DESCRIBING FUNCTION ANALYSIS:

- The frequency response method is a powerful tool for the analysis and design of linear control systems.
- It is based on describing a linear system by a complex-valued function, the frequency response, instead of differential equation.
- The power of the method comes from a number of sources.
- First, graphical representations can be used to facilitate analysis and design.
- Second, physical insights can be used, because the frequency response functions have clear physical meanings.
- Finally, the method's complexity only increases mildly with system order.
- Frequency domain analysis, however, can not be directly applied to nonlinear systems because frequency response functions (FRF) cannot be defined for nonlinear systems.

For some nonlinear systems, an extended version of the frequency response method, called the **describing function method**, can be used to approximately analyze and predict nonlinear behavior.

Even though it is only an approximation method, the desirable properties it inherits from the frequency response method, and the shortage of other systematic tools for nonlinear system analysis, make it an indispensable component of the bag of tools of practicing control engineers.

The main use of describing function method is for the prediction of limit cycles in nonlinear systems.

APPLICATIONS OF DESCRIBING FUNCTIONS (DF):

Simply speaking, any system which can be transformed into the configuration in below Figure can be studied using describing functions. There are at least two important classes of systems in this category.

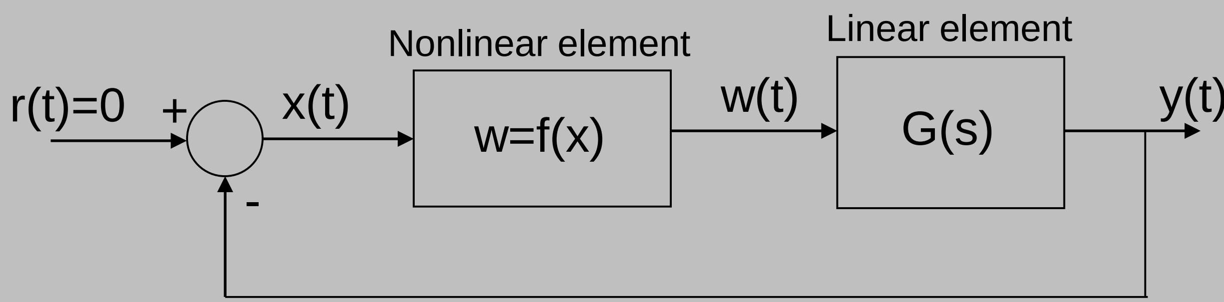


Figure 1 . A nonlinear system.

The first important class consists of "almost" linear systems. By "almost" linear systems, we refer to systems which contain **hard nonlinearities** in the control loop but are otherwise linear. Such systems arise when a control system is designed using linear control but its implementation involves hard nonlinearities, such as motor saturation, actuator or sensor dead-zones, Coulomb friction, or hysteresis in the plant. An example is shown in below Figure, which involves hard nonlinearitie in the actuator.

Consider the control system shown in below Figure. The plant is linear and the controller is also linear. However, the actuator involves a hard nonlinearity. This system can be rearranged as shown in Figure 1 by regarding $G_pG_1G_2$ as the linear component G, and the actuator nonlinearity as the nonlinear element.

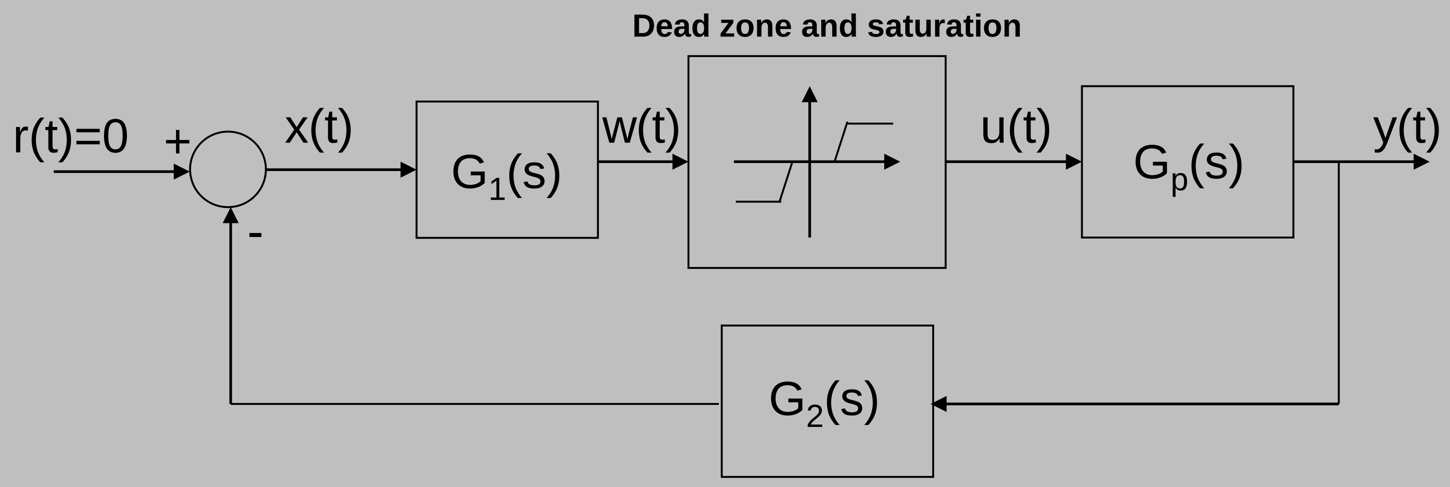


Figure 2 . A control system with hard nonlinearity.

"Almost" linear systems involving sensor or plant nonlinearities can be similarly rearranged into the form of Figure 1.

The second class of systems consists of genuinely nonlinear systems whose dynamic equations can actually be rearranged into the form of Figure 1.

For systems such as the one in Figure 2, limit cycles can often occur due to the nonlinearity.

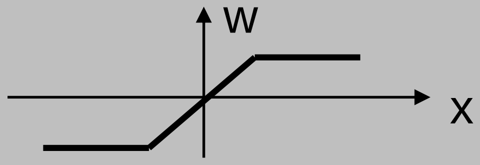
However, linear control cannot predict such problems. Describing functions, on the other hand, can be conveniently used to discover the existence of limit cycles and determine their stability, regardless of whether the nonlinearity is "hard" or "soft".

The applicability to limit cycle analysis is due to the fact that the form of the signals in a limit-cycling system is usually approximately sinusoidal.

Basic Assumptions in Describing Function Analysis:

Consider a nonlinear system in the general form of Figure 1. In order to develop the basic version of the describing function method, the system has to satisfy the following four conditions:

1. Ther is only a single nonlinear components
2. The nonlinear component is time invariant (saturation, backlash, Coulomb friction, etc.)
3. Corresponding to a sinusoidal input x=sin(ωt), only the fundamental component $w_1(t)$ in the output w(t) has to be considered. $|G(\omega i)| >> |G(n\omega i)| \quad \text{for} \quad n = 2,3,\ldots$
4. The nonlinearity is odd (symmetry about the origin).



Saturation

Basic Definitions:

Let us now discuss how to represent a nonlinear component by a describing function. Let us consider a sinusoidal input to the nonlinear element, of amplitude A and frequency ω, i.e., x(t)=Asin(ωt) as shown in Figure 3.

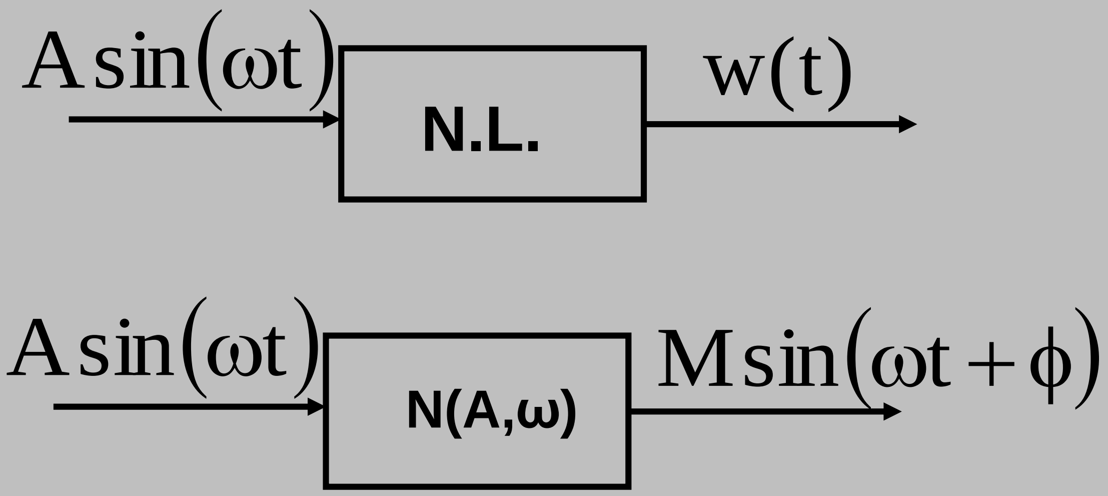


Figure 3. A nonlinear element and its describing function representation

The output of a nonlinear component w(t) is often a periodic, though generally non-sinusoidal, function. Note that this is always the case if the nonlinearity f(x) is single-valued, because the output is f[Asin(w(t+2π/ω))]=f[Asin(ωt)].

Using Fourier series, the periodic function w(t) can be expanded as

$$w(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \left[a_n \sin\left(n\omega t\right) + b_n \cos\left(n\omega t\right) \right]$$

where the Fourier coefficients a_i's and b_i's are generally functions of A and ω, determined by

$$b_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} w(t)\, d(\omega t)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} w(t) \cos\left(n\omega t\right) d(\omega t)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} w(t) \sin\left(n\omega t\right) d(\omega t)$$

Due to the fourth assumtion above, one as b_0=0. Furthermore, the third assumption implies that we only need to consider the fundamental component $w_1(t)$, namely

$$w(t) = w_1(t) = a_1 \sin(\omega t) + b_1 \cos(\omega t) = M \sin(\omega t + \varphi)$$

where

$$M(A,\omega) = \sqrt{a_1^2 + b_1^2} \quad and \quad \varphi(A,\omega) = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

Expression for w(t) indicates that the fundamental component corresponding to a sinusoidal input is a sinusoid at the same frequency. In complex representation, this sinusoid can be written as

$$w_1 = Me^{i(\omega t + \phi)}$$

Similarly to the concept of frequency response function, which is the frequency-domain ratio of the sinusoidal input and the sinusoidal output of a system, we define the describing function of the nonlinear element to be the complex ratio of the fundamental component of the nonlinear element by the input sinusoid, i.e.,

$$N(A,\omega) = \frac{Me^{i(\omega t+\phi)}}{Ae^{i\omega t}} = \frac{M}{A}e^{i\phi}$$

With a describing function representing the nonlinear component, the nonlinear element, in the presence of sinusoidal input, can be treated as if it were a linear element with a frequency response function $N(A,\omega)$ as shown in Figure 3.

