Let $s = j\omega$, $\therefore\ G(j\omega) = \dfrac{50}{j\omega\,(1 + j\omega)\,(1 + j0.2\omega)}$

MAGNITUDE PLOT

The corner frequency are, $\omega_{c1} = 1$ rad/sec and $\omega_{c2} = 1/0.2 = 5$ rad/sec.

The various terms of $G(j\omega)$ are listed in table-6. Also the table shows the slope contributed by each term and the change in slope at the corner frequencies.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{50}{j\omega}$	–	–20	–
$\dfrac{1}{1+j\omega}$	$\omega_{c1} = 1$	–20	$-20 + (-20) = -40$
$\dfrac{1}{1+j0.2\omega}$	$\omega_{c2} = \dfrac{1}{0.2} = 5$	–20	$-40 - 20 = -60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 10$ rad/sec

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l,\quad A = 20\log\left|\frac{50}{j\omega}\right| = 20\log\frac{50}{0.5} = 40 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\quad A = 20\log\left|\frac{50}{j\omega}\right| = 20\log\frac{50}{1} = 34 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log\frac{5}{1} + 34 = 6 \text{ db}$$

$$\text{At } \omega = \omega_h,\quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log\frac{10}{5} + 6 = -12 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b, c and d. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 6.6.2.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}0.2\omega$$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1.0	5	10
ϕ deg	−96°	−122	−146	−214	−238

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.6.2.

Step-3 : Determine the phase margin

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency.

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -224°$

Now, $\gamma = 180° + \phi_{gc} = 180° - 224° = -44°$

The phase margin of the system is negative and so the system is unstable. Hence lead compensation is required to make the system stable and to have a phase margin of 20°.

Step-4 : Find ϕ_m

The desired phase margin, $\gamma_d \geq 20°$

Let additional phase lead required, $\epsilon = 5°$

Maximum lead angle, $\phi_m = \gamma_d - \gamma + \epsilon = 20° - (-44°) + 5° = 69°$

Since the lead angle required is greater than 60° we have to realise the lead compensator as cascade of two lead compensators with each compensator providing half of the required phase lead angle.

$$\therefore \phi_m = \frac{69°}{2} = 34.5°$$

Step-5 : Determine the transfer function of lead compensator

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin 34.5°}{1 + \sin 34.5°} = 0.28$$

$$\left.\begin{array}{l}\text{The db magnitude}\\ \text{corresponding to}\end{array}\right\} \omega_m = -20\log\frac{1}{\sqrt{a}} = -20\log\frac{1}{\sqrt{0.28}} = -5.5 \text{ db.}$$

From the bode plot of uncompensated system the frequency, ω_m corresponding to a db gain of −5.5 db is found to be 7.8 rad/sec.

$$\therefore \omega_m = 7.8 \text{ rad/sec.}$$

$$\text{Now, } T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{7.8\sqrt{0.28}} = 0.24$$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{the lead compensator}\end{array}\right\} G_c(s) = \frac{\left(s+\frac{1}{T}\right)^2}{\left(s+\frac{1}{\alpha T}\right)^2} = \alpha^2 \frac{(1+sT)^2}{(1+s\alpha T)^2}$$

$$= (0.28)^2 \frac{(1+0.24s)^2}{(1+0.28\times 0.24s)^2} = 0.0784 \frac{(1+0.24s)^2}{(1+0.067s)^2}$$

Step-6 : Open loop transfer function of compensated system.

The block diagram of the compensated system is shown in fig 6.6.1.

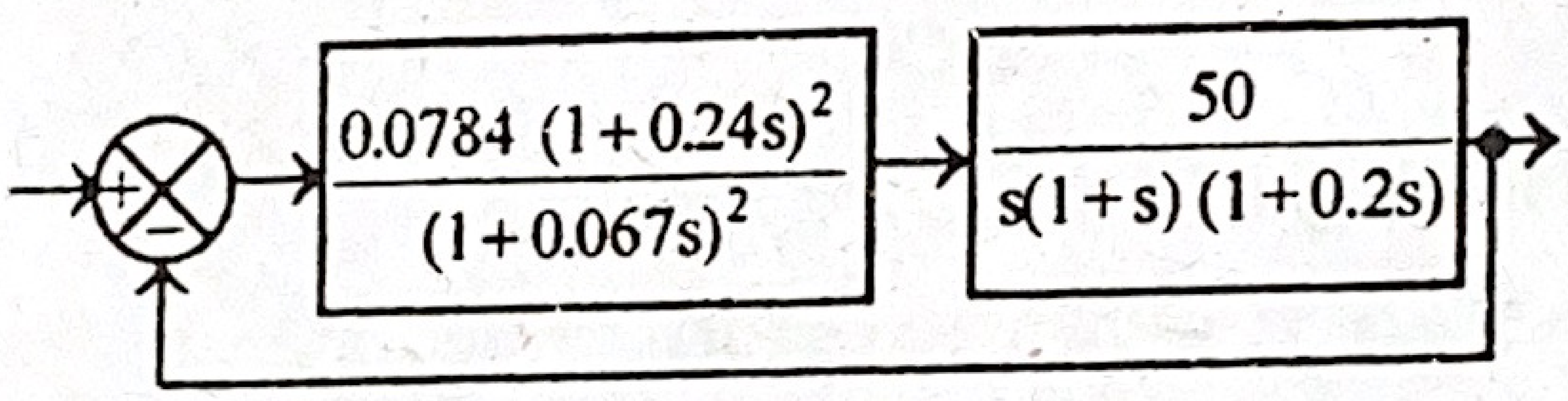


Fig 6.6.1 : Block diagram of lead compensated system.

The attenuation provided by the compensator can be retained to reduce the large value of open loop gain, so that the unstable system can be easily brought to stable region.

Let $G_0(s)$ be open loop transfer function of compensated system.

$$G_0(s) = \frac{0.0784\,(1+0.24s)^2}{(1+0.067s)^2} \times \frac{50}{s\,(1+s)\,(1+0.2s)} = \frac{4\,(1+0.24s)^2}{s\,(1+s)\,(1+0.2s)\,(1+0.067s)^2}$$

Step-7 : Draw the bode plot of compensated system to verify the design.

$$\text{Put, } s = j\omega \text{ in } G_0(s), \;\therefore G_0(j\omega) = \frac{4\,(1+j0.24\omega)^2}{j\omega\,(1+j\omega)\,(1+j0.2\omega)\,(1+j0.067\omega)^2}$$

MAGNITUDE PLOT

The corner frequencies are ω_{c1}, ω_{c2}, ω_{c3} and ω_{c4}.

$$\omega_{c1} = 1\,; \quad \omega_{c2} = \frac{1}{0.24} = 4.2\,; \quad \omega_{c3} = \frac{1}{0.2} = 5\,; \quad \omega_{c4} = \frac{1}{0.06}$$

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c4}$.

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 30$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A_0 = 20\log\left|\frac{4}{j\omega}\right| = 20\log\frac{4}{0.5} = 18 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A_0 = 20\log\left|\frac{4}{j\omega}\right| = 20\log\frac{4}{1} = 12 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A_0 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_0 \text{ at } (\omega = \omega_{c1})$$

$$= -40 \times \log\frac{4.2}{1} + 12 = -13 \text{ db}$$

$$\text{At } \omega = \omega_{c3}, \quad A_0 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_0 \text{ at } (\omega = \omega_{c2})$$

$$= 0 \times \log\frac{5}{4.2} + (-13) = -13 \text{ db}$$

$$\text{At } \omega = \omega_{c4}, \quad A_0 = \left[\text{slope from } \omega_{c3} \text{ to } \omega_{c4} \times \log\frac{\omega_{c4}}{\omega_{c3}}\right] + A_0 \text{ at } (\omega = \omega_{c3})$$

$$= -20 \times \log\frac{15}{5} + (-13) = -22.5 \text{ db} \approx -23 \text{ db}$$

$$\text{At } \omega = \omega_{h}, \quad A_0 = \left[\text{slope from } \omega_{c4} \text{ to } \omega_{h} \times \log\frac{\omega_{h}}{\omega_{c4}}\right] + A_0 \text{ at } (\omega = \omega_{c4})$$

$$= -60 \times \log\frac{30}{15} + (-23) = -41 \text{ db}$$

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$4/j\omega$	–	–20	–
$\frac{1}{1+j\omega}$	$\omega_{c1} = 1$	–20	$-20 - 20 = -40$
$(1+j0.24\omega)^2$	$\omega_{c2} = \frac{1}{0.24} = 4.2$	+40	$-40 + 40 = 0$
$\frac{1}{1+j0.2\omega}$	$\omega_{c3} = \frac{1}{0.2} = 5$	–20	$0 - 20 = -20$
$\frac{1}{(1+j0.067\omega)^2}$	$\omega_{c4} = \frac{1}{0.067} = 15$	–40	$-20 - 40 = -60$

Let the points e, f, g, h, i and j be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3}, ω_{c4} and ω_h respectively on the magnitude plot of compensated system. The magnitude plot of compensated system is drawn on the same semilog graph sheet by using the same scales as shown in fig 6.6.2.

