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Modern Control Theory (ICE 3153)

Diagonal Canonical Form & Diagonalization

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What is canonical form..?

- The state model having minimum number of non-zero elements are called as canonical forms.
- There are two types,
  - Diagonal canonical form
  - Jordan canonical form

- This method is based on the concept of partial fraction expansion.
- This form is useful in checking controllability, observability, state transition matrix and also in model order reduction techniques.

Case-1:-Degree of denominator is greater than degree of numerator (strictly proper transfer function).

- Consider the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s^1 + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \text{ ----(1)}$$

$$\frac{Y(s)}{U(s)} = \frac{c_1}{(s+\lambda_1)} + \frac{c_2}{(s+\lambda_2)} + \frac{c_3}{(s+\lambda_3)} \text{ ---(2)}$$

$$\frac{Y(s)}{U(s)} = \frac{c_1(s+\lambda_2)(s+\lambda_3)+c_2(s+\lambda_1)(s+\lambda_3)+c_3(s+\lambda_1)(s+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)+} \text{-(3)}$$

$$b_2 s^2 + b_1 s^1 + b_0 = c_1(s+\lambda_2)(s+\lambda_3) + c_2(s+\lambda_1)(s+\lambda_3) + c_3(s+\lambda_1)(s+\lambda_2)$$

- Replace $s = -\lambda_2$ for c_2

- $c_2 = \frac{b_0 - b_1\lambda_2 + b_2\lambda_2^2}{(-\lambda_2 + \lambda_1)\,(-\lambda_2 + \lambda_3)}$

- Replace $s = -\lambda_3$ for c_3

- $c_3 = \frac{b_0 - b_1\lambda_3 + b_2\lambda_3^2}{(-\lambda_3 + \lambda_1)\,(-\lambda_3 + \lambda_2)}$

- Replace $s = 0$ for c_1

- $c_1 = \frac{b_0 - c_2\lambda_1\lambda_3 + c_3\lambda_1\lambda_2}{\lambda_3\lambda_2}$

- From equation (2)

$$Y(s) = \frac{c_1}{(s+\lambda_1)}U(s) + \frac{c_2}{(s+\lambda_2)}U(s) + \frac{c_3}{(s+\lambda_3)}U(s)$$

- $X_1(s) = \frac{U(s)}{(s+\lambda_1)}$; $sX_1(s) + \lambda_1 X_1(s) = U(s)$; $\dot{x_1} = -\lambda_1 x_1 + u$
- $X_2(s) = \frac{U(s)}{(s+\lambda_2)}$; $sX_2(s) + \lambda_2 X_2(s) = U(s)$; $\dot{x_2} = -\lambda_2 x_2 + u$
- $X_3(s) = \frac{U(s)}{(s+\lambda_3)}$; $sX_3(s) + \lambda_3 X_3(s) = U(s)$; $\dot{x_3} = -\lambda_3 x_3 + u$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

