2) Discuss the extreme values of the function
$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

Ans:

$f_x = 3x^2 - 63 + 12y$ $\qquad$ $f_y = 3y^2 - 63 + 12x$

$A = f_{xx} = 6x$ $\qquad$ $B = f_{xy} = 12$ $\qquad$ $C = f_{yy} = 6y$

Necessary condition for extremum value

$f_x = 0$ & $f_y = 0$

$3x^2 - 63 + 12y = 0 \to ①$ $\qquad$ $3y^2 - 63 + 12x = 0 \to ②$

eq① − ② ⇒

$$3(x^2 - y^2) + 12(y - x) = 0$$

÷3 $\quad x^2 - y^2 + 4(y - x) = 0$

$$(x+y)(x-y) - 4(x-y) = 0$$

$$(x-y)[x+y-4] = 0$$

$x - y = 0$ or $x + y - 4 = 0$

$\boxed{x = y}$ $\qquad$ $\boxed{y = 4 - x}$

When $x = y$ in ①, we get

$$3y^2 - 63 + 12y = 0$$

÷3 $\quad y^2 + 4y - 21 = 0$

$$y = 3, -7$$

Since $x = y$, $(3, 3)$, $(-7, -7)$

when $y = 4 - x$ in eqn ①, we get

$$3x^2 - 63 + 12(4 - x) = 0$$

$$3x^2 - 12x - 15 = 0$$

$\div 3$, $x^2 - 4x - 5 = 0$

$$x = 5, -1$$

Sine $y = 4 - x$,

$x = 5$, $y = -1$ $(5, -1)$

$x = -1$, $y = 5$ $(-1, 5)$

$\therefore$ $(3,3)$, $(-7,7)$ $(5,-1)$ $(-1,5)$ are the stationary points.

St. Pts	A	$AC - B^2$	
$(3,3)$	$18 > 0$	$180 > 0$	Min
$(-7,-7)$	$-42 < 0$	$1620 > 0$	Max
$(5,-1)$	30	-324	Saddle points.
$(-1,5)$	-6	-324	

$A = 6x$
$B = 12$
$C = 6y$

Maximum value $= f(-7,-7) = 784$

Minimum value $= f(3,3) = -216$

3) In a plane Δ^{le} ABC, find the maximum value of $\cos A \cos B \cos C$.

In Δ^{le} ABC, $A+B+C=\pi$

$C = \pi-(A+B)$

$\cos C = \cos(\pi-(A+B)) = -\cos(A+B)$

Let

$$\boxed{f(x,y) = -\cos x \cos y \cos(x+y)}$$

$$f_x = -\cos y\left[\cos x(-\sin(x+y)) + \cos(x+y)(-\sin x)\right]$$

$$= \cos y\left[\cos x \sin(x+y) + \cos(x+y)\sin x\right]$$

$$f_x = \cos y(\sin(2x+y))$$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$$f_y = \cos x \sin(x+2y)$$

$$A = f_{xx} = 2\cos y \cos(2x+y)$$

$$B = f_{xy} = \cos y \cos(2x+y) + \sin(2x+y)(-\sin y)$$

$$= \cos(2x+2y)$$

$$C = 2\cos x \cos(x+2y)$$

Necessary condition for extremum value $f_x=0$, $f_y=0$

$$\cos y \sin(2x+y) = 0 \quad —(1)$$

$$\cos x \sin(x+2y) = 0 \quad —(2)$$

From (1) $y = \pi/2$ or $2x+y=\pi$

When $y=\pi/2$ eqn (2) $\Rightarrow$ $\cos x \sin(x+\pi) = 0$

$(\frac{2}{2})\cos x(-\sin x) = 0$

$\sin 2x = 0 \Rightarrow 2x = \pi \Rightarrow x = \pi/2$ Not possible

$$\therefore \boxed{2x + y = \pi} \quad \text{--- (3)}$$

From eq (2) $\quad x = \pi/2 \quad$ or $\quad x + 2y = \pi$

when $x = \pi/2$ in (1) $\quad \cos y \sin(\pi + y) = 0$

$$\sin 2y = 0$$

$$2y = \pi \Rightarrow y = \pi/2$$

Not possible.

