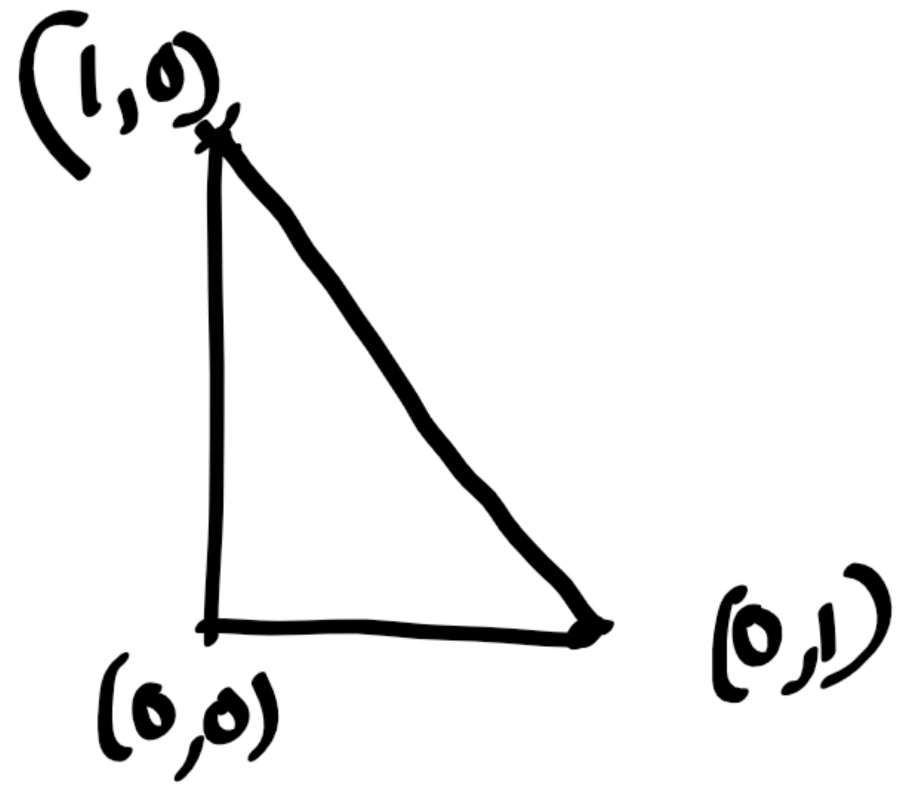
(iv)



