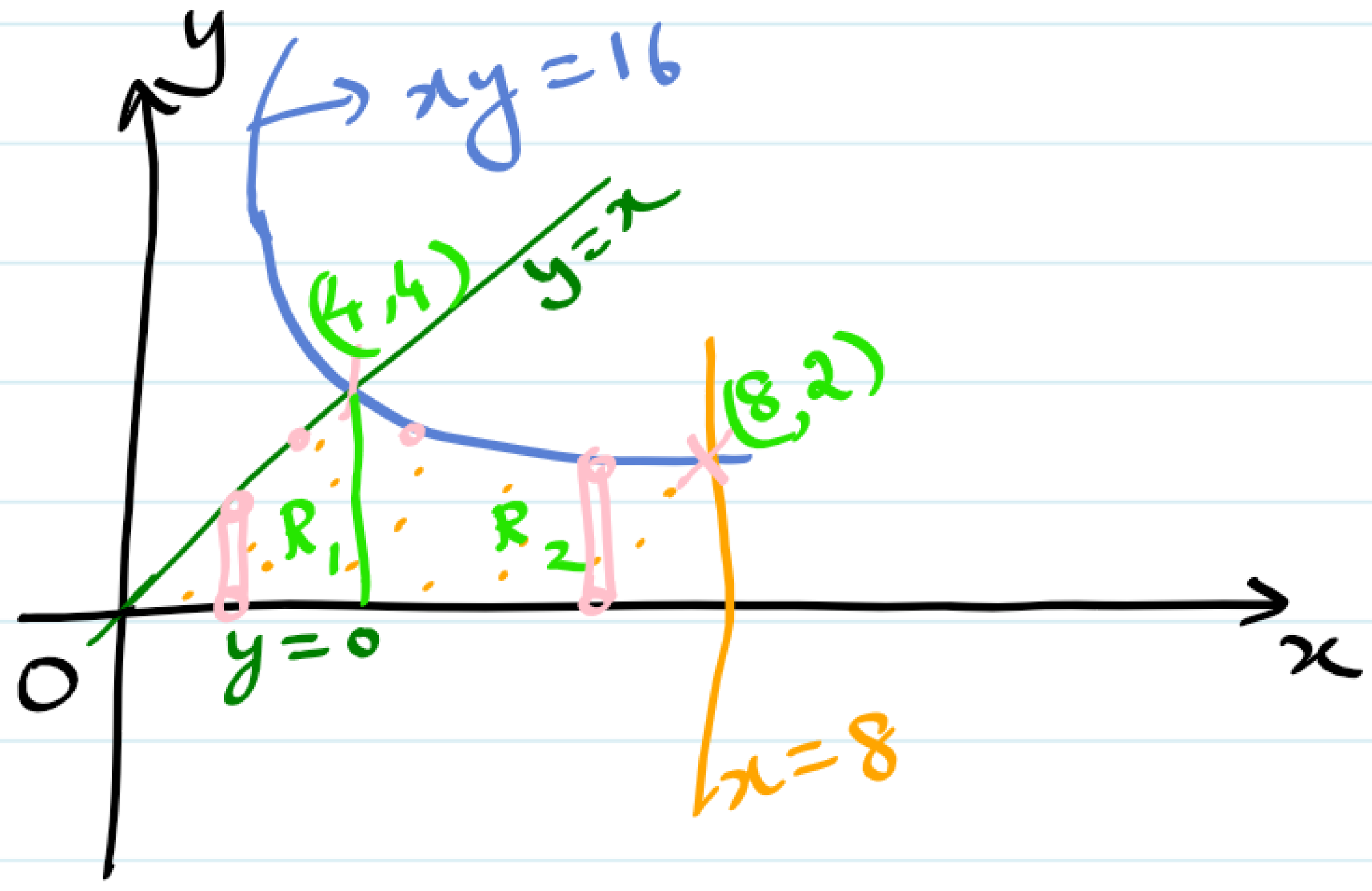
5) Evaluate $\iint_R x^2\,dx\,dy$, where R is the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x$, $y=0$, $x=8$.



$$I=\iint_{R_1} x^2\,dy\,dx+\iint_{R_2} x^2\,dy\,dx$$

$$=\int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx+\int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$$

