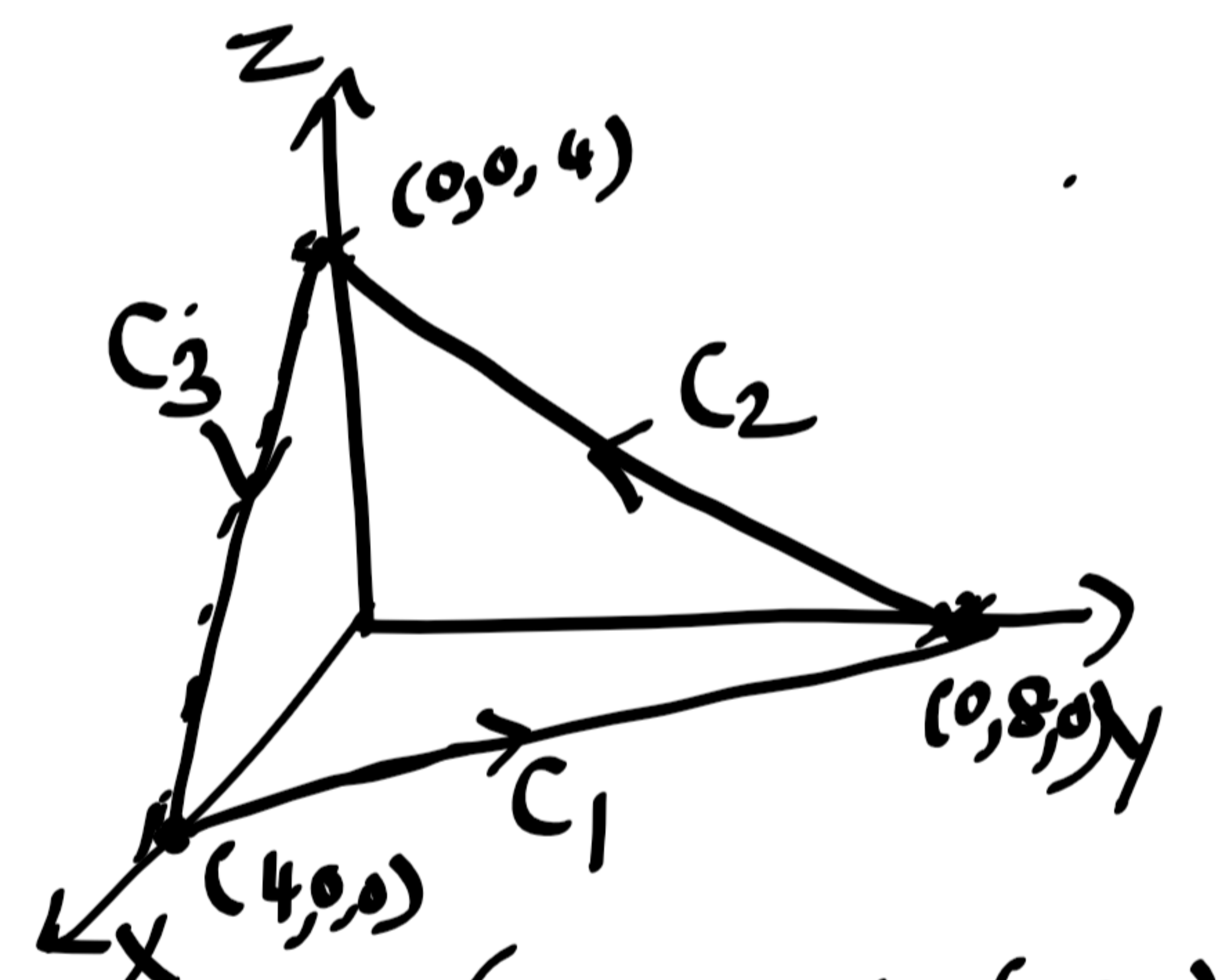
$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds = \iint_S k \cdot n\, ds = \iint_R k \cdot n \frac{dx\,dy}{|n \cdot k|}$$

$$= \iint_R dx\,dy = \text{Area of } x^2 + y^2 = 1$$

$$= \underline{\underline{\pi}}$$

Hence the verification

2. verify Stoke's theorem for $\vec{F} = xz\,i - y\,j + x^2 y\,k$ where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x + y + 2z = 8$ which is not included in the xz- plane.



