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Modern Control Theory (ICE 3153)

Phase Plane Analysis

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Basic Idea

-To generate motion trajectories corresponding to various initial conditions in the phase plane.

-To examine the qualitative features of the trajectories.

- In such a way, information concerning stability and other motion patterns of the system can be obtained.

Feature

1. A graphical method: to visualize what goes on in a nonlinear system without solving the nonlinear equations analytically.

2. Applying to strong nonlinearities and to “hard” nonlinearities.

3. Limitation: limited for second-order (or first –order) dynamic system; however, some practical control systems can be approximated as second-order systems.

Concepts of Phase Plane Analysis

- The phase plane method is concerned with the graphical study of second-order autonomous systems described by

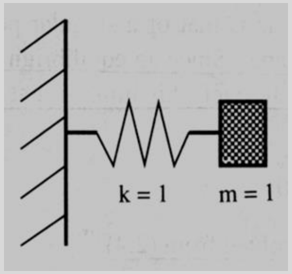
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

- where *x1* and *x2* are the states of the system, and f1, and f2 are nonlinear functions of the states.
- Geometrically, the state space of this system is a plane having x1, and *x2* as coordinates.
- We will call this plane the *phase plane*.

- Given a set of initial conditions x(0) = *x0*, the above Equation defines a solution x(t).
- With time t varied from zero to infinity, the solution *x(t)* can be represented geometrically as a curve in the phase plane.
- Such a curve is called a phase plane *trajectory*.
- A family of phase plane trajectories corresponding to various initial conditions is called a *phase portrait* of a system.

Phase Portrait of a Mass-spring System

Mass-spring system



$$\ddot{x} + x = 0 \qquad , \quad x(0) = x_0$$

$$\dot{x}(0) = 0$$

Assume that the mass is initially at rest, at length *xo*

Solution: $$x(t) = x_0 \cos t$$

$$\dot{x}(t) = -x_0 \sin t$$

Eliminating t from above equation, Equation of the trajectories:

$$x^2 + \dot{x}^2 = x_0{}^2$$

- This represents a circle in the phase plane.

- Corresponding to different initial conditions, circles of different radii can be obtained.

- Plotting these circles on the phase plane, we obtain a phase portrait for the mass-spring system.

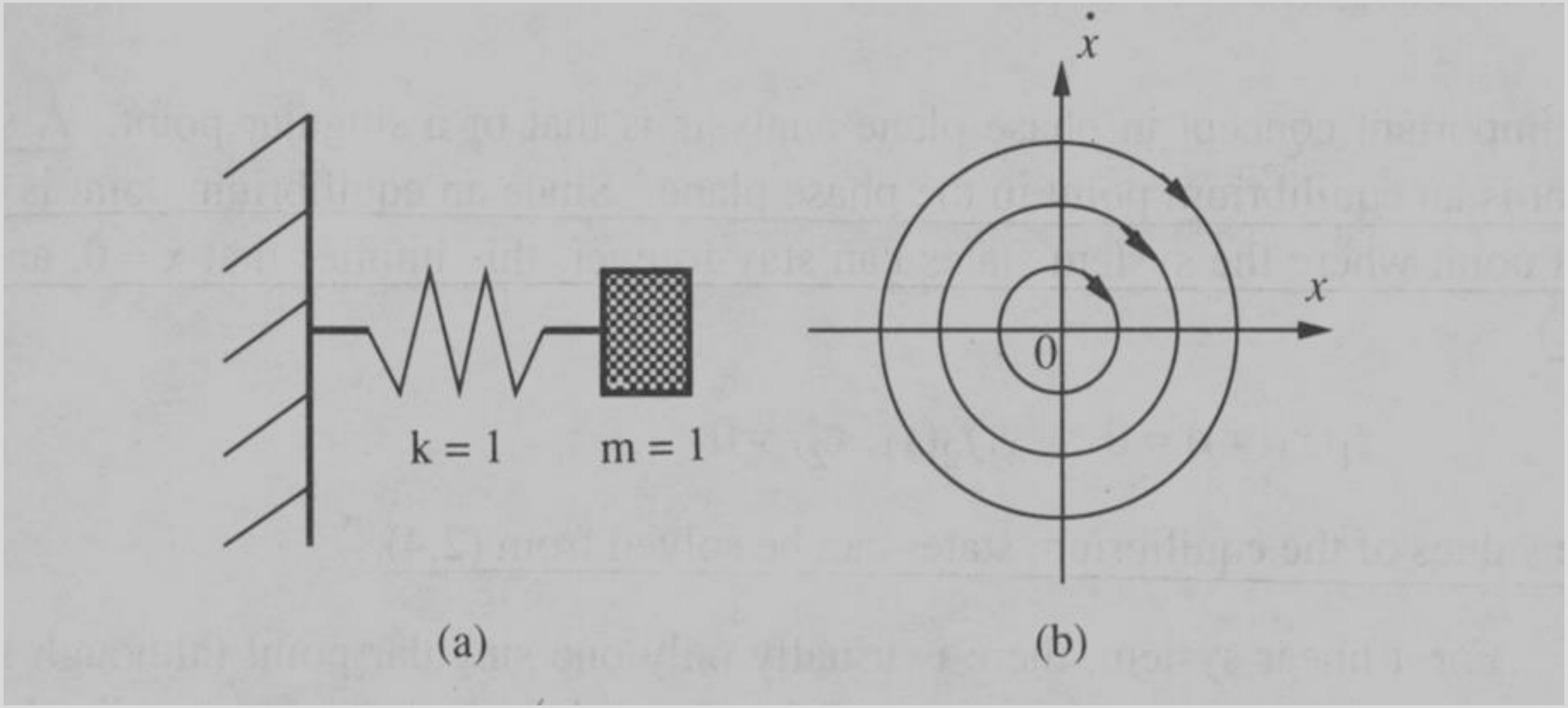


Figure: A mass-spring system and its phase portrait

-Importance of Phase plane:

-Once the phase portrait is obtained the nature of the system response corresponding to various initial conditions is directly displayed on the phase plane.

-For this example:

- The system trajectories neither converge to the origin nor diverge to infinity.

- They simply circle around the origin, indicating the marginal nature of the system's stability.

