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Modern Control Theory (ICE 3153)

Lyapunov Stability Analysis

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Lyapunov Stability-Basics

- *The General Problem of Motion Stability*, includes two methods for stability analysis (the so-called linearization method and direct method)

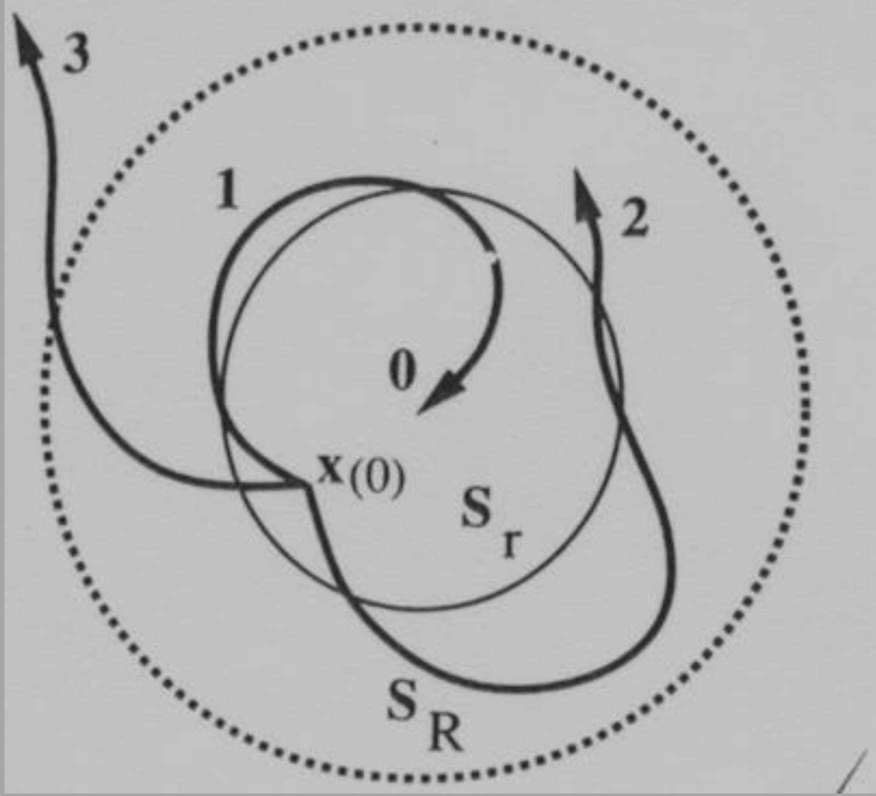
- The linearization method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation.

- The direct method is not restricted to local motion, and determines the stability properties of a nonlinear system by constructing a scalar "energy-like" function for the system and examining the function's time variation.

Stability and Instability

Definition : *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *is said to be stable (or Lyapunov stable) if , for any* $\boldsymbol{R > 0}$, *there exists* $r > 0$, *such that if* $||\mathbf{x}(0)|| < r$, *then* $||\mathbf{x}(t)|| < \boldsymbol{R}$ *for all* $t \geq 0$ *. Otherwise, the equilibrium point is unstable.*

Figure : Concepts of stability



curve 1 - asymptotically stable

curve 2 - marginally stable

curve 3 - unstable

Definition : *An equilibrium point* 0 *is asymptotically stable if it is stable, and if, in addition, there exists some r > 0 such that* $||\mathbf{x}(0)|| < r$ implies that $\mathbf{x}(t) \rightarrow \mathbf{0}$ *as* $t \rightarrow \infty$

—**Asymptotic stability** means that the equilibrium is stable, and that in addition, started close to **0** actually converge to **0** as time t goes to ∞.

—*marginally stable: A*n equilibrium point which is Lyapunov stable but not asymptotically stable.

—*Domain of attraction* of the equilibrium point: the **largest region** such that trajectories initiated at the points in the region eventually converge to the origin.