$\frac{1}{z-2}$ is analytic inside and on the triangle

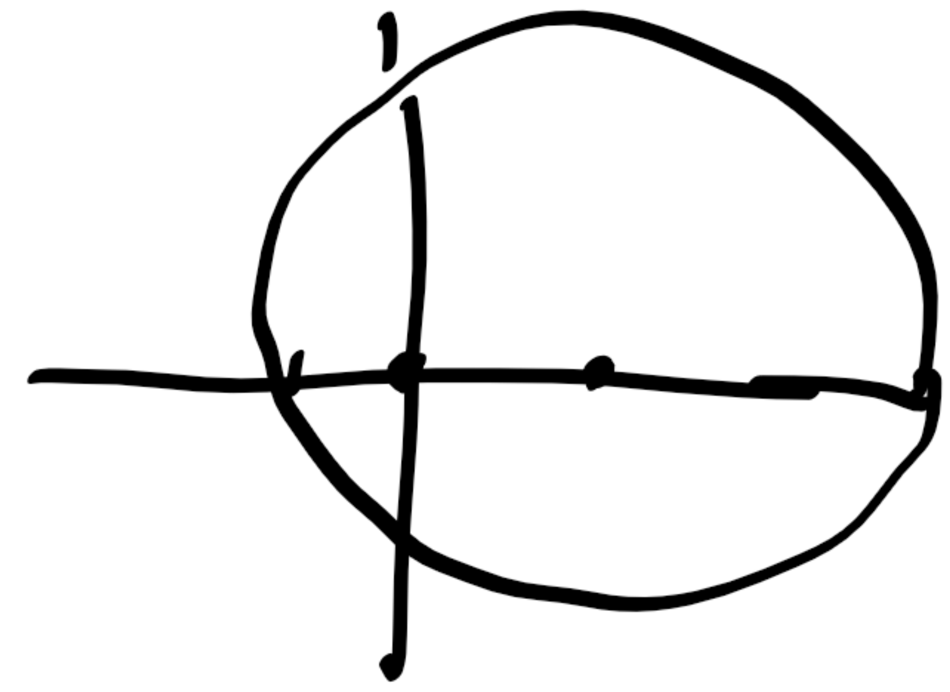
$$\therefore \int_C f(z)\,dz = 0$$

3) Evaluate $\int \frac{dz}{z+4}$ where $C: |z| = 1$

4) Evaluate $\int_C \frac{z^2 - z + 1}{z-2}\,dz$ where $|z-1| = 1/2$.

5) Evaluate $\int_{|z|=1} \frac{z^3}{z^2+5z+6}\,dz$

2) Evaluate $\oint_C \frac{dz}{z-2}$ around



(i) the circle $|z-2| = 4$

(ii) $C: |z-1| = 5$

(iii) Rectangle with vertices $3 \pm 2i$, $-2 \pm 2i$

(iv) triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$

(i) $f(z) = \frac{1}{z-2}$ is not analytic at $z=2$ which lies inside the circle $|z-2| = 4$

$$|z-2| = 4 \quad \therefore \quad z-2 = 4e^{i\theta} d\theta$$

$$dz = 4ie^{i\theta} d\theta$$

$$\int_C \frac{1}{z-2} dz = \int_0^{2\pi} \frac{1}{4e^{i\theta}} 4ie^{i\theta} d\theta = \underline{2\pi i}$$

$z-1 = 5e^{i\theta}$

(ii) $$\int_C \frac{1}{z-2} dz = \int_0^{2\pi} \frac{1}{(5e^{i\theta}-1)} 5ie^{i\theta} d\theta$$

