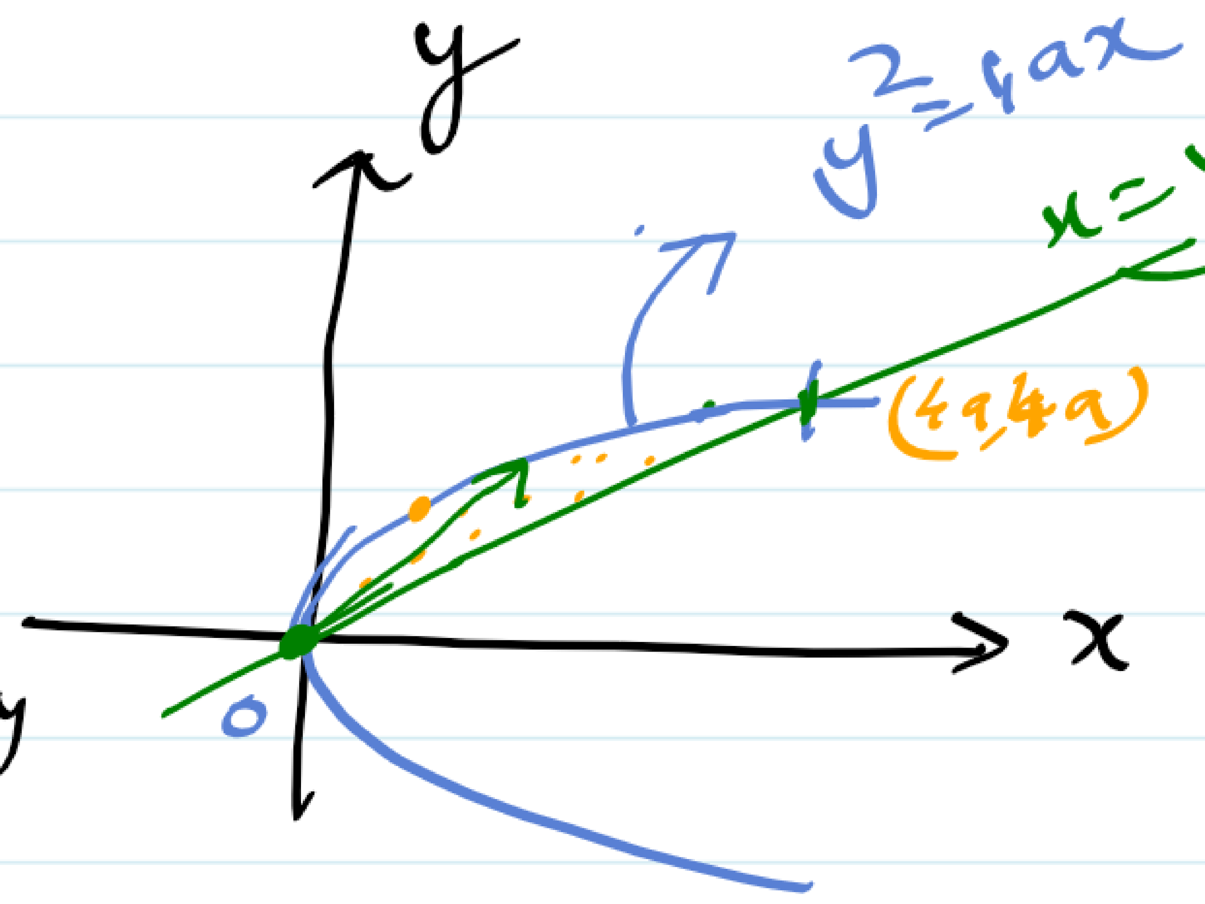
$$I = \int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy,$$



The region is bounded by $y^2 = 4ax$, $x = y$

$y = 0$, $y = 4a$

By using polara coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $J = r$

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}\, r\, dr\, d\theta$$

$y^2 = 4ax$ ↳ cartisean form

↓

$r^2\sin^2\theta = 4ar\cos\theta$

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

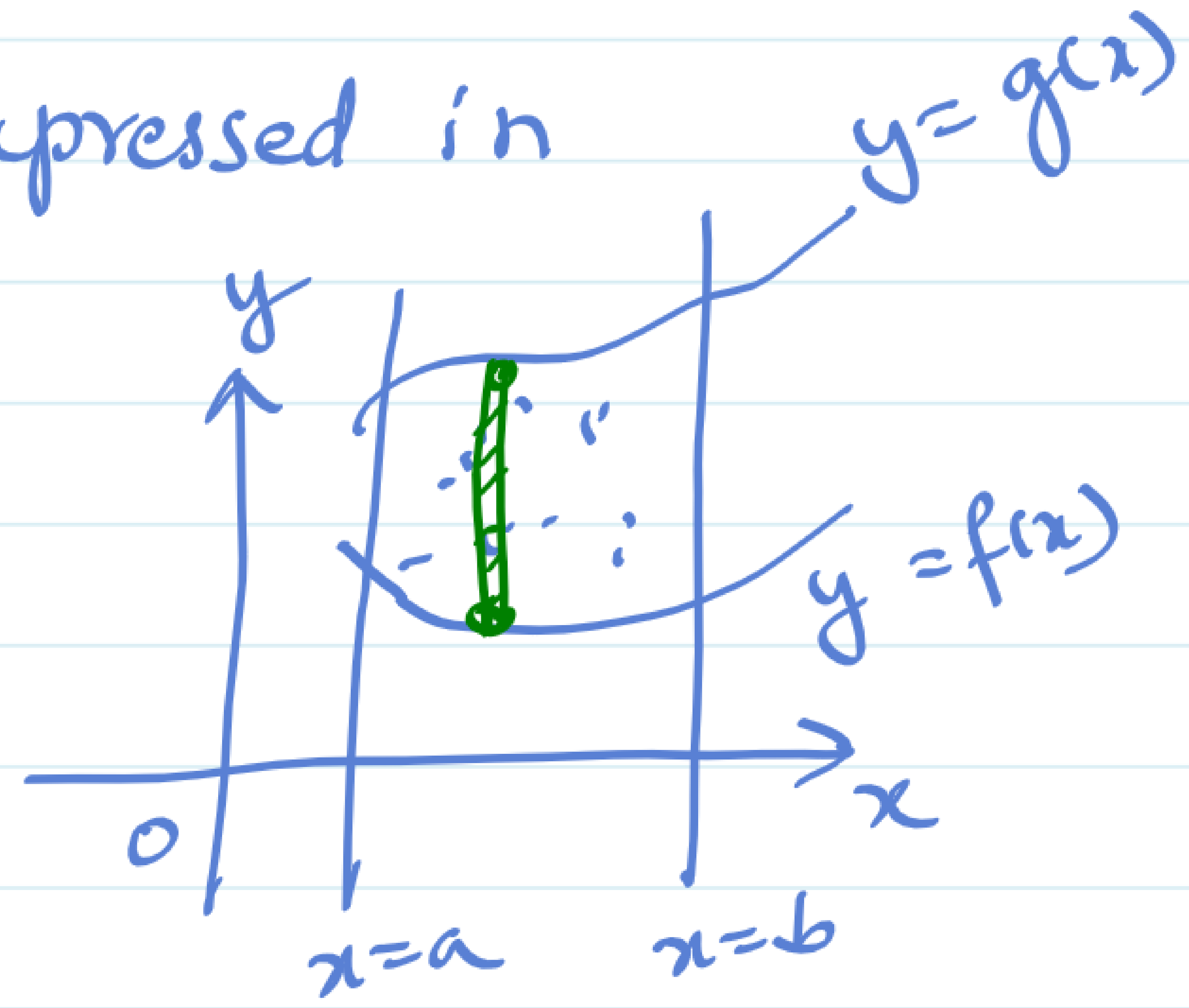
$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \left(\cos^2\theta - \sin^2\theta\right) r\, dr\, d\theta$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

