$$\left[\because z(na^n) = \frac{az}{(z-a)^2}\right]$$

$$= \frac{1}{25}(-2)^n - \frac{1}{25}(3)^n + \frac{1}{15} z^{-1}\left(\frac{-3z}{(z+3)^2}\right)$$ [Mutipy and divide by -3].

$$= \frac{1}{25}(-2)^n - \frac{1}{25}(3)^n + \frac{1}{15} n(-3)^n$$

HW

① $y_{n+2} + 2y_{n+1} + y_n = n$, $y_0 = y_1 = 0$.

② $y_{n+1} - 2y_n = 1$, $y_0 = 0$.

Probability Distributions:

$n \to$ finite ← Binomial – pmf } DRV

$n \to \infty$ ← Poission

Normal

$x = 0, 1, 2, \ldots n$

Poission distribus in Biomial not their ~~Dont~~ not be asked

Binomial Distrubution:

Let x be a DRV the assumtions used to define a Binomial Distrubution are ① there are only 2 possible outcomes (success and failure)

② The probability of succes in each trail is same

③ There are 'n' trails.

Probability of (succes) = P

Probability of (failure) = $q = 1-p$.

Let 'x' be a Binomial RV with parameters n and p if its pmf is given by $n_{C_k} p^k q^{n-k}$.

$$P(X=k) = n_{C_k} \cdot p^k \cdot q^{n-k}, \quad k = 0, 1, 2, 3 \ldots k.$$

n = no of trails, p is p(succes) & p(failure).

$X \sim B(n, p) \rightarrow$ means x has binomial distrubution

Mean (or) Expectation:

$$E(X) = \sum_{k=0}^{n} k P(X=k)$$

$$= \sum_{k=0}^{n} k \cdot n_{C_k} \cdot p^k \cdot q^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$$

$$= \sum_{k=0,1}^{n} \frac{n!}{(n-k)!(k-1)!} p^k q^{n-k}$$

put $k-1 = s$

$$E(x) = \sum_{s=0}^{n-1} \frac{n(n-1)!}{(n-s-1)!\, s!} \cdot p^{s+1} q^{n-s-1}$$

$$= \sum_{s=0}^{n-1} \frac{n(n-1)!}{(n-s-1)!\, s!} \cdot p^s \cdot p \cdot q^{n-s-1}$$

$$= np \sum_{s=0}^{n-1} \frac{n(n-1)!}{(n-1-s)!\, s!} \cdot p^s \cdot q^{n-s-1}$$

$$\left[(a+b)^n = \sum_{k=0}^{n} nc_k a^k b^{n-k} \right].$$

$$= np(p+q)^{n-1}$$

$$\boxed{E(x) = np}$$

Variance:

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum_{k=0}^{n} k^2 P(x=k)$$

$$= \sum_{k=0}^{n} [k(k-1)+k] P(x=k)$$

$$= \sum_{k=0}^{n} k(k-1) P(x=k) + \sum_{k=0}^{n} k P(x=k)$$

$$= \sum_{k=0}^{n} k(k-1) \frac{n!}{(n-k)!\, k!} + np.$$

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$$= \sum_{k=2}^{n} \frac{n!}{(n-k)!\,(k-2)!} p^k q^{n-k} + np.$$

$k-2 = s.$

$$= \sum_{s=0}^{n-2} \frac{n!}{(n-2-s)!\, s!} p^{s+2} \cdot q^{n-2-s} + np$$

$$= n(n-1)p^2 \sum_{s=0}^{n-2} \frac{(n-2)!}{(n-2-s)!\, s!} p^s q^{n-2-s} + np.$$

$$= n(n-1)p^2 (p+q)^{n-2} + np.$$

$$= n(n-1)p^2 + np \quad (\because p+q=1).$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= n(n-1)p^2 + np - n^2p^2 = -np^2 + np = -np^2 + np$$

$$\boxed{V(x) = npq}$$

19) In tossing of 6 coins find probability of getting ① exactly 3 heads, ② at most 3 head, ③ at least 1 head.

A) X: No. of heads

$X \sim B(6, 1/2)$

① $P(X=3) = {}^6C_3 (1/2)^3 (1/2)^{6-3}$

$= 20 \times \frac{1}{64}$

$= \frac{5}{16}$.

② At most 3 heads

$P(X \le 3) = \sum_{k=0}^{3} {}^6C_k (1/2)^k (1/2)^{6-k}$

$= \frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{20}{64}$

$= \frac{42}{64} = \frac{21}{32}$

③ $P(X \ge 1) = \cancel{\sum_{k=1}^{6} {}^6C_k (1/2)^k (1/2)^{6-k}}$

$= 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - \frac{1}{64}$

$= \frac{63}{64}$

Poisson Distribution: Let x be a DRV assuming the possible values $0, 1, 2, \dots \infty$. If $P(X=k)$

$$P(X=k) = \frac{e^{-\alpha} \alpha^k}{k!} \quad k = 0, 1, 2 \dots$$

then we say that x has Poisson distribution with parameter α.