$$= \underline{0}$$

Let $5e^{i\theta} - 1 = t$

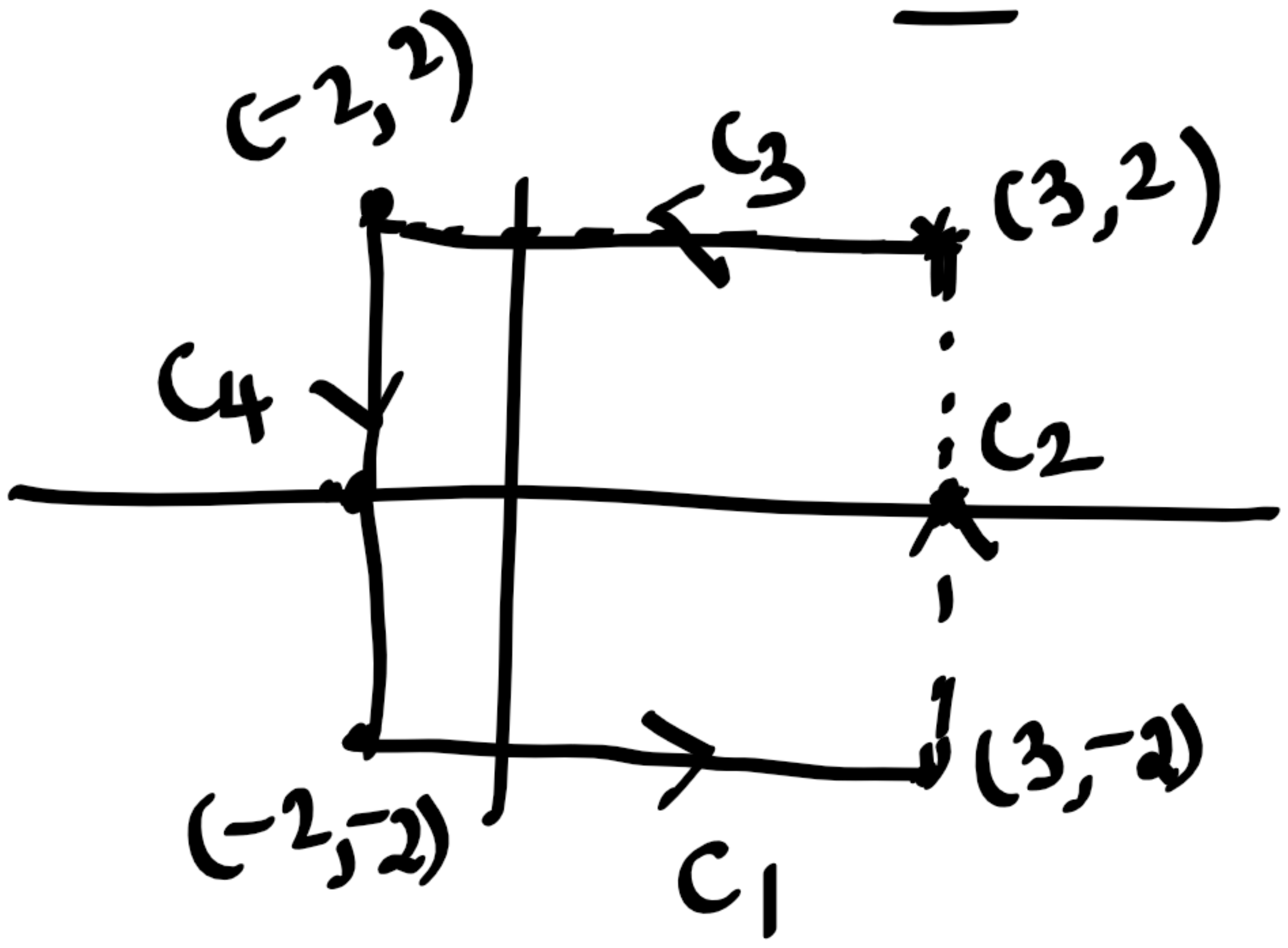
$5ie^{i\theta} d\theta = dt$

$\int \frac{1}{t} dt = \log t$

$= \{\log(5e^{i\theta}-1)\}_0^{2\pi}$

$= \log\{5e^{i2\pi}-1\} - \log(5-1) = \underline{0}$

(iii)



Along C_3, x varies from 1 to -1 and $y = 1$.

$$z = x + iy = x + i, \qquad dz = dx$$

$$z^2 = (x+i)^2 = x^2 + 2ix - 1$$

$$\int_{C_3} z^2 dz = \int_1^{-1} (x^2 + 2ix - 1)\,dx = \left[\frac{x^3}{3} + ix^2 - x\right]_1^{-1}$$

$$= -\frac{1}{3} + i + 1 - \frac{1}{3} - i + 1 = 4/3$$

Along C_4: $x = -1$, y varies from 1 to 0

$$z = -1 + iy, \quad dz = i\,dy$$

$$z^2 = (-1+iy)^2 = 1 - 2iy - y^2$$

$$\int_{C_4} z^2 dz = \int_1^0 (1 - 2iy - y^2)\,i\,dy = i\left[y - iy^2 - y^3/3\right]_1^0$$

