Z-Transform

Department of Instrumentation and Control

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Lesson Plan

L. No./ T. No.	Topics	Course Outcome Addressed
L0	Introduction to the subject	CO1
L1	Overview of systems – Introduction to signal processing, signals, systems, classification of signals, introduction to transforms	CO1
L2	Operations on signals, Digital signal Processing, advantages, limitations	CO1
L3	Discrete time fourier transform, convlution and correlation	CO1
L4	TUTORIAL-Convolution of different types of signals, correlation	CO1
L5	Relationship between Laplace transform and z transform	CO1
L6	Representation in Z plane , ROC and its significance	CO1
L7	Z transform of causal and anticausal sequences and the corresponding ROC	CO1
L8	TUTORIAL- Z Transform and ROC	CO1
L9	Z-transform and its properties	CO1
L10	Inverse Z transform	CO1
L11	Analysis of LTI systems using Z transform	CO1
L12	TUTORIAL- analysis of discrete time LTI systems using z transform	CO1

Discrete-Time Fourier Transform

A discrete-time signal can be represented in the frequency domain using discrete-time Fourier transform. Therefore, the Fourier transform of a discretetime sequence is called the *discrete-time Fourier transform (DTFT)*.

Mathematically, if $x(n)$ is a discrete-time sequence, then its discrete-time Fourier transform is defined as –

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

The discrete-time Fourier transform X(ω) of a discrete-time sequence $x(n)$ represents the frequency content of the sequence $x(n)$. Therefore, by taking the Fourier transform of the discrete-time sequence, the sequence is decomposed into its frequency components. For this reason, the DTFT X(ω) is also called the **signal spectrum.**

Condition for Existence of Discrete-Time Fourier Transform

The Fourier transform of a discrete-time sequence $x(n)$ exists if and only if the sequence $x(n)$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

The discrete-time Fourier transform (DTFT) of the exponentially growing sequences do not exist, because they are not absolutely summable.

Also, the DTFT method of analysing the systems can be applied only to the asymptotically stable systems and it cannot be applied for the unstable systems, i.e., the DTFT can only be used to analyse the systems whose transfer function has poles inside the unit circle.

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Example 1: $\mathbf{x[n] = a^n\ u(n)}$ where a>1

$$\sum_{\mathbf{n=-\infty}}^{\infty} |\mathbf{x(n)}| = \sum_{\mathbf{n=0}}^{\infty} \mathbf{a^n} \to \infty$$

DTFT does not exist

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Relation between Z-transform and DTFT

Taking a look at the equations describing the Z-Transform and the Discrete-Time Fourier Transform:

Discrete-Time Fourier Transform

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-(j\omega n)}$$

Z-Transform

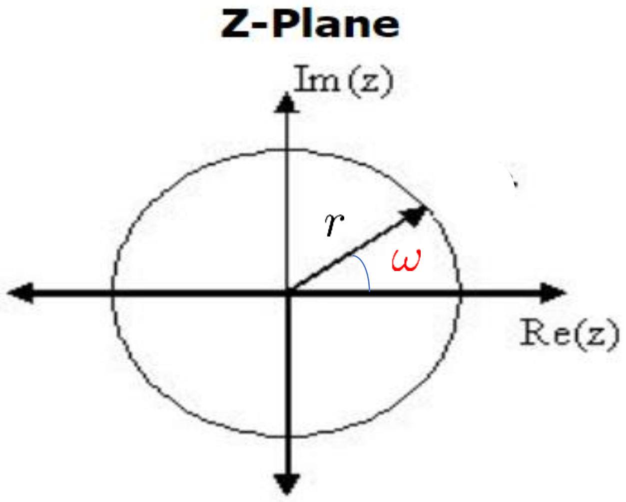
$$\mathbf{X[z] = \sum_{-\infty}^{\infty} x[n]z^{-n}} \quad \text{where } \mathbf{z = r\ e^{j\omega}}$$

$$\mathbf{X[z] = \sum_{-\infty}^{\infty} x[n](r\ e^{j\omega})^{-n}} \qquad \text{when } \mathbf{r = 1} \quad \text{DTFT} = \text{Z Transform}$$