PHASE PLOT

The phase angle of $G_0(j\omega)$ as a function of ω is given by,

$$\phi_0 = \angle G_0(j\omega) = 2\tan^{-1}0.24\omega - 90° - \tan^{-1}\omega - \tan^{-1}0.2\omega - 2\tan^{-1}0.067\omega.$$

The phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table - 4.

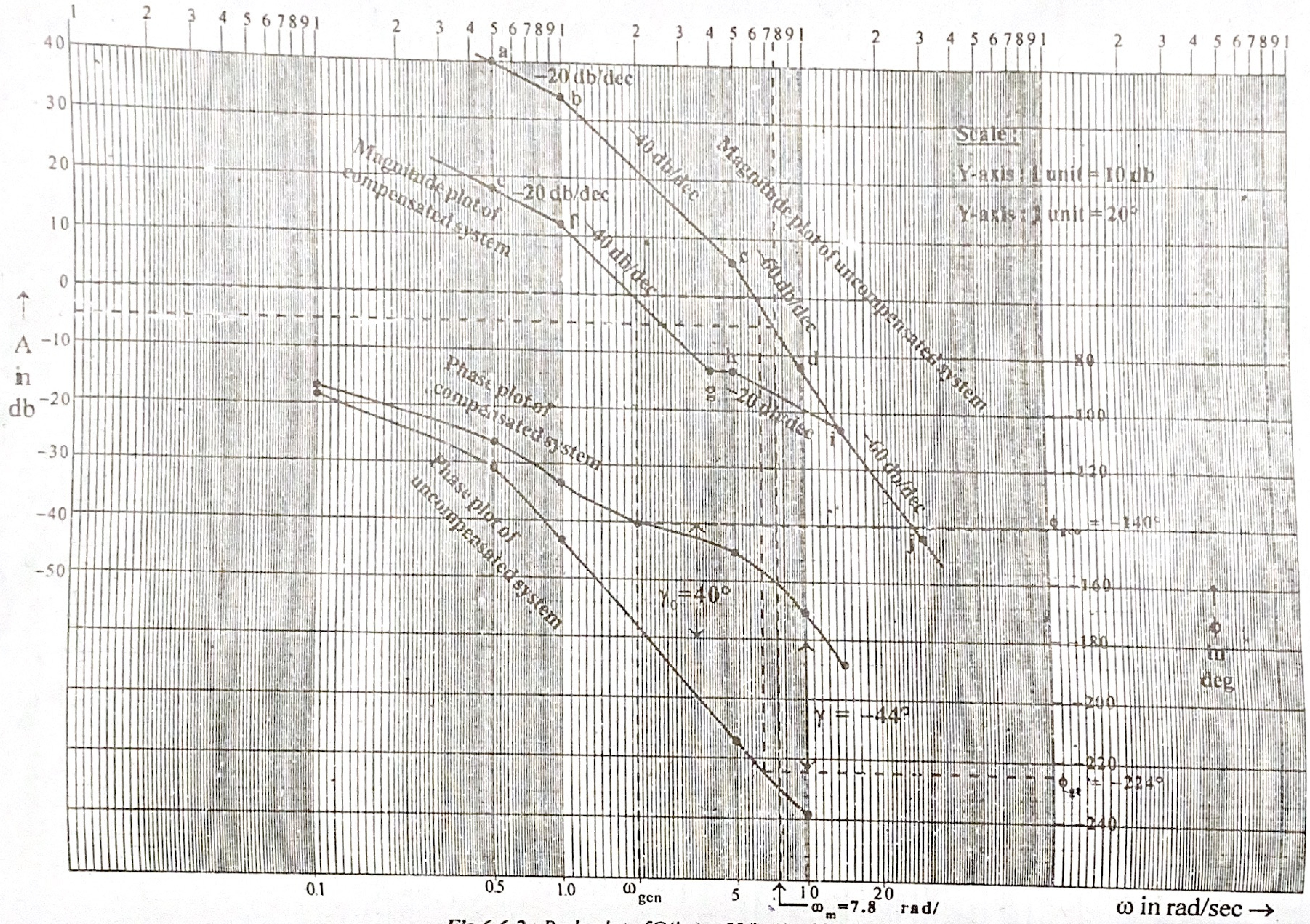


Fig 6.6.2 : Bode plot of $G(j\omega) = 50/j\omega(1 + j0.5\omega)(1 + j0.2\omega)$.

TABLE-4

ω rad/sec	0.1	0.5	6.0	2.0	5	10	15
$\angle G_0(j\omega)$ deg	−94	−112	−127°	−139 ≈ −140	−150	−171°	−189 ≈ −188

In the same semilog sheet and by using the same scales, the phase plot of compensated system is sketched as shown in fig 6.6.2.

Let, ϕ_{gc0} = Phase of $G_0(j\omega)$ at new gain crossover frequency (ω_{gcn}).

and γ_0 = Phase margin of compensated system.

From the bode plot of compensated system we get, $\phi_{gc0} = -140°$

Now, $\gamma_0 = 180° + \phi_{gc0} = 180° - 140° = 40°$

CONCLUSION

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.

RESULT

$$\left.\begin{array}{l}\text{The transfer function}\\ \text{of lead compensator}\end{array}\right\} G_c(s) = \frac{0.0784\,(1+0.24s)^2}{(1+0.067s)^2} = \frac{(s+4.17)^2}{(s+14.92)^2}$$

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{4\,(1+0.24s)^2}{s\,(1+s)\,(1+0.2s)\,(1+0.067s)^2}$$

EXAMPLE 6.7

Consider a unity feedback system with open loop transfer function, $G(s) = K/s(s+8)$. Design a lead compensator to meet the following specifications. (i) Percentage peak overshoot = 9.5%. (ii) Natural frequency of oscillation, ω_n = 12 rad/sec. (iii) Velocity error constant, $K_v \geq 10$.

SOLUTION

Step-1 : Determine the dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Given that, ω_n = 12 rad/sec and $\%M_p$ = 9.5%.

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$ $\therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.095$.

On taking natural log we get, $-\zeta\pi/\sqrt{1-\zeta^2} = \ln(0.095)$

On squaring we get, $\dfrac{\zeta^2\pi^2}{1-\zeta^2} = (ln\ 0.095)^2 = 5.54$.

$\therefore \zeta^2\pi^2 = 5.54 - 5.54\zeta^2 \quad \Rightarrow \quad \zeta^2\pi^2 + 5.54\zeta^2 = 5.54 \quad \Rightarrow \quad \zeta^2(\pi^2 + 5.54) = 5.54$

$$\therefore \zeta = \sqrt{\frac{5.54}{\pi^2 + 5.54}} = 0.6$$

$$\therefore s_d = -(0.6 \times 12) \pm j12 \times \sqrt{1-0.6^2} = -7.2 \pm j9.6$$

Step-2 : Draw the pole-zero plot

The pole-zero plot of open loop transfer function is shown fig 6.7.1. Poles are represented by the symbol "x". The pole at point P is the dominant pole, s_d.

Step-3 : To find the angle to be contributed by lead network.

Let ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

$$\text{Now, } \phi = \begin{pmatrix} \text{sum of angles} \\ \text{contributed by poles} \\ \text{of uncompensated system} \end{pmatrix} - \begin{pmatrix} \text{sum of angles} \\ \text{contributed by zeros} \\ \text{of uncompensated system} \end{pmatrix} \pm n180°$$

From fig 6.7.1, we get,

$$\left.\begin{matrix} \text{Sum of angles contributed} \\ \text{by poles of uncompensated system} \end{matrix}\right\} = \theta_1 + \theta_2 = 127° + 85° = 212°$$

Since there is no finite zero in uncompensated system, there is no angle contribution by zeros.

$\therefore \phi = 212° \pm n180°$

Let $n = 1$, $\quad \therefore \phi = 212° - 180° = 32°$.

Step-4 : To find the pole and zero of the compensator

Draw a line AP parallel to x-axis as shown in fig 6.7.1. The bisector PC is drawn to bisect the angle APO. The angles CPD and BPC are constructed as shown in fig 6.7.1. Here $\angle CPD = \angle BPC = \phi/2 = 32°/2 \doteq 16°$.

From fig 6.7.1.