Linearization and Local Stability

Consider the autonomous system $\dot{x} = f(x)$ assuming $\mathbf{f}(\mathbf{x}) \in C^{'}$ is continuously differentiable and f(0)=0. Then the system dynamics can be written as

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \mathbf{f}_{h.\,o.\,t.}(\mathbf{x})$$

where $\mathbf{f}_{h.\,o.\,t.}$ stands for higher-order terms in **x**.

—Let **A** denote

$$\left(\frac{\partial f}{\partial x}\right)_{x=0}$$

Then, the system

$$\dot{x} = Ax$$

is called the *linearization* (or *linear approximation*) of the original nonlinear system at the equilibrium point **0**.

—For non-autonomous nonlinear system with a control input **u**

$$\dot{x} = f(x, u)$$

such that f(0,0)=0, we can write

$$\dot{x} = \left(\frac{\partial f}{\partial x}\right)_{(x=0,u=0)} x + \left(\frac{\partial f}{\partial u}\right)_{(x=0,u=0)} u + f_{h.o.t(x,u)}$$

— Let

$$A = \left(\frac{\partial f}{\partial x}\right)_{(x=0,u=0)} \quad B = \left(\frac{\partial f}{\partial u}\right)_{(x=0,u=0)}$$

— $\dot{x} = Ax + Bu$

: the linearization (or linear approximation) of the original nonlinear system at (**x = 0, u = 0**).

Alternative Linear Approximation

—Considering the autonomous closed-loop system

$$\dot{x} = f(x, u(x)) = f_1(x)$$

and linearizing the function f_1 with respect to x at its equilibrium point **x = 0**.

Theorem: (Lyapunov's linearization method)

- *If the linearized system is strictly stable (i.e ,if all eigenvalues of* **A** *are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*

- *If the linearized system is unstable (i.e , if at least one eigenvalue of* **A** *is strictly in the right-half complex plane), then the equilibrium point is unstable(for the nonlinear system).*

• *If the linearized system is marginally stable (i. e, all eigenvalues of* **A** *are in the left-half complex plane, but at least one of them is on the* $j\omega$ *axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system).*

Example: Consider the following system and comment on the stability of the system using Lyapunov linearization method

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$

$$\dot{x}_2 = x_2 + (x_1 + 1)\, x_1 + x_1 \sin x_2$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}$$

Example 3.6 : Consider the first order system

$$\dot{x} = ax + bx^5$$

—The origin 0 is an equilibrium point

— The linearization around 0:

$$\dot{x} = ax$$

—The stability properties of the nonlinear system:

$a < 0$: asymptotically stable;

$a > 0$: unstable;

$a = 0$: cannot tell from linearization.

—What shall we do when a=0?

Lyapunov’s Direct Method

Basic idea from Physical Observation

If the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *where linear or nonlinear*, must eventually settle down to an equilibrium point.

—Example: Consider the nonlinear mass-damper-spring system:

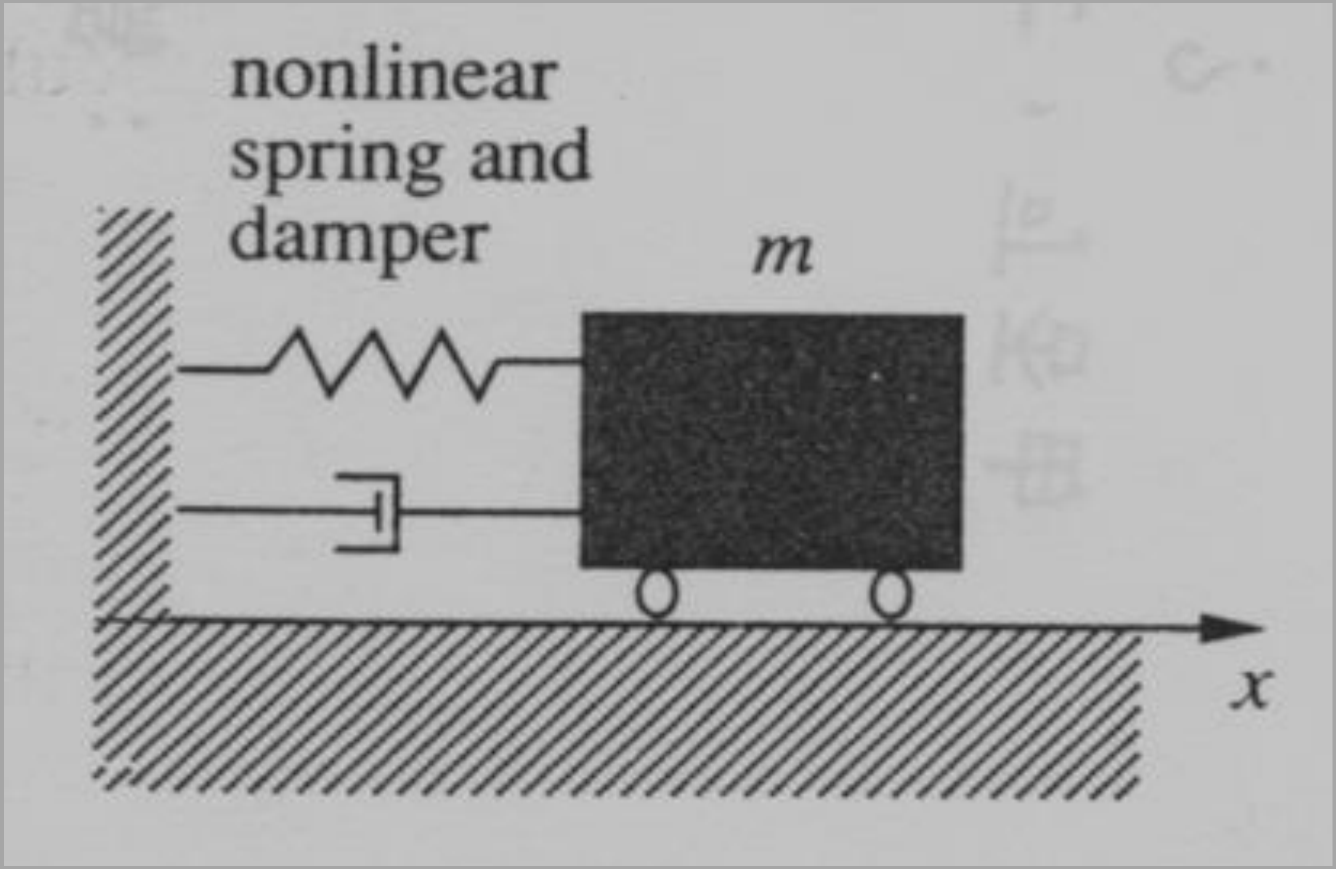
$$m\ddot{x} + b\,\dot{x}\left|\dot{x}\right| + k_0 x + k_1 x^3 = 0$$

$b\,\dot{x}\left|\dot{x}\right|$:nonlinear dissipation or damping

$(k_0 x + k_1 x^3)$:nonlinear spring term

—Assume the mass is pulled away from the natural length of the spring by a large distance, and then released $\Longrightarrow$(i) linearization method does not apply (ii) the linearized system is marginally stable only

Figure 3.6 : A nonlinear mass-damper-spring system



–The total mechanical energy of the system :

$$V(\mathrm{x}) = \frac{1}{2} m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3) dx$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$$

– Comparing the definitions of stability and mechanical energy:

·zero energy corresponds to the equilibrium point $(x = 0, \dot{x} = 0)$

·asymptotic stability $\Rightarrow$ the convergence of mechanical energy to zero

-- instability is related to the growth of mechanical energy

–These relations indicate

(i) the mechanical energy indirectly reflects the magnitude of the state vector

(ii) the stability properties can be characterized by the variation of the mechanical energy

–The rate of energy variation:

$$\dot{V}(\mathrm{x}) = m\,\dot{x}\,\ddot{x} + (k_0 x + k_1 x^3)\,\dot{x}$$

$$= \dot{x}(-b\,\dot{x}\left|\dot{x}\right|) = -b\left|\dot{x}\right|^3$$

shows the energy of the system is continuously dissipated by the damper until $\dot{x} = 0$.

