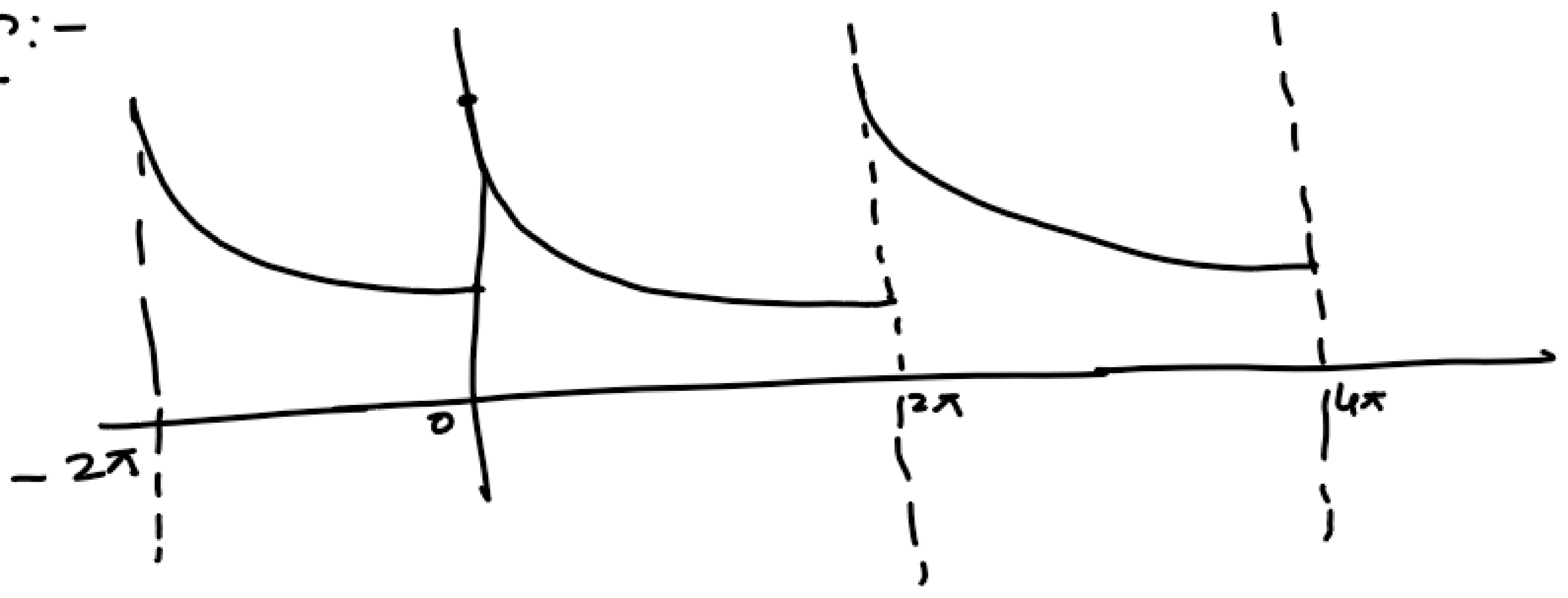
Exercise:

Obtain the Fourier Series of the following functions.

1) $f(x) = e^{-x}$, $x \in (0, 2\pi)$. $f(x+2\pi) = f(x)$

Soln:-



F.S. expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_1 a_n \cos \frac{n\pi x}{c} + \sum b_n \sin \frac{n\pi x}{c} \quad \text{where } c = \frac{2\pi - 0}{2} = \pi.$$

$$= \frac{a_0}{2} + \sum_1 a_n \cos nx + \sum_1 b_n \sin nx.$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left.\frac{e^{-x}}{-1}\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left[(-1)\cos nx + n \cancel{\sin nx}^{\,0}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{1}{n^2+1}\left[e^{-2\pi}(-1) - e^{0}(-1)\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi[n^2+1]}$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left[(-1)\cancel{\sin nx}^{\,0} - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{n(1 - e^{-2\pi})}{\pi(n^2+1)}.$$