Z-Transform

$$\mathbf{X[z] = \sum_{-\infty}^{\infty} x[n] z^{-n}} \qquad \text{where } \mathbf{z = r\ e^{j\omega}}$$

The Z-plane is a complex plane with an imaginary and real axis referring to the complex-valued variable Z



Z-Transform and ROC

G. S. $\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

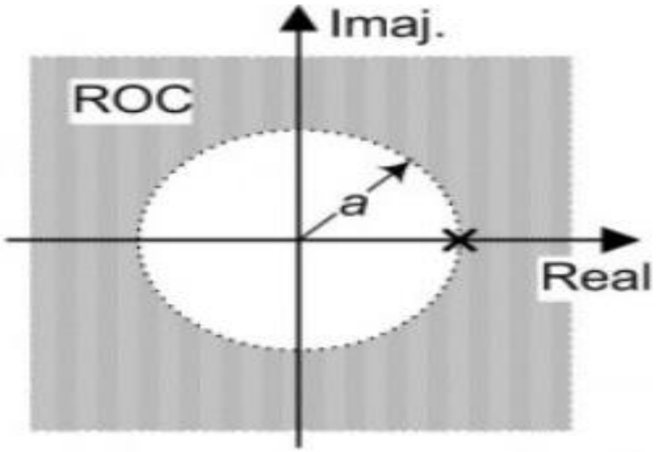
$$\mathbf{X[z]} = \sum_{\mathbf{n=-\infty}}^{\infty} \mathbf{x[n]z^{-n}} \quad \text{where } \mathbf{z = r\ e^{j\omega}}$$

Example 1: $\mathbf{x[n] = a^n\ u(n)}$ where a>1

$$\mathbf{X[z]} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{a^n z^{-n}} = \sum_{\mathbf{n=0}}^{\infty} (\mathbf{az^{-1}})^{\mathbf{n}}$$

when will this G.S. converge ??

when $\frac{\mathbf{a}}{|\mathbf{z}|} < 1$ or $|\mathbf{z}| > \mathbf{a}$

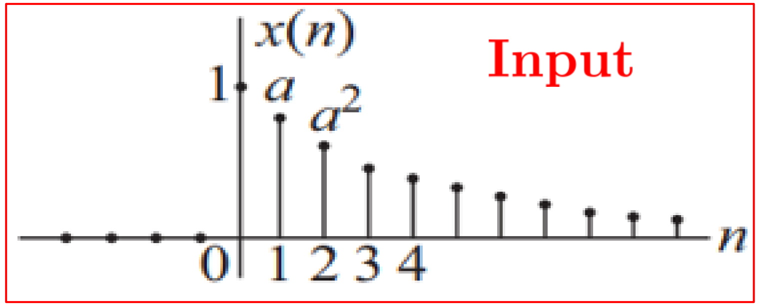


Right-Sided Exponential Sequence

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{because} \quad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



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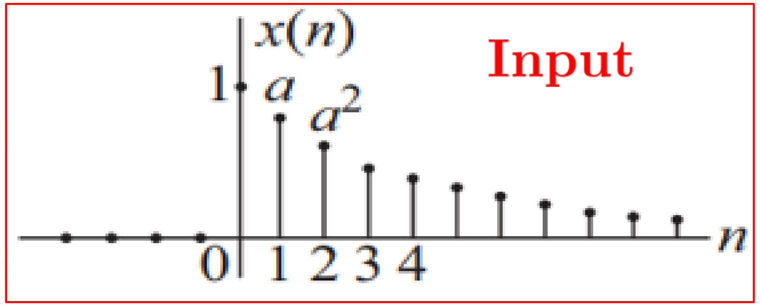

$$\therefore \quad Z[x(n)] = Z[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