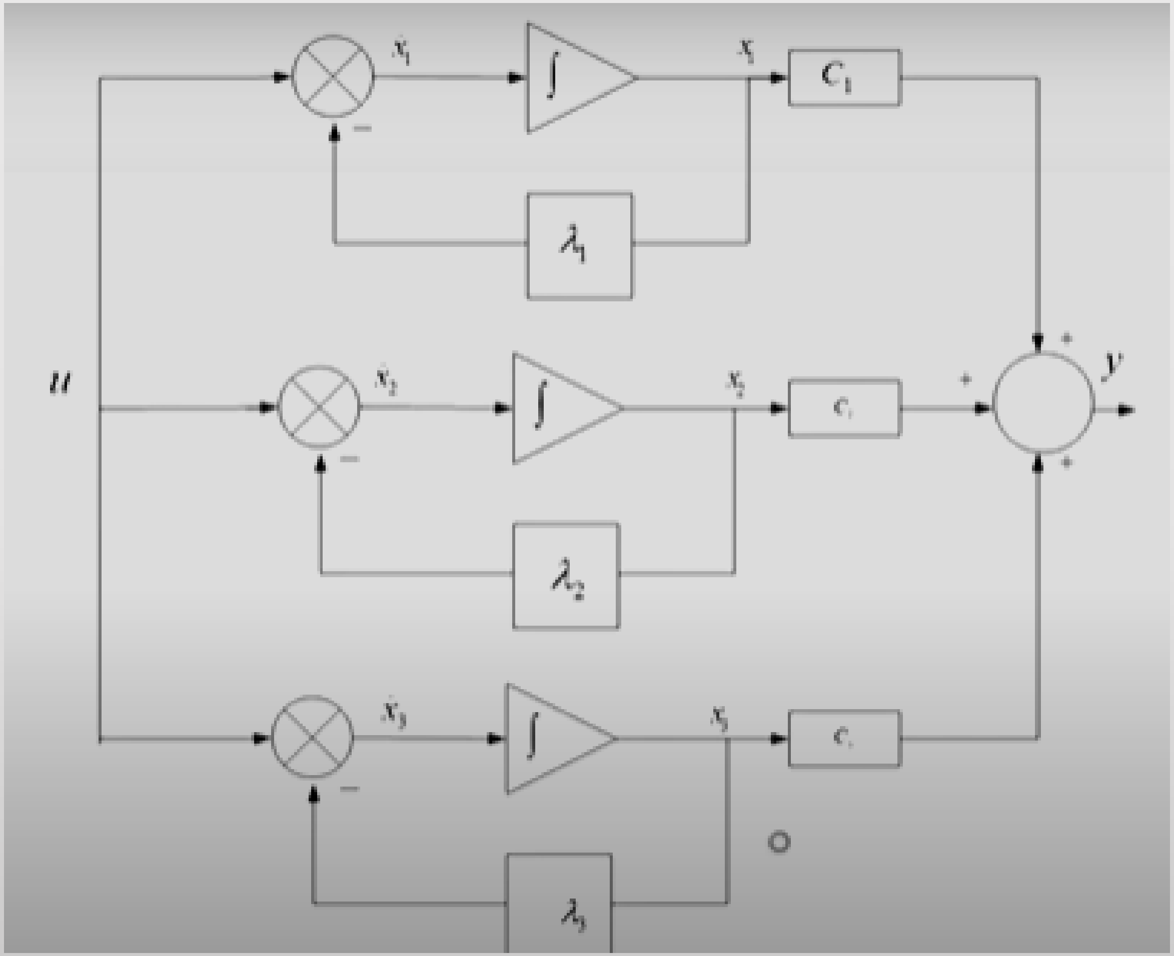
$$Y = CX + DU$$

$$Y(s) = c_1X_1(s)+c_2X_2(s) + c_3X_3(s)$$

Apply inverse LT,

$$y = c_1x_1+c_2x_2 + c_3x_3$$

- $y = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$



u
$\dot{x}_1$
x_1
C_1
λ_1
$\dot{x}_2$
x_2
λ_2
$\dot{x}_3$
x_3
λ_3
y

Case-2:-Degree of denominator is same degree of numerator (proper transfer function).

- Consider the transfer function

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0};$$

- The TF can be expressed as,

$$\frac{Y(s)}{U(s)} = b_3 + \frac{(b_2 - b_3 a_2)s^2 + (b_1 - b_3 a_1)s + (b_0 - b_3 a_0)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}$$

- TF can be written as partial fraction expansion as,

$$\frac{Y(s)}{U(s)} = b_3 + \frac{c_1}{(s+\lambda_1)} + \frac{c_2}{(s+\lambda_2)} + \frac{c_3}{(s+\lambda_3)}$$

- Comparing the above two equations,

$$c_2 = \frac{\left((b_0 - b_3 a_0) - \lambda_2 (b_1 - b_3 a_1) + \lambda_2^2 (b_2 - b_3 a_2)\right)}{(-\lambda_2 + \lambda_1)(-\lambda_2 + \lambda_3)};$$

$$c_3 = \frac{\left((b_0 - b_3 a_0) - \lambda_3 (b_1 - b_3 a_1) + \lambda_3^2 (b_2 - b_3 a_2)\right)}{(-\lambda_3 + \lambda_1)(-\lambda_3 + \lambda_2)};$$