Applications of Multiple Integrals –

① Representation of Area as double integral –

(i) Area enclosed by plane curves expressed in Cartesian co-ordinates :

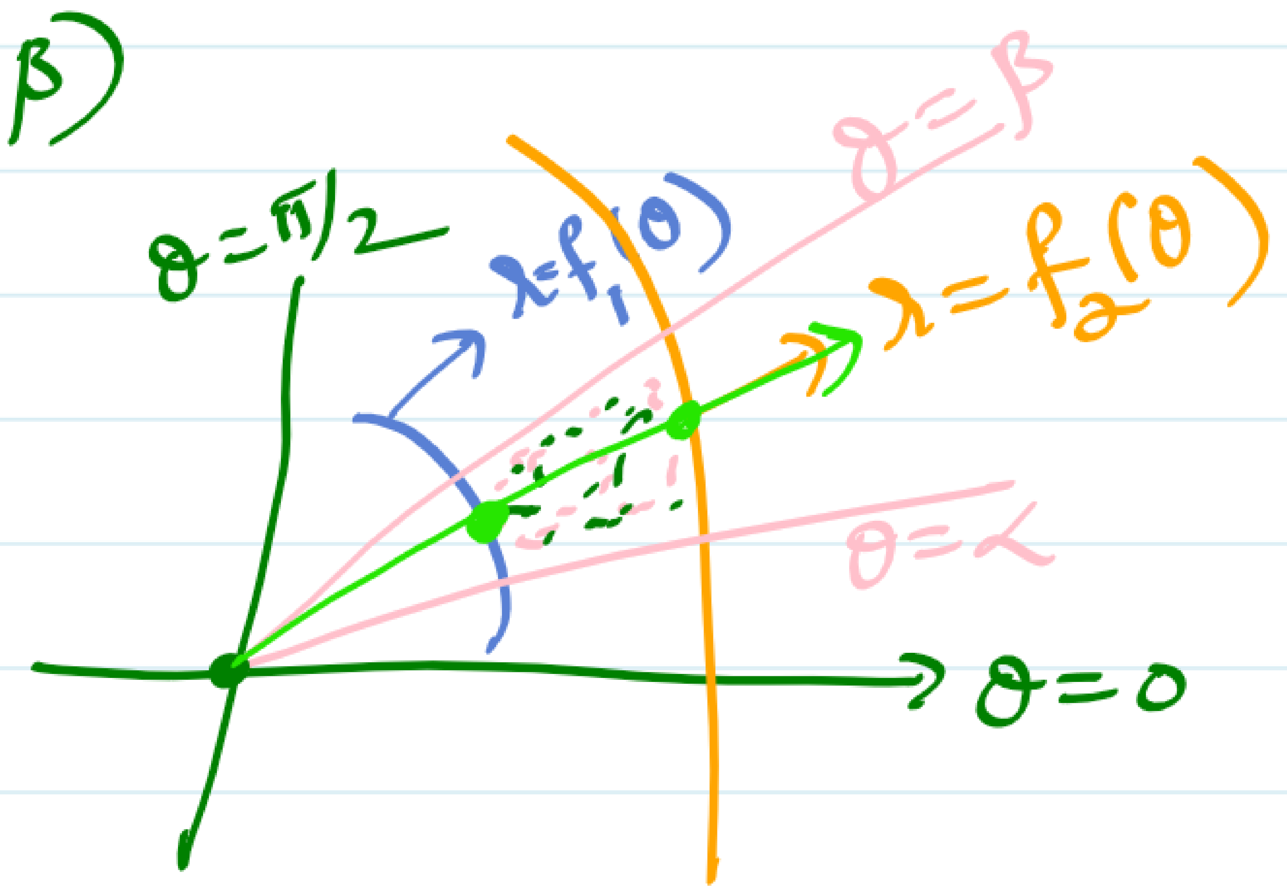


Consider the area enclosed by the curves $y = f(x)$, $y = g(x)$ and ordinates $x = a$, $x = b$.

$$\text{Area} = \int_{x=a}^{b} \int_{y=f(x)}^{g(x)} dx\,dy$$

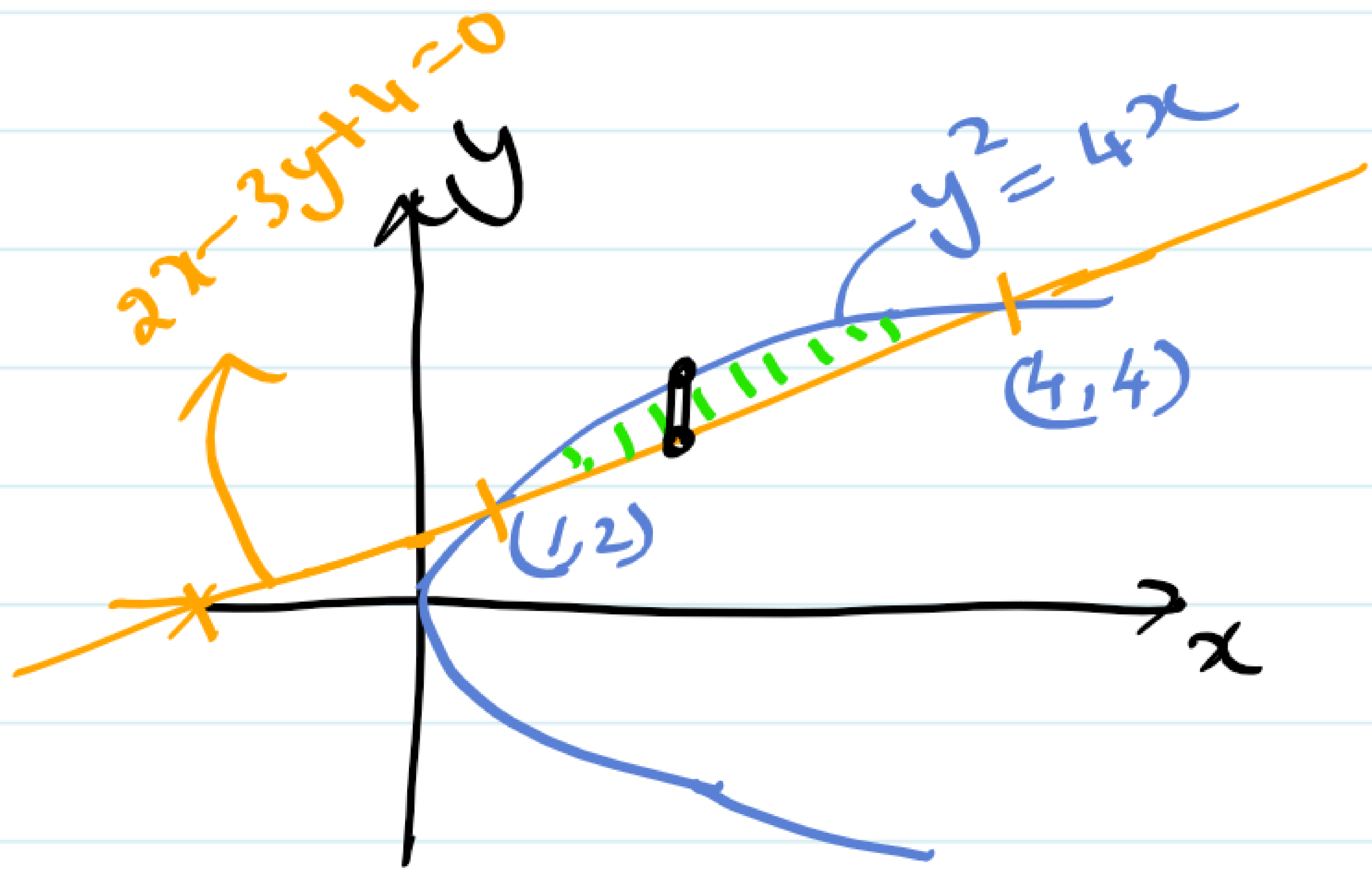
(ii) Area enclosed by plane curves expressed in polar co-ordinates –