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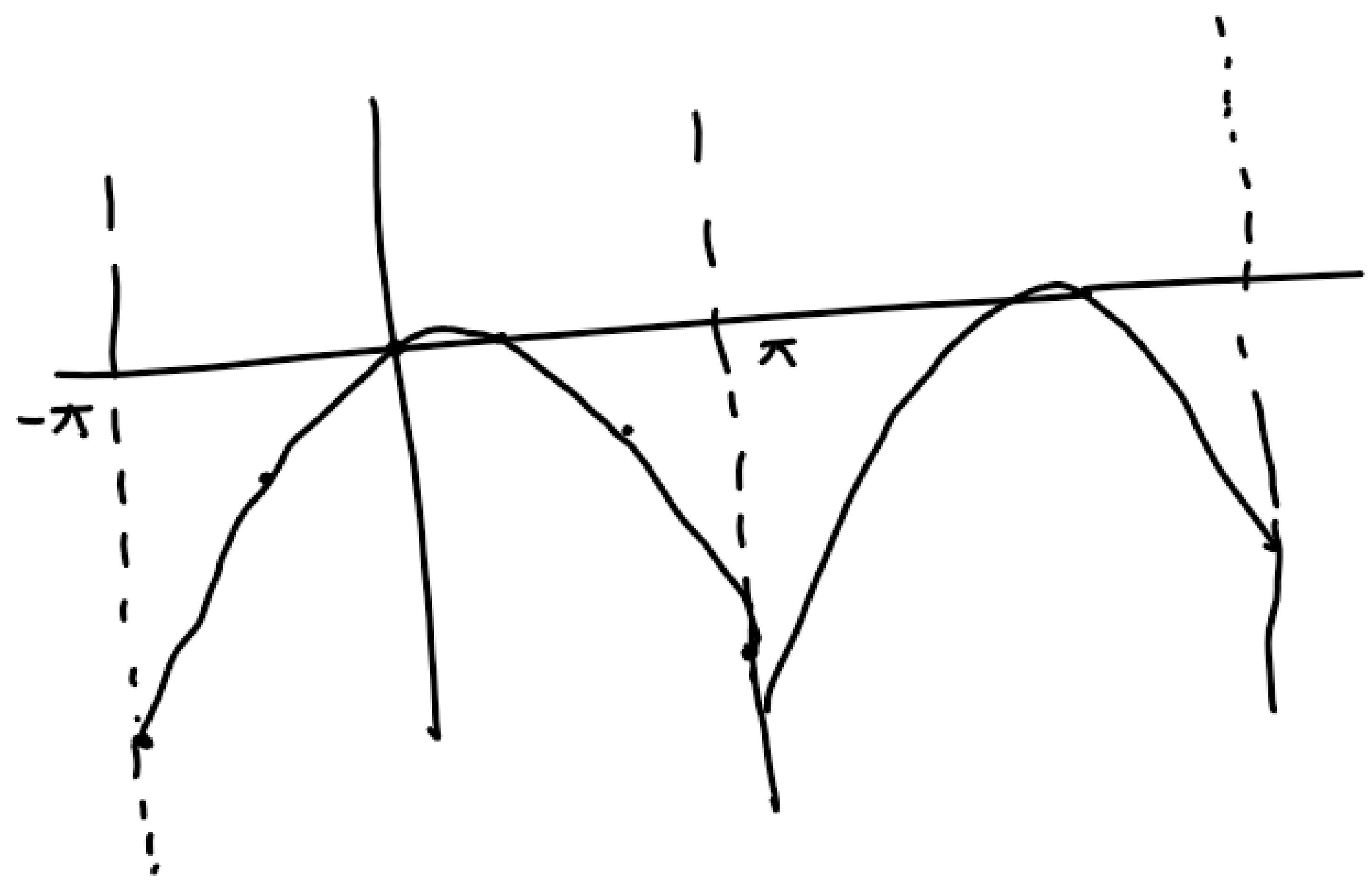
2) $f(x) = x - x^2$, $x \in (-\pi, \pi)$ $f(x+2\pi) = f(x)$.

Deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$.

