Methods for finding Particular Integral:

I. Inverse Differential operator:

Consider the DE

$$(D^n + b_1 D^{n-1} + \dots + b_n) y = R(x)$$

$$f(D) y = R(x), \quad f(D) = D^n + b_1 D^{n-1} + \dots + b_n.$$

$f(D)$ is called a differential operator.

Properties:

① $\boxed{f(D) e^{ax} = e^{ax} f(a)}$

Proof:

$$D e^{ax} = a e^{ax} \qquad D = \frac{d}{dx} \qquad D e^{ax} = \frac{d}{dx}(e^{ax})$$

$$D^2 e^{ax} = a^2 e^{ax} \qquad \cancel{e^{ax} D = e^{ax} \frac{d}{dx}}$$

$$\vdots$$

$$D^n e^{ax} = a^n e^{ax}$$

$$f(D) e^{ax} = (D^n + b_1 D^{n-1} + \dots + b_n) e^{ax} = D^n e^{ax} + b_1 D^{n-1} e^{ax} + \dots + b_n e^{ax}$$

$$= (a^n + b_1 a^{n-1} + \dots + b_n) e^{ax}$$

$$= f(a) e^{ax}$$

② $\boxed{f(D) e^{ax} y = e^{ax} f(D+a) y}$

$$D e^{ax} y = \frac{d}{dx}(e^{ax} y) = \frac{dy}{dx} e^{ax} + a e^{ax} y$$

$$= e^{ax}(Dy + ay)$$

$$= e^{ax}(D+a) y$$

$$D^2(e^{ax} y) = D(D e^{ax} y) = D(e^{ax}(D+a) y)$$

$$= e^{ax}(D+a)(D+a) y$$

$$= e^{ax}(D+a)^2 y$$

Illy $D^n(e^{ax} y) = e^{ax}(D+a)^n y$

$$\therefore f(D) e^{ax} y = e^{ax} f(D+a) y \qquad \frac{1}{f(D)} e^{ax}$$

③ $\boxed{(D-a)^k e^{ax} x^j = \begin{cases} 0, & j = 1, 2, \dots, k-1 \\ e^{ax} k!, & j = k \end{cases}}$

$$\frac{1}{\psi(D)(D-a)^k} e^{ax}$$

$$\frac{1}{\psi(a)(D-a)^k} e^{ax} = \frac{1}{\psi(a)} \frac{x^k e^{ax}}{k!}$$

Proof:

$$(D-a)^k e^{ax} x^j = e^{ax}(D+a-a)^k x^j = e^{ax} D^k x^j = \begin{cases} e^{ax} k!, & j = k \\ 0, & j = 1, 2, \dots, k-1 \end{cases}$$

$$\frac{d^k}{dx^k}(x^k) = k!, \quad \frac{d^k}{dx^k}(x^j) = 0, \text{ if } j < k$$

Inverse Differential operators Method:

Consider $f(D)\, y = R(x)$ —— (1)

$\frac{1}{f(D)} R(x)$ is that function of x, not containing arbitrary constants, which when operated upon $f(D)$ gives $R(x)$.

i.e. $f(D)\left\{\frac{1}{f(D)} R(x)\right\} = R(x)$.

Thus $\frac{1}{f(D)} R(x)$ satisfies equation (1) and hence is a particular integral of (1).

ie $\boxed{P.I = \frac{1}{f(D)} R(x)}$

$f(D)$ and $\frac{1}{f(D)}$ are inverse operators of each other.

Result 1: $\frac{1}{D} R(x) = \int R(x)\, dx$

Proof: Let

$$y = \frac{1}{D} R(x)$$

$$Dy = D\left(\frac{1}{D} R(x)\right)$$

$$\frac{dy}{dx} = R(x)$$

$$dy = R(x)\, dx$$

$$y = \int R(x)\, dx$$

$$\frac{1}{D} R(x)\, dx = \int R(x)\, dx$$

Result 2: $\frac{1}{D-a} R(x) = e^{ax} \int R(x)\, e^{-ax}\, dx$

$$y = \frac{1}{D-a} R(x)$$

$$(D-a)\, y = R(x)$$

$$\frac{dy}{dx} - ay = R(x), \qquad IF = e^{-\int a\, dx} = e^{-ax}$$

$$y\,(e^{-ax}) = \int R(x)\, e^{-ax}\, dx$$

$$y = e^{ax} \int R(x)\, e^{-ax}\, dx$$

$$\frac{1}{D-a} R(x) = e^{ax} \int R(x)\, e^{-ax}\, dx.$$

Finding P I :