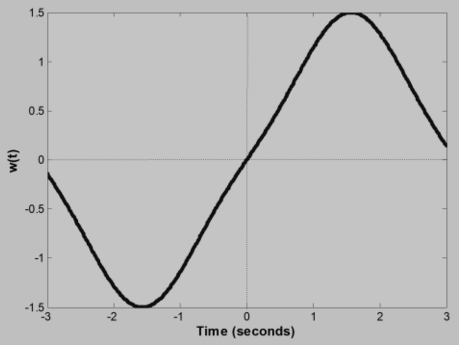
Example: Describing function of a hardening spring

The characteristics of a hardening spring are given by

$$\mathrm{w} = \mathrm{x} + \frac{\mathrm{x}^3}{2}$$

with x being the input and w being the output. Given an input x(t)=Asin(ωt), the output

$$\mathrm{w(t)} = \mathrm{A}\sin(\omega \mathrm{t}) + \frac{\mathrm{A}^3}{2}\sin^3(\omega \mathrm{t}) \Rightarrow$$



The output can be expanded as a Fourier series, with the fundamental being

$$w(t) = a_1 \sin(\omega t) + b_1 \cos(\omega t)$$

Because w(t) is an odd function, one has $b_1=0$ and the coefficient a_1 is

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[A\sin(\omega t) + \frac{A^3}{2}\sin^3(\omega t)\right]\sin(\omega t)\,d\omega t = A + \frac{3}{8}A^3$$

Therefore the fundamental is

$$w_1 = \left(A + \frac{3}{8} A^3 \right) \sin(\omega t) \qquad w_1 = N(A, \omega) x(t)$$

and the describing function ofthis nonlinear component is

$$N(A, \omega) = N(A) = 1 + \frac{3}{8} A^2$$

$$w_1 = \left(1 + \frac{3}{8} A^2 \right) A \sin(\omega t)$$

Note that due to the odd nature of this nonlinearity, the describing function is real, being a function only of the amplitude of the sinusoidal input.

Saturation

On-off nonlinearity

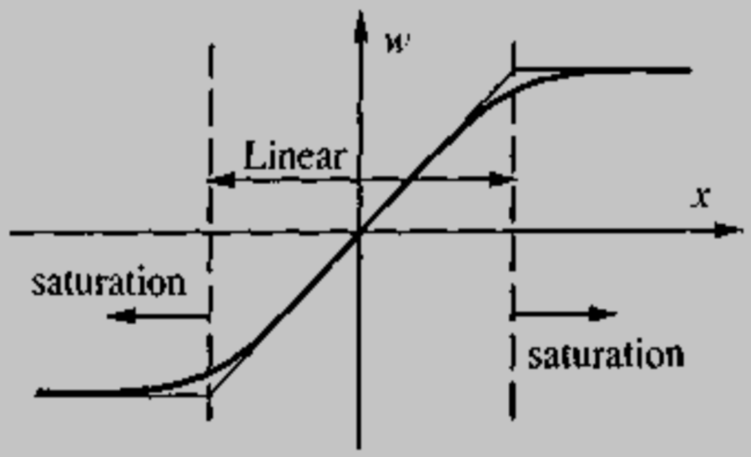
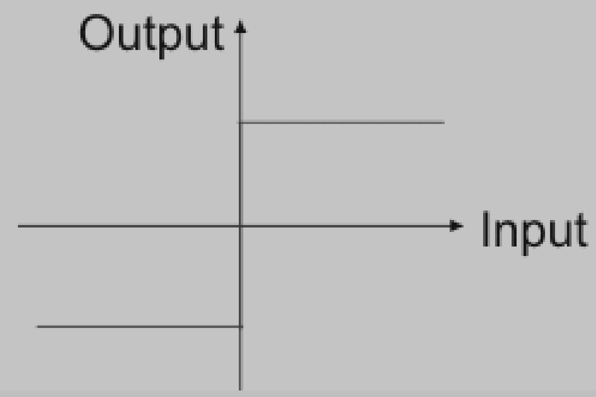
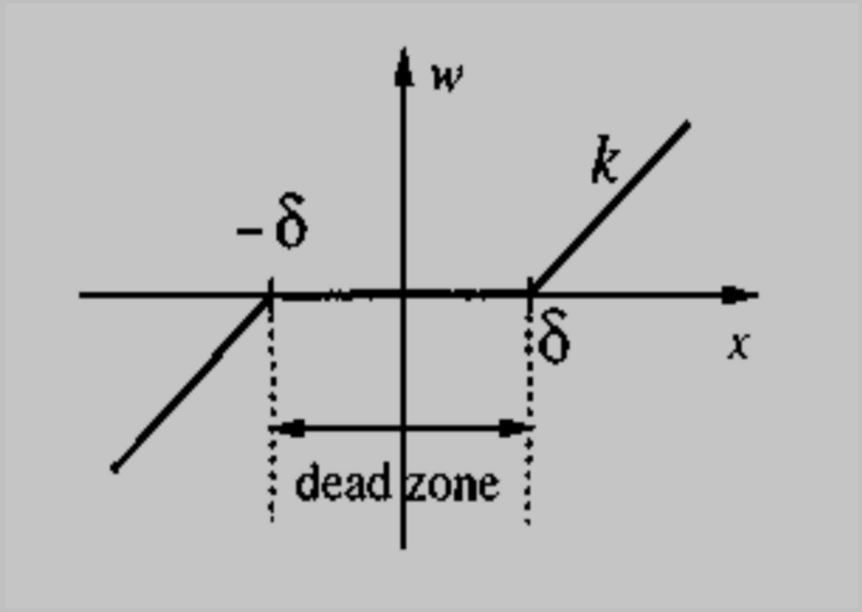


Figure 5.8 : A saturation nonlinearity



Dead-zone



Backlash and hysteresis

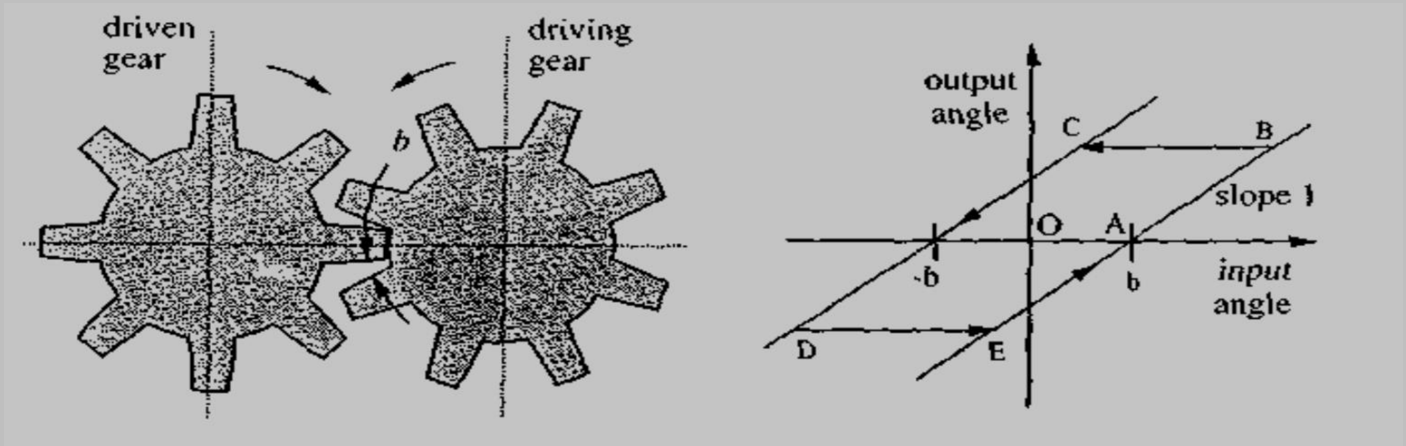
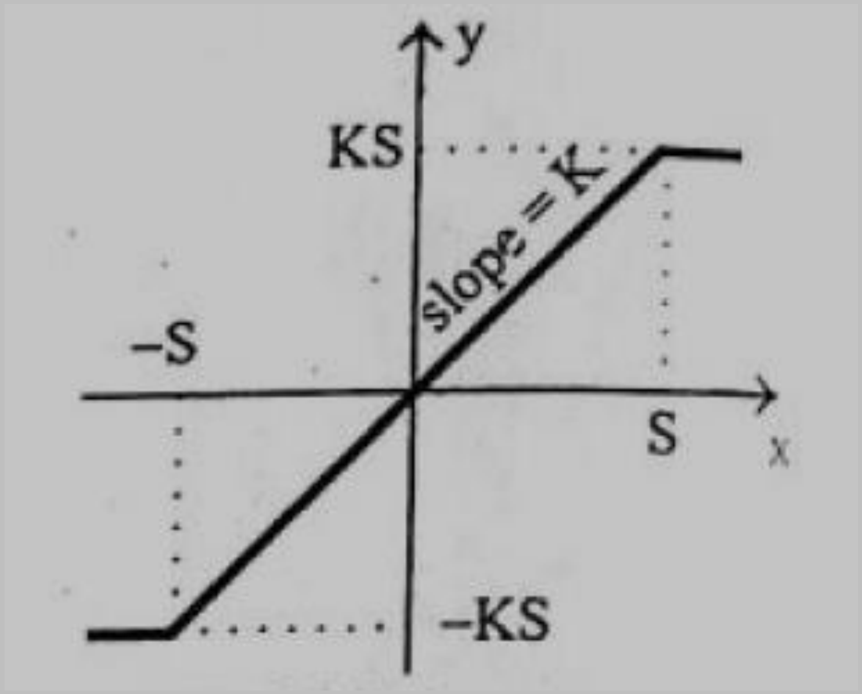


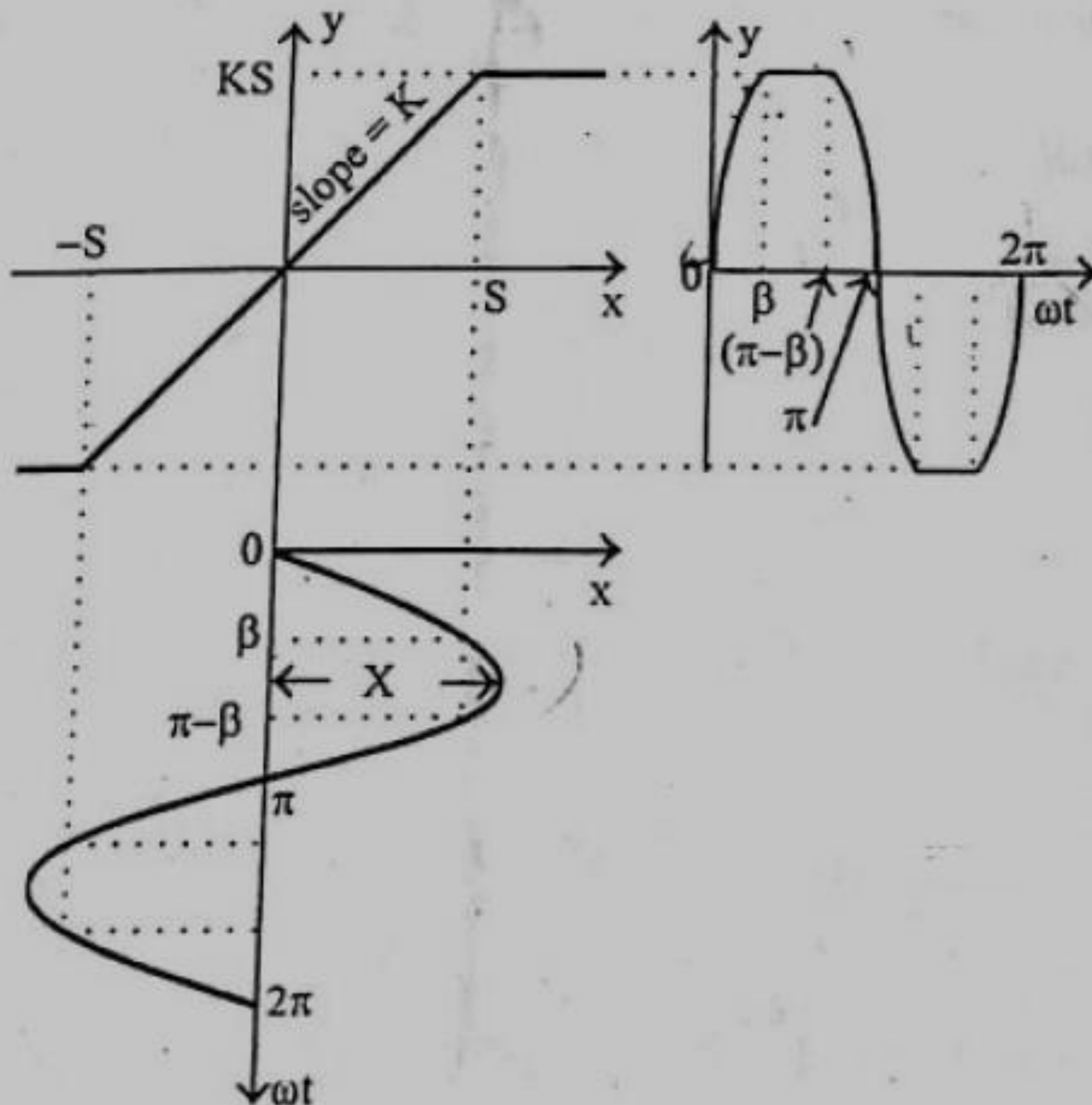
Figure 5.10 : A backlash nonlinearity

Describing Functions of Saturation Nonlinearity

The input-outp4ut relationship for a saturation nonlinearity is shown below, with *S* and *k* denoting the range and slope of the linearity.