Area enclosed by the polar curve $r = f_1(\theta)$, $r = f_2(\theta)$ and line $\theta = \alpha$, $\theta = \beta$ $(\alpha < \beta)$



$$\text{Area} = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

① Find the area between the curves $y^2=4x$ & $2x-3y+4=0$



$$\text{Area} = \iint dx\,dy.$$

$$= \int_{x=1}^{4}\int_{y=\frac{2x+4}{3}}^{2\sqrt{x}} dy\ dx$$

$$= \int_{1}^{4} (y)_{\frac{2x+4}{3}}^{2\sqrt{x}}\ dx = \int_{1}^{4}\left[2\sqrt{x} - \left(\frac{2x+4}{3}\right)\right] dx$$

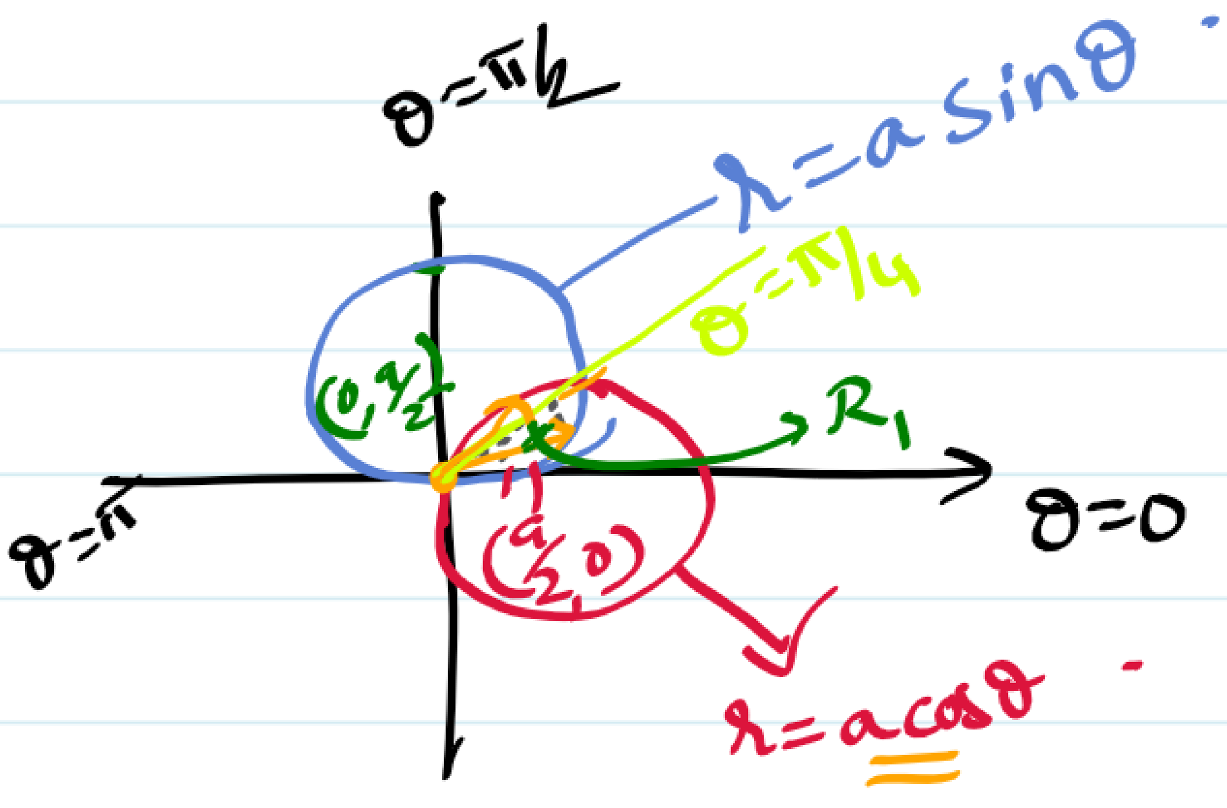
$$= \left[2\,\frac{x^{3/2}}{3/2} - \frac{2}{3}\cdot\frac{x^2}{2} + \frac{4}{3}x\right]_1^4$$

$$= \underline{\underline{1/3}}$$

$\boxed{\text{OR}}$

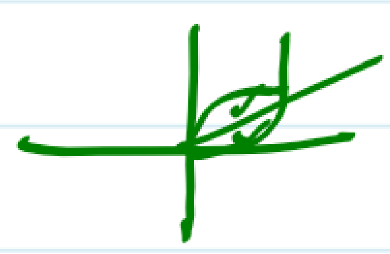
$$A = \int_{y=2}^{4}\int_{x=y^2/4}^{\frac{3y-4}{2}} dx\,dy = \underline{\underline{1/3}}$$

② Find the common area between the curves $r = a\sin\theta$, $r = a\cos\theta$.



$$\text{Area} = \iint r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta + \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{a\cos\theta} r\,dr\,d\theta$$