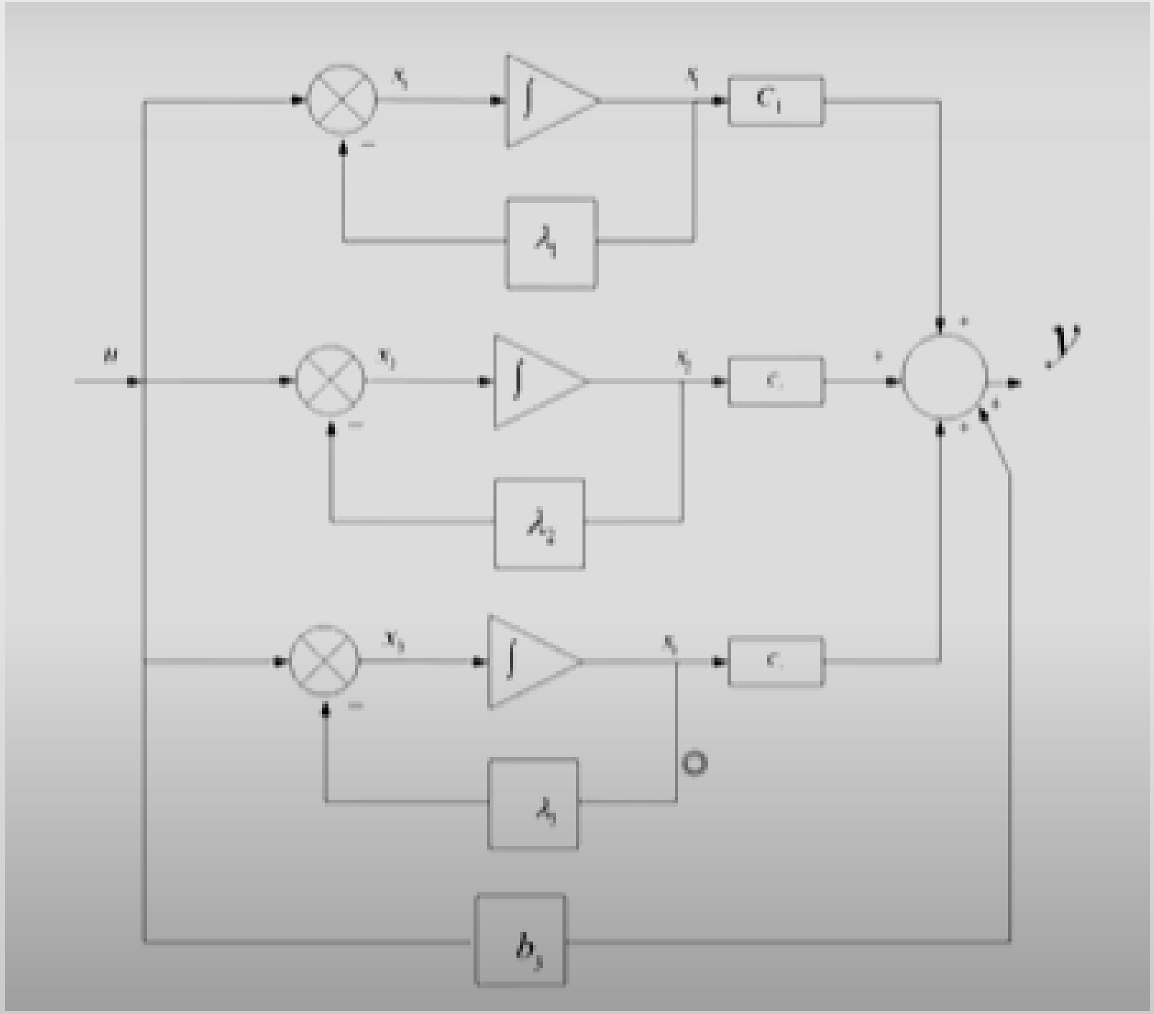
$$c_1 = \frac{\left((b_0 - b_3 a_0) - c_2 (\lambda_1 \lambda_3) - c_3 (\lambda_1 \lambda_2)\right)}{\lambda_2 \lambda_3} \quad \circ$$

$$or \quad c_1 = \frac{\left((b_0 - b_3 a_0) - \lambda_1 (b_1 - b_3 a_1) + \lambda_1^2 (b_2 - b_3 a_2)\right)}{(-\lambda_1 + \lambda_3)(-\lambda_1 + \lambda_2)}$$

- As discussed in Case-1 the SS model can be derived as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$



u
C_1
λ_1
λ_2
λ_3
b_3
y

Advantages..