Pole of the compensator, $p_c = -16.25$

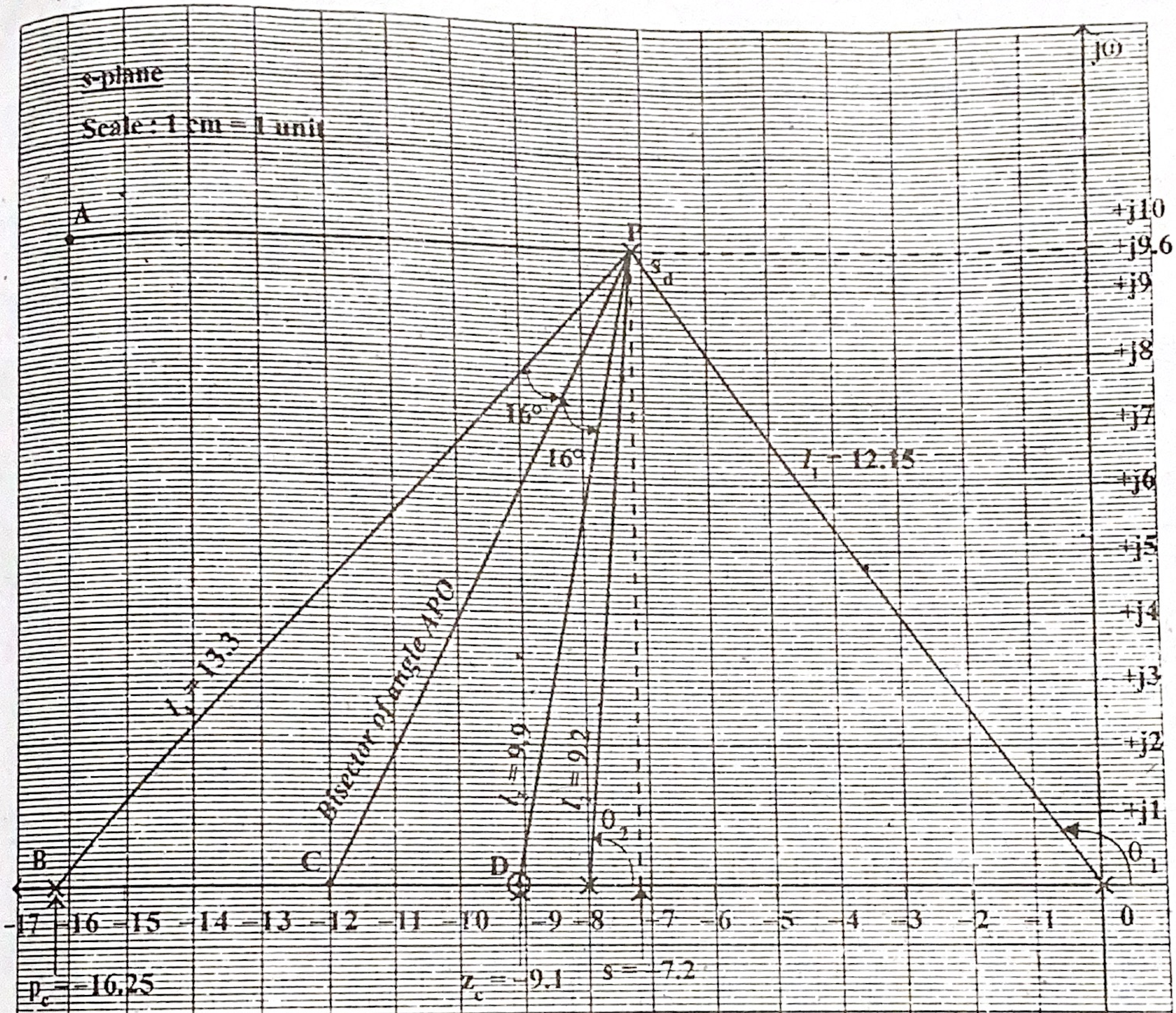
Zero of the compensator, $z_c = -9.1$

We know that, $z_c = -1/T \qquad \therefore T = 1/9.1 = 0.11$

We know that, $p_c = -1/\alpha T \qquad \therefore \alpha T = 1/16.25$ (or) $\alpha = 1/(T \times 16.25) = 0.56$.

Step-5 : Determine the transfer function of lead compensator

$$\left.\begin{matrix} \text{Transfer function of} \\ \text{lead compensator} \end{matrix}\right\} G_c(s) = \frac{(s + \frac{1}{T})}{(s + \frac{1}{\alpha T})} = \frac{(s + 9.1)}{(s + 16.25)}$$



$$\theta_1 = 180° - \tan^{-1}\frac{9.6}{7.2} = 127°$$

$$\theta_2 = \tan^{-1}\frac{9.6}{0.8} = 85°$$

$$K_1 = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{12.15 \times 9.2 \times 13.3}{9.9} = 150$$

Fig 6.7.1. : Pole zero plot of open loop transfer function.

Step-6 : Determine the open loop transfer function of lead compensated system.

The block diagram of lead compensated system is shown in fig 6.7.2.

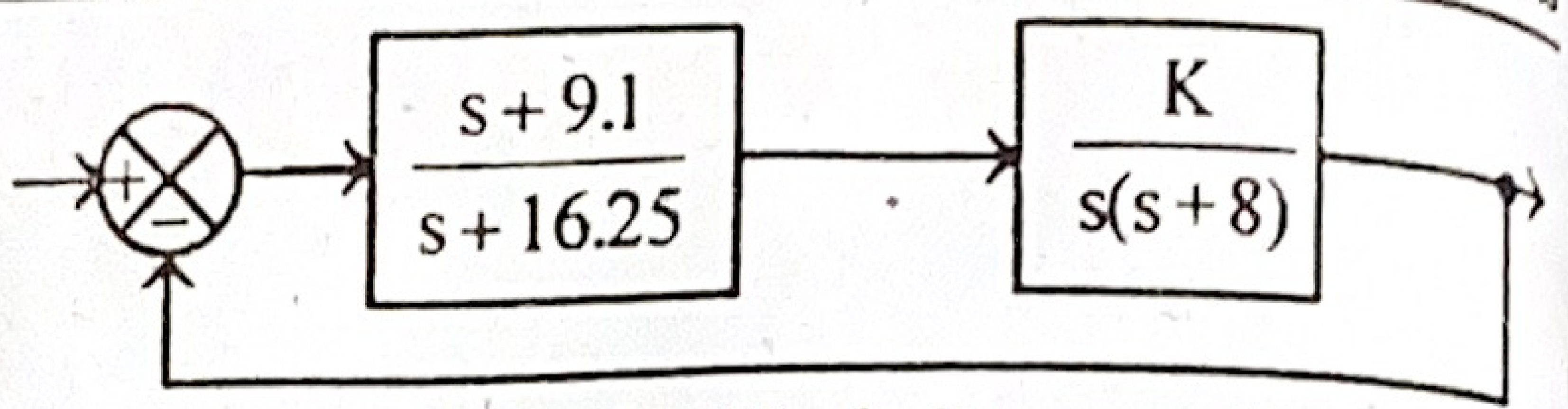


Fig 6.7.2 : Block diagram of lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{(s+9.1)}{(s+16.25)} \times \frac{K}{s(s+8)} = \frac{K(s+9.1)}{s(s+8)\,(s+16.25)}$$

Here the value of K is given by the value of gain at the dominant pole, s_d on the root locus. From magnitude condition K is given by,

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

From fig 6.7.1, we get,

$$K = \frac{l_1 \times l_2 \times l_4}{l_3} = \frac{12.15 \times 9.2 \times 13.3}{9.9} = 150$$

$$\therefore\ G_0(s) = \frac{150\,(s+9.1)}{s(s+8)\,(s+16.25)}$$

Step-7 : Check for error requirement

For the compensated system, the velocity error constant is given by,

$$K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\,G_0(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\frac{150\,(s+9.1)}{s\,(s+8)\,(s+16.25)} = \frac{150 \times 9.1}{8 \times 16.25} = 10.5$$

CONCLUSION

Since the velocity error constant of the compensated system, satisfies the requirement, the design is acceptable.

RESULT

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_c(s) = \frac{(s+9.1)}{(s+16.25)} = 0.56\,\frac{(1+0.11s)}{(1+0.06s)}$$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensated system}\end{array}\right\} G_0(s) = \frac{150\,(s+9.1)}{s\,(s+8)\,(s+16.25)} = \frac{10.5\,(1+0.11s)}{s\,(1+0.125s)\,(1+0.06s)}$$

EXAMPLE 6.8

Design a lead compensator for a unity feedback system with open loop transfer function $G(s) = K/s\,(s+4)\,(s+7)$ to meet the following specifications. (i) % Peak overshoot = 12.63%. (ii) Natural frequency of oscillation, $\omega_n = 8$ rad/sec. (iii) Velocity error constant, $K_v \geq 2.5$.

SOLUTION

Step-1: Determine the dominant pole, s_d.

Dominant pole, $s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Given that, ω_n = 8 rad/sec and $\%M_p$ = 12.63%.