Mean | Expectation.

$$E(X) = \sum_{k=0}^{\infty} k\, P(X=k)$$

$$= \sum_{k=0}^{\infty} k . \frac{e^{-\alpha}\alpha^k}{k!}$$

$$= \sum_{k=1}^{\infty} \cancel{k} . \frac{e^{-\alpha}\alpha^k}{\cancel{k}(k-1)!}$$

$k-1 = s$

$$= \sum_{s=0}^{\infty} \frac{e^{-\alpha}\alpha^{s+1}}{s!}$$

$$= \alpha e^{-\alpha} \sum_{s=0}^{\infty} \frac{\alpha^s}{s!}$$

$$= \alpha e^{-\alpha} e^{\alpha} = \alpha.$$

Variance.

$$V(X) = E(X^2) - [E(X)]^2.$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2\, P(X=k)$$

$$= \sum_{k=0}^{\infty} k^{\cancel{2}} \frac{e^{-\alpha}\alpha^k}{k(k-1)!}$$

$$= \sum_{s=0}^{\infty} (s+1) \frac{e^{-\alpha}\alpha^{s+1}}{s!}$$

$$= \alpha\left(\sum_{s=0}^{\infty} \frac{s e^{-\alpha}\alpha^s}{s!} + \sum_{s=0}^{\infty} e^{-\alpha}\alpha^s/s!\right).$$

$$= \alpha\{E(X) + 1\}$$

$$= \alpha^2 + \alpha.$$

→ Binomial distribution tends to poisson distribution under following conditions

1) n is very large ($n \to \infty$)
2) p is very small ($p \to 0$).
3) There exist a constant $\alpha > 0$ such that $np = \alpha$

$$\lim_{n\to\infty} {}^nC_k\, p^k\, q^{n-k} = \frac{e^{-\alpha}\alpha^k}{k!} \quad \boxed{\alpha = np}$$

Q2) Suppose that a container contains 10,000 particles. the probability that the particle escapes from the container is 0.0004 what is the probability that more than 5 such escapes occur.

Ans)

$n = 10{,}000$

$P = 0.0004$

$X =$ no. of escapes

$P(X > 5) = 1 - P(X \leq 5)$

$\alpha = np = 10000 \times 0.0004 = 4.$

$= 1 - \sum_{k=0}^{5} \frac{e^{-4}(4)^k}{k!}$

$= 1 - [e^{-4} + e^{-4} \times 4 + e^{-4} \times 8 + e^{-4}\frac{32}{3} + e^{-4}\frac{32}{3}$
$e^{-4}\frac{128}{15}]$

$= 1 - [e^{-4}(1 + 4 + 8 + \frac{32+32+640}{3})]$

$= 1 - [e^{-4}(\frac{3+12+24+704}{3})]$

$= 1 - 0.01(247.66)$

$=$

$= 0.68$

Q3) If $X \sim P(\alpha)$

① $P(X \text{ is even}) = \frac{1+e^{-\alpha}}{2}$

② $P(X \text{ is odd}) = \frac{1-e^{-\alpha}}{2}$

Ans)

$P(X = \text{even}) = \sum_{k=\text{even}} \frac{e^{-\alpha}\alpha^k}{k!}$

$= P(X=0) + P(X=2) + P(X=4) \ldots$

$= e^{-\alpha} + \frac{e^{-\alpha}\alpha^2}{2!} + \frac{e^{-\alpha}\alpha^4}{4!} + \ldots$

$$e^{\alpha} = 1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}$$

$$e^{-\alpha} = 1-\alpha+\frac{\alpha^2}{2!}-\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\cdots$$

$$= e^{-\alpha}\left(1+\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}+\cdots\right)$$

$$= e^{-\alpha}\left(\frac{e^{\alpha}+e^{-\alpha}}{2}\right)$$

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In playing with an opponent of equal ability. which is more probable?

i) winning 3 games out of 4 or 5 out of 8.

ii) winning atleast 3 games out of 4 or atleast 5 out of 8.

Sol.

$P = \frac{1}{2}$ $q = 1/2$

i) 3 games out of 4

$n=4$ $x=3$

$P(X=3) = 4C_3 (1/2)^3 (\frac{1}{2})^1$

$= \frac{4}{16} = 1/4$

5 out of 8

$n=8$ $x=5$

$P(X=5) = 8C_5 (P)^5 (q)^3$

$= \frac{56}{256} = \frac{7}{32}$

$P(X=3) > P(X=5)$

$\therefore$ winning 3 out of 4 is more probable.

Q) In a certain frequency producing blades, there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packets of 10. Calculate the approx number of packets containing (i) no defective blade.

(ii) 1 defective blade in a consignment of 10000 packets.