$xy=16$ & $y=x$

$x x=16$

$x=4$, $y=4$

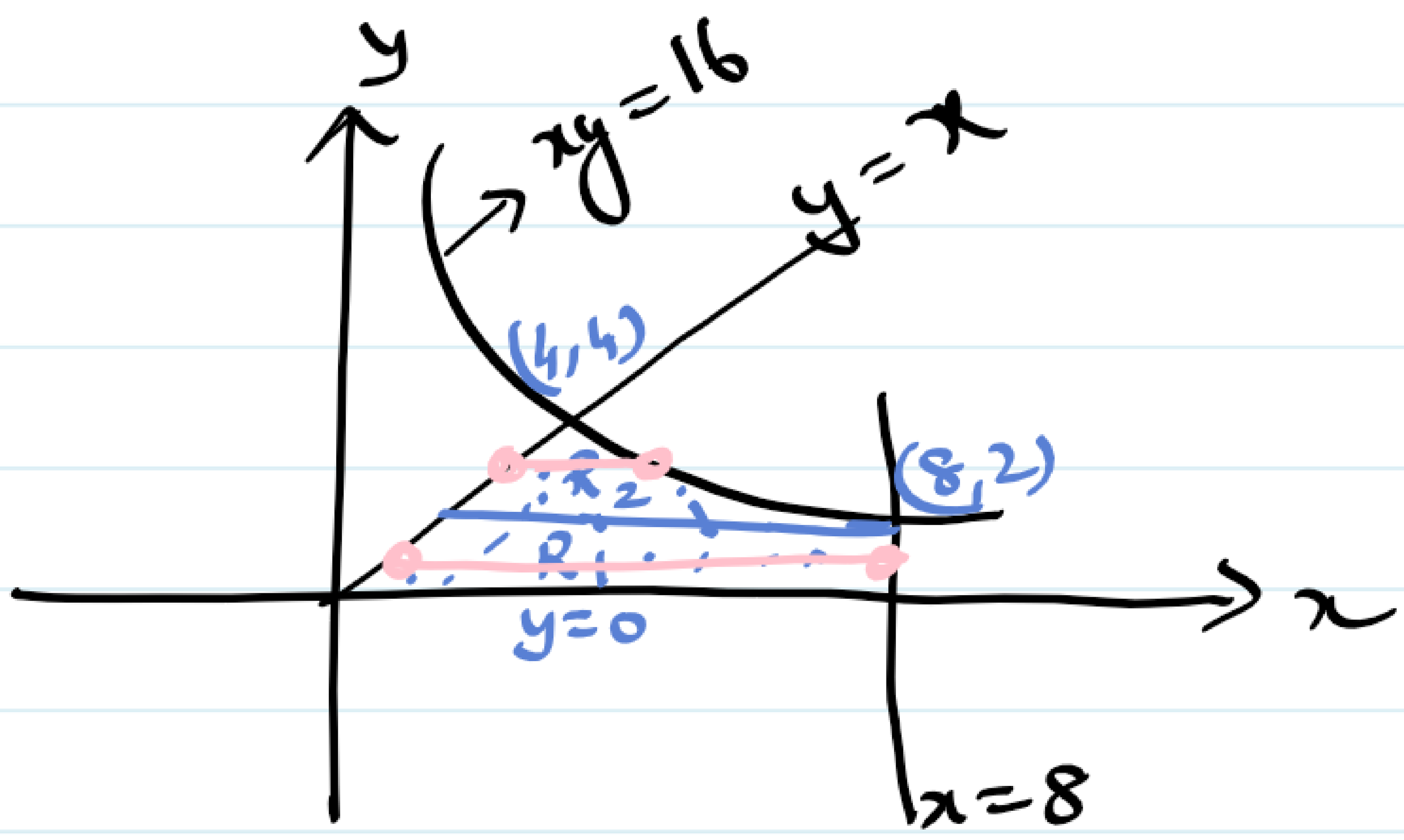
$x=8$, $xy=16$

$y=2$

$$=\int_0^4 x^2 y\Big]_0^x\,dx+\int_4^8 \left(x^2y\right)_0^{16/x}\,dx$$

$$=\int_0^4 x^3\,dx+16\int_4^8 x\,dx$$

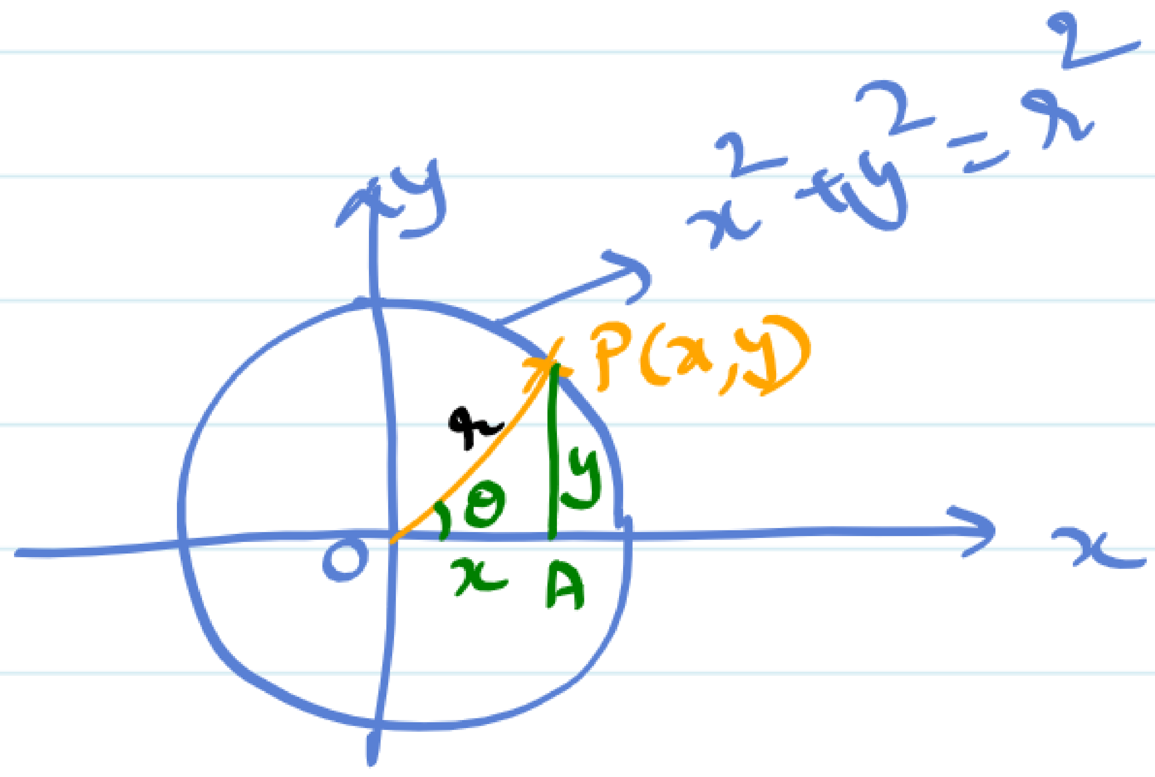
$$=\frac{x^4}{4}\Big|_0^4+16\frac{x^2}{2}\Big|_4^8 \qquad =\underline{\underline{448}}$$



$$I=\iint_{R_1} x^2\,dx\,dy+\iint_{R_2} x^2\,dx\,dy$$

$$=\int_{y=0}^{2}\int_{x=y}^{8} x^2\,dx\,dy+\int_{y=2}^{4}\int_{x=y}^{16/y} x^2\,dx\,dy.$$