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$; $\therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.1263$

On taking natural log we get, $-\zeta\pi / \sqrt{1-\zeta^2} = ln\,(0.1263)$

On squaring we get, $\frac{\zeta^2\,\pi^2}{1-\zeta^2} = (ln\,0.1263)^2 = 4.28.$

$\therefore \zeta^2\pi^2 = 4.28 - 4.28\zeta^2 \quad \Rightarrow \quad \zeta^2\pi^2 + 4.28\zeta^2 = 4.28 \quad \Rightarrow \quad \zeta^2(\pi^2 + 4.28) = 4.28$

$\therefore \zeta = \sqrt{\frac{4.28}{\pi^2 + 4.28}} = 0.55$

$\therefore s_d = -(0.55 \times 8) \pm j8 \times \sqrt{1 - 0.55^2} = -4.4 \pm j6.68 = -4.4 \pm j6.7$

Step-2 : Draw the pole-zero plot

The pole-zero plot of open loop transfer function is shown fig 6.8.1. Poles are represented by the symbol "x". The pole at point P is the dominant pole, s_d.

Step-3 : To find the angle to be contributed by lead network.

Let ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

Now, $\phi = \begin{pmatrix}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{pmatrix} - \begin{pmatrix}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{pmatrix} \pm n180$

From fig 6.8.1, we get,

$\left.\begin{array}{l}\text{Sum of angles contributed}\\ \text{by poles of uncompensated system}\end{array}\right\} = \theta_1 + \theta_2 + \theta_3 = 123° + 93° + 69° = 285°$

Since there is no finite zero in uncompensated system, there is no angle contribution by zeros.

$\therefore \phi = 285° \pm n180°$

Let n = 1,

$\therefore \phi = 285° - 180° = 105°.$

Since the angle contribution is more than 60°, the lead compensator is realised as cascade of two compensators with each compensator, contributing half of the required angle.

$\therefore \phi = 105/2 = 52.5° \approx 52°$

Step-4 : To find the poles and zeros of the compensator

Draw a line AP parallel to x-axis is as shown in fig 6.8.6. The bisector PC is drawn to bisect the angle APO. The angles CPD and BPC are constructed as shown in fig 6.8.6. Here $\angle CPD = \angle BPC = \phi/2 = 52°/2 = 26°$.

From fig 6.8.1.

Pole of the compensator, $p_c = -13.55$

Zero of the compensator, $z_c = -4.65$

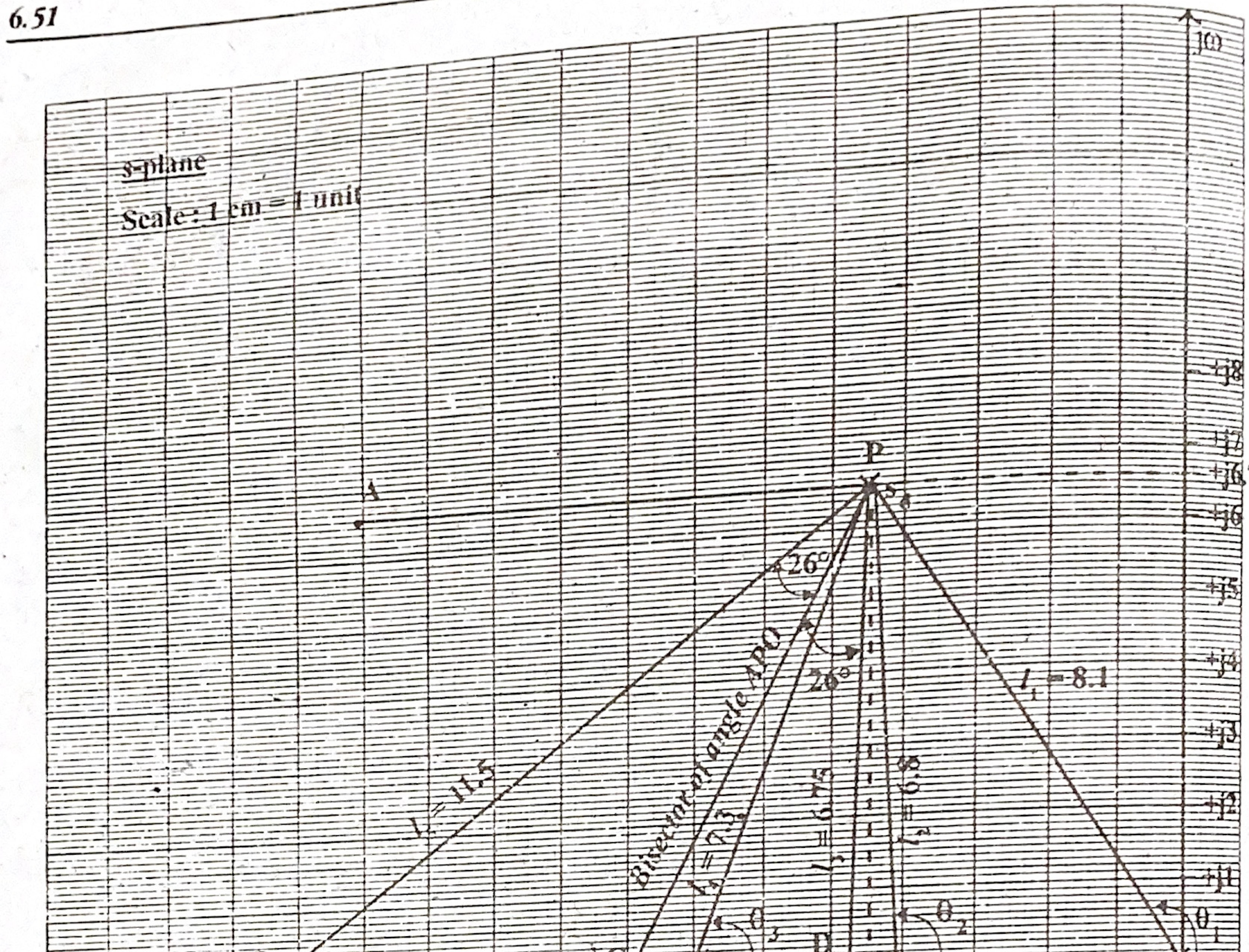


Fig 6.8.1 : Pole zero plot of lead compensated system.

$$\theta_1 = 180^\circ - \tan^{-1}\frac{6.7}{4.4} = 123^\circ$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{6.7}{0.4} = 93^\circ$$

$$\theta_3 = \tan^{-1}\frac{6.7}{2.6} = 69^\circ$$

$$K_1 = \frac{l_1 \times l_2 \times l_4 \times 2l_5}{2l_3} = \frac{8.1 \times 6.8 \times 7.3 \times 2 \times 11.5}{2 \times 6.75} = 685$$

We know that, $z_c = -1/T$; $\therefore T = 1/4.65 = 0.215$

We know that, $p_c = -1/\alpha T$; $\therefore \alpha T = 1/13.55$ (or) $\alpha = 1/T \times 13.55 = 0.343$.

Step-5 : Determine the transfer function of lead compensator

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_C(s) = \frac{\left(s + \frac{1}{T}\right)^2}{\left(s + \frac{1}{\alpha T}\right)^2} = \frac{(s + 4.65)^2}{(s + 13.55)^2}$$

Step-6 : Determine the open loop transfer function of lead compensated system.

Block diagram of lead compensated system is shown in fig.6.8.2.

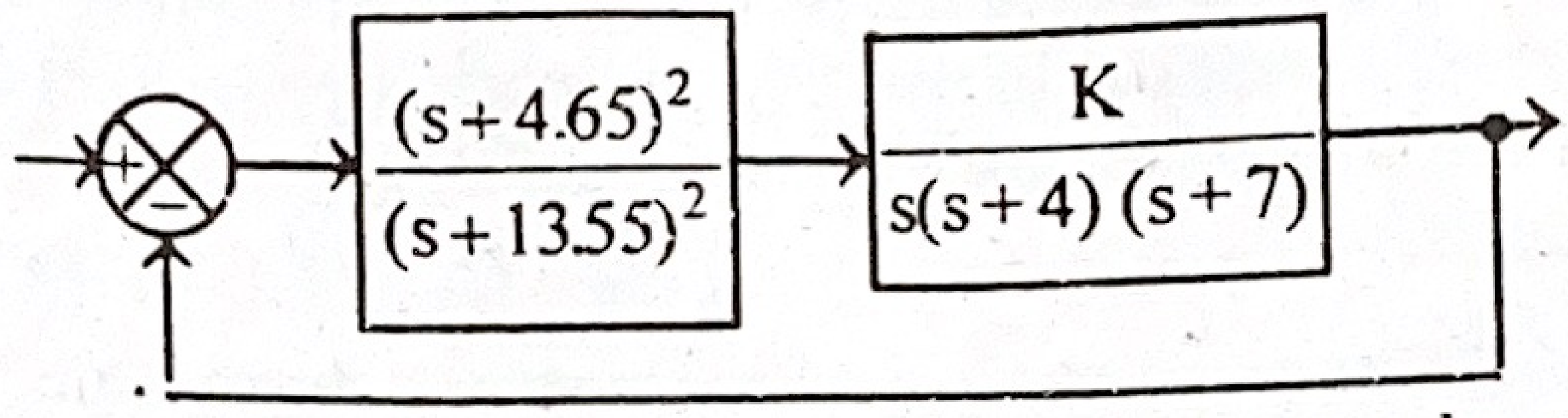


Fig 6.8.2 : Block diagram of lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{(s + 4.65)^2}{(s + 13.55)^2} \times \frac{K}{s(s + 4)(s + 7)}$$

