Beta and Gamma Functions

Gamma functions – The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad , n>0$$

Properties of gamma functions –

① $$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

Proof: We have, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$x = t^2$
$dx = 2t\, dt$

when $x=0,\ t=0$
$x=\infty,\ t=\infty$

$$= \int_0^\infty e^{-t^2} (t^2)^{n-1}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2} t^{2n-2} \cdot t\, dt$$

$$\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$$

$$\boxed{\therefore \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dt}$$

$$\int f(x)\,dx = \int f(t)\,dt$$

② $\Gamma(1) = 1$

We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = -e^{-x}\Big]_0^\infty$$

$$= 0-(-1)$$

$$= 1$$

③ Reduction formula for gamma function -

$\Gamma(n+1) = n\Gamma(n)$

Pf:
We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\Gamma(n+1) = \int_0^\infty \underset{2}{e^{-x}} \underset{1}{x^n} dx$$

$$= \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x}) n x^{n-1} dx$$

$$= (0-0) + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

$x^n e^{-x}$

$\lim_{x\to\infty} \frac{x^n}{e^x}$

Appling L' Hospital rule 'n' times

$\lim_{x\to\infty} \frac{x^n}{e^x} = \frac{n!}{e^x}$
$= \frac{n!}{\infty}$
$= 0$

$\lim_{x\to 0} \frac{x^n}{e^x} = \frac{0}{1} = 0$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\Gamma(n-2)$

$\vdots$

$\Gamma(3) = 2\Gamma(2)$

$\Gamma(2) = 1\,\Gamma(1)$

But $\Gamma(1) = 1$

$\Gamma(2) = 1$, $\Gamma(3) = 2$

$\therefore\ \Gamma(n+1) = n\,\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$\vdots$

$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$$\boxed{\Gamma(n+1) = n!}$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n) = n!}$$

$\Gamma(5) = 4! = 24$

$\Gamma(9) = 8!$

④ $\Gamma(0) = \infty$

Proof: $n\,\Gamma(n) = \Gamma(n+1)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(0) = \frac{\Gamma(0+1)}{0} = \frac{1}{0} = \infty$$

⑤ $\Gamma(1/2) = \sqrt{\pi}$

By definition, $\Gamma(1/2) = \int_0^\infty e^{-x} x^{1/2-1}\,dx$

$= \int_0^\infty e^{-x} x^{-1/2}\,dx$

Put $x = u^2$

$dx = 2u\,du$

$x=0,\ u=0$

$x=\infty,\ u=\infty$

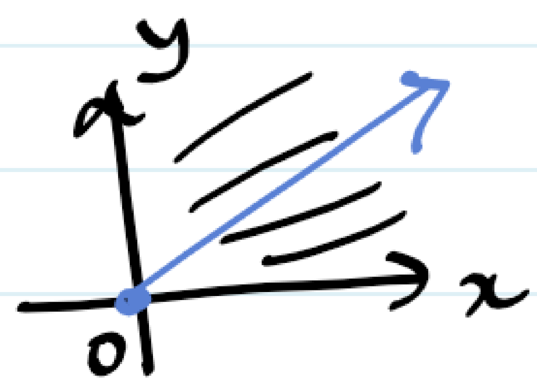
$= \int_0^\infty e^{-u^2} u^{-1}\, 2u\,du$

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\,du$$

Now $$\left[\Gamma(1/2)\right]^2 = \Gamma(1/2)\cdot\Gamma(1/2)$$

$$= 2\int_0^\infty e^{-x^2}\,dx \times 2\int_0^\infty e^{-y^2}\,dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$



Using polar co-ordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta .$$

$$= 4\int_0^{\pi/2} d\theta \times \int_{r=0}^{\infty} e^{-r^2}\,\frac{(-2)r}{(-2)}\,dr$$

$$= 4 \times \frac{\pi}{2} \times \left(\frac{-1}{2}\right)\left(e^{-r^2}\right)_0^\infty$$

$$= -\pi\,(0-1) = \pi$$

$$\left[\Gamma(1/2)\right]^2 = \pi$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