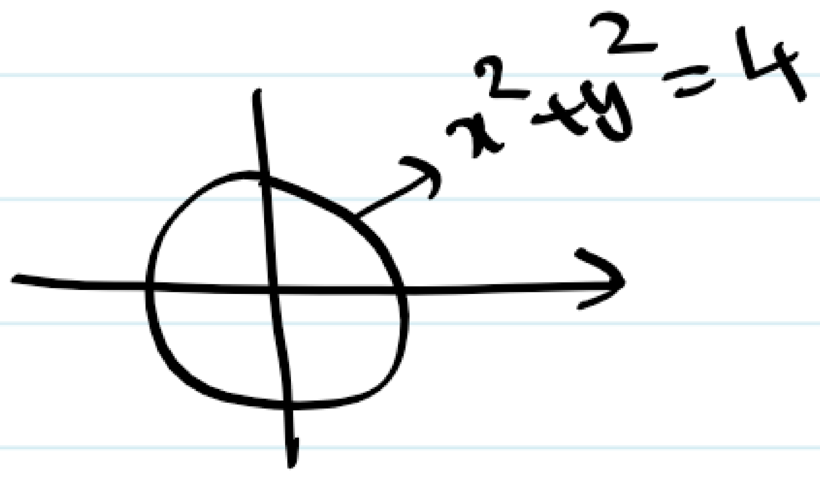
6) Evaluate $\iint_R r^2 \sin\theta \, dr\, d\theta$, where R is the semi circle $r = 2a\cos\theta$ above the initial line.



$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$x^2+y^2 = r^2$



Cartesian form $\leftarrow$ $x^2+y^2 = 4$.

$(r\cos\theta)^2 + (r\sin\theta)^2 = 4$

$r^2(\cos^2\theta + \sin\theta) = 4$

$r^2 = 4 \Rightarrow r = 2$ — is eqn of cle centre at the origin with radius 2

$\downarrow$ polar form.

$r = 2a\cos\theta$ ✓ polar form

$r^2 = 2a\, r\cos\theta$

$x^2+y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 + y^2 = a^2$

$(x-a)^2 + y^2 = a^2 \longrightarrow$ Cartesian form

$\hookrightarrow$ circle centre at x-axis & radius $= a$.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2$

$r = a\sin\theta$

$r^2 = a\,r\sin\theta$

$x^2+y^2 = ay$

$x^2+y^2-ay = 0$

$x^2+y^2-ay+\left(\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$

$x + \left(y-\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 \rightarrow$ Circle centre $(0, a/2)$ on the y-axis

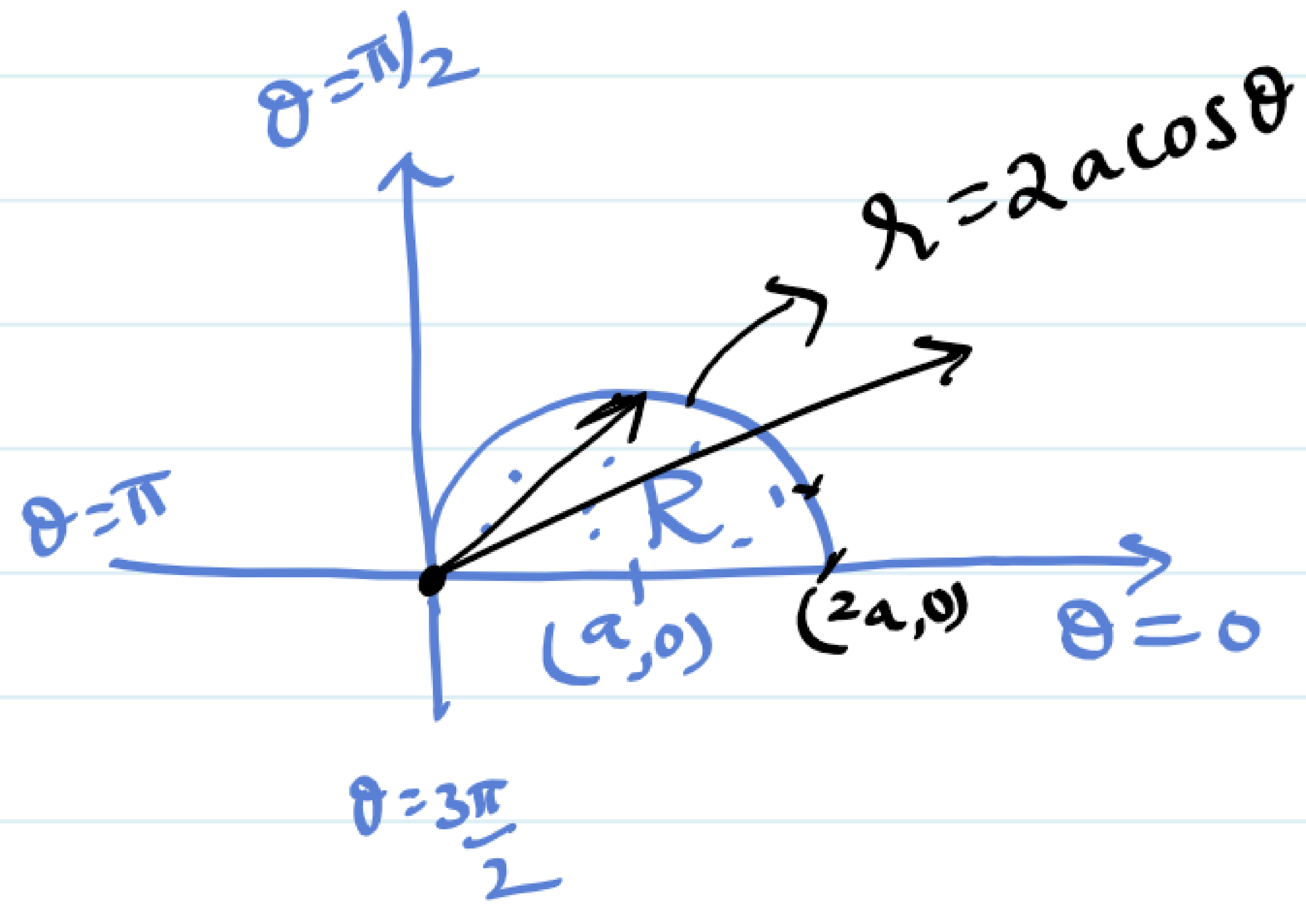
& radius $a/2$.