$f(D)\, e^{ax} = f(a)\, e^{ax}$

Case I : Let $R(x) = e^{ax}$

Then

$$PI = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \quad \text{provided } f(a) \neq 0.$$

If $f(a) = 0$, then

$$PI = \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}, \quad f'(a) \neq 0$$

If $f'(a) = 0$, then

$$PI = \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}, \quad f''(a) \neq 0 \quad \text{and so on.}$$

Solve the following:

① $(D^2 + 5D + 6)y = e^x$.

To find CF :

A.E. is

$m^2 + 5m + 6 = 0$

$m = -2, -3$

$CF = c_1 e^{-2x} + c_2 e^{-3x}$

To find PI :

$$PI = \frac{1}{D^2 + 5D + 6} e^x \qquad a = 1$$

$$= \frac{1}{1^2 + 5\times 1 + 6} e^x$$

$$= \frac{e^x}{12}$$

$\therefore$ Complete solution is

$$y = CF + PI = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$$

② $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1-e^x)^2$ $\qquad (D^2+D+1)y = (1-e^x)^2$

To find CF:

$m^2+m+1=0$

$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$ $\qquad CF = e^{-x/2}\left[C_1 \cos\frac{\sqrt{3}x}{2} + C_2 \sin\frac{\sqrt{3}x}{2}\right]$

To find P.I:

$(1-e^x)^2$
$= 1 + (e^x)^2 - 2e^x$
$= 1 + e^{2x} - 2e^x$

$PI = \frac{1}{D^2+D+1}(1-e^x)^2 = \frac{1}{D^2+D+1}(1+e^{2x}-2e^x)$

$= \frac{1}{D^2+D+1}e^{0x} + \frac{1}{D^2+D+1}e^{2x} - 2\frac{1}{D^2+D+1}e^x$ $\qquad 1 = e^{0x}$

$= \frac{1}{0^2+0+1}e^{0x} + \frac{1}{2^2+2+1}e^{2x} - 2\frac{1}{1^2+1+1}e^x$

$= 1 + \frac{e^{2x}}{7} - \frac{2}{3}e^x$

Hence the complete solution is

$y = CF + PI$

$= e^{-x/2}\left[C_1 \cos\frac{\sqrt{3}x}{2} + C_2 \sin\frac{\sqrt{3}x}{2}\right] + 1 + \frac{e^{2x}}{7} - \frac{2}{3}e^x$

③ $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^{2x} + e^x$

$(D^3+6D^2+11D+6)y = e^{2x}+e^x$

To find CF:

$m=-1$ is a root.

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$m^3+6m^2+11m+6=0$

$m=-1, \quad m^2+5m+6=0$

$m=-1, \quad m=-2,-3$

$CF = C_1e^{-x} + C_2e^{-2x} + C_3e^{-3x}$

To find PI:

$PI = \frac{1}{D^3+6D^2+11D+6}(e^{2x}+e^x) = \frac{1}{D^3+6D^2+11D+6}e^{2x} + \frac{1}{D^3+6D^2+11D+6}e^x$

$= \frac{1}{2^3+6\times2^2+11\times2+6}e^{2x} + \frac{1}{1^3+6\times1^2+11\times1+6}e^x = \frac{e^{2x}}{60} + \frac{e^x}{24}$

④ $(D-1)^3 y = e^x + 2^x - \frac{3}{2}$

To find CF:

$(m-1)^3 = 0$

$m = 1, 1, 1$

$CF = (c_1 x^2 + c_2 x + c_3) e^x$

$2^x = e^{\log 2^x} = e^{x \log 2}$

To find PI:

$\frac{1}{(D-1)^3}\left(e^x + 2^x - 3/2\right) = \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3} 2^x - \frac{3}{2} \frac{1}{(D-1)^3} \cdot 1$

$= x \frac{1}{3(D-1)^2} e^x + \frac{1}{(D-1)^3} e^{(\log 2)x} - \frac{3}{2} \frac{1}{(D-1)^3} e^{0x}$

$= \frac{x}{3} \cdot x \frac{1}{2(D-1)} e^x + \frac{1}{(\log 2 - 1)^3} e^{(\log 2)x} - \frac{3}{2} \frac{1}{(0-1)^3} \cdot e^{0x}$