Sign Definiteness

Definition 3.7 : *A scalar continuous function* $V(\mathrm{x})$ *is said to be locally positive definite if* $V(0) = 0$ *and, in a ball* $\mathbf{B}_{R_0}$

$$\mathbf{x} \neq \mathbf{0} \quad => \quad V(\mathbf{x}) > 0$$

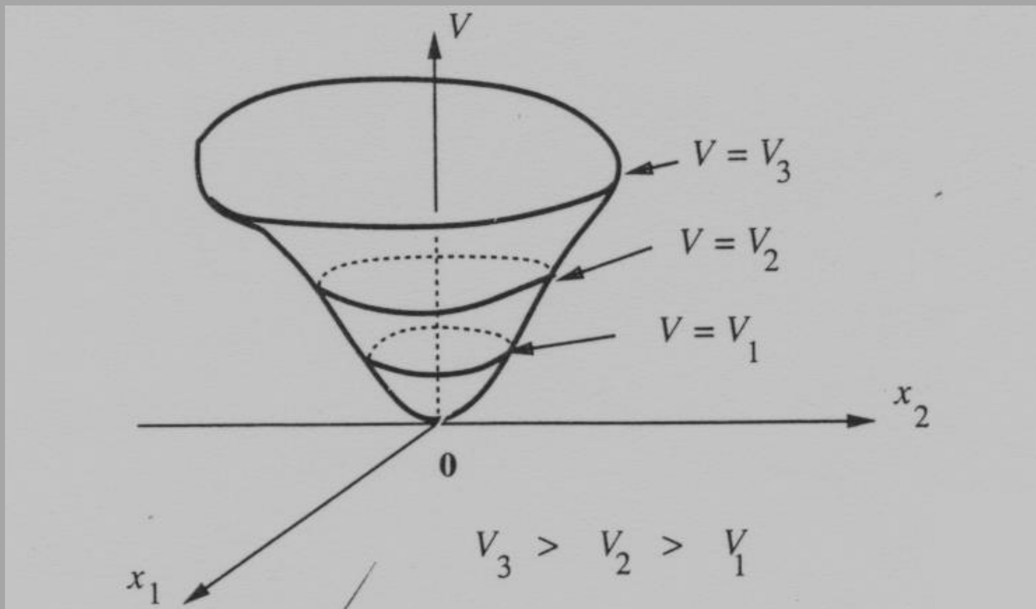
If $V(0)$ *and the above property holds over the whole state space, then* $V(\mathrm{x})$ *is said to be globally positive definite.*

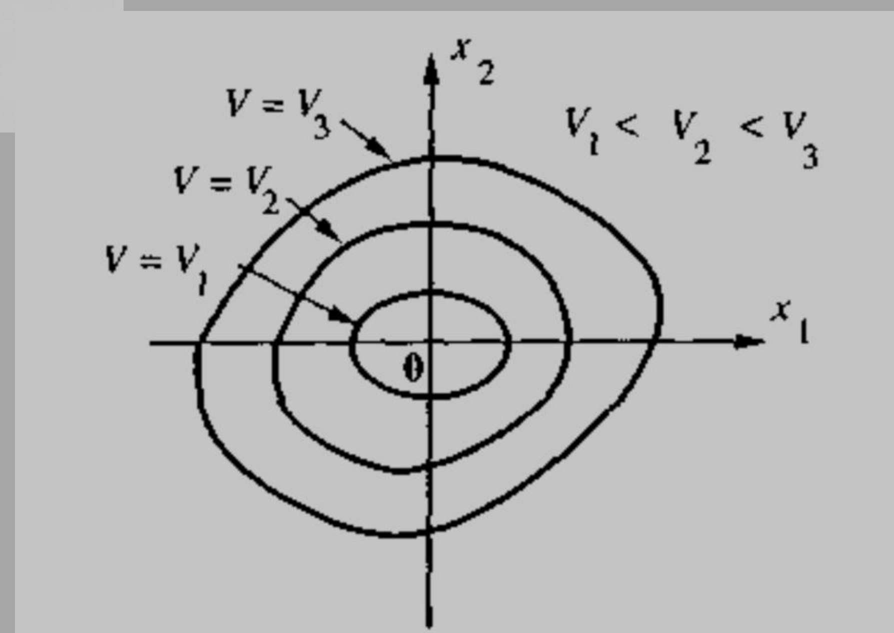
– A function V(x) is *negative definite* if ($-V(\mathbf{x})$) is positive definite