$r = 3 \rightarrow$ Cetre $=(0,0)$ $r=3$

$r = a\cos\theta \rightarrow$ centre $(a/2, 0)$ $r = a/2$

$r = a\sin\theta \rightarrow$ centre $(0, a/2)$ $r = a/2$.

$r = 2a\cos\theta$



$I = \iint r^2 \sin\theta \, dr\, d\theta$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} \sin\theta\; r^2\, dr\; d\theta$$

$r = 2a\cos\theta$

$\cancel{r=0}$

$\theta = 0, \quad r = 2a$

$\theta = \pi/6, \quad r = \cancel{2}a\frac{\sqrt{3}}{\cancel{2}}$

$\theta = \pi/2 \quad r = 0$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta = \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta\cos^3\theta\, d\theta$$

put $\cos\theta = t$

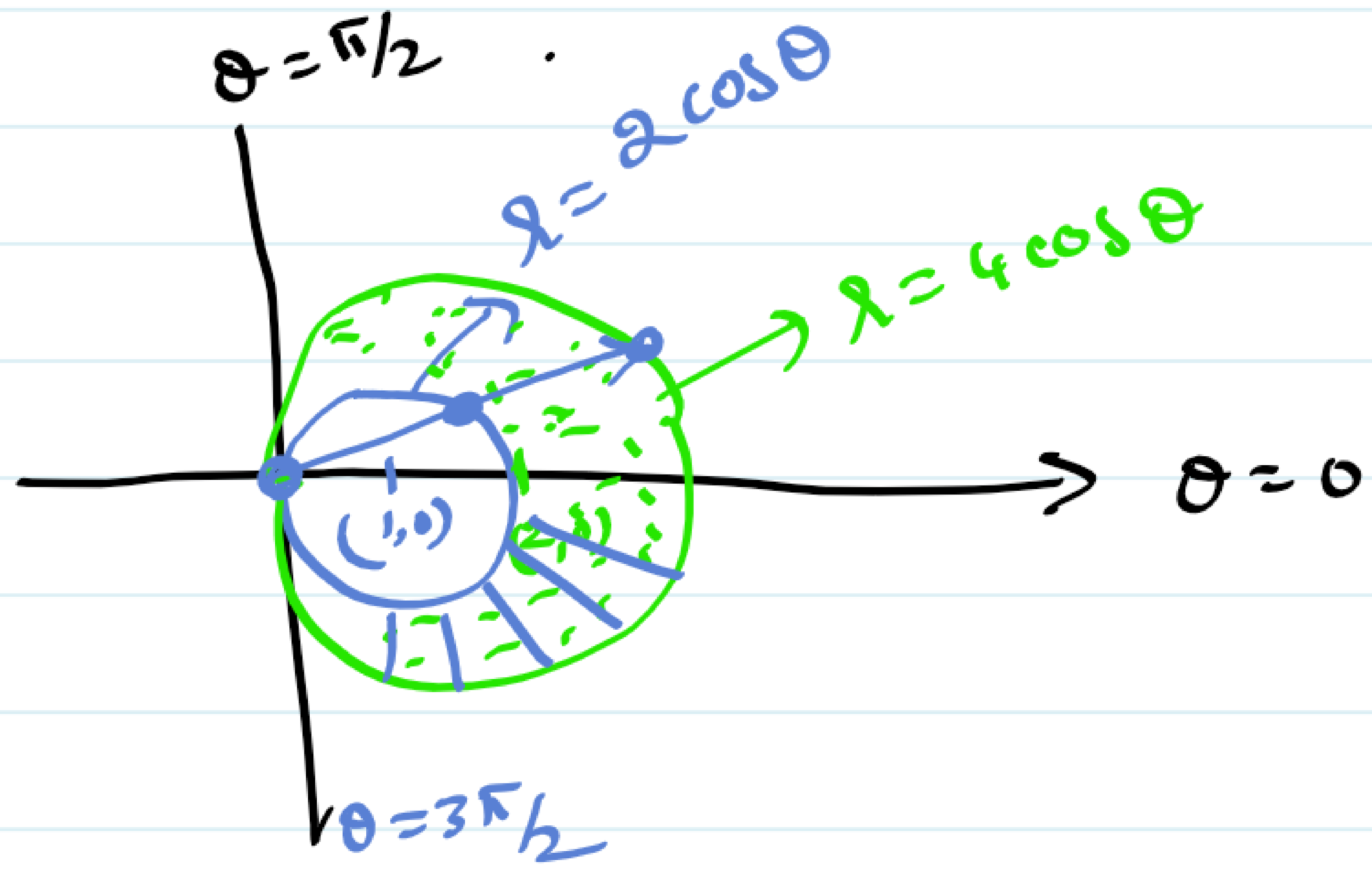
$-\sin\theta\, d\theta = dt$

$\theta = 0, \; t = 1$

$\theta = \pi/2, \; t = 0$

$$= \frac{8a^3}{3}\int_0^1 t^3 dt = \frac{8a^3}{3}\left(\frac{t^4}{4}\right)_0^1 = \frac{8a^3}{3}\times\frac{1}{4} = \frac{2a^3}{3}$$

7) Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.



$$I = 2\int_{\theta=0}^{\pi/2}\int_{r=2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{2}{4}\int_0^{\pi/2} 4^4\cos^4\theta - 2^4\cos^4\theta\,d\theta$$

$$= \frac{1}{2}(256-16)\int_0^{\pi/2}\cos^4\theta\,d\theta \quad = 120\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$n=4$, even