General Description of Second-Order System

A major class of second-order systems can be described by differential equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0$$

- In state space form :

$$\begin{aligned} \dot{x}_1 &= x_2 = f_1(x_1, x_2) \\ \dot{x}_2 &= -f(x_1, x_2) = f_2(x_1, x_2) \end{aligned} \qquad (2.3)$$

with $\quad x_1 = x \quad$ and $\quad x_2 = \dot{x}$

Phase Plane Method

The phase plane method is developed for the dynamics (2.3), and the phase plane is defined as the plane having x and $\dot{x}$ as coordinates.

Equilibrium Point and Singular Points

- Equilibrium point is defined as a point where the system states can stay forever, this implies that $\dot{x} = \mathbf{0}$, and using (2.3)

$$\begin{aligned} f_1(x_1, x_2) &= x_2 = 0 \\ f_2(x_1, x_2) &= -f(x_1, x_2) = 0 \end{aligned} \qquad (2.4)$$

The values of the equilibrium states can be solved from (2.4)

A singular point is an equilibrium point in the phase plane.

Example : A nonlinear second-order system

- The system

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

- has two singular points: (0, 0) and (-3, 0).
- The trajectories move towards the point (0,0) while moving away from the point (-3,0).

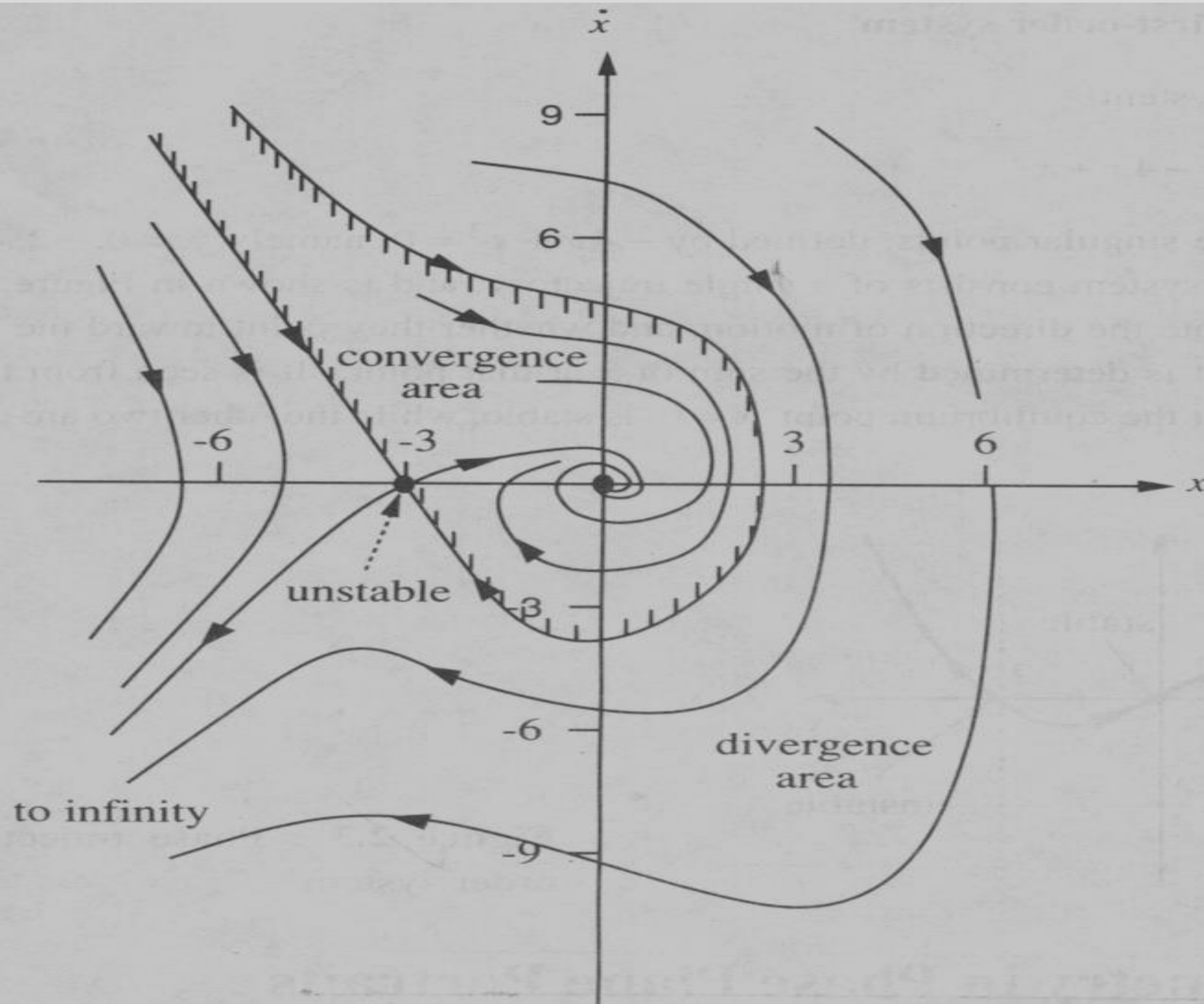


Figure 2.2 : The phase portrait of a nonlinear system

Why Called Singular Point

- The slope of the phase trajectories:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \qquad (2.5)$$

•With the functions f1 and *f2* assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane.

•This implies that the phase trajectories will not intersect.

- The value of the slope is 0/0 ,however, at singular points *i. e.*, the slope is indeterminate.

•This indeterminacy of the slope accounts for the adjective "singular".

Symmetry in Phase Plane Portraits

- Since symmetry of the phase portraits also implies symmetry of the slopes (equal in absolute value but opposite in sign), we can identify the following situations:

- **Symmetry about the x_1 axis:**

$$f(x_1, x_2) = f(x_1, -x_2)$$

- **Symmetry about the x_2 axis:**

$$f(x_1, x_2) = -f(-x_1, x_2)$$

- **Symmetry about the x_1 and x_2 axis:**

$$f(x_1, x_2) = f(x_1, -x_2) = -f(-x_1, x_2)$$

only one quarter of it has to be explicitly considered.

