LTI SYSREMS

System Properties

$$x(t) \longrightarrow \boxed{\quad} \longrightarrow y(t)$$
$$x[n] \longrightarrow \boxed{\quad} \longrightarrow y[n]$$

- Memory
- Invertibility
- Causality
- Stability
- Time Invariance
- Linearity

Time-Invariance

C-T:

$$x(t) \rightarrow y(t)$$

Then

$$x(t-t_0) \rightarrow y(t-t_0) \quad \text{any } t_0$$

D-T:

$$x[n] \longrightarrow y[n]$$

$$x[n-n_0] \longrightarrow y[n-n_0] \quad \text{any } n_0$$

Linearity

$$\phi_k \rightarrow \psi_k$$

Then

$$a_1\phi_1 + a_2\phi_2 + \ldots \rightarrow a_1\psi_1 + a_2\psi_2 + \ldots$$

STRATEGY:

- decompose input signal into a linear combination of basic signals
- choose basic signals so that response easy to compute

LT I Systems

delayed impulses $\Longleftrightarrow$ Convolution

complex exponentials $\Longleftrightarrow$ Fourier Analysis

Representation of arbitrary DT sequence in terms of impulses:

- **Basic idea**: use a (infinite) set of discrete time impulses to represent any signal

- Consider any discrete input signal $x[n]$. This can be written as the linear sum of a set of unit impulse signals:

$$x[-1]\delta[n+1] = \begin{cases} x[-1] & n = -1 \\ 0 & n \neq -1 \end{cases}$$

$$x[0]\delta[n] = \begin{cases} x[0] & n = 0 \\ 0 & n \neq 0 \end{cases}$$

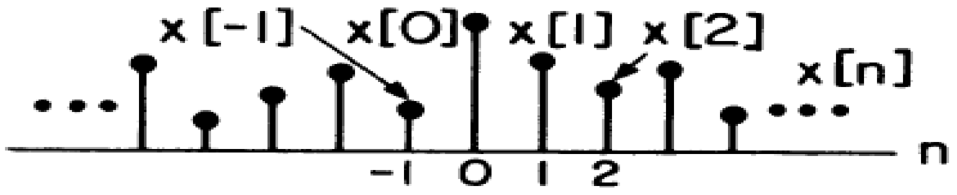
$$x[1]\delta[n-1] = \begin{cases} x[1] & n = 1 \\ 0 & n \neq 1 \end{cases}$$

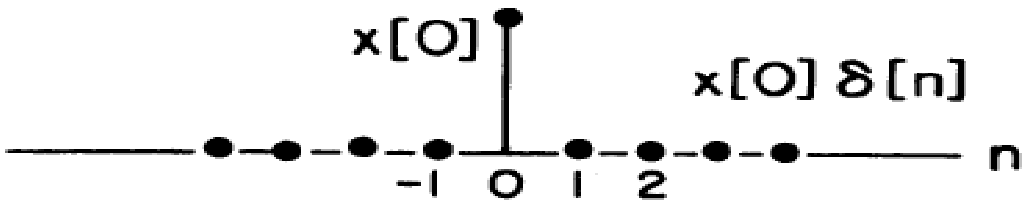
$$x[-1] \quad \delta[n+1]$$

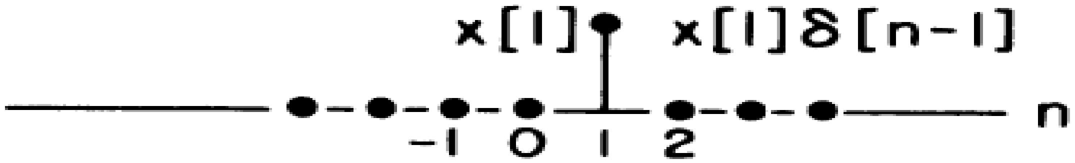
actual value

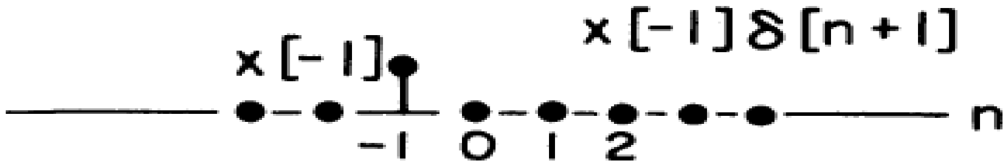
Impulse, time shifted signal

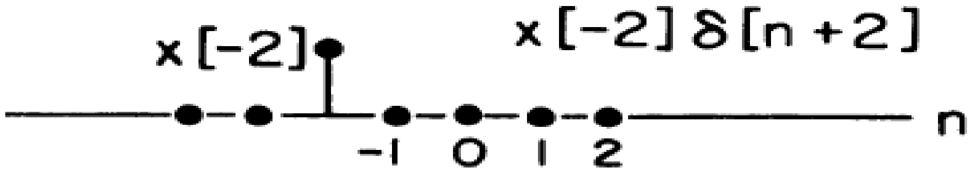
Representation of arbitrary DT sequence in terms of impulses:











Therefore, the signal can be expressed as:

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

In general, any discrete signal can be represented as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The **sifting property**

Introduction to Convolution

- Convolution is an operator that takes an input signal and returns an output signal, based on knowledge about the system's unit impulse response $h[n]$.

$$x[n] = \delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] = h[n]$$

$$x[n] \longrightarrow \boxed{\text{System: } h[n]} \longrightarrow y[n]$$

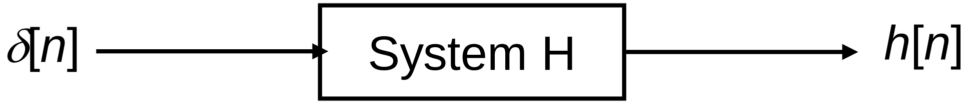
- The basic idea behind convolution is to use the system's response to a simple input signal to calculate the response to more complex signals
- This is possible for LTI systems because they possess the **superposition** property

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \cdots$$

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \cdots$$

Discrete, Unit Impulse System Response

- A very important way to analyse a system is to study the output signal when a unit impulse signal is used as an input



- This is so common, a specific notation, $h[n]$, is used to denote the output signal, rather than the more general $y[n]$.

- The output signal can be used to infer properties about the system's structure and its parameters $\mathbf{H}$.

Linear Time Invariant Systems

- When system is linear, **time invariant**, the unit impulse responses are all time-shifted versions of each other:

$$h_k[n] = h_0[n-k]$$

- It is usual to drop the 0 subscript and simply define the unit impulse response *h*[*n*] as:

$$h[n] = h_0[n]$$

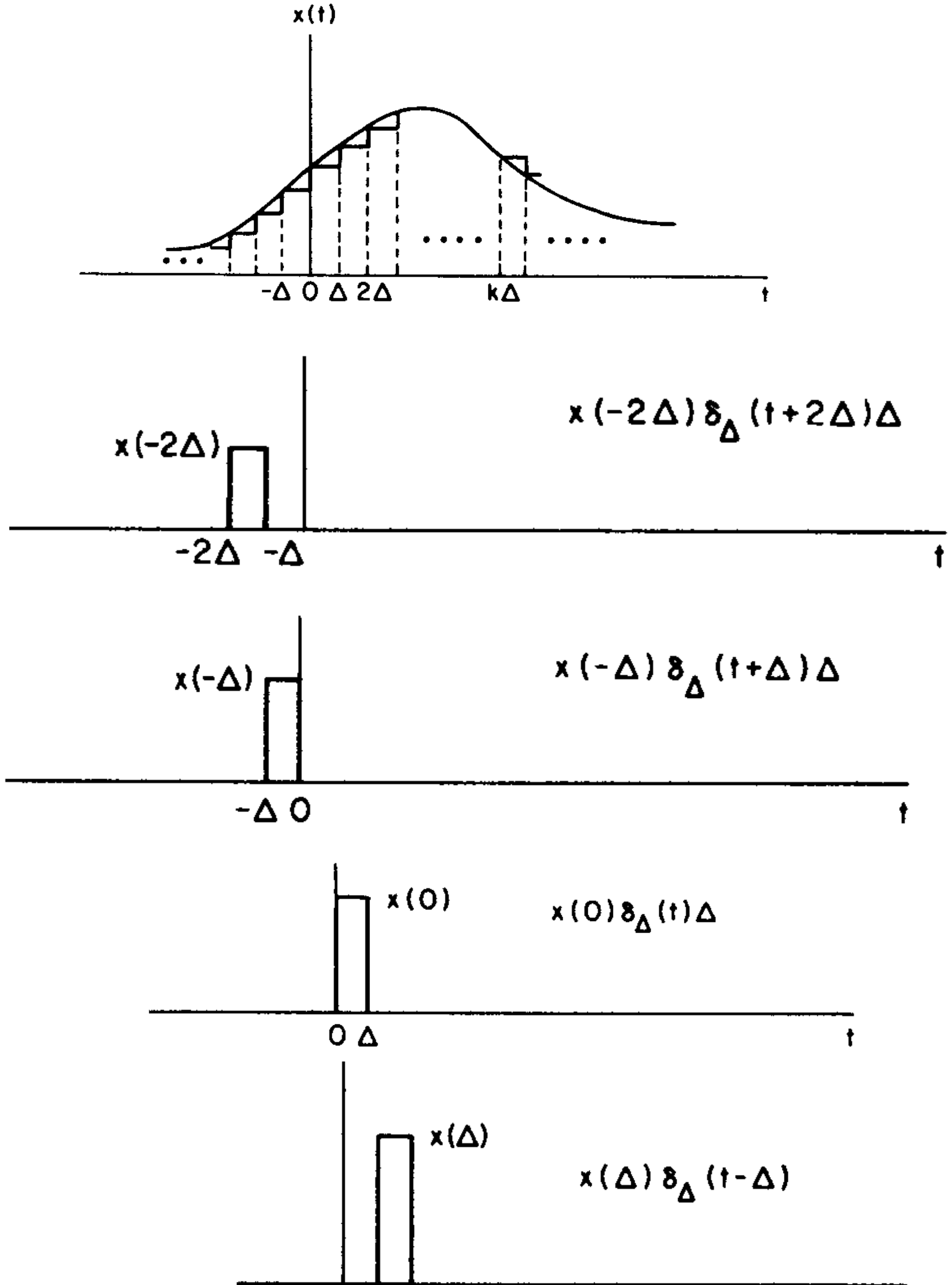
- In this case, the convolution sum for LTI systems is:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- It is called the convolution sum (or superposition sum) because it involves the convolution of two signals *x*[*n*] and *h*[*n*], and is sometimes written as:

$$y[n] = x[n] * h[n]$$

Representation of arbitrary CT signal in terms of impulses:



$$x(t) \cong x(0)\,\delta_\Delta(t)\,\Delta + x(\Delta)\,\delta_\Delta(t-\Delta)\,\Delta + x(-\Delta)\,\delta_\Delta(t+\Delta)\,\Delta + \ldots$$

$$x(t) \cong \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Response of LTI Systems - Convolution Integral

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\ \delta_\Delta(t - k\Delta)\ \Delta$$

Linear System:

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\ h_{k\Delta}(t)\ \Delta \qquad = \int_{-\infty}^{+\infty} x(\tau)\ h_\tau(t)\ d\tau$$

If Time-Invariant:

$$h_{k\Delta}(t) = h_o(t - k\Delta)$$

$$h_\tau(t) = h_o(t - \tau)$$

LTI:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\ h(t - \tau)\ d\tau}$$

Convolution Integral

Properties of Linear Time Invariant Systems

LTI Systems and Impulse Response

- Any continuous/discrete-time LTI system is completely described by its impulse response through the convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

This only holds for LTI systems as follows:

- **Example**: The discrete-time impulse response

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Is completely described by the following LTI

$$y[n] = x[n] + x[n-1]$$

- However, the following systems also have the same impulse response

$$y[n] = \left(x[n] + x[n-1]\right)^2$$
$$y[n] = \max\left(x[n], x[n-1]\right)$$

- Therefore, if the system is non-linear, it is not completely characterised by the impulse response

Commutative Property

- Convolution is a commutative operator (in both discrete and continuous time),

$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

- For example, in discrete-time:

$$x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{r=-\infty}^{\infty}x[n-r]h[r]=h[n]*x[n]$$

and similar for continuous time.

- Therefore, when calculating the response of a system to an input signal $x[n]$, we can imagine the signal being convolved with the unit impulse response $h[n]$, or vice versa, whichever appears the most straightforward.

