CHAPTER SIX

6 LINEAR SYSTEM DESIGN

6.1 INTRODUCTION TO DESIGN USING COMPENSATORS

The control systems are designed to perform specific tasks. The requirements of a control syst are usually specified as performance specifications. The specifications are generally related to accura relative stability and speed of response.

In time domain, the transient state performance specifications are given in terms of rise tin maximum overshoot, settling time and/or damping ratio. In frequency domain, the transient sta performance specifications are given in terms of phase margin, gain margin, resonant peak and/ bandwidth. The steady state requirement are given in terms of error constants.

When performance specifications are given for single input, single output linear time invaria systems, then the system can be designed by using root locus or frequency response plots. To be mor precise, when time domain specifications are given, root locus technique is employed in designing th system. If frequency domain specifications are given, frequency response plots like Bode plots are use in designing the system.

The first step in design is the adjustment of gain to meet the desired specifications. In practica systems, adjustment of gain alone will not be sufficient to meet the given specifications. In many cases increasing the gain may result in poor stability or instability. In such cases, it is necessary to introduc additional devices or components in the system to alter the behaviour and to meet the desired specifications

Such a redesign or addition of a suitable device is called compensation. A device inserted into th system for the purpose of satisfying the specifications is called compensator. The compensators basicall introduce pole and/or zero in open loop transfer function to modify the performance of the system.

The design problem may be stated as follows : *When a set of specifications are given for a system, then a suitable compensator should be designed so that the overall system will meet the given specification.*

The compensation schemes used for feedback control system is either series compensation or parallel (feedback) compensation.

In series compensation, the compensator with transfer function, $G_c(s)$ is placed in series with plant. In feedback compensation, the signal from some element is feedback to the input and a compensator with transfer function $G_c(s)$ is placed in the resulting inner feedback path. The series and parallel compensation schemes are shown in fig 6.1 and 6.2 respectively.

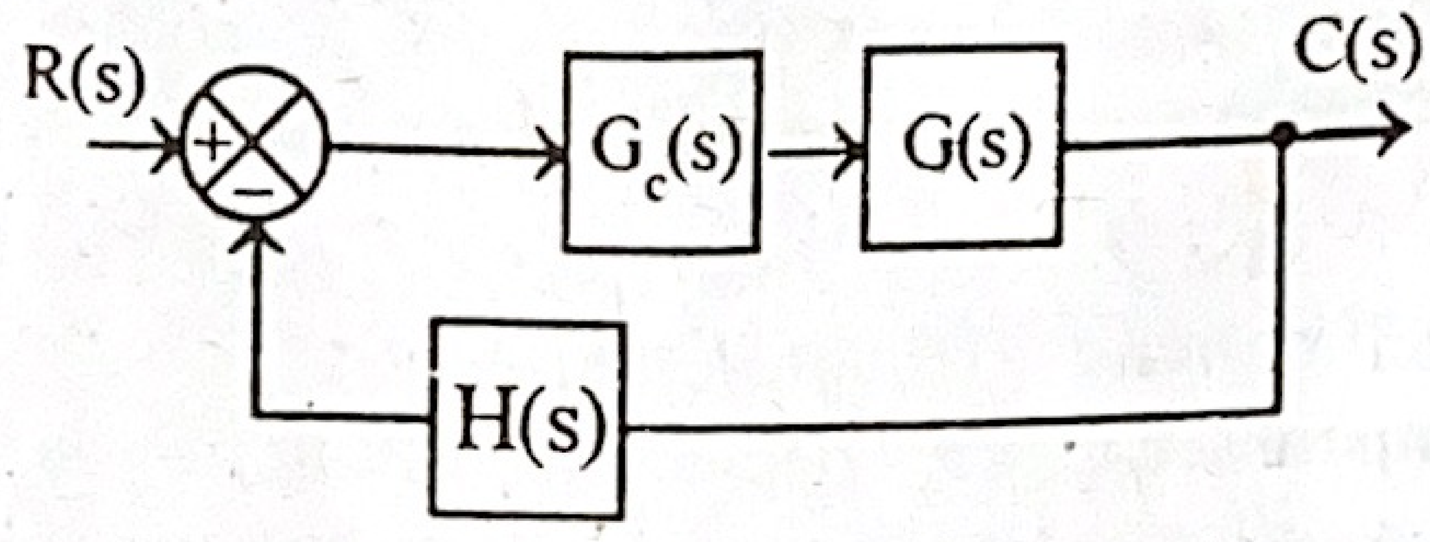


Fig 6.1 : Series compensation.

The choice between series compensation and parallel compensation depends on

1. *Nature of signals in the system.*
2. *Power levels at various points.*
3. *Components available.*
4. *Designer's experience..*
5. *Economic considerations and so on.*

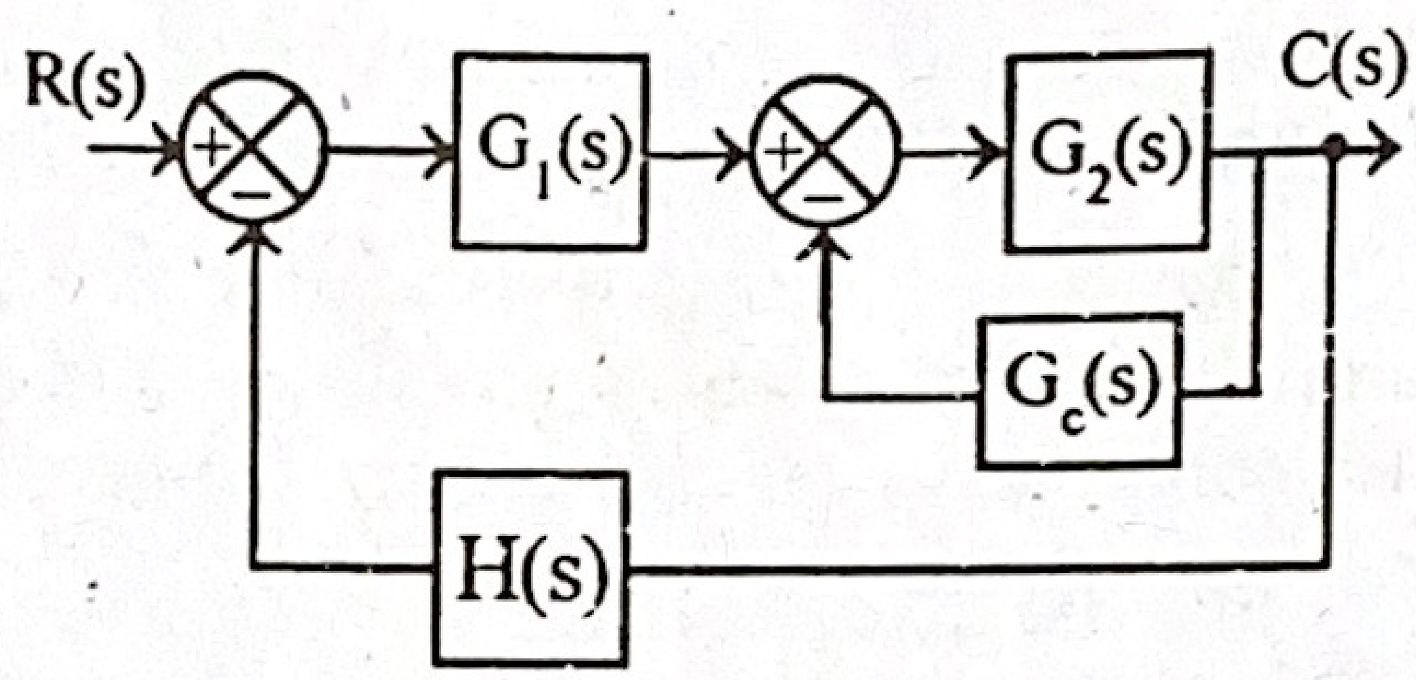


Fig 6.2 : Feedback/parallel compensation.

The compensator may be electrical, mechanical, hydraulic, pneumatic or other type of device or network. Usually, an electric network or electronic device serves as compensator in many control systems. The different types of electrical or electronic compensators used are ***Lag compensator, Lead compensator and Lag-lead compensator.***

In control systems, compensation is required in the following situations,

1. *When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.*
2. *When the system is stable, compensation is provided to obtain the desired performance.*

The systems with type number 2 and above are usually absolutely unstable systems. Hence for systems with type number 2 and above, lead compensation is required, because the lead compensator increases the margin of stability.

In systems with type number 1 or 0, stability is achieved by adjusting the gain. In such cases any of the three compensators-lag, lead and lag-lead may be used to obtain the desired performance. The particular choice of compensator is based on the factors that are discussed in the following sections.

ROOT LOCUS APPROACH TO CONTROL SYSTEM DESIGN

In design using root locus, the desired behaviour is specified in terms of transient response specifications and steady - state error requirement. The steady - state error is usually specified in terms of error constants for standard inputs, while the transient response requirement is specified in terms of peak overshoot, settling time, rise time, etc., for a step input. The transient response specifications can be translated into desired locations for a pair of dominant closed loop poles.