$$\oint_C \vec{F} \cdot dr = \int_{C_1} \vec{F} \cdot dr + \int_{C_2} \vec{F} \cdot dr + \int_{C_3} F \cdot dr$$

$C_1 : (4,0,0)$ to $(0,8,0)$

$$\frac{x-4}{-4} = \frac{y-0}{8}$$

$$8x - 32 = -4y$$

$$y = 8 - 2x$$

$$\int_{C_1} \vec{F} \cdot dr = \int_{x=4}^{0} (2x-8)(-2dx)$$

$$= -2\left[x^2 - 8x\right]_4^0 = -2[0 - 16 + 32]$$

$$= \underline{\underline{-32}}$$

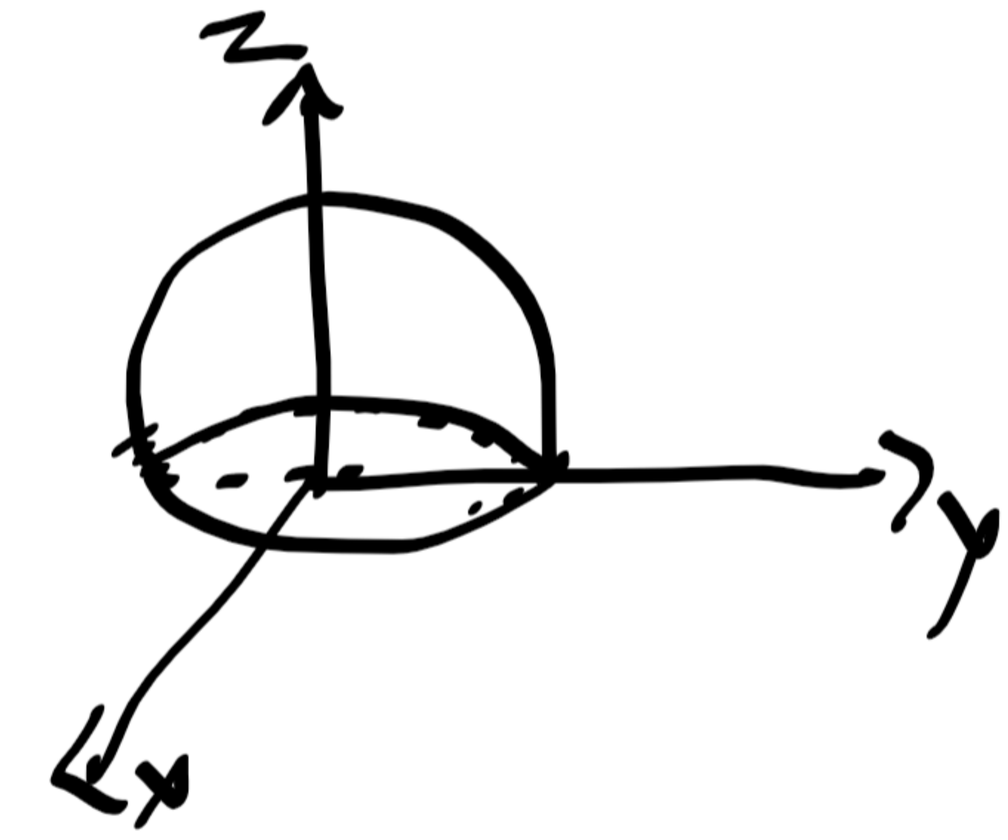
Stokes theorem

Suppose S is an open two sided surface, bounded by a simple closed curve C and suppose A is a vector function with continuous derivatives.