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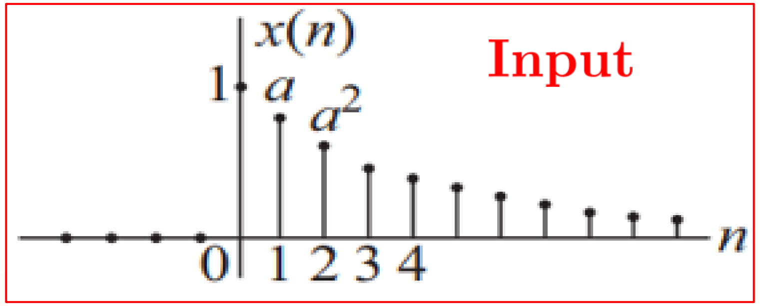

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This is a geometric series of infinite length, and converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$.

$$\therefore \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}; \text{ ROC; } |z| > |a|$$

which implies that the ROC is exterior to the circle of radius a as shown in Figure 3.1(a)

$$\boxed{a^n u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - az^{-1}} = \frac{z}{z - a}; \text{ ROC; } |z| > a}$$

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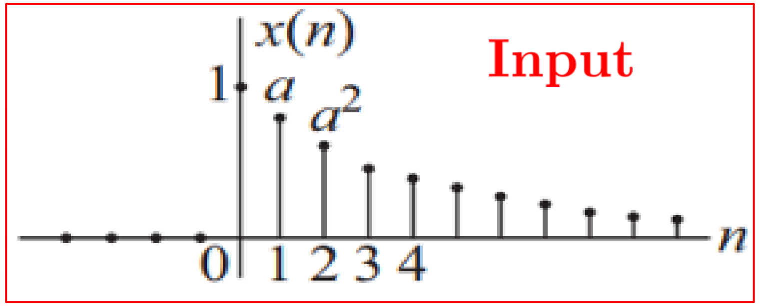

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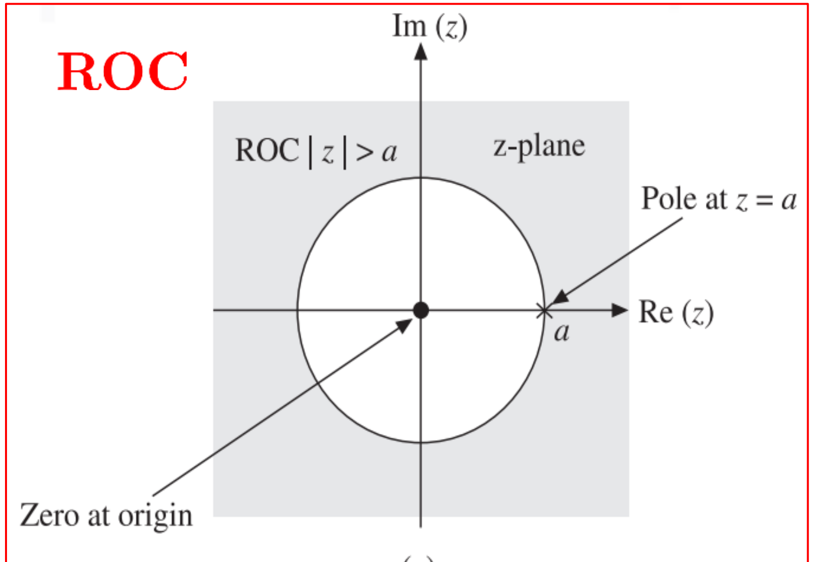


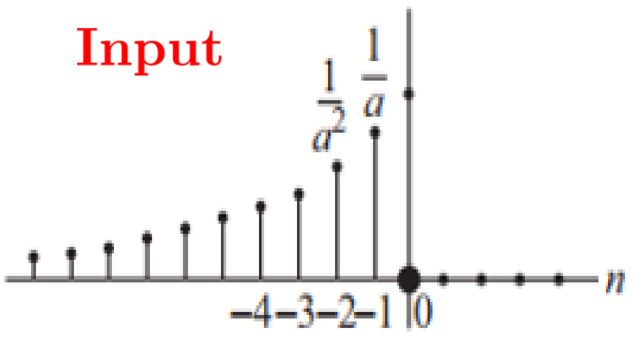
(a)

Left-Sided Exponential Sequence

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Next consider the *left-sided* discrete time signal $x_2[n] = -a^n u[-n-1]$

$|a| > 1$, a real-valued



$$X_2(z) \equiv -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m \quad \textbf{where } \mathbf{m = -n}$$