In order to meet the desired specifications, the root loci are reshaped so that they pass through the points where the dominant closed loop poles are located. The root loci are reshaped by introducing a compensator. The compensator will add a pole and/or a zero in the open loop transfer function of the system.

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, which reduce the relative stability of the system and increase the settling time. The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left which make the system more stable and reduce the settling time.

When a system is either unstable or stable but has undesirable transient response characteristics a lead compensator can be employed to modify the root locus. The transfer function of lead compensator will have a zero (compensating zero) and a pole (compensating pole).

The compensator zero can be placed on the real axis by trial-and-error to satisfy transient response specifications. The introduction of zero will amplify high frequency noise which is eliminated by the compensating pole. The compensating pole is located on real axis such that it makes negligible effect on the root locus in the region where the two dominant closed loop poles are located.

If the pole is located far away from zero then it will not be effective in suppressing the noise. If the pole is too close to zero then it will not allow the zero to do its job. In order to avoid this conflict, the pole is located at 3 to 10 times the value of zero location.

The lag compensator is employed when a stable system has satisfactory transient response characteristics but unsatisfactory steady state characteristics, i.e., error requirement. The transfer function of lag compensator will have a zero (compensating zero) and a pole (compensating pole).

In order to preserve the transient response characteristics the compensating pole and zero should have negligible effect on shape of root locus. This is achieved by placing the compensating pole and zero very close to each other. If the pole and zero are located close to the origin then the error constant will increase which will reduce the steady state error.

The lag-lead compensator is employed when both the transient and steady-state characteristics are not satisfactory. The lead compensation will improve the transient response and lag compensation will reduce the steady state error.

The advantage in design using root locus technique is that the information about closed loop transient response and frequency response are directly obtained from the pole-zero configuration of the system in the s-plane.

FREQUENCY RESPONSE APPROACH TO CONTROL SYSTEM DESIGN

The objective of frequency domain design is to reshape the frequency-response characteristics so that the desired specifications are satisfied. The frequency domain design can be carried out using Nyquist plot, Bode plot or Nichols chart. But Bode plot are popularly used for design because they are easier to draw and modify.

In design using bode plots the desired performance specifications are given in terms of frequency domain specifications and steady-state error requirement.

The stability requirement is specified in terms of phase margin and resonant peak. The transient response requirements are specified in terms of gain crossover frequency, bandwidth and resonant frequency. The error requirement is specified in terms of static error constants (K_p, K_v or K_a).

Note : *In case the transient response specifications are given in time domain, we can translate them into frequency domain specifications using the following formulae.*

Phase margin, $\gamma = \tan^{-1} \dfrac{2\zeta}{\left[\sqrt{4\zeta^4+1}-2\zeta^2\right]^{\frac{1}{2}}} \approx 100\zeta$

Resonant peak, $M_r = 1\Big/\left[2\zeta\sqrt{1-\zeta^2}\right]$

Gain crossover frequency, $\omega_{gc} = \omega_n \left[\sqrt{4\zeta^4+1}-2\zeta^2\right]^{\frac{1}{2}}$

Resonant frequency, $\omega_r = \omega_n \sqrt{1-2\zeta^2}$

Bandwidth, $\omega_b = \omega_n \left[(1-2\zeta^2) + \sqrt{4\zeta^4-4\zeta^2+2}\right]^{\frac{1}{2}}$

The low frequency region of bode plot provides information regarding the steady state performance and high frequency region provides information regarding the transient-state performance. The medium frequency (or mid-frequency) range provides information regarding relative stability. Therefore, the low frequency region of the bode plot is reshaped by lag compensation to improve steady state performance. The high frequency region of the bode plot is reshaped by lead compensation to improve transient-state performance.

When the system requires improvement in both steady-state and transient state, lag-lead compensation can be employed to alter both the low and high frequency regions of bode plot.

The primary function of lead compensator is to reshape the frequency response curve to provide sufficient phase lead angle to offset the excessive phase lag associated with the components of the plant. The primary function of lag compensator is to provide attenuation in the high frequency region to achieve sufficient phase margin.

The advantage in frequency-domain design is that the effects of disturbances, sensor noise and plant uncertainties are relatively easy to visualize and assess in frequency domain. Another advantage of using frequency response is the ease with which experimental information can be used for design purposes.

A disadvantage of frequency-response design is that it gives us information on closed-loop system's transient response indirectly, while the root locus design gives this information directly.

6.2 LAG COMPENSATOR

A compensator having the characteristics of a lag network is called a lag compensator. If a sinusoidal signal is applied to a lag network, then in steady state the output will have a phase lag with respect to input.

Lag compensation results in a large improvement in steady state performance but results in slower response due to reduced bandwidth. The attenuation due to the lag compensator will shift the gain crossover frequency to a lower frequency point where the phase margin is acceptable. Thus, the lag compensator will reduce the bandwidth of the system and will result in slower transient response.

Lag compensator is essentially a low pass filter and so high frequency noise signals are attenuated. If the pole introduced by the compensator is not cancelled by a zero in the system , then lag compensator increases the order of the system by one.

S-PLANE REPRESENTATION OF LAG COMPENSATOR

The lag compensator has a pole at $s = -1/\beta T$ and a zero at $s = -1/T$. The pole-zero plot of lag compensator is shown in fig 6.3. Here, $\beta>1$, so the zero is located to the left of the pole on the negative real axis. The general form of lag compensator transfer function is given by equation (6.1).

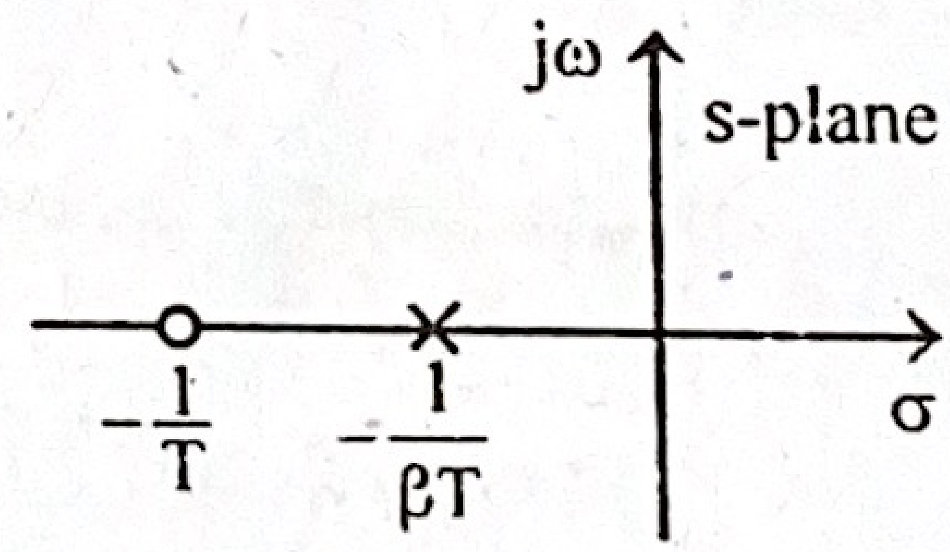


Fig 6.3 : Pole-zero plot of lag compensator.

$$\text{Transfer function of lag compensator, } G_c(s) = \frac{s+z_c}{s+p_c} = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad .....(6.1)$$

where, $T > 0$ and $\beta>1$

$$\text{The zero of lag compensator, } z_c = \frac{1}{T} \quad .....(6.2)$$

$$\text{The pole of lag compensator, } p_c = \frac{1}{\beta T} = \quad .....(6.3)$$

From equation(6.2) we get, $T = \frac{1}{z_c}$ .....(6.4)

From equation(6.3) we get, $\beta = \frac{z_c}{p_c}$ .....(6.5)

REALISATION OF LAG COMPENSATOR USING ELECTRICAL NETWORK

The lag compensator can be realised by the R-C network shown in fig 6.4.

Let, $E_i(s)$ = Input voltage

$E_o(s)$ = Output voltage

In the network shown in fig 6.4, the input voltage is applied to the series combination of R_1, R_2 and C. The output voltage is obtained across series combination of R_2 and C.