① Evaluate $\Gamma\left(\frac{5}{2}\right)$ $\qquad$ $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{3}{2}\Gamma(3/2) \quad = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2} \times \frac{1}{2}\Gamma(1/2)$$

$$= \frac{3}{4}\sqrt{\pi}$$

② $\Gamma(11/2) = \Gamma(9/2+1)$

$$= 9/2 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma(1/2)$$

$$= 9/2 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$$

③ Evaluate $\Gamma(-5/3)$ $\qquad$ $\Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-5/3) = \frac{\Gamma\left(-\frac{5}{3}+1\right)}{-5/3} = \frac{-3}{5}\Gamma(-2/3)$$

$$= \frac{-3}{5} \times \frac{\Gamma(-2/3+1)}{-2/3}$$

$$= \left(\frac{-3}{5}\right)\left(\frac{-3}{2}\right)\Gamma(1/3)$$

$$= \frac{9}{10}\Gamma(1/3)$$

Transformation of gamma function –

① We know that, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

Put $x = ky$

$dx = k\,dy$

$x = 0,\ y = 0$

$x = \infty,\ y = \infty$

$$\Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1}\, k\, dy$$

$$= \int_0^\infty e^{-ky} k^{n-1} y^{n-1}\, k\, dy$$

$$\Gamma(n) = k^n \int_0^\infty e^{-ky} y^{n-1}\, dy$$

$$\boxed{\therefore \int_0^\infty e^{-ky} y^{n-1}\, dy = \frac{\Gamma(n)}{k^n}}$$

Eg: $\int_0^\infty e^{-5x} x^8\, dx = \frac{\Gamma(9)}{5^9} = \frac{8!}{5^9}$

② We know that, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

Put $x^n = y$

$n x^{n-1}\, dx = dy$

$x = y^{1/n}$

when $x = 0,\ y = 0$

$x = \infty,\ y = \infty$

$$\Gamma(n) = \int_0^\infty e^{-y^{1/n}} \frac{dy}{n}$$

$$\boxed{\int_0^\infty e^{-y^{1/n}}\, dy = n\Gamma(n) = \Gamma(n+1)}$$

Eg: Put $n = 1/2$

$$\int_0^\infty e^{-y^2} dy = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

③ We know that, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\Gamma(n) = \int_1^0 \cancel{y} \left(\ln(1/y)\right)^{n-1} \frac{dy}{-\cancel{y}}$$

$$\boxed{\Gamma(n) = \int_0^1 \left[\ln(1/y)\right]^{n-1} dy}$$

$e^{-x} = y$

$-e^{-x} dx = dy$

$dx = \frac{dy}{-e^{-x}}$

$dx = \frac{dy}{-y}$

$e^{-x} = y$ ✓

$e^{x} = 1/y$

$x = \ln(1/y)$

when $x = 0$, $y = 1$

$x = \infty$, $y = 0$

Additional Result –

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad \text{if} \quad 0 < p < 1$$

$$\Gamma(1/4)\cdot\Gamma(3/4) = \Gamma(1/4)\Gamma(1-1/4) \qquad 0 < p = 1/4 < 1$$

$$= \frac{\pi}{\sin(\pi/4)} = (\sqrt{2})\pi$$

Problems -

① Evaluate $I = \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

$$I = \int_0^\infty (x)^{1/4} e^{-\sqrt{x}}\, dx$$

$\sqrt{x} = y$

$x = y^2$

$dx = 2y\, dy$

when $x=0,\ y=0$

$x=\infty,\ y=\infty$

$$= \int_0^\infty (y^2)^{1/4} e^{-y}\, 2y\, dy$$

$$= 2\int_0^\infty e^{-y} y^{3/2}\, dy$$

$$= 2\,\Gamma(5/2) = 2\,\Gamma(3/2+1)$$

$$= 2 \times \frac{3}{2} \times \frac{1}{2}\Gamma(1/2) = \frac{3}{2}\sqrt{\pi}$$

② Evaluate $\int_0^\infty \frac{dx}{3^{4x^2}}$

$$I = \int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{\ln(3^{-4x^2})} dx = \int_0^\infty e^{-4x^2 \ln 3} dx$$

$$= \int_0^\infty e^{-(4\ln 3)x^2} dx$$

Put $x^2 = t$

$2x\, dx = dt$

when $x=0,\ t=0$

$x=\infty,\ t=\infty$

$$= \int_0^\infty e^{-(4\ln 3)t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-(4\ln 3)t}\, t^{-1/2}\, dt$$

$$I = \frac{1}{2}\int_0^\infty e^{-(4\ln 3)t}\, t^{-1/2}\, dt$$

$$\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma(n)}{k^n}.$$

$$= \frac{1}{2}\,\frac{\Gamma(1/2)}{(4\ln 3)^{1/2}} = \frac{1}{2}\times\frac{\sqrt{\pi}}{2\sqrt{\ln 3}} = \frac{1}{4}\,\frac{\sqrt{\pi}}{\sqrt{\ln 3}}$$