$=$

OR

Area = 2 Area in R_1

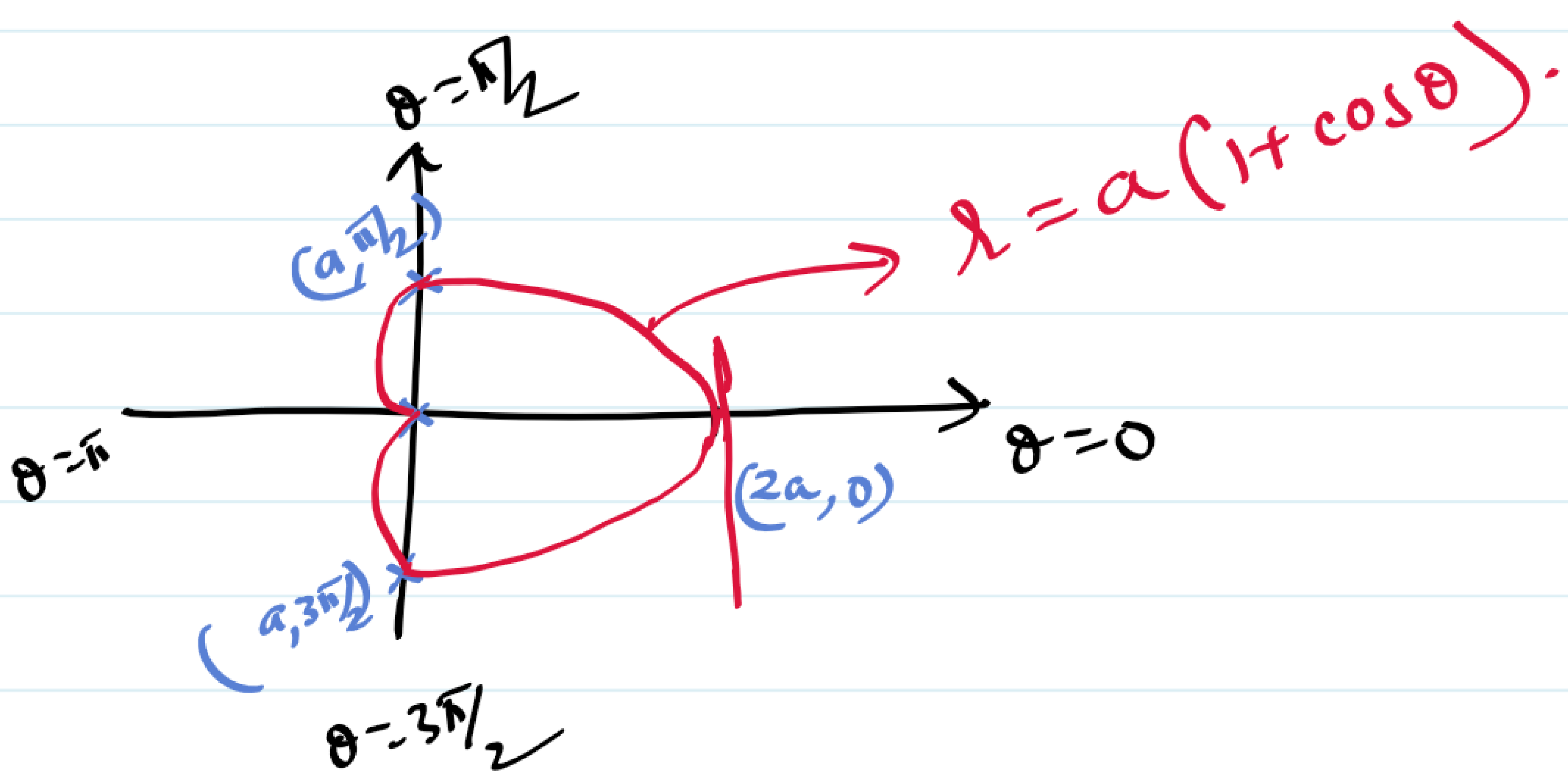
$$= 2\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sin\theta}$$

$$= \int_0^{\pi/2} a^2\sin^2\theta\,d\theta$$

$$= \left(\frac{\pi}{4}-\frac{1}{2}\right)\frac{a^2}{2}$$

③ Find the total Area bounded between the two ~~curves~~ cardiode's $r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$

