FORMULA BOOK – EI

COMPLEX VARIABLES

1. A complex number $z = x + iy$ where x and y are real numbers and $i = \sqrt{-1}$

2. Modulus of z, $|z| = \sqrt{x^2 + y^2}$

3. Complex conjugate $\bar{z} = x - iy$

4. Polar form $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

5. Cauchy-Riemann Equations: If $f(z) = u + iv$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

6. If $f(z) = u + iv$ is analytic function, then $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

7. Cauchy-Riemann Equations in polar form: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\,\frac{\partial v}{\partial r}$

8. Cauchy's Integral Formula: Let $f(z)$ be analytic in a simply connected domain D. Let C be any simple closed curve in D enclosing any point $'a'$ in D. Then $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, where C is traversed in counter clockwise direction.

POWER SERIES:

9. A power series at center at a point $'a'$ is an infinite series of the form $\sum_{n=0}^{\infty} c_n (z-a)^n = c_0(z-a)^1 + c_1(z-a)^2 + \cdots$ where z is a variable , for each n, c_n is a constant called coefficient of power series.

   Laurent's series: Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with the center $'a'$ and the annulus between them. Then $f(z)$ can be represented by the Laurent series, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$. The coefficients of the Laurent series are given by the integrals $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}} dz$ and $b_n = \frac{1}{2\pi i}\oint_C (z-a)^{n-1} f(z) dz$ taken counter clockwise around any simple closed path C that lies in the annulus and encircles the inner circle.

10. $Res_{z=a}\, f(z) = coefficient\ of\ \frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$.

11. If $'a'$ is a simple pole: $Res_{z=a}\, f(z) = \lim_{z \to a} (z-a) f(z)$

12. If $f(z) = \frac{P(z)}{Q(z)}$, where $Q(z)$ has a simple zero at $z = a$, then $Res_{z=a}\, f(z) = \frac{P(a)}{Q'(a)}$

13. If $z = a$ is a pole of order n of $f(z)$, then $Res_{z=a}\, f(z) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}[(z-a)^n f(z)]}{dz^{n-1}}$

FOURIER SERIES AND TRANSFORMS

A function $f(t)$ is said to be periodic if there exists a real number T such that $f(t+T)=f(t)$ for all t. Smallest such number is called period of the function.

Note: If $f(t)$ is periodic with period T, then $f(t+nT)=f(t)$ for all integers n.

If $f(x)$ is a periodic function with period $2l$ and is known in the interval $\alpha<x<\alpha+2l$, then the Fourier series expansion of $f(x)$ in terms of an infinite sum of sines and cosines is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

where $a_0=\frac{1}{2l}\int_{\alpha}^{\alpha+2l}f(x)\,dx$, $a_n=\frac{1}{l}\int_{\alpha}^{\alpha+2l}f(x)\cos\frac{n\pi x}{l}dx$, $b_n=\frac{1}{l}\int_{\alpha}^{\alpha+2l}f(x)\sin\frac{n\pi x}{l}\,dx$

These expressions for a_0,a_n,b_n are known as Euler's formulae.

Note: The graph of an even function is symmetric about y –axis, whereas the graph of an odd function is symmetric about the origin.

HALF RANGE SERIES:

While solving various physical and engineering problems, there is a practical need for expanding functions defined over a finite range. Such an expansion is possible if functions under consideration can be extended to a periodic function which is either even or odd.

Consider a piecewise continuous function $f(x)$, defined in a finite interval $(0,l)$. Then it is possible to extend $f(x)$ to a periodic function, which is even or odd.

Consider the function $g(x)$ defined as follows:

$$g(x)=\begin{cases}f(x), & 0<x<l\\ f(-x), & -l<x<0\end{cases};\ g(x+2l)=g(x).$$

Then $g(x)$ is called an even periodic extension of $f(x)$. The function $g(x)$ can be expanded as Fourier cosine series $g(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{l}x$

where $a_0=\frac{1}{l}\int_0^l f(x)dx$ and $a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx$.

Such an expansion of $f(x)$ is called the half range Fourier Cosine series expansion of $f(x)$.

Also, if $g(x)=\begin{cases}f(x), & 0\le x\le l\\ -f(-x), & -l<x<0\end{cases};\ g(x+2l)=g(x).$

Then $g(x)$ is called an odd periodic extension of $f(x)$. The function $g(x)$ can be expanded as Fourier Sine series

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{l}x$$

where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$.

FOURIER INTEGRAL REPRESENTATION

Let $f(x)$ be a piecewise continuous and absolutely integrable function of x. Then $f(x)$ can be represented by an integral as

$f(x) = \frac{1}{\pi}\int_0^{\infty}(A(s)\cos sx + B(s)\sin nx)ds$ ………………………………………………… (1)

where $A(s) = \int_{-\infty}^{\infty} f(t)\cos st\, dt, B(s) = \int_{-\infty}^{\infty} f(t)\sin st\, dt$.

Such an integral representation is called the Fourier integral.

FOURIER TRANSFORMS

Consider the Fourier integral representation of the function $f(x)$ given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(x-t)}dt\, ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt\right]ds$$

Let $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$ ……………………………………………………………… (3)

Then $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$ ……………………………………………………………… (4)

The integral defined by (3) is called the Fourier transform of the function $f(x)$ and is denoted by $F\{f(x)\}$.

Given $F(s) = F\{f(x)\}$, the formula (4) defined $f(x)$, which is called the inverse Fourier transform of $F(s)$and is denoted by $F^{-1}\{(F(s)\}$.

Note: A function $f(x)$ is said to be self-reciprocal under Fourier transforms if $F\{f(x)\} = f(s)$.

Properties of the Fourier transforms:

If $F\{f(x)\} = F(s)$, then

1. $F\{e^{iax}f(x)\} = F(s-a)$.
2. $F\{f(x-a)\} = e^{isa}F(s)$.
3. $F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), a > 0$.
4. $F\{\overline{f(-x)}\} = \overline{F(s)}$.
5. $F\{ f(-x)\} = F(-s)$.
6. $F\{\overline{f(x)}\} = \overline{F(-s)}$.
7. $F\{ x^n f(x)\} = i^n \frac{d^n}{ds^n} F(s)$.
8. $F\{ f^n(x)\} = (is)^n F(f(x))$.
9. $F\{ f(x)cosax\} = \frac{1}{2} (F(s+a) + F(s-a))$.
10. $F\{ f(x)sinax\} = \frac{i}{2} \left(F(s+a) - F(s-a)\right)$

Convolution: For functions $f(x)\,\&\,g(x)$, we define the convolution of $f(x)\,\&\,g(x)$ denoted by $(f*g)(x)$ as $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$; provided the integral exists.

Note that $f * g = g * f$.

Convolution Theorem:

$F\{ (f * g)(x)\} = F\{ f(x)\}F\{ g(x)\}$.

