Interpolation:

Let $y = f(x)$.

If f is known explicitly interms of x, then we can compute value of y for a given value of x.

In fact, given the values $x_0, x_1, \ldots, x_n$ of x, we can compute the corresponding values $y_0, y_1, \ldots, y_n$ of y.

Think of the converse:

Given a set of values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying the relation $y = f(x)$, where the explicit nature of $f(x)$ is not known.

Can we find $f(x)$?

Answer is "No"

But we can find a simpler function, say $\varphi(x)$, such that $\varphi(x)$ and $f(x)$ agree at the tabulated points.

i.e. $\varphi(x_i) = f(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n.$

At an intermediate point $\varphi(x)$ gives an approximate value of y.

Given a set of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying $y = f(x)$, where f is not known explicitly, the process of obtaining the value of y for an $x \in (x_0, x_n)$ is called interpolation. The function $\varphi(x)$ that approximates $f(x)$ is called interpolating function.

If $\varphi(x)$ is a polynomial, then it is called interpolating polynomial and the process is called polynomial interpolation.

The process of finding the value of y for an x outside the interval $[x_0, x_n]$ is called extrapolation.

Weierstrass Theorem:

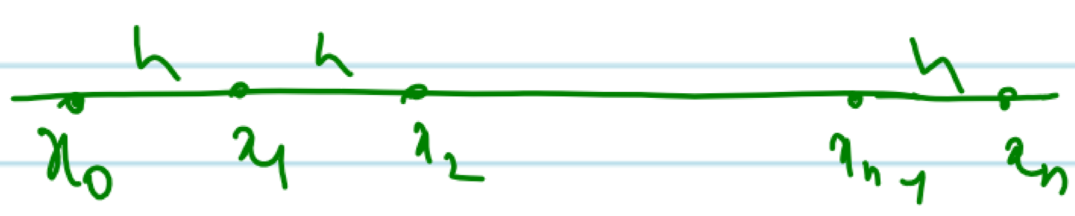
If $f(x)$ is continuous in $x_0 \le x \le x_n$, then given any $\epsilon > 0$, there exist a polynomial $P(x)$, such that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \in (x_0, x_n).$$

Interpolation With Equally Spaced Points:

Consider a set of values

$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$



where $x_i = x_0 + ih, \quad i = 1, 2, \ldots, n, \; h > 0$

We define the following differences:

(1) Forward Differences

(2) Backward Differences.

Forward Differences:

The differences

$$y_1 - y_0, \quad y_2 - y_1, \quad \ldots, \quad y_n - y_{n-1}$$

are called first order forward differences and are respectively denoted by

$$\Delta y_0, \quad \Delta y_1, \quad \ldots, \quad \Delta y_{n-1}.$$

$$\therefore \quad \Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \ldots, \quad \Delta y_{n-1} = y_n - y_{n-1}$$

The differences of the first order forward differences are called second order forward differences and are respectively denoted by

$$\Delta^2 y_0, \quad \Delta^2 y_1, \quad \ldots, \quad \Delta^2 y_{n-2}.$$

$$\therefore \quad \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

In a similar way, we can define third and higher order differences.

Examples:

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0.$$

In general $\Delta^r y_k = {}^rC_0\, y_{k+r} - {}^rC_1\, y_{k+r-1} + {}^rC_2\, y_{k+r-2} - \cdots + {}^rC_r (-1)^r y_k$

$$\Delta^4 y_1 = y_5 - {}^4C_1\, y_4 + {}^4C_2\, y_3 - {}^4C_3\, y_2 + {}^4C_4\, y_1$$
$$= y_5 - 4y_4 + 6y_3 - 4y_2 + y_1$$

'Δ' is called the forward difference operator.

The above differences can be tabulated in the form of a table as follows:

		Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Backward Differences:

The differences $y_1 - y_0$, $y_2 - y_1$, $\dots y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$

respectively, are called first order backward differences.

$$\therefore \nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \quad \dots, \quad \nabla y_n = y_n - y_{n-1}$$

The differences of the first order differences are called second order backward differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3, \nabla^2 y_4, \dots\dots$

In a similar way, we can define third and higher order of backward differences.

Examples:

$$\nabla^2 y_3 = y_3 - 2y_2 + y_1$$

$$\nabla^4 y_4 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

In general,

$$\nabla^n y_n = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - \dots + (-1)^n y_0 .$$

'∇' is called backward difference operator.

These differences can be tabulated in the form of a table as follows:

x	y				
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Shift Operator:

The shift operator E is defined by the equation,

$$E(y_r) = y_{r+1}$$

$$E^2 y_r = E(E y_r) = E y_{r+1} = y_{r+2}$$

In general $E^k y_r = y_{r+k}$

The inverse shift operator is defined by the relation,

$$E^{-1} y_r = y_{r-1}$$

In general $E^{-k} y_r = y_{r-k}$

Relation between the operators:-

① $E = I + \Delta$.