cardioide : $r = a(1+\cos\theta)$



$\theta = 0$, $r = 2a$

$\theta = \pi/4$ $r = (1+1/\sqrt{2})a$

$\theta = \pi/3$ $r = 3/2\, a$

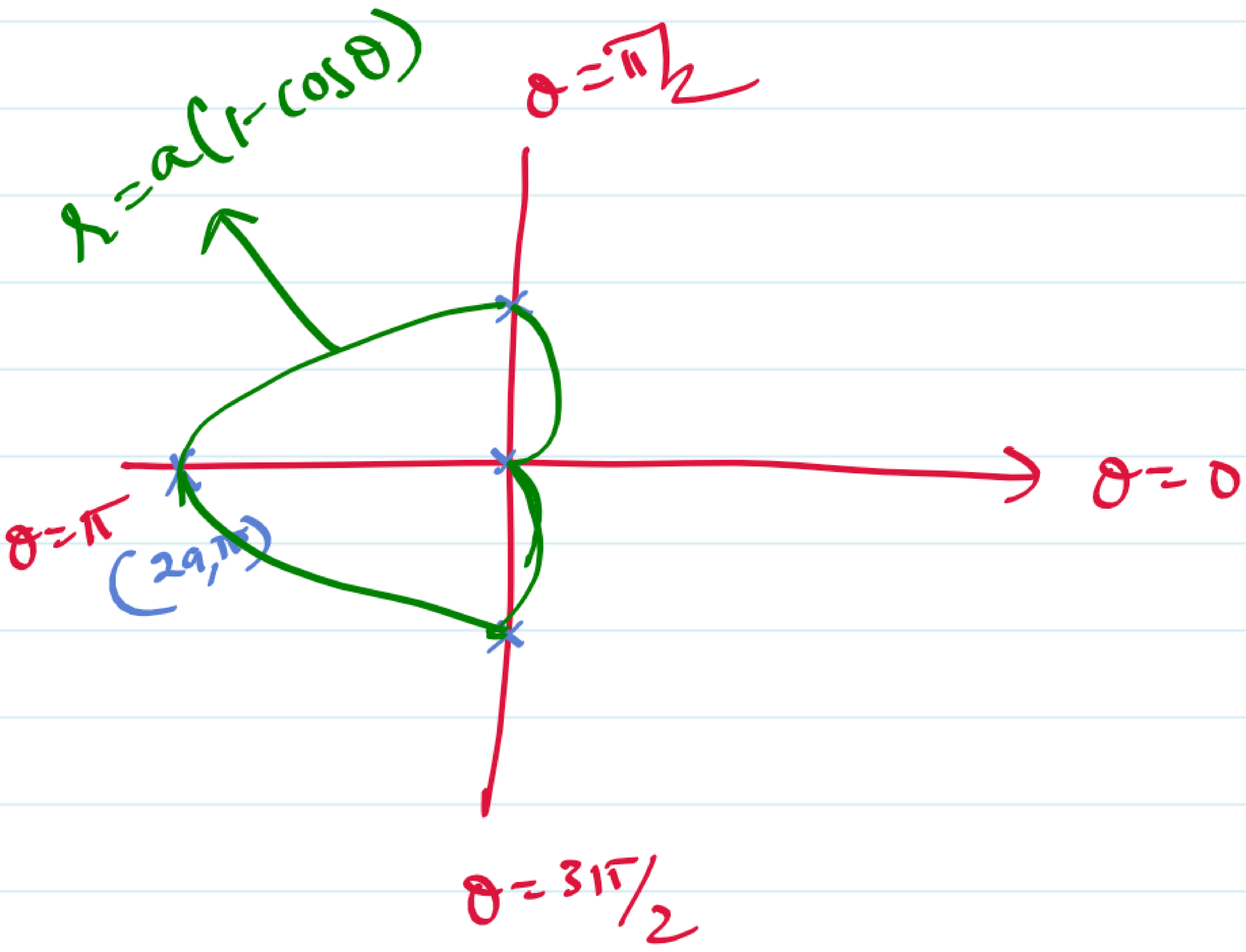
$\theta = \pi/2$, $r = a$

$\theta = \pi$ $r = 0$

$\theta = 3\pi/2$ $r = a$

$\theta = 2\pi$, $r = 2a$

$r = a(1-\cos\theta)$



$\theta = 0 \quad r = 0$

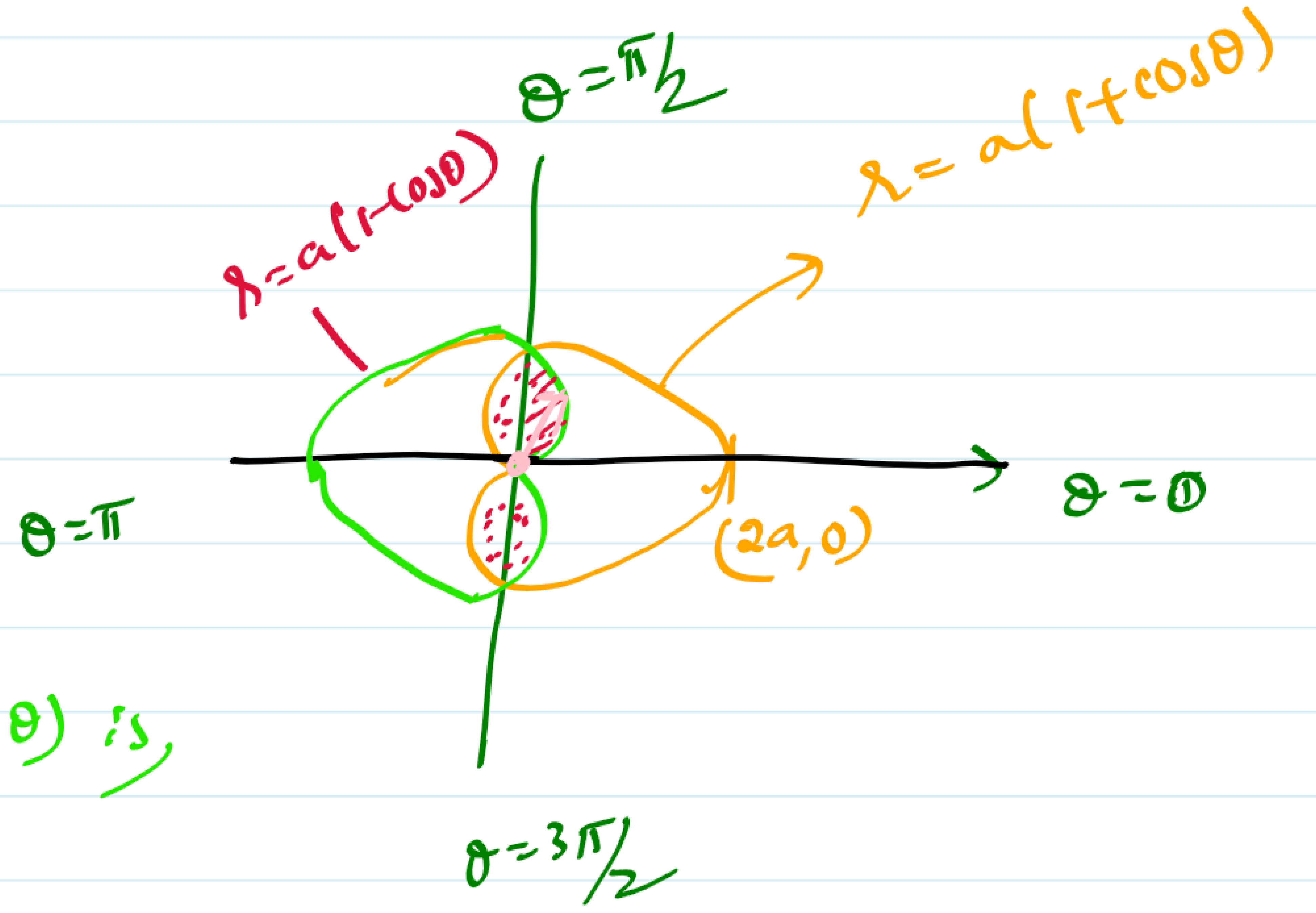
$\theta = \pi/2 \quad r = a$

$\theta = \pi \quad r = 2a$

$\theta = \frac{3\pi}{2} \quad r = a$

$r = a(1+\sin\theta) \qquad r = a(1-\sin\theta)$

Ans:



Area bounded between

$r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ is,

Area = 4 Area in 1st Q.

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

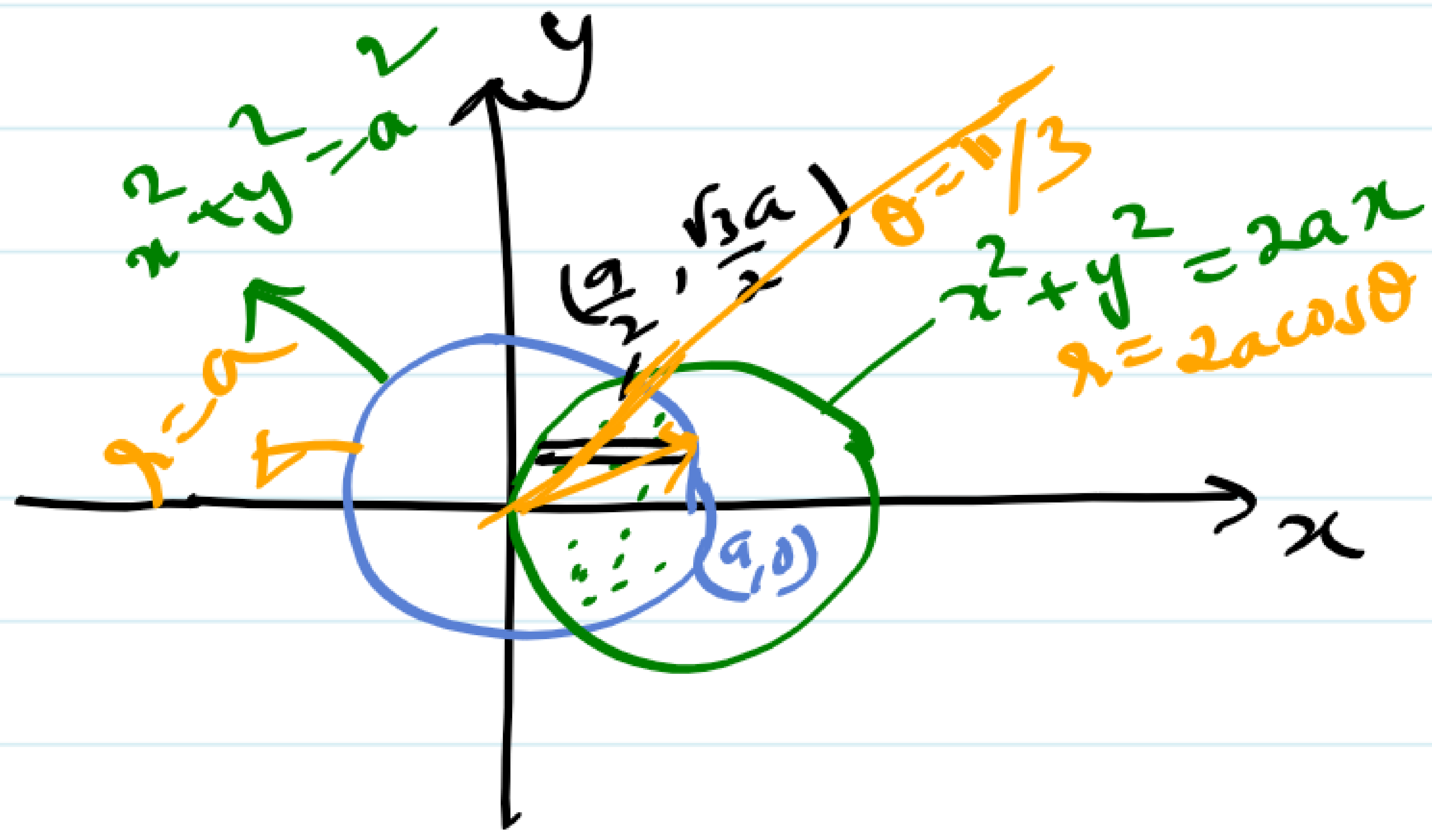
$$= 4\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta \qquad = 4\int_0^{\pi/2}\frac{1}{2}(a^2)(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\int_0^{\pi/2} 1+\cos^2\theta - 2\cos\theta \; d\theta$$

