(HW)

1. $\Delta^2 y_n - 3\Delta y_n + 4y_n = n + 2^n$
2. $y_{n+1} - 3y_n = 3^n(n+2)$
3. $y_{n+2} - (2\cos\frac{1}{2})y_{n+1} + y_n = \sin(\frac{n}{2})$
4. $y_{n+2} - 6y_{n+1} + 9y_n = n2^n + 3^n + 7$
5. $(E^2 - 3E + 2)y_n = n^2 + 2n - 1$

Z-Transforms

If funtion u_n is defined for discrete values $n = 0, 1, 2, 3 \ldots$ and $u_n = 0$, $n < 0$ then its z-transform is defined $Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$Z^{-1}[U(z)] = u_n.$$

Standard z-transforms:

(1) $Z(a^n) = \frac{z}{z-a}$

(2) $Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})$ [(p is +ve integer)]

[Consider $Z(n^{p-1}) = \sum n^{p-1} z^{-n}$].

$$= \sum_{n=0}^{\infty} n^p z^{-n} \rightarrow (1)$$

$$\frac{d}{dz}(Z(n^{p-1})) = \sum n^{p-1}(-n) z^{-(n+1)} \rightarrow (2)$$

$$Z(n^p) = -z \cdot \frac{d}{dz}(Z(n^{p-1})),$$

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If $a = 1$

$Z(1) = \frac{z}{z-1}$ (from (A))

from (B)

$$Z(n) = -z \cdot \frac{d}{dz} Z(1)$$

$$= -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