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Modern Control Theory (ICE 3153)

Controllability & Observability

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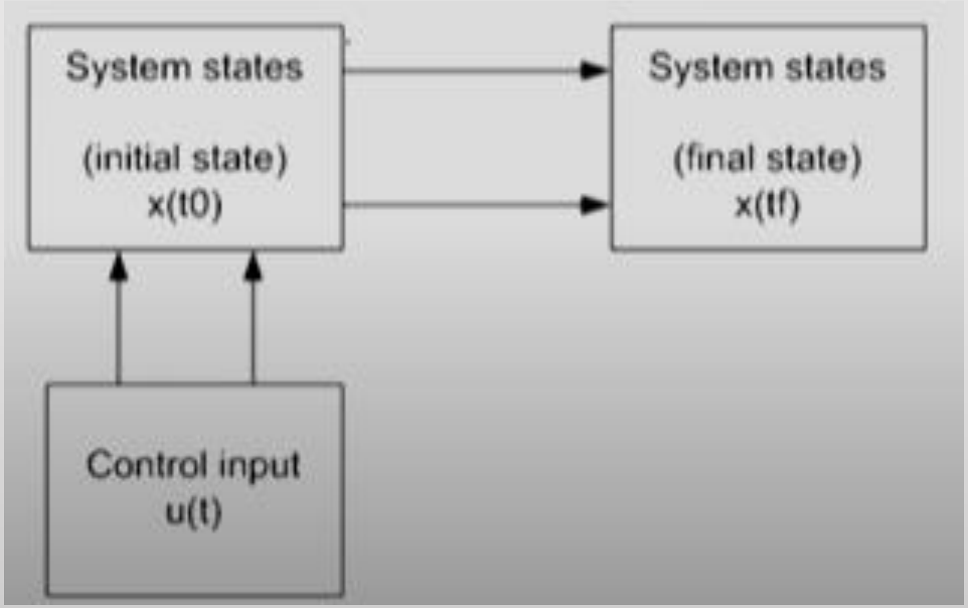
Controllability

A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state *x(t0)* to any other state in a finite interval of time.



Complete State Controllability

Consider the continuous-time system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

The system described above is said to be state controllable at *t* = *to* if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \leq t \leq t_1$. If every state is controllable, then the system is said to be completely state controllable.

Testing of Controllability

- **Kalman's Test:**
- The system represented as $\dot{x} = Ax + Bu$

is completely state controllable if and only if the vectors $B, AB \ldots . A^{n-1}B$ are linearly independent or $n\ x\ n$ matrix,

$C = [B \vdots\ AB \vdots\ \ldots . \vdots A^{n-1}B]$ is or rank n.

Or we can say $|C| \neq 0$

The matrix C is called controllability matrix.

Disadvantage:-Uncontrollable states cannot be identified.

Question: 1
Ex 11-10, MCE 4th Edition K. Ogata

Check the controllability of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Gilbert test for controllability

- It is based on the concept of diagonalization.

$$\dot{x} = Ax + B\mathrm{u}$$

- After diagonalization, we get $\dot{z} = P^{-1}APz + P^{-1}Bu$
- $\dot{z} = A = \tilde{A}z + \tilde{B}u$
- $\begin{bmatrix} \dot{z_1} \\ \dot{z_2} \\ \vdots \\ \dot{z_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \cdots & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} \widetilde{b_1} \\ \widetilde{b_2} \\ \vdots \\ \widetilde{b_n} \end{bmatrix} u$

For the system to be completely controllable the elements $\widetilde{b_1}, \widetilde{b_2} \ldots . \widetilde{b_n}$ has to be non zero (for distinct eigenvalues)

- When u is an r vector,

$$\dot{z}_1 = \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \cdots + f_{1r}u_r$$

$$\dot{z}_2 = \lambda_2 z_2 + f_{21}u_1 + f_{22}u_2 + \cdots + f_{2r}u_r$$

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$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_2 + \cdots + f_{nr}u_r$$

If the elements of any one row of the $\boldsymbol{n} \times \mathrm{r}$ matrix are all zero, then the corresponding state variable cannot be controlled by any of the input u.

Hence, the condition of complete state controllability is that if the eigenvectors of A are distinct, then the system is completely state controllable if and only if no row of $P^{-1}B$ has all zero elements.

- If the A matrix does not possess distinct eigenvectors, then diagonalization is impossible. In such a case, we may transform A into a Jordan canonical form.

$$
\mathbf{J} = \left[\begin{array}{ccc|cc|cccc}
\lambda_1 & 1 & 0 & & & & & & 0 \\
0 & \lambda_1 & 1 & & & & & & \\
0 & 0 & \lambda_1 & & & & & & \\
\hline
 & & & \lambda_4 & 1 & & & & \\
 & & & 0 & \lambda_4 & & & & \\
\hline
 & & & & & \lambda_6 & & & \\
 & & & & & & \cdot & & \\
 & & & & & & & \cdot & \\
 & & & & & & & & \cdot \\
0 & & & & & & & & \lambda_n
\end{array}\right]
$$

$$
\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z} + \mathbf{S}^{-1}\mathbf{B}\mathbf{u} \\
&= \mathbf{J}\mathbf{z} + \mathbf{S}^{-1}\mathbf{B}\mathbf{u}
\end{aligned}
$$

- The system is completely state controllable if and only if
- (1) no two Jordan blocks in J of above Equation are associated with the same eigenvalues,
- (2) the elements of any row of $S^{-1}B$ that correspond to the last row of each Jordan block are not all zero,
- and (3) the elements of each row of $S^{-1}B$ that correspond to distinct eigenvalues are not all zero.