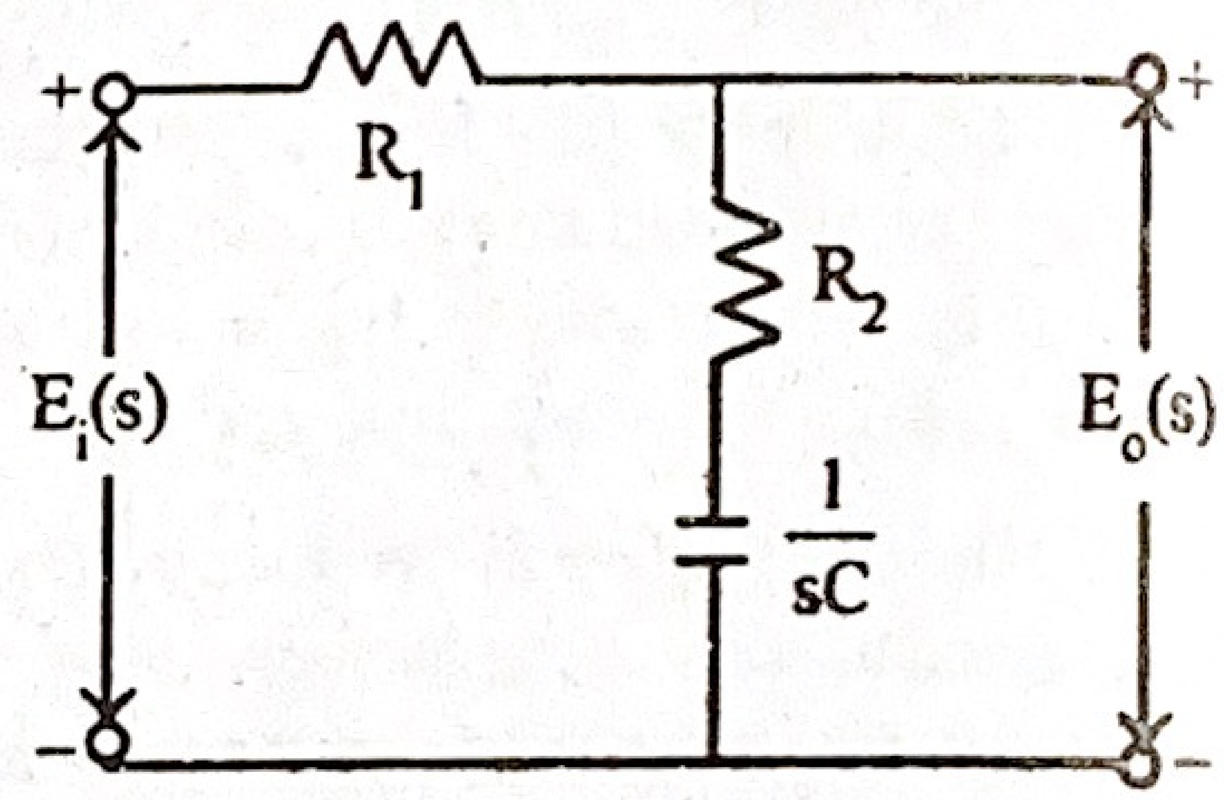


Fig 6.4 : Electrical lag compensator.

- By voltage division rule,

$$E_o(s) = E_i(s)\,\frac{(R_2 + 1/sC)}{(R_1 + R_2 + 1/sC)} = E_i(s)\,\frac{(sCR_2 + 1)/sC}{[sC(R_1 + R_2) + 1]/sC} = E_i(s)\,\frac{(sCR_2 + 1)}{[sC(R_1 + R_2) + 1]}$$

The transfer function of the electrical network is the ratio of output voltage to input voltage,

$$\left.\begin{array}{r}\text{Transfer function}\\ \text{of electrical network}\end{array}\right\}\frac{E_o(s)}{E_i(s)} = \frac{CR_2(s + 1/CR_2)}{C(R_1 + R_2)\left[s + 1/C(R_1+R_2)\right]}$$

$$= \frac{\left(s + \frac{1}{R_2C}\right)}{\left(\frac{R_1 + R_2}{R_2}\right)\left[s + \frac{1}{((R_1 + R_2)/R_2)\,R_2C}\right]} \quad .....(6.6)$$

But the transfer function of lag compensator is given by,

$$G_c(s) = \frac{(s + \frac{1}{T})}{(s + \frac{1}{\beta T})} \quad .....(6.7)$$

On comparing equations (6.6) and (6.7) we get,

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta}\,\frac{(s + \frac{1}{T})}{(s + \frac{1}{\beta T})} \quad .....(6.8)$$

where, $T = R_2C$ and $\beta = (R_1+R_2)/R_2$

The transfer function of RC network as given by equation(6.8) is similar to the general form with an attenuation of $1/\beta$ (since $\beta > 1, (1/\beta) < 1$). If the attenuation is not required then an amplifier with gain β can be connected in cascade with RC network to nullify the attenuation.

FREQUENCY RESPONSE OF LAG COMPENSATOR

Consider the general form of lag compensator,

$$G_c(s) = \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\beta T}\right)} = \frac{(sT+1)/T}{(s\beta T+1)/\beta T} = \beta\frac{(1+sT)}{(1+s\beta T)} \quad .....(6.9)$$

Sinusoidal transfer function of lag compensator is obtained by letting $s = j\omega$ in equation (6.9).

$$\therefore G_c(j\omega) = \frac{\beta(1+j\omega T)}{(1+j\omega\beta T)} \quad .....(6.10)$$

When $\omega = 0$, $G_c(j\omega) = \beta$ .....(6.11)

From equation(6.11) we can say that the lag compensator provides a dc gain of β (here $\beta > 1$). If the dc gain of the compensator is not desirable then it can be eliminated by a suitable attenuation.

Let us assume that the gain β is eliminated by a suitable attenuation network. Now, $G_c(j\omega)$ is given by,

$$G_c(j\omega) = \frac{1+j\omega T}{1+j\omega\beta T} = \frac{\sqrt{1+(\omega T)^2}\angle\tan^{-1}\omega T}{\sqrt{1+(\omega\beta T)^2}\angle\tan^{-1}\omega\beta T} \quad .....(6.12)$$

The sinusoidal transfer function shown in equation(6.12) has two corner frequencies and they are denoted as ω_{c1} and ω_{c2}.

Here, $\omega_{c1} = \frac{1}{\beta T}$ and $\omega_{c2} = \frac{1}{T}$.

Since, $\beta T > T$, $\omega_{c1} < \omega_{c2}$.

Let, $A = |G_c(j\omega)|$ in db $= 20\log\frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\omega\beta T)^2}}$ .....(6.13)

At very low frequencies i.e., upto ω_{c1}, $\omega T << 1$ and $\omega\beta T << 1$.

$$\therefore A \approx 20\log 1 = 0$$

In the frequency range from ω_{c1} to ω_{c2}, $\omega T << 1$ and $\omega\beta T >> 1$.

$$\therefore A \approx 20\log\frac{1}{\sqrt{(\omega\beta T)^2}} = 20\log\frac{1}{\omega\beta T}$$

At very high frequencies i.e., after ω_{c2}, $\omega T >> 1$ and $\omega\beta T >> 1$.

$$\therefore A \approx 20\log\frac{\sqrt{(\omega T)^2}}{\sqrt{(\omega\beta T)^2}} = 20\log\frac{1}{\beta}$$

The approximate magnitude plot of lag compensator is shown in fig 6.5. The magnitude plot of Bode plot of $G_c(j\omega)$ is a straight line through 0 db upto ω_{c1}, then it has a slope of –20 db/dec upto ω_{c2} and after ω_{c2} it is a straight line with a constant gain of 20 log $(1/\beta)$.

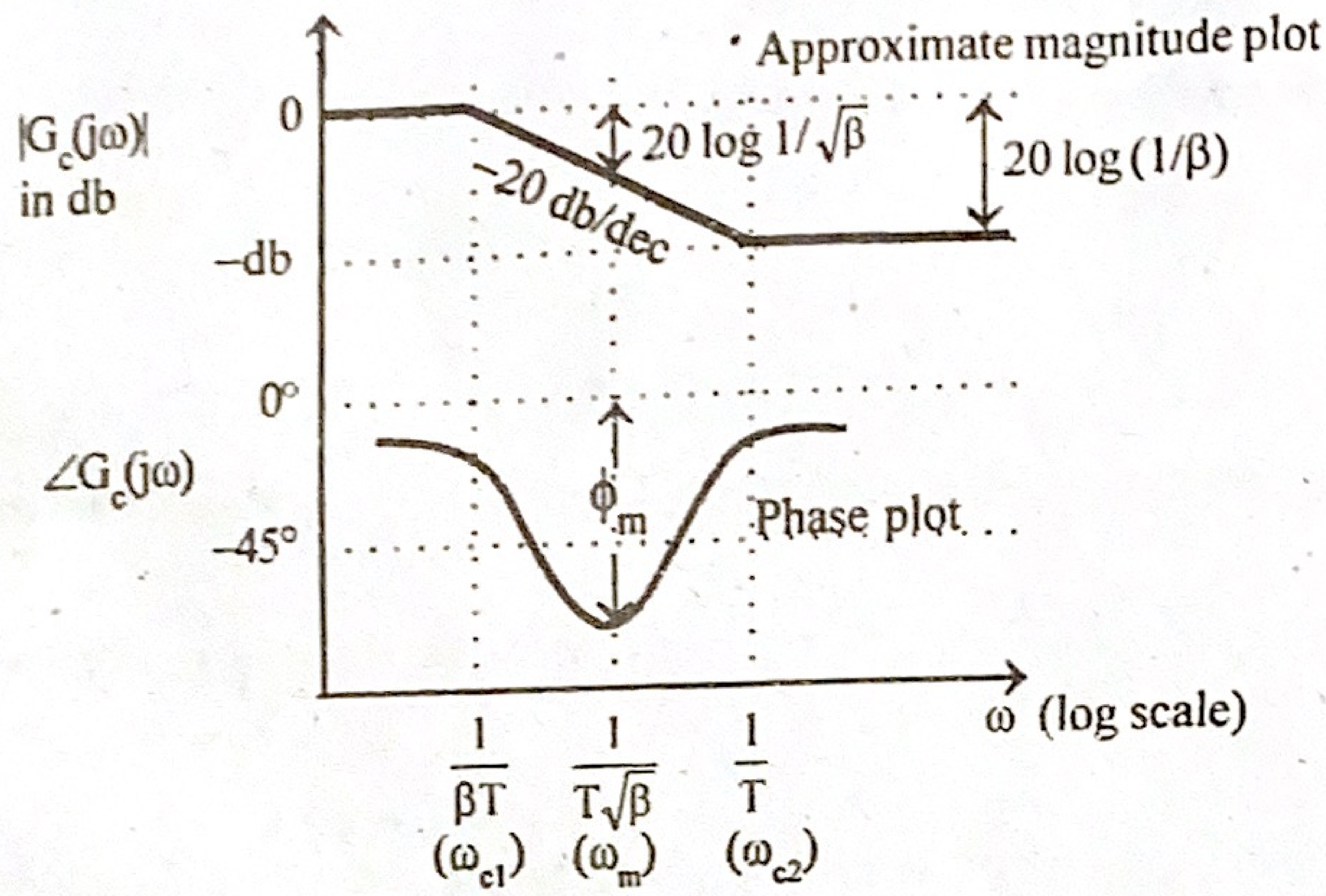


Fig 6.5 : Bode plot of lag compensator.