Ans)

for 1 packet

$n = 10$

X : no. of det. blade

p = prob of det blade $= \frac{1}{500}$

$\lambda = np = \frac{100}{500} = 0.02.$

$P(X=0) = \frac{e^{-\lambda}(\lambda)^k}{k!} = \frac{e^{-0.02}(0.02)^0}{0!}$

$= 0.98.$

i) No. of packets containing no defective blade

$= 10000 \times 0.98 = 9800.$

(ii) $P(X=1) = \frac{e^{-0.02}(0.02)^1}{1!} = 0.0196.$

No. of packets having 1 det. blade

$= 0.0196 \times 10000$

$= 196$ packets.

Continuous distributions:

Normal distribution (Gaussian distribution):

The RV X assuming all real values from $-\infty$ to ∞ has a normal distribution, if its pdf is of the form

$$f(x) = \frac{1}{(\sqrt{2\pi})\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \rightarrow (1)$$

$-\infty < \mu < \infty$

$\sigma > 0.$

$E(X) = \mu$

$X \sim N(\mu, \sigma^2)$

$V(X) = \sigma^2$

Mean :

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \cdot e^{-1/2 \left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\mu + \sigma z) \cdot e^{-\frac{z^2}{2}} (\sigma)\, dz$$

let $\frac{x-\mu}{\sigma} = z$

$x = \mu + \sigma z$

$dx = \sigma dz$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_{0}^{\infty} \mu \cdot e^{-\frac{z^2}{2}} \cdot dz + \sigma \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2}} \cdot dz$$

$$= \sqrt{2/\pi}\; \mu \int_0^{\infty} e^{-\frac{z^2}{2}} dz + 0.$$

let $\frac{z^2}{2} = t.$

$z^2 = 2t$

$2z\,dz = 2 \cdot dt \Rightarrow \quad dz = \frac{dt}{z} = \frac{dt}{\sqrt{2t}}.$

$$= \sqrt{\frac{2}{\pi}} \cdot \mu \int_0^{\infty} e^{-t} \cdot \frac{dt}{\sqrt{2t}} \quad \cancel{dz = dt}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \mu \int_0^{\infty} e^{-t} \cdot t^{-1/2} \cdot dt \qquad \Gamma(n) = \int_0^{\infty} e^{-t} \cdot t^{n-1}\, dt$$

$$= \frac{1}{\sqrt{\pi}}\, \mu\; \Gamma(1/2)$$

$$= \frac{1}{\sqrt{\pi}}\, \mu \sqrt{\pi} = \mu.$$

Variance:

$$V(X) = E[X - E(X)]^2.$$

$$= E(X-\mu)^2 = \frac{2}{\sqrt{\pi}}\, \sigma^2\, \Gamma(3/2).$$

$$V(X) = \frac{2}{\sqrt{\pi}}\, \sigma^2 \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

$$V(X) = \sigma^2.$$

Standard normal variate:

$X \sim N(0,1)$ i.e mean $= 0$

variance $= 1$.

Sub $\mu = 0$ $\sigma = 1$ in (1)

we get $\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$.