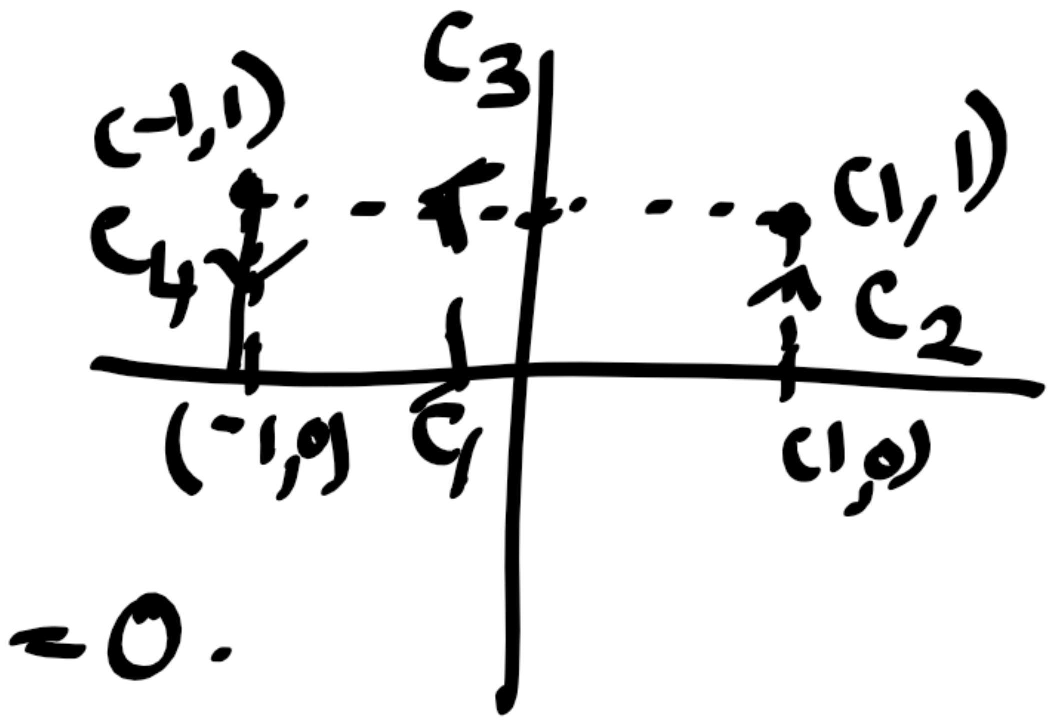
$$= 0 - i(1 - i - 1/3)$$

$$= -1 - \frac{2}{3}i$$

$$\therefore \oint_C f(z)\,dz = \frac{2}{3} - 1 + \frac{2}{3}i + \frac{4}{3} - 1 - \frac{2}{3}i = \underline{\underline{0}}$$

Examples

1. Verify Cauchy's theorem for the function z^2 with C as the boundary of the rectangle with vertices $-1,\ 1,\ 1+i,\ -1+i$.



The function $f(z)=z^2$ is analytic in the given rectangle. $\therefore$ By CIT $\int_C f(z)\,dz=0$.

$C: C_1\cup C_2\cup C_3\cup C_4$

$$\int_C f(z)\,dz=\int_{C_1} f(z)\,dz+\int_{C_2} f(z)\,dz+\int_{C_3} f(z)\,dz+\int_{C_4} f(z)\,dz$$

Along C_1: x varies from -1 to 1, $y=0$

$$z^2=(x+iy)^2=x^2, \qquad z=x+iy, \quad dz=dx$$

$$\int_{C_1} z^2\,dz=\int_{-1}^{1} x^2\,dx=2\int_0^1 x^2\,dx=\frac{2}{3}$$

Along C_2: $x=1$, y varies from 0 to 1

$$z=x+iy=1+iy, \quad dz=i\,dy$$

$$z^2=(1+iy)^2=1+2iy-y^2$$

$$\int_{C_2} z^2\,dz=\int_0^1\left(1+2iy-y^2\right)i\,dy=i\left(y+\frac{2iy^2}{2}-\frac{y^3}{3}\right)\Big|_0^1$$