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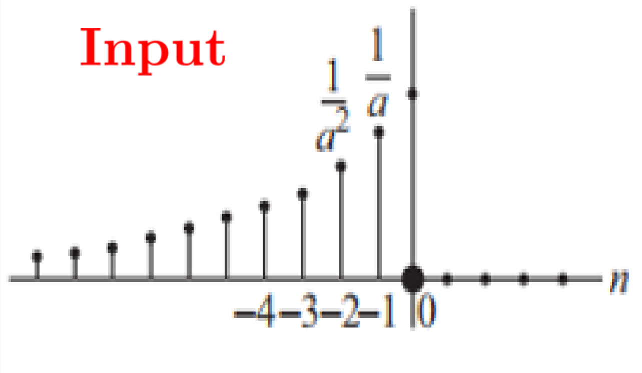

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$$\text{for} \quad \left|\frac{z}{a}\right| < 1 \quad \Rightarrow |z| < |a|$$

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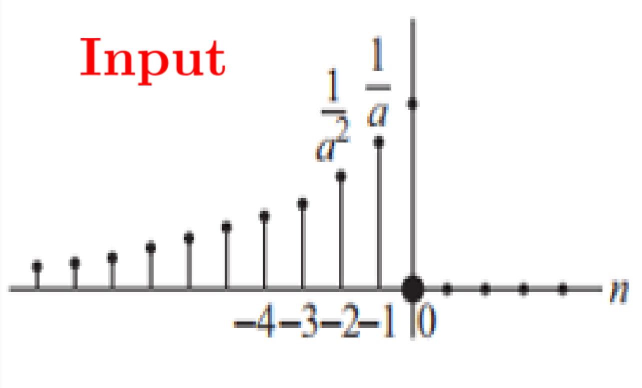

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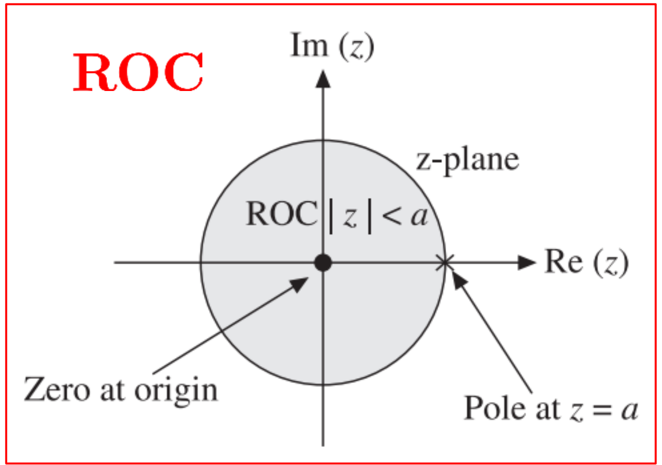
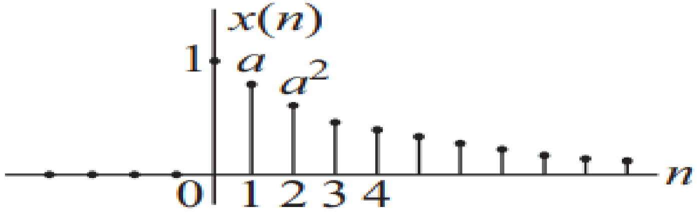


ROC provides the information about the signal in time domain

Right-Sided Exponential Sequence

$$a^n u(n) \longleftrightarrow \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

The ROC for $X(z)$ is $|z| > |a|$, as shown in the shaded area in Figure.