– $V(\mathbf{x})$ is *positive semi-definite* if $V(\mathbf{x}) \geq 0$ and V(x)=0 for some $x \neq 0$

– $V(\mathbf{x})$ is *negative semi-definite* if ($-V(\mathbf{x})$) is positive semi-definite.

Figure: Typical shape of a positive definite function $V(x_1, x_2)$





– $V(\mathbf{x})$ represents an implicit function of time t.

– Assuming that $V(\mathbf{x})$ is differentiable:

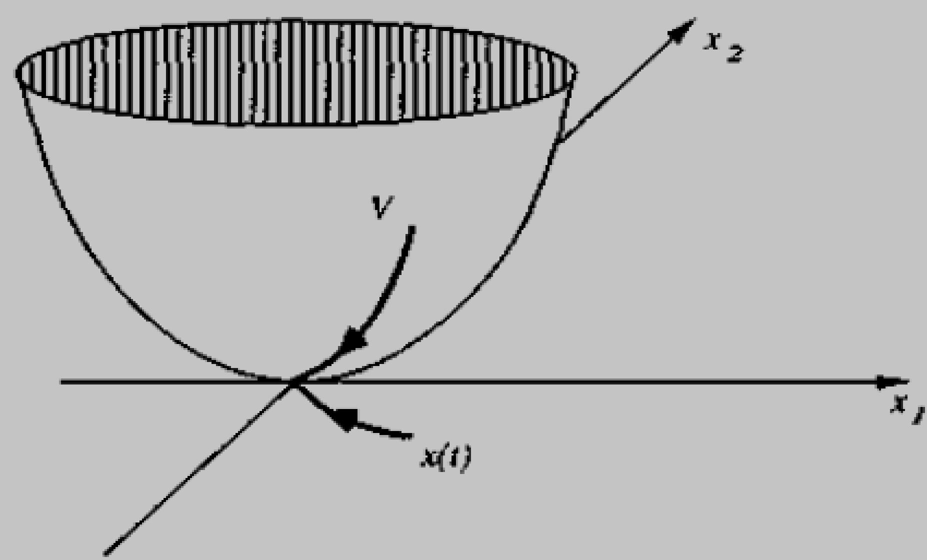
$$\dot{V} = \frac{dV(\mathrm{x})}{dt} = \frac{\partial V}{\partial \mathrm{x}}\dot{x} = \frac{\partial V}{\partial \mathrm{x}}\mathrm{f}(\mathrm{x})$$