- System matrix A is always in diagonal form
- In this approach each state equation is of first order irrespective of the order of the transfer function. These equation can be solved independently. Moreover, the decoupling of the state equation is also possible.
- Due to diagonal nature of the system matrix, it is useful in determination of state transition matrix controllability, observability and stabilizability and detectability.

Disadvantage:-

- Similar to companion form, diagonal canonical form are not the physical variable of the system, that is similar to companion form, diagonal canonical form are not the physical variable of the system. Therefore, they are difficult for measurement and control purpose practically.

Jordan Canonical Form

- Consider the third order proper TF,

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0};$$

- The above TF can be written as,

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{(s+\lambda_1)^2 (s+\lambda_3)};$$

- Which can be further represented in simplified form as,

$$\frac{Y(s)}{U(s)} = b_3 + \frac{c_1}{(s+\lambda_1)^2} + \frac{c_2}{(s+\lambda_1)} + \frac{c_3}{(s+\lambda_3)}$$

$$Y(s) = b_3 U(s) + \frac{c_1}{(s+\lambda_1)^2} U(s) + \frac{c_2}{(s+\lambda_1)} U(s) + \frac{c_3}{(s+\lambda_3)} U(s)$$

$$Y(s) = b_3 U(s) + c_1 X_1(s) + c_2 X_2(s) + c_3 X_3(s)$$

- From the above equation we can write,

$$X_1(s) = \frac{1}{(s+\lambda_1)^2} U(s),$$

$$X_2(s) = \frac{1}{(s+\lambda_1)} U(s),$$

$$X_3(s) = \frac{1}{(s+\lambda_3)} U(s)$$

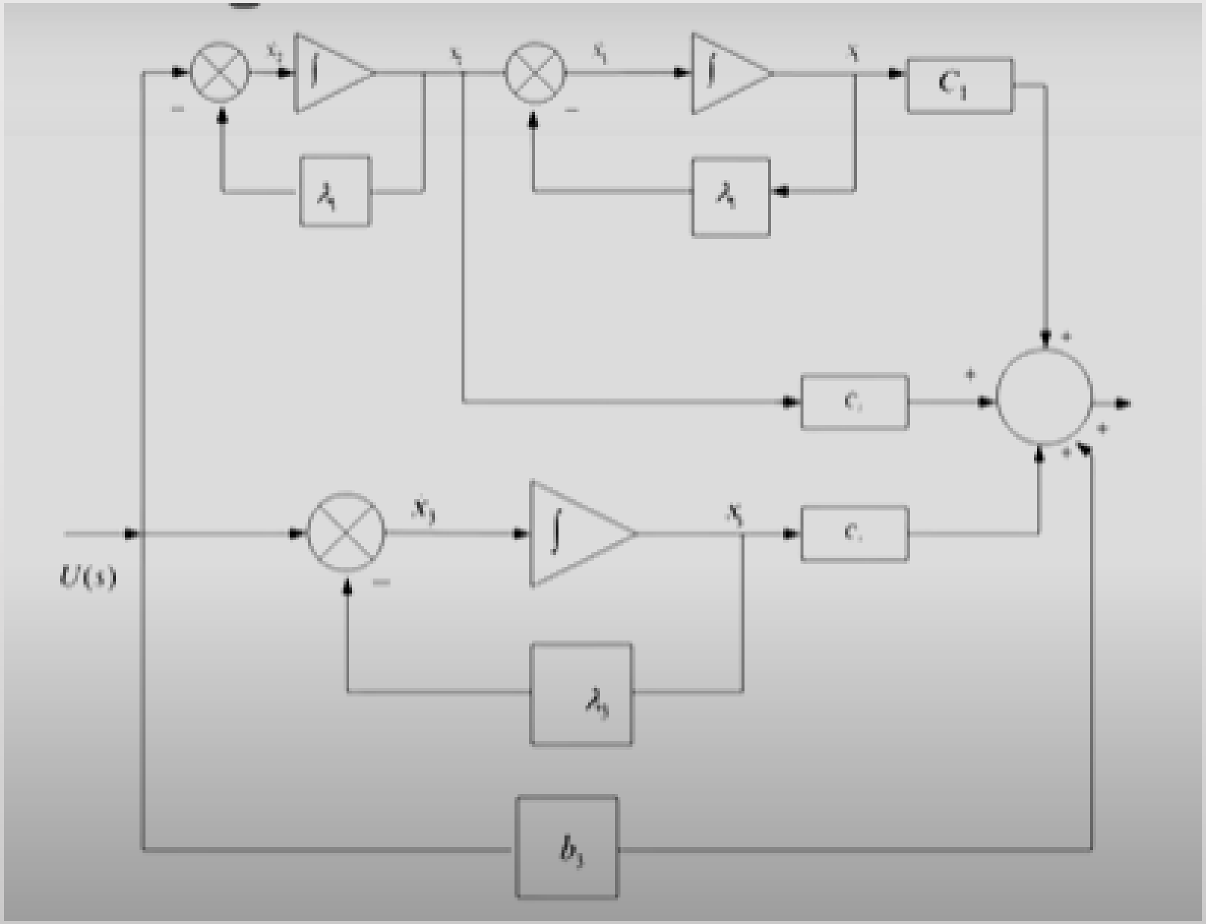
$$\frac{X_1(s)}{X_2(s)} = \frac{1}{(s+\lambda_1)}, \dot{x}_1 = -\lambda_1 x_1 + x_2$$