$$= 2a^2\left(\frac{3\pi}{4} - 2\right)$$

4) Find the area common to the circles $x^2+y^2=a^2$ and $x^2+y^2=2ax$

$$(x-a)^2+y^2=a^2 \Rightarrow x-a=\sqrt{a^2-y^2}$$
$$x=a+\sqrt{a^2-y^2}$$



Required Area = Area in the upper half of the plane

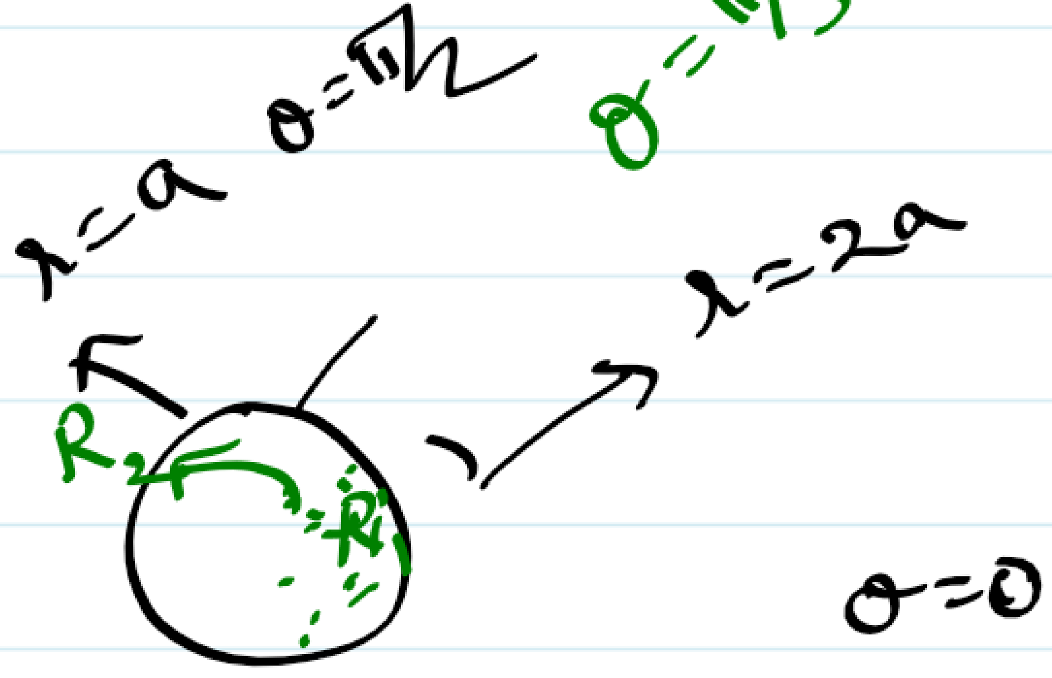
$$= 2\int_{y=0}^{\sqrt{3}a/2}\int_{a+\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dx\, dy.$$

Eqn of both the circles in polar co-ordinates

$r=a$, $r=2a\cos\theta$

$a=2a\cos\theta$

$\boxed{\theta=\pi/3}$



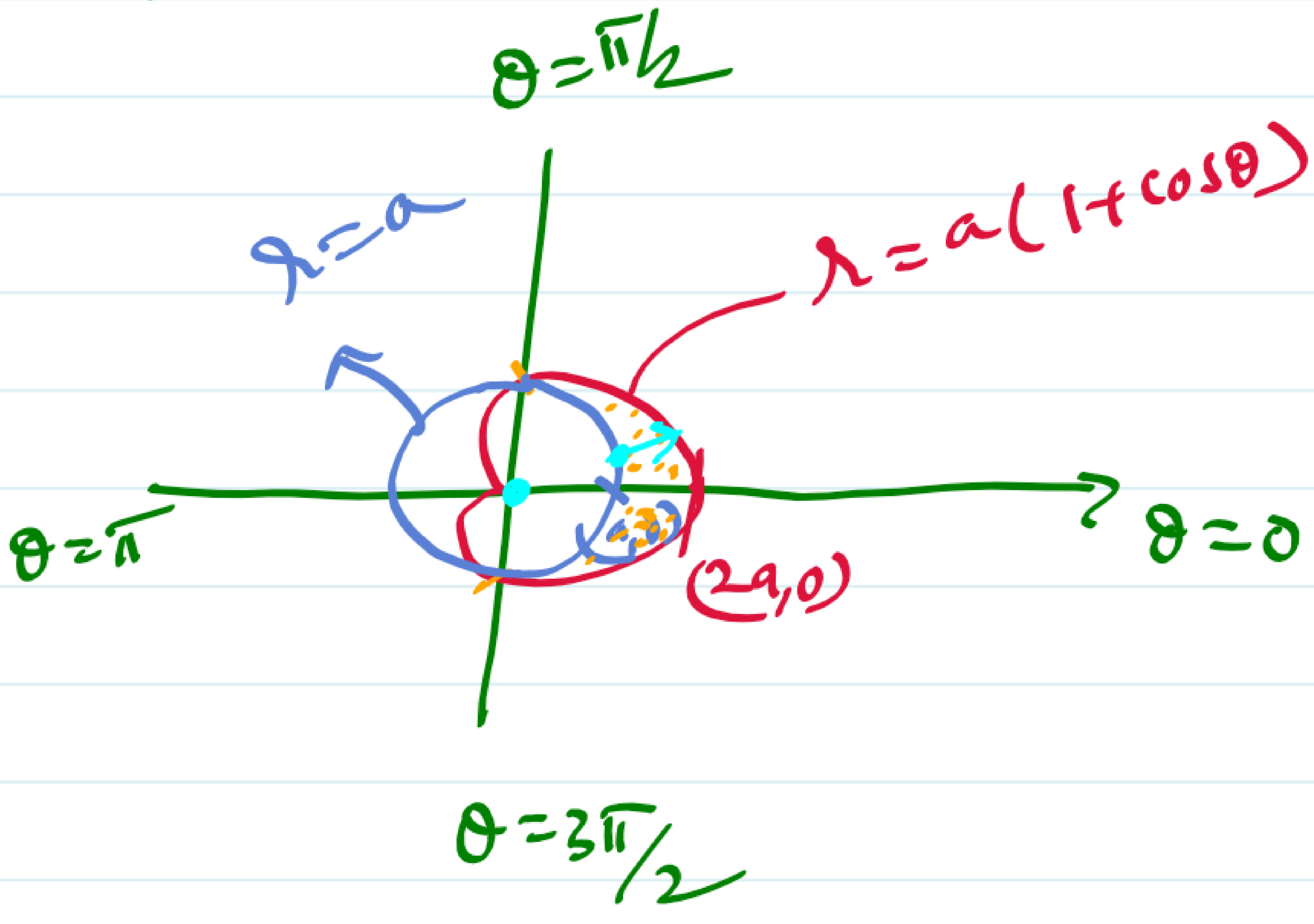
$$\text{Area} = 2\left\{\iint_{R_1} r\,dr\,d\theta + \iint_{R_2} r\,dr\,d\theta\right\}$$