The cdf of standard normal variate is given by

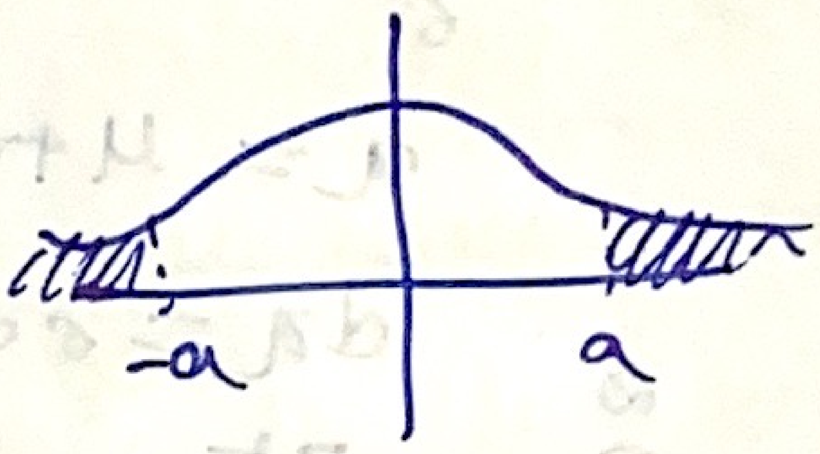
$\phi(a) = P(z \le a)$

$\phi(-a) = P(z \le -a)$

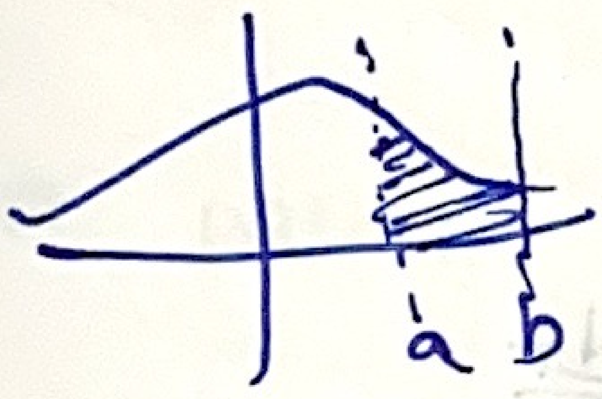
$= P(z > a)$

$= 1 - P(z \le a)$

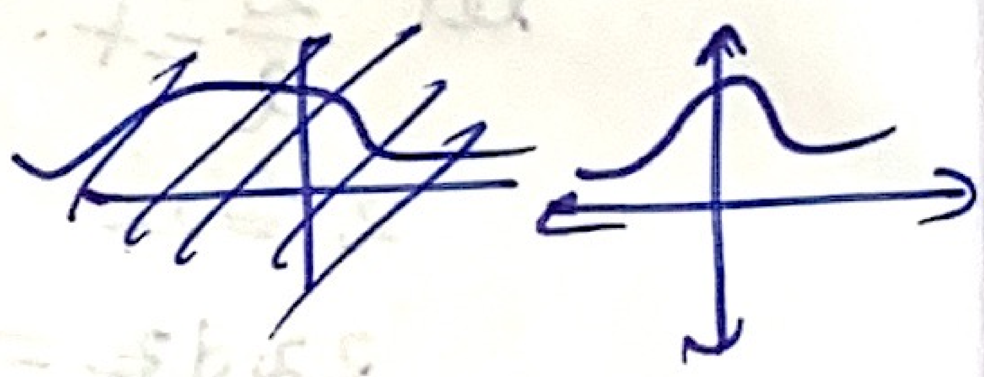
$\phi(-a) = 1 - \phi a$



$P(a < z < b) = \phi(b) - \phi(a)$



Graph of normal distribution is symmetrical abt x-axis



If x has

$X \sim N(\mu, \sigma^2)$ then, $z = \frac{x - \mu}{\sigma} \sim N(0,1)$

then $P(a < x < b) = P\left(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}\right)$

$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$

normal variate
Paper 4M
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Binomial
Poisson 3M

Q) Suppose the 'x' has normal distribution with Mean 75 and variance 100 find i) $P(X < 60)$

ii) $P(70 < x < 100)$.

Ans)

Let $z = \frac{x - \mu}{\sigma} \sim N(0,1) \rightarrow$ 0.5 Mars ★

$\mu = 75$, $\sigma = 10$.

$z = \frac{x - 75}{10} \sim N(0,1)$

(i) $P(X<60) = P\left(z < \frac{60-75}{10}\right)$.

$= P(z < -1.5)$

$= \phi(-1.5)$

$= 1 - \phi(1.5)$

$= 1 - 0.9332$.

$= 0.0668$.

(ii) $P(70 < X < 100) = P\left(\frac{70-75}{10} < z < \frac{100-75}{10}\right)$

$= P\left(-\frac{1}{2} < z < 2.5\right)$. $= P(-0.5 < z < 2.5)$.

$= \phi(v) - \phi(a)$

$= \phi(2.5) - \phi(-0.5)$

$= \phi(2.5) - (1 - \phi(0.5))$.

$= 0.9938 - 1 + 0.6915$

$= 0.6853$.

Q) If x has Normal distribution $X \sim N(1, 4)$.
Find $P(|x| > 4)$.

Ans) $P(|X| > 4) = 1 - P(|X| \leq 4)$

$= 1 - P(-4 \leq X \leq 4)$.

$z = \frac{X - \mu}{\sigma} = \frac{X - 1}{2} \sim N(0,1)$.

$P\left(\frac{-4-1}{2} \leq z \leq \frac{4-1}{2}\right) = P\left(\frac{-5}{2} \leq z \leq \frac{3}{2}\right)$

$= P(-2.5 \leq z \leq 1.5)$

$= 1 - (\phi(1.5) - \phi(-2.5))$

$= \cancel{1} - (\phi(1.5) - \cancel{1} + \phi(2.5))$

$= \phi(2.5) - \phi(1.5)$

$= 0.9938 - 0.9332$

$= 0.0606$.

Q) If $X \sim N(75, 25)$
find $P(X>80 \mid X>77)$

Ans) $\dfrac{P\{(X>80) \cap (X>77)\}}{P(X>77)} = \dfrac{P(X>80)}{P(X>77)}$

$= \dfrac{1-P(X\le 80)}{1-P(X\le 77)} = \dfrac{1-P\left(Z\le \frac{80-75}{5}\right)}{1-P\left(Z\le \frac{77-75}{5}\right)}$

$= \dfrac{1-P(Z\le 1)}{1-P(Z\le 0.4)} = \dfrac{1-\phi(1)}{1-\phi(0.4)}$

$= \dfrac{1-0.8413}{1-0.6554} \approx 0.460533.$

Q) In the normal distribution 31% of items are under 45 & 8% are over 64. find the Mean and Standard deviation of distribution.