Parseval's Identity:

If $F\{f(x)\} = F(s)$, then $\int_{-\infty}^{\infty} | f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$.

FOURIER COSINE AND SINE TRANSFORMS:

Consider the Fourier cosine integral representation of a function $f(x)$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\cos st \cos sx \; dtds, \quad x \geq 0$$

$$= \sqrt{\frac{2}{\pi}}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos st \; dt\right]\cos sx \; ds$$

Let $F_c\{ f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos st \; dt = F_c\,(s)$ ……………………………………………….(5)

Then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c\{ f(x)\}\cos sx \; ds$ ……………………………………………………………(6)

The transform $F_c\{f(x)\}$ defined by (5) is called the Fourier cosine transform of $f(x)$. The formula (6) is called the inverse Fourier cosine transform of $F_c\{f(x)\} = F_c(s)$ and is denoted by $f(x) = F_c^{-1}\{F_c(s)\}$.

Similarly, using the Fourier sine integral representation of $f(x)$ given by $f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin st \sin sx \; dt \; ds$, we can define the Fourier sine transform of $f(x)$ denoted by $F_s\{f(x)\}$ as $F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin st \; dt = F_s(s)$.

Then the inverse Fourier sine transform of $F_s(s)$ is defined as $f(x) = F_s^{-1}\{F_s(s)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx \; ds$.

Definition: A function f(x) is said to be self-reciprocal under Fourier cosine (sine) transform if

$F_c\{f(x)\} = f(s) \;\; \left(F_s\{f(x)\} = f(s)\right)$.

Properties of Fourier cosine/ sine transforms:

1. Both Fourier cosine and sine transforms are linear.
   $F_c\{c_1 f(x) + c_2 g(x)\} = c_1 F_c\{f(x)\} + c_2 F_c\{g(x)\}$
   $F_s\{c_1 f(x) + c_2 g(x)\} = c_1 F_s\{f(x)\} + c_2 F_s\{g(x)\}$ where c_1 and c_2 are constants.
2. $F_c\{f(x)cosax\} = \frac{1}{2}\left(F_c(s+a) + F_c(s-a)\right)$

   $F_c\{f(x)sinax\} = \frac{1}{2}\left(F_s(s+a) + F_s(a-s)\right)$
3. $F_s\{f(x)cosax\} = \frac{1}{2}\left(F_s(s+a) + F_s(s-a)\right)$

   $F_s\{f(x)sinax\} = \frac{1}{2}\left(F_c(s-a) - F_s(s+a)\right)$
4. $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{s}{a}\right)$

$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

5. If $f(x) \to 0$ as $x \to \infty$, then

   $F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + sF_s(s)$ and $F_s\{f'(x)\} = -sF_c(s)$

6. $F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}} f'(0) - s^2 F_c(s)$ and $F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} sf\ (0) - s^2 F_s(s)$ provided $f(x)$ and $f'(x) \to 0$ as $x \to \infty$.

7. $F_c\{xf(x)\} = \frac{dF_s}{ds}$ and $F_s\{xf(x)\} = -\frac{dF_c}{ds}$.

8. If $F_c\{f(x)\} = F_c(s), F_c\{g(x)\} = G_c(s), F_s\{f(x) = F_s(s)$ and $F_s\{g(x)\} = G_s(s)$ exist, then $\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty f(x)g(x)dx$ and

   $\int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |F_s(s)|^2\, ds = \int_0^\infty |f(x)|^2\, dx$ which is called Parseval's Identity.

PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equation is an equation involving partial derivatives of an unknown functions of two or more independent variables.

The order of the highest derivative is called order of the equation.

The degree of the Partial differential equation is the degree of highest derivative after clearing the fractional power.

Partial differential equation is linear if it is of the first degree in the dependent variable and its partial derivatives.

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	One dimensional Homogeneous wave equation
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	One dimensional Homogeneous heat equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Two dimensional Homogeneous Laplace equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	Two dimensional non Homogeneous with (f$\neq$ 0) Poisson equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	Three dimensional Laplace equation

METHOD OF SEPARATION OF VARIABLES:

It involves a solution which breaks up into a product of functions each of which contain only one of the variable.

Then, the solution for the partial differential equation in is in the form $u(x,t) = X(x) \,.\, T(t)$ where X and T are the function of x and t respectively.

Substitute the partial derivatives involved in the equation and separate the variables, and equate to a common constant c^2.

That is, $F(x) = T(t) = c^2$.

Solving the differential equations $F(x) = c^2$ and $T(t) = c^2$, we get, $X(x)$ and $T(t)$. The required solution is, $u(x,t) = X(x) \,.\, T(t)$

D'ALEMBERT'S SOLUTION OF WAVE EQUATION:

D'Alembert's solution for wave equation $u_{tt} = c^2\, u_{xx}$ in the general form is

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

When $u(x,0) = f(x), \frac{\partial u\,(x,\ 0)}{\partial t} = 0$, D'Alembert's solution for wave equation $u_{tt} = c^2\, u_{xx}$ is

$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

1. One-dimensional Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{m}$

   T-tension of the string, *m*-mass per unit length of the string.

2. Solution of the One-dimensional wave equation:
   $u = (C_1 cos\ px + C_2 \sin px)(C_3 cos\ cpt + C_4 \sin cpt)$.

3. One-dimensional heat-flow equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{k}{s\rho}$.

   ρ-density, *s*-the specific heat, *k*-thermal conductivity.

4. Solution of one-dimensional heat-flow equation:

   $u = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t}$.

VECTOR CALCULUS

VECTOR DIFFERENTIATION:

1) The differential operator ∇ is defined as $\nabla = \frac{\partial}{\partial x}\,\boldsymbol{i} + \frac{\partial}{\partial y}\,\boldsymbol{j} + \frac{\partial}{\partial z}\,\boldsymbol{k}$
2) Let $\emptyset = \emptyset(x, y, z)$ is the scalar field, the gradient of $\emptyset$ at the point (x, y, z) is $grad\emptyset = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\,\boldsymbol{i} + \frac{\partial\emptyset}{\partial y}\,\boldsymbol{j} + \frac{\partial\emptyset}{\partial z}\,\boldsymbol{k} = \sum\frac{\partial\emptyset}{\partial x}\,\boldsymbol{i}$
3) Let $\vec{f} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ is the vector field, the divergence of $\vec{f}$ at the point (x, y, z) is, $div\vec{f} = \nabla.\,\vec{\boldsymbol{f}} = \frac{\partial \boldsymbol{f_1}}{\partial x} + \frac{\partial \boldsymbol{f_2}}{\partial y} + \frac{\partial \boldsymbol{f_3}}{\partial z} = \sum\frac{\partial \boldsymbol{f_1}}{\partial x}$
4) If $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are the two surfaces, then the angle between their surfaces at (x_1, y_1, z_1) is
$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}.$$
5) If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is the given direction vector, then the directional derivative of ϕ along $\hat{n}$ is $\nabla\phi.\hat{n}$, where $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$.
6) If $\phi(x, y, z) = c$ be the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then
   a) equation of tangent plane at P is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.
   b) Equation of normal line at P is $\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$,
   where $A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)}, B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)}, C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}$.
7) The directional derivative of a scalar function ϕ at any point is maximum along $\nabla\phi$.
8) The Laplacian operator ∇^2 is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
9) The Laplacian of a scalar function $\emptyset$ as, $\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$
10) Let $\vec{f} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ is the vector field, the curl of $\vec{f}$ at the point (x, y, z) is $curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$
11) If $\nabla \times \vec{f} = \mathbf{0},$ then the vector field $\vec{f}$ is irrotational
12) If $\nabla.\,\vec{f} = \mathbf{0},$ then the vector field $\vec{f}$ is solenoidal
13) $div(\vec{f} \times \vec{g}) = \vec{g}.\boldsymbol{curl}\vec{f} - \vec{f}.\boldsymbol{curl}\vec{g}$
14) $curl\,\boldsymbol{\nabla}\emptyset = \mathbf{0} = div(curl\vec{f})$
15) $div(\nabla\phi) = \nabla^2\phi$.
16) $\nabla \times (\nabla \times \vec{f}) = \nabla(\nabla.\vec{f}) - \nabla^2\vec{f}$.
17) $\nabla.(\phi\vec{f}) = \phi(\nabla.\vec{f}) + (\nabla\phi).\vec{f}$
18) $\nabla \times (\phi\vec{f}) = \phi(\nabla \times \vec{f}) + \nabla\phi \times \vec{f}$
19) $\nabla(\overrightarrow{f_1} \times \overrightarrow{f_2}) = \overrightarrow{f_2}.(\nabla \times \overrightarrow{f_1}) - \overrightarrow{f_1}.(\nabla \times \overrightarrow{f_2})$

20) $\nabla \times (\overrightarrow{f_1} \times \overrightarrow{f_2}) = \overrightarrow{(\nabla.\overrightarrow{f_2})f_1} - (\nabla.\overrightarrow{f_1})\overrightarrow{f_2} + (\overrightarrow{f_2}.\nabla)\overrightarrow{f_1} - (\overrightarrow{f_1}.\nabla)\overrightarrow{f_2}$

VECTOR INTEGRATION:

LINE INTEGRALS

Suppose $\mathrm{r(u)} = \mathrm{x(u)}\mathbf{i} + \mathrm{y(u)}\mathbf{j} + \mathrm{z(u)}\mathbf{k}$ is the position vector of points P(x,y,z) and suppose **r**(u) defines a curve C joining points P_1 and P_2 where $u = u_1$ and $u = u_2$ respectively. Let $\mathbf{A}(\mathrm{x,y,z}) = \mathrm{A}_1\mathbf{i} + \mathrm{A}_2\mathbf{j} + \mathrm{A}_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the line integral is defined by $\int_{P_1}^{P_2} A \bullet dr = \int_C A \bullet dr = \int_C A_1 dx + A_2 dy + A_3 dz$.

1) If **A** is the force **F** on a particle moving along C, then the line integral represents the work done by the force Work done= $\int_C F \bullet dr$
2) If C is a closed curve, the integral around C is often denoted by $\oint A \bullet dr = \oint A_1 dx + A_2 dy + A_3 dz$
3) In Aerodynamics and fluid dynamics, this line integral is called the circulation of **A** about C, where **A** represents the velocity of a fluid.

SURFACE INTEGRALS

4) Let S be a two- sided surface. A unit normal **n** to any point of the positive side of S is called a positive or outward drawn unit normal. Associate with the differential of surface area dS a vector d**S** and whose direction is that of **n.** Then d**S** = **n** dS. The integral $\iint_S A \bullet dS = \iint_S A \bullet \mathrm{n dS}$ is called surface integral or the flux of **A** over S.
5) Suppose that the surface S has projection R on the xy-plane, then
$$\iint_S A \bullet dS = \iint_R A \bullet n \frac{dxdy}{|n \bullet k|}$$
6) Suppose that the surface S has projection R on the yz-plane, then
$$\iint_S A \bullet dS = \iint_R A \bullet n \frac{dydz}{|n \bullet i|}$$
7) Suppose that the surface S has projection R on the xz-plane, then
$$\iint_S A \bullet dS = \iint_R A \bullet n \frac{dxdz}{|n \bullet j|}$$

VOLUME INTEGRALS

8) Consider a closed surface in space enclosing a volume V. Then $\iiint_V A\,dV$ and $\iiint_V \phi\,dV$ are called volume integrals or space integrals. Here ϕ is a scalar function.

9) **Divergence Theorem of Gauss :** Suppose V is the volume bounded by a closed surface S and A is a vector function of position with continuous derivatives. Then

$$\iiint_V \nabla \bullet A\,dV = \iint_S A \bullet \mathrm{n}\,\mathrm{dS} = \oiint_S A \bullet dS$$

Where **n** is the positive (outward drawn) normal to S.

10) **Stoke's theorem:** Suppose S is an open, two- sided surface bounded by a closed, non-intersecting curve C (simple closed curve), and suppose **A** is a vector function of position with continuous derivatives. Then

$\oint_C A \bullet dr = \iint_S (\nabla \times A) \bullet n\,dS = \iint_S (\nabla \times A) \bullet dS$ where C is traversed in the positive direction.

11) **Green's Theorem in the plane:** Suppose R is a closed region in the xy-plane bounded by a simple closed curve C, and suppose M and N are continuous functions of x and y having continuous derivatives in R. Then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$ where C is traversed in the positive (counter- clockwise) direction.

PROBABILITY

Addition rule

If A and B are two events of an experiment having sample space S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The conditional probability of an event B, given that the event A already taken place is

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

Baye's Theorem

Let $B_1, B_2, ...B_k$ are the partitions of S with $P(B_i) \neq 0,\ i = 1, 2, ..., k$ and A be any event of S, then

$$P(B_i / A) = \frac{P(A/B_i)P(B_i)}{\sum_{i=1}^{k} P(A/B_i)P(B_i)}.$$

The multiplicative rule of probability: $P(A \cap B) = \begin{cases} P(A)P(B|A), & if\ P(A) \neq 0 \\ P(B)P(A|B), & if\ P(B) \neq 0 \end{cases}$

If $P(A \cap B) = P(A)P(B)$, then A and B are independent.

Random Variable: Let S be the sample space of a random experiment. Suppose with each element s of S, a unique real number X is associated according to some rule then X is called random variable. There are two types of random variable, i) Discrete and ii) Continuous.