Let, $\phi = \angle G_c(j\omega)$

$\therefore \phi = \tan^{-1}\omega T - \tan^{-1}\omega\beta T$

As $\omega \rightarrow 0,\ \phi \rightarrow 0$

As $\omega \rightarrow \infty,\ \phi \rightarrow 0$

As ω is varied from 0 to ∞, the phase angle decreases from 0 to a negative maximum value of ϕ_m at $\omega = \omega_m$, then increases from this maximum value to 0. The phase plot of lag compensator is shown in fig 6.5. It can be shown that the frequency at which maximum phase lag occurs is the geometric mean of the two corner frequencies.

$$\text{Frequency of maximum phase lag, } \omega_m = \sqrt{\omega_{c1}\,\omega_{c2}} = \sqrt{\frac{1}{\beta T}\cdot\frac{1}{T}} = \frac{1}{T\sqrt{\beta}}$$

From the bode plot of lag compensator , we observe that lag compensator has a dc gain of unity while it offers a high frequency gain of (1/β). [In decibels , it is 20 log (1/β)]. It means that the high frequency noise is attenuated in passing through the network and so the signal to noise ratio is improved. A typical choice of β = 10.

DETERMINATION OF ω_m AND ϕ_m

The frequency ω_m can be determined by differentiating ϕ with respect to ω and equating dϕ/dω to zero as shown below.

From equation (6.12) we get,

$$\text{Phase of } G_c(j\omega),\ \phi = \angle G_c(j\omega) = \tan^{-1}\omega T - \tan^{-1}\omega\beta T \quad .....(6.14)$$

On differentiating the equation (6.14) we get,

$$\boxed{\frac{d}{d\theta}(\tan^{-1}\theta) = \frac{1}{1+\theta^2}}$$

$$\frac{d\phi}{d\omega} = \frac{1}{1+(\omega T)^2}T - \frac{1}{1+(\omega\beta T)^2}\beta T \quad .....(6.15)$$

When $\omega = \omega_m$, $d\phi/d\omega = 0$

Hence, replace ω by ω_n in equation (6.15) and equate to zero.

$$\therefore \frac{1}{1+(\omega_m T)^2}T - \frac{1}{1+(\omega_m\beta T)^2}\beta T = 0 \quad \Rightarrow \quad \frac{T}{1+(\omega_m T)^2} = \frac{\beta T}{1+(\omega_m\beta T)^2}$$

On cross multiplication we get,

$$1+(\omega_m \beta T)^2 = \beta\left[1+(\omega_m T)^2\right] \quad \Rightarrow \quad (\omega_m \beta T)^2 - \beta(\omega_m T)^2 = \beta - 1$$

$$\therefore \; \beta(\omega_m T)^2(\beta-1) = (\beta-1) \quad \Rightarrow \quad \omega_m^2 = \frac{1}{T^2\beta}$$

$$\therefore \omega_m = \frac{1}{T\sqrt{\beta}}$$

Frequency corresponding to maximum phase lag, $\omega_m = \dfrac{1}{T\sqrt{\beta}}$ .....(6.16)

The maximum phase angle ϕ_m can be calculated from the knowledge of β and viceversa. The relations between ϕ_m and β are derived below.

From equation (6.14) we get, $\angle G_c(j\omega) = \phi = \tan^{-1}\omega T - \tan^{-1}\omega\beta T$

On taking tan on either side we get,

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}}$$

$$\tan\phi = \tan\left[\tan^{-1}\omega T - \tan^{-1}\omega\beta T\right]$$

$$= \frac{\tan(\tan^{-1}\omega T) - \tan(\tan^{-1}\omega\beta T)}{1+\tan(\tan^{-1}\omega T)\,.\,\tan(\tan^{-1}\omega\beta T)} = \frac{\omega T - \omega\beta T}{1+\omega^2 T^2 . \beta} = \frac{\omega T(1-\beta)}{1+\beta(\omega T)^2}$$

$$\boxed{\omega_m = \frac{1}{T\sqrt{\beta}}}$$

$$\text{At } \omega = \omega_m, \; \phi \to \phi_m, \quad \therefore \tan\phi_m = \frac{\omega_m T(1-\beta)}{1+\beta(\omega_m T)^2} = \frac{\frac{1}{T\sqrt{\beta}}T(1-\beta)}{1+\beta\,.\,\frac{1}{T^2\beta}T^2} = \frac{(1-\beta)/\sqrt{\beta}}{1+1} = \frac{(1-\beta)}{2\sqrt{\beta}}$$

$$\therefore \text{Maximum lag angle, } \phi_m = \tan^{-1}\left[\frac{1-\beta}{2\sqrt{\beta}}\right] \qquad .....(6.17)$$

To find the value of β from ϕ_m

From the equation (6.17), it is evident that $(1-\beta)$ and $2\sqrt{\beta}$ are the two sides of right angled triangle. Hence construct a right angle triangle as shown in fig 6.6.

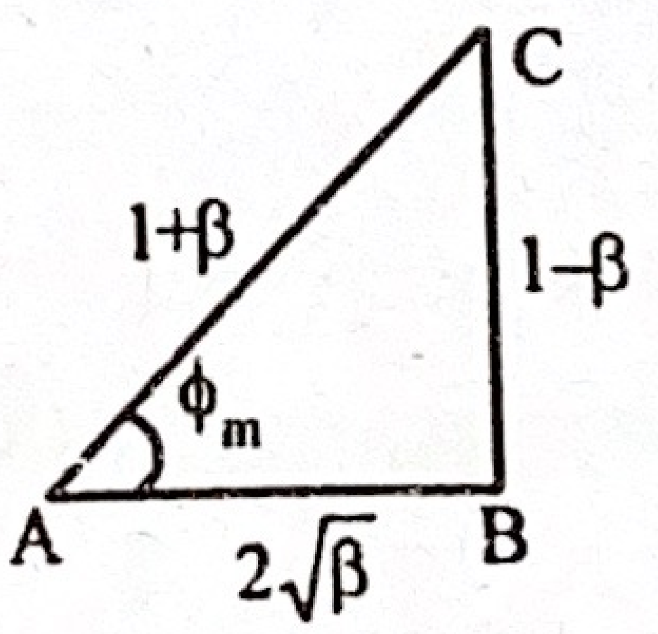
With reference to figure 6.6, $\sin\phi_m = \dfrac{1-\beta}{1+\beta}$

$$\sin\phi_m(1+\beta) = (1-\beta)$$

$$\sin\phi_m + \beta\sin\phi_m = 1-\beta$$

$$\beta\sin\phi_m + \beta = 1-\sin\phi_m$$

$$\beta(\sin\phi_m + 1) = 1-\sin\phi_m$$



$$AC = \sqrt{BC^2 + AB^2} = \sqrt{(1-\beta)^2 + (2\sqrt{\beta})^2} = \sqrt{1+\beta^2 - 2\beta + 4\beta} = \sqrt{1+\beta^2+2\beta} = \sqrt{(1+\beta)^2} = 1+\beta$$

Fig 6.6

$$\therefore \beta = \frac{1-\sin\phi_m}{1+\sin\phi_m} \qquad .....(6.18)$$

PROCEDURE FOR THE DESIGN OF LAG COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lag compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : Choose the value of K in uncompensated system to meet the steady state error requirement.

Step-2 : Sketch the bode plot of uncompensated system. *[Refer Chapter-4 for the procedure to sketch bode plot].*

Step-3 : Determine the phase margin of the uncompensated system from the bode plot. If the phase margin does not satisfy the requirement then lag compensation is required.

Step-4 : Choose a suitable value for the phase margin of the compensated system.

Let γ_d = Desired phase margin as given in specifications.

γ_n = Phase margin of compensated system.

Now, $\gamma_n = \gamma_d + \in$

where $\in$ = Additional phase lag to compensate for shift in gain crossover frequency.

Choose an initial value of $\in = 5°$.