Then $$\oint_C \vec{A} \cdot dr = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds$$

Examples

1. Verify Stoke's theorem for $\vec{A} = (2x-y)i - yz^2 j - y^2 z k$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary. Let R be the projection of S on the xy-plane.



$$\oint_C \vec{A} \cdot dr = \oint_C (2x-y)dx - yz^2 dy - y^2 z dz$$

$$= \oint_C (2x-y)dx = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta d\theta)$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta) d\theta = \underline{\underline{\pi}}$$

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = k$$

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} \int_{x=0}^{2} (4xy - 2y + 8xz)\, dx\, dz\, dy$$

$$= \int_{y=0}^{3} \int_{0}^{\sqrt{9-y^2}} \left(2x^2y - 2xy + 4x^2z\right)_0^2 dz\, dy$$

$$= \int_{z=0}^{3} \int_{0}^{\sqrt{9-z^2}} (8y - 4y + 16z)\, dy\, dz$$

$$= \int^{3} 4y^2 - 2y^2 + 16zy \Big|_0^{\sqrt{9-z^2}} dz$$

$$9 - z^2 = t$$

$$z\,dz = \frac{-1}{2}dt$$

$$\frac{-1}{2}\int \sqrt{t}\,dt$$

$$= \frac{-1}{2}\, t^{3/2} \cdot \frac{2}{3}$$

$$= \int_0^3 \left[2(9-z^2) + 16z\sqrt{9-z^2}\right] dz$$

$$= \left[18z - \frac{2z^3}{3} - \frac{16}{3}(9-z^2)^{3/2}\right]_0^3$$

$$= 54 - 18 - \frac{16}{3}(0 - 27) = 54 - 18 + 144$$

$$= \underline{\underline{180}}$$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\,dS_3 = \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$\sin 3A = 3\sin A - 4\sin^3 A$

$$= 48\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta$$

$$= 48\int_0^{2\pi} \left\{\left(\frac{1+\cos 2\theta}{2}\right) - \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)\right\} d\theta$$

$$= 48\left\{\frac{1}{2}\theta + \frac{\sin 2\theta}{4} - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right\}_0^{2\pi}$$

$$= 48\left\{\pi - \frac{1}{4}(0)\right\} = \underline{\underline{48\pi}}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = 0 + 36\pi + 48\pi = \underline{\underline{84\pi}}$$

Hence <u>the verification</u>

2. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ by divergence theorem where $\vec{A} = 2x^2y\,i - y^2 j + 4xz^2 k$, S is the surface in the first octant bounded by $y^2+z^2=9$ and $x=2$.

By Divergence theorem, $\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{A}\,dV.$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{S_1} \vec{A}\cdot\hat{n}\,ds_1 + \iint_{S_2} \vec{A}\cdot\hat{n}\,ds_2 + \iint_{S_3} \vec{A}\cdot\hat{n}\,ds_3$$

On S_1: $z=0$, $n=-k$, $\vec{A} = 4xi - 2y^2j$

$$\vec{A}\cdot\hat{n} = (4xi - 2y^2j)\cdot -k = 0$$

$$\iint_{S_1} \vec{A}\cdot\hat{n}\,ds_1 = 0$$

On S_2: $z=3$, $\hat{n}=k$, $A = 4xi - 2y^2j + 9k$

$$\vec{A}\cdot\hat{n} = 9$$

$$\iint_{S_2} 9\,ds_2 = 9\iint_{S_2} ds_2 = 9\times 4\pi = \underline{\underline{36\pi}}$$

On S_3: $x^2+y^2=4$

$$\hat{n} = \frac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|} = \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{xi+yj}{2}$$

$$\vec{A}\cdot\hat{n} = (4xi - 2y^2j + z^2k)\cdot\left(\frac{xi+yj}{2}\right)$$

$$= 2x^2 - y^3$$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\,ds_3 = \iint_{S_3} (2x^2 - y^3)\,ds_3$$

Put $x = r\cos\theta = 2\cos\theta$, $y = r\sin\theta = 2\sin\theta$.