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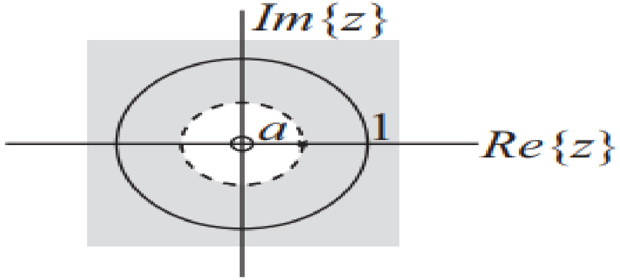

(b)

$$-a^n u(-n-1) \longleftrightarrow \frac{1}{1 - az^{-1}}, \qquad |z| < |a|$$

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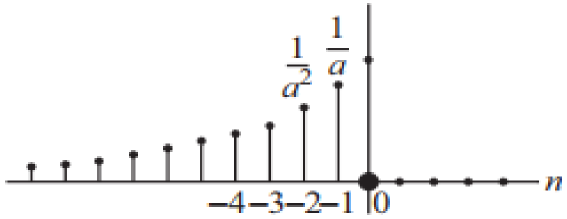

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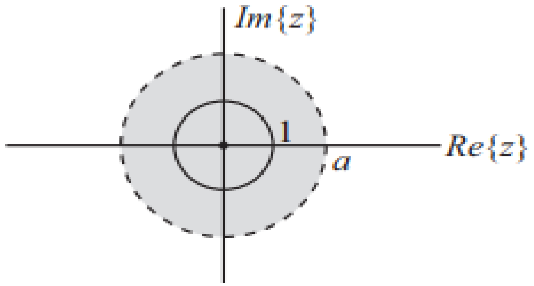

(b)

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EXAMPLE 3.6 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, -2, 3, 5, 4\}$$

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For the given sample values,

$$X(z) = x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + x(4)\,z^{-4} + x(5)\,z^{-5}$$

$$\therefore \qquad \mathrm{Z}\,[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

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The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 3.8 Find the Z-transform and ROC of the anticausal sequence.

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We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2) z^2 + x(-1)\, z + x(0)$$

$$\therefore \qquad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

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$\therefore$ $$Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

EXAMPLE 3.9 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

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$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Example 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Example 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$X_1[z] = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC: entire Z-plane except Z=0

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X_2[z] = z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

ROC: entire Z-plane except Z=0 and Z=∞

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

$$X_3[z] = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$ **ROC: entire Z-plane except Z=0**

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$$X_4[z] = 2 + 4z^{-1} + 5z^{-2} + 7z^{-3} + z^{-4}$$

ROC: entire Z-plane except Z=0

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(f) $x_6(n) = \delta(n-k),\ k > 0$

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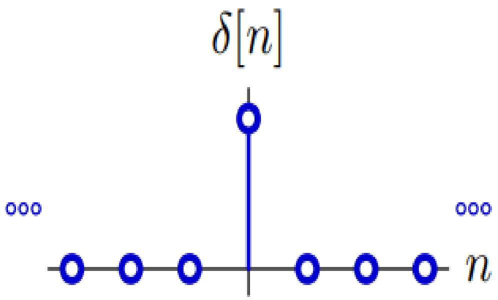
$$X_6[z] = z^{-k}$$

ROC: entire Z-plane except Z=0

Simple Z transforms

Find the Z transform of the unit-sample signal.



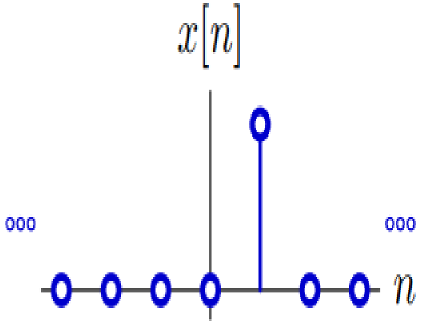
$x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0} = 1$$

ROC=?

Simple Z transforms

Find the Z transform of a delayed unit-sample signal.



$x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[1]z^{-1} = z^{-1}$$

ROC=?