Here the value of K is given by the value of gain at the dominant pole, s_d on the root locus. From magnitude condition K is given by,

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

From fig 6.8.1, we get,

$$K = \frac{l_1 \times l_2 \times l_4 \times l_5^2}{l_3^2} = \frac{8.1 \times 6.8 \times 7.3 \times 11.5^2}{6.75^2} = 1167$$

$$\therefore G_0(s) = \frac{1167\,(s + 4.65)^2}{s(s + 4)(s + 7)(s + 13.55)^2}$$

Step-7 : Check for error requirement

For the compensated system, the velocity error constant is given by,

$$K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\,G_0(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\,\frac{1167\,(s + 4.65)^2}{s\,(s + 4)(s + 7)(s + 13.55)^2} = \frac{1167 \times 4.65^2}{4 \times 7 \times c13.55^2} = 4.91$$

CONCLUSION

Since the velocity error constant of the compensated system, satisfies the requirement, the design is acceptable.

RESULT

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_C(s) = \frac{(s + 4.65)^2}{(s + 13.55)^2} = 0.1178\,\frac{(1 + 0.215s)^2}{(1 + 0.0738s)^2}$$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensated system}\end{array}\right\} G_0(s) = \frac{1167\,(s + 4.65)^2}{s\,(s + 4)(s + 7)(s + 13.55)^2}$$

$$= \frac{4.91\,(1 + 0.215s)^2}{s\,(1 + 0.25s)(1 + 0.143s)(1 + 0.0738s)^2}$$

6.4 LAG-LEAD COMPENSATOR

A compensator having the characteristics of lag-lead network is called lag-lead compensator. In a lag-lead network when sinusoidal signal is applied, both phase lag and phase lead occurs in the output, but in different frequency regions. Phase lag occurs in the low frequency region and phase lead occurs in the high frequency region (i.e) the phase angle varies from lag to lead as the frequency is increased from zero to infinity.

A lead compensator basically increases bandwidth and speeds up the response and decreases the maximum overshoot in the step response. Lag compensation increases the low frequency gain and thus improves the steady state accuracy of the system, but reduces the speed of responses due to reduced bandwidth.

If improvements in both transient and steady state response are desired, then both a lead compensator and lag compensator may be used simultaneously, rather than introducing both a lead and lag compensator as separate elements. However it is economical to use a single lag-lead compensator.

A lag-lead compensation combines the advantages of lag and lead compensations. Lag-lead compensator possess two poles and two zeros and so such a compensation increases the order of the system by two, unless cancellation of poles and zeros occurs in the compensated system.

S-PLANE REPRESENTATION OF LAG-LEAD COMPENSATOR

The s-plane representation of lag-lead compensator is shown in fig 6.15. The lag section has one real pole and one real zero with pole to the right of zero. The lead section also has one real pole and one real zero but the zero is to the right of the pole.

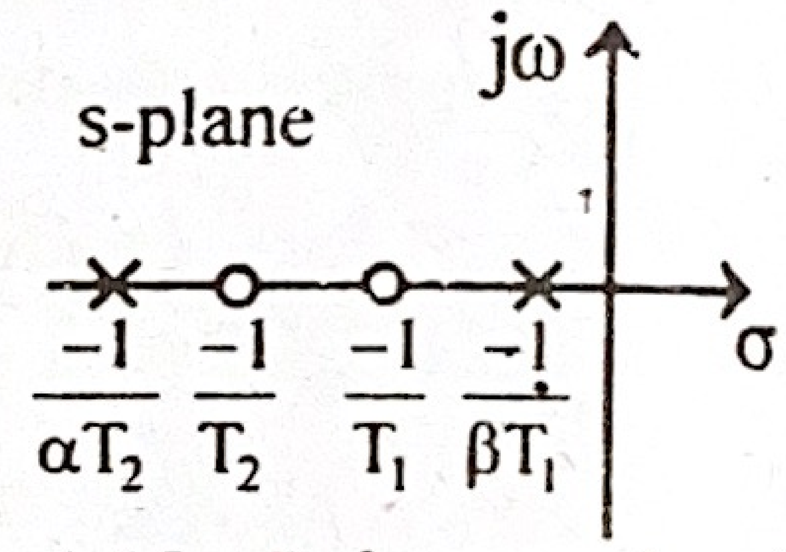


Fig 6.15 : Pole-zero plot of lag-lead compensator.

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lag - lead compensator}\end{array}\right\} G_c(s) = \underbrace{\frac{(s + 1/T_1)}{(s + 1/\beta T_1)}}_{\text{lag section}} \underbrace{\frac{(s + 1/T_2)}{(s + 1/\alpha T_2)}}_{\text{lead section}} \quad .....(6.36)$$

where $\beta > 1$ and $0 < \alpha < 1$

REALISATION OF LAG-LEAD COMPENSATOR USING ELECTRICAL NETWORK

The lag-lead compensator can be realised by the R-C network shown in fig 6.16.

Let $E_i(s)$ = Input voltage

$E_0(s)$ = Output voltage.

In the network shown in fig 6.16, the input voltage is applied to the series combination of $R_1 \,||\, C_1$, R_2 and C_2. The output voltage, is obtained across series combination of R_2 and C_2. By voltage division rule,

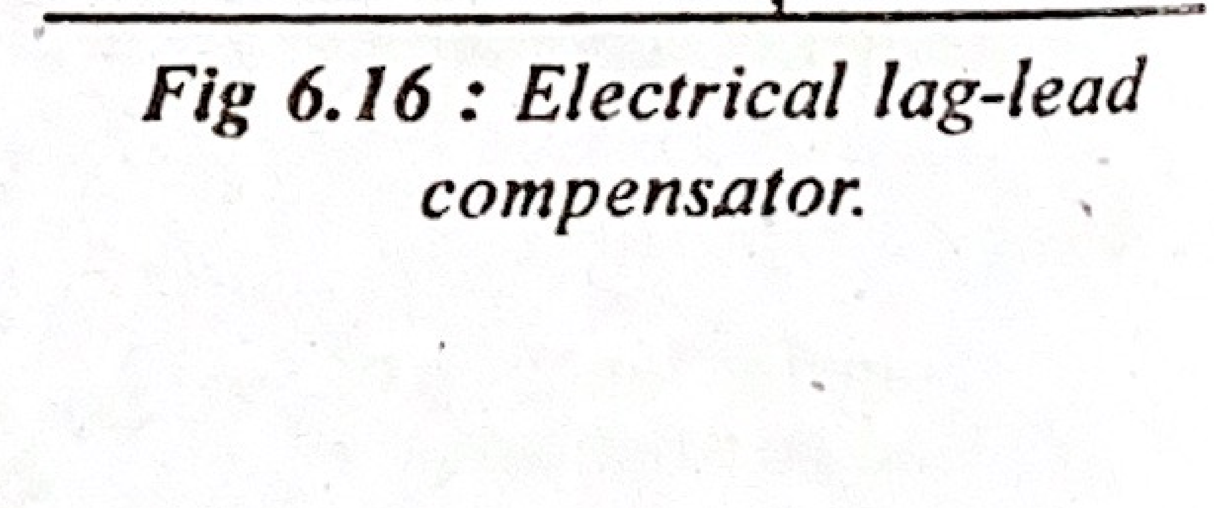


Fig 6.16 : Electrical lag-lead compensator.

$$E_0(s) = E_i(s)\,\frac{R_2 + \frac{1}{sC_2}}{(R_1 \,||\, \frac{1}{sC_1}) + R_2 + \frac{1}{sC_2}}$$

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{\dfrac{sR_2C_2 + 1}{sC_2}}{\dfrac{R_1 \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} + \dfrac{sR_2C_2 + 1}{sC_2}} = \frac{\dfrac{sR_2C_2 + 1}{sC_2}}{\dfrac{R_1}{sR_1C_1 + 1} + \dfrac{sR_2C_2 + 1}{sC_2}}$$

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{\dfrac{sR_2C_2+1}{sC_2}}{\dfrac{sR_1C_2+(sR_1C_1+1)(sR_2C_2+1)}{(sR_1C_1+1)\,sC_2}} = \frac{(sR_1C_1+1)(sR_2C_2+1)}{sR_1C_2+(sR_1C_1+1)(sR_2C_2+1)}$$

$$= \frac{R_1C_1R_2C_2(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}{sR_1C_2+R_1C_1R_2C_2\,(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}$$