Step-5 : Determine the new gain crossover frequency, ω_{gcn}. The new ω_{gcn} is the frequency corresponding to a phase margin of γ_n on the bode plot of uncompensated system.

Let, ϕ_{gcn} = Phase of $G(j\omega)$ at new gain crossover frequency, ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$ (or) $\phi_{gcn} = \gamma_n - 180°$

The new gain crossover frequency, ω_{gcn} is given by the frequency at which the phase of $G(j\omega)$ is ϕ_{gcn}.

Step-6 : Determine the parameter, β of the compensator. The value of β is given by the magnitude of $G(j\omega)$ at new gain crossover frequency, ω_{gcn}. Find the db gain (A_{gcn}) at new gain crossover frequency, ω_{gcn}

Now, $A_{gcn} = 20\log\beta$ (or) $\dfrac{A_{gcn}}{20} = \log\beta$, $\quad \therefore \beta = 10^{A_{gcn}/20}$

Step-7 : Determine the transfer function of lag compensator.

Place the zero of the compensator arbitrarily at 1/10th of the new gain crossover frequency, ω_{gcn}.

$\therefore$ Zero of the lag compensator, $z_c = \dfrac{1}{T} = \dfrac{\omega_{gcn}}{10}$

Now, $T = \dfrac{10}{\omega_{gcn}}$.

Pole of the lag compensator, $p_c = 1/\beta T$

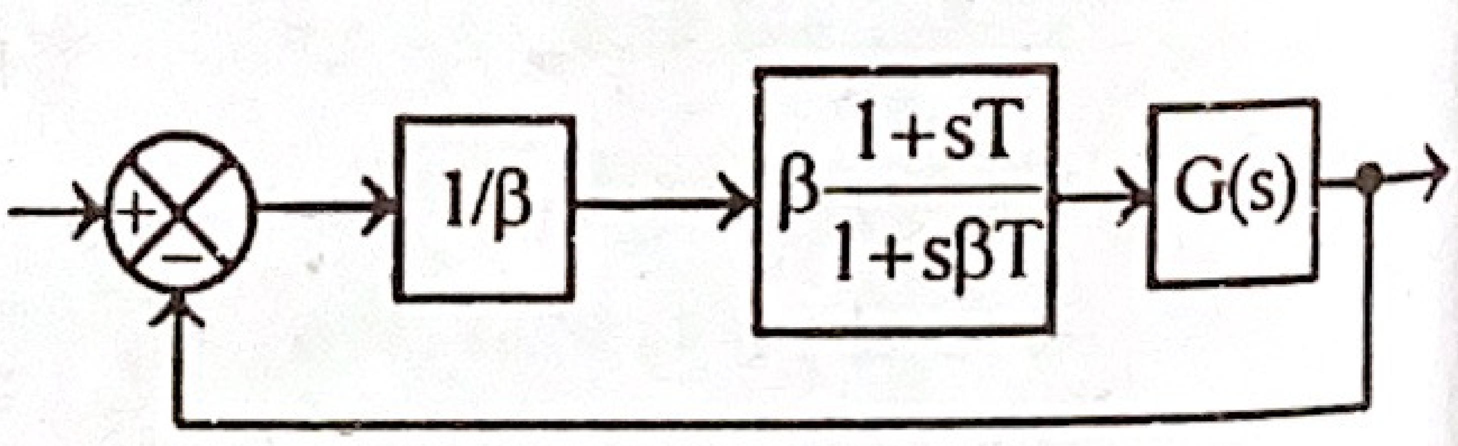


Fig 6.7 : Block diagram of lag compensated system.

Transfer function of lag compensator $\Big\}\; G_c(s) = \dfrac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\left(\dfrac{1+sT}{1+s\beta T}\right)$

Step-8 : Determine the open loop transfer function of compensated system. The lag compensator is connected in series with plant as shown in fig 6.7.

When the lag compensator is inserted in series with plant, the open loop gain of the system is amplified by the factor $\beta(\because \beta>1)$. If the gain produced is not required then attenuator with gain $1/\beta$ can be introduced in series with the lag compensator to nullify the gain produced by lag compensator.

The open loop transfer function of the compensated system,

$$G_o(s) = \frac{1}{\beta}.G_c(s).G(s) = \frac{1}{\beta}.\beta\frac{(1+sT)}{(1+s\beta T)}.G(s) = \frac{(1+sT)}{(1+s\beta T)}\,.\,G(s)$$

Step-9 **:** Determine the actual phase margin of compensated system. Calculate the actual phase angle of the compensated system using the compensated transfer function at new gain crossover frequency, ω_{gcn}.

Let, ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

Actual phase margin of the compensated system, $\gamma_o = 180° + \phi_{gco}$

If the actual phase margin satisfies the given specification then the design is accepted. Otherwise repeat the procedure from step 4 to 9 by taking $\in$ as 5° more than previous design.

PROCEDURE FOR DESIGN OF LAG COMPENSATOR USING ROOT LOCUS

The following steps may be followed to design a lag compensator using root locus and to be connected in series with the transfer function of uncompensated system.

Step-1 **:** Draw the root locus of uncompensated system. *[Refer chapter-5 for the procedure to construct root locus]*.

Step-2 **:** Determine the dominant pole, s_d. Draw a straight line through the origin with an angle $\cos^{-1}\zeta$ with respect to negative real axis. The intersection point of the straight line with root locus gives the dominant pole, s_d.

Step-3 **:** Determine the open loop gain of the uncompensated system at $s = s_d$. Let this gain be K. The open loop gain K at $s = s_d$ on root locus is given by,

$$K = \frac{\text{Product of vector lengths from } s_d \text{ to open loop poles}}{\text{Product of vector lengths from } s_d \text{ to open loop zeros}} \quad \text{(vector length measured to scale)}$$

Step-4 **:** Calculate the parameter, β of the compensator.

Let, K_{vu} = Velocity error constant of uncompensated system.

K_{vd} = Desired velocity error constant.

$$K_{vu} = \underset{s\to 0}{Lt}\; sG(s)$$

Let A be the factor by which the velocity error constant of the system has to be increased,

where, $A = K_{vd}/K_{vu}$

Choose β such that it is 10 to 20% greater than A.

$\therefore \beta = (6.1 \text{ to } 6.2) \times A.$

Step-5 **:** Determine the transfer function of lag compensator. The zero of the lag compensator (1/T) is chosen to be 10% of the second pole of uncompensated system.

$\therefore$ Zero of the compensator, $z_c = (-1/T) = 0.1 \times$ (second pole of G(s))

Now, $T = 1/[-0.1 \times$ (second pole of G(s))]

Pole of the lag compensator, $p_c = -1/\beta T$.

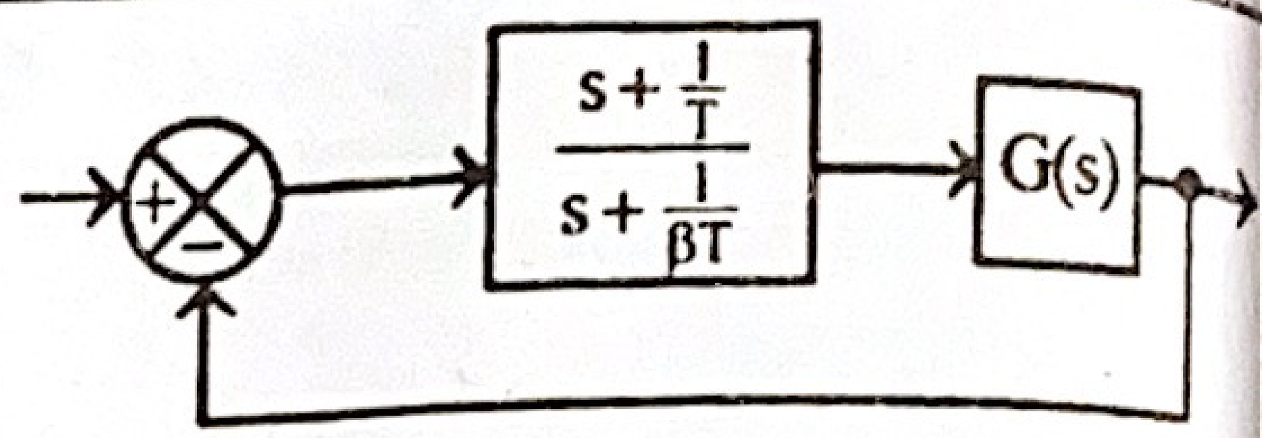


Fig 6.8 : Block diagram of lag compensated system.

Transfer function of lag compensator
$$G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta\frac{(1+sT)}{(1+s\beta T)}$$

Step-6 : Determine the open loop transfer function of the compensated system. The lag compensator is connected in series with the plant as shown in fig 6.8

Open loop transfer fuction of compensated system
$$G_o(s) = G_c(s).G(s) = \frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})}.G(s)$$

Step-7 : Check whether the compensated system satisfies the steady state error requirement. If it is satisfied, then the design is accepted otherwise repeat the design by modifying the locations of poles and zeros of the compensator.

EXAMPLE 6.1

A unity feedback system has an open loop transfer function, $G(s) = K/s(1+2s)$. Design a suitable lag compensator so that phase margin is 40° and the steady state error for ramp input is less than or equal to 0.2.