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(g) $x_7(n) = \delta(n+k), k > 0$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

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$$X_7[z] = z^k$$

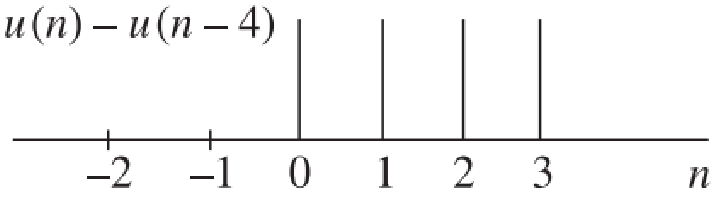
ROC: entire Z-plane except Z=∞

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

EXAMPLE 3.10 Find the Z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$

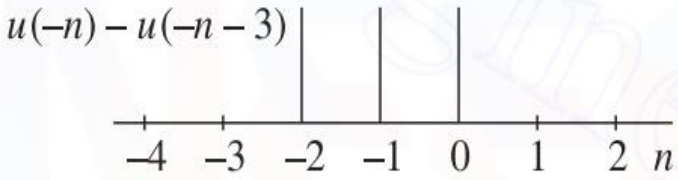
(a)



(a) ANS: $X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$

The ROC is entire z-plane except at $z = 0$.

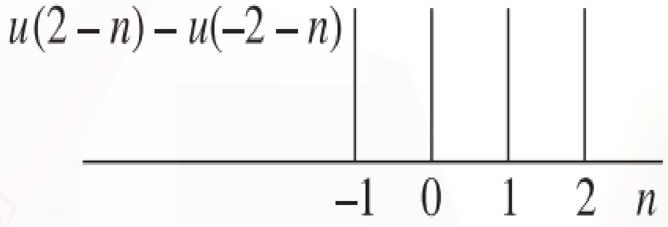
(b)



(b) ANS: $X(z) = 1 + z + z^{2}$

The ROC is entire z-plane except at $z = \infty$.

(c)



(c) ANS: $X(z) = z + 1 + z^{-1} + z^{-2}$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

Z-Transform of Unit Step Function

The *unit step signal* or *unit step sequence* is defined as −

$$x(n) = u(n) = \{ 1 \quad \text{for } n \geq 0 \; 0 \quad \text{for } n < 0 \}$$

Therefore, the Z-transform of unit step function is given by,

$$Z[x(n)] = X(z) = Z[u(n)]$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} (1) \cdot z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

$$\Rightarrow X(z) = \frac{1}{(1 - z^{-1})} = \frac{z}{z - 1}$$

$\mathrm{ROC} \rightarrow |z| > 1$

Z-Transform of Unit Ramp Sequence

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}; \quad a < 1$$

The unit ramp sequence is defined as -

$$x(n) = r(n) = \{ n \quad \text{for } n \geq 0 \; 0 \quad \text{for } n < 0 \}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} r(n) z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} n z^{-n} = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots$$

$$\Rightarrow X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}$$

$$\text{ROC} \rightarrow |z| > 1$$

Sum of Two Exponential Sequences

Example 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Sum of Two Exponential Sequences

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Determine the z-transform of the signal

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$$\mathbf{X[z] = \frac{1}{1 - az^{-1}} - \frac{1}{1 - bz^{-1}}}$$

$\mathbf{ROC}\ |\mathbf{z}| > |\mathbf{a}| \cap |\mathbf{z}| < |\mathbf{b}|$

Sum of Two Exponential Sequences

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Determine the z-transform of the signal