On dividing the numerator and denominator by $R_1C_1\ R_2C_2$ we get,

$$\frac{E_0(s)}{E_i(s)} = \frac{(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}{\frac{s}{R_2C_1}+(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})} = \frac{(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}{s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}} \quad .....(6.37)$$

The transfer function of lag-lead compensator is given by,

$$G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{(s+1/\beta T_1)(s+1/\alpha T_2)} = \frac{(s+1/T_1)(s+1/T_2)}{s^2+s\,((1/\beta T_1)+(1/\alpha T_2))+1/\alpha\beta T_1T_2} \quad .....(6.38)$$

On comparing equations (6.37) and (6.38) we get,

$$T_1 = R_1\,C_1 \quad .....(6.39)$$

$$T_2 = R_2\,C_2 \quad .....(6.40)$$

$$R_1R_2\,C_1C_2 = \alpha\beta\ T_1T_2 \quad .....(6.41)$$

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1} = \frac{1}{\beta T_1}+\frac{1}{\alpha T_2} \quad .....(6.42)$$

From equation (6.41) we get, $\alpha\beta = \dfrac{R_1R_2C_1C_2}{T_1T_2}$ .....(6.43)

From equations (6.39), (6.40) and (6.43), we can say that, $\alpha\beta = 1$ .....(6.44)

The equation (6.44) implies that a single lag-lead network does not allow an independent choice of α and β. (But separate lag and lead network will allow independent choice of α and β). Hence in the transfer function of electrical lag-lead compensator replace α by $1/\beta$ as shown below.

$$\therefore \frac{E_0(s)}{E_i(s)} = G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{(s+1/\beta T_1)(s+\beta/T_2)} \quad .....(6.45)$$

Where $\beta>1$, $T_1 = R_1C_1$, $T_2 = R_2\,C_2$ and $\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_2}+\dfrac{1}{R_2C_1} = \dfrac{1}{\beta T_1}+\dfrac{\beta}{T_2}$

FREQUENCY RESPONSE OF LAG-LEAD COMPENSATOR

Consider the transfer function of lag-lead compensator.

$$G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{(s+1/\beta T_1)(s+1/\alpha T_2)} = \alpha\beta\frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} \quad .....(6.46)$$

The sinusoidal transfer function of lag-lead compensator is obtained by letting $s = j\omega$ in equation (6.46).

$$\therefore\ G_c(j\omega) = \alpha\beta\frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)(1+j\omega\alpha T_2)} \quad .....(6.47)$$

For a single lag-lead compensator, $\alpha\beta = 1$. Hence from equation (6.47) we can say that the lag-lead compensator provides a dc gain of unity.

$$\therefore\ G_c(j\omega) = \frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)(1+j\omega\alpha T_2)} \quad .....(6.48)$$

The sinusoidal transfer function shown in equation (6.48) has four corner frequencies and they are ω_{c1}, ω_{c2}, ω_{c3} and ω_{c4}, where $\omega_{c1} < \omega_{c2} < \omega_{c3} < \omega_{c4}$.

Here $\omega_{c1} = \dfrac{1}{\beta T_1}$; $\omega_{c2} = \dfrac{1}{T_1}$; $\omega_{c3} = \dfrac{1}{T_2}$ and $\omega_{c4} = \dfrac{1}{\alpha T_2}$

By an analysis similar to that of lag compensator the bode plot of lag-lead compensator is sketched as shown in fig 6.17.

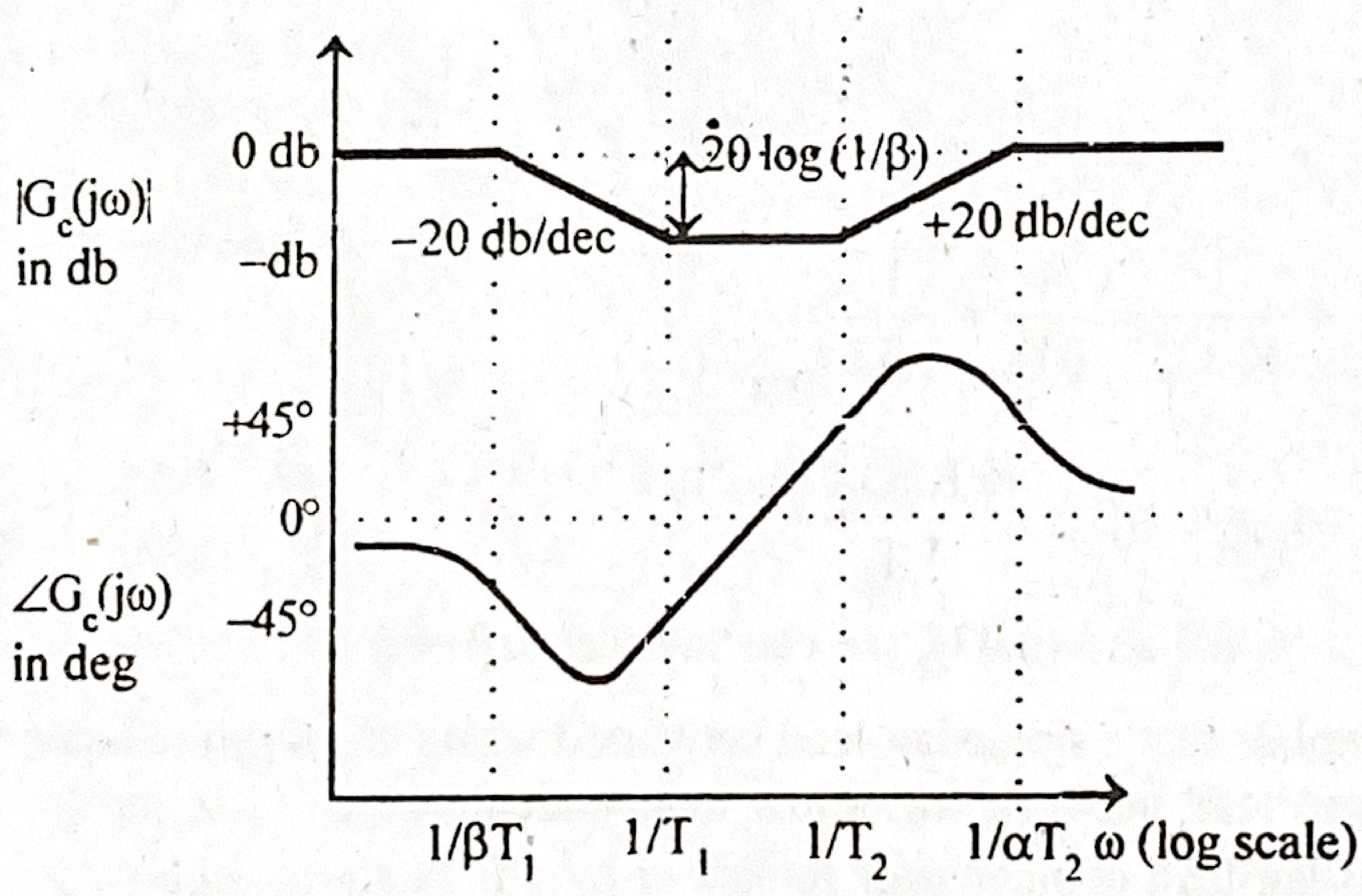


Fig 6.17 : Bode plot of lag-lead compensator.

PROCEDURE FOR DESIGN OF LAG-LEAD COMPENSATOR USING BODE PLOT

The lag-lead compensator is employed only when a large error constant and a large bandwidth are required. First design a lag section and then take $\alpha = 1/\beta$ and design a lead section. The step by step procedure for the design of lag-lead compensator is given below.

Step-1 : Determine the open loop gain K of the uncompensated system to satisfy the specified error requirement.

Step-2 : Draw the bode plot of uncompensated system.

Step-3 : From the bode plot determine the gain margin of the uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency.

γ = Phase margin of uncompensated system.

Now, $\gamma = 180° + \phi_{gc}$

If the gain margin is not satisfactory then compensation is required.

Step-4 : Choose a new phase margin

Let, γ_d = Desired phase margin

Now, new phase margin, $\gamma_n = \gamma_d + \epsilon$

Choose an initial value of $\epsilon = 5°$.

Step-5 : From the bode plot, determine the new gain crossover frequency, which is the frequency corresponding to a phase margin of γ_n.

Let, ω_{gcn} = New gain crossover frequency

ϕ_{gcn} = Phase of $G(j\omega)$ at ω_{gcn}

$\gamma_n = 180° + \phi_{gcn}$ (or) $\phi_{gcn} = \gamma_n - 180°$

In the phase plot of uncompensated system, the frequency corresponding to a phase of ϕ_{gcn} is the new gain crossover frequency ω_{gcn}.