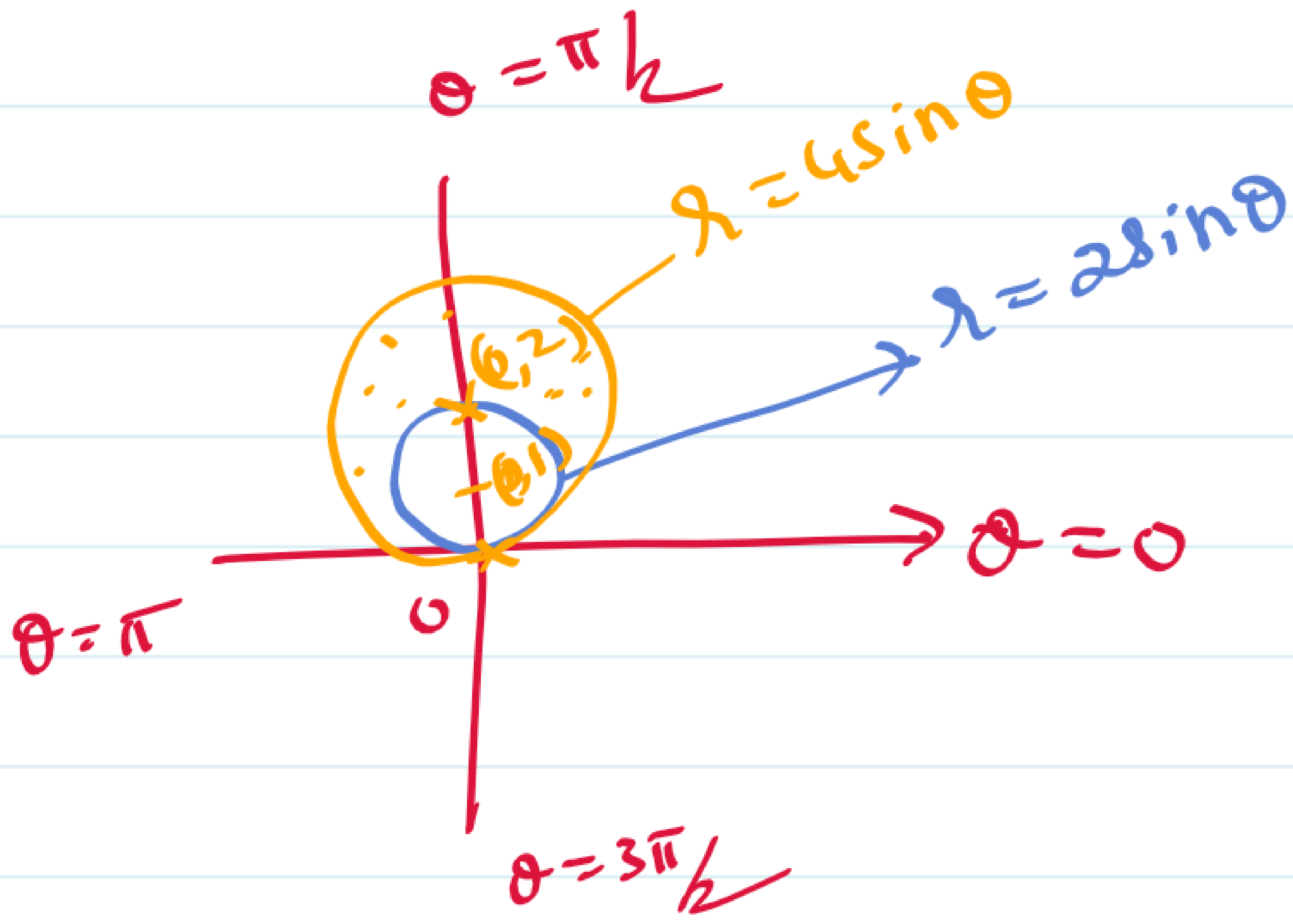
$$= 120\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

Reduction formula:

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases}\frac{(n-1)}{n}\cdot\frac{(n-3)}{n-2}\cdots\cdots\frac{2}{3}, & n \text{ odd}\\[4pt] \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{1}{2}\cdot\pi/2, & n \text{ even}\end{cases}$$

8) Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$



$$I = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta .$$

$r = 2\sin\theta$

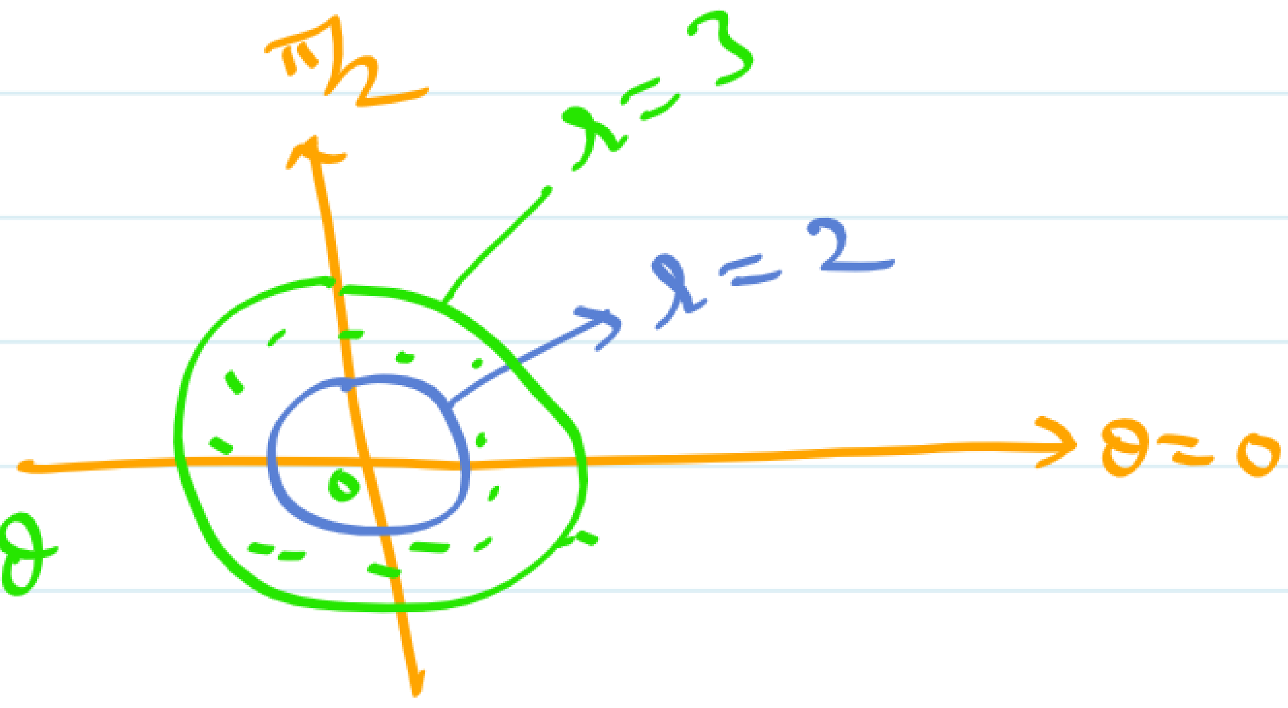
$\theta = 0 \quad r = 0$

$\theta = \pi/2 \quad r = 2\sin\pi/2$

$r = 2.$

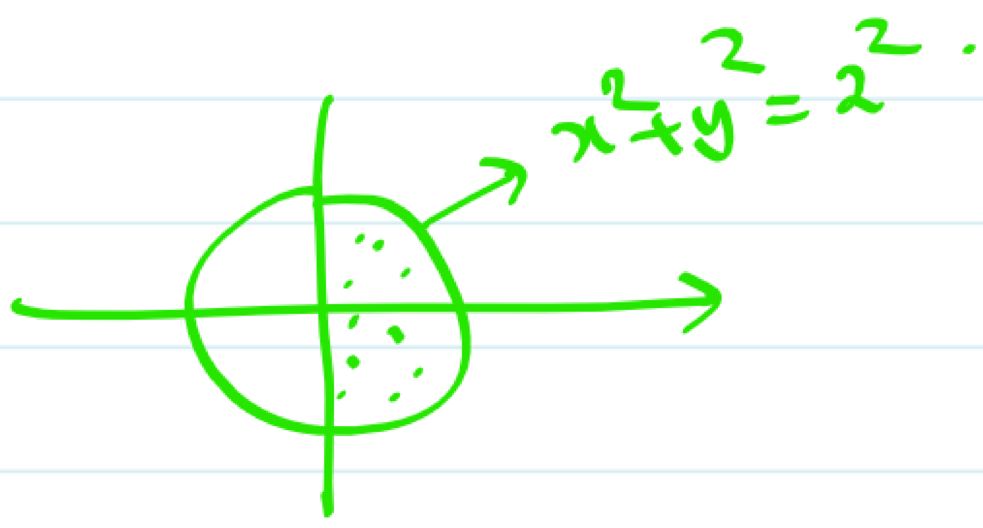
$\theta = \pi \quad r = 0$

9) Evaluate $\iint_R r^2\,dr\,d\theta$, where R is the region between $r=2$ and $r=3$.



$$I = \int_{\theta=0}^{2\pi} \int_{r=2}^{3} r^2\,dr\,d\theta = 4\int_{0}^{\pi/2} \int_{r=2}^{3} r^2\,dr\,d\theta$$

$$= \frac{38\pi}{3}$$



$x = r\cos\theta \quad , \quad y = r\sin\theta$

$x^2+y^2 = 2^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 2^2$

$r^2 = 4 \Rightarrow r = 2$

Change of order of Integration —

In the double integral $\int\{\int f(x,y)dy\}dx$ is difficult or even impossible to integrate w.r.to y first, but can be easily integrated w.r.to x first. In such an event, it becomes necessary to reverse (or change) the order of integration in the double integral.

$$\text{i.e.,}\quad \int\left\{\int f(x,y)dx\right\}dy$$

Similarly, if $I = \int\{\int f(x,y)dx\}dy$ is difficult to integrate w.r.to x, then by changing the order of integration,

we get, $I = \int\{\int f(x,y)dy\}dx$

Method: Let $I = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right) dx$

— Given region R is bounded by $x=a$, $x=b$, $y=f_1(x)$ & $y=f_2(x)$

* find the point of intersections
* Sketch the region of integration R and check whether it is correct or wrong by drawing a vertical strip.

— Now to reverse the order, draw a horizontal strip cutting through the region R. Writedown the given integral I with order of integration reversed as

$$I = \int\left\{\int f(x,y)dx\right\}dy$$

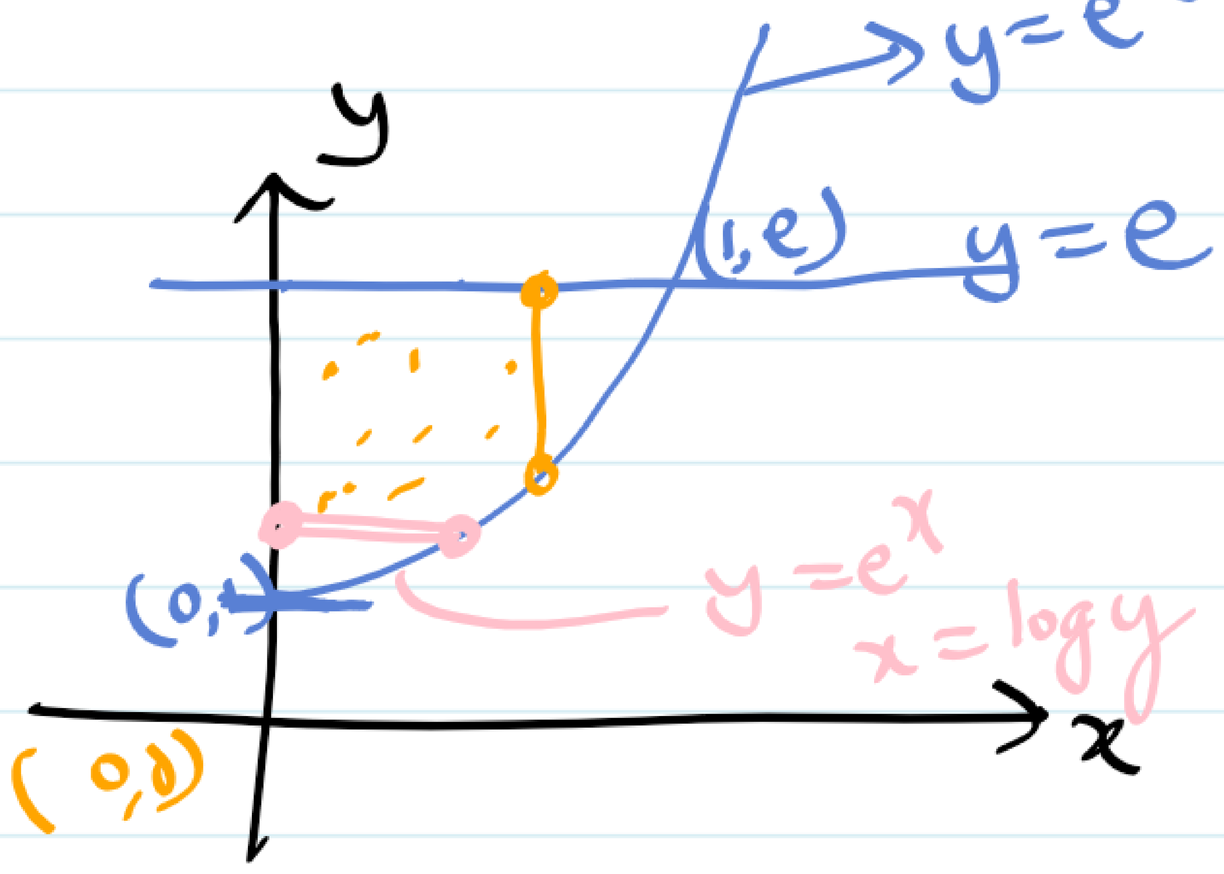
- Find the new limits for x: say $x = g_1(y)$ to $x = g_2(y)$