$$=2\left\{\int_{\theta=0}^{\pi/3}\int_{r=0}^{a} r\,dr\,d\theta + \int_{\theta=\pi/3}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\,dr\,d\theta\right\}$$

$$=2\int_0^{\pi/3}\left.\frac{r^2}{2}\right)_0^{\pi/3} d\theta + 2\int_{\pi/3}^{\pi/2}\left.\frac{r^2}{2}\right)_0^{2a\cos\theta} d\theta.$$

$$=\frac{2\pi a^2}{3} + 4a^2\int_{\pi/3}^{\pi/2}\cos^2\theta\, d\theta = \frac{2\pi a^2}{3}+4a^2\int_{\pi/3}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta.$$

6) Find the area lying inside the cardiod $r = a(1+\cos\theta)$ and outside the circle $r = a$.



$A = 2$ Area in I^{st} Q.

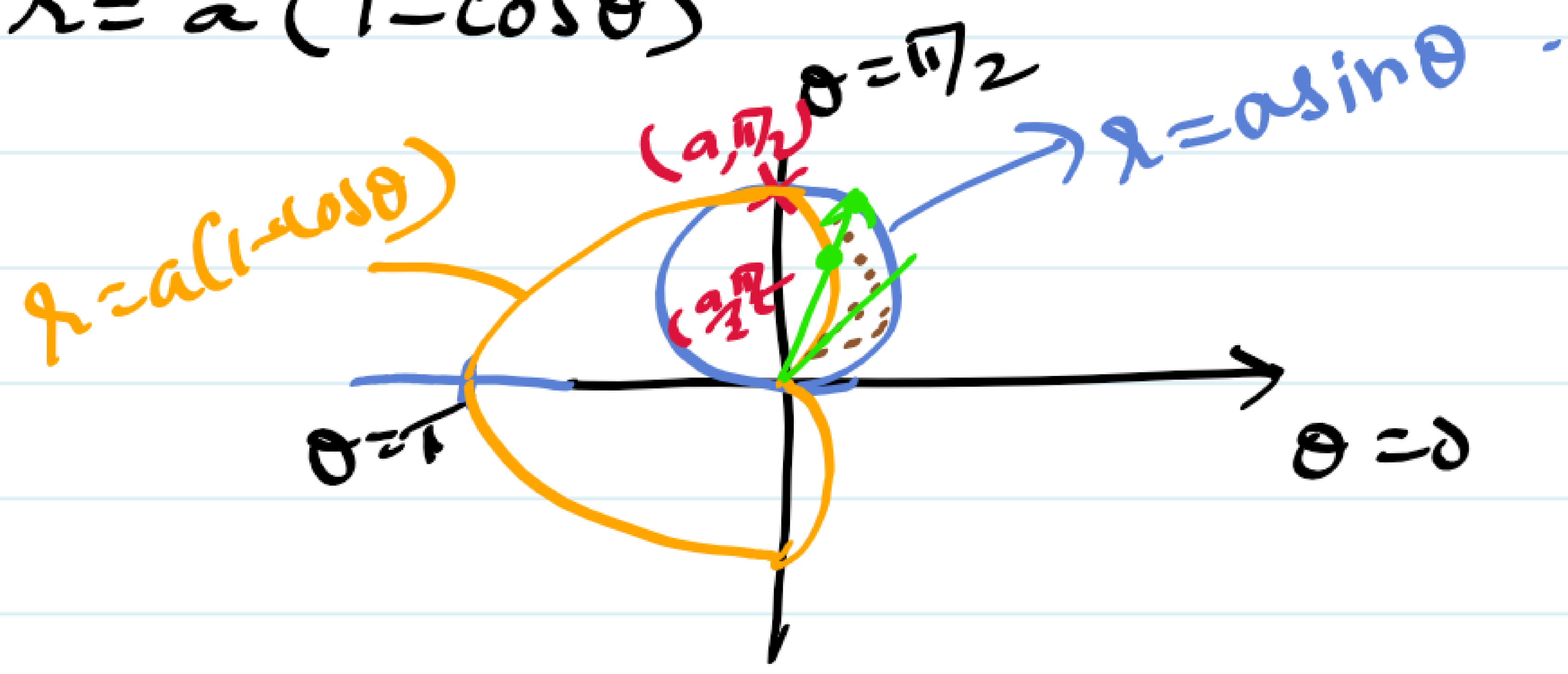
$$= 2\int_{\theta=0}^{\pi/2}\int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= \cancel{2}\int_0^{\pi/2} \frac{a^2(1+\cos^2\theta + 2\cos\theta) - a^2}{\cancel{2}}\,d\theta.$$

$$= a^2\left(\frac{\pi}{4}+2\right)$$

7) Find the area lying inside the circle $r = a\sin\theta$ and outside the cardiod $r = a(1-\cos\theta)$