- **Symmetry about the origin:**

$$f(x_1, x_2) = -f(-x_1, -x_2)$$

Phase Plane Analysis of Linear Systems

- Consider the second-order linear system

$$\ddot{x} + a\dot{x} + bx = 0$$

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$ is the general solution

Where the eigenvalues λ_1 and λ_2 are the solutions of the characteristic equation

$$s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0$$

$$\ni \lambda_1 = \frac{(-a + \sqrt{a^2 - 4b})}{2}, \lambda_2 = \frac{(-a - \sqrt{a^2 - 4b})}{2}$$

There is only one singular point (assuming $b \neq 0$), namely the origin.

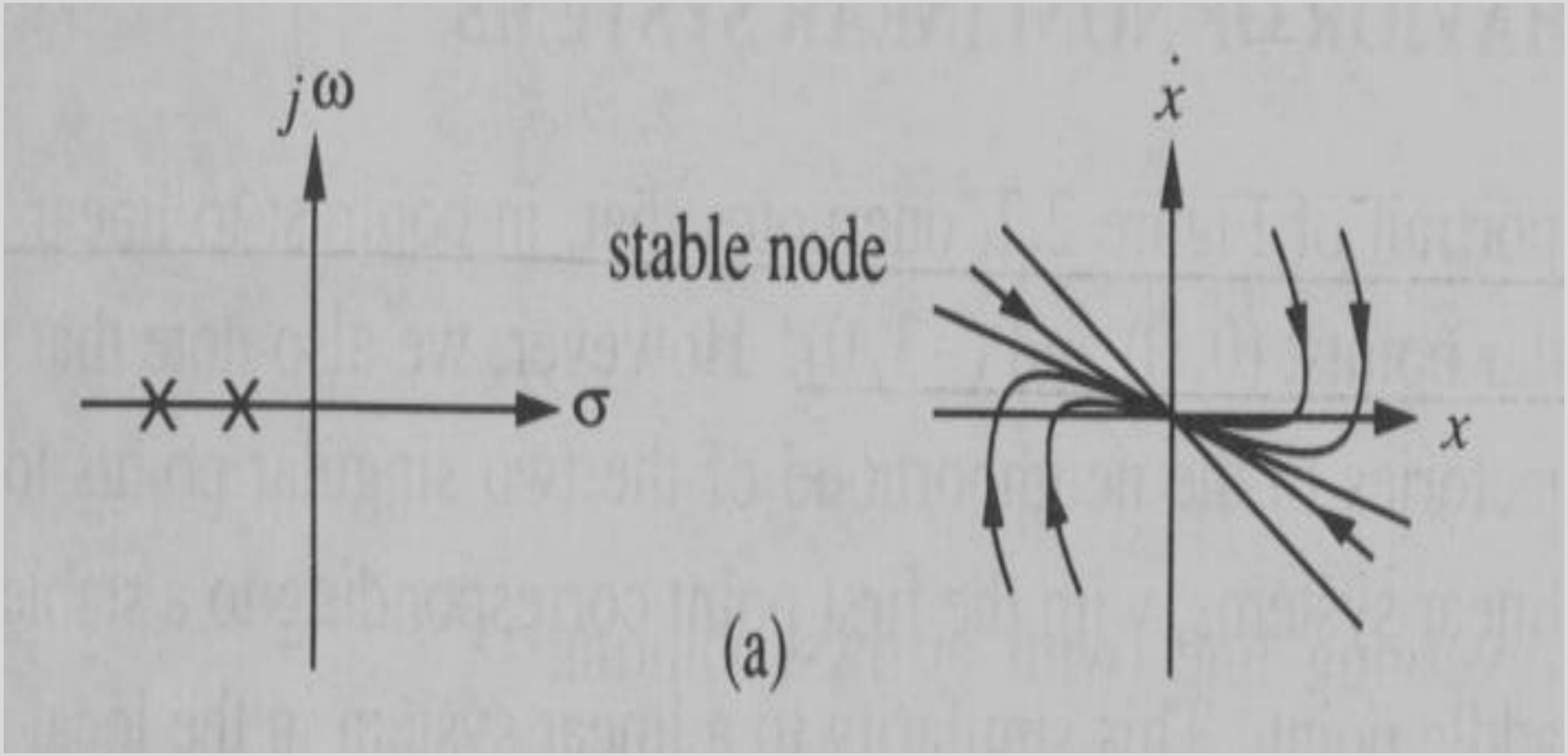
1. λ_1 and λ_2 are both real and have the same sign (positive or negative)

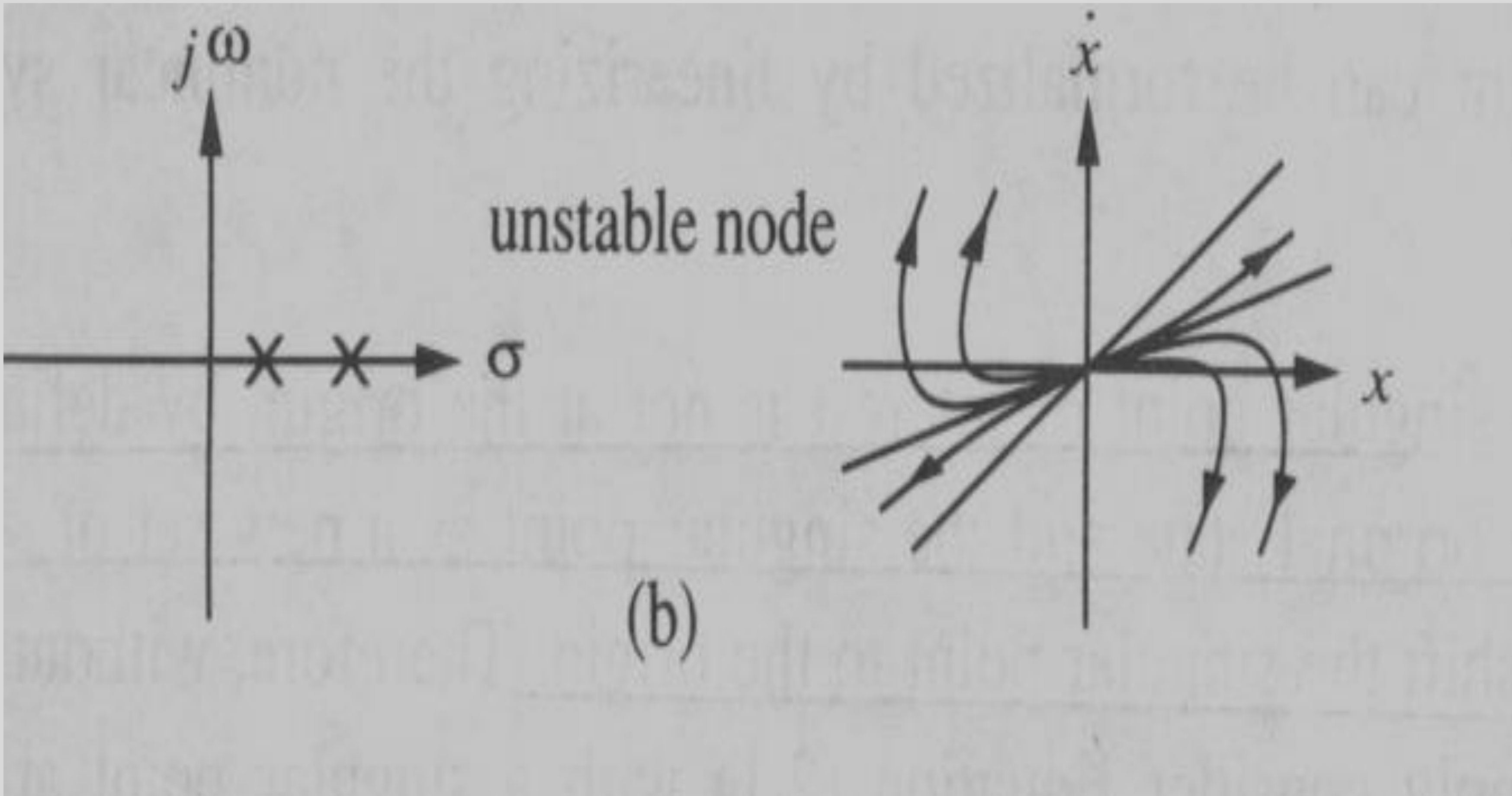
2. λ_1 and λ_2 are both real and have opposite signs (saddle point)