Choose the gain crossover frequency of the lag compensator, ω_{gcl}, somewhat greater than ω_{gcn} (i.e., choose ω_{gcl} such that $\omega_{gcl} > \omega_{gcn}$).

Step-6 : Calculate β of lag compensator.

Let, $A_{gcl} = |G(j\omega)|$ in db at $\omega = \omega_{gcl}$

From the bode plot find A_{gcl}

Now, $A_{gcl} = 20 \log \beta$ (or) $\beta = 10^{(A_{gcl}/20)}$

Step-7 : Determine the transfer function of lag section

The zero of the lag compensator is placed at a frequency one-tenth of ω_{gcl}.

$\therefore$ Zero of lag compensator, $z_{cl} = 1/T_1 = \omega_{gcl}/10$

Now, $T_1 = 10/\omega_{gcl}$

Pole of lag compensator, $p_{cl} = 1/\beta T_1$

$$\left.\begin{array}{l}\text{Transfer function}\\ \text{of lag section}\end{array}\right\} G_1(s) = \frac{(s+1/T_1)}{(s+1/\beta T_1)} = \beta\frac{(1+sT_1)}{(1+s\beta T_1)}$$

Step-8 : Determine the transfer function of lead section

Take, $\alpha = 1/\beta$

From the bode plot find ω_m which is the frequency at which the db gain is $-20\log(1/\sqrt{\alpha})$

$$\text{Now } T_2 = \frac{1}{\omega_m\sqrt{\alpha}}$$

$$\left.\begin{array}{l}\text{Transfer function}\\ \text{of lead section}\end{array}\right\} G_2(s) = \frac{(s+1/T_2)}{(s+1/\alpha T_2)} = \alpha\frac{(1+sT_2)}{(1+s\alpha T_2)}$$

Step-9 : Determine the transfer function of lag-lead compensator.

Transfer function of lag - lead compensator, $G_c(s) = G_1(s) \times G_2(s) = \beta \dfrac{(1+sT_1)}{(1+s\beta T_1)} \times \alpha \dfrac{(1+sT_2)}{(1+s\alpha T_2)}$

Since $\alpha = \dfrac{1}{\beta}$,

$$G_c(s) = \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)}$$

Step-10 : Determine the open loop transfer function of compensated system.

The lag-lead compensator is connected in series with G(s) as shown in fig 6.18.

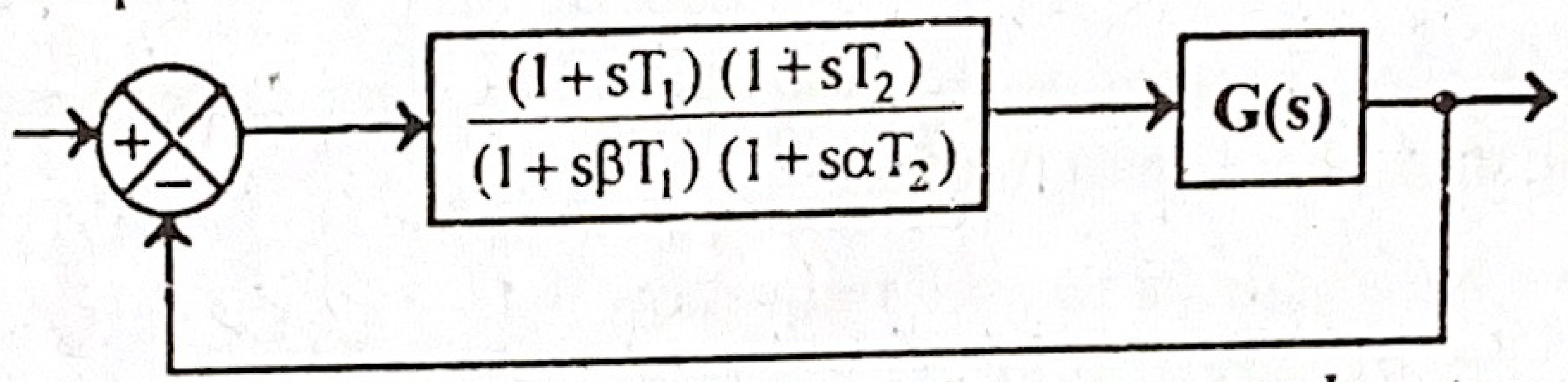


Fig 6.18 : Block diagram of lag-lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of compensated system}\end{array}\right\} G_0(s) = \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} \times G(s)$$

Step-11 : Draw the bode plot of compensated system and verify whether the specifications are satisfied or not. If the specifications are not satisfied then choose another choice of α such that, $\alpha < 1/\beta$ and repeat the steps 8 to 11.

PROCEDURE FOR DESIGN OF LAG-LEAD COMPENSATOR USING ROOT LOCUS

The lag-lead compensation is employed to improve both the transient and steady state responses of a system. First design a lead section to realize the required ζ and ω_n for the dominant closed loop poles. Then determine the error constant of lead compensated system. If it is satisfactory then only lead compensation will meet the requirement. If the error constant has to be increased then design a lag section. The step-by-step procedure for the design of lag-lead compensator is given below.

Step-1 : Determine the dominant pole, s_d

$$s_d = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

where, ζ = Damping ratio ; ω_n = Natural frequency of oscillation, rad/sec.

Step-2 : Mark the poles and zeros of open loop transfer function and the dominant pole on the s-plane. Let the dominant pole be point P.

Step-3 : Find the angle to be contributed by lead network to make the point P as a point on root locus.

Let, ϕ = Angle to be contributed by lead network to make point, P as a point on root locus.

Draw vectors from all open loop poles and zeros to point P. Measure the angle contributed by the vectors. [For the procedure to find angle contribution by vectors refer root locus in Chapter-5].

$$\text{Now, } \phi = \left(\begin{array}{c}\text{sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{array}\right) - \left(\begin{array}{c}\text{sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{array}\right) \pm n180^\circ$$

where n is an odd integer, so that n180° is nearest to the difference between angles contributed by poles and zeros.

Step-4 : Determine the pole and zero of the lead section.

Let point O be the origin of s-plane and point P be the dominant pole. Draw straight lines OP and AP such that AP is parallel to x-axis as shown in fig.6.19. Draw a line PC so as to bisect the angle APO [∠APO] where the point C is on the real axis. With line PC as reference, draw angles BPC and CPD such that each equal to $\phi/2$. Here the points B and D are located on the real axis.

Now the point B is the location of the pole of the compensator $(-1/\alpha T_2)$ and the point D is the location of the zero of the compensator $(-1/T_2)$. Compute T_2 and α from the values of point D and B.

