$$\nabla\varphi_2 = -\frac{1}{\sqrt6}i - \frac{1}{\sqrt6}j - k$$

Let θ be the angle between φ_1 and φ_2

then $\nabla\varphi \cdot \nabla\varphi_2 = |\nabla\varphi_1||\nabla\varphi_2|\cos\theta$

$$\left(\frac{1}{\sqrt6}i + \frac{1}{\sqrt6}j - k\right)\cdot\left(-\frac{1}{\sqrt6}i - \frac{1}{\sqrt6}j - k\right) = \left(\sqrt{\frac16+\frac16+1}\right)\left(\sqrt{\frac16+\frac16+1}\right)\cos\theta$$

$$\left(-\frac16 - \frac16 + 1\right) = \left(\frac16 + \frac16 + 1\right)\cos\theta$$

$$\frac{2}{\cancel{3}} = \frac{4}{\cancel{3}}\cos\theta$$

$$\cos\theta = 2/4 = 1/2 \Rightarrow \theta = \cos^{-1}\frac12 = \underline{\underline{60^\circ}}$$

$$\nabla\phi = (2xyz-4yz^2)i+(x^2z-4xz^2)j+(x^2y-8xyz)k$$

At $(1,3,1)$ $\nabla\phi = -6i-3j-21k.$

Unit vector in the direction of $2i-j-2k$, $\hat{a}=\frac{2i-j-2k}{3}$

$\therefore$ The directional derivative is $\nabla\phi\cdot\hat{a} = -4+1+14 = \underline{\underline{11}}$

9) Find the angle between the surfaces $z=x^2+y^2$ and $z=(x-1/\sqrt{6})^2+(y-1/\sqrt{6})^2$ at the point $\left(\frac{\sqrt{6}}{12},\frac{\sqrt{6}}{12},1/12\right)$.

Angle between two surfaces at the point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2+y^2-z$, $\phi_2=(x-1/\sqrt{6})^2+(y-1/\sqrt{6})^2-z.$

A normal to ϕ_1 is
$$\nabla\phi_1 = 2xi+2yj-k$$

A normal to ϕ_2 is
$$\nabla\phi_2 = 2(x-1/\sqrt{6})i+2(y-1/\sqrt{6})j-k$$

$\frac{\sqrt{6}}{12} = \frac{\sqrt{6}}{6\times 2} = \frac{1}{2\sqrt{6}}$

$2(x-1/\sqrt{6}) = 2\left(\frac{1}{2\sqrt{6}}-1/\sqrt{6}\right)$

At $\left(\frac{\sqrt{6}}{12},\frac{\sqrt{6}}{12},1/12\right)$
$$\nabla\phi_1 = \frac{1}{\sqrt{6}}i+\frac{1}{\sqrt{6}}j-k$$

<u>Note</u> :- The maximum directional derivative (i.e, the maximum value of $\frac{d\varphi}{ds}$) takes place when $\nabla\varphi$ and $\frac{dr}{ds}$ have the same direction (so the maximum directional derivative takes place in the direction of $\nabla\varphi$) and its magnitude is $|\nabla\varphi|$.

7). Let $\varphi = x^2y^3z^6$.

(a) In what direction from the point $P(1,1,1)$ is the directional derivative of φ a maximum?

(b) What is the magnitude of this maximum?

<u>Ans</u> :- $\nabla\varphi = 2xy^3z^6\,i + 3x^2y^2z^6\,j + 6x^2y^3z^5\,k$

At $(1,1,1)$

$$\nabla\varphi = 2i + 3j + 6k.$$

(a) The directional derivative is a maximum in the direction of $2i + 3j + 6k$

(b) Magnitude of this maximum $= |\nabla\varphi| = \sqrt{2^2+3^2+6^2}$

$$= \sqrt{4+9+36} = \sqrt{49} = \underline{\underline{7}}$$

8) Let $\varphi = x^2yz - 4xyz^2$. Find the directional derivative of φ at $P(1,3,1)$ in the direction of $2i - j - 2k$.

At $(1,2,1)$ $\nabla\varphi = -4\hat{i}-3\hat{j}-14k$

Then $4\hat{i}+3\hat{j}+14k$ is normal to the surface at $(1,2,1)$.