SOLUTION

Step-1: Calculation of gain, K.

Given that, $e_{ss} \le 0.2$ for ramp input. Let $e_{ss} = 0.2$

We know that, $e_{ss} = 1/K_v$ for ramp input.

$$\therefore \text{Velocity error constant, } K_v = \frac{1}{e_{ss}} = \frac{1}{0.2} = 5.$$

By definition of velocity error constant, $K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\,G(s)H(s)$.

Since the system is unity feedback system, $H(s) = 1$.

$$\therefore K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\,G(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\frac{K}{s(1+2s)} = K \qquad \therefore K = 5$$

Step-2: Bode plot of uncompensated system.

Given that, $G(s) = 5/s(1+2s)$

Let $s = j\omega$, $\therefore G(j\omega) = 5/j\omega(1+j2\omega)$.

MAGNITUDE PLOT

The corner frequency is, $\omega_c = 1/2 = 0.5$ rad/sec

The various terms of $G(j\omega)$ are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{5}{j\omega}$		-20	
$\frac{1}{1+j2\omega}$	$\omega_c = \frac{1}{2} = 0.5$	-20	$-20-20=-40$

Choose a low frequency ω_l such that $\omega_l < \omega_c$ and choose a high frequency ω_h such that $\omega_h > \omega_c$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|\frac{5}{j\omega}\right| = 20\log\frac{5}{0.1} = 34 \text{ db}$$

$$\text{At } \omega = \omega_c, \quad A = 20\log\left|\frac{5}{j\omega}\right| = 20\log\frac{5}{0.5} = 20 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_c \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_c}\right] + A_{(\text{at } \omega=\omega_c)} = -40 \times \log\frac{10}{0.5} + 20 = -32 \text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_c and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on respective region. Magnitude plot is shown in fig 6.1.1.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by, $\phi = \angle G(j\omega) = -90° - \tan^{-1}2\omega$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	6.0	5	10
ϕ deg	-101	-135	-153	-174	-177

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.1.1.

Step-3 **:** Determination of phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

and γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -162°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 162° = 18°$

The system requires a phase margin of 40°, but the available phase margin is 18° and so lag compensation should be employed to improve the phase margin.

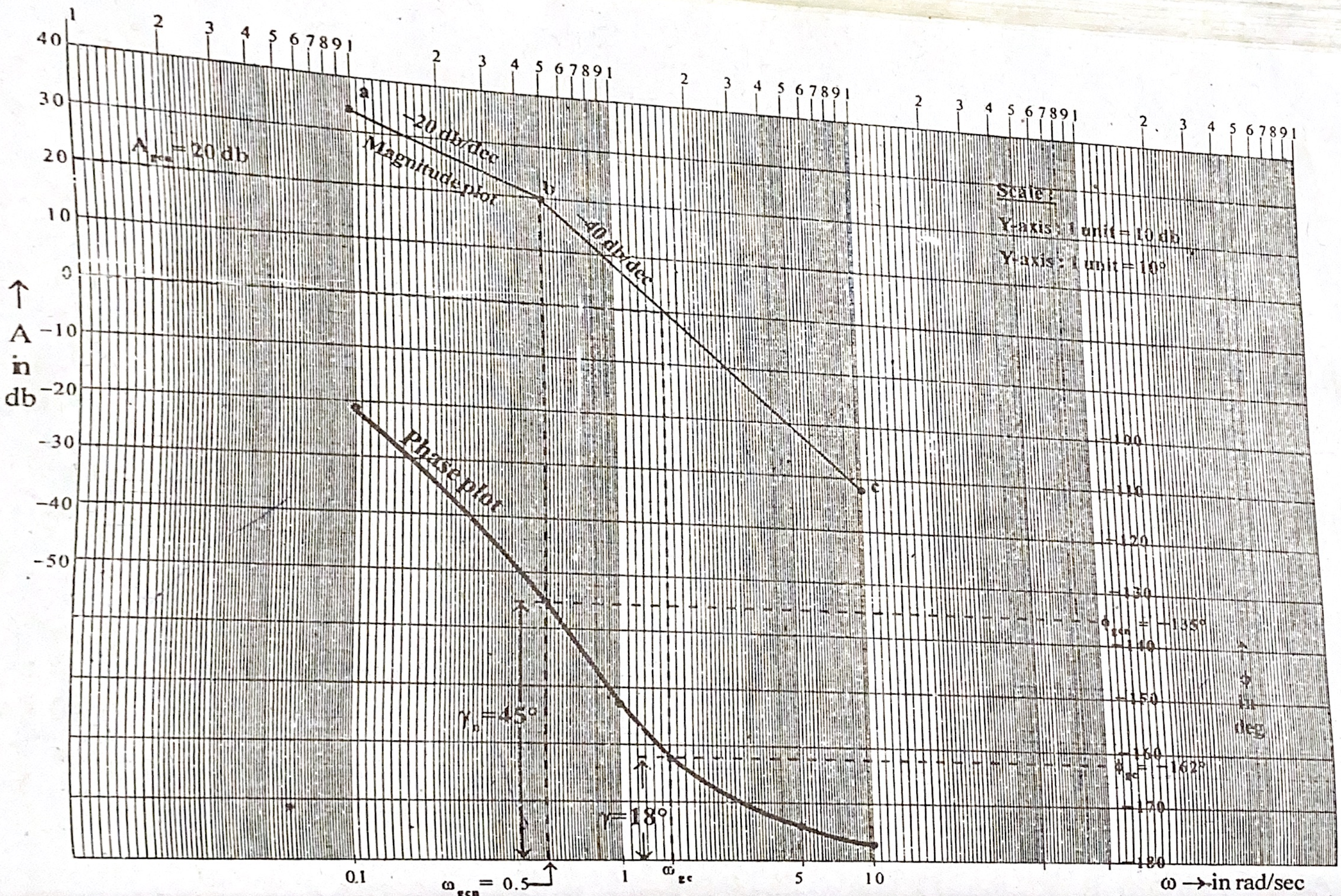


Fig 6.1.1 : Bode plot of $G(j\omega) = 5 / j\omega\,(1 + j2\omega)$

Step-4 : Choose a suitable value for the phase margin of compensated system.

The desired phase margin, $\gamma_d = 40°$.

$\therefore$ Phase margin of compensated system, $\gamma_n = \gamma_d + \epsilon$

Let initial choice of $\epsilon = 5°$

$$\therefore \gamma_n = \gamma_d + \epsilon = 40° + 5° = 45°$$

Step 5 : Determine new gain crossover frequency.

Let ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$

$$\therefore \phi_{gcn} = \gamma_n - 180° = 45° - 180° = -135°$$

From the bode plot we found that, the frequency corresponding to a phase of $-135°$ is 0.5 rad/sec.

$\therefore$ New gain crossover frequency, $\omega_{gcn} = 0.5$ rad/sec.

Step-6 : Determine the parameter, β

From the bode plot we found that, the db magnitude at ω_{gcn} is 20 db.

$$\therefore |G(j\omega)| \text{ in db at } (\omega = \omega_{gcn}) = A_{gcn} = 20 \text{ db}$$

Also, $A_{gcn} = 20 \log \beta$; $\therefore \beta = 10^{A_{gcn}/20} = 10^{20/20} = 10$.

Step-7 : Determine the transfer function of lag compensator.

The zero of the compensator is placed at a frequency one-tenth of ω_{gcn}.

$$\therefore \text{Zero of the lag compensator, } z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$$

$$\text{Now, } T = \frac{10}{\omega_{gcn}} = \frac{10}{0.5} = 20$$

$$\text{Pole of the lag compensator, } p_c = \frac{1}{\beta T} = \frac{1}{10 \times 20} = \frac{1}{200} = 0.005$$

$$\text{Transfer function of lag compensator, } G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \beta \frac{1 + sT}{1 + s\beta T} = 10 \frac{(1+20s)}{(1+200s)}$$

Step-8 : Determine the open loop transfer function of compensated system.

The block diagram of the compensated system is shown in fig 6.1.2. The gain of compensator is nullified by introducing an attenuator in series with compensator, as shown in fig 6.1.2.