Consider we apply a sinusoidal input to the nonlinearity, $x(t) = A\sin(\omega t)$



y
KS
slope = K
–S
S
x
0
β
X
π–β
π
2π
ωt
(π–β)

The output of the nonlinearity can be divided into three regions in a period of $0 \text{ to } \pi$. The output equation of the three segments are given by,

$$y = \begin{cases} Kx, & 0 \le \omega t \le \beta \\ KS, & \beta \le \omega t \le (\pi - \beta) \\ Kx, & (\pi - \beta) \le \omega t \le \pi \end{cases}$$

The output has half wave and quarter wave symmetries, b_1=0 and the coefficient a_1 is

$$a_1 = \frac{2}{\pi/2} \int_0^{\pi/2} y \sin(\omega t) d(\omega t)$$

Output is given by two different expression in the period of $0 \text{ to } {}^{\pi}/_{2}$, hence the expression for a_1 can be written as,

$$a_1 = \frac{4}{\pi} \int_0^{\beta} Kx \sin(\omega t) d(\omega t) + \frac{4}{\pi} \int_{\beta}^{\pi/2} KS \sin(\omega t) d(\omega t)$$

On substituting for $x(t) = A\sin(\omega t)$

$$a_1 = \frac{4KA}{\pi} \int_0^{\beta} sin^2(\omega t) d(\omega t) + \frac{4KS}{\pi} \int_{\beta}^{\pi/2} \sin(\omega t) d(\omega t)$$

$$a_1 = \frac{4KA}{\pi}\int_0^{\beta}\frac{1 - cos2\ \omega t}{2}d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\pi/2}\sin(\omega t)d(\omega t)$$

$$a_1 = \frac{2KA}{\pi}\left[\omega t - \frac{sin2\omega t}{2}\right]_0^{\beta} + \frac{4KS}{\pi}[-cos\omega t]_{\beta}^{\pi/2}$$

$$= \frac{2KA}{\pi}\left[\beta - \frac{sin2\beta}{2}\right] + \frac{4KS}{\pi}\left[-cos\,{}^{\pi}/_{2} + cos\beta\right]$$

$$= \frac{2KA}{\pi}\left[\beta - \frac{sin2\beta}{2}\right] + \frac{4KS}{\pi}[cos\beta]$$

From the I/O relation graph, when $\omega t = \beta, x = S$. Therefor we can write the input equation as

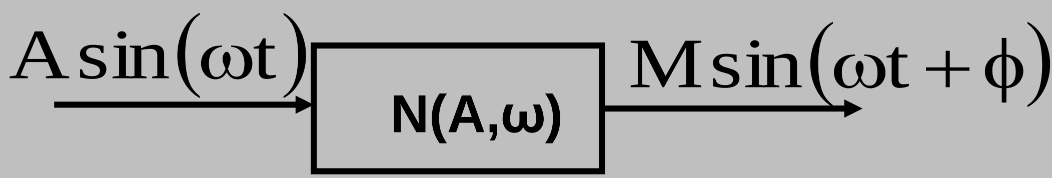
$$S = A\ sin\beta \qquad \frac{S}{A} = sin\beta \qquad \beta{=}sin^{-1}(S/A)$$

$$a_1 = \frac{2KA}{\pi}\left[\beta - \frac{sin2\beta}{2}\right] + \frac{4K}{\pi}A\ sin\beta cos\beta$$

$$a_1 = \frac{2KA}{\pi}\left[\beta - \frac{2sin\beta cos\beta}{2}\right] + \frac{4KA}{\pi}\ sin\beta cos\beta$$

$$= \frac{2KA}{\pi}[\beta - sin\beta cos\beta + 2sin\beta cos\beta]$$

$$= \frac{2KA}{\pi}[\beta + sin\beta\ cos\beta]$$



$$M(A,\omega) = \sqrt{a_1^2 + b_1^2}$$

$$and \quad \varphi(A,\omega) = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

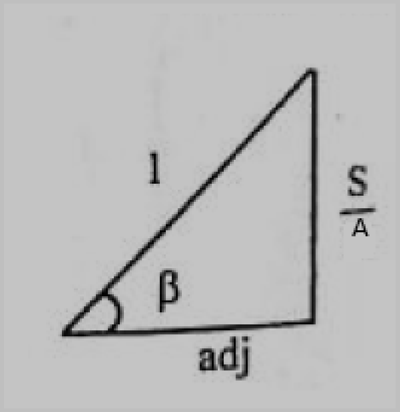
$M = \frac{2KA}{\pi}[\beta + sin\beta\ cos\beta]$ and $\varphi = 0$

The describing function will be, $$N(A,\omega) = \frac{Me^{i(\omega t+\phi)}}{Ae^{i\omega t}} = \frac{M}{A}e^{i\phi}$$

$$N(A,\omega) = \frac{2K}{\pi}[\beta + sin\beta\ cos\beta]$$

Describing function can also be written in another form with relation,

$$S = A\ sin\beta \qquad \frac{S}{A} = sin\beta \qquad \beta = sin^{-1}(S/A)$$



$$cos\beta = \sqrt{1 - \left(\frac{S}{A}\right)^2})$$

$$N(A,\omega) = \frac{2K}{\pi}[sin^{-1}(s/A) + \frac{S}{A}\sqrt{1 - \left(\frac{S}{A}\right)^2}]$$

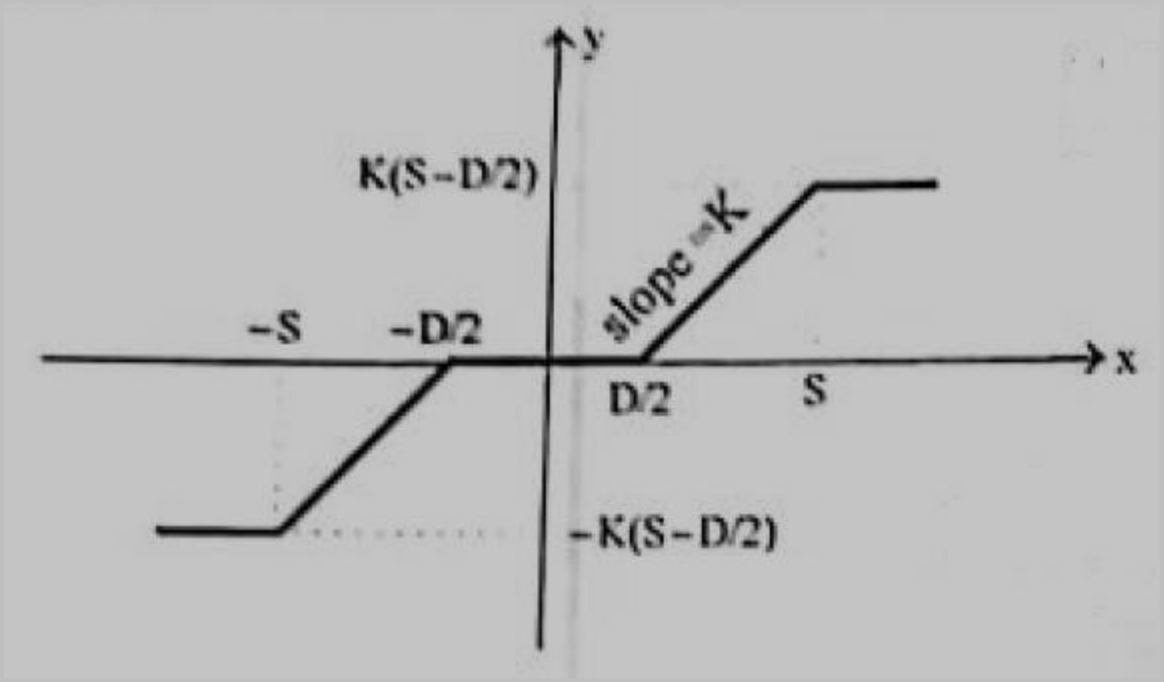
Depending upon the maximum values of input the describing function can be written as,

$$If\ A < S\ then\ \ \beta = \frac{\pi}{2} and N(A, \omega) = K$$

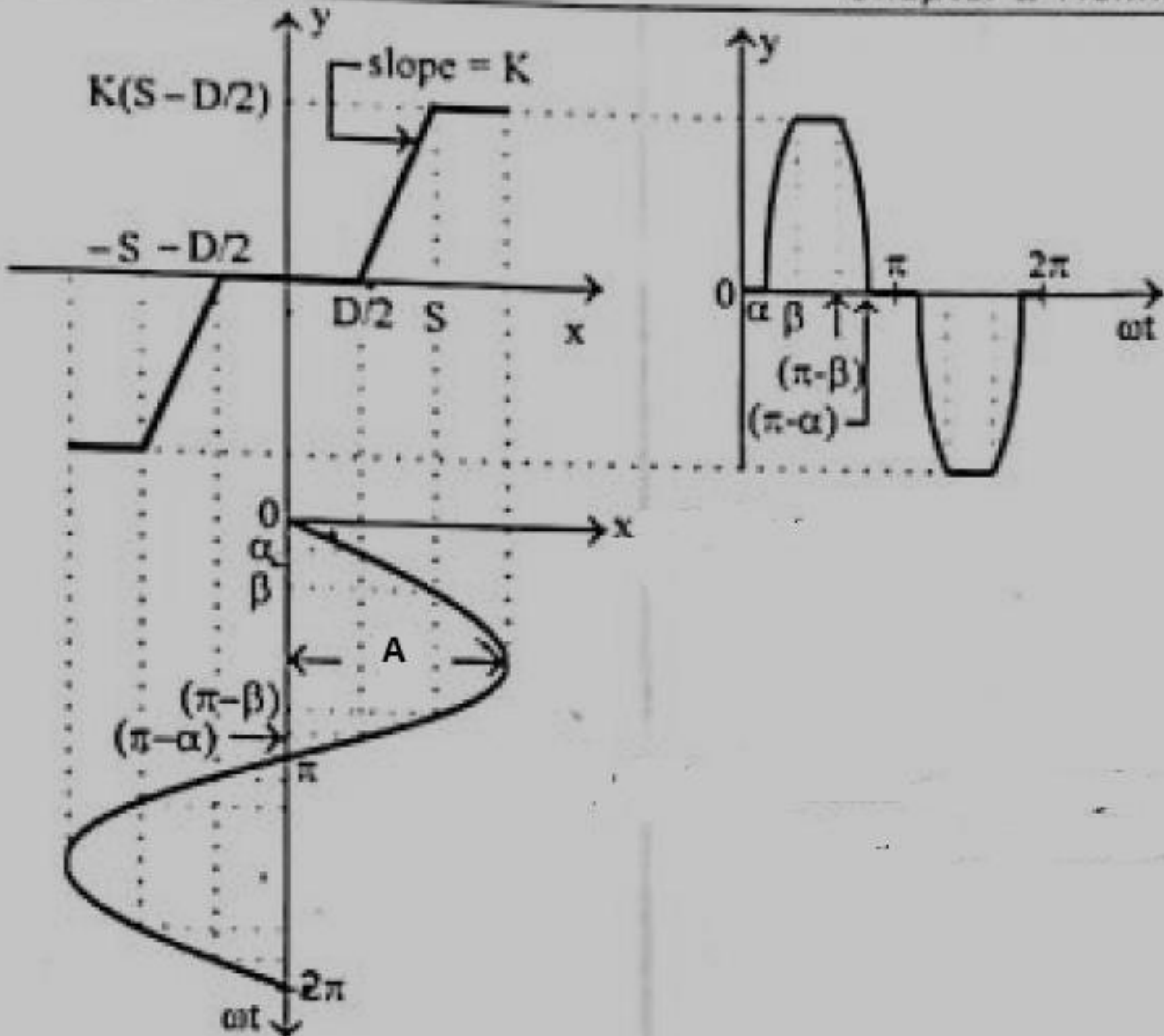
$$If\ A > S\ then\ \ \beta = \frac{\pi}{2} and\ N(A, \omega) = \frac{2K}{\pi}[\beta + sin\beta\ cos\beta]$$

Describing Functions of Dead zone and Saturation Nonlinearity

The input-outp4ut relationship for a dead zone and saturation nonlinearity is shown below, with D, *S* and *k* denoting the range and slope of the linearity and dead zone.