3. λ_1 and λ_2 are complex conjugate with non-zero real parts

4. λ_1 and λ_2 are complex conjugates with real parts equal to zero (center point)

Figure： Phase-Portraits of linear systems





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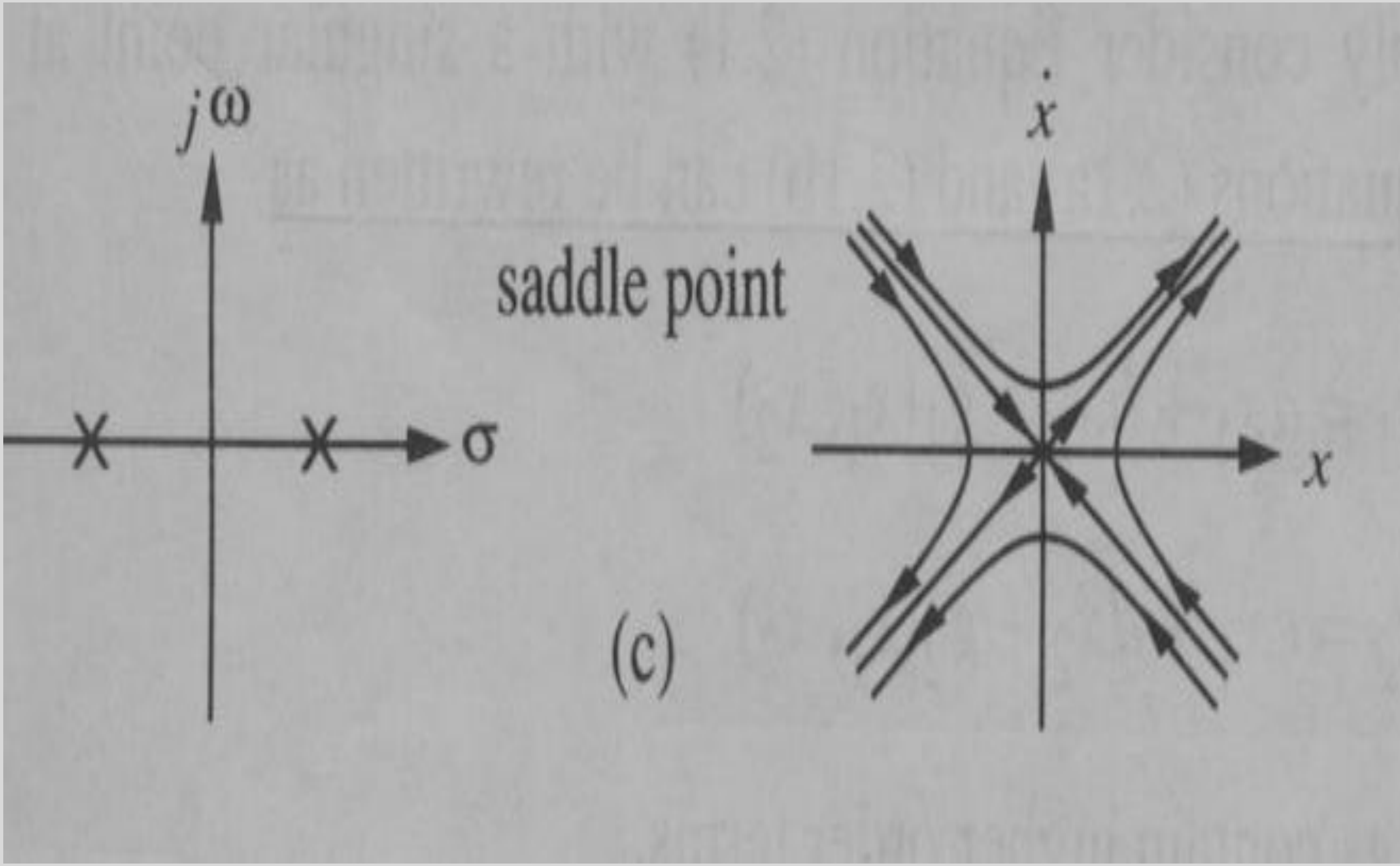
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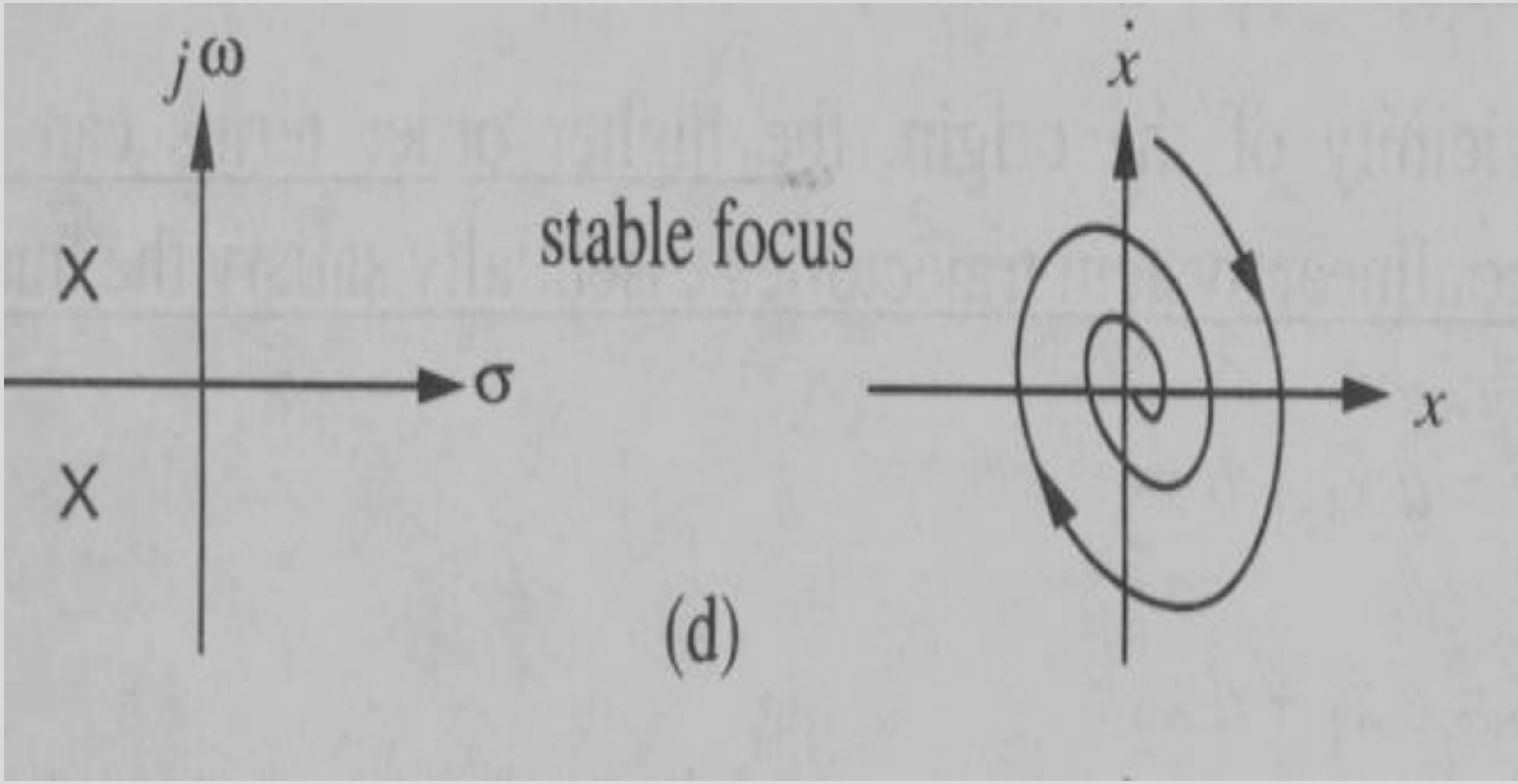
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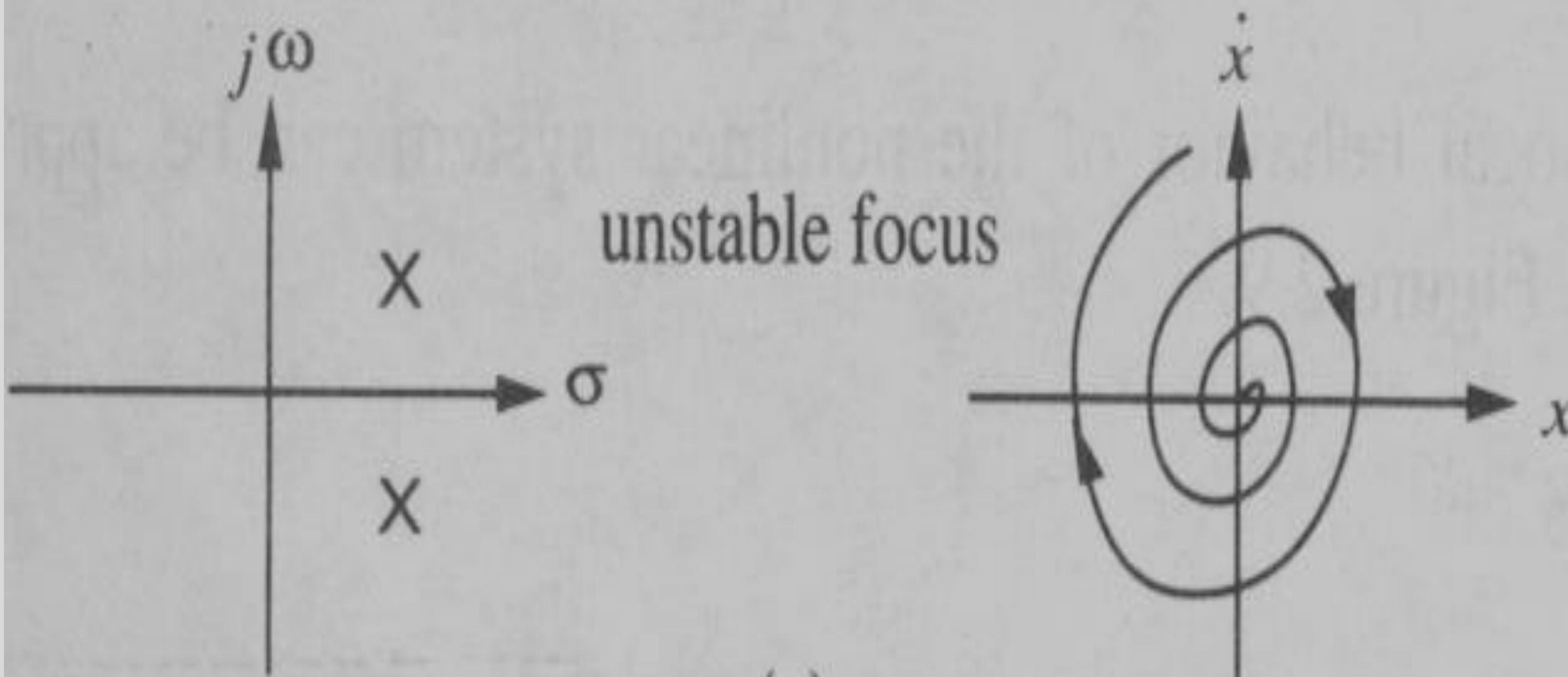