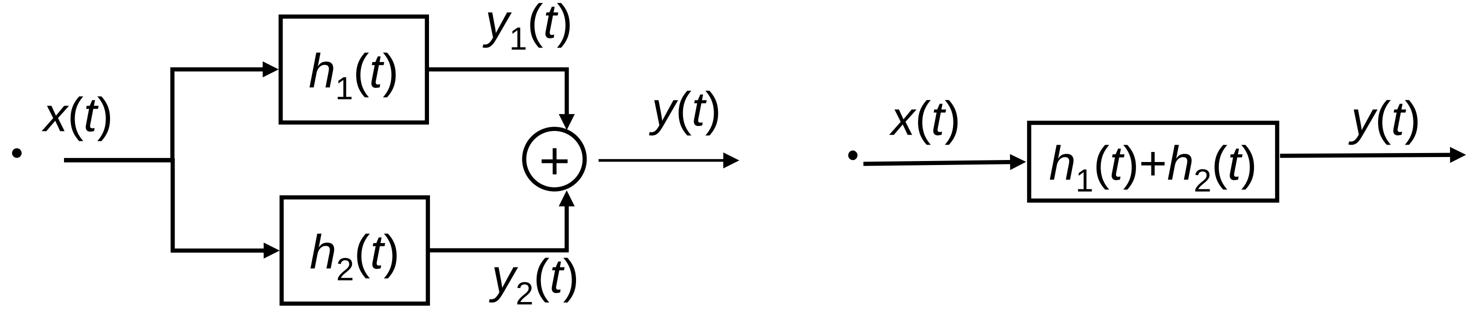
Distributive Property (Parallel Systems)

- Another property of convolution is the distributive property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n] = y_1[n]+y_2[n]$$
$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t)+x(t)*h_2(t) = y_1(t)+y_2(t)$$

  This can be easily verified

- Therefore, the two systems:



- are equivalent. The convolved sum of two impulse responses is equivalent to considering the two equivalent parallel system (equivalent for discrete-time systems)

Example: Distributive Property

- Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = 0.5^n u[n] + 2^n u[-n]$$
$$h[n] = u[n]$$

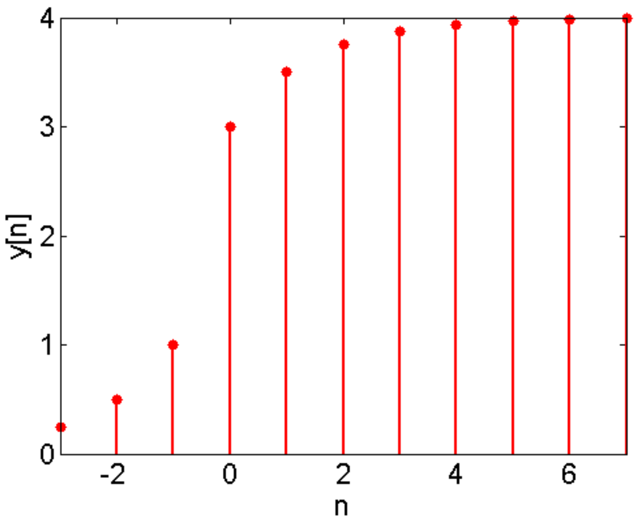
- We will use the **distributive** property to express $y[n]$ as the sum of two simpler convolution problems. Let $x_1[n] = 0.5^n u[n]$, $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]$$

- and $y[n] = y_1[n]+y_2[n]$, where $y_1[n] = x_1[n]*h[n]$, $y_1[n] = x_1[n]*h[n]$.

$$y_1[n] = \left(\frac{1-0.5^{n+1}}{1-0.5}\right)u[n]$$

$$y_2[n] = \begin{cases} 2^{n+1} & n \le 0 \\ 2 & n \ge 1 \end{cases}$$



Associative Property (Serial Systems)

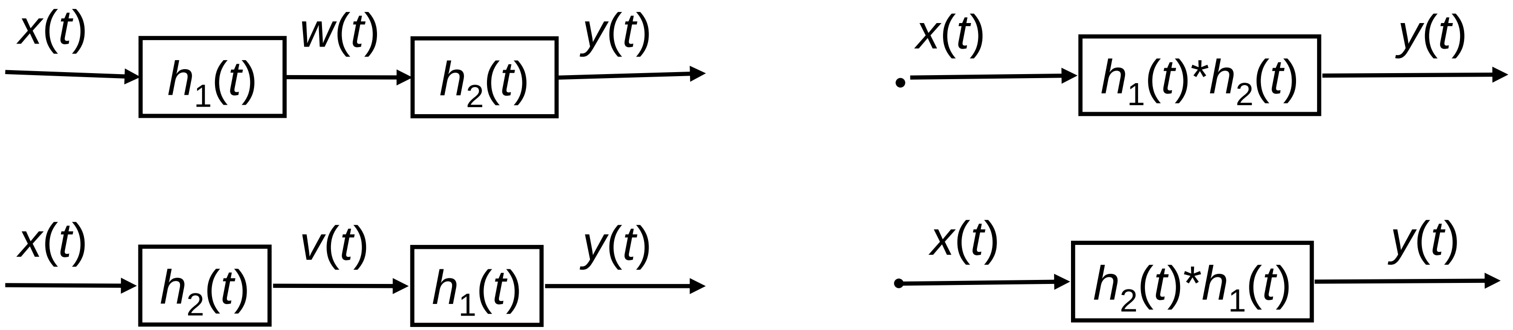
Another property of (LTI) convolution is that it is associative