A) 31% → ↓45

8% → ↑64

$P(X<45) = 0.31$

$P(X>64) = 0.8$

$X \sim N(\mu, \sigma^2)$

let $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$P(X<45) = P\left(Z<\dfrac{45-\mu}{\sigma}\right) = 0.31$

$\phi\left(\dfrac{45-\mu}{\sigma}\right) = 0.31$

$\phi\left(\dfrac{\mu-45}{\sigma}\right) = 1-0.31$

$\phi\left(\dfrac{\mu-45}{\sigma}\right) = 0.69$

$\dfrac{\mu-45}{\sigma} = 0.5$

$\mu - 0.5\sigma = 45 \rightarrow (1)$

$\phi\left(\dfrac{64-\mu}{\sigma}\right) = 0.92$

$\dfrac{64-\mu}{\sigma} = 1.41$

$64-\mu-1.41\sigma = 0 \rightarrow (2)$

Q) The height of 500 soliders are found to have normal distribution of them 258 are found to be within 2 cm of mean height is 170 cm. Find the standard deviation of height.

A) Let,

$X = $ Height

$X \sim N(170, \sigma^2)$.

$$P(170-2 \leq X \leq 170+2) = \frac{258}{500}$$

$$P(168 \leq X \leq 172) = 0.516.$$

$$P\left(\frac{168-170}{\sigma} \leq z \leq \frac{172-170}{\sigma}\right) = 0.516$$

$$P\left(-\frac{2}{\sigma} \leq z \leq \frac{2}{\sigma}\right) = 0.516$$

$$\phi\left(\frac{2}{\sigma}\right) - \phi\left(-\frac{2}{\sigma}\right) = 0.516$$

$$\phi\left(\frac{2}{\sigma}\right) - 1 + \phi\left(\frac{2}{\sigma}\right) = 0.516$$

$$2\phi\left(\frac{2}{\sigma}\right) = 1.516$$

$$\phi\left(\frac{2}{\sigma}\right) = \phi(0.7)$$

$$\sigma = \frac{2}{0.7} = 2.85$$

HW

Q1) If $X \sim N(0,1)$, find $E(X^2)$ and $V(X^2)$.

Q2) Suppose that temp is normally distributed with mean 50 and variance 4. what is probality that temp lies b/w 48°c to 53°c

Q3) $X \sim N(\mu, \sigma^2)$. Find c as a function of μ and σ. Such that $P(X<C) = 2P(X>C)$.

Q4) Suppose that life lengths of 2 electronic devices. say d_1 and d_2 have distributions $N(40,36)$ & $N(45,9)$ respectively. If the device is to be used for atleast 45 hrs. which device use to be prefer

Hey!!

Exponential Distribution:

A CRV 'X' is said to have exponential distribution with parameter $\alpha > 0$ if its pdf is given by $\alpha e^{-\alpha x}$.

$$f(x) = \begin{cases} \alpha e^{-\alpha x}; & x > 0, \alpha > 0. \\ 0; & \text{elsewhere} \end{cases}$$

Mean and variance:

Mean $E(X) = \frac{1}{\alpha}$ variance $V(X) = \frac{1}{\alpha^2}$

Gamma distribution:

$$f(x) = \begin{cases} \frac{\alpha}{\Gamma(\gamma)} e^{-\alpha x} (\alpha x)^{\gamma - 1} & x > 0, \gamma > 0, \alpha > 0. \\ 0, & \text{elsewhere} \end{cases}$$

Mean and variance:

$E(X) = \frac{\gamma}{\alpha}$ $V(X) = \frac{\gamma}{\alpha^2}$

Chi-square distribution:

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{\frac{n}{2}-1}, \quad x > 0, n > 0$$

$X \sim \chi^2(n)$.

Mean and variance:

$E(X) = n$, $V(X) = 2n$.

Moment generating function

Let 'x' be a DRV with pmf $p(x_i) = P(X = x_i)$ the function $M_X(t)$ called as Moment generation function of X, is defined as $M_X(t) = \sum_{i=0}^{\infty} e^{tx_i} p(x_i). = E(e^{tX})$.

If X is CRV then Mgf of X is,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= E(e^{tX}).$$

Q1) suppose x, has Binomial $X \sim B(n, p)$, $P(x) = n_{C_x} p^x q^{n-x}$

$$M_x(t) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} n_{C_x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} n_{C_x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

HW
Q2) find mgf of poission distribution. $X \sim P(\alpha)$

$$M_x(t) = e^{\alpha(e^t - 1)}$$

Q3) Find the mgf of normal distribution. $X \sim N(\mu, \sigma^2)$.