Discrete Random Variable: A random variable X is said to be discrete, if the number of possible values of X is finite or countably infinite. The probability distribution function (pdf) is named as probability mass function (PMF). Let X be a random variable, hence the range space R_X consists of atmost a countably infinite number of values. The probability mass function is defined as $p(x_i) = \Pr\{X = x_i\}$, satisfying the conditions

i) $p(x_i) \geq 0\ for\ all\ i$

ii) $\sum_{i=1}^{k} p(x_i) = 1$.

Continuous Random Variable: A random variable X is said to be continuous if it can take all possible values between certain limits, here the range space of X is infinite. Therefore the probability distribution function named for such random variable is probability density function (PDF), which is defined as the function $f(x)$ satisfying the following properties

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) $\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$ for any a, b such that $-\infty < a < b < \infty$.

Note:

1. If X is a continuous random variable with pdf f(x), then

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_a^b f(x)dx.$$

2. $P(X = a) = 0$, if X is a continuous random variable.

Cumulative distribution function: Let X be random variable (discrete or continuous), we define F to be the cumulative distribution function of a random variable X given by $F(x) = \Pr\{X \leq x\}$.

Case 1. If X is discrete random variable then $F(t) = \Pr\{X \leq t\} = P(x_1) + P(x_2) + \cdots + P(t)$

Case 2. If x is a continuous random variable then $F(x) = \Pr\{X \leq x\} = \int_{-\infty}^{x} f(x)dx$.

Two dimensional random variable: Let E be an experiment and S be a sample space associated with E. Let X = X(s) and Y = Y(s) be two functions each assigning a real number to each outcome s of S. We call (X, Y) to be two dimensional random variable.

Discrete two dimensional random variable: If the possible values of (X, Y) are finite or countably infinite then (X, Y) is called discrete and it is defined as $P(x_i, y_j)$ satisfying the following condition,

i) $P(x_i, y_j) \geq 0$ and

ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P(x_i, y_j) = 1$.

The function $P(x_i, y_j)$ defined is called as Joint probability distribution function (Jpdf).

Continuous two dimensional random variable: If (X, Y) is a continuous random variable assuming all values in some region R of the Euclidean plane, then the Joint probability density function $f(x, y)$ is a function satisfying the following conditions

i) $f(x, y) \geq 0$ for all (x, y) $\epsilon\, R$

ii) $\iint f(x, y) dx\, dy = 1$ over the region R.

Marginal Probability distribution: The marginal probability distribution is defined as

Case 1. In the discrete (X, Y), it is defined as $p(x_i) = P\{X = x_i\} = \sum_{j=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of X. Similarly $q(y_j) = P\{Y = y_j\} = \sum_{i=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of Y.

Case 2. In the continuous (X, Y), it is defined as the marginal probability function of X is defined as $g(x) = \int_{-\infty}^{\infty} f(x, y) dy$ and the marginal probability function of Y is defined as $h(y) = \int_{-\infty}^{\infty} f(x, y) dx$.

To calculate the conditional probability:

Case 1. Discrete: Probability of x_i given y_j is defined as $\frac{P(x_i,\ y_j)}{q(y_j)}$, $q(y_j) > 0$

Probability of y_j given x_i is defined as = $\frac{P(x_i,\ y_j)}{p(x_i)}$, $p(x_i) > 0$

Case 2. Continuous: The pdf of X for given Y = y is $\frac{f(x,\ y)}{h(y)},\ h(y) > 0$

The pdf off Y for given X = x is $\frac{f(x,\ y)}{g(x)},\ g(x) > 0$.

Independent random variable:

Discrete: If $P(x_i, y_j) = p(x_i). q(y_j)$ for all I and j, then X and Y are independent random variables.

Continuous: If $f(x, y) = g(x). h(y)$ for all x and y, then X and Y are independent random variables.

Mathematical Expectation: If X is a discrete random variable with pmf p(x), then the expectation of X is given by $E(X) = \sum_x xp(x)$, provided the series is absolutely convergent.

If X is continuous with pdf f(x), then the expectation of X is given by $E(X) = \int xf(x)dx$, provided $\int |x| f(x)dx < \infty$.

Variance of X is given by $V(X) = E(X - E(X))^2 = E(X^2) - \left(E(X)\right)^2$.

Distributions

Binomial distribution : If $X \sim B(n, p)$,

Pdf: $P(x) = {}^nC_k\, p^k (1-p)^{n-k},\ \ k = 0, 1,\ 2,\ \ldots,\ n$.

Mean = $E(x) = np$ and Variance = $V(x) = np(1-p)$.

Poisson's distribution: If $X \sim P(\alpha)$,

Pdf: $P(x) = \dfrac{e^{-\alpha}\alpha^k}{k!},\ \ k = 0,1,2,\ldots;\ \alpha > 0$

Mean = $E(x) = \alpha = np$ and Variance = $V(x) = \alpha = np$.

Uniform distribution: If $X \sim U(a,b)$,

Pdf: $$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$$

Mean = $E(x) = \dfrac{b+a}{2}$ and Variance = $V(x) = \dfrac{(b-a)^2}{12}$.

Normal distribution: If $X \sim N(\mu, \sigma^2)$,

Pdf: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$.

Mean = $E(x) = \mu$ and Variance = $V(x) = \sigma^2$.

Exponential distribution: If $X \sim E(\lambda)$,

Pdf: $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$$

Mean = $E(x) = \dfrac{1}{\lambda}$ and Variance = $E(x) = \dfrac{1}{\lambda^2}$.

Gamma distribution: If $X \sim G(r, \alpha)$, Pdf: $$f(x)=\begin{cases}\dfrac{x^{r-1}e^{-\alpha x}\alpha^{r}}{\Gamma(r)}, & x>0,\ \alpha,r>0\\ 0, & elsewhere\end{cases}$$

Mean = $E(x)=\dfrac{r}{\alpha}$ and Variance = $V(x)=\dfrac{r}{\alpha^2}$.

Chi-square distribution: If $X\sim\chi^2(n)$,

Pdf: $$f(x)=\begin{cases}\dfrac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x>0\\ 0, & elsewhere\end{cases}$$

Mean = $E(x)=n$ and Variance = $V(x)=2n$.

Uniform distribution on a two dimensional set: If R is a set in the two-dimensional plane, and R has a finite area, then we may consider the density function equal to the reciprocal of the area of R inside R, and equal to 0 otherwise, i.e., $f(x,y)=\begin{cases}\frac{1}{area\ R}; & if\ (x,y)\in R\\ 0 & Otherwise\end{cases}$.

Chebyshev's inequality:

Let X be random variable with mean μ and variance σ^2 then for any positive real number k (k > 0),

$P\{|X-\mu|\ge k\}\le\dfrac{\sigma^2}{k^2}$ (Upper bound)

$P\{|X-\mu|<k\}>1-\dfrac{\sigma^2}{k^2}$ (Lower bound)

Note: Some other forms

1. $P\{|X-\mu|\ge k\sigma\}\le\dfrac{1}{k^2}$ and $P\{|X-\mu|<k\sigma\}\ge1-\dfrac{1}{k^2}$.
2. $P\{|X-\mu|\ge\in\}\le\dfrac{1}{\in^2}E(\mathrm{X}-c)^2$ and $P\{|X-\mu|<\in\}\ge1-\dfrac{1}{\in^2}E(\mathrm{X}-c)^2$.

