Curves and regions in the complex plane

Equation of a circle with centre at $z = a$ and radius r is given by

$$C : |z-a| = r$$

$|z-a| < r$ ① denotes the interior of the circle and $|z-a| > r$ ② denotes the exterior of the circle

$z = x+iy$

$a = a_1 + ia_2$

$z-a = (x-a_1) + i(y-a_2)$

$|z-a|^2 = [(x-a_1)+i(y-a_2)][(x-a_1)-i(y-a_2)]$

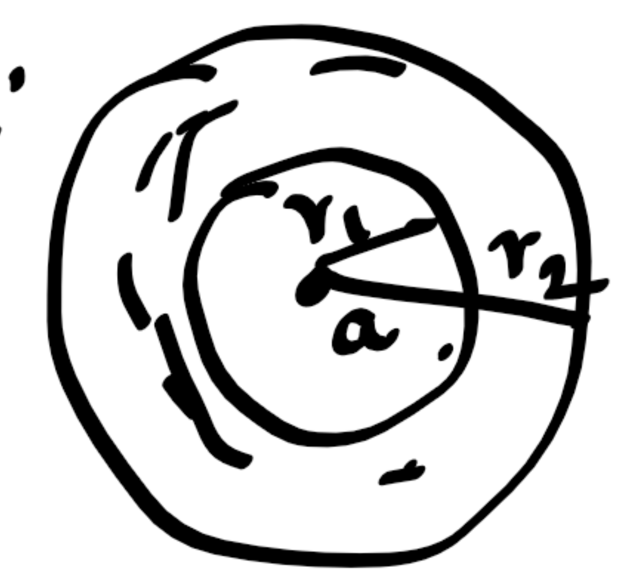
$= (x-a_1)^2 + (y-a_2)^2$

$= r^2$

① is called an open circular disk

closed circular disk is of the form $|z-a| \le r$

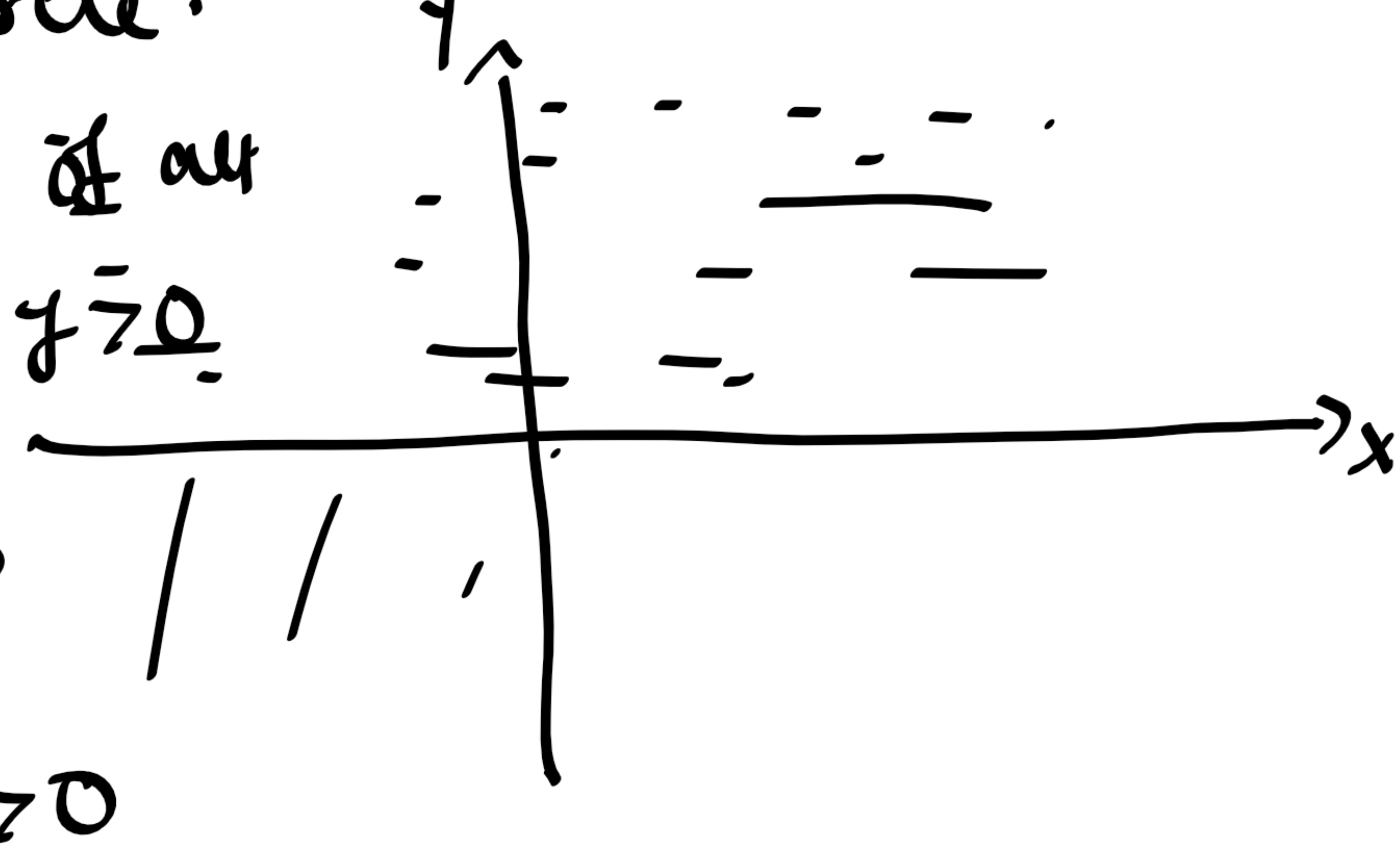
The region between two concentric circles is called an open annulus $r_1 < |z-a| < r_2$.