$$X_2(s) = \frac{1}{(s+\lambda_1)} U(s), \dot{x}_2 = -\lambda_1 x_2 + u$$

$$X_3(s) = \frac{1}{(s+\lambda_3)} U(s); \dot{x}_3 = -\lambda_3 x_3 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$



C_1
λ_1
λ_3
b_3
$U(s)$

Diagonalization

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

- Sometimes it is difficult to analyze that system.
- Sometimes difficult to design a controller
- Sometimes internal property of the system cannot be easily studied.

- We need to transfer a system into different form in a such a manner that, that properties should not change.

- The concept of diagonalization is that the approach or method of transforming a general state space model into canonical form.

- It is useful for investigation of system properties.
- It is useful for evaluation of time response.
- It is useful in checking a controllability and observability and finally, control design.
- It is useful in model order reduction techniques.

Role of Vander Monde Matrix

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$ Controllable canonical form or companion from or phase variable form

$$\mathrm{P} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

In general, if Eigenvalues are $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

- When A has repeated Eigen values, ???
- Suppose if A has Eigen values as, $\lambda_1, \lambda_1, \lambda_3$

- Then $x = Sz$ and S $= \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$

- And this will give you,

$$\mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Invariance of Eigenvalues.

- To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials of $|\lambda\mathbf{I} - \mathbf{A}|$ and $|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}|$ are identical.

Tutorial -3

Question 1:
Ex 5.6, Advanced Control Theory , Nagoor Kani

- Find the Eigen values, Eigenvectors and comment on the stability of the given system.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u \; ; \; y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 & -1 \\ 2 & \lambda + 3 & 0 \\ 0 & -2 & \lambda + 3 \end{bmatrix}$$

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -4$$

- $\text{Eigenvector} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix}$
- System is stable as all the Eigen values are negative

Question 2:

Ex 5.6, CSE , 4th Edition Norman S Nice

- For the following system represented in state space find the Eigen values, Eigenvectors and comment on the stability of the system.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \; ; \; y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Question 3:
Ex 4.11, Advanced Control Theory , Nagoor Kani

- A feedback system whose closed loop transfer function is given below. Construct the state model in diagonal canonical form and draw the block diagram.