Covariance:

$$Cov(\mathrm{X},\mathrm{Y})=E(\mathrm{XY})-E(\mathrm{X})E(\mathrm{Y})$$

Correlation coefficient:

$$\rho_{xy} = \rho = \frac{E(\text{XY}) - E(\text{X})E(\text{Y})}{\sqrt{V(\text{X})\,V(\text{Y})}}.$$

Properties:

1. $E(c) = c$, where c is a constant.
2. $V(c) = 0$, where c is a constant.
3. If $E(\text{XY}) = 0$ then X and Y are orthogonal.
4. $V(AX + b) = A^2 V(\text{X})$ when AX + B is linear function of X.
5. If $\rho = 0$ then X and Y are uncorrelated.
6. $V(AX + BY) = A^2\,V(\text{X}) + B^2\,V(\text{Y}) + 2AB\;COV(\text{X},\,Y)$, where A and B are constants.

Functions of one dimensional random variables

Let S be a sample space associated with a random experiment E, then it is known that a random variable X on S is a real valued function, i.e., X : $S \to R$, for each element s $\in$ S, there is a real number associated.

Let X be a random variable defined on S. Let $y = H(x)$ is a real valued function of x. Then $Y = H(X)$ is a random variable on S. i.e., for each element s $\in$ S, there is a real number associated, say $y = H\big(X(s)\big)$. Here Y is called a **function of the random variable** X.

Notations:

1. R_X – the set of all possible values of the function X, called the **range space** of the random variable X.
2. R_Y – the set of all possible values of the function $Y = H(X)$, called the **range space** of the random variable Y.

Equivalent Events: Let C be an event associated with the range space R_Y. Let $B \subset R_X$ defined by $B = \{x \in R_X; H(x) \in C\}$, then B and C are called equivalent events.

Distribution function of functions of random variables

Case 1: Let X be a discrete random variable with p.m.f. $p(x_i) = P(X = x_i)$ for $i = 1,2,3, \dots$. Let $Y = H(X)$ then Y is also a discrete random variable. If $Y = H(X)$ is a one to one function then the probability distribution of Y is as follows:

For the possible values of $y_i = H(x_i)$ for $i = 1,2,3, \dots$. The p.m.f. of $Y = H(X)$ is $q(y_i) = P(Y = y_i) = P(X = x_i) = p(x_i)$ for $i = 1,2,3, \dots$.

Case 2: Let X be a discrete random variable with p.m.f. $p(x_i) = P(X = x_i)$ for $i = 1,2,3,\ldots$. Let $Y = H(X)$ then Y is also a discrete random variable. Suppose that for one value of $Y = y_i$ there corresponds several values of X say $x_{i_1}, x_{i_2}, \ldots, x_{i_j}, \ldots$ then the p.m.f. of $Y = H(X)$ is

$$q(y_i) = P(Y = y_i) = p\left(x_{i_1}\right) + p\left(x_{i_2}\right) + \cdots + p\left(x_{i_j}\right) + \cdots$$

Case 3: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = H(X)$ be a discrete random variable. Then if the set $\{Y = y_i\}$ is equivalent to an event $B \subseteq R_X$ then the p.m.f. of Y is

$$q(y_i) = P(Y = y_i) = \int_B f(x)\, dx$$

Case 4: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = H(X)$ be a continuous random variable. Then the p.d.f. of Y, say g is obtained by the following procedure:

Step 1: Obtain the c.d.f. of Y, $\quad G(y) = P(Y < y)$, by finding the event

$A \subseteq R_X$, which is equivalent to the event $\{Y = y_i\}$.

Step 2: Differentiate $G(y)$ with respect to y to get $g(y)$.

Step 3: Determine those values of y in R_Y for which $g(y) > 0$.

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$ where $f(x) > 0$ for $a < x < b$. Suppose that $Y = H(X)$ is strictly monotonic function on $[a, b]$. Then the p.d.f. of the random variable $Y = H(X)$ is given by $g(y) = f(x)\left|\frac{dx}{dy}\right|$

If $Y = H(X)$ is strictly increasing then $g(y) > 0$ for $H(a) < y < H(b)$.

If $Y = H(X)$ is strictly decreasing then $g(y) > 0$ for $H(b) < y < H(a)$.

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = X^2$ then the p.d.f. of Y is

$$g(y) = \frac{1}{2\sqrt{y}}\left[f\left(\sqrt{y}\right) + f\left(-\sqrt{y}\right)\right]$$

Functions of two dimensional random variables

Let (X, Y) be a two dimensional continuous random variable. Let $Z = H(X, Y)$ be a continuous function of X and Y then $Z = H(X, Y)$ is a continuous one dimensional random variable.

To find the p.d.f. of Z, we introduce another suitable random variable say,

$W = G(X, Y)$ and obtain the joint p.d.f. of the two dimensional random variable (Z, W), say $k(z, w)$. From this distribution, the p.d.f. of Z can be obtained by integrating k with respect to w.

Theorem: Suppose (X, Y) is a two dimensional continuous random variable with joint p.d.f. $f(x, y)$ defined on a region R of the XY-plane. Let $Z = H_1(X, Y)$ and $W = H_2(X, Y)$. Suppose that H_1 and H_2 satisfies the following conditions;

(i) $z = H_1(x, y)$ and $w = H_2(x, y)$ may be uniquely solved for x, y in terms of $z \,\&\, w$ say, $x = G_1(z, w)$ and $y = G_2(z, w)$.

(ii) The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous

Then the joint p.d.f. of (Z, W) say $k(z, w)$ is given by,

$$k(z, w) = f[G_1(z, w), G_2(z, w)]|J(z, w)|$$

where $J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$ is called the Jacobian of the transformation $(x, y) \mapsto (z, w)$. Also, $k(z, w) > 0$ for those values of (z, w) corresponding to the values of (x, y) for which $f(x, y) > 0$.

Moment generating function (m.g.f.) of one dimensional random variables

Let X be any one dimensional random variable then the mathematical expectation $E(e^{tX})$ if exists then it is called the moment generating function (m.g.f.) of X, i.e., $M_X(t) = E(e^{tX})$.

In particular, if X is discrete then, $M_X(t) = \sum_{i=1}^{i=\infty} e^{tx_i}\, P(X = x_i)$.

If X is continuous then, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$.

Properties of m.g.f.