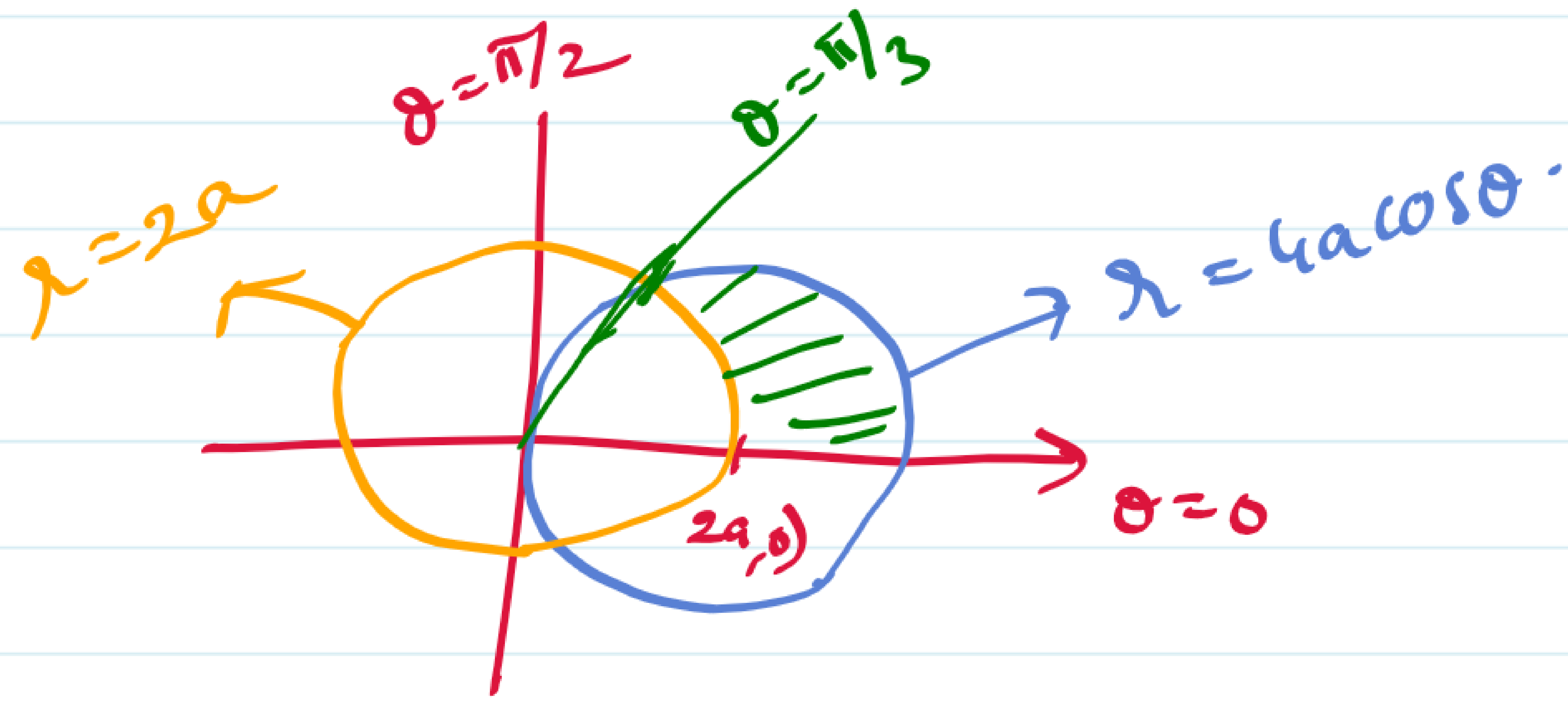


$$\text{Area} = \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= a^2\left(1-\frac{\pi}{4}\right) \text{ sq. units.}$$

8) Find the area of the plane region in the first quadrant which is inside the circle $r = 4a\cos\theta$ and outside the circle $r = 2a$.

$$\text{Area} = \int_{\theta=0}^{\pi/3}\int_{r=2a}^{4a\cos\theta} r\,dr\,d\theta$$



$$= \int_0^{\pi/3} \left[\frac{r^2}{2}\right]_{2a}^{4a\cos\theta} d\theta = \int_0^{\pi/3} \frac{1}{2}\left(16a^2\cos^2\theta - 4a^2\right) d\theta$$

$r = 2a \quad r = 4a\cos\theta$

$2a = 4a\cos\theta$

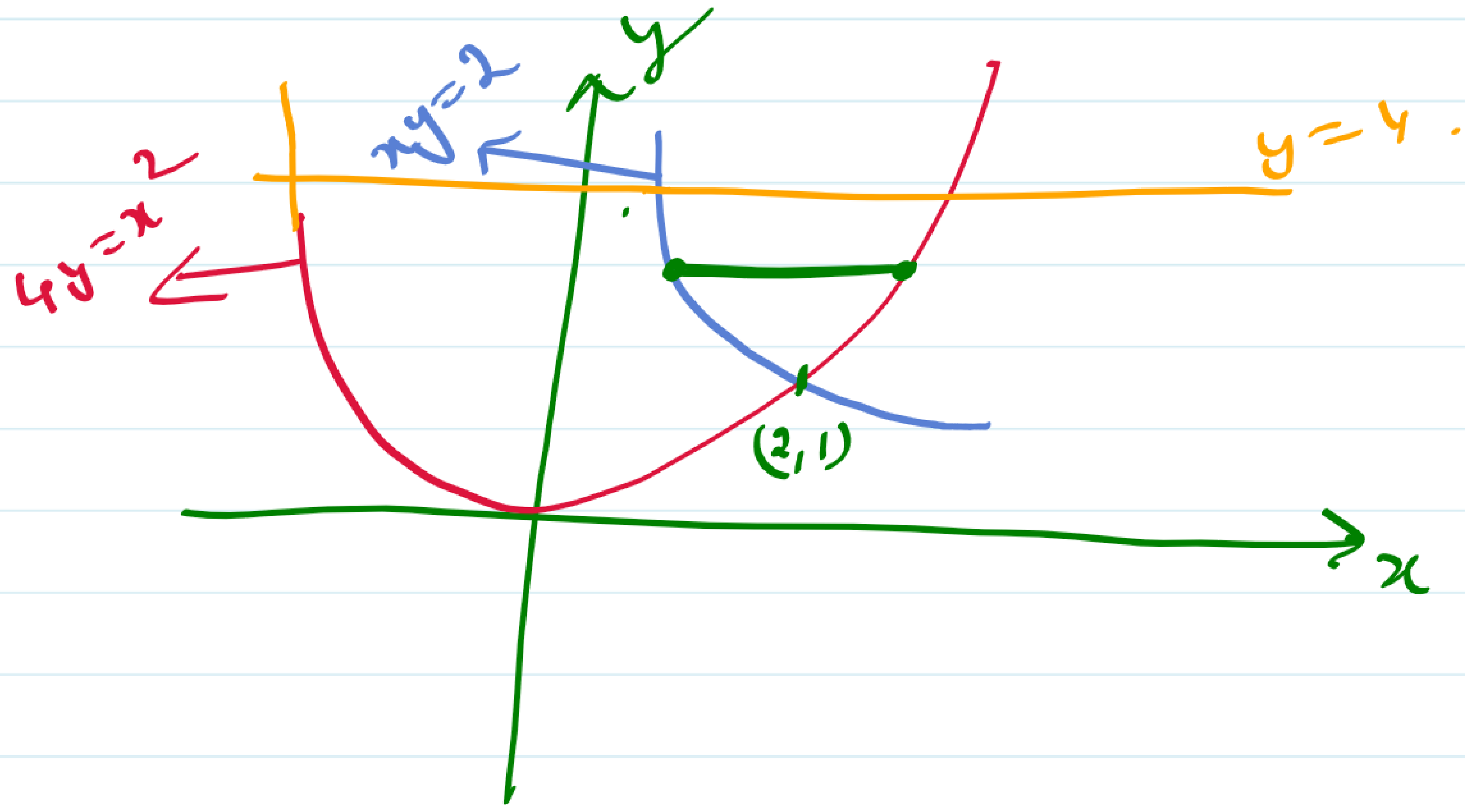
$\cos\theta = 1/2$

$\theta = \pi/3$

$$= \left(\frac{2\pi + 3\sqrt{3}}{3}\right) a^2$$

8) Find the area bounded by $xy = 2$, $4y = x^2$, $y = 4$

$$A = \int_{y=1}^{4}\int_{x=2/y}^{2\sqrt{y}} dx\,dy$$



$$= \frac{28}{3} - 2\ln 4$$

Practice questions –

① Find the area between the curve $y+8 = x^2 - 2x$ and x-axis.

(Ans: 36)

② Find the area between the curves $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and $x + y = a$

(Ans: $a^2/3$)

③ Find the area between the curves $y = \sin x$, $y = \cos x$, $x = 0$ in first quandrant.

(Ans: $\sqrt{2} - 1$)

(4) Find the common area bounded by $r = 3a/2$ and $r = a(1+\cos\theta)$

Ans: $\left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right)a^2$