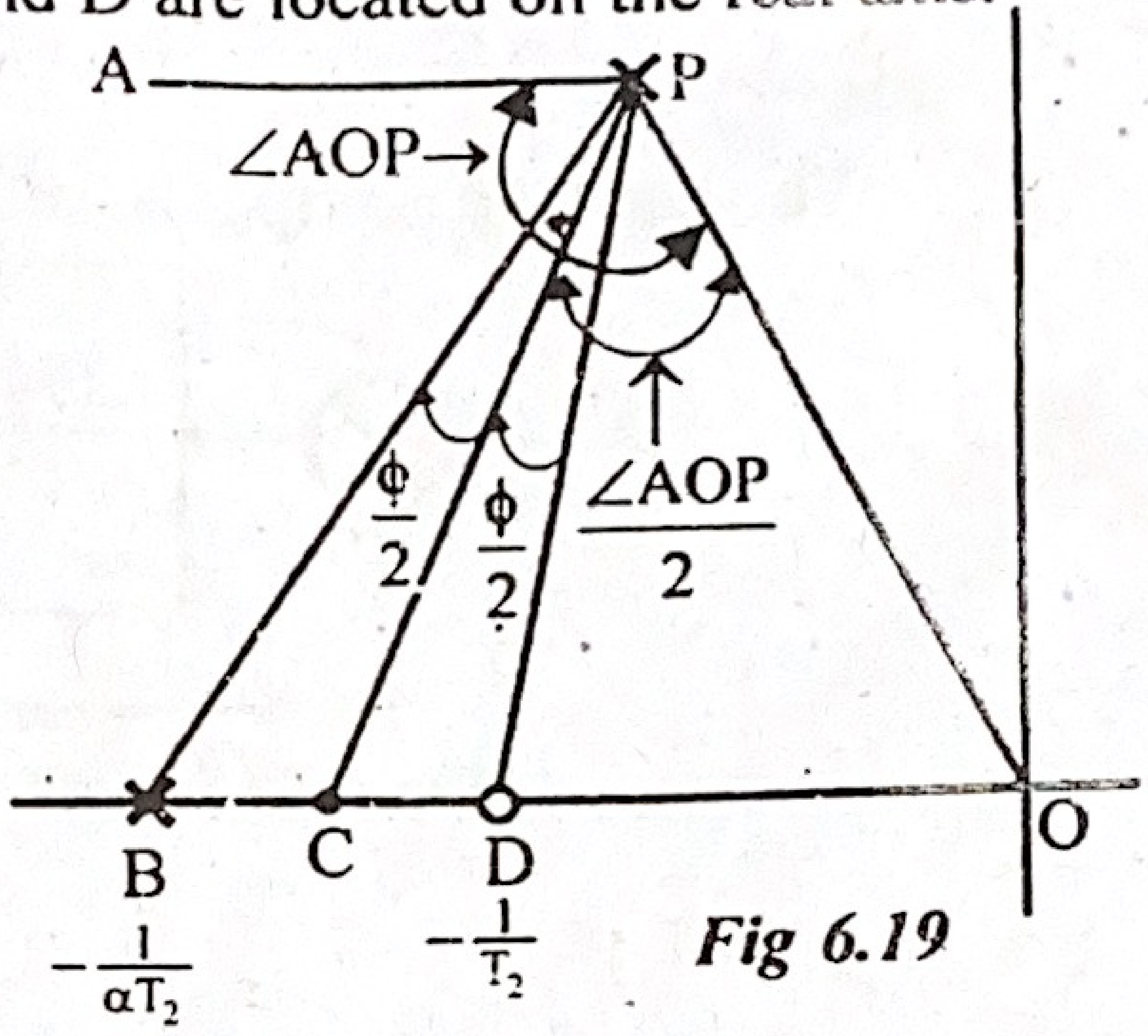


Fig 6.19

Step-5 : Determine the transfer function of lead section.

$$\left.\begin{array}{l}\text{Transfer function}\\ \text{of lead section}\end{array}\right\} G_2(s) = \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_2}\right)}$$

Step-6 : Determine the open loop gain, K.

The open loop gain K is the value of gain at $s = s_d$. The value of gain, K is determined from pole-zero plot of lead compensated system and by using the magnitude condition given below.

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

***Note :** The length of vectors should be measured to scale. For details of magnitude condition refer root locus in Chapter-5.*

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_{02}(s) = G_2(s) \times G(s)$$

Step-7 : Determine the velocity error constant of lead compensated system.

$$\left.\begin{array}{l}\text{Velocity error constant of}\\ \text{lead compensated system}\end{array}\right\} K_{v2} = \underset{s\to 0}{\text{Lt}}\ s\,G_{02}(s)$$

If K_{v2} satisfies the requirement then only lead compensation is sufficient but if K_{v2} is less than the desired value then provide lag compensation.

Step-8 : Determine the parameter, β of lag section.

Let, K_{vd} = Desired velocity error constant ; A = Factor by which K_v is increased.

Now, $A = K_{vd}/K_{v2}$. Select β, such that $\beta > A$. [i.e., $\beta = (1.1 \text{ to } 1.2) \times A$]

Step-9 : Determine the transfer function of lag section. Choose the zero of lag section as 10% of the second pole of uncompensated system.

$\therefore$ Zero of lag section, $z_{c1} = 0.1 \times$ second pole of G(s)

Also, $z_{c1} = \frac{-1}{T_1}$; $\therefore T_1 = \frac{-1}{z_{c1}}$; $\therefore$ Pole of lag section, $p_{c1} = \frac{-1}{\beta T_1}$

$$\text{Transfer function of lag section, } G_1(s) = \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{1}{\beta T_1}\right)}$$

Step-10 : Determine the transfer function of lag-lead compensator and compensated system

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lag - lead compensator}\end{array}\right\} G_c(s) = G_1(s) \times G_2(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)}$$

The lag-lead compensator is connected in series with G(s) as shown in fig 6.20.

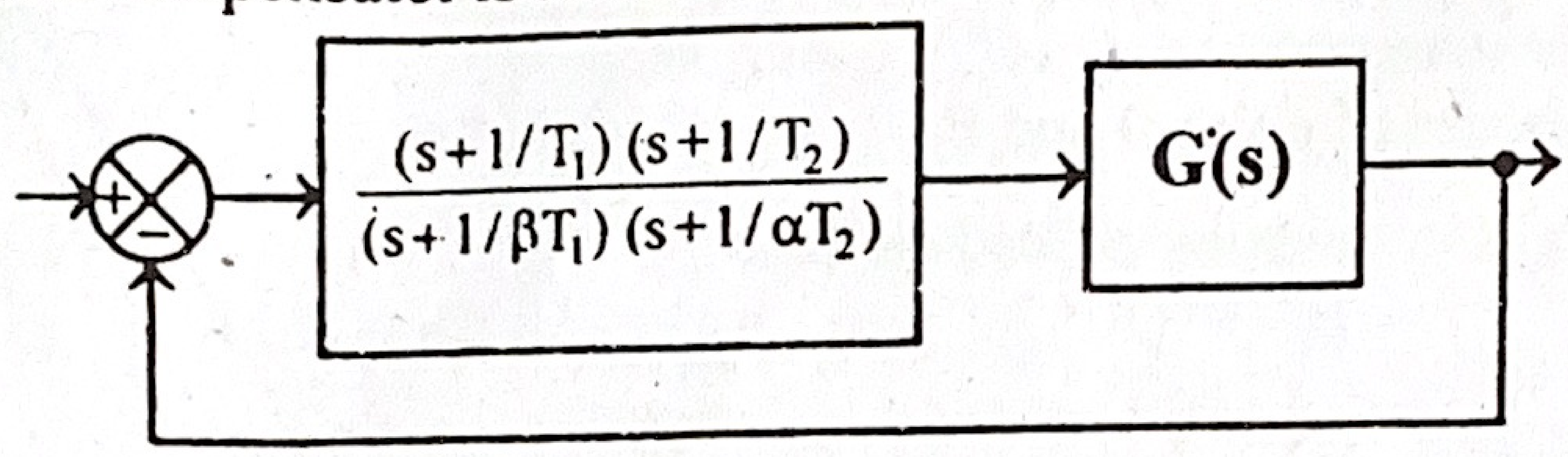


Fig 6.20 : Block diagram of lag-lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function of}\\ \text{lag - lead compensated system}\end{array}\right\} G_0(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)} \times G(s)$$

Step-11 : Check the velocity error constant of compensated system, if it is satisfactory then the design is accepted, otherwise repeat the design by modifying the locations of poles and zeros of the compensator.

EXAMPLE 6.9

Consider the unity feedback system whose open loop transfer function is $G(s) = K/s(s+3)(s+6)$. Design a lag-lead compensator to meet the following specifications. (i) Velocity error constant, $K_v = 80$. (ii) Phase margin, $\gamma \geq 35°$.

SOLUTION

Step-1 : Determine K

For unity feedback system,

Velocity error constant, $K_v = \underset{s\to 0}{\mathrm{Lt}}\ sG(s)$

Given that, $K_v = 80$.

$$\therefore \underset{s\to 0}{\mathrm{Lt}}\ s\,G(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\frac{K}{s(s+3)(s+6)} = 80 \quad \Rightarrow \quad \frac{K}{3\times 6} = 80 \quad \Rightarrow \quad K = 80\times 3\times 6 = 1440$$

$$\therefore G(s) = \frac{1440}{s(s+3)(s+6)} = \frac{1440}{s\times 3(1+s/3)\times 6(1+s/6)} = \frac{80}{s(1+0.33s)(1+0.167s)}$$

Step-2: Bode plot of uncompensated system.

In G(s), put $s = j\omega$

$$\therefore G(j\omega) = \frac{80}{j\omega(1+j0.33\omega)(1+j0.167\omega)}$$

MAGNITUDE PLOT

The corner frequencies are ω_{c1} and ω_{c2}.

Here $\omega_{c1} = 1/0.33 = 3$ rad/sec and $\omega_{c2} = 1/0.167 = 6$ rad/sec.

The various terms of $G(j\omega)$ are listed in table-1. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{80}{j\omega}$	–	–20	–
$\frac{1}{1+j0.33\omega}$	$\omega_{c1} = \frac{1}{0.33} = 3$	–20	$-20-20=-40$
$\frac{1}{1+j0.167\omega}$	$\omega_{c2} = \frac{1}{0.167} = 6$	–20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 20$ rad/sec.

Let $A = |G(j\omega)|$ in db

$$\text{At } \omega = \omega_l, \quad A = 20\log\frac{80}{\omega} = 20\log\frac{80}{0.5} = 44 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\frac{80}{\omega} = 20\log\frac{80}{3} = 28.5 \text{ db} \approx 28 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ at } (\omega = \omega_{c1})$$

$$= -40 \times \log\frac{6}{3} + 28 = 16 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A \text{ at } (\omega = \omega_{c2})$$

$$= -60 \times \log\frac{20}{6} + 16 = -15 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b, c and d. Join the points by straight lines and mark the slope on the respective region. The magnitude plot is shown in fig 6.9.2.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}0.33\omega - \tan^{-1}0.167\omega.$$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.