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \cdot \frac{1}{\sqrt{2\pi}\sigma}$$

$\frac{x-\mu}{\sigma} = z$

$$M_x(t) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} \cdot e^{-z^2/2} \sigma \, dz$$

$$\cancel{M_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - t\sigma z - t\mu)} dz}$$

$$M_x(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z)} \cdot dz.$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{e^{t\mu + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2} dz$$

$z - t\sigma = u$
$dz = du$

$$= \frac{e^{t\mu + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du.$$

$$= e^{t\mu + \frac{\sigma^2 t^2}{2}} \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$\boxed{M_x(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}}$$

① Consider a Mclauren series expansion of, $e^{tx} = ?$

$$e^{tx} = 1 + tx + \frac{1}{2!}(tx)^2 + \frac{1}{3!}(tx)^3 + \frac{1}{4!}(tx)^4 + \dots + \frac{1}{n!}(tx)^n + \dots$$

$$M_x(t) = E(e^{tx}).$$

$$= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \dots + \frac{t^n}{n!}E(x^n) + \dots$$

$E(x^n)$ is coefficient of $\frac{t^n}{n!}$ in the expansion of $M_x(t)$

$E(x^n) \to n^{th}$ moment of RV. X

$$M_x'(t) = E(x) + tE(x^2) + \frac{3t^2}{3!}E(x^3) + \dots$$

$$M_x''(t) = E(x^2) + \frac{6t}{3!}E(x^3) + \dots$$

$$M_x'(0) = E(x)$$

$$M_x''(0) = E(x^2)$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= M_x''(0) - [M_x'(0)]^2.$$

② Theorem.:

suppose RV 'x' has mgf $M_x(t)$ then mgf of RV 'y' $= \alpha x + \beta$ is $M_y(t) = E(e^{ty})$.

$$= E(e^{t(\alpha x + \beta)}).$$

$$= E(e^{\alpha tx} \cdot e^{\beta t}).$$

$$= e^{\beta t} E(e^{\alpha t x})$$

$$\boxed{M_y(t) = e^{\beta t} M_x(\alpha t).}$$

Ex: If mgf of 'x' is $M_x(t) = (0.4e^t + 0.6)^8$. find the mgf of $y = 3x + 2$ and hence find $E(Y)$.

$$M_y(t) = e^{2t} \cdot M_x(3t)$$

$$= e^{2t} \cdot (0.4e^{3t} + 0.6)^8.$$

$$E(Y) = M_y'(0) \Rightarrow \left\{ 2e^{2t}(0.4e^{3t} + 0.6)^8 + 8e^{2t}(0.4e^{3t} + 0.6)^7 \times 1.2e^{3t} \right\}_{t=0}$$

$$= 2 + 9.6 = 11.6.$$

③ Let X, Y are independent RV

If $M_X(t) = M_Y(t)$

then X & Y has same prob distribution.

④ suppose X & Y are independent RV $Z = X+Y$. Mgf of

Let $M_X(t)$, $M_Y(t)$, $M_Z(t)$ be mgf of X, Y & Z, then,

$$M_Z(t) = M_X(t) \cdot M_Y(t)$$

$$M_Z(t) = E(e^{t(X+Y)})$$

$$= E(e^{tX} e^{tY})$$

$$= E(e^{tX}) \cdot E(e^{tY})$$

$$= M_X(t) \cdot M_Y(t)$$

Reproductive property:

If 2 or more independent RV having a certain distribution are added the resulting RV has a distribution of the same type. As that of RV. is called reproductive property.

Ex suppose X & Y are independent,

$N(\mu_1, \sigma_1^2)$ $N(\mu_2, \sigma_2^2)$

$$Z = X + Y$$

$$M_Z(t) = M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$$

$\therefore \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

Q) If x is RV taking values 0, 1, 2, 3 ... and $P(x) = ab^x$. where a, b are +ve and $a+b=1$. Find mgf of x. If $E(x) = m_1$ & $E(x^2) = m_2$ show that $m_2 = m_1(2m_1+1)$.

Ans)
$$M_x(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} P(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} . ab^x$$
$$= a \sum_{x=0}^{\infty} (be^t)^x$$
$$= \frac{a}{1-be^t}$$
$$M_x(t) = \frac{1-b}{1-be^t}$$

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$$M_x'(t) = \frac{b(1-b)e^t}{(1-be^t)^2}$$
$$M_x''(t) = \frac{(1-be^t)^2 b(1-b)e^t - b(1-b)e^t 2(1-be^t)(-be^t)}{(1-be^t)^4}$$
$$M_x'(0) = \frac{b}{1-b} = E(X)$$
$$m_1 = \frac{b}{1-b}.$$
$$M_x''(0) = \frac{b(1-b)^3 + 2b^2(1-b)^2}{(1-b)^4}$$
$$= \frac{b(1-b) + 2b^2}{(1-b)^2}$$
$$E(x^2) = \frac{b(1+b)}{(1-b)^2}.$$
$$m_2 = \frac{b(1+b)}{(1-b)^2}.$$
$$m_1(2m_1+1) = \frac{b}{1-b}\left(\frac{2b}{1-b}+1\right) = m_2$$

(4)

Q) suppose mgf of 3 independent RV is X_1, X_2, X_3 are $e^{2t(1+t)}$, $e^{3t(1+t)}$, $e^{4t(1+t)}$ respectively, then find the pdf of $Z = 4X_1 + X_2 + 2X_3$ & obtain $E(Z/2)$, $V(Z/2) = ?$

Ans).

$$M_Z(t) = M_{4X_1}(t) \cdot M_{X_2}(t) \cdot M_{2X_3}(t)$$

$$= M_{X_1}(4t) \cdot M_{X_2}(t) \cdot M_{X_3}(2t)$$

$$= e^{8t(1+4t)} \cdot e^{3t(1+t)} \cdot e^{8t(1+2t)}$$

$$= e^{19t + 51t^2} \quad \text{compare} \leftrightarrow \left\{ e^{\mu t + \frac{\sigma^2 t^2}{2}} \right\}$$

$\therefore Z \sim (\mu, \sigma^2)$.