Proof: Consider $(1+\Delta) y_k = y_k + \Delta y_k$

$$= y_k + (y_{k+1} - y_k)$$

$$\Delta y_k = y_{k+1} - y_k$$

$$= y_{k+1}$$

$(1+\Delta) y_k = E(y_k)$, for all k.

$\therefore 1+\Delta = E$ & $E = 1+\Delta$

② $\nabla = 1 - E^{-1}$

$$(1 - E^{-1}) y_k = y_k - E^{-1} y_k$$

$$= y_k - y_{k-1}$$

$(1 - E^{-1}) y_k = \nabla y_k$, for all k.

$\therefore 1 - E^{-1} = \nabla$.

$y_k \xrightarrow{E} y_{k+1} \quad y_{k+2} \cdots$

$y_{k+1} \xrightarrow{E^{-1}} y_k$

③ $\Delta = \nabla E = E \nabla$

$\nabla E y_k = \nabla (y_{k+1}) = y_{k+1} - y_k = \Delta y_k$, for all k.

$\Delta = \nabla E$.

$E \nabla y_k = E(y_k - y_{k-1}) = E y_k - E y_{k-1} = y_{k+1} - y_k = \Delta y_k$, for all k

$\therefore E \nabla = \Delta$

④ $\Delta^r y_k = \nabla^r y_{k+r}$

$\nabla^r y_{k+r} = (1 - E^{-1})^r y_{k+r}$ $\qquad \nabla = 1 - E^{-1}$

$$= \left(1 - \frac{1}{E}\right)^r y_{k+r}$$

$$= (E-1)^r E^{-r} y_{k+r}$$

$E^{-r} y_{k+r} = y_{k+r-r} = y_k$

$$= (E-1)^r y_{k+r-r}$$

$$= (E-1)^r y_k.$$

$$\boxed{(x+y)^n = \sum_{i=0}^{n} {}^n c_i \, x^i \, y^{n-i}}$$

⑤ $\nabla^r y_k = \Delta^r y_{k-r}$

⑥ $\Delta^k y_r = \sum_{i=1}^{k} {}^k c_i \, (-1)^{k-i} \, y_{r+i}$

Proof: $\Delta^k y_r = (E-1)^k y_k = \left(\sum_{i=0}^{k} {}^k c_i \, E^i \, (-1)^{k-i} \right) y_k = \sum_{i=0}^{k} (-1)^{k-i} \, {}^k c_i \, y_{k+i}$

$\because E^i y_k = y_{k+i}$

Notation: $\Delta y_x = f(x+h)-f(x)$, $\nabla y_k = f(x)-f(x-h)$.

Properties:

① **Linearity Property:** For any two constants a & b, and any two functions $f(x)$ and $g(x)$,

$$\Delta(af(x) \pm bg(x)) = a\,\Delta f(x) \pm b\,\Delta g(x).$$

$$\nabla(af(x) \pm bg(x)) = a\,\nabla f(x) \pm b\,\nabla g(x).$$

② **Index Law:**

For any positive integers m and n,

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$$

$$\nabla^m \nabla^n f(x) = \nabla^{m+n} f(x).$$

③ The first order difference of a polynomial of degree n is again a polynomial of degree $n-1$. The n^{th} order difference is a constant.

Proof:

$$\left[\begin{aligned} \Delta y_0 = y_1 - y_0 &= f(x_1) - f(x_0) \\ &= f(x_0+h) - f(x_0) \end{aligned}\right]$$

$y = f(x)$
$y_1 = f(x_1)$
$y_0 = f(x_0)$

Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n, \quad a_0 \neq 0$$

be a polynomial of degree n.

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
&= \{a_0(x+h)^n + a_1(x+h)^{n-1} + \dots + a_n\} - \{a_0 x^n + a_1 x^{n-1} + \dots + a_n\} \\
&= \{a_0(x^n + n x^{n-1} h + \dots) + a_1(x^{n-1} + (n-1)x^{n-2} h + \dots) + \dots + a_n\} \\
&\quad - \{a_0 x^n + a_1 x^{n-1} + \dots + a_n\} \\
&= a_0 n h x^{n-1} + \text{lower degree terms}, \quad a_0 n h \neq 0
\end{aligned}$$

This is a polynomial of degree $n-1$.

$\therefore$ first order difference of an n^{th} degree polynomial is again a polynomial of degree $n-1$.

$$\begin{aligned}
\Delta^2 f(x) = \Delta(\Delta f(x)) &= \Delta(a_0 n h x^{n-1} + \text{lower degree terms}) \\
&= \{a_0 n h (x+h)^{n-1} + \text{lower degree terms}\} - \{a_0 n h x^{n-1} + \text{lower degree terms}\} \\
&= a_0 n h\{x^{n-1} + {}^{n-1}C_1 x^{n-2} h + \text{lower degree terms}\} \\
&\quad - \{a_0 n h x^{n-1} + \text{lower degree terms}\} \\
&= a_0 n h (n-1) h x^{n-2} + \text{lower degree terms} \\
&= a_0 h^2 n(n-1) x^{n-2} + \text{lower degree terms.}
\end{aligned}$$

Similarly $\Delta^3 f(x) = a_0 h^3 n(n-1)(n-2) x^{n-3} + \text{lower degree terms.}$

$$\therefore \Delta^n f(x) = a_0 h^n\, n(n-1)\cdots(2)(1)$$
$$= a_0 h^n n!$$

which is a constant.