Laypunov function

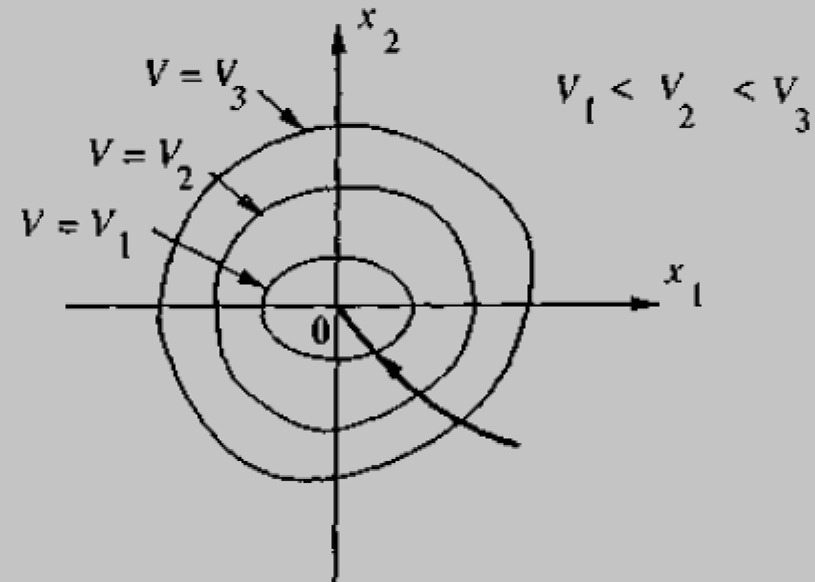
Definition 3.8 *If, in a ball* B_{R_0} *, the function V(**x**) is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system* $\dot{x} = f(x)$ *is negative semi-definite, i. e.,*

$$\dot{V}(\mathrm{x}) \leq 0$$

*then V(**x**) is said to be a Laypunov function for the system* $\dot{x} = f(x)$.



Illustrating Definition 3.8 for $n = 2$



Sylvester's Criterion

- $V(x)$ is in a quadratic form in the x_i^s if $V(x)$ is in the form, $V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j$
- Which can be written as, $V(x) = x^T Q x$
- Where Q can be written as, $Q = \begin{bmatrix} q_{11} & q_{12} \cdots & q_{1n} \\ q_{21} & q_{21} \cdots & q_{1n} \\ \vdots & \vdots & \vdots \\ q_{n1} & q_{n2} \ldots & q_{nn} \end{bmatrix}$

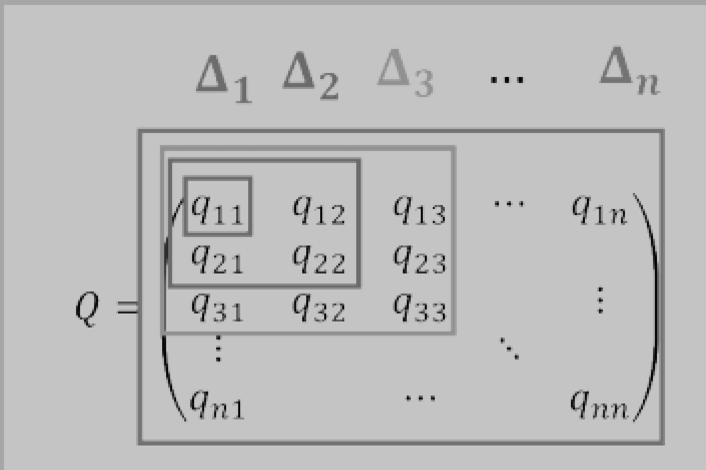
$$V(x) = [x_1 \quad x_2 \ldots \ldots . x_n] \begin{bmatrix} q_{11} & q_{12} \cdots & q_{1j} \\ q_{21} & q_{22} \cdots & q_{2j} \\ \vdots & \vdots & \vdots \\ q_{i1} & q_{i2} \ldots & q_{ij} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$q_{ij} = k_{ij} \; where \; i = j$$

$$q_{ij} = \frac{1}{2}(k_{ij} + k_{ji}) = q_{ji} \text{ for } i \neq j$$

- Sylvester's criterion provides an approach to testing positive definiteness or positive semi definiteness of a matrix.

A symmetric matrix Q is *positive definite* if and only if $\det(\Delta_1), \det(\Delta_2), \dots, \det(\Delta_n)$ are *positive*, where $\Delta_1, \Delta_2, \dots, \Delta_n$ are submatrices defined as in the drawing below. These determinants are called the *leading principal minors* of the matrix Q.
There are always n leading principal minors.



- A quadratic function $V(x) = x^T Qx$ is positive definite (pd) if and only if all the principle minors are positive.
- $V(x)$ is negative definite if $-V(x)$ is positive definite.