limits for y: constant, say, $y = c$ to $y = d$.

$\therefore$ By changing the order of integration,

$$I = \int_{y=c}^{d} \left\{ \int_{x=g_1(y)}^{g_2(y)} f(x,y)\, dx \right\} dy$$

① Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration.



Here region R is bounded by

$y=e^x,\ y=e,\ x=0,\ x=1$

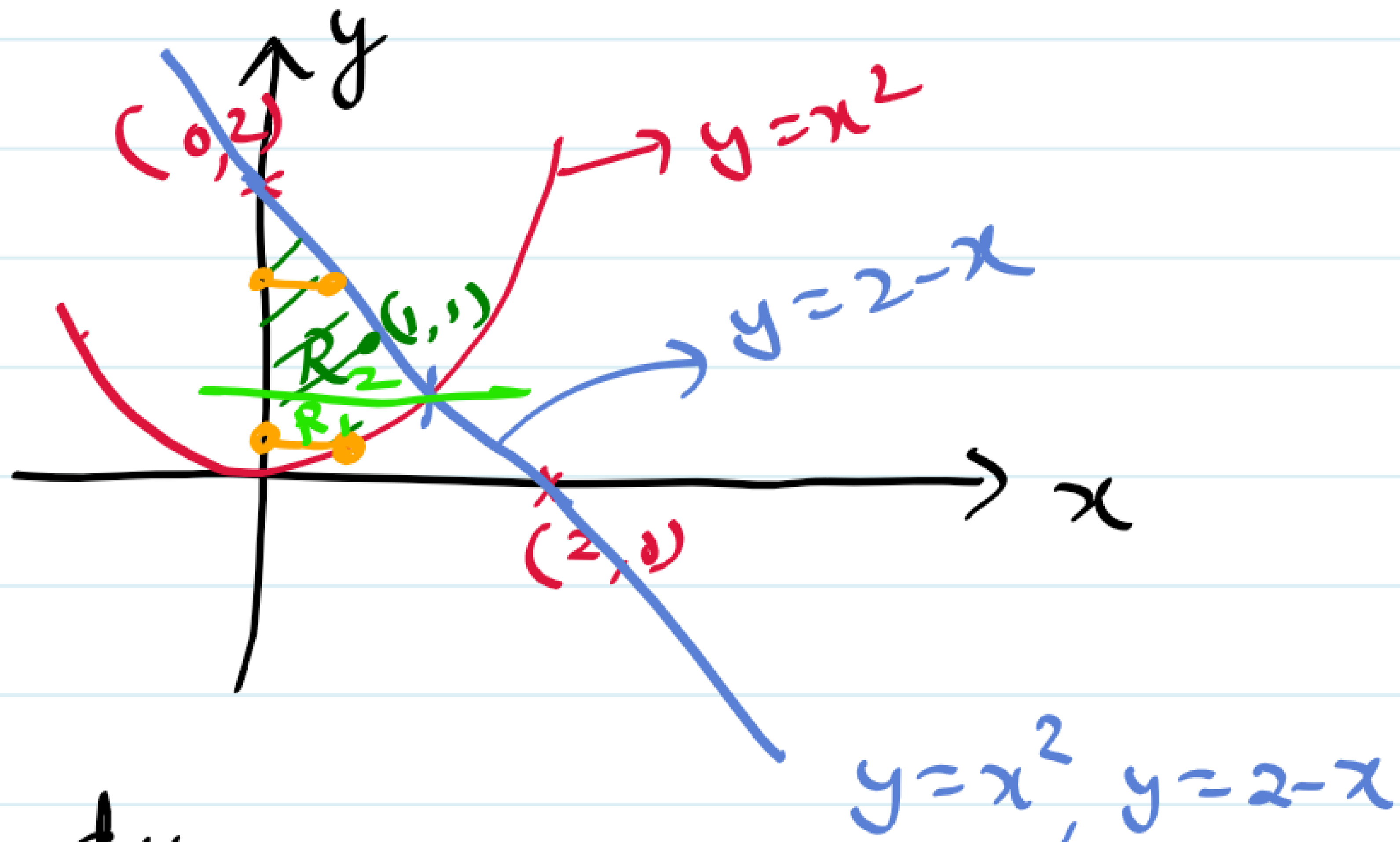
$$I=\int_{y=1}^{e}\int_{x=0}^{\log y}\left(\frac{1}{\log y}\,dx\right)dy$$

$$=\int_1^e \frac{1}{\log y}(x)\Big|_{x=0}^{\log y}dy=\int_1^e \frac{\log y}{\log y}\,dy=\int_{y=1}^{e} dy$$

$$=y\Big)_1^e=\underline{\underline{e-1}}$$

2) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$ by changing the order of integration.