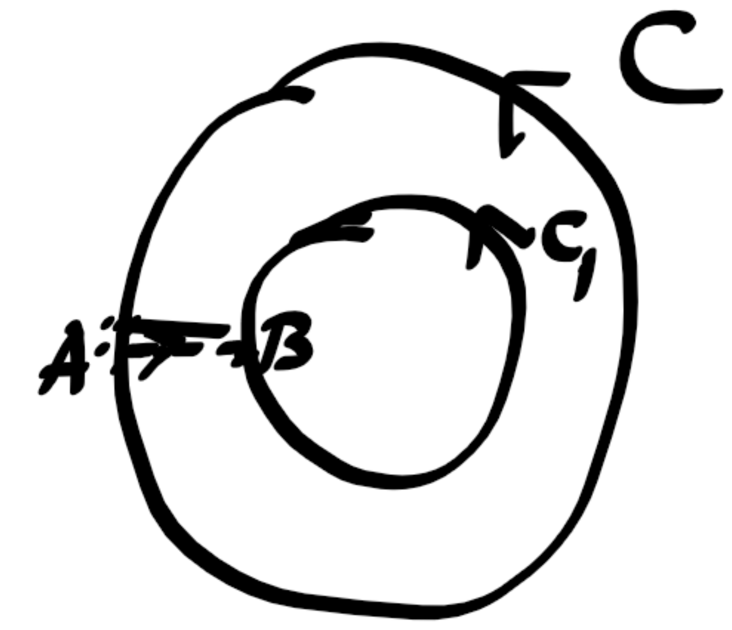
$$=i\left(1+i-\tfrac{1}{3}\right)$$

$$=-1+\frac{2}{3}i$$

Cauchy's theorem for multiply connected Domains.

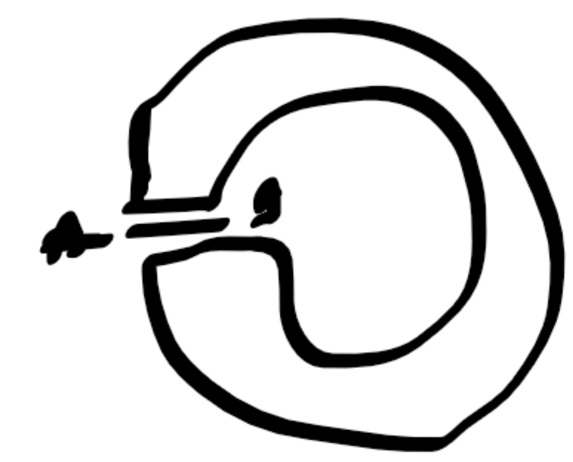
If $f(z)$ is analytic in the domain D between two simple closed curves C and C_1 then

$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz$$



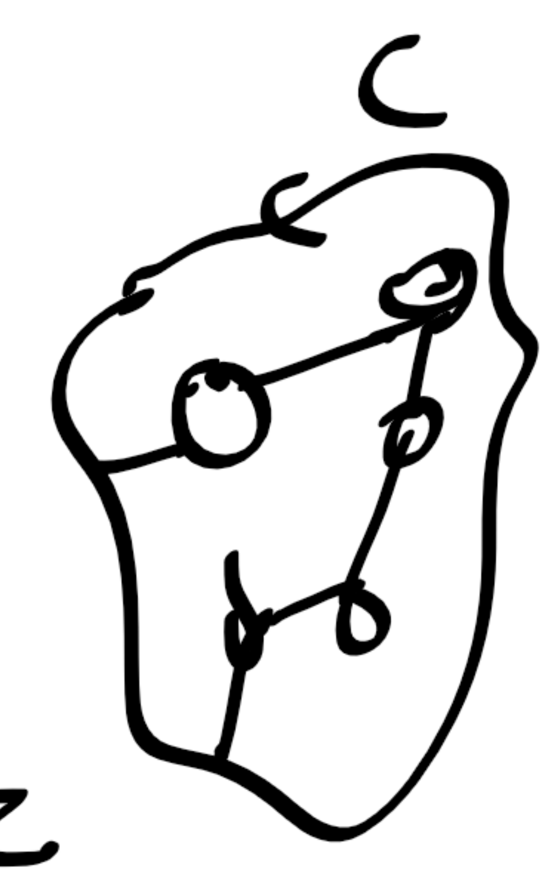
since $f(z)$ is analytic

$$\int_{ABC_1BAC} f(z)\,dz = 0$$



$$\cancel{\int_{AB} f(z)\,dz} + \int_{C_1} f(z)\,dz + \cancel{\int_{BA} f(z)\,dz} + \int_{C} f(z)\,dz = 0$$

$$\int_{C} f(z)\,dz = -\int_{C_1} f(z)\,dz = \int_{C_1} f(z)\,dz.$$



Note If $C_1, C_2, C_3, \dots C_n$ are any closed curves within C, then

$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$$

Ex:- 1 $\oint_C e^z \, dz = 0$ for any simple closed curve C because e^z is analytic function.