Let X be any one dimensional random variable and $M_X(t)$ be the m.g.f. of X then

1. $M_X^n(0) = E(X^n)$ where $M_X^n(0)$ is the n[th] derivative of $M_X(t)$ at $t = 0$.
   i.e.; $M_X'(0) = E(X)$
   $M_X''(0) = E(X^2)$
2. $V(X) = M_X''(0) - \left(M_X'(0)\right)^2$
3. Let X be any one dimensional random variable and $M_X(t)$ be the m.g.f. of X. Let $Y = \alpha X + \beta$. Then the m.g.f. of Y is $M_Y(t) = e^{\beta t} M_X(\alpha t)$.
4. Suppose that X and Y are independent random variables. Let $Z = X + Y$. Let $M_X(t), M_Y(t)$ and $M_Z(t)$ be the m.g.f.'s of the random variables X, Y and Z respectively. Then $M_Z(t) = M_X(t) M_Y(t)$
5. Let $X_1, X_2, \ldots, X_n$ be n independent random variables which follows a normal distribution $N(\mu_i, \sigma_i^2)$ for $i = 1,2,3, \ldots, n$. Let $Z = X_1 + X_2 + \cdots + X_n$ then
   $Z \to N(\mu_1 + \mu_2 + \cdots + \mu_n,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)$.
6. Let $X_1, X_2, \ldots, X_n$ be n independent random variables which follows a Poisson distribution with parameter α_i for $i = 1,2,3, \ldots, n$. Let $Z = X_1 + X_2 + \cdots + X_n$ then Z has a Poisson distribution with parameter $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
7. Let $X_1, X_2, \ldots, X_k$ be k independent random variables which follows a Chi-square distribution with degrees of freedom n_i for $i = 1,2,3, \ldots, k$. Let $Z = X_1 + X_2 + \cdots + X_k$ then Z has a Chi-square distribution with degrees of freedom $n = n_1 + n_2 + \cdots + n_k$.
8. Let $X_1, X_2, \ldots, X_k$ be k independent random variables, each having distribution $N(0,1)$. Then $S = X_1^2 + X_2^2 + \cdots + X_k^2$ has a Chi-square distribution with degrees of freedom k.

9. Let $X_1, X_2, \dots, X_r$ be r independent random variables, each having exponential distribution with the same parameter α. Let $Z = X_1 + X_2 + \cdots + X_r$ then Z has a Gamma distribution with parameters α and r.
10. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variable with c.d.f.'s $F_1, F_2, \dots, F_n, \dots$ and m.g.f.'s $M_1, M_2, \dots, M_n, \dots$ Suppose that $\lim_{n\to\infty} M_n(t) = M(t)$, where $M(0) = 1$. Then $M(t)$ is the m.g.f. of the random variable X whose c.d.f is $F = \lim_{n\to\infty} F_n(t)$.

MGF of some standard distributions:

1. Binomial Distributions: $M_X(t) = M_X(t) = \ (pe^t + q)^n$
2. Poisson Distributions: $M_X(t) = e^{\propto(e^t - 1)}$
3. Normal Distributions: $M_X(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$
4. Exponential Distributions: $M_X(t) = \frac{\propto}{\propto - t}$
5. Gamma Distributions: $M_X(t) = \frac{\propto^r}{(\propto - t)^r}$
6. Chi square Distributions: $M_X(t) = (1 - 2t)^{-n/2}$

Sampling

In statistical investigation, the characteristics of a large group of individuals (called population) is studied. Sampling is a study of the relationship between a population and samples drawn from it.

The population mean and the population variance are denoted by μ and σ^2 respectively.

Sample mean and sample variance: Let X be the random variable which denotes the population with mean μ and variance σ^2. Let $(X_1, X_2, \dots, X_n)$ be a random sample of size n from X. Then,

sample mean, $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ and sample variance, $s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$.

- If $X \to N(\mu, \sigma^2)$ then $\bar{X}$ and s^2 are independent random variables.
- Let X be the random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Let $(X_1, X_2, \dots, X_n)$ be a random sample of size n from X. Then, $E(\bar{X}) = \mu$ and $V(\bar{X}) = \frac{\sigma^2}{n}$.
- Let $X \to N(\mu, \sigma^2)$ then $\bar{X} \to N(\mu, \frac{\sigma^2}{n})$ and $s^2 \to \chi^2(n-1)$.

Central Limit Theorem: Let $X_1, X_2, \dots, X_n$ be n independent random variables all of which have the same distribution. Let $\mu = E(X_i)$ and $\sigma^2 = V(X_i)$ be the common expectation and variance. Let $S = \sum_{i=1}^{n} X_i$ then $E(S) = n\mu$ and $V(S) = n\sigma^2$ then for large values of n, the random variable $T_n = \frac{S - E(S)}{\sqrt{V(S)}}$ has approximately the distribution $N(0,1)$.

NUMERICAL METHODS

Numerical Solution of Boundary Value Problems

Boundary value problems are of great importance in science and engineering. In this section, we have elaborated Numerical methods based on finite difference scheme for the solution of following problems:

1. Boundary value problems in second order ordinary differential equations
2. Boundary value problems governed by linear second order partial differential equations: Laplace equation and Poisson equation.
3. Initial boundary value problems governed by linear second order partial differential equations: One dimensional heat and wave equation.

Boundary value problems governed by second order ordinary differential equations

For our discussion, we shall consider only the linear second order ordinary differential equations

$$y'' + p(x)\,y'(x) + q(x)\,y = r(x), \quad x \in [a, b] \qquad \ldots \qquad (1)$$

Since the ordinary differential equation is of second order, we need to prescribe two conditions to obtain a unique solution of the problem. If the conditions are prescribed at the end points x = a and x = b, then it is called a two point boundary value problem. The two conditions required to solve (1), can be prescribed in one of the following three boundary conditions:

Boundary conditions of the first kind : $y(a) = A, \quad y(b) = B$ … (2)

Boundary conditions of the second kind : $y'(a) = A, \quad y'(b) = B$ … (3)

Boundary conditions of the third (or mixed kind) :

$$a_0\, y(a) - a_1\, y'(a) = A, \quad b_0\, y(b) + b_1\, y'(b) = B, \qquad \ldots. \qquad (4)$$

Where a_0, b_0, a_1 , b_1, A and B are constants such that

$$a_0\, a_1 \ge 0,\ |a_0| + |a_1| \ne 0 \ ;\ b_0\, b_1 \ge 0,\ |b_0| + |b_1| \ne 0 \ \text{ and } \ |a_0| + |b_0| \ne 0$$

Finite Difference Methods for Ordinary Differential Equation

These are the explicit or implicit relations between the derivatives and function values at the adjacent nodal points. The nodal points on an interval may be defined by

$x_i = x_0 + i\,h$, i = 0, 1, …., N, where $a = x_0$, $b = x_N$ and $h = (b-a)/N$

Finite Difference Approximations to derivatives are given below:

Approximations to $y'(x_i)$ at $x = x_i$ **:**

(i) Forward difference approximation of first order or O(*h)* approximation:

$$y'(x_i) = \frac{1}{h}\left[y(x_{i+1}) - y(x_i)\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{h}\left[y_{i+1} - y_i\right] \qquad \ldots (5)$$

(ii) Backward difference approximation of first order or O(*h)* approximation:

$$y'(x_i) = \frac{1}{h}\left[y(x_i) - y(x_{i-1})\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{h}\left[y_i - y_{i-1}\right] \qquad \ldots (6)$$

(iii) Central difference approximation of second order or O(h^2) approximation:

$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right] \qquad \ldots (7)$$

Approximations to $y''(x_i)$ at $x = x_i$:

Central difference approximation of second order or O(h^2) approximation:

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2\,y(x_i) + y(x_{i-1})\right] \quad , \text{or} \quad y''_i = \frac{1}{h^2}\left[y_{i+1} - 2\,y_i + y_{i-1}\right] \qquad \ldots (8)$$

The finite difference solution of a boundary value problem (1) is obtained by replacing the differential equation at each nodal point by difference equations (5) to (8) along with given boundary conditions (2) to (4).

Boundary value problems governed by linear second order partial differential equations

Over a two dimensional Cartesian domain, let u be the dependent variable. Then a general second order partial differential equation may be written as,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F(x, y, u, u_x, u_y) + G = 0 \qquad \text{-----------} \qquad (9)$$

where A, B, C are functions of (x, y) and F may be non-linear function, then equation (9) is called a quasi-linear partial differential equation. If F is also a linear function then equation (9) is called a linear partial differential equation. If $G = 0$ then equation (9) is homogeneous otherwise non-homogeneous.

Classification of PDE:

Equation (9) is said to be elliptic, parabolic and hyperbolic depending on $B^2 - 4AC < 0$, $B^2 - 4AC = 0$ and $B^2 - 4AC > 0$ at a point or in a domain.

Finite Difference Methods for Partial Differential Equation:

Consider a rectangular region R in the XY –plane. Divide the region into rectangular network of sides $\Delta x = h$ and $\Delta y = k$. Writing $u(x, y) = u(ih, jk)$ as simply $u_{i,j}$, the finite difference approximations for the first order partial derivatives are as follows:

Forward: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i+1,j} - u_{i,j}\right) + O(h)$

Backward: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i,j} - u_{i-1,j}\right) + O(h)$

Central: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h}\left(u_{i+1,j} - u_{i-1,j}\right) + O(h^2)$

Similarly,

Forward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) + O(k)$

Backward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j} - u_{i,j-1}\right) + O(k)$

Central: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j-1}\right) + O(k^2)$

The finite difference approximations for the second order partial derivatives are as follows.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + O(h^2)$$

Similar expressions can be written for $\left(\frac{\partial^2 u}{\partial y^2}\right), \left(\frac{\partial^2 u}{\partial x \partial y}\right)$.

Elliptic partial differential equation (Laplace equation & Poisson equation):

Most relevant examples of elliptic PDE are Laplace equation and Poisson equation.

The Poisson equation in Cartesian coordinate system is

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ or $\nabla^2 u = f(x, y),\ \ a \leq x \leq b,\ \ c \leq y \leq d.$

subject to boundary condition: u(x, y) = g(x, y) (Dirichlet boundary condition).

The Laplace equation is a special case of Poisson equation with $f(x, y) = 0$.

Solution for Laplace Equation (for uniform mesh size h =k)**:**

<u>Standard 5-point formula :</u>

$$u_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1})$$

<u>Diagonal 5-point formula</u> **:**

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})$$

Solution to Poisson equation:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

Parabolic Partial Differential Equation (One – dimensional heat conduction equation)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le l, \quad t > 0$$

In order that the solution of the problem exists and is unique, we need to prescribe the following conditions:

(i) Initial conditions : $u(x, 0) = f(x), \quad 0 \le x \le l$

(ii) Boundary Conditions : $u(0, t) = g(t), \quad u(l, t) = h(t), \quad t > 0$

(in this study we restricted to simple boundary condition, i.e., temperature at the ends of the bar is prescribed) where c^2 is the diffusivity of the substance, $u(x,t)$ is a temperature function which is defined for values of x from 0 to l (length of the bar) and for values of time t from 0 to ∞.

Solution of one dimensional heat equation

<u>Explicit Method :</u>

Schmidt Method :

$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j}$ where $\lambda = \frac{kc^2}{h^2}$; $0 < \lambda \le \frac{1}{2}$.

Where λ is called the mesh ratio parameter, h mesh length along x-axis and k mesh length along t-axis.

Bender – Schmidt Method (*particular case when* $\lambda = 1/2$) **:**

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

Note : For $\lambda = \frac{1}{6}$, the error in Schmidt's formula is least.

<u>Implicit Method:</u> ***Crank – Nicolson's method:***

$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

where $\lambda = \frac{kc^2}{h^2}$

For $\lambda = 1$: $-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$ *(particular case)*

Hyperbolic Partial Differential Equation (One dimensional wave equation)

All vibration problems arising in science and engineering are governed by wave equation.

Consider the problem of vibrations of an elastic string governed by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le l, \quad t > 0$$

Subject to the initial and boundary conditions:

(i) Initial conditions: $u(x, 0) = f(x), \quad 0 \le x \le l$ (initial displacement)

$$\frac{\partial u}{\partial t}(x,0) = g(x)\ , \quad 0 \le x \le l \text{ (initial velocity)}$$

(ii) Boundary Conditions: u(0, t) = 0, u(*l*, t) = 0, t >0

(we consider the case when the ends of the string are fixed)

Where c^2 is a constant and depends on the material properties of the elastic string.

Solution of one dimensional Wave equation:

Explicit Method:

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2k^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$= r^2\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

Where $r = \frac{kc}{h}$ is called mesh ratio parameter, h is the mesh length along x-axis and k is the mesh length along t-axis.

The above method is stable when $r \le 1$, higher order method uses the value of r = 1

Therefore, *we* have $\boldsymbol{u_{i,j+1} = \left[u_{i+1,j} + u_{i-1,j} - u_{i,j-1}\right]}$, is the explicit finite difference method for wave equation at higher level.

To get the value at first level, we have $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right] + kg_i$.

If the initial condition is prescribed as $\frac{\partial u}{\partial t}(x,0) = 0$, that is g(x) = 0, then above formula simplifies to

$\boldsymbol{u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]}$ **gives the value of 'u' at first level.**

DIFFERENCE EQUATION

A **difference equation** is a relation between the differences of an unknown function at one or more general values of an argument.

Order of difference equation = $\frac{largest\ argument - smallest\ argument}{unit\ of\ increment}$.

Solution of a difference equation is an expression which satisfies the difference equation.

General solution is a solution in which the number of arbitrary constants is equal to the order of the difference equation.