The given region bounded by $y=x^2$, $y=2-x$, $x=0$, $x=1$.



$$I = \iint_{R_1} xy \, dx \, dy + \iint_{R_2} xy \, dx \, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_0^1 y \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 y \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{1}{2} y (2-y)^2 dy = \underline{\underline{3/8}}$$

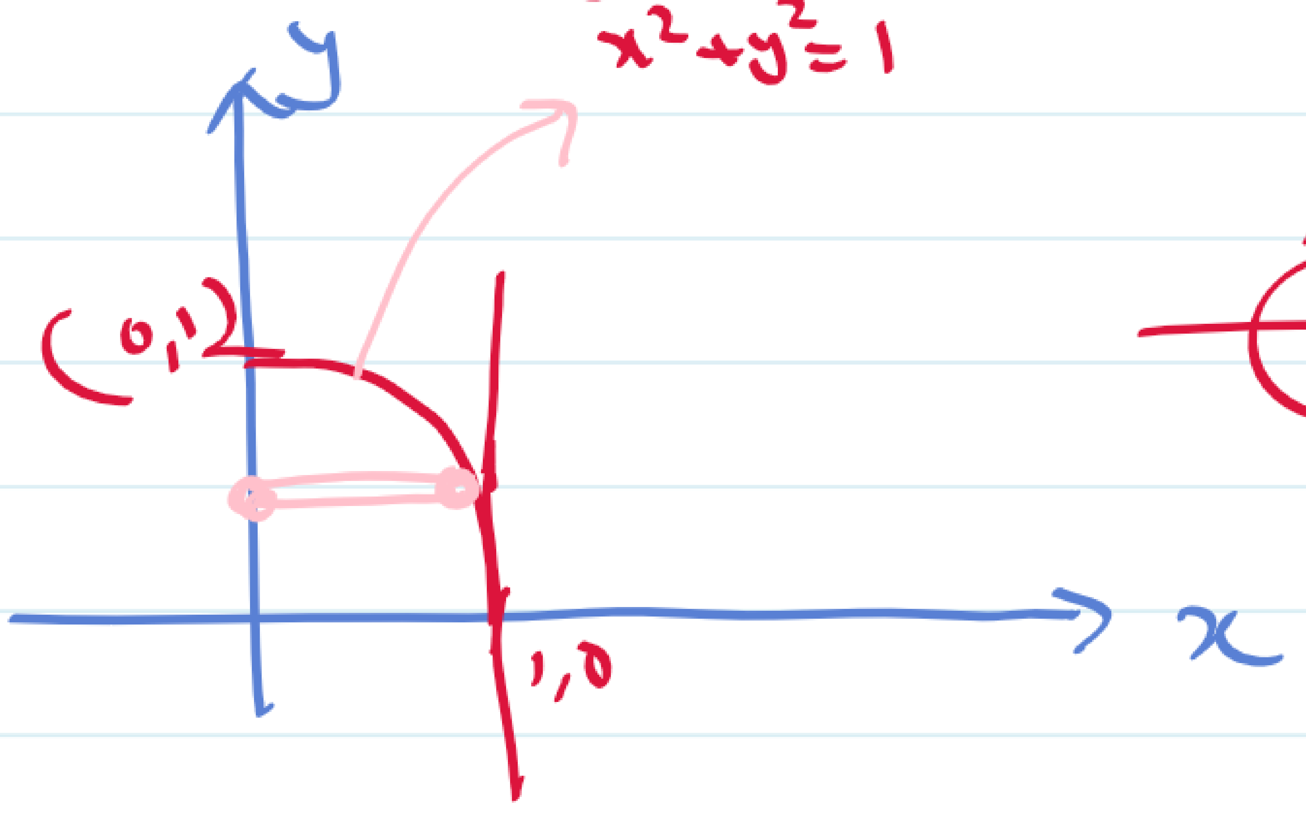
③ Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dx\,dy}{(1+e^y)\sqrt{1-x^2-y^2}}$

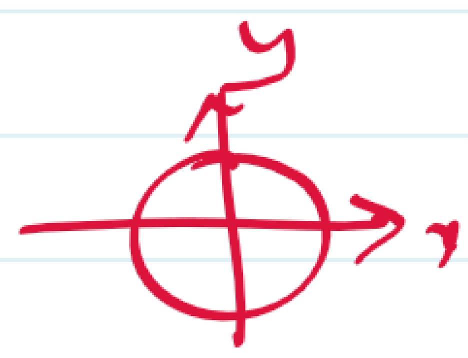
Here integration w.r.to y first is difficult, so we change the order of integration.

The given region is bounded $y=0$, $y=\sqrt{1-x^2}$, $x=0$, $x=1$

$y^2 = 1-x^2$

$x^2+y^2=1$





$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{dx\,dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

$$= \int_0^1 \frac{1}{1+e^y} \left(\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\, dx \right) dy$$

$$\frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{\text{write } 1-y^2 = b^2 \qquad \sqrt{1-y^2} = b}$$

$$= \int_0^1 \frac{1}{1+e^y} \sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\Bigg|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{1}{1+e^y}\left(\frac{\pi}{2}-0\right) dy = \frac{\pi}{2}\int_0^1 \frac{(1+e^y)-e^y}{1+e^y}\, dy$$

$$= \frac{\pi}{2}\int_0^1 \left(1 - \frac{e^y}{1+e^y}\right) dy$$

$$= \frac{\pi}{2}\Big[y - \log(1+e^y)\Big]_0^1$$

$$= \frac{\pi}{2}\left(1-\log(1+e)+\log 2\right) = \frac{\pi}{2}\left(\log e + \log 2 - \log(1+e)\right)$$

$$= \frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)$$

Practice Questions

① Change the order of integration $\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx$ and evaluate.

(Ans: $\frac{4}{7}a^4$)

② Change the order of integration $\int_0^a \int_y^a \frac{x \, dx \, dy}{x^2+y^2}$ and evaluate.

(Ans: $\frac{a\pi}{4}$)