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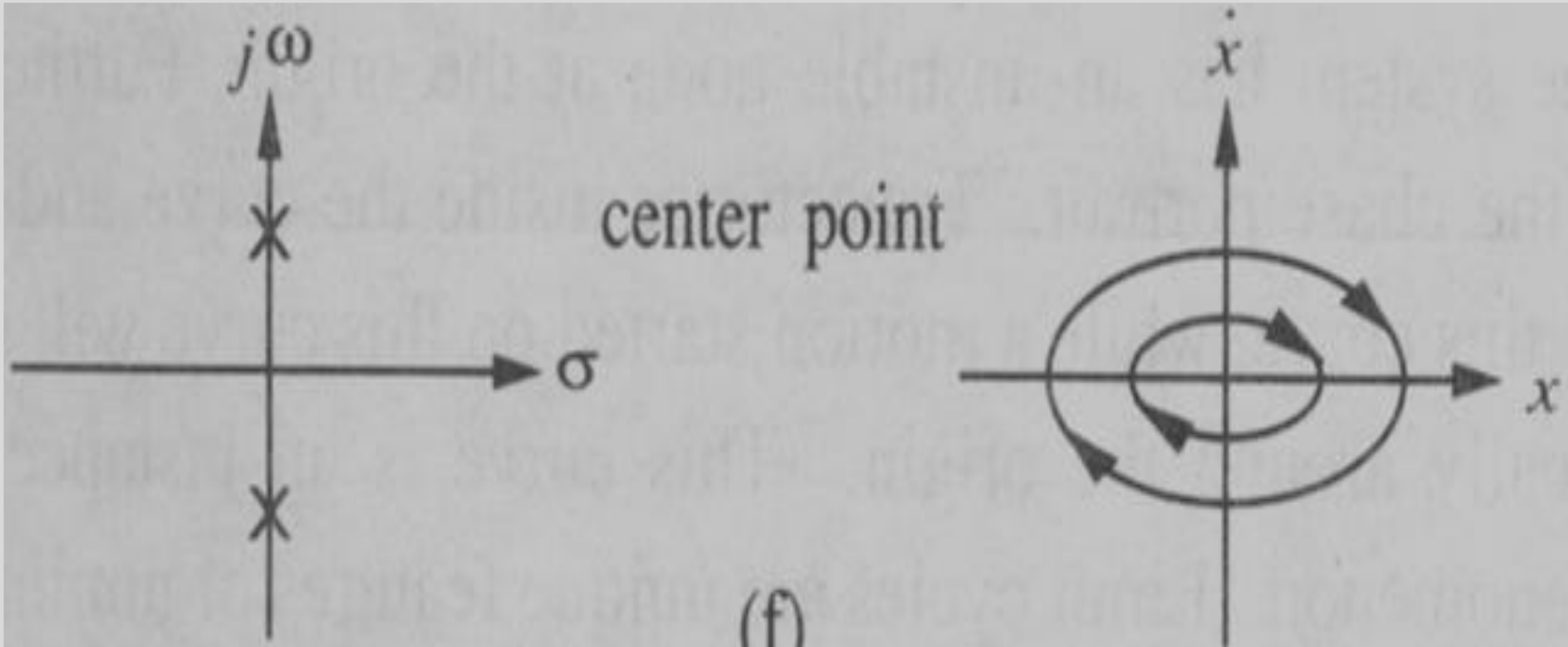

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Phase Plane Analysis of Nonlinear Systems

- In discussing the phase plane analysis of nonlinear systems, two points should be kept in mind.
- Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of a nonlinear system can be approximated by the behaviour of a linear system.
- Nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles

Local behavior of Nonlinear Systems

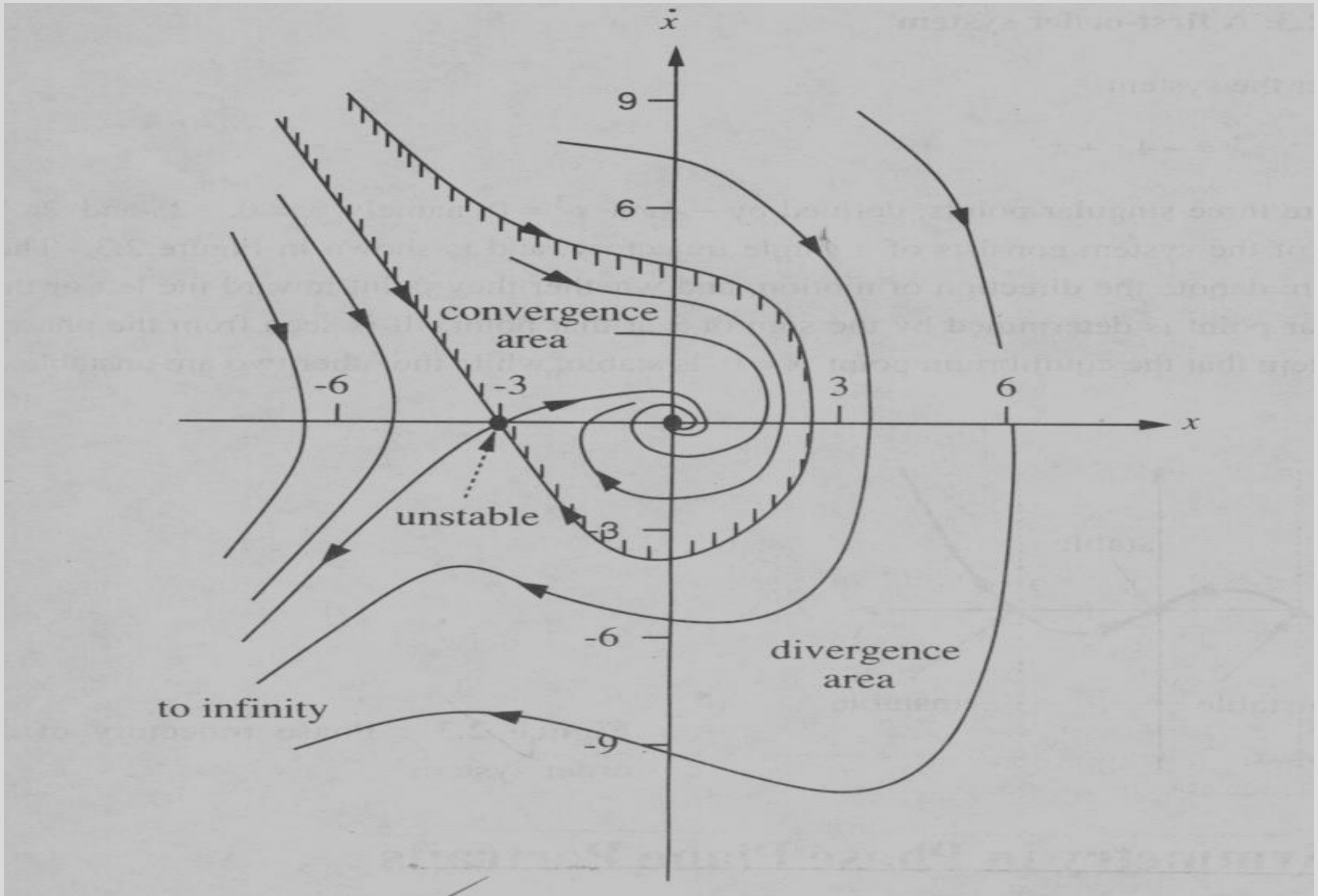

Figure 2.2 : The phase portrait of a nonlinear system

- Multiple equilibrium point : $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$

- If the singular point of interest is not at the origin : defining the difference between the original state and the singular point as a new set of state variables

$\Rightarrow$ shift the singular point to the origin.

The phase portrait of a nonlinear system around a singular point can be obtained like the phase portrait of a linear system.

Limit Cycles

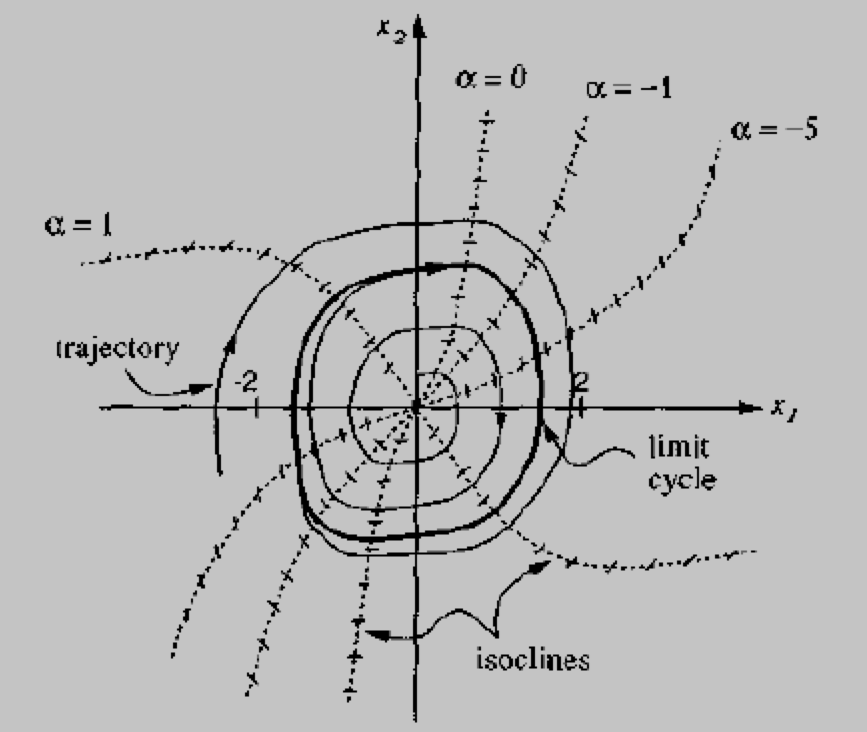

Figure 2.8 : Phase portrait of the Van der Pol equation

In the phase plane, *a limit cycle is defined as an isolated closed curve*.

The trajectory has to be both closed, indicating the periodic nature of the motion,

and isolated, indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it).

•Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, one can distinguish three kinds of limit cycles

1. **Stable Limit Cycles**： all trajectories in the vicinity of the limit cycle converge to it as

$$t \rightarrow \infty$$

2.**Unstable Limit Cycles**： all trajectories in the vicinity of the limit cycle diverge from it as $t \rightarrow \infty$

3. Semi-Stable Limit Cycles： some of the trajectories in the vicinity converge to it, while the other diverge from it as $t \longrightarrow \infty$