$\therefore$ n^{th} difference of a polynomial of degree n is constant.

Hence $(n+1)^{th}$ and higher order differences are zero.

Example:

① $\Delta^{10}(1-x)(1-2x^2)(1-3x^3)(1-4x^4)$

$= \Delta^{10}\left((-1)(-2)(-3)(-4)x^{10} + \text{lower degree terms}\right)$

$= \Delta^{10}\left(24x^{10} + \text{lower degree terms}\right)$

$= 24\Delta^{10}(x^{10}) + 0$ $\qquad$ [$h=1$ of h^{10}].

$= 24(10)!$ $\qquad$ [$24\,(10)!\,h^{10}$]

② $\Delta^3(1-x)(1-2x)(1-3x) = -6\Delta^3(x^3) + \Delta^3(\text{lower degree terms})$

$= -6\cdot 3! + 0$

$= -6\times 6$

$= -36.$

$a+bx$, $\quad a,b$ $\quad 2$

$a+bx+cx^2$, $\quad a,b,c$, $\quad 3$

$a+bx+cx^2+dx^3$, $\quad a,b,c,d$, $\quad 4$.

To get an n^{th} degree polynomial, we need $(n+1)$ points.

Remark: Converse of the above is also true.
i.e. if the n^{th} differences of a tabulated function are constant when the values of the independent variable are equally spaced, then the function is a polynomial of degree n.

Example 1: With suitable assumptions find the missing terms in the following table:

x:	1	2	3	4	5	6	7
y:	103.4	97.6	122.9	?	179.0	?	195.8

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	103.4						
		-5.8					
2	97.6		31.1				
		25.3		$a-179.3$			
3	122.9		$a-148.2$		$629.4-4a$		
		$a-122.9$		$450.1-3a$		$b+10a-1739.4$	
4	a		$301.9-2a$		$b+6a-1110$		.....
		$179-a$		$b+3a-659.9$		$2502.7-5b-10a$	
5	179.0		$b+a-358$		$1392.7-4b-4a$		
		$b-179$		$732.8-3b-a$			
6	b		$374.8-2b$				
		$195.8-b$					
7	195.8						

We assume that y is a polynomial in x of degree 4. Hence 4^{th} order differences are constants. (we know 5 values of y). 5^{th} order differences are zero. $\therefore$

$$\left.\begin{aligned} b+10a-1739.4 &= 0 \\ 2502.7-5b-10a &= 0 \end{aligned}\right\}$$ solving it, we get

$a = 154.858$

$b = 190.825$

Example 2: Find the missing term:

x:	0	1	2	3	4
y:	1	3	9	—	81

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		$a-19$	
2	9		$a-15$		$124-4a$
		$a-9$		$105-3a$	
3	a		$90-2a$		
		$81-a$			
4	81				

Since 4 values of y are known, we assume that y is a polynomial in x of degree 3. ∴ Third order differences must be constant. 4th order difference must be zero.

$$\Rightarrow \quad 124-4a=0$$

$$a=\frac{124}{4}=31$$

Problems:

u_n:
$n=1, ②, 3, ④, 5, 6$

① Given $u_1=30.6$, $u_3=45.7$, $u_5=73.6$, $u_6=89.2$ find u_2 & u_4 under suitable assumptions.

② If $u_{-1}=10$, $u_1=8$, $u_2=10$, $u_4=50$, find u_0 and u_3.

③ Given $y_0=3$, $y_1=12$, $y_2=81$, $y_3=200$, $y_4=100$, $y_5=8$, without forming the difference table find $\Delta^5 y_0$ & $\Delta^4 y_1$.

$$\Delta^5 y_0=(E-1)^5 y_0=(E^5-5E^4+10E^3-10E^2+5E-1)y_0$$
$$=E^5y_0-5E^4y_0+10E^3y_0-10E^2y_0+5Ey_0-y_0$$
$$=y_5-5y_4+10y_3-10y_2+5y_1-y_0$$
$$=755$$

$$E^r y_k=y_{k+r}$$

$$\Delta^4 y_1=(E-1)^4 y_1=(E^4-4E^3+6E^2-4E+1)y_1$$
$$=E^4y_1-4E^3y_1+6E^2y_1-4Ey_1+y_1$$
$$=y_5-4y_4+6y_3-4y_2+y_1$$
$$=496$$