Example: Check the sign definiteness of the following quadratic functions.

$$V(x) = 6x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

$$V(x) = -x_1^2 - 3x_2^2 - 11x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_1x_3$$

Theorem 3.2 (Lyapunov Theorem for Local Stability)

If, in a ball B_{R_0} *, there exists a scalar function* $V(\mathbf{x})$ *with continuous first partial derivatives such that*

V(x) is *positive definite (locally in* B_{R_0} *).*

$\dot{V}(x)$ *is negative semi-definite (locally in* B_{R_0} *).*

then the equilibrium point **0** *is stable. If, actually, the derivative* $\dot{V}(x)$ *is locally negative definite in* B_{R_0}*, then the stability is asymptotic*.

Example 3.8：Asymptotic stability

Consider the nonlinear system

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

Define the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

– $\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$

–Locally negative definite in

$$\mathrm{B}_2 = \left\{ \left(x_1, x_2 \right) \middle| x_1^2 + x_2^2 < 2 \right\}$$

$\Rightarrow$ the origin is asymptotically stable.

Theorem 3.3 (Lyapunov Theorem for Global Stability) *Assume that there exists a scalar function V of the state* $\mathbf{x}$, *with continuous first order derivatives such that*

- $V(\mathbf{x})$ *is positive definite*
- $\dot{V}(\mathrm{x})$ *is negative definite*
- $V(\mathrm{x}) \to \infty$ *as* $\|\mathrm{x}\| \to \infty$ *(V(x) must be radially unbounded)*

then the equilibrium at the origin is globally asymptotically stable.

•Lyapunov Functions for Linear Time-Invariant Systems

Consider a linear system $\dot{\mathrm{x}} = \mathrm{Ax}$, let the Lyapunov function candidate be

$$V = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

P: a symmetric positive definite matrix.

If $\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$

then $\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} = \mathbf{-Q}$ is the Lyapunov equation.

Lyapunov Equation

–Symmetric matrix **Q** defined by $\mathbf{A}^T\mathbf{P}+\mathbf{PA}=\mathbf{-Q}$ is *p. d.* $\Rightarrow$ the origin is globally asymptotically stable.

–This “natural” approach may lead to inconclusive result, *i. e.*, **Q** may be not positive definite even for stable systems.

Tutorial -9

Question:1

Example 3.17 , Nonlinear Control Theory , Slotine

Consider a second-order linear system whose **A** matrix is. Find a Lyapunov function.

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

If we take **P = I,** then

$$-Q = \mathbf{PA} + \mathbf{A}^T\mathbf{P} = \begin{bmatrix} 0 & -4 \\ -4 & -24 \end{bmatrix}$$

Q is not positive definite $\Rightarrow$ don’t know whether the system is stable or not.

Thinking in Opposite Direction

–To derive a positive definite matrix **P** from a given positive definite matrix **Q,** *i. e.*,

- choose a positive definite matrix **Q**
- *solve* for **P** from the Lyapunov equation
- check whether **P** is *p. d*
- If **P** is *p. d.*, global asymptotical stability is guaranteed.

Theorem 3.6：*A necessary and sufficient condition for a LTI system* $\dot{\mathbf{x}} = \mathbf{Ax}$ *to be strictly stable is that, for any symmetric p. d. matrix* **Q,** *the unique matrix* **P,** *solution of the Lyapunov equation (3.19), be symmetric positive definite.*

Question:2

Example 4.13 , Nonlinear Systems , H. K. Khalil

Consider a second-order linear system whose **A** matrix is. Find a Lyapunov function.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Take **Q = I, and** $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$

$$\mathrm{P} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

P is positive definite the system is strictly stable

Question:3

Exercise 4.4 , Nonlinear Systems , H. K. Khalil

For the given second-order nonlinear systems use a quadratic Lyapunov function to show that the origin is asymptotically stable

1.

$$\dot{x}_1 = -x_1 - x_2$$

$$\dot{x}_2 = 2x_1 - x_2^3$$

2.

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$