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

$$x(t)*(h_1(t)*h_2(t)) = (x(t)*h_1(t))*h_2(t)$$

Again this can be easily verified by manipulating the summation/integral indices

Therefore, the following four systems are all equivalent and $y[n] = x[n]*h_1[n]*h_2[n]$ is unambiguously defined.



This is not true for non-linear systems ($y_1[n] = 2x[n]$, $y_2[n] = x^2[n]$)

LTI System Memory

- An LTI system is memoryless if its output depends only on the input value at the same time,

$$y[n] = kx[n]$$
$$y(t) = kx(t)$$

- For an impulse response, this can only be true if

$$h[n] = k\delta[n]$$
$$h(t) = k\delta(t)$$

- and the output of dynamic engineering, physical systems depend on:

Preceding values of $x[n-1]$, $x[n-2]$, …

Past values of $y[n-1]$, $y[n-2]$, …

for discrete-time systems, or derivative terms for continuous-time systems

System Invertibility

- Does there exist a system with impulse response $h_1(t)$ such that $y(t)=x(t)$?

$$\xrightarrow{x(t)} \boxed{h(t)} \xrightarrow{w(t)} \boxed{h_1(t)} \xrightarrow{y(t)}$$

- Widely used concept for:
- **control** of physical systems, where the aim is to calculate a control signal such that the system behaves as specified
- **filtering** out noise from communication systems, where the aim is to recover the original signal x(t)
- The aim is to calculate “inverse systems” such that

$$h[n]h_1[n] = \delta[n]$$

$$h(t)h_1(t) = \delta(t)$$

- The resulting **serial** system is therefore **memoryless**

Example: Accumulator System

- Consider a DT LTI system with an impulse response

  $h[n] = u[n]$

- Using convolution, the response to an arbitrary input $x[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- As $u[n\text{-}k] = 0$ for $n\text{-}k<0$ and 1 for $n\text{-}k\geq 0$, this becomes $y[n] = \sum_{k=-\infty}^{n} x[k]$

- i.e. it acts as a running sum or accumulator. Therefore an inverse system can be expressed as: $y[n] = x[n] - x[n-1]$

- A first difference (differential) operator, which has an impulse response

$$h_1[n] = \delta[n] - \delta[n-1]$$

Causality for LTI Systems

- Remember, a causal system response depends only on present and past values of the input signal. We do not use knowledge about future information.

For a discrete LTI system, convolution tells us that $h[n] = 0$ for $n<0$

- as $y[n]$ must not depend on $x[k]$ for $k>n$, as the impulse response must be zero before the pulse!

$$x[n]*h[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

$$x(t)*h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

- Both the integrator and its inverse in the previous example are causal
- This is strongly related to inverse systems as we generally require our inverse system to be causal. If it is not causal, it is difficult to manufacture!

LTI System Stability

- A system is stable if every bounded input produces a bounded output
- Therefore, consider a bounded input signal $|x[n]| < B$ for all n

- Applying convolution and taking the absolute value:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

- Using the triangle inequality (magnitude of a sum of a set of numbers is no larger than the sum of the magnitude of the numbers):

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\le B \sum_{k=-\infty}^{\infty} |h[k]|$$

- Therefore a DT **LTI system is stable** if and only if its impulse response is absolutely summable, ie

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Continuous-time system $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

Example: System Stability

- Are the DT and CT pure time shift systems stable?

$$h[n] = \delta[n - n_0]$$
$$h(t) = \delta(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\delta[k - n_0]| = 1 < \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |\delta(\tau - t_0)| d\tau = 1 < \infty$$

Therefore, both the CT and DT systems are **stable**: all finite input signals produce a finite output signal

- Are the discrete and continuous-time integrator systems stable?

$$h[n] = u[n - n_0]$$
$$h(t) = u(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |u[k - n_0]| = \sum_{k=n_0}^{\infty} |u[k]| = \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |u(\tau - t_0)| d\tau = \int_{t_0}^{\infty} |u(\tau)| d\tau = \infty$$

Therefore, both the CT and DT systems are **unstable**: at least one finite input causes an infinite output signal

References:

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