$Z \sim (19, 102)$

Pdf of Z, $f(z) = \frac{1}{\sqrt{2\pi}\sqrt{102}} e^{-1/2\left(\frac{z-19}{\sqrt{102}}\right)^2}$

$$E(Z/2) = \frac{19}{2} = \frac{1}{2}E(Z)$$

$$V(Z/2) = \frac{1}{4}V(Z) \Rightarrow \frac{\sigma^2}{4} = \frac{1}{4} \times 102.$$

Q) show that for Normal distribution with mean μ and variance σ^2 $[N(\mu, \sigma^2)]$.

$$E[(x-\mu)^{2n}] = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}.$$

Ans)

$E[(x-\mu)^{2n}]$ is coefficient of $\frac{t^{2n}}{2!}$ is the expansion of $M_{x-\mu}(t)$.

$$M_{x-\mu}(t) = e^{-\mu t} M_x(t)$$

$$= e^{-\mu t} \cdot \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right).$$

$\left(e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right)$

$$\Rightarrow e^{\sigma^2 t^2/2} = 1 + \frac{\sigma^2 t^2}{2} + \frac{1}{2!}\left(\frac{\sigma^2 t^2}{2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{\sigma^2 t^2}{2}\right)^n + \ldots$$

$$= \frac{1}{n!} \frac{\sigma^{2n} t^{2n}}{2^n}$$

multiply & divide by $2n!$

$$= \frac{t^{2n}}{2n!}\left[\frac{2n!\,\sigma^{2n}}{2^n\, n!}\right] \Rightarrow E((X-\mu)^{2n}) = \frac{2n!\,\sigma^{2n}}{2^n.\,n!}$$

$$= \frac{1.2.3\ldots 2n.\,\sigma^{2n}}{n!\,2^n}$$

$$= 1.3\ldots(2n-1)$$

$$= \frac{1.3.5\ldots(2n-1).\,2.4.6\ldots 2n\ \sigma^{2n}}{n!\ 2^n}$$

$$= \frac{1.3.5\ldots(2n-1)\,[(2\times1)(2\times2)(2\times3)\ldots(2\times n)]\ \sigma^{2n}}{(n!\,2^n)}$$

$$= 1.3.5\ldots(2n-1)\,\sigma^{2n}$$

1Q) hw

$$f(x) = \lambda e^{-\lambda(x-a)}, \quad x \geq a$$

(i) find mgf of X $\quad \frac{\lambda e^{at}}{\lambda - t}$ $\forall\ t<\lambda$ $\qquad E(x^2) = \frac{\lambda^2 a^2 + 2a\lambda}{\lambda^2}$

(ii) using mgf find $E(X)$ & $V(X)$

$\frac{a\lambda+1}{\lambda}$ $\qquad \frac{1}{\lambda^2}$.

Q2) $f(x) = \frac{1}{2}e^{-|x|}$, $\quad -\infty < x < \infty$

(i) find mgf of X $\qquad M_X(t) = \frac{1}{1-t^2}$

(ii) find $E(X)$ & $V(X)$. $\quad E(X)=0$

$E(X^2) = M_X''(0)$

$= 2$

$V(X) = 2$

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function of RV:

Let x be a rv suppose $y = H(x)$ is a real valued function of x they y is a RV.

Let R_x be the range space of x and R_y be the range space of Y for any event $c \in R_y$. we define probability of,

$$P(C) = P\{x \in R_x : H(x) \in C\}.$$

Ex: Let 'x' be a CRV with pdf $f(x) = e^{-x}, x > 0$ we define a RV $Y = H(X) = 2X+1$.

$$R_X = \{x, x > 0\}$$

$$R_Y = \{y, y > 1\}$$

Suppose c is defined as $C = \{y \geq 5\}$ $P(C) = ?$

$$P(C) = P(Y \geq 5).$$
$$= P(2X+1 \geq 5).$$
$$= P(X \geq 2).$$
$$P(X \geq 2) = \int_2^{\infty} e^{-x} dx.$$

→ If x is DRV then y is callo a DRV.