Consider we apply a sinusoidal input to the nonlinearity, $x(t) = A\sin(\omega t)$


y
slope = K
K(S – D/2)
–S – D/2
D/2
S
x
0
α
β
A
(π–β)
(π–α)
π
2π
ωt
y
0
α
β
(π-β)
(π-α)
π
2π
ωt

The output of the nonlinearity can be divided into five regions in a period of $0 \text{ to } \pi$. The output equation of the five segments are given by,

$$y = \begin{cases} 0, & 0 \le \omega t \le \alpha \\ K\left(x - \frac{D}{2}\right), & \alpha \le \omega t \le \beta \\ K\left(S - \frac{D}{2}\right), & \beta \le \omega t \le (\pi - \beta) \\ K\left(x - \frac{D}{2}\right), & (\pi - \beta) \le \omega t \le (\pi - \alpha) \\ 0, & (\pi - \alpha) \le \omega t \le \pi \end{cases}$$

The output has half wave and quarter wave symmetries, b_1=0 and the coefficient a_1 is

$$a_1 = \frac{2}{\pi/2} \int_0^{\pi/2} y \sin(\omega t) d(\omega t)$$

Output is given by three different expression in the period of $0 \text{ to } \pi/2$, hence the expression for a_1 can be written as,

$$a_1 = \frac{4}{\pi} \int_0^{\alpha} y \sin(\omega t) d(\omega t) + \frac{4}{\pi} \int_{\alpha}^{\beta} y \sin(\omega t) d(\omega t) + \frac{4}{\pi} \int_{\beta}^{\pi/2} y \sin(\omega t) d(\omega t)$$

Substitute for $x(t) = A \sin(\omega t)$

$$a_1 = \frac{4K}{\pi}\left[\int_{\alpha}^{\beta}\left(A\ sin\omega t - \frac{D}{2}\right)\sin(\omega t)d(\omega t\right) + \int_{\beta}^{\pi/2}\left(S - \frac{D}{2}\right)\sin(\omega t)d(\omega t\Big)\right]$$

$$= \frac{4K}{\pi}\left[\left(\int_{\alpha}^{\beta} A\ \ sin^2(\omega t)d(\omega t) - \int_{\alpha}^{\beta}\frac{D}{2}\sin(\omega t)d(\omega t\right) + \int_{\beta}^{\pi/2}\left(S - \frac{D}{2}\right)\sin(\omega t)d(\omega t\Big)\right]$$

$$= \frac{4K}{\pi}\left[\frac{A}{2}\int_{\alpha}^{\beta}(1 - cos2\omega t\ )d(\omega t) - \frac{D}{2}\int_{\alpha}^{\beta}\sin(\omega t)d(\omega t\Big) + \left(S - \frac{D}{2}\right)\int_{\beta}^{\pi/2}\sin(\omega t)d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{A}{2}\left(\beta - \alpha - \frac{\sin 2\beta}{2} + \frac{\sin 2\,\alpha}{2}\right) - \frac{D}{2}\cos\alpha + S\cos\beta\right]$$

From the I/O relation graph, when $\omega t = \alpha, x = \frac{D}{2}$. Therefor we can write the input equation as

$$\frac{D}{2} = A\sin\alpha \qquad \frac{D}{2A} = \sin\alpha \qquad \alpha{=}sin^{-1}(\frac{D}{2A})$$

From the I/O relation graph, when $\omega t = \beta, x = S$. Therefor we can write the input equation as

$$S = A\, sin\beta \qquad \frac{S}{A} = sin\beta \qquad \beta = sin^{-1}(S/A)$$

Use the above results to replace D/2 and S in the above equation for a1.

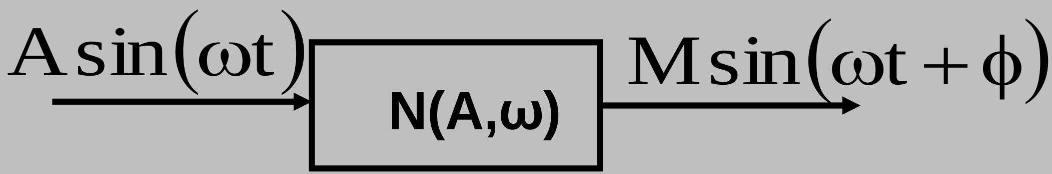
$$a_1 = \frac{4K}{\pi}\left[\frac{A}{2}\left(\beta - \alpha - \frac{\sin 2\beta}{2} + \frac{\sin 2\,\alpha}{2}\right) - A\sin\alpha\cos\alpha + A\, sin\beta\ \cos\beta\right]$$

$$= \frac{4KA}{\pi}\left[\frac{\beta}{2} - \frac{\alpha}{2} - \frac{\sin 2\beta}{4} + \frac{\sin 2\,\alpha}{4} - \frac{\sin 2\alpha}{2} + \frac{\sin 2\beta}{2}\right]$$

$$= \frac{4KA}{\pi}\left[\frac{1}{2}(\beta - \alpha) + \frac{\sin 2\beta}{4} - \frac{\sin 2\,\alpha}{4}\right]$$

$$= \frac{2KA}{\pi}\left[(\beta - \alpha) + \frac{\sin 2\beta}{2} - \frac{\sin 2\,\alpha}{2}\right]$$

$$= \frac{KA}{\pi}[2(\beta - \alpha) + \sin 2\beta - \sin 2\,\alpha]$$



$$M(A,\omega) = \sqrt{a_1^2 + b_1^2}$$

$$and \quad \varphi(A,\omega) = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

$$M = \frac{KA}{\pi}[2(\beta - \alpha) + \sin 2\beta - \sin 2\,\alpha] \text{ and } \varphi = 0$$

The describing function will be, $$N(A,\omega) = \frac{Me^{i(\omega t + \phi)}}{Ae^{i\omega t}} = \frac{M}{A}e^{i\phi}$$

$$N(A,\omega) = \frac{K}{\pi}[2(\beta - \alpha) + \sin 2\beta - \sin 2\,\alpha] \text{ and } \varphi = 0$$

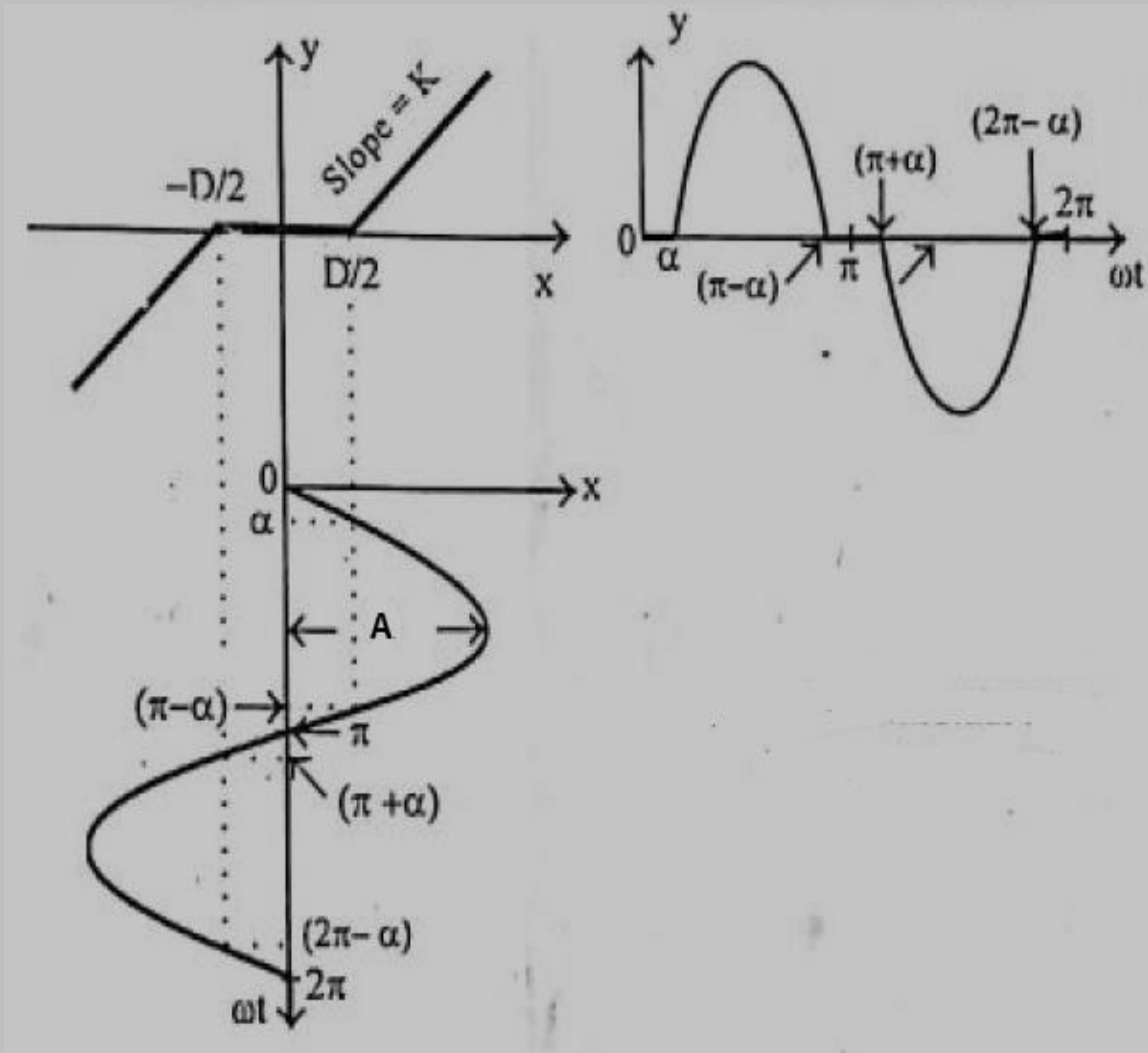
Depending upon the maximum values of input the describing function can be written as,

$$If\ A < \frac{D}{2} then\ \ \alpha = \beta = \frac{\pi}{2}\ and\ N(A,\omega) = 0$$

$$If\ \frac{D}{2} < A < S\ then\ \ \beta = \frac{\pi}{2}\ and\ N(A,\omega) = K\left[1 - \frac{2}{\pi}(\alpha + sin\alpha\ cos\alpha)\right]$$

$$If\ A > S\ then\ \ N(A,\omega) = \frac{K}{\pi}[2(\beta - \alpha) + \sin 2\beta - \sin 2\,\alpha]$$

