27.04.21

Lecture 9

ERRORS AND APPROXIMATIONS

Let $z = f(x, y)$ be any function.

If Δx and Δy are small increments in x and y respectively and Δz, the corresponding increment in z, then we have

$$z + \Delta z = f(x + \Delta x, y + \Delta y).$$

$$\Rightarrow \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Using Taylor's expansion of 2 variables,

$$\Delta z = f(x, y) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + \cdots\cdots - f(x, y)$$

Neglecting partial derivatives of 2nd and higher order, we get

$$\boxed{\Delta z = \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}}$$

The value Δz is called the error in $z = f(x, y)$ due to errors Δx & Δy in x, y respectively.

$\frac{\Delta z}{z}$ is called the relative error in z.

$\frac{\Delta z}{z} \times 100$ is called the percentage error in z

Note: If $z = f(x, y)$ then

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$ is known as

Total differential of z or exact differential of z

1. If $PV^2 = k$ and if the relative errors in P is 0.05 and in V is 0.025, then find the error in k.

$$\frac{\Delta P}{P} = 0.05 \qquad \frac{\Delta V}{V} = 0.025$$

$$PV^2 = k$$

$$\ln(PV^2) = \ln k \implies \ln P + 2\ln V = \ln k$$

differentiate the expression

$$\frac{1}{P}\Delta P + \frac{2}{V}\Delta V = \frac{1}{K}\Delta K$$

$$0.05 + 2\times 0.025 = \frac{\Delta K}{K}$$

$$\frac{\Delta K}{K} = 0.1$$

$$\frac{\Delta K}{K}\times 100 = 10\%$$

2 Find the % error in the area of an ellipse when an error of 1% is made in measuring the major and minor axes.

$$A = \pi a b$$

$$\ln A = \ln(\pi a b)$$

$$\ln A = \ln \pi + \ln a + \ln b$$

diff

$$\frac{\Delta A}{A} = 0 + \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

$$\frac{\Delta A}{A}\times 100 = \frac{\Delta a}{a}\times 100 + \frac{\Delta b}{b}\times 100$$

$$= 1\% + 1\%$$

$$= 2\%$$

3. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in the measurement is 0.1 cm. Find approximately the possible error in volume and lateral surface

$$V = \pi r^2 h \qquad\qquad S = 2\pi r h$$

$$\ln V = \ln \pi + 2\ln r + \ln h$$

diff

$$\frac{\delta V}{V} = 0 + 2\frac{\delta r}{r} + \frac{\delta h}{h}$$

$$\Rightarrow \delta V = V\left\{\frac{2\times 0.1}{2} + \frac{0.1}{6}\right\} = \pi \times 4\times 6\left\{0.1 + \frac{0.1}{6}\right\}$$

$$= 8.7969 \text{ cm}^3$$

$S = 2\pi r h$

$\ln S = \ln 2\pi + \ln r + \ln h$

diff $\frac{\Delta S}{S} = 0 + \frac{\Delta r}{r} + \frac{\Delta h}{h}$

$$\Delta S = S\left\{\frac{\Delta r}{r} + \frac{\Delta h}{h}\right\}$$

$$= 2\pi \times 2 \times 6 \left\{\frac{0.1}{2} + \frac{0.1}{6}\right\}$$

$$= \underline{\underline{5.0265}}$$

4. If the kinetic energy T is given by $T = \frac{1}{2} m V^2$, find approximately the change in T as ~~m~~ changes from 49 to 49.5 and V from 1600 to 1590

$\delta m = 49.5 - 49 = 0.5$

$\delta V = 1590 - 1600 = -10$

$T = \frac{1}{2} m V^2$ $\qquad \delta T = ?$

$\ln T = \ln \frac{1}{2} + \ln m + 2\ln V$

$\frac{\delta T}{T} = 0 + \frac{\delta m}{m} + \frac{2\delta V}{V}$ $\qquad T = \frac{1}{2} \times 49 \times 1600^2$

$\delta T = T\left(\frac{0.5}{49} + \frac{2\times(-10)}{1600}\right)$ $\qquad =$

$= -144000$

5. In estimating the cost of a pile of bricks measured as $2m \times 15m \times 1.2m$, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cubic metre and bricks cost ₹ 530 per 1000, find an approximate error in the cost.