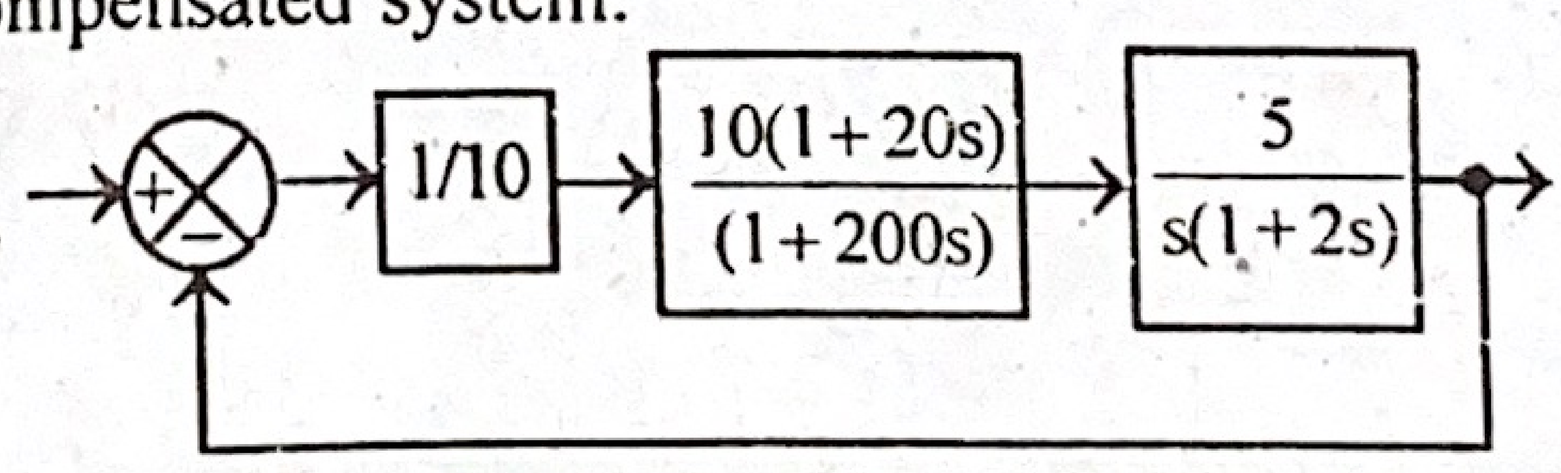


Fig 6.1.2 : Block diagram of lag compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of compensated system}\end{array}\right\} G_o(s) = \frac{1}{10} \times \frac{10\,(1+20s)}{(1+200s)} \times \frac{5}{s(1+2s)} = \frac{5(1+20s)}{s(1+200s)\,(1+2s)}$$

Step-9 : Determine the actual phase margin of compensated system.

On substituting $s = j\omega$ in $G_o(s)$ we get, $G_o(j\omega) = \dfrac{5\,(1+j20\omega)}{j\omega\,(1+j200\omega)\,(1+j2\omega)}$

Let, ϕ_o = Phase of $G_o(j\omega)$

ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

$\phi_o = \tan^{-1} 20\omega - 90° - \tan^{-1} 200\omega - \tan^{-1} 2\omega$

At $\omega = \omega_{gcn}$, $\phi_o = \phi_{gco} = \tan^{-1} 20\,\omega_{gcn} - 90° - \tan^{-1} 200\omega_{gcn} - \tan^{-1} 2\,\omega_{gcn}$

$\therefore \phi_{gco} = \tan^{-1}(20 \times 0.5) - 90° - \tan^{-1}(200 \times 0.5) - \tan^{-1}(2 \times 0.5) = -140°.$

Actual phase margin of compensated system, $\gamma_o = 180° + \phi_{gco} = 180° - 140° = 40°$

CONCLUSION

The actual phase margin of the compensated system satisfies the requirement. Hence the desi is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = \dfrac{10(1+20s)}{(1+200s)} = \dfrac{(s+0.05)}{(s+0.005)}$

Open loop transfer function of compensated system, $G_o(s) = \dfrac{5\,(1+20s)}{s\,(1+200s)\,(1+2s)}$

EXAMPLE 6.2

The open loop transfer function of certain unity feedback control system is given $G(s) = K/s(s+4)(s+80)$. It is desired to have the phase margin to be atleast 33° and the velocity err constant $K_v = 30\ \text{sec}^{-1}$. Design a phase lag series compensator.

SOLUTION

Step-1 : Calculation of gain, K

Given that, $K_v = 30\ \text{sec}^{-1}$

By definition of velocity error constant, $K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\,G(s)\,H(s)$

Since the system is unity feedback system, $H(s) = 1$.

$$\therefore K_v = \underset{s\to 0}{\mathrm{Lt}}\ sG(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\,\frac{K}{s(s+4)(s+80)} = \frac{K}{4\times 80}$$

$$\text{i.e.,}\quad \frac{K}{4\times 80} = 30, \qquad \therefore K = 30\times 80\times 4 = 9600.$$

Step-2 : Bode plot of uncompensated system

$$G(s) = \frac{9600}{s(s+4)(s+80)} = \frac{9600/4\times 80}{s(1+\frac{s}{4})(1+\frac{s}{80})} = \frac{30}{s(1+0.25s)(1+0.0125s)}$$

Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{30}{j\omega\,(1 + j0.25\omega)\,(1 + j0.0125\omega)}$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = 1/0.25 = 4$ rad/sec and $\omega_{c2} = 1/0.0125 = 80$ rad/sec. The various terms of $G(j\omega)$ are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{30}{j\omega}$	–	–20	–
$\dfrac{1}{1+j0.25\omega}$	$\omega_{c1} = \dfrac{1}{0.25} = 4$	–20	$-20-20=-40$
$\dfrac{1}{1+j0.0125\omega}$	$\omega_{c2} = \dfrac{1}{0.0125} = 80$	–20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c2}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.
Let $\omega_l = 1$ rad/sec and $\omega_h = 100$ rad/sec.

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|\frac{30}{j\omega}\right| = 20\log\frac{30}{1} = 29.5 \text{ db} \approx 30 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|\frac{30}{j\omega}\right| = 20\log\frac{30}{4} = 17.5 \text{ db} \approx 18 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$$

$$= -40\log\frac{80}{4} + 18 = -34 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$$

$$= -60 \times \log\frac{100}{80} + (-34) = -40 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b, c and d. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 6.2.1.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}0.25\omega - \tan^{-1}0.0125\omega$$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	1	4	10	50	80	100
φ deg	−104	−138	−165 ≈ −164	−207 ≈ −208	−222	−229 ≈ −230

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.2.1.

Step-3: Determination of phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we found that, $\phi_{gc} = -168°$.

Now. $\gamma = 180° + \phi_{gc} = 180° - 168° = 12°$

The system requires a phase margin of atleast 33°, but the available pha e margin is 12° and so lag compensation should be employed to improve the phase margin.

Step-4 : Choose a suitable value for the phase margin of compensated system.

The desired phase margin, $\gamma_d = 33°$

$\therefore$ Phase margin of compensated system, $\gamma_n = \gamma_d + \in$

Let initial choice of $\in = 5°$; $\therefore \gamma_n = \gamma_d + \in = 33° + 5° = 38°$

Step 5: Determine new gain crossover frequency.

Let ω_{gcn} = New gain crossover frequency and ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

Now, $\gamma_n = 180° + \phi_{gcn}$

$\therefore \phi_{gcn} = \gamma_n - 180° = 38° - 180° = -142°$

From the bode plot we found that, the frequency corresponding to a phase of −142° is 4.7 rad/sec.

$\therefore$ New gain crossover frequency, $\omega_{gcn} = 4.7$ rad/sec.

Step-6 : Determine the parameter, β

From the bode plot we found that, the db magnitude at ω_{gcn} as 16 db.

$\therefore |G(j\omega)|$ in db at $(\omega = \omega_{gcn}) = A_{gcn} = 16$ db

Also, $A_{gcn} = 20 \log \beta$; $\therefore \beta = 10^{A_{gcn}/20} = 10^{16/20} = 6.3$.

Step-7 : Determine the transfer function of lag compensator.

The zero of the compensator is placed at a frequency one-tenth of ω_{gcn}.

$$\therefore \text{ Zero of the lag compensator, } z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$$