$$\therefore \boxed{x + 2y = \pi} \quad \text{--- (4)}$$

Solving (3) & (4)

$$2x + y = \pi$$
$$x + 2y = \pi$$

$$x = \pi/3, \quad y = \pi/3$$

At $(\pi/3, \pi/3)$, $\quad A = f_{xx} = -1 < 0$

$$AC - B^2 = 3/4 > 0$$

$\therefore (\pi/3, \pi/3)$ is maximum point & Max. value $= f(\pi/3, \pi/3) = \underline{\underline{1/8}}$

Practice Questions –

① Find the maximum and minimum values of
$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Ans: Min – $f(6,0) = 108$
Max – $f(4,0) = 112$.

② Find the extreme value of $f(x,y) = xy(a-x-y)$

Ans: $f(a/3, a/3) = a^2/2$ is max if $a>0$
$f(a/3, a/3) = a^2/2$ is min if $a<0$

③ Find the extreme values of $f(x,y) = xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$

Ans: Min value – $f(3,3) = 27$

④ Find the extreme value of $f(x,y) = \sin x + \sin y + \sin(x+y)$, $0 < x, y < \frac{\pi}{2}$.

Max – $f(\pi/3, \pi/3) = 2\sqrt{3}$

Lagrange's Method of Undetermined Multipliers –

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some relations.

Usually, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method.

When such a procedure becomes impracticable, Lagrange's method proves very convenient.

Let $U = f(x, y, z)$ be a function of 3 variables x, y, z are connected by the relation $\phi(x,y,z) = 0$.

Working Rule:

1. Write $F = f(x,y,z) + \lambda\,\phi(x,y,z)$, '$\lambda$' Lagrange's undetermined multiplier.
2. Obtain the eqns $F_x = 0$, $F_y = 0$, $F_z = 0$
3. Solve the above eqns together with $\phi(x,y,z) = 0$
4. The values of (x, y, z) so obtained will give stationary value of $f(x,y,z)$.

1) A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Ans: Let x, y, z be the edges of the box & S be its Surface area.

$\therefore S = xy + 2yz + 2zx$

and $V = 32$

$xyz = 32$.

$\Rightarrow xyz - 32 = 0$

By Lagrange's method.

$$F = f + \lambda\phi$$

$$= S + \lambda\phi$$

$$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

$F_x = 0 \Rightarrow y + 2z + \lambda(yz) = 0$ — (1)

$F_y = 0 \Rightarrow x + 2z + \lambda(xz) = 0$ — (2)

$F_z = 0 \Rightarrow 2y + 2x + \lambda(xy) = 0$ — (3)

Eq(1) × x — Eq(2) × y $\Rightarrow$

$$\cancel{xy} + 2xz + \lambda(\cancel{xyz}) = 0$$

$$- \quad \cancel{xy} + 2zy + \lambda(xyz) = 0$$

$$2xz - 2zy = 0$$

$$2z(x - y) = 0$$

$z = 0$, $x - y = 0$

$$\boxed{x = y}$$

The value $z = 0$ is neglected, as $xyz = 32$.

$Eq ② \times y \;-\; Eq ③ \times z \Rightarrow$

$$xy + \cancel{2yz} + \lambda xyz = 0$$

$$- \quad \cancel{2yz} + 2xz + \lambda xyz = 0$$

$$xy - 2xz = 0$$

$$x(y - 2z) = 0$$

$$y - 2z = 0$$

$(x \neq 0)$ $\qquad y = 2z$

$v = 32$

$$\therefore \; x = y = 2z.$$

$\phi(x, y, z) = 0 \Rightarrow xyz - 32 = 0$

$$xyz = 32$$

$$x(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64 \Rightarrow \underline{\underline{x = 4}}$$

$$\therefore \; x = 4, \quad \underline{\underline{y = 4}}, \quad z = 2$$