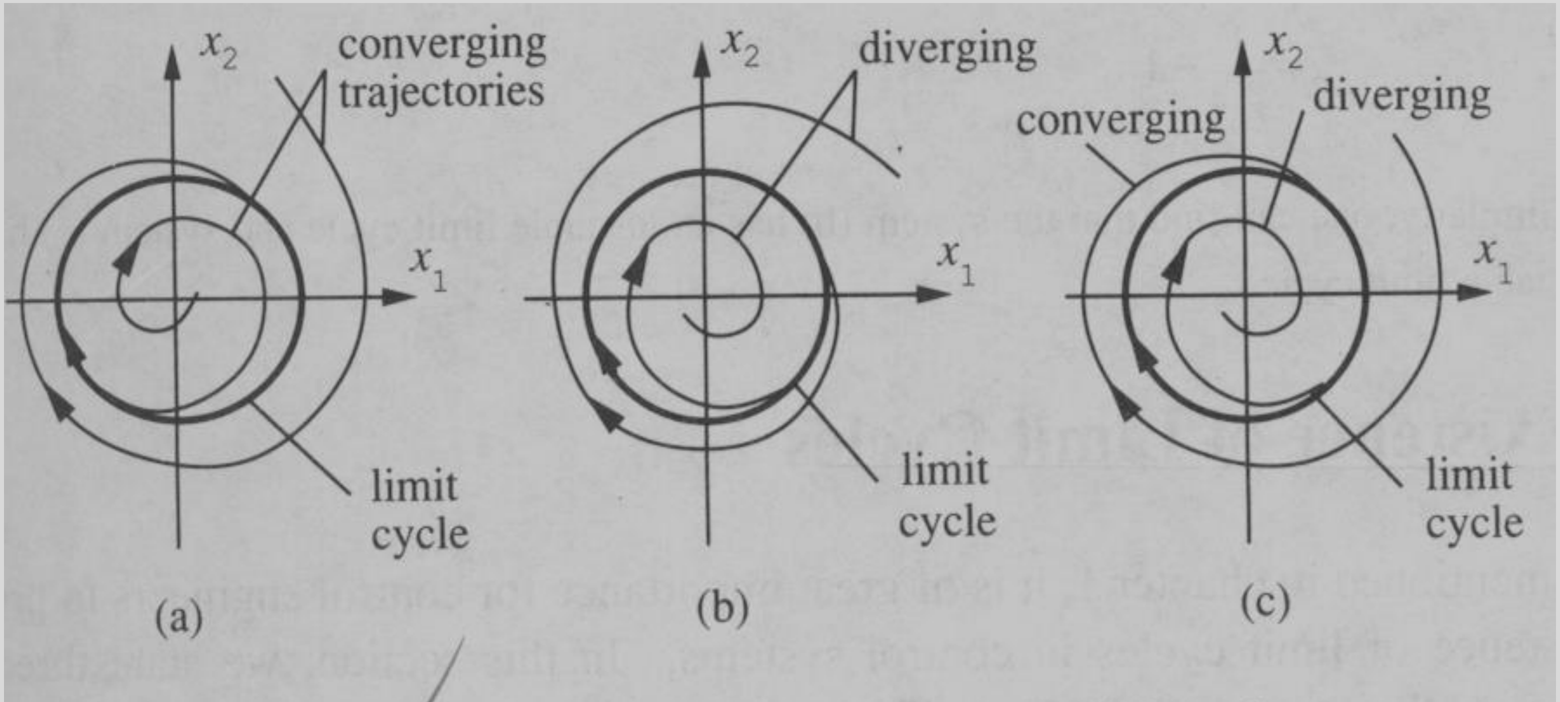

Figure 2.10 : Stable, unstable, and semi-stable limit cycles

Existence of Limit Cycles

- The first theorem to be presented reveals a simple relationship between the existence of a limit cycle and the number of singular points it encloses.
- In the statement of the theorem, we use N to represent the number of nodes, centers, and foci enclosed by a limit cycle, and S to represent the number of enclosed saddle points.

- **Poincare:** If *a limit cycle exists in the second-order autonomous system (2.1), then* $N = S + 1$.

Existence of Limit Cycles

- **Poincare-Bendixson:** *If a trajectory of the second-order autonomous system remains in a finite region* Q, *then one of the following is true:*

(a) the trajectory goes to an equilibrium point

(b) the trajectory tends to an asymptotically stable limit cycle

(c) the trajectory is itself a limit cycle

Existence of Limit Cycles

•**Bendixson:** *For the nonlinear system (2.1), no limit cycle can exist in a region Q. of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Jacobian Linearization

Consider the autonomous system $\dot{x} = f(x)$ assuming f(x) is continuously differentiable and f(0)=0. Then the system dynamics can be written as

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x=0}} \mathrm{x} + \mathrm{f}_{h.\,o.\,t.}(\mathrm{x})$$

where $\mathrm{f}_{h.\,o.\,t.}$ stands for higher-order terms in x.

Let A denote $\left(\dfrac{\partial f}{\partial x}\right)_{x=0}$

$$\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Then, the system $\dot{x} = Ax$

is called the *linearization* (or *linear approximation*) of the original nonlinear system at the equilibrium point 0.

For non-autonomous nonlinear system with a control input **u** $\dot{x} = f(x,u)$

such that f(0,0)=0, we can write

$$\dot{x} = \left(\frac{\partial f}{\partial x}\right)_{(x=0,u=0)} x + \left(\frac{\partial f}{\partial u}\right)_{(x=0,u=0)} u + f_{h.o.t(x,u)}$$

Let $A = \left(\frac{\partial f}{\partial x}\right)_{(x=0,u=0)}$ $B = \left(\frac{\partial f}{\partial u}\right)_{(x=0,u=0)}$

$$\dot{x} = Ax + Bu$$

: the linearization (or linear approximation) of the original nonlinear system at (**x = 0, u = 0**).

Tutorial -6

Question:1
Exercise 2.1 , Nonlinear systems , 3rd Edition H. K Khalil

For the nonlinear systems given below find the equilibrium points and determine the type of each isolated equilibrium point.

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

Equilibrium points are, (0,0) , (1,-1), (-1,1)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1+6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j \Rightarrow (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \pm \sqrt{8} \Rightarrow (1,-1) \text{ is a saddle}$$

Similarly, $(-1,1)$ is a saddle.

Question:2
Exercise 2.1 , Nonlinear systems , 3rd Edition H. K Khalil

For the nonlinear systems given below find the equilibrium points and determine the type of each isolated equilibrium point.

$$\dot{x}_1 = x_1 + x_1 x_2$$

$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

Equilibrium points are, (0,0) , (0,1) and (1,-1)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 + x_2 & x_1 \\ x_2 - 3x_1^2 & -1 + 2x_2 + x_1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 1,\ -1 \Rightarrow (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,1)} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2,\ 1 \Rightarrow (0,1) \text{ is unstable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j\sqrt{3} \Rightarrow (1,-1) \text{ is a stable focus}$$

Question:3
Exercise 2.20 , Nonlinear systems , 3rd Edition H. K Khalil

For the nonlinear systems given below check for the existence of limit cycle using Bendixson theorem .

$$\dot{x}_1 = -x_1 + x_1^3 + x_1 x_2^2$$

$$\dot{x}_2 = -x_2 + x_2^3 + x_1^2 x_2$$