Equations Reducible to Exact Differential Equations:

II.

(i) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then $IF = e^{\int f(x)\,dx}$

(ii) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = g(y)$, a function of y alone, then $IF = e^{\int g(y)\,dy}$

Proof (i): Consider

$$M\,dx + N\,dy = 0.$$

Suppose that $v = v(x)$, a function of x alone, is an IF the equation.

Then

$$Mv\,dx + Nv\,dy = 0$$

is exact. ($\because$ when you multiply an equation by IF, it reduces to exact).

$$\Rightarrow \frac{\partial (Mv)}{\partial y} = \frac{\partial (Nv)}{\partial x}$$

$$v\frac{\partial M}{\partial y} = v\frac{\partial N}{\partial x} + N\frac{dv}{dx}$$

(v is a function of x alone $\frac{\partial v}{\partial x} = \frac{dv}{dx}$)

$$v\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{dv}{dx}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx = \frac{dv}{v}$$

Integrating we get

$$\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx = \log v$$

$$v = e^{\int\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)dx} = e^{\int f(x)\,dx}, \text{ where } f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Solve the following differential equations.

$\int_{y\,\text{const}} M\,dx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

① $(1 + 3x\sin y)\,dx - x^2\cos y\,dy = 0.$

Solution: $M = 1 + 3x\sin y$, $N = -x^2\cos y$

$$\frac{\partial M}{\partial y} = 3x\cos y, \quad \frac{\partial N}{\partial x} = -\cos y(2x) = -2x\cos y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x\cos y - (-2x\cos y) = 5x\cos y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{5x\cos y}{-x^2\cos y} = \frac{-5}{x}$$

$$IF = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

Multiplying the equation by $\frac{1}{x^5}$, we get

$$\left(\frac{1}{x^5} + \frac{3}{x^4}\sin y\right)dx - \frac{1}{x^3}\cos y\,dy = 0$$

which is exact.

Solution is

$$\int\left(\frac{1}{x^5} + \frac{3}{x^4}\sin y\right)dx + 0 = -c$$

$$\frac{-1}{4x^4} + 3\frac{1}{-3x^3}\sin y = -c$$

$$\frac{1}{4x^4} + \frac{\sin y}{x^3} = c$$

② $(3\tan x - 2\cos y)\sec^2 x\,dx + \tan x \sin y\,dy = 0$

Solution: $M = (3\tan x - 2\cos y)\sec^2 x$, $N = \tan x \sin y$

$$\frac{\partial M}{\partial y} = \sec^2 x(0 - 2(-\sin y)) = 2\sin y \sec^2 x \qquad \frac{\partial N}{\partial x} = \sin y \sec^2 x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2\sin y\sec^2 x - \sin y \sec^2 x = \sin y \sec^2 x$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\sin y \sec^2 x}{\tan x \sin y} = \frac{\sec^2 x}{\tan x}$$

$$IF = e^{\int \frac{\sec^2 x}{\tan x}dx} = e^{\int \frac{dz}{z}} = e^{\log z} = z = \tan x$$

$\tan x = z$
$\sec^2 x\,dx = dz$

Multiplying the equation by IF,

$\tan x(3\tan x - 2\cos y)\sec^2 x\,dx + \tan^2 x \sin y\,dy = 0$ which is exact

verification

$$\frac{\partial}{\partial y}\left[\tan x\,(3\tan x - 2\cos y)\sec^2 x\right] = \tan x \sec^2 x\,(0 + 2\sin y)$$

$$\frac{\partial}{\partial x}(\tan^2 x \sin y) = \sin y \cdot 2\tan x \cdot \sec^2 x.$$

Solution is

$$\int \tan x(3\tan x - 2\cos y)\sec^2 x\,dx + 0 = C$$

$\tan x = t$
$\sec^2 x\,dx = dt$

$$\int t(3t - 2\cos y)\,dt = C$$

$$\int (3t^2 - 2\cos y\, t)\,dt = C$$

$$t^3 - 2\cos y \cdot \frac{t^2}{2} = C$$

$$t^3 - t^2\cos y = C$$

$$\tan^3 x - \tan^2 x \cos y = C.$$

③ $\cos y \sin 2x\,dx + (\cos^2 y - \cos^2 x)\,dy = 0.$

$M = \cos y \sin 2x$ $\qquad N = \cos^2 y - \cos^2 x$

$$\frac{\partial M}{\partial y} = \sin 2x(-\sin y) = -\sin y \sin 2x \qquad \frac{\partial N}{\partial x} = 0 - 2\cos x(-\sin x) = 2\sin x\cos x = \sin 2x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\sin y \sin 2x - \sin 2x = -(\sin y + 1)\sin 2x$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-(\sin y + 1)\sin 2x}{-\cos y \sin 2x} = \frac{\sin y + 1}{\cos y} = \tan y + \sec y$$