$= \frac{x^2}{6} \cdot x \frac{1}{1} e^x + \frac{1}{(\log 2 - 1)^3} 2^x + 3/2$

$= \frac{x^3 e^x}{6} + \frac{2^x}{(\log 2 - 1)^3} + \frac{3}{2}$

Problems for Practice:

① $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

② $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x$

③ $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$

④ $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

⑤ $(D^2 + D + 1)y = \cosh 2x$

⑥ $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

⑦ $(D^2 - 4)y = \sinh^2 x$

Hyperbolic Functions:

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\coth x = \frac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$\operatorname{cosech} x = \frac{1}{\sinh x}$.

$\boxed{\cosh^2 x - \sinh^2 x = 1}$

Case 2: Let $R(x) = \sin(ax+b)$ or $R(x) = \cos(ax+b)$.

$$y = \sin ax$$
$$Dy = a\cos ax$$
$$D^2y = -a^2 \sin ax \longrightarrow D^2(\sin ax) = -a^2 \sin ax$$
$$D^3y = -a^3 \cos ax$$
$$D^4y = a^4 \sin ax \longrightarrow D^4(\sin ax) = (-a^2)^2 \sin ax$$
$$D^5y = a^5 \cos ax$$
$$D^6 y = -a^6 \sin ax \longrightarrow D^6(\sin ax) = (-a^2)^3 \sin ax$$

$$PI = \frac{1}{f(D^2)} \sin(ax+b)$$
$$= \frac{1}{f(-a^2)} \sin(ax+b), \quad \text{provided} \quad f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, then

$$PI = \frac{1}{f(D^2)} \sin(ax+b)$$
$$= x \frac{1}{f'(-a^2)} \sin(ax+b), \quad f'(-a^2) \neq 0$$

you have to differentiate f w.r.t. D.

If $f'(-a^2) = 0$, then

$$PI = x^2 \frac{1}{f''(-a^2)} \sin(ax+b), \quad f''(-a^2) \neq 0 \text{ and so on.}$$

Similar results hold if $R(x) = \cos(ax+b)$.

Solve the following:

① $(D^2+4)y = 3\cos x$

To find CF:

$m^2+4=0, \quad m = \pm 2i$

$CF = c_1 \cos 2x + c_2 \sin 2x$

To find PI:

$$PI = \frac{1}{D^2+4} 3\cos x = 3 \frac{1}{D^2+4} \cos x = 3 \frac{1}{-1+4} \cos x$$
$$= \cos x.$$

$a=1$

$-a^2 = -1$

$\therefore$ Solution $y = c_1 \cos 2x + c_2 \sin 2x + \cos x.$

② $(D^2+9)y = \cos 3x$

To find CF:

$$m^2+9=0$$

$$m = \pm 3i \qquad (\alpha=0,\ \beta=3)$$

$$CF = c_1 \cos 3x + c_2 \sin 3x$$

To find PI:

$$PI = \frac{1}{D^2+9}\cos 3x \qquad a=3,\quad -a^2=-9$$

$$= x\frac{1}{2D}\cos 3x \qquad \frac{1}{D}\cos 3x = \int \cos 3x\,dx$$

$$= \frac{x}{2}\int \cos 3x\,dx$$

$$= \frac{x}{2}\left[\frac{\sin 3x}{3}\right] = \frac{x\sin 3x}{6}$$

Complete solution is $y = c_1\cos 3x + c_2 \sin 3x + \frac{x\sin 3x}{6}$

③ $(D^2-6D+10)y = \cos 2x + e^{-3x}$

CF: $m^2-6m+10=0$

$$m = \frac{6\pm\sqrt{36-40}}{2} = \frac{6\pm 2i}{2} = 3\pm i \qquad \alpha=3,\ \beta=1$$

$$CF = e^{3x}(c_1\cos x + c_2\sin x)$$

To find PI:

$$PI = \frac{1}{D^2-6D+10}(\cos 2x + e^{-3x})$$

$$= \frac{1}{D^2-6D+10}\cos 2x + \frac{1}{D^2-6D+10}e^{-3x}$$

$$(a=2) \qquad\qquad (a=-3)$$

$$= \frac{1}{-4-6D+10}\cos 2x + \frac{1}{(-3)^2-6(-3)+10}e^{-3x}$$

$$= \frac{1}{6-6D}\cos 2x + \frac{1}{37}e^{-3x}$$

$$= \frac{1}{6}\frac{1}{1-D}\cos 2x + \frac{1}{37}e^{-3x}$$

$$= \frac{1}{6}\cdot\frac{1}{5}\left[\cos 2x - 2\sin 2x\right] + \frac{1}{37}e^{-3x} = \frac{\cos 2x - 2\sin 2x}{30} + \frac{e^{-3x}}{37}$$

$$\boxed{D = \frac{d}{dx}}$$

$$\frac{1}{1-D}\cos 2x$$

$$= \frac{1+D}{(1+D)(1-D)}\cos 2x$$

$$= \frac{1+D}{1-D^2}\cos 2x$$

$$= \frac{1+D}{1-(-4)}\cos 2x$$

$$= \frac{1}{5}\left[\cos 2x + D\cos 2x\right]$$

$$= \frac{1}{5}\left[\cos 2x - 2\sin 2x\right]$$

④ $(D^2+3D+2)y = \sin 2x$

CF:

$$m^2+3m+2=0$$

$$m=-2,-1$$

$$CF = c_1 e^{-x} + c_2 e^{-2x}$$

PI:

$$PI = \frac{1}{D^2+3D+2}\sin 2x$$

$a = 2$

$-a^2 = -4$

$$= \frac{1}{-4+3D+2}\sin 2x$$

$$= \frac{1}{3D-2}\sin 2x$$

$$= \frac{3D+2}{(3D+2)(3D-2)}\sin 2x$$

$$= \frac{(3D+2)}{9D^2-4}\sin 2x$$

$$= \frac{3D+2}{9(-4)-4}\sin 2x$$

$$= \frac{1}{-40}(3D+2)\sin 2x$$

$$= -\frac{1}{40}\left[3\,D\sin 2x + 2\sin 2x\right]$$

$D\sin 2x = \frac{d}{dx}(\sin 2x)$

$= 2\cos 2x$

$$= -\frac{1}{40}\left[3\cdot 2\cos 2x + 2\sin 2x\right]$$

$$= -\frac{1}{40}\left[6\cos 2x + 2\sin 2x\right]$$

$$PI = -\frac{1}{20}\left[3\cos 2x + \sin 2x\right]$$

Hence the complete solution is

$$y = CF + PI = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{20}\left[3\cos 2x + \sin 2x\right]$$

⑤ $(D^4+18D^2+81)y = \cos^3 x$

To find CF:

$$m^4+18m^2+81=0$$

$$(m^2+9)^2=0$$

$$m^2=-9, -9$$

$$m=\pm 3i, \pm 3i$$

$$CF = (C_1x+C_2)\cos 3x + (C_3x+C_4)\sin 3x$$

To find PI:

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\cos^3 x = \frac{1}{4}\left[\cos 3x + 3\cos x\right]$$

$$PI = \frac{1}{D^4+18D^2+81}\cos^3 x$$

$$= \frac{1}{D^4+18D^2+81}\,\frac{\cos 3x+3\cos x}{4}$$

$$= \frac{1}{4}\left\{\frac{1}{D^4+18D^2+81}\cos 3x + 3\frac{1}{D^4+18D^2+81}\cos x\right\}$$

$$= \frac{1}{4}\left\{x\frac{1}{4D^3+36D}\cos 3x + 3\frac{1}{(-1)^2+18(-1)+81}\cos x\right\}$$

$$= \frac{1}{4}\left\{x\frac{1}{4D\cdot D^2+36D}\cos 3x + \frac{3\cos x}{64}\right\}$$

$$= \frac{1}{4}\left\{x\cdot x\frac{1}{12D^2+36}\cos 3x + 3\frac{\cos x}{64}\right\}$$

$$= \frac{1}{4}\left\{x^2\frac{1}{12(-9)+36}\cos 3x + 3\frac{\cos x}{64}\right\}$$

$$= \frac{1}{4}\left\{\frac{-x^2\cos 3x}{72} + \frac{3\cos x}{64}\right\}.$$

Problems for Practice:

① $(D^2+1)y = \sin x \sin 2x$

② $(D^2+2D+1)y = 4\sin 2x$

③ $(D^3+1)y = \cos(2x-1)$

④ $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = e^{2x}-\cos^2 x$

⑤ $(D^3+2D^2+D)y = e^{-x}+\sin 2x$