Soln:
Let x, y, z be the length, breadth and height of the pile so that its volume

$$V = xyz$$

$$\ln V = \ln x + \ln y + \ln z$$

$$\frac{\Delta V}{V} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z}$$

Given $\frac{\Delta x}{x} \times 100 = \frac{\Delta y}{y} \times 100 = \frac{\Delta z}{z} \times 100 = 1\%$

$$\Rightarrow \frac{\Delta x}{x} = \frac{\Delta y}{y} = \frac{\Delta z}{z} = \frac{1}{100}$$

$V = xyz$
$= 2 \times 15 \times 1.2$
$= 36 \ m^3$

$$V = 36 \ m^3$$

$$\Rightarrow \Delta V = V\left(\frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z}\right)$$

$$= 36\left(\frac{3}{100}\right) = 1.08 \ m^3$$

$\therefore$ Number of bricks in $\Delta V = 1.08 \times 450 = 486$

Thus the error in cost

$1000 \longrightarrow 530$
$486 \longrightarrow ?$

$$\frac{486 \times 530}{1000} = 257.58$$, a loss to the brick seller.

6. If the sides and angles of a plane ΔABC vary in such a way that its circum radius remains constant. P.T $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$

where δa, δb and δc denote small increments in sides a, b and c respectively.

Soln: Let R be the circum-radius of ΔABC. Then

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C$$

differential

$$\delta a = 2R \cos A \cdot \delta A, \quad \delta b = 2R \cos B \, \delta B, \quad \delta c = 2R \cos C \, \delta C$$

$\Rightarrow \quad \frac{\delta a}{\cos A} = 2R\,\delta A, \quad \frac{\delta b}{\cos B} = 2R\,\delta B, \quad \frac{\delta c}{\cos C} = 2R\,\delta C$

Adding these we get

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2R\left[\delta A + \delta B + \delta C\right]$$

$$= 2R\,\delta\left[A + B + C\right]$$

$$= 2R\,\delta(\pi)$$

$$= \underline{0}$$

7\. The indicated horse power I of an engine is computed from $I = \frac{PLAN}{33000}$ where $A = \frac{\pi d^2}{4}$.

Assuming that errors of r percent may have been made in measuring P, L, N and d, find the greatest possible error in I.

$$I = \frac{PLAN}{33\,000} = \frac{PL\pi d^2 N}{4 \times 33{,}000}$$

$$\ln I = \ln P + \ln L + \ln \pi + 2\ln d + \ln N - \ln(4\times 33{,}000)$$

$$\frac{\Delta I}{I} = \frac{\Delta P}{P} + \frac{\Delta L}{L} + 0 + \frac{2\Delta d}{d} + \frac{\Delta N}{N} - 0$$

$$\frac{\Delta I}{I}\times 100 = r\% + r\% + 2r\% + r\%$$

$$= 5r\%$$

$\therefore$ The greatest possible error in I is $5r\%$.

HW i) In an experiment to determine the value of g, using a simple pendulum of length l, errors of 1.5% and 0.5% are possible in the values of l and T (period of of pendulum) respectively. Show that the error in the calculated value of g is 0.5%. The time of oscillation is given by $T = 2\pi\sqrt{\frac{l}{g}}$.

2 The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies as $w l^3 d^{-4}$. What is the increase in the deflection corresponding to $p\%$ increase in w, $q\%$ decrease in l and $r\%$ increase in d ?

3. The work that must be done to propel a ship of displacement D for a distance S in time t is proportional to $\frac{S^2 D^{2/3}}{t^2}$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time diminished by 1% and distance diminished by 3%. Ans $-\frac{10}{3}\%$