A closed annulus is $r_1 \le |z-a| \le r_2$

$|z| = 1 \longrightarrow$ Unit circle.

Upper half plane $\rightarrow$ The set of all points $z = x+iy$ for which $y > 0$



Lower half plane $\rightarrow$ $y < 0$

Right half plane $\longrightarrow$ $x > 0$

Left half plane $\longrightarrow$ $x < 0$

In Polar form, $z = x + iy$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$ → Modulus of Z

$\theta = \tan^{-1} y/x$ → Arguement of Z.

Complex Variables

A complex no: z can be written as

$$z = x + iy = (x, y)$$

Two complex nos: $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are said to be equal if $x_1 = x_2$ and $y_1 = y_2$.

Addition $\quad z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$$= (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication: $\quad z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= x_1 x_2 + i x_1 y_2 + i x_2 y_1 - y_1 y_1$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Division $\quad \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \times \dfrac{x_2 - iy_2}{x_2 - iy_2}$

$$= \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + i \frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Subtraction $\quad z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$$= (x_1 - x_2) + i(y_1 - y_2)$$

If $z = x + iy$ then $\bar{z} = x - iy$ is complex conjugate of z.

Exercises

1. Find $F_s\left(e^{-ax}\right)$, $a>0$ and hence find

(i) $\displaystyle\int_0^\infty \frac{s \sin sx}{a^2+s^2}\,ds$ (ii) $F_s\left(\frac{x}{a^2+x^2}\right)$

2. Find $F_s\left(\frac{e^{-ax}}{x}\right)$

3. Find the $F_c\left(e^{-ax}\cos ax\right)$ and $F_c\left(e^{-ax}\sin sx\right)$.

$$F_c\left(x^{a-1}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{a-1} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{s^a} \cos \frac{\pi}{2} a$$

$$F_s\left(x^{a-1}\right) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{s^a} \sin \frac{\pi}{2} a$$

Put $a = \frac{1}{2}$

$$F_c\left(1/\sqrt{x}\right) = \cancel{\sqrt{\frac{2}{\pi}}} \frac{\cancel{\Gamma(\frac{1}{2})}}{s^{1/2}} \cancel{\cos \frac{\pi}{4}} = 1/\sqrt{s}$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

similarly $F_s\left(1/\sqrt{x}\right) = 1/\sqrt{s}$

$\therefore \frac{1}{\sqrt{x}}$ is self reciprocal under sine and cosine transforms.