③ Evaluate $\int_0^1 (x\log x)^4\, dx$

$\log x = -t$

$x = e^{-t}$

$dx = -e^{-t}dt$

when $x=0$, $t=\infty$

$x=1$, $t=0$

$$I = \int_0^1 x^4 (\log x)^4\, dx$$

$$= \int_\infty^0 (e^{-t})^4 (-t)^4 (-e^{-t})\, dt$$

$$= +\int_0^\infty e^{-5t}\, t^4\, dt = \frac{\Gamma(5)}{5^5} = \frac{4!}{5^5}$$

Beta function –

The beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \; m>0,\; n>0. \quad \to (1')$$

Eg: $\int_0^1 x^2(1-x)^{1/2}\,dx = \beta(3, 3/2)$

Properties of beta function

(1) $\beta(m,n) = \beta(n,m)$

Pf: $x = 1-y$ in (1')

$dx = -dy$

when $x=0$, $y=1$

$x=1$, $y=0$

(1') $\Rightarrow$

$$\beta(m,n) = -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

(2) $\int_0^1 x^m(1-x)^n\,dx = \beta(m+1, n+1)$

(3) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta \; d\theta$

Proof: Put $x = \sin^2\theta$ in eq (1')

$dx = 2\sin\theta\cos\theta\,d\theta$

when $x=0$, $\theta=0$

$x=1$, $\theta=\pi/2$

eqn (1')

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta\,d\theta$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$$

$2m-1=p$ $\quad$ $2n-1=q$

$m=\frac{p+1}{2}$ $\quad$ $n=\frac{q+1}{2}$

$$* \quad \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Also $\displaystyle\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$

$$\int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{p+1}{2}\right) = \frac{1}{2}\left(\frac{p+1}{2}, \frac{1}{2}\right)$$

Relation between beta and gamma functions -

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Pf: We know that $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\,dx$

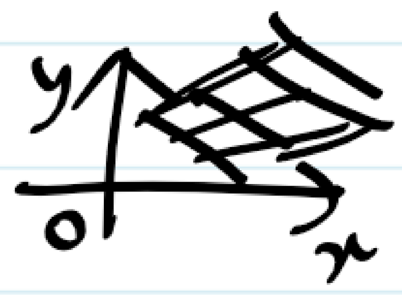
$$\checkmark \quad \Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \qquad \text{by } 1^{st} \text{ property}$$

$|||^{ly}$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \times \int_0^\infty e^{-y^2} x^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$



Using polar co-ordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\,dr\,d\theta.$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r^{2m+2n-1} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta \times 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr$$

$$\Gamma(m)\cdot\Gamma(n) = \beta(n,m) \times \Gamma(m+n)$$

$$\boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$p=0, \; q=0$

$$\int_0^{\pi/2} d\theta = \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)}$$

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma(1/2)\right)^2$$

$$\left(\Gamma(1/2)\right)^2 = \pi$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

① Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$

$x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

$dx = \dfrac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2\sqrt{\tan\theta}} \cancel{\sec^2\theta} \, d\theta}{\cancel{1+\tan^2\theta}} = \int_0^{\pi/2} \frac{1}{2}(\tan\theta)^{-1/2} d\theta$$

$x=0, \; \theta=0$

$x=\infty, \; \theta=\pi/2$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \times \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4} \times \frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1/4+3/4)} = \frac{1}{4}\cdot\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{1} = \frac{1}{4}\sqrt{2}\,\pi$$

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma(1-1/4) = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$$

Practice questions – Evaluate the following integrals

1. $\int_0^{\pi/2} \sin^8\theta \cos^4\theta \, d\theta$ (Ans: $\frac{5\pi}{32}$)

2. $\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$

3. $\int_0^\infty x^c c^{-x}\,dx$, where 'c' is constant Ans: $\frac{\Gamma(c+1)}{(\ln c)^{c+1}}$

4. $\int_0^\infty a^{-bx^2}\,dx$ (Ans: $\frac{\sqrt{\pi}}{2\sqrt{b\log a}}$)

5. $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$ (Ans: $\pi/2$)

6. $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$ (Ans: π)