$$IF = e^{\int(\tan y + \sec y)dy} = e^{\log \sec y + \log(\sec y + \tan y)} = e^{\log(\sec y(\sec y + \tan y))} = \sec y(\sec y + \tan y)$$

multiplying the equation by I.F, we get

$$\sec y(\sec y+\tan y)\cos y\sin 2x\,dx + \sec y(\sec y+\tan y)(\cos^2 y-\cos^2 x)\,dy=0$$

which is exact.

$$(\sec y+\tan y)\sin 2x\,dx + (\sec^2 y+\sec y\tan y)(\cos^2 y-\cos^2 x)\,dy=0.$$

Solution is

$$\int_{y\text{ const}}(\sec y+\tan y)\sin 2x\,dx + \int(\sec^2 y+\sec y\tan y)\cos^2 y\,dy = c.$$

$$(\sec y+\tan y)\left(\frac{-\cos 2x}{2}\right) + \int(\sec^2 y\cos^2 y + \sec y\tan y\cos^2 y)\,dy = c$$

$$\frac{1}{\cos y}\frac{\sin y}{\cos y}\cos^2 y = \sin y$$

$$-\frac{\cos 2x(\sec y+\tan y)}{2} + y - \cos y = c.$$

④ $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy=0.$

$M = xy^3+y \qquad N = 2x^2y^2+2x+2y^4$

$\frac{\partial M}{\partial y} = 3xy^2+1 \qquad \frac{\partial N}{\partial x} = 4xy^2+2$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = (3xy^2+1)-(4xy^2+2) = -(xy^2+1)$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M} = \frac{-(xy^2+1)}{-(xy^3+y)} = \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y}$$

$$I.F = e^{\int\frac{1}{y}dy} = e^{\log y} = y$$

Multiplying by I.F, we get

$$(xy^4+y^2)\,dx + 2y(x^2y^2+x+y^4)\,dy=0$$

which is exact.

Solution is

$$\int_{y\text{ constant}}(xy^4+y^2)\,dx + \int 2y\cdot y^4\,dy = \frac{c}{6}$$

$2yy^4 = 2y^5$

Integral is

$2\frac{y^6}{6}$

$$y^4\cdot\frac{x^2}{2} + y^2\cdot x + 2\frac{y^6}{6} = \frac{c}{6}$$

$$3x^2y^4 + 6xy^2 + 2y^6 = c.$$

⑤ $(y\log y)\,dx + (x-\log y)\,dy = 0.$

Linear Differential Equations:

A linear differential equation of order one and degree one in y is of the form

$$\frac{dy}{dx} + Py = Q \text{ ——— } (1)$$

where P and Q are functions of x alone.

This DE is also called Leibnitz's Linear equation.

Equation linear in x is of the form

$$\frac{dx}{dy} + Px = Q \longrightarrow$$ Solution is $x\,(IF) = \int Q\,(IF)\,dy + c$

$IF = e^{\int P\,dy}$

where P and Q are functions of y alone.

Suppose that $v = v(x)$, a function of x alone, is an IF of (1).

Then

$$v\frac{dy}{dx} + Pvy = Qv$$

is exact.

ie. $(Pvy - Qv)\,dx + v\,dy = 0$

is exact.

$P, Q, v \to$ are functions of x alone.

$$\Rightarrow \frac{\partial}{\partial y}(Pvy - Qv) = \frac{\partial}{\partial x}(v).$$

$$Pv\frac{\partial y}{\partial y} - \frac{\partial}{\partial y}(Qv) = \frac{dv}{dx}$$

$$Pv - 0 = \frac{dv}{dx}$$

$$P\,dx = \frac{dv}{v}$$

$$\int P\,dx = \log v$$

$$\therefore v = e^{\int P\,dx} \quad \text{ie. } IF = e^{\int P\,dx}$$

Multiplying (1) by IF, we get

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}.Py = Q\,e^{\int P\,dx}$$

$$\frac{d}{dx}\left(y\,e^{\int P\,dx}\right) = Q\,e^{\int P\,dx}$$

$$\frac{d}{dx}\left(e^{\int P\,dx}\right) = e^{\int P\,dx}.\frac{d}{dx}\left(\int P\,dx\right) = e^{\int P\,dx}.P.$$

Integrating, we get,

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

$$\boxed{y\,(IF) = \int Q\,(IF)\,dx + c}$$