Soln:-



$$f(x) = \frac{a_0}{2} + \sum_1 a_n \cos\frac{n\pi x}{c} + \sum_1 b_n \sin\frac{n\pi x}{c} \quad \text{where } c = \frac{\pi - (-\pi)}{2} = \pi$$

$$= \frac{a_0}{2} + \sum_1 a_n \cos nx + \sum_1 b_n \sin nx.$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{2\pi^3}{3}\right] = -\frac{2\pi^2}{3},$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

ILATE
$u \quad v$

$$\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$a_n = \frac{1}{\pi}\left[\cancel{(x-x^2)\frac{\sin nx}{n}} - (1-2x)\cdot\frac{1}{n}\left(-\frac{\cos nx}{n}\right) + \cancel{(0-2)\left(-\frac{1}{n^2}\right)\left(\frac{\sin nx}{n}\right)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,\frac{1}{n^2}\left[(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi\right]$$

$$= \frac{1}{\pi n^2}(-4\pi)\cos n\pi = -\frac{4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - \cancel{(1-2x)\left(-\frac{1}{n}\right)\left(\frac{\sin nx}{n}\right)} + (-2)\left(-\frac{1}{n^2}\right)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left[(\pi-\pi^2)\cos n\pi - (-\pi-\pi^2)\cos n\pi\right] - \frac{2}{n^3}\left(\cos n\pi - \cos n\pi\right)\right]$$

$$= -\frac{2\pi}{\pi n}\cos n\pi = -\frac{2}{n}(-1)^n.$$

∴ F.S expn of $f(x) = x - x^2$ is

$$x - x^2 = \frac{1}{2}\left[-\frac{2\pi^2}{3}\right] + \sum_{n=1}^{\infty}\frac{(-4)}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty}\frac{(-2)}{n}(-1)^n\sin nx.$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \cdots\right] \quad \text{— (1)}$$

Put $x = 0$ in (1)

$$0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \cdots = \frac{1}{4}\left[\frac{\pi^2}{3}\right] = \frac{\pi^2}{12}$$