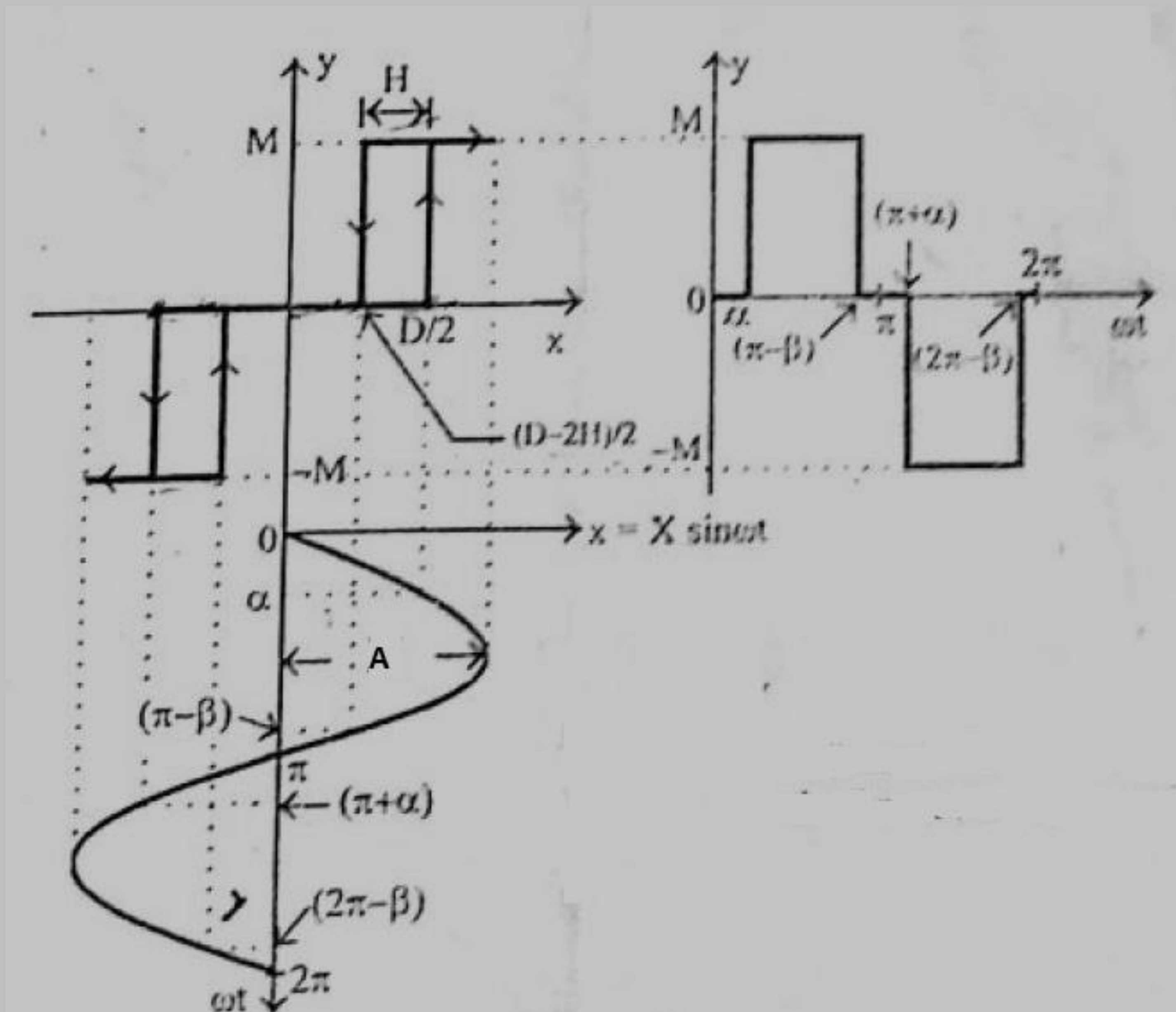
Describing Functions of Dead zone Nonlinearity



$$y = \begin{cases} 0, & 0 \le \omega t \le \alpha \\ K\left(x - \frac{D}{2}\right), & \alpha \le \omega t \le (\pi - \alpha) \\ 0, & (\pi - \alpha) \le \omega t \le \pi \end{cases}$$

$$N(A,\omega) = K\left[1 - \frac{2}{\pi}(\alpha + sin\alpha\ cos\alpha)\right] \text{and } \varphi = 0$$

Describing Functions of Relay with Dead zone and Hysteresis Nonlinearity



$$y = \begin{cases} 0, & 0 \leq \omega t \leq \alpha \\ M, & \alpha \leq \omega t \leq (\pi - \beta) \\ 0, & (\pi - \beta) \leq \omega t \leq (\pi + \alpha) \\ -M, & (\pi + \alpha) \leq \omega t \leq (2\pi - \beta) \\ 0, & (2\pi - \beta) \leq \omega t \leq 2\pi \end{cases}$$

$$a_1 = \frac{2M}{\pi}[\cos\alpha + \cos\beta] \qquad b_1 = \frac{2M}{\pi}[\sin\beta - \sin\alpha]$$

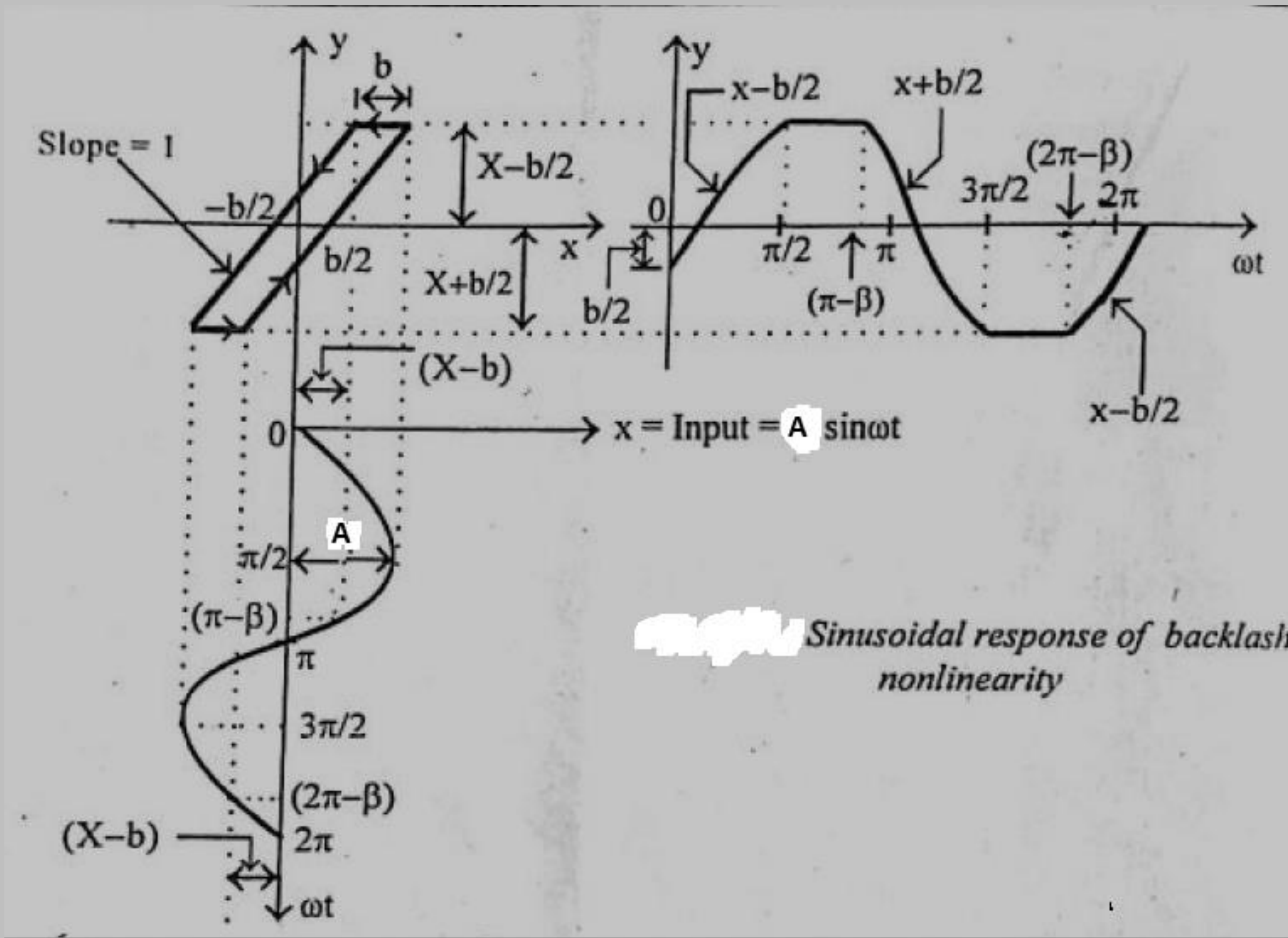
$$M = \left[\frac{4M^2}{\pi^2}\{[\cos\alpha + \cos\beta]^2 + [\sin\beta - \sin\alpha]^2\}\right]^{1/2}$$

and $\varphi = tan^{-1}\left[\frac{[\sin\beta - \sin\alpha]}{[\cos\alpha + \cos\beta]}\right]$

The describing function will be, $$N(A,\omega) = \frac{Me^{i(\omega t + \phi)}}{Ae^{i\omega t}} = \frac{M}{A}e^{i\phi}$$

Ideal Relay $$N(A,\omega) = \frac{2M}{\pi A}$$

Describing Functions of Backlash Nonlinearity



Sinusoidal response of backlash nonlinearity

$$y = \begin{cases} x - b/2, & 0 \le \omega t \le \pi/2 \\ A - b/2, & \pi/2 \le \omega t \le (\pi - \beta) \\ x + b/2, & (\pi - \beta) \le \omega t \le 3\pi/2 \\ -A + b/2, & 3\pi/2 \le \omega t \le (2\pi - \beta) \\ x - b/2, & (2\pi - \beta) \le \omega t \le 2\pi \end{cases}$$

$$a_1 = \frac{A}{\pi}\left[\frac{\pi}{2} + \beta + \frac{1}{2} sin\, 2\beta\right] \qquad b_1 = \frac{-A}{\pi}\ cos^2\, \beta$$

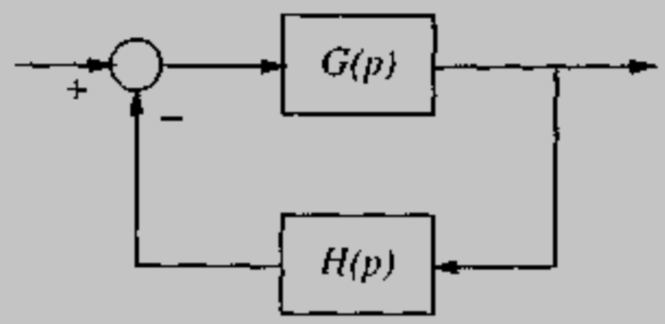
$$M = \frac{A}{\pi}\left[\left(\frac{\pi}{2} + \beta + \frac{1}{2} sin\, 2\beta^2\right)^2 + cos^4\, \beta\ \right]^{1/2}$$

and $\varphi = tan^{-1}\left[\dfrac{\frac{-A}{\pi}\ cos^2\, \beta}{\frac{A}{\pi}\left[\frac{\pi}{2} + \beta + \frac{1}{2} sin\, 2\beta\right]}\right]$