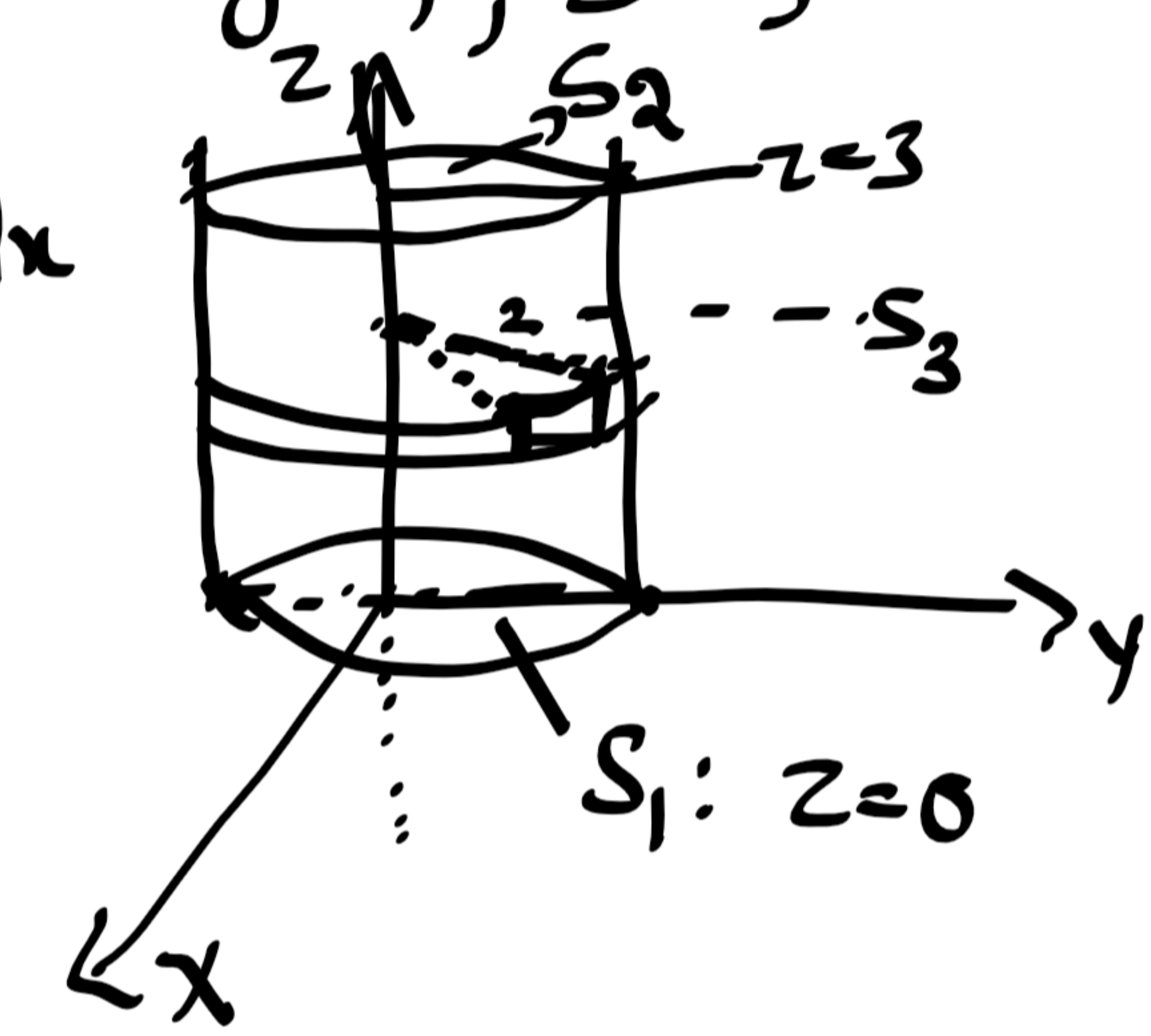
$ds_3 = 2\,d\theta\,dz$

Divergence Theorem

$$\iiint_V \nabla\cdot\vec{A}\,dV = \iint_S \vec{A}\cdot\hat{n}\,dS$$

Example

1. Verify divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$, $z=3$



$$\iiint_V \nabla\cdot\vec{A}\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dy\,dx = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}21\,dy\,dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx = 84\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left\{\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}x/2\right\}_0^2$$

$$= 84\times 2\times\frac{\pi}{2} = \underline{\underline{84\pi}}$$