4. Find $F_c\left(e^{-x}\right)$ and hence find $F_c\left(\frac{1}{1+x^2}\right)$ and $F_s\left(\frac{x}{1+x^2}\right)$

$$F_c\left(e^{-x}\right) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

$$F_c\left(\sqrt{\frac{2}{\pi}} \frac{1}{1+x^2}\right) = e^{-s}$$

$$F_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}} e^{-s}$$

$$F_s\left(\frac{x}{1+x^2}\right) = -\frac{d}{ds} F_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}} e^{-s}$$

Put $a = 1/\sqrt{2}$

$$F_s\left(x e^{-x^2/2}\right) = s e^{-s^2/2}$$

$\therefore x e^{-x^2/2}$ is self-reciprocal under the sine transform.

3 Find $F_c\left(x^{a-1}\right)$ and $F_s\left(x^{a-1}\right)$; $0<a<1$

Deduce that $\frac{1}{\sqrt{x}}$ is self reciprocal under the sine and cosine transforms.

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\,dt$$

$bx = t$, $dx = \frac{dt}{b}$

$$\int_0^\infty e^{-t} \frac{t^{a-1}}{b^{a-1}} \frac{dt}{b}$$

$$= \frac{1}{b^a}\Gamma(a)$$

$\frac{1}{i} = -i$

from Gamma function

$$\int_0^\infty e^{-bx} x^{a-1}\,dx = \frac{\Gamma(a)}{b^a}$$

Set $b = is$

$$\int_0^\infty e^{-isx} x^{a-1}\,dx = \frac{\Gamma(a)}{(is)^a} = \frac{(-i)^a\,\Gamma(a)}{s^a} = \frac{e^{-\frac{i\pi a}{2}}\Gamma(a)}{s^a}$$

$$\int_0^\infty (\cos sx - i\sin sx) x^{a-1}\,dx = \left(\cos\frac{\pi}{2}a - i\sin\frac{\pi}{2}a\right)\frac{\Gamma(a)}{s^a}$$

Equate real and imaginary parts,

$$\int_0^\infty \cos sx\, x^{a-1}\,dx = \frac{\Gamma(a)}{s^a}\cdot\cos\frac{\pi}{2}a$$

$$\int_0^\infty \sin sx\, x^{a-1}\,dx = \frac{\Gamma(a)}{s^a}\sin\frac{\pi}{2}a$$

2. Find the $F_c\left(e^{-a^2x^2}\right)$ and $F_s\left(x e^{-a^2x^2}\right)$

$$F_c\left(e^{-a^2x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$\left(\text{F.T of } e^{-a^2x^2}\right)$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2}\, a}$$

Put $a = 1/\sqrt{2}$

$$F_c\left(e^{-x^2/2}\right) = e^{-s^2/2}$$

$\therefore$ $e^{-x^2/2}$ is self-reciprocal under the Fourier cosine transform.

$$F_s\left(x e^{-a^2x^2}\right) = -\frac{d}{ds} F_c\left(e^{-a^2x^2}\right)$$

$$= -\frac{d}{ds} \frac{e^{-s^2/4a^2}}{\sqrt{2}\, a}$$

$$= -\frac{1}{\sqrt{2}\, a} e^{-s^2/4a^2} \cdot \left(-\frac{s}{2a^2}\right) = \frac{1}{2\sqrt{2}\, a^3} s e^{-s^2/4a^2}$$

$$F_c(xe^{-ax}) = \frac{d}{ds} F_s(e^{-ax})$$

$$= \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}\right)$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{(a^2+s^2)(1) - s \cdot 2s}{(a^2+s^2)^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2-s^2}{(a^2+s^2)^2} \right]$$

$$F_s(xe^{-ax}) = -\frac{d}{ds} F_c(e^{-ax})$$

$$= -\frac{d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$= -\sqrt{\frac{2}{\pi}} \left\{ \frac{(a^2+s^2)(0) - a \cdot 2s}{(a^2+s^2)^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$$

Examples

1. Find the sine and cosine transforms of xe^{-ax}.

$$F_c(xf(x)) = \frac{d}{ds} F_s(f(x))$$

$$F_s(xf(x)) = -\frac{d}{ds} F_c(f(x))$$

Let $f(x) = e^{-ax}$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}(-a\cos sx + s\sin sx)}{a^2+s^2} \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}.$$

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}(-a\sin sx - s\cos sx)}{a^2+b^2} \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$