The describing function will be, $N(A, \omega) = \dfrac{Me^{i(\omega t + \phi)}}{Ae^{i\omega t}} = \dfrac{M}{A} e^{i\phi}$

Describing Function Analysis of Nonlinear Systems

The Nyquist Criterion and Its Extension



Consider the linear system in the above Figure. The characteristic equation of this system is

$$\delta(p) = 1 + G(p)\,H(p) = 0$$

Note that it is, often called the *loop transfer function*, is a rational function of *p*, with its zeros being the poles of the closed-loop system, and its poles being the poles of the open-loop transfer function *G(p) H(p).*

Let us rewrite the characteristic equation as

$$G(p)\,H(p) = -1$$

Nyquist criterion procedure can be summarised as,

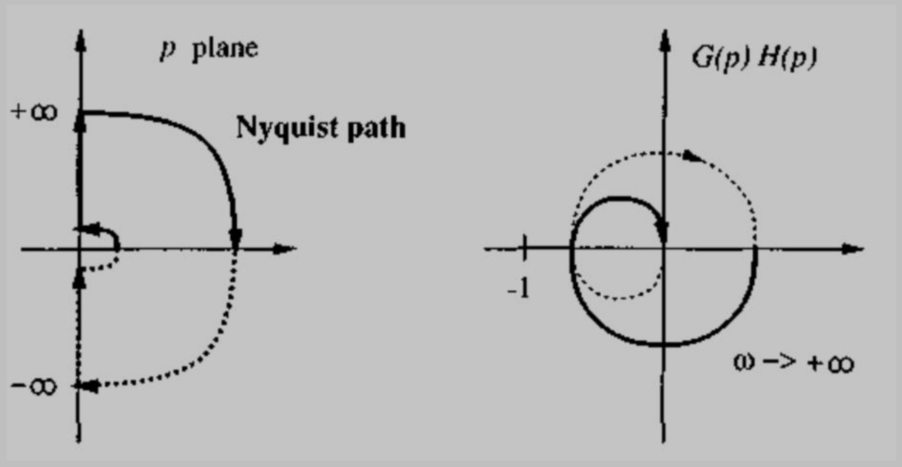
1. draw, in the *p* plane, a so-called Nyquist path enclosing the right-half plane

2. map this path into another complex plane through *G(p)H(p)*

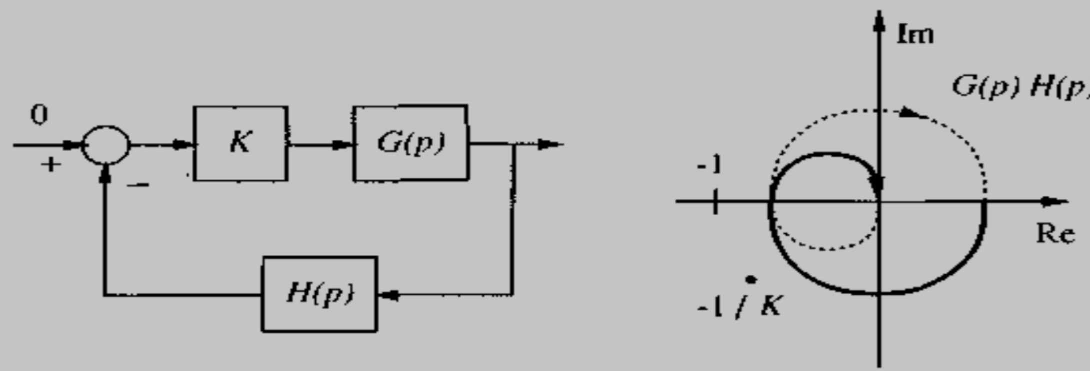
3. determine N, the number of clockwise encirclements of the plot of *G(p)H(p)* around the point (- 1,0)

4. compute Z, the number of zeros of the loop transfer function in the right-half *p* plane, by Z = *N* + *P* , where *P* is the number of unstable poles of loop transfer function

Then the value of Z is the number of unstable poles of the closed-loop system.



A simple formal extension of the Nyquist criterion can be made to the case when a constant gain *K* (possibly a complex number) is included in the forward path in Figure below.
This modification will be useful in interpreting the stability analysis of limit cycles using the describing function method.



The loop transfer function becomes

$$\delta(p) = 1 + K\,G(p)\,H(p)$$

with the corresponding characteristic equation

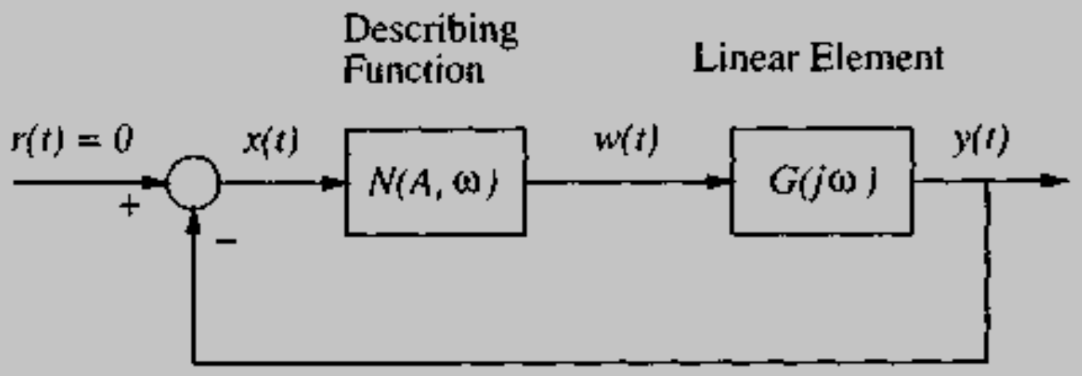
$$G(p)\,H(p) = -\,1/K$$

The same arguments as used in the derivation of Nyquist criterion suggest the same procedure for determining unstable closed-loop poles, with the minor difference that now Z *represents the number of clockwise encirclements of the G(p) H(p) plot around the point - l/K*

Existence of Limit Cycles

Let us now assume that there exists a self-sustained oscillation of amplitude *A* and frequency omega in the system.

Then the variables in the loop must satisfy the following relations:



$$x = -y$$

$$w = N(A,\omega)x$$

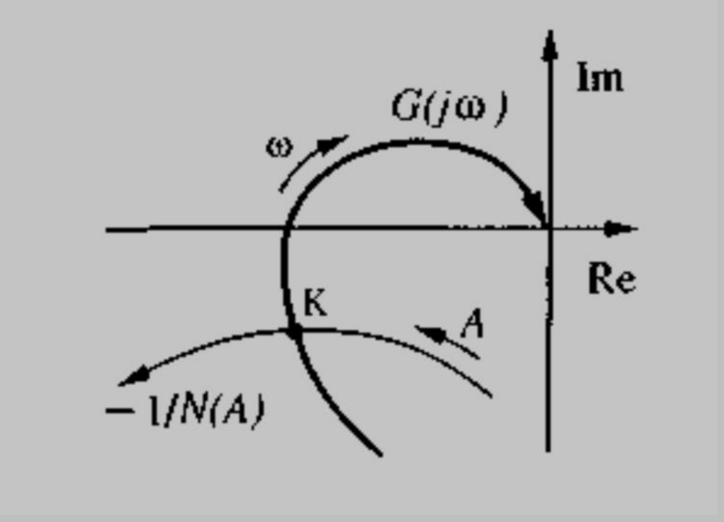
$$y = G(j\omega)w$$

$$G(j\omega)N(A,\omega) + 1 = 0$$

$$G(j\omega) = -\frac{1}{N(A,\omega)}$$

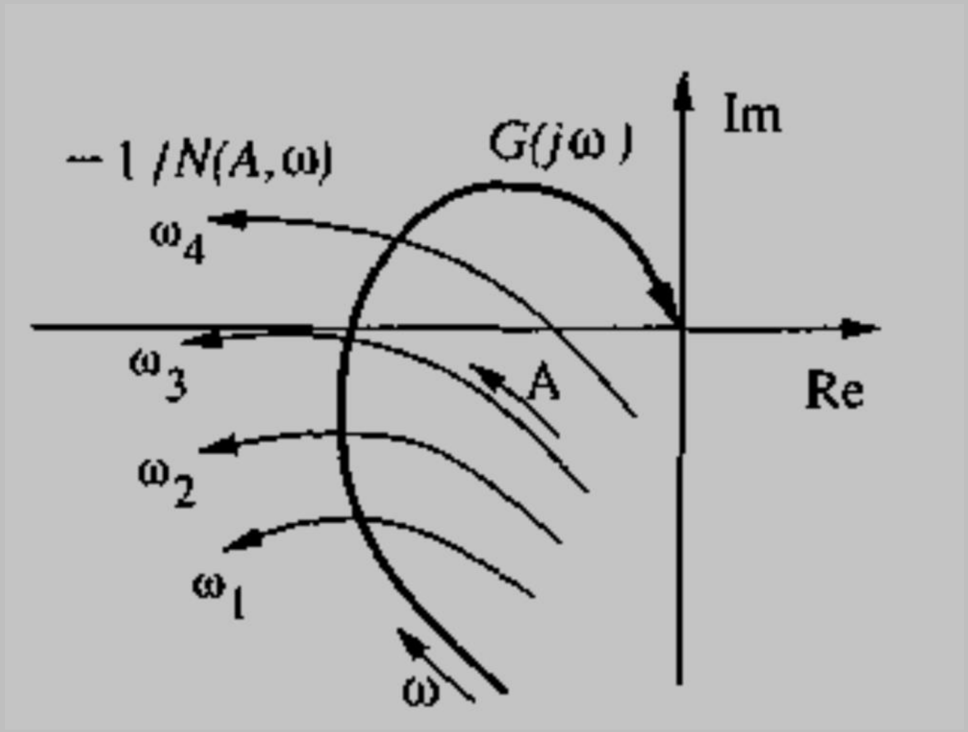
FREQUENCY-INDEPENDENT DESCRIBING FUNCTION

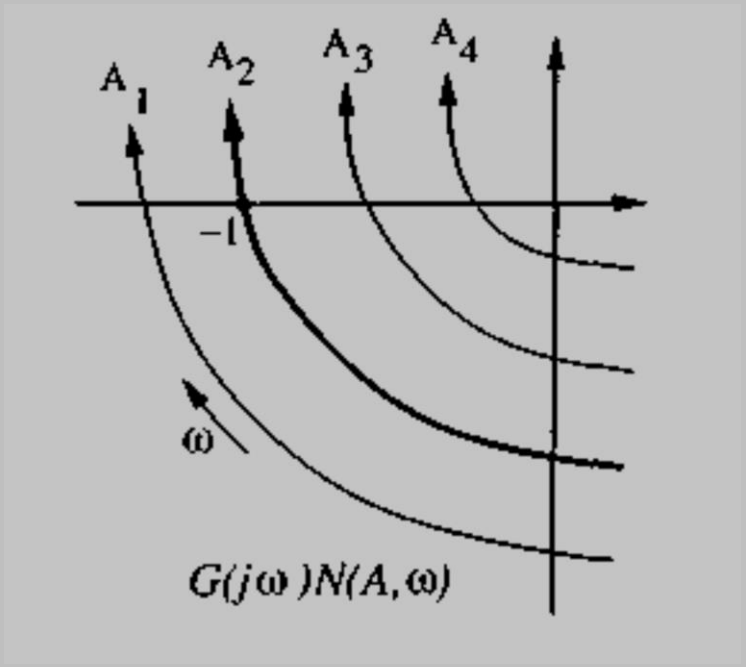
$$G(j\omega) = -\frac{1}{N(A)}$$



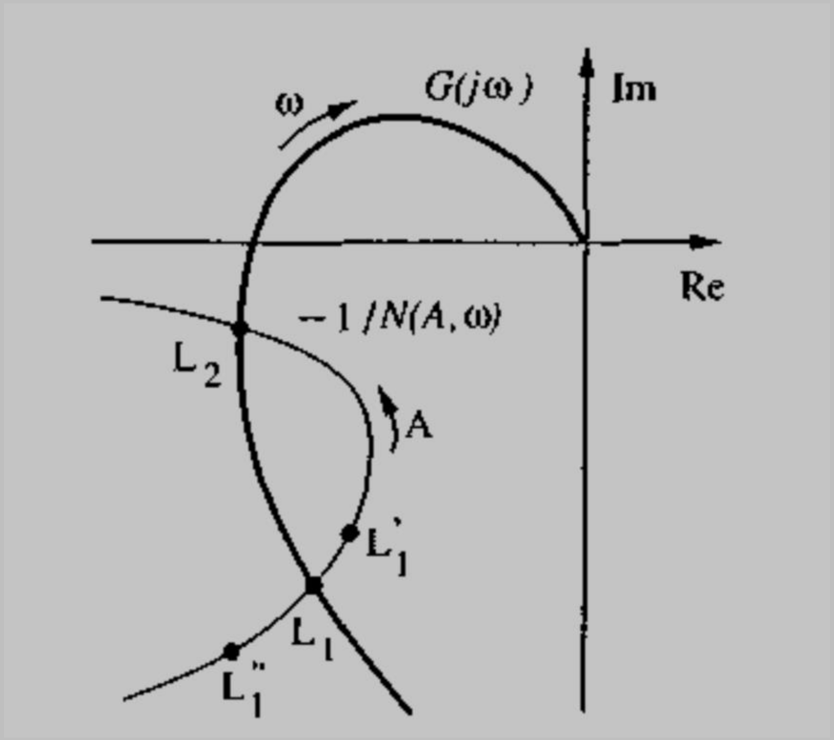
Note that for single-valued nonlinearities, *N* is real and therefore the plot of - *l/N* always lies on the real axis.

FREQUENCY-DEPENDENT DESCRIBING FUNCTION





Stability of Limit Cycles



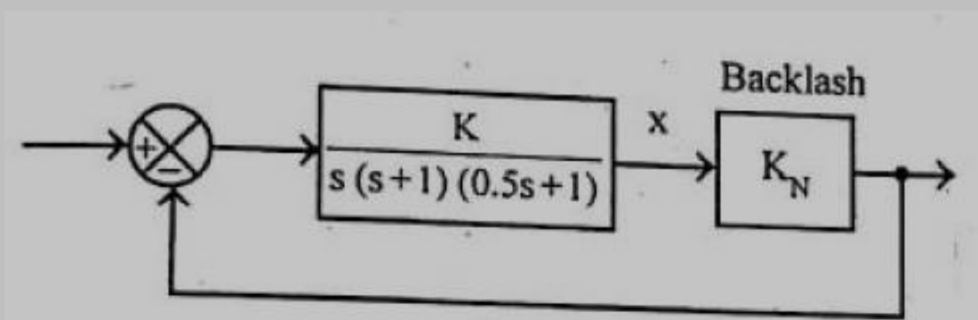
Each intersection point of the curve G(jw) and the curve -l/N(A) corresponds to a limit cycle. If points near the intersection and along the increasing A side of the curve -1/N(A) are not encircled by the curve G(jw) , then the corresponding limit cycle is stable. Otherwise, the limit cycle is unstable.

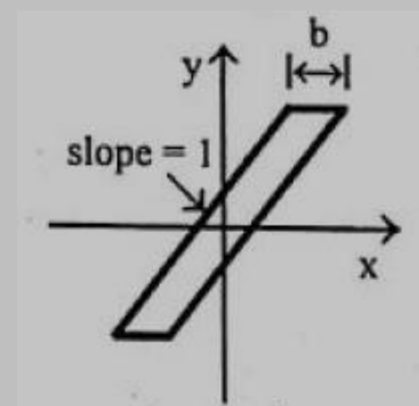
Tutorial -7

Question:1

Example 2.2 , Advanced Control Theory , Nagoor Kani

A servo system is used for positioning a load has backlash characteristics as shown in below diagram. The magnitude and phase of the describing function for various values of b/X are listed in table below, Show that the system is stable if K=1. Also show that the limit cycle exists when K=2. Find the stability of these limit cycles and determine their frequency and b/X.





b/X	0	0.2	0.4	1.0	1.4	1.6	1.8	1.9	2.0
$\|K_N\|$	1	0.954	0.882	0.592	0.367	0.248	0.125	0.064	0
$\angle K_N$	0	−6.7°	−13.4°	−32.5°	−46.6°	−55.2°	−66°	−69.8°	−90°

$G(j\omega)$ when $K=1$

$$G(s) = \frac{K}{s\,(1+s)\,(1+0.5s)}$$

Let $K = 1$ and $s = j\omega$

$$\therefore G(j\omega) = \frac{1}{j\omega(1+j\omega)\,(1+j0.5\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\,\sqrt{1+0.25\omega^2}}$$

$$\angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}0.5\omega$$

ω rad/sec	0.1	0.15	0.2	0.25	0.5	0.75	1.0	1.25
$\lvert G(j\omega)\rvert$	9.94	6.57	4.88	3.85	1.74	1.0	0.63	0.42
$\angle G(j\omega)$ deg.	−99	−103	−107	−111	−131	−147	−162	−173
$G_R(j\omega)$	−1.6	−1.5	−1.4	−1.4	−1.1	−0.8	−0.6	−0.4
$G_I(j\omega)$	−9.8	−6.4	−4.7	−3.6	−1.3	−0.5	−0.2	−0.05

$G(j\omega)$ when $K=2$

$$|G(j\omega)| = \frac{2}{\omega\,\sqrt{1+\omega^2}\,\sqrt{1+0.25\omega^2}}$$

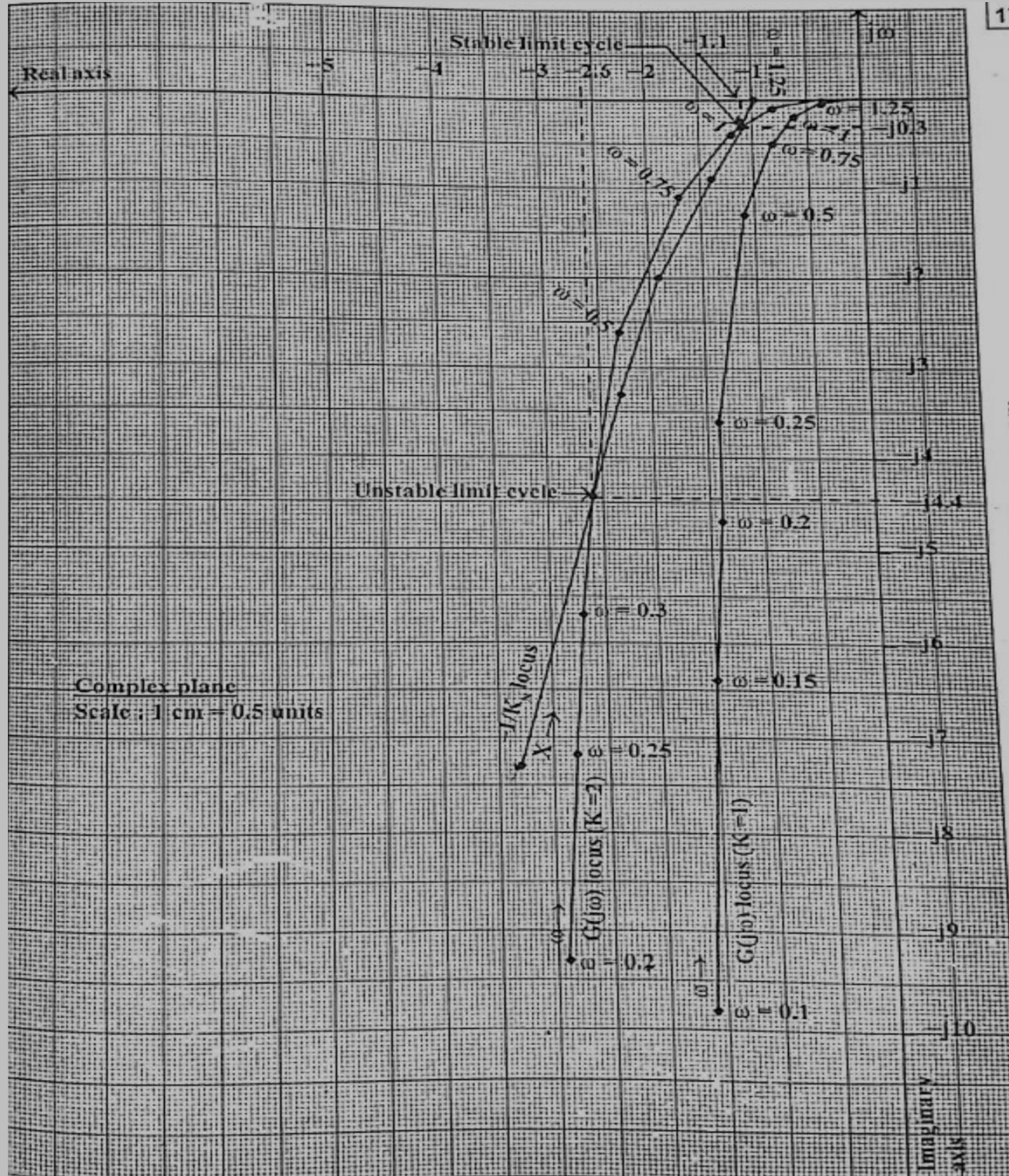
ω rad/sec	0.2	0.25	0.3	0.5	0.75	1.0	1.25
$\lvert G(j\omega)\rvert$	9.76	7.7	6.31	3.48	2.0	1.26	0.84
$\angle G(j\omega)$ deg	−107	−111	−115	−131	−147	−162	−173
$G_R(j\omega)$	−2.9	−2.8	−2.7	−2.3	−1:7	−1.2	−0.8
$G_I(j\omega)$	−9.3	−7.2	−5.7	−2.6	−1.1	−0.4	−0.1

Polar plot of $-1/K_N$

$$-1/K_N = -1 \times \frac{1}{K_N} = 1\angle -180 \times \frac{1}{|K_N|\angle K_N}$$

$$\therefore \; |-1/K_N| = \frac{1}{|K_N|} \qquad \angle(-1/K_N) = -180^\circ - \angle K_N$$

b/X	0	0.2	0.4	1.0	1.4	1.6	1.8	1.9	2.0
$\lvert K_N \rvert$	1	0.954	0.882	0.592	0.367	0.248	0.125	0.064	0
$\angle K_N$	0	−6.7°	−13.4°	−32.5°	−46.6°	−55.2°	−66°	−69.8°	−90°
$\lvert -1/K_N \rvert$	1	1.05	1.13	1.69	2.72	4.03	8.0	15.63	∞
$\angle(-1/K_N)$	−180°	−173°	−166°	−148°	−133°	−125°	−114°	−110°	−90°
Real part of $-1/K_N$	−1.0	−1.04	−1.1	−1.4	−1.9	−2.3	−3.3	−5.3	0
Ima. part of $-1/K_N$	0	−0.1	−0.3	−0.9	−2.0	−3.3	−7.3	−14.7	∞


jω
Stable limit cycle
−1.1
Real axis
−5
−4
−3
−2.5
−2
−1
ω = 1.25
ω = 1.25
−j0.3
ω = 1
ω = 1
ω = 0.75
ω = 0.75
−j1
ω = 0.5
−j2
ω = 0.5
−j3
ω = 0.25
−j4
Unstable limit cycle
−j4.4
ω = 0.2
−j5
ω = 0.3
−j6
ω = 0.15
Complex plane
Scale : 1 cm = 0.5 units
−1/K_N locus
X
−j7
ω = 0.25
G(jω) locus (K=2)
G(jω) locus (K=1)
−j8
−j9
ω = 0.2
ω = 0.1
−j10
Imaginary axis