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$$

$$\frac{Y(s)}{U(s)} = \frac{c_1}{s} + \frac{c_2}{(s+1)} + \frac{c_3}{(s+3)}$$

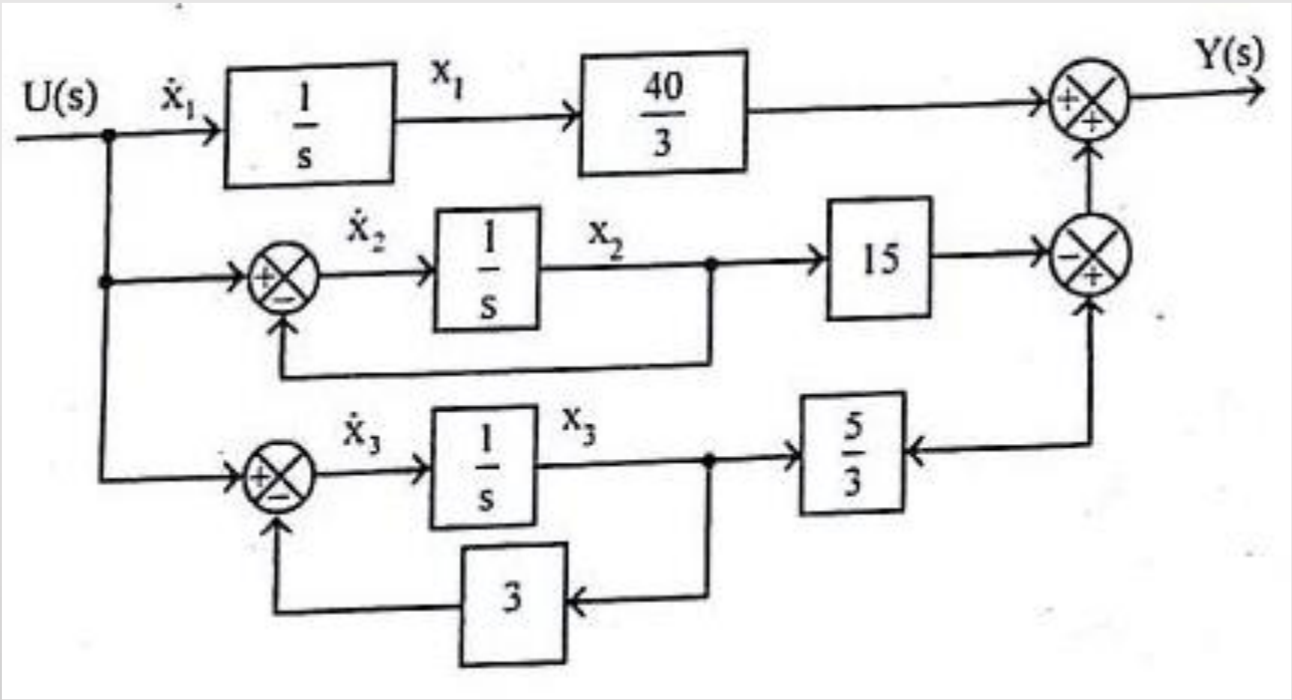
Substitute s=0, we get c_1=40/3

Substitute s=-1, we get c_2=-15

Substitute s=-3, we get c_3=5/3

$$\dot{x_1} = u$$
$$\dot{x_2} = -x_2 + u$$
$$\dot{x_3} = -3x_3 + u$$
$$y = \frac{40}{3}x_1 - 15x_2 + 5/3x_3$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \; ; \; y = \begin{bmatrix} 40/3 \\ -15 \\ 5/3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$



U(s)
$\dot{x}_1$
$\frac{1}{s}$
x_1
$\frac{40}{3}$
Y(s)
$\dot{x}_2$
$\frac{1}{s}$
x_2
15
$\dot{x}_3$
$\frac{1}{s}$
x_3
$\frac{5}{3}$
3

Question 4:
Exercise B-11-13, MCE , 4th Edition Ogata

- Consider the system represented by the state space model given. Perform diagonalization and obtain the canonical form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u \; ; y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$