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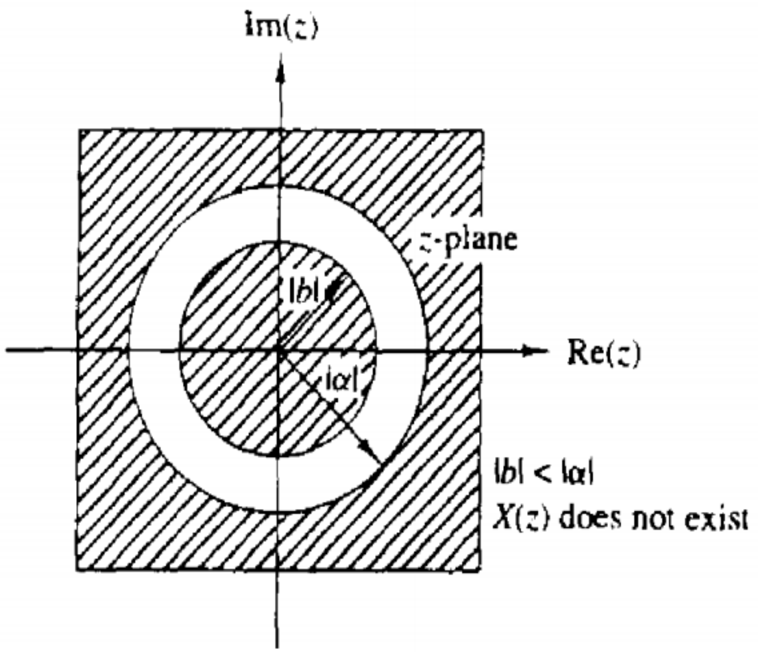
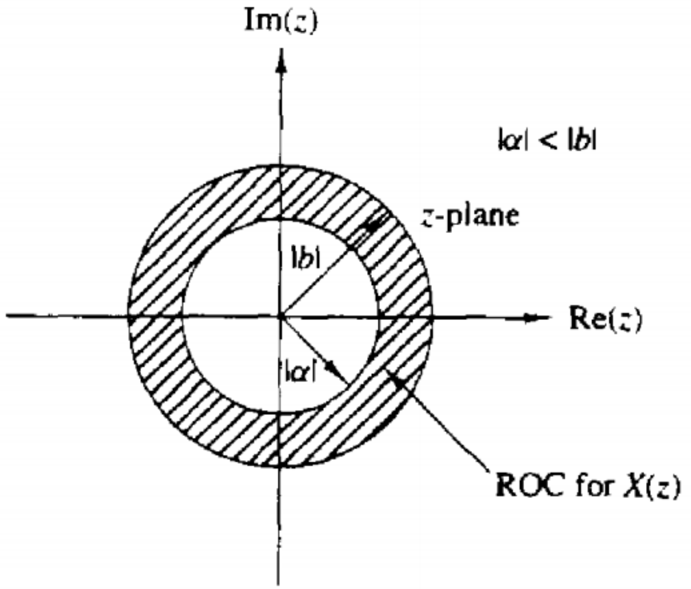
ROC $\mathbf{|z| > |a| \cap |z| < |b|}$

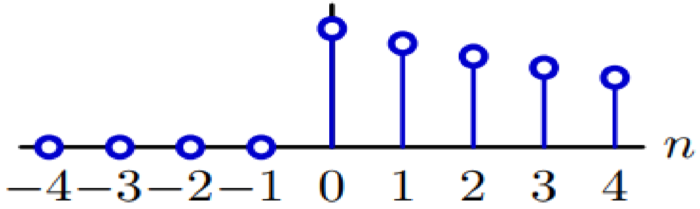




What is the Z transform of the following signal.

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8} z^{-1}}$$

What is the Z transform of the following signal.

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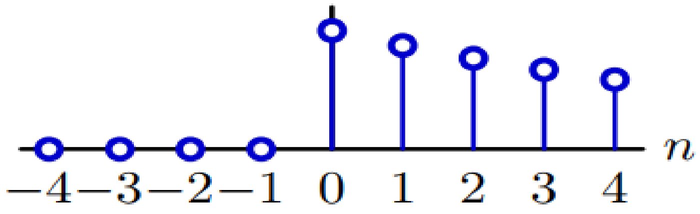