$\therefore$ Equation of the tangent plane is

$$4(x-1)+3(y-2)+14(z-1)=0$$

$$4x+3y+14z=24$$

6). Let $\varphi(x,y,z)$ and $\varphi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the temperatures at two neighbouring points $P(x,y,z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ of a certain region. Then show that $\frac{d\varphi}{ds} = \nabla\varphi\cdot\frac{d\vec{r}}{ds}$ where s is the distance between P and Q.

$\frac{d\varphi}{ds}$ represents the rate of change of temperature with respect to the distance at P in a direction towards Q. This is also called directional derivative of φ.

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial\varphi}{\partial x}i + \frac{\partial\varphi}{\partial y}j + \frac{\partial\varphi}{\partial z}k\right)\cdot\left(\frac{dx}{ds}i+\frac{dy}{ds}j+\frac{dz}{ds}k\right)$$

$$= \nabla\varphi\cdot\frac{d\vec{r}}{ds}.$$

$d\varphi = 0$

$\therefore \nabla\varphi \cdot d\vec{r} = 0 \Rightarrow \nabla\varphi$ is perpendicular to $d\vec{r}$ and therefore to the surface.

4) Find a unit normal to the surface $-x^2yz^2 + 2xy^2z = 1$ at the point $(1,1,1)$

Let $\varphi(x,y,z) = -x^2yz^2 + 2xy^2z$

$$\nabla\varphi = (-2xyz^2 + 2y^2z)\hat{i} + (-x^2z^2 + 4xyz)\hat{j} + (-2x^2yz + 2xy^2)k$$

At $(1,1,1)$

$$\nabla\varphi = 3\hat{j} \qquad |\nabla\varphi| = |3\hat{j}| = 3$$

Unit normal to the surface $= \dfrac{\nabla\varphi}{|\nabla\varphi|} = \dfrac{3\hat{j}}{3} = \hat{j}$

5) Find an equation to the tangent plane to the surface $x^2yz - 4xyz^2 = -6$ at $(1,2,1)$

Eqn: of a plane which passes through the point (x_1,y_1,z_1) is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ where (a,b,c) are direction ratios of normal to the plane.

$$\nabla\varphi = \nabla(x^2yz - 4xyz^2)$$
$$= (2yz - 4yz^2)\hat{i} + (x^2z - 4xz^2)\hat{j} + (x^2y - 8xyz)k$$

$$= -\frac{x i + y j + z k}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{|\vec{r}|^3}$$

2) Show that $\nabla|\vec{r}|^n = n|\vec{r}|^{n-2}\vec{r}$

$$|\vec{r}|^n = (x^2+y^2+z^2)^{n/2}$$

$$\nabla|\vec{r}|^n = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot 2x\hat{i}$$

$$+ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\, 2y j$$

$$+ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\, 2z k$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x i + y j + z k)$$

$$= n|\vec{r}|^{n-2}\vec{r}$$

3) Show that $\nabla\varphi$ is a vector perpendicular to the surface $\varphi(x,y,z) = c$ where c is a constant.

Ans:- Let $\vec{r} = x i + y j + z k$ be the position vector to any point $P(x,y,z)$ on the surface.

$d\vec{r} = dx\, i + dy\, j + dz\, k$ lies in the tangent plane to the surface at P.

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}j + \frac{\partial\varphi}{\partial z}k\right)\cdot(dx\, i + dy\, j + dz\, k)$$

$$= \nabla\varphi \cdot d\vec{r}$$

Gradient of a Scalar function $\varphi(x,y,z)$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x} i + \frac{\partial\varphi}{\partial y} j + \frac{\partial\varphi}{\partial z} k$$

$\nabla\varphi \cdot \hat{a} \longrightarrow$ Directional derivative, $\hat{a} \to$ unit vector

Examples

1. Find $\nabla\varphi$ if (a) $\varphi = \log|\vec{r}|$

(b) $\varphi = \frac{1}{|\vec{r}|}$

$$\vec{r} = xi + yj + zk$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

(a) $$\varphi = \log|\vec{r}| = \frac{1}{2}\log(x^2+y^2+z^2)$$

$$\therefore \nabla\varphi = \frac{1}{2}\left(\frac{2x}{x^2+y^2+z^2} i + \frac{2y}{x^2+y^2+z^2} j + \frac{2z}{x^2+y^2+z^2} k\right)$$

$$= \frac{xi+yj+zk}{x^2+y^2+z^2} = \frac{\vec{r}}{|\vec{r}|^2}$$

(b) $$\varphi = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\nabla\varphi = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\, i - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2y\, j - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2z\, k$$