Coordinates corresponding to stable limit cycle is $-1.1 + j0.3$.

$$-1.1 + j0.3 = 1.14\angle -165^0$$

$$\angle G(j\omega) = -90^0 - tan^{-1}\omega - tan^{-1}0.5\omega = -165$$

$$tan^{-1}\omega + tan^{-1}0.5\omega = 75$$

Take tan on bot sides, $tan(tan^{-1}\omega + tan^{-1}0.5\omega) = \tan 75$

Expand using tan (A+B) $\dfrac{\omega + 0.5\omega}{1 - \omega \times 0.5\omega} = 3.732$ $\omega = 1.07\ rad/sec$

$$-\frac{1}{K_N} = 0.3 = 1.14\angle -165^0 \quad but\ -\frac{1}{K_N} = 1\angle -180^0 \times \frac{1}{K_N} \quad K_N = 0.877\angle -15^0$$

From the DF of backlash nonlinearity, $\angle K_N = tan^{-1}\left[\dfrac{cos^2\beta}{\left[\frac{\pi}{2} + \beta + \frac{1}{2}sin\,2\beta\right]}\right] = -15$

Take tan on bot sides, $\left[\dfrac{cos^2\beta}{\left[\frac{\pi}{2} + \beta + \frac{1}{2}sin\,2\beta\right]}\right] = \tan -15$

$$\left[\frac{\pi}{2} + \beta + \frac{1}{2}sin\,2\beta\right] = 3.732cos^2\beta$$

$$[K_N] = \frac{1}{\pi}\left[\left(\frac{\pi}{2} + \beta + \frac{1}{2} sin\, 2\beta \quad\right)^2 + cos^4\, \beta\ \right]^{1/2} = 0.877$$

$$\frac{1}{\pi}[(3.732 cos^2\, \beta\ )^2 + cos^4\, \beta\ ]^{1/2} = 0.877$$

$$[(3.732 cos^2\, \beta\ )^2 + cos^4\, \beta\ ]^{1/2} = \pi *\ 0.877$$

$$\beta = 32.4\ degree$$

$$\beta = sin^{-1}(1 - \frac{b}{X}) \qquad \frac{b}{X} = 0.464$$

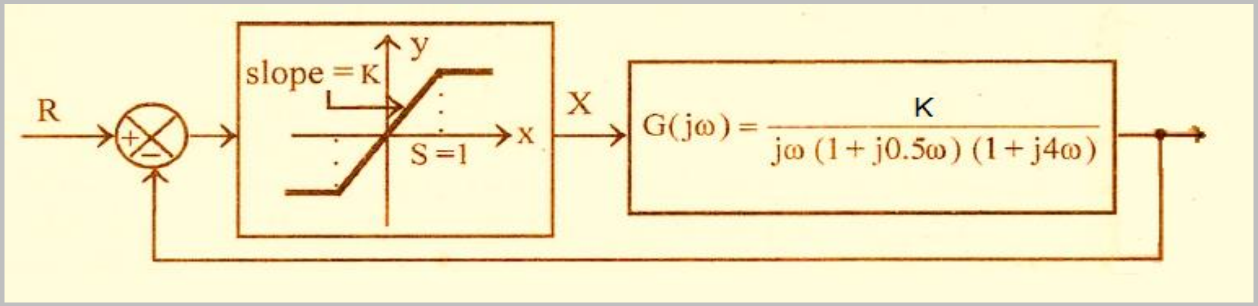
1. The unstable limit cycle exist when $b/X = 0.316$ and the frequency of oscillation is 0.36 rad/sec.
2. The stable limit cycle exist when $b/X = 0.464$ and the frequency of oscillation is 1.07 rad/sec.

Tutorial -8

Question:2

Example 2.3 , Advanced Control Theory , Nagoor Kani

The unity feedback system shown have saturating amplifier with gain K. Determine the maximum value of K for system to stay stable. What would be the frequency and nature of limit cycle for gain of K=2.5?



$$G(j\omega) = \frac{K}{j\omega(1 + j0.5\omega)(1 + j4\omega)}$$

$G(j\omega) when\ K = 1$:

$$G(j\omega) = \frac{1}{j\omega(1 + j0.5\omega)(1 + j4\omega)}$$

$$= \frac{1}{\omega\angle 90^0\sqrt{1 + 0.25\omega^2}\angle tan^{-1}0.5\omega\sqrt{1 + 16\omega^2}\angle tan^{-1}4\omega}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1 + 0.25\omega^2}\ \sqrt{1 + 16\omega^2}}$$

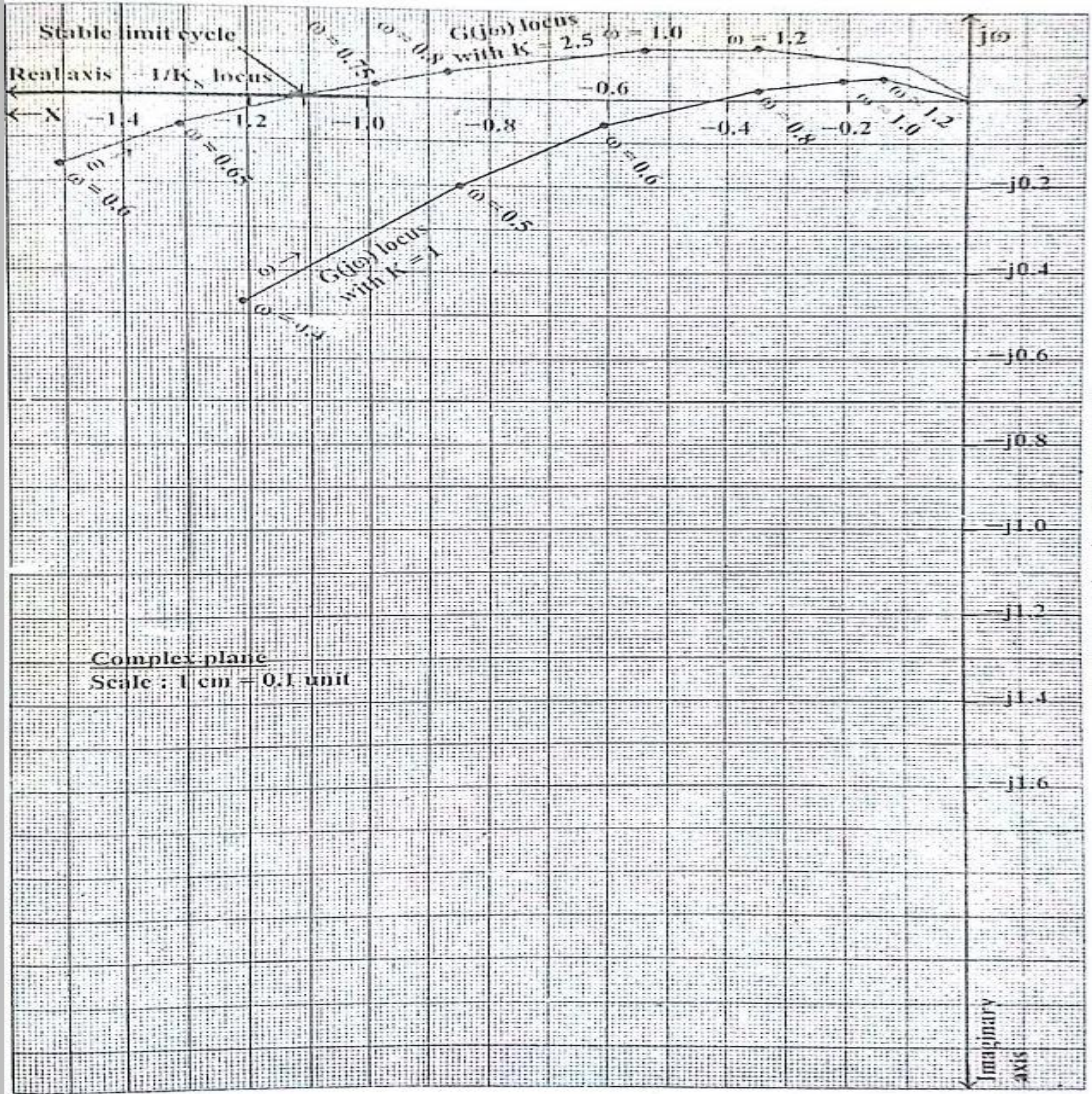
$$\angle G(j\omega) = -90^0 - tan^{-1}0.5\omega - tan^{-1}4\omega$$

ω rad/sec	0.4	0.5	0.6	0.8	1.0	1.2
$\|G(j\omega)\|$	1.299	0.868	0.614	0.346	0.216	0.145
$\angle G(j\omega)$	-159^o	-167^o	-174^o	-184^o	-192^o	-199^o
$G_R(j\omega)$	−1.21	−0.85	−0.61	−0.35	−0.21	−0.14
$G_I(j\omega)$	−0.47	−0.2	−0.06	0.02	0.04	0.05

$when\ K = 2.5$: $$|G(j\omega)| = \frac{2.5}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}$$

ω rad/sec	0.6	0.65	0.75	0.8	1.0	1.2
$\|G(j\omega)\|$	1.535	1.313	0.987	0.865	0.54	0.363
$\angle G(j\omega)$	−174	−177	−182	−184	−192	−199
$G_R(j\omega)$	−1.52	−1.31	−0.99	−0.87	−0.53	−0.34
$G_I(j\omega)$	−0.16	−0.07	0.03	0.06	0.11	0.12

For single-valued nonlinearities, *N* is real and therefore the plot of - *l/N* always lies on the real axis.



$when\ K = 1$:

$G(j\omega)$ locus does not encloses the locus -1/N(A), hence the system is stable

To find maximum value of K for stability:

When $\mathrm{K}\, is\ increased\ the\ G(j\omega)$ locus shift upwards. For a particular value of K, the $G(j\omega)$ locus crosses the starting point $(ie\ -1+j0)$ of -1/N locus and this value K is The limiting value of K for stability.

If $G(j\omega)$ locus crosses the negative real axis at $-1+j0, \mathrm{then} G(j\omega) = -1 = 1\angle -180^0$

$$|G(j\omega)| = 1 \qquad\qquad \angle G(j\omega) = -180$$

$$from\ the\ angle\ equation\ we\ have\ \angle G(j\omega) = -90^0 - tan^{-1}0.5\omega - tan^{-1}4\omega = -180$$

$$tan^{-1}0.5\omega + tan^{-1}4\omega = 90$$

Take tan on bot sides, $\tan(\,tan^{-1}0.5\omega + tan^{-1}4\omega) = \tan 90$

Expand using tan (A+B) $\dfrac{0.5\omega + 4\omega}{1 - 0.5\omega \times 4\omega} = \infty$

For the above equation to be infinity denominator has to be zero $1 - 0.5\omega \times 4\omega = 0$ and $\omega = 1/\sqrt{2}$

$$|G(j\omega)| = 1 = \frac{K}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}} \qquad solving\ we\ get\ K < 2.25\ \ for\ stability$$

$when\ K = 2.5$: $G(j\omega)$ locus intersect -1/N at $-1.11 + j0$ and limit cycle exist.

Frequency of limit cycle is $\omega = \frac{1}{\sqrt{2}} = 0.707\ rad/sec$

When K = 1, the system is stable

The system remains stable if K < 2.25

When K = 2.5, a stable limit cycle occurs, whose frequency of oscillation is 0.707 rad/sec.