$|z-a| = r$
$z-a = re^{i\theta}$
$|z| = 1$
$z = e^{i\theta}$
$dz = ie^{i\theta} d\theta$

Ex:- $\int_{C:|z|=1} \frac{1}{z^2} dz = \int_0^{2\pi} \frac{1}{e^{2i\theta}} i e^{i\theta} d\theta$

$$= i\int_0^{2\pi} e^{-i\theta} d\theta = i \cdot \left. \frac{e^{-i\theta}}{-i} \right]_0^{2\pi}$$

$$= -\left[e^{-2\pi i} - 1 \right]$$

$$= -\left[\cos 2\pi - i \sin 2\pi - 1 \right] = \underline{\underline{0}}$$

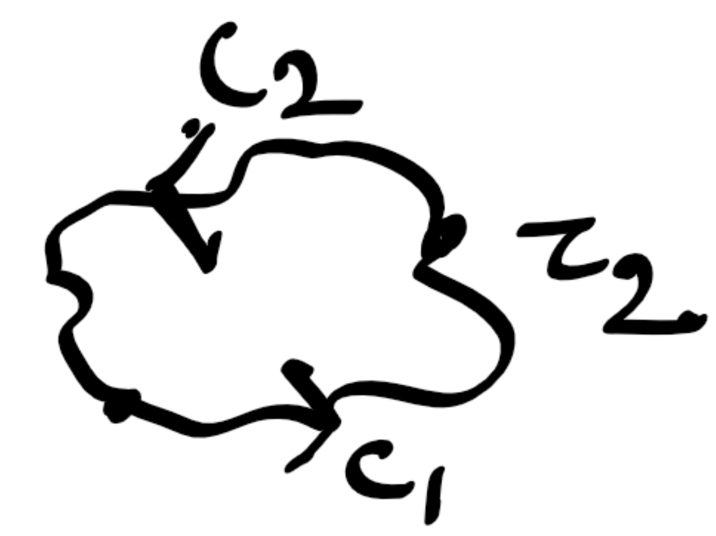
$\frac{1}{z^2}$ is not analytic at $z=0$

$\therefore \oint_C f(z) dz = 0 \not\Rightarrow f(z)$ is analytic.

Independence of Path

Let $f(z)$ be analytic in a simply connected domain D. Let C_1 and C_2 be any two paths in D joining any two points z_1 and z_2 and having no further points in common.

Then $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$



Cauchy's Integral theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)\,dz = 0$

Proof

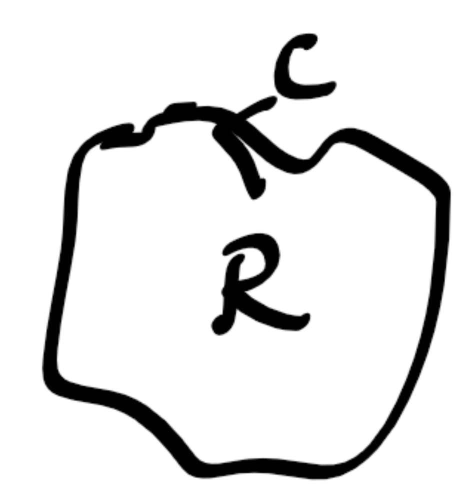
$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int u\,dy + v\,dx$$

Since $f(z)$ is analytic, u and v have continuous partial derivatives.

Apply Green's theorem in the plane.

ie, if $M(x,y)$ and $N(x,y)$ are continuous in a region R of the xy-plane bounded by a closed curve C, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$



$$\therefore \oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C u\,dy + v\,dx$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dy\,dx + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dy\,dx + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$= 0$ (using C-R eqns)