$$\text{Now, } T = \frac{10}{\omega_{gcn}} = \frac{10}{4.7} = 2.13$$

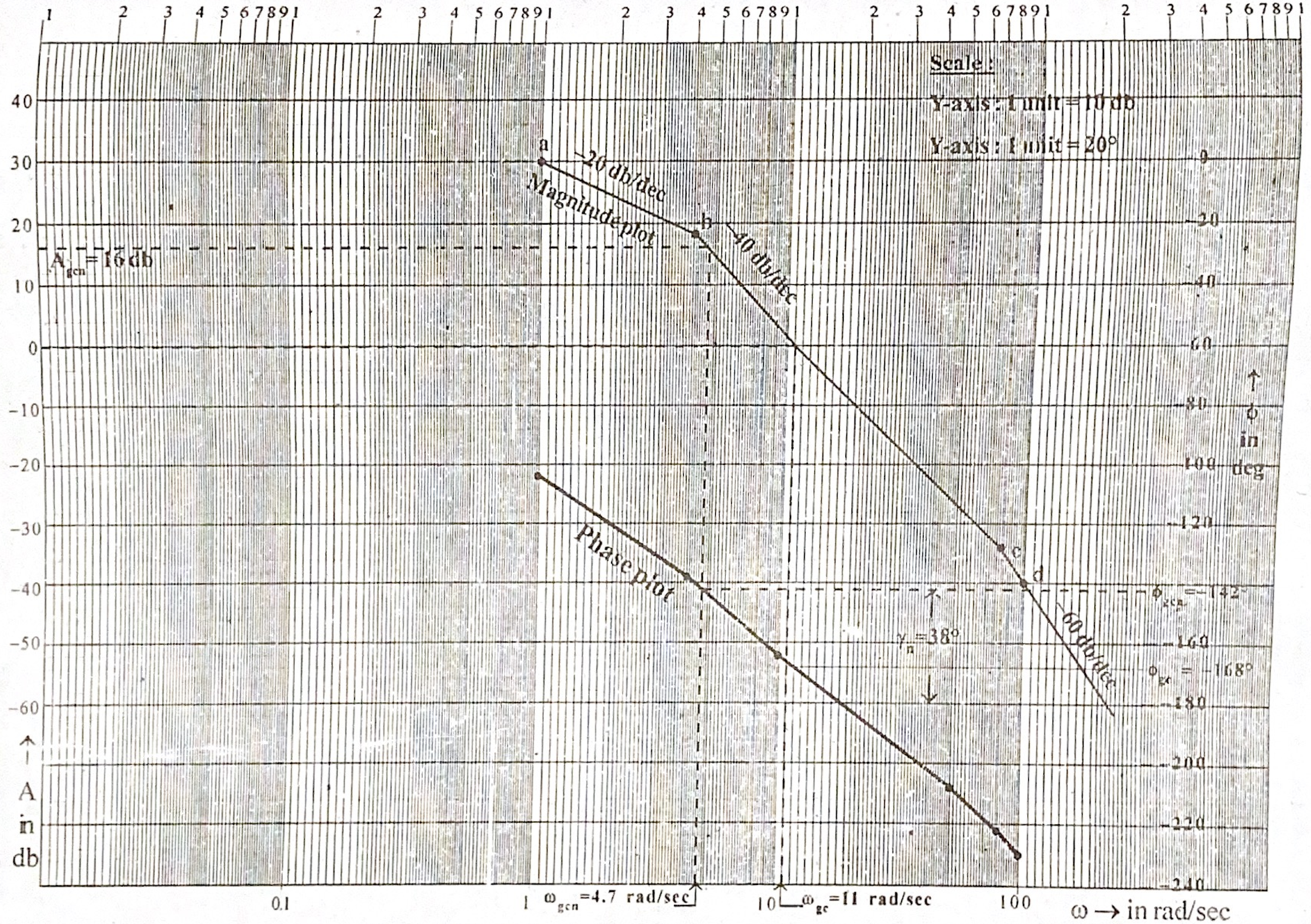


Fig 6.2.1 : Bode plot of $G(j\omega) = 30 / j\omega\,(1 + j0.25\omega)\,(1 + j0.0125\omega)$

Pole of the lag compensator, $p_c = \frac{1}{\beta T} = \frac{1}{6.3 \times 2.13} = \frac{1}{13.419}$

Transfer function of lag compensator $\left.\right\}$ $G_c(s) = \dfrac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \beta \dfrac{1+sT}{1+s\beta T} = 6.3 \dfrac{(1+2.13s)}{(1+13.419s)}$

Step-8 : Determine the open loop transfer function of the compensated system..

The block diagram of the compensated system is shown in fig 6.2.2. The gain of the compensator is nullified by introducing an attenuator in series with the compensator, as shown in fig 6.2.2.

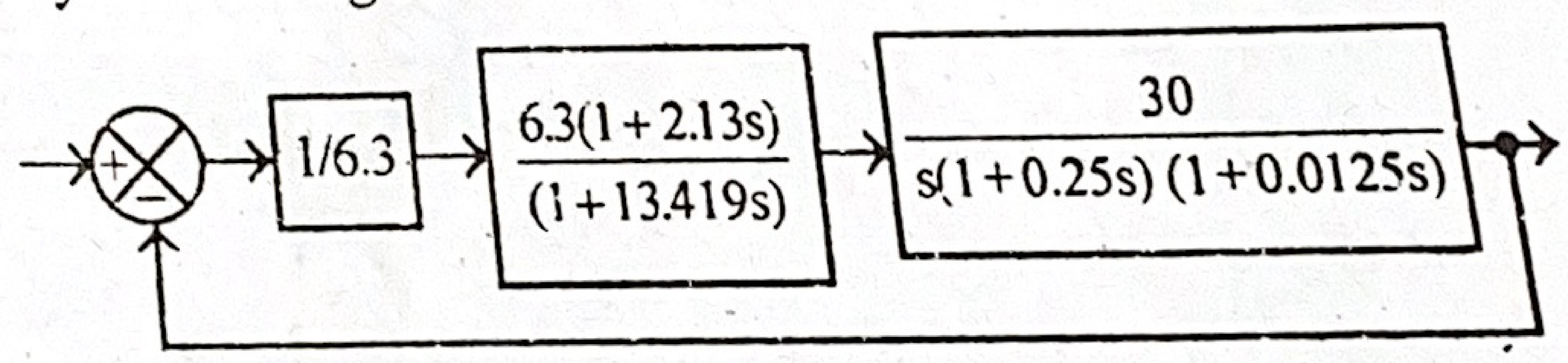


Fig 6.2.2 : Block diagram of lag compensated system.

Open loop transfer function of compensated system $\left.\right\}$ $G_o(s) = \dfrac{1}{6.3} \times \dfrac{6.3\,(1+2.13s)}{(1+13.419s)} \times \dfrac{30}{s(1+0.25s)\,(1+0.0125s)}$

$$= \frac{30\,(1+2.13s)}{s\,(1+13.419s)\,(1+0.25s)\,(1+0.0125s)}$$

Step-9 : Determine the actual phase margin of compensated system.

On substituting $s = j\omega$ in $G_o(s)$ we get,

$$G_o(j\omega) = \frac{30\,(1+j2.13\omega)}{j\omega\,(1+j13.419\omega)\,(1+j0.25\omega)\,(1+j0.0125\omega)}$$

Let ϕ_o = Phase of $G_o(j\omega)$ and ϕ_{gco} = Phase of $G_o(j\omega)$ at $\omega = \omega_{gcn}$

$$\phi_o = \tan^{-1} 2.13\omega - 90^\circ - \tan^{-1} 13.419\omega - \tan^{-1} 0.25\omega - \tan^{-1} 0.0125\omega$$

$$\text{At } \omega = \omega_{gcn},\ \phi_o = \phi_{gco} = \tan^{-1} 2.13\,\omega_{gcn} - 90^\circ - \tan^{-1} 13.419\omega_{gcn} - \tan^{-1} 0.25\omega_{gcn} - \tan^{-1} 0.0125\,\omega_{gcn}$$

$$\therefore \phi_{gco} = \tan^{-1}(2.13 \times 4.7) - 90^\circ - \tan^{-1}(13.419 \times 4.7) - \tan^{-1}(0.25 \times 4.7) - \tan^{-1}(0.0125 \times 4.7) = -147^\circ.$$

Actual phase margin of compensated system, $\gamma_o = 180^\circ + \phi_{gco} = 180^\circ - 147^\circ = 33^\circ$

CONCLUSION

The actual phase margin of the compensated system satisfies the requirement. Hence the design is acceptable

RESULT

Transfer function of lag compensator, $G_c(s) = \dfrac{6.3\,(1+2.13s)}{(1+13.419s)} = \dfrac{(s+0.469)}{(s+0.074)}$

Transfer function of lag compensated system $\left.\right\}$ $G_o(s) = \dfrac{30\,(1+2.13s)}{s\,(1+13.419s)\,(1+0.25s)\,(1+0.0125s)}$

EXAMPLE 6.3

The forward path transfer function of a certain unity feedback control system is given by $G(s) = K/s\,(s+2)\,(s+8)$. Design a suitable lag compensator so that the system meets the following specifications: (i) Percentage overshoot $\leq 16\%$ for unit step input, (ii) Steady state error ≤ 0.125 for unit ramp input.

SOLUTION

Step-1 : Sketch the root locus of uncompensated system.

To find poles of open loop system

Given that, $G(s) = K / s(s+2)\,(s+8)$.

The poles of open loop transfer function are the roots of the equation, $s(s+2)\,(s+8) = 0$.

$\therefore$ The poles are lying at $s = 0, -2, -8$.

Let us denote poles by, p_1, p_2 and p_3. Here, $p_1 = 0$, $p_2 = -2$ and $p_3 = -8$.

To find root locus on real axis

The segments of real axis between $s = 0$ and $s = -2$ and the segment of real axis between $s = -8$ and $s = -\infty$ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros.

To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is no finite zero and so all the three root locus branches will meet the zeros at infinity. Hence the number of asymptotes required is three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}\ ; \qquad q = 0, 1, 2, \ldots\ldots n-m.$$

Here $n = 3$ and $m = 0$. $\quad \therefore q = 0, 1, 2, 3$.

$$\text{When, } q = 0, \text{ Angles } = \frac{\pm 180^\circ}{3} = \pm 60^\circ$$

$$\text{When, } q = 1, \text{ Angles } = \frac{\pm 180^\circ (2+1)}{3} = \pm 180^\circ$$

$\therefore$ Angles of asymptotes are, $+60^\circ, -60^\circ$ and $\pm 180^\circ$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2-8}{3} = -3.33$$

To find breakaway point

$$\text{Closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+2)\,(s+8)}}{1+\dfrac{K}{s\,(s+2)\,(s+8)}} = \frac{K}{s\,(s+2)\,(s+8)\,+K}$$

The characteristic equation is, $s(s+2)\,(s+8) + K = 0$.

$$\therefore K = -s(s+2)\,(s+8) = -(s^3 + 10s^2 + 16s)$$