Ex 'X' assums values $-1, 0, 1$ with probability $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$ respectively then $Y = 3X+1$.

$$R_X = \{-1, 0, 1\}$$
$$R_Y = \{-2, 1, +4\}$$
$$P(Y=-2) = P(X=-1) = 1/3$$
$$P(Y=1) = P(X=0) = 1/2$$
$$P(Y=4) = P(X=1) = 1/6.$$

If the range space of 'x' is countabley infinite to evaluate the $P(Y = y_j)$ find the equivalent events in terms of x and then add all the corresponding probabilities.

CRV endsem

Suppose,

$P(X=n) = \frac{1}{2^n} \quad n = 1, 2, 3, \ldots$

Let $Y = \begin{cases} 1 & \text{if } x \text{ is even.} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$

common ratio $= \frac{1}{4}$.

$A = 1^{st}$ term.

$P(Y=1) = \frac{1}{4} + \frac{1}{16} + \cdots \qquad \left(\frac{A}{1-r}\right) \left(\frac{1}{1-1/4}\right) = \frac{1}{3}$

$P(Y=-1) = 1 - P(Y=1)$

$= 1 - 1/3$.

$\boxed{\text{sum (infinite series)} = \frac{A}{1-r}}$

If 'x' is CRV,

case (i): 'y' is DRV.

Ex: 'x', assumes all real values while Y assumes,

$Y = 1$ if $X \geq 0$.

$Y = -1$ if $X < 0$.

$P(Y=1) = P(X \geq 0)$. $\int_{-\infty}^{\infty}$ (pdf of this)

$P(X=-1) = P(X<0)$.

case (ii): 'y' is CRV (Both are countinuous).

The general procedure to obtain $p(Y)$ is

step(1): obtain $G(y)$ (is cdf of 'y').

$G(y) = P(Y \leq y)$; By finding an event in the range space of 'x' which is equivalent to the event $\{Y \leq y\}$.

step(2): Differentiate $G(y)$ wrt y inorder to obtain the pdf $[g(y)]$.

step(3): Determine those values of y in the range space of 'y' for which $g(y) > 0$.

Ex: $f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$ find pdf of $Y = 3X+1$.

Ans) $G(y) = P(Y \leq y)$.

$= P(3X+1 \leq y)$.

$= P\left(X \leq \frac{y-1}{3}\right)$.

$$= \int_0^{y-1/3} 2x\,dx$$
$$= \left[x^2\right]_0^{y-1/3}$$
$$= \frac{(y-1)^2}{9}.$$

Pdf of Y, $P(y) = G'(y)$

$$= \frac{2}{9}(y-1), \quad 1 < y < 4$$

→ Suppose $Y = H(X)$ is strictly Monotone (increasing) (decreasing) function of 'X' then Pdf of 'y' is non zero for those values of 'y' satisfying

$H(a) < y < H(b)$. , $x \in (a,b) \rightarrow$ [increasing $H(x)$]

$H(b) < y < H(a)$, $x \in (a,b) \rightarrow$ [$H(x)$ is decreasing]. g is non-zero for those values ↑

→ If $Y = H(X)$ is strictly Mono tone function then pdf is given by $\boxed{g(y) = f(x)\left|\frac{dx}{dy}\right|}$

Same ex: $Y = 3X+1 \Rightarrow X = \frac{Y-1}{3} \Rightarrow \frac{dx}{dy} = \frac{1}{3}$.

$g(y) = 2x \{\cancel{\frac{dx}{dy}} | 1/3)$

$g(y) = \frac{2x}{3}$.

$g(y) = \frac{2(y-1)}{3(3)}$

$g(y) = \frac{2y-2}{9}$. // $\quad 1 < y < 4$.

Q) $f(x) = \begin{cases} 2x, & 0 \le x \le 1. \\ 0, & \text{else}. \end{cases}$ $\quad Y = e^{-X}$. $\quad$ g(y) = Pdf = ?

~~$G(y) = P(Y \le y)$~~

~~$= (P(e^{-X} \le y)$~~

$g(y) = 2x \left| -\frac{1}{y} \right|$

$g(y) = \frac{2x}{y}$.

$g(y) = -\frac{2\log y}{y}$. $\quad \frac{1}{e} < y < 1$.

$y = e^{-x}$

$x = -\log y$

$\frac{dx}{dy} = \frac{-1}{y}$

(HW)

① If x is uniformly distributed $(1,3)$ find pdf of $y = 3x+4$ and $y = e^{x}$.

② If x is uniformly distributed $(0,1)$ find pdf of $y = x^2+1$ and $y = \frac{1}{x+1}$

③ If 'x' has pdf $f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{else} \end{cases}$ $\quad y = 8x^3$.

④ If 'x' has uniform distribution $(-\frac{\pi}{2}, \frac{\pi}{2})$ find pdf of $y = \tan x$.

⑤ If 'x' cauchy's distribution show that '$\frac{1}{x}$' also has same distribution

cauchy's ds = $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$

finite differ -12m

Solve, ordinary DE

ODE

Heat, wave, laplace / poission

sq's.

up Bayes weight class

1D, 2D, correlation

Distribution - 13m.