Particular solution is a solution obtained from general solution by giving particular values to the constants.

A linear difference equation with constant coefficient is of the form

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \cdots + a_r y_n = f(n) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

where $a_1, a_2, \ldots, a_n$ are constants.

Let $u_1(n),\ u_2(n), \ldots, u_r(n)$ be the r independent solutions of $y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \cdots + a_r y_n = 0$.

$U_n = c_1 u_1(n) + c_2 u_2(n) + \cdots + c_r u_r(n)$, where $c_1, c_2, \ldots, c_r$ are constants, is a solution of the above equation.

If V_n is the particular solution of (i), then the complete solution of (i) is

$y_n = U_n + V_n$, U_n is called complementary function and V_n the particular solution.

Shift Operator: $E^1 y_0 = y_1$, $E^2 y_0 = y_2, \ldots$, $E^n y_0 = y_n$, $E^n y_r = y_{r+n}$

Rules for finding Complementary Function (CF):

For equation (*i*), $m^r + a_1 m^{r-1} + a_2 m^{r-2} + \cdots + a_r = 0$, is called auxiliary equation.

Let its r roots be $\lambda_1, \lambda_2, \ldots, \lambda_r$.

1. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are real and distinct, $CF = c_1 {\lambda_1}^n + c_2 {\lambda_2}^n + \cdots + c_r {\lambda_r}^n$,
2. If two roots say λ_1 and λ_2 are real and equal, all other real and distinct,
   $CF = (c_1 + c_2 n){\lambda_1}^n + c_3 {\lambda_3}^n + \cdots + c_r {\lambda_r}^n$.
   If $\lambda_1 = \lambda_2 = \lambda_3$, then CF= $(c_1 + c_2 n + c_3 n^2){\lambda_1}^n + c_4 {\lambda_4}^n + \cdots + c_r {\lambda_r}^n$.
3. If the two roots are complex, say $\alpha \pm i\beta$,
   then CF = $r^n (c_1 \cos n\theta + c_2 \sin n\theta) + c_3 {\lambda_3}^n + \ldots + c_r {\lambda_r}^n$ where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Rules for finding Particular Solution:

Case 1. Suppose $f(n) = a^n$.

Particular solution = $\frac{1}{f(E)} a^n = \frac{1}{f(a)} a^n$, if $f(a) \neq 0$.

If $f(a) = 0$, then (a) for $(E-a)y_n = a^n$, PI = $\frac{1}{E-a} a^n = n a^{n-1}$.

(b) for $(E-a)^2 y_n = a^n$, PI = $\frac{1}{(E-a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$.

(c) for $(E-a)^3 y_n = a^n$, PI = $\frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)}{2!} a^{n-3}$.

Case 2. Suppose $f(n) = \sin kn = \frac{e^{ikn} - e^{-ikn}}{2i} = \frac{1}{2i}[a^n - b^n]$, where $a = e^{ik}$, $b = e^{-ik}$.

If $f(E) = \cos kn = \frac{e^{ikn} + e^{-ikn}}{2} = \frac{1}{2i}[a^n + b^n]$, where $a = e^{ik}$, $b = e^{-ik}$.

Proceed as in case 1.

Case 3. When f(n) = n^p,

Particular solution = $\frac{1}{\phi(E)}n^p = \frac{1}{\phi(1+\Delta)}n^p$

1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by binomial theorem as far as the term in Δ^p.

2) Write n^p in the factorial form and operate on it with each term of the expansion.

Note: While expanding in ascending powers, the following formulae are useful.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \qquad \frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Factorial Notation:

A product of the form x(x - 1)(x - 2)…(x – r + 1) is denoted by $[x]^r$ and is called a factorial.

Thus, [x] = x, $[x]^2$ = x (x - 1), $[x]^3$ = x (x - 1) (x - 2) and so on.

The operator Δ is used similar to D, when a polynomial is written in factorial notation.

For example, $\Delta[x] = 1$, $\Delta[x]^2 = 2[x]$, $\Delta^2[x]^2 = \Delta 2[x] = 2$, $\Delta[x]^3 = 3[x]^2$, $\Delta^2[x]^3 = 6[x]$, $\Delta^3[x]^3 = 6$.

Case 4. Suppose $(n) = a^n n^p$.

Particular solution = $\frac{1}{f(E)}\, a^n n^p = a^n \frac{1}{f(aE)} n^p$, proceed as in case 3.

Z TRANSFORMS

If {u_n} is a sequence, then its Z-transform is defined as $Z_T\{u_n\} = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, whenever the infinite series converges.

Z-transforms of some standard functions:

u_n	$Z(u_n) = U(z)$
1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$
n^p	$-z\frac{d}{dz}Z(n^{p-1}), \qquad p \in Z_+$
na^n	$\frac{az}{(z-a)^2}$
n^2a^n	$\frac{az^2+za^2}{(z-a)^3}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$a^n \cos n\theta$	$\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
$a^n\, si\; n\; n\theta$	$\frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$

$n^m u_n$	$\left(-z\frac{d}{dz}\right)^m U(z)$

Inverse Z transforms

If $Z_T\{u_n\} = U(z)$, then $\{ u_n\}$ is called inverse Z transform of U(z) and we denote this by $Z^{-1}[U(z)] = \{u_n\}$.

Inverse Z transforms of some standard functions:

$U(z)$	$u_n = Z^{-1}[U(z)]$
$\frac{1}{z-a}$	a^{n-1}
$\frac{1}{z+a}$	$(-a)^{n-1}$
$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
$\frac{z}{z-a}$	a^n
$\frac{z}{z+a}$	$(-a)^n$
$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$

Properties of Z transforms:

I. Linearity Property: If $\{u_n\}$ *and* $\{v_n\}$ are any two sequences then

$$Z_T\{c_1 u_n + c_2\, v_n\} = c_1\, Z_T\{u_n\} + c_2\, Z_T\{v_n\}$$

II. Damping Property: If $Z_T\{u_n\} = U$(z) , then $Z_T\{k^n u_n\} = U\left(\frac{z}{k}\right)$, k ≠ 0

III. Right shifting rule: If $Z_T\{u_n\} = U$(z) then $Z_T\{u_{n-k}\} = z^{-k}U$(z), k > 0

IV. Left shifting rule: If $Z_T\{u_n\} = U$(z) then $Z_T\{u_{n+k}\} = z^k\,\{U(z) - \sum_{r=0}^{k-1} u_r\, z^{-r}\}$

V. Initial Value theorem If $Z_T\{u_n\} = U$(z) then $\lim_{z\to\infty} U(z) = u_0$

VI. Final Value theorem If $Z_T\{u_n\} = U$(z) then $\lim_{z\to 1}[\,(z-1)U(z)] = \lim_{n\to\infty} u_n$

VII. Convolution Theorem : If $Z^{-1}[(U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_m v_{n-m} = u_n * v_n.$$