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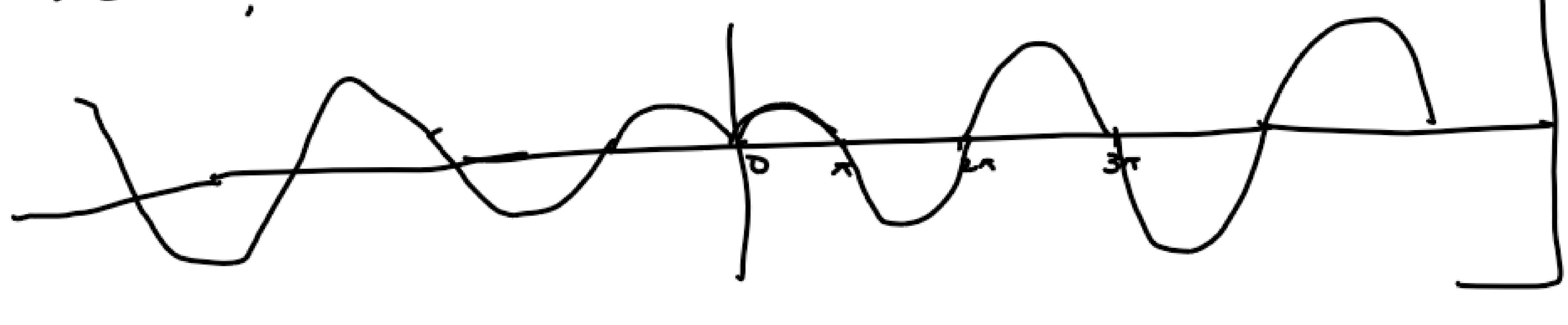
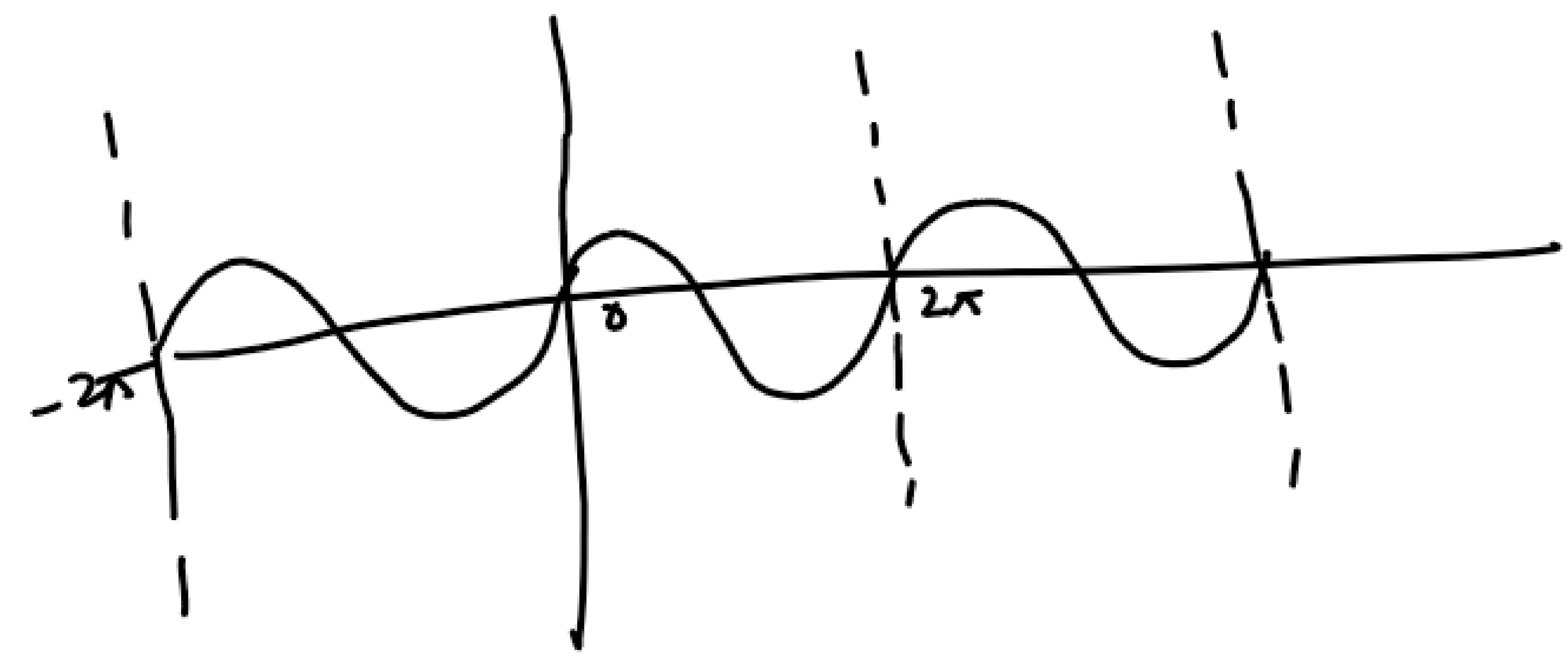
3) $f(x) = x\sin x$, $x \in (0, 2\pi)$ $\quad f(x+2\pi) = f(x)$

Graph of $x\sin x$, $-\infty < x < \infty$





$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c}, \quad c = \frac{2\pi - 0}{2} = \pi$$

$$= \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1\cdot\cancel{(-\sin x)}^{\,0}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi)\right] = -2$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x + \sin(1-n)x\right]dx.$$

$$= \frac{1}{2\pi}\left\{x\left(\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right) + 1\cdot\left[\cancel{\frac{\sin(1+n)x}{(1+n)^2}}^{\,0} + \cancel{\frac{\sin(1-n)x}{(1-n)^2}}^{\,0}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}(-2\pi)\left[\frac{\cos(1+n)2\pi}{1+n} + \frac{\cos(1-n)2\pi}{1-n}\right]$$

$$= -\left[\frac{(-1)^{2+2n}}{1+n} + \frac{(-1)^{2-2n}}{1-n}\right]$$

$$= -\left[\frac{1-n+1+n}{1-n^2}\right] = -\frac{2}{1-n^2}, \quad n \neq 1.$$

when $n=1$

$$a_1 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{\pi x}{c}\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\cdot\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(-\tfrac{1}{2}\right)\cancel{\frac{\sin 2x}{2}}^{\,0}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\,\frac{2\pi}{-2}\cos 4\pi = -\frac{1}{2}$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left(\cancel{\frac{\sin(1-n)x}{1-n}}^{\,0} - \cancel{\frac{\sin(1+n)x}{1+n}}^{\,0}\right) - 1\cdot\left[\frac{-\cos(1-n)x}{(1-n)^2} + \frac{\cos(1+n)x}{(1+n)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\overset{=1}{\cos 2\pi(1-n)} - \overset{=1}{\cos 0}}{(1-n)^2} - \frac{\overset{=1}{\cos 2\pi(1+n)} - \overset{=1}{\cos 0}}{(1+n)^2}\right]$$