Question: 2

Ex 11-12, MCE 4th Edition K. Ogata

- Comment on the controllability of the following systems using Gilberts method.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

Completely state controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$

Completely state controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc:cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hdashline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Completely state controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

not completely state controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc:cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hdashline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} u$$

not completely state controllable

Condition for Complete State Controllability in the *s* Plane

- The necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the system cannot be controlled in the direction of the canceled mode.

Output Controllability

- Complete state controllability is neither necessary nor sufficient for controlling the output of the system.
- Consider the system,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The system described above is said to be completely output controllable if it is possible to construct an unconstrained control vector $u(t)$ that will transfer any given initial output $y(t_0)$ to any final output $y(t_1)$ in a finite time interval $t_0 \leq t \leq t_1$.

Condition for complete output controllability is as follows:

- The system described by above equation is completely output controllable if and only if the $m\ X\ (n+1)r$ matrix C_{out} Is of rank m

$$C_{out} = [CB \vdots\ CAB \vdots\ CA^2B \vdots\ \ldots. \vdots CA^{n-1}B \vdots\ D]$$

C_{out} is called output controllability matrix

Stabilizability

- For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable.
- Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

- The stable mode is uncontrollable
- But Unstable mode is controllable
- Such a system can be made stable by the use of a suitable feedback. Thus this system is stabilizable.

OBSERVABILITY

- Consider the unforced system described by the following equations:

$$\dot{x} = Ax \; ; y = Cx$$

- The system described above is said to be completely observable if every stae $x(t_0)$ can be determined from the observation of $y(t)$ over a finite time interval $t_0 \leq t \leq t_1$.
- The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector.
- The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time.
- The concept of observability is very important because, in practice, the difficulty encountered with state feedback control is that some of the state variables are not accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasurable state variables in order to construct the control signals.

Testing of Observability

• Kalman's Test:

- The system represented as $\dot{x} = Ax + Bu$

is completely observable if and only if the $n\ x\ nm$ matrix O is of rank n or has *n* linearly independent column vectors.

$$O = [C^* \vdots\ A^*C^* \vdots\ \ldots. \vdots\ A^{*(n-1)}C^*]$$

Or we can say $|O| \neq 0$

This matrix is called the *observability matrix.*

Disadvantage:-Unobservable states cannot be identified.

Question: 3

Ex 11-14, MCE 4th Edition K. Ogata

Check the output controllability & observability of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[\mathbf{CB} \ \vdots\ \mathbf{CAB}] = [0 \quad 1]$$

$$[\mathbf{C}^* \ \vdots\ \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Completely output controllable & observable

Gilbert test for Observability

- Consider the system,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Case1-Distict Eigenvalues

- After diagonalization we get, $y = CPz$
- The system is completely observable if none of the columns of the m X ***n*** matrix **CP** consists of all zero elements.
- This is because, if the i th column of **CP** consists of all zero elements, then the state variable $z_i(0)$ will not appear in the output equation and therefore cannot be determined from observation of y(t).
- Case2-Repeated Eigenvalues
- After transformation we get, $y = CSz$
- The system is completely observable if
- (1) no two Jordan blocks in J are associated with the same eigenvalues,
- (2) no columns of **CS** that correspond to the first row of each Jordan block consist of zero elements,
- and (3) no columns of **CS** that correspond to distinct eigenvalues consist of zero elements.

Question: 4

Ex 11-16, MCE 4^{th} Edition K. Ogata

- Comment on the observability of the following systems using Gilberts method.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Completely observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Completely observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Completely observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

not completely observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

not completely observable

Conditions for Complete Observability in the s Plane

- The conditions for complete observability can also be stated in terms of transfer functions or transfer matrices.
- The necessary and sufficient conditions for complete observability is that no cancellation occur in the transfer function or transfer matrix.
- If cancellation occurs, the canceled mode cannot be observed in the output.

- The transfer function has no cancellation if and only if the system is completely state controllable and completely observable. This means that the canceled transfer function does not carry along all the information characterizing the dynamic system.

Relationship between controllability and observability

- **Principle of Duality:**
- Consider the system S1 described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

x = state vector (n-vector), **u** = control vector (r-vector), y = output vector (m-vector), **A** = *n* X *n* matrix, **B** = *n* X r matrix, **C** = m X *n* matrix

and the dual system *S2* defined by

$$\dot{z} = A^*z + C^*v$$
$$n = B^*z$$

- **z** = state vector (n-vector), v = control vector (m-vector), n = output vector (r-vector), **A*** = conjugate transpose of **A, B*** = conjugate transpose of **B, C*** = conjugate transpose of **C**
- The principle of duality states that the system S1 is completely state controllable (observable) if and only if system *S2* is completely observable (state controllable).

- For system *S1*:

1. A necessary and sufficient condition for complete state controllability is that the rank of the *n* X *nr* matrix be n.

$$C_{S1} = [B \vdots\ AB \vdots\ \dots . \vdots A^{n-1}B]$$

2. A necessary and sufficient condition for complete observability is that the rank of the *n* X *nm* matrix be n.

$$O_{S1} = [C^* \vdots\ A^*C^* \vdots\ \dots . \vdots A^{*(n-1)}C^*]$$

For system *S2:*

1. A necessary and sufficient condition for complete state controllability is that the rank of the *n* X *nm* matrix be n.

$$C_{S1} = [C^* \vdots\ A^*C^* \vdots\ \dots . \vdots A^{*(n-1)}C^*]$$

2. A necessary and sufficient condition for complete observability is that the rank of the *n* X *nr* matrix be n.

$$O_{S1} = [B \vdots\ AB \vdots\ \dots . \vdots A^{n-1}B]$$

- **Detectability.**
- For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable.
- Note that the concept of detectability is dual to the concept of stabilizability.