$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8} z^{-1}}$$

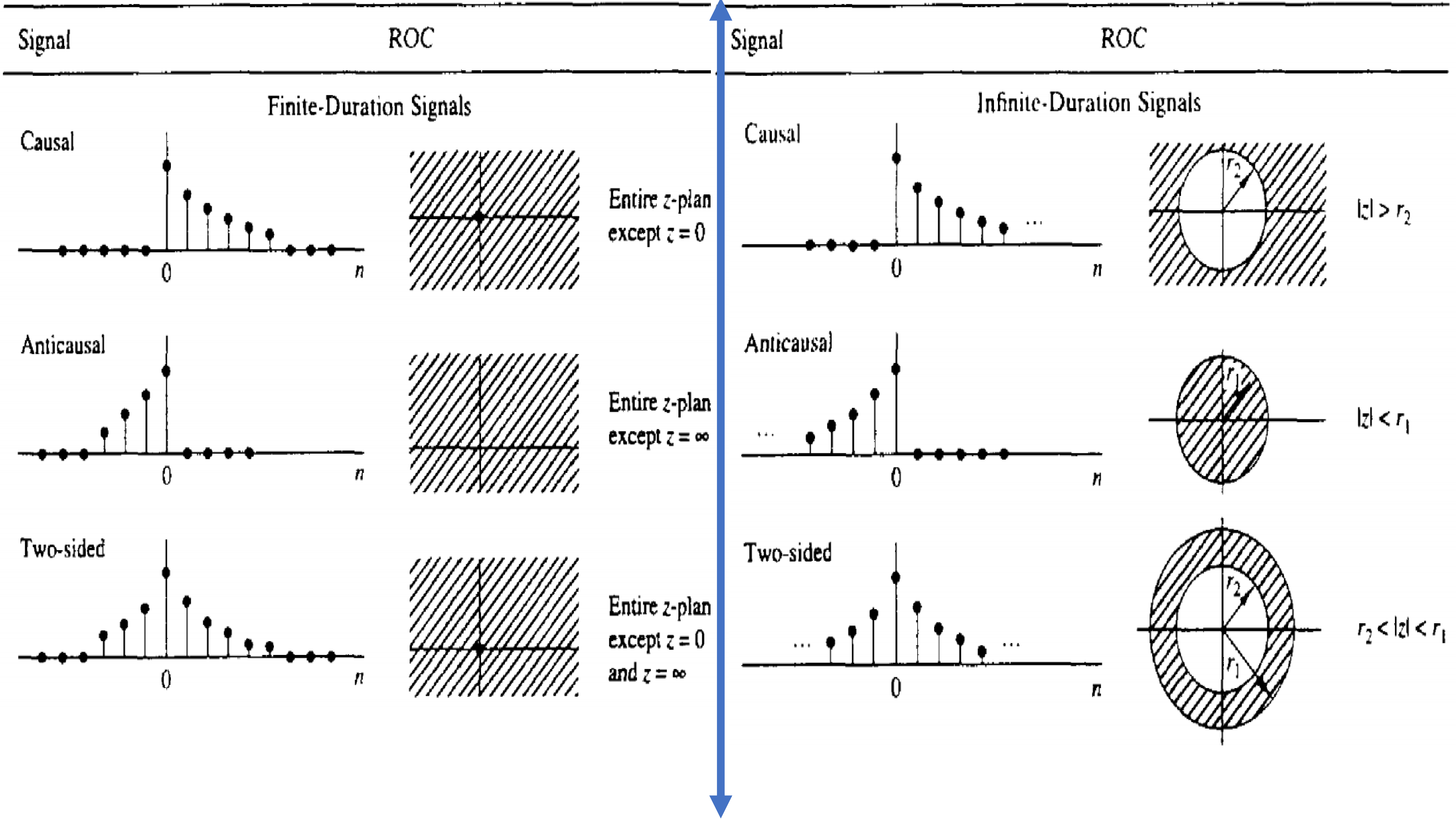
Regions of Convergence

The Z transform $X(z)$ is a function of z defined for all z inside **Region of Convergence (ROC)**.

$$x[n] = \left(\frac{7}{8}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1 - \frac{7}{8} z^{-1}}\,; \quad |z| > \frac{7}{8}$$

ROC: $|z| > \dfrac{7}{8}$

Properties of ROC of Z-Transforms


Signal
ROC
Finite-Duration Signals
Causal
Entire z-plan except z = 0
Anticausal
Entire z-plan except z = ∞
Two-sided
Entire z-plan except z = 0 and z = ∞
Signal
ROC
Infinite-Duration Signals
Causal
|z| > r2
Anticausal
|z| < r1
Two-sided
r2 < |z| < r1

3.4 PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.
   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.
   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.
5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

SUMMATION FORMULAS FOR GEOMETRIC SERIES

G. S. $\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

1. $$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$$

2. $$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a}; \quad |a| < 1$$

3. $$\sum_{k=n}^{\infty} a^k = \frac{a^n}{1 - a}; \quad |a| < 1$$

4. $$\sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1 - a}; \; n_2 > n_1$$

5. $$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1 - a)^2}; \quad a < 1$$