$$= 0, \quad n \neq 1.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[1-\cos 2x\right] dx$$

$$= \frac{1}{2\pi}\left\{x\left[x - \cancel{\frac{\sin 2x}{2}}^{\,0}\right] - 1\cdot\left[\frac{x^2}{2} + \cancel{\frac{\cos 2x}{4}}^{\,0}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[4\pi^2 - 2\pi^2\right]$$

$$= \frac{2\pi^2}{2\pi} = \pi \,//.$$

$$\therefore x\sin x = \frac{-2}{2} - \frac{1}{2}\cos x + \left(\sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx\right) + \pi \sin x.$$

Formulae:

1) $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

2) $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

3) $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

4) $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

5) $\sin^2 A = \frac{1}{2}\left[1-\cos 2A\right]$

6) $\cos^2 A = \frac{1}{2}\left[1+\cos 2A\right]$

7) $\sin^3 A = \frac{3\sin A - \sin 3A}{4}$

8) $\cos^3 A = \frac{\cos 3A + 3\cos A}{4}$

Functions having points of discontinuity:

If in the interval $(\alpha, \alpha+2c)$, $f(x)$ is defined by

$$f(x) = \begin{cases} \phi(x). & \alpha < x < a \\ \psi(x). & a < x < \alpha + 2c. \end{cases}$$

where a is the point of discontinuity, then

$$a_0 = \frac{1}{c}\left[\int_\alpha^a \phi(x)\,dx + \int_a^{\alpha+2c} \psi(x)\,dx\right]$$

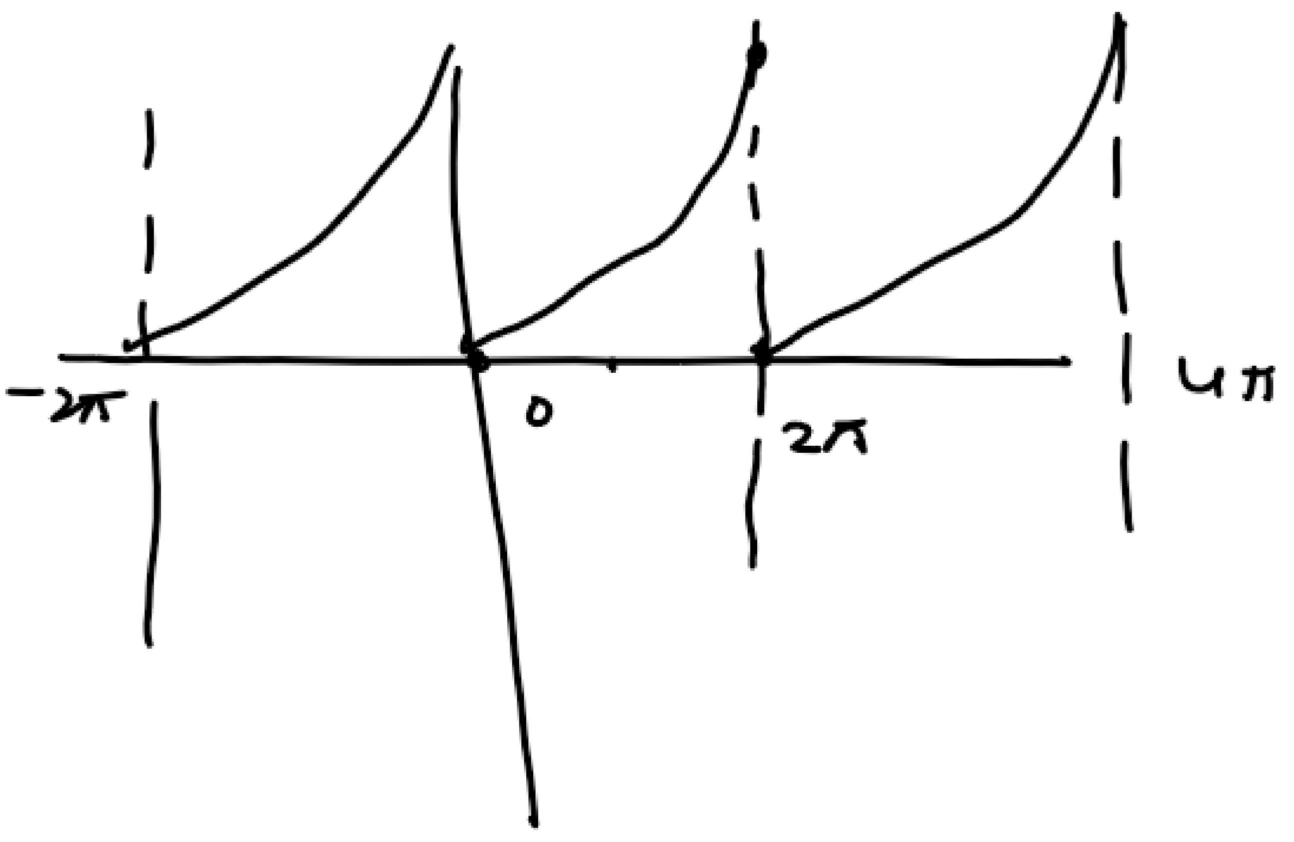
$$a_n = \frac{1}{c}\left[\int_\alpha^a \phi(x)\cos\frac{n\pi x}{c}\,dx + \int_a^{\alpha+2c}\psi(x)\cos\frac{n\pi x}{c}\,dx\right]$$

$$b_n = \frac{1}{c}\left[\int_\alpha^a \phi(x)\sin\frac{n\pi x}{c}\,dx + \int_a^{\alpha+2c}\psi(x)\sin\frac{n\pi x}{c}\,dx\right]$$

At a, $f(a) = \frac{f(a^+) + f(a^-)}{2}.$

Exercise:

1) The sum of Fourier Series of the function $f(x) = x^2$, $0 \le x \le 2\pi$, $f(x+2\pi) = f(x)$ at $x = 2\pi$ is ______ .



$$f(2\pi) = \frac{0+4\pi^2}{2} = \underline{\underline{2\pi^2}}.$$

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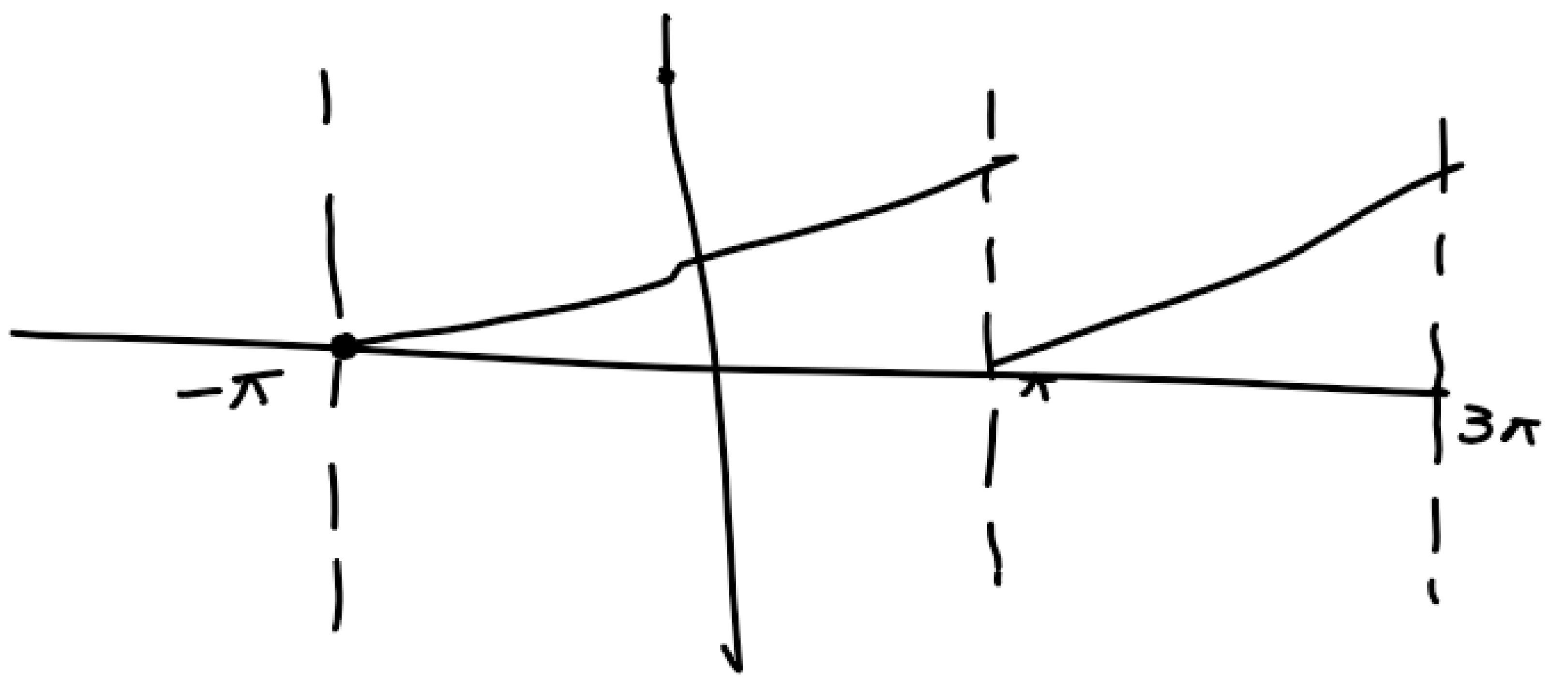
2) The sum of Fourier Series of $f(x) = x+\pi$ $\; -\pi \le x \le \pi$, $f(x+2\pi) = f(x)$ at $x=\pi$ is ____.



$$f(\pi) = \frac{\pi+\pi-0}{2} = \pi_{//}.$$

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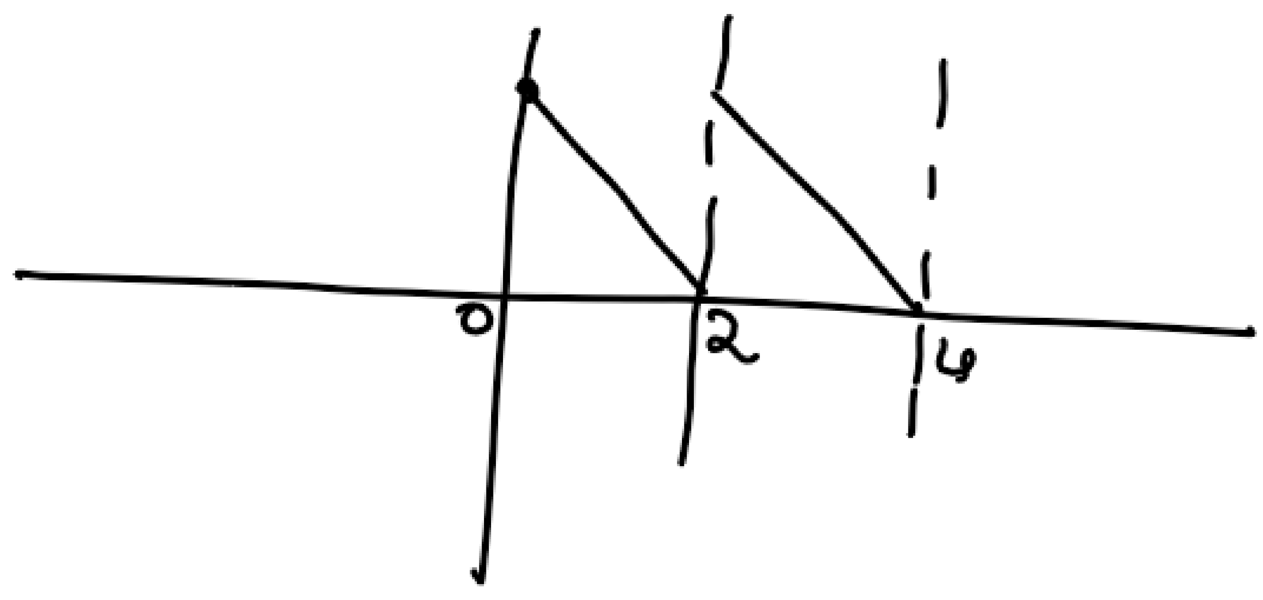
3) The sum of Fourier series of the function $f(x) = 2-x$, $0<x<2$, $f(x+2) = f(x)$ at $x=2$ is __



$$f(2) = \frac{0+2}{2} = 1_{//}$$

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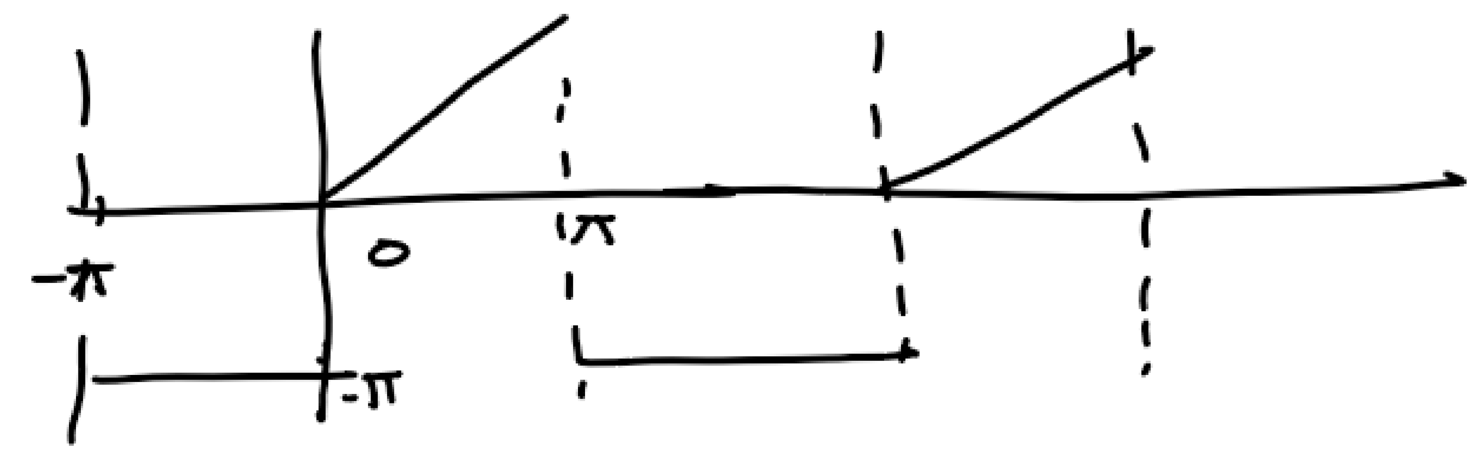
4) Find the Fourier Series expansion of following functions

(i) $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, $f(x+2\pi) = f(x)$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Soln:-



$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c}$$

$$c = \frac{\pi-(-\pi)}{2} = \pi$$

$$= \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx.$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0}\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(0+\pi) + \pi^2/2\right] = -\pi/2$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\cancel{\frac{\sin nx}{n}}^{0}\Big|_{-\pi}^{0} + \left[x\cdot\cancel{\frac{\sin nx}{n}}^{0} - 1\cdot\frac{(-\cos nx)}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos n\pi - 1\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\frac{(-\cos nx)}{n}\Big|_{-\pi}^{0} + \left[x\frac{(-\cos nx)}{n} - 1\cdot\cancel{\frac{(-\sin nx)}{n^2}}^{0}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos 0 - \pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{1 - 2\cos n\pi}{n}.$$

$$\therefore f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}\cos nx + \sum \frac{1-2(-1)^n}{n}\sin nx$$

$$= -\frac{\pi}{4} + \frac{-2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

$$+ \frac{3\sin x}{1} - \frac{\sin 2x}{2} + \cdots$$

Put $x = 0$, $\quad f(0) = \frac{-\pi + 0}{2} = -\pi/2$

$$\therefore -\pi/2 = -\pi/4 - 2/\pi\left[1/1^2 + 1/3^2 + \cdots\right]$$

$$\therefore 1/1^2 + 1/3^2 + \cdots = -\frac{\pi}{2}\left[-\pi/2 + \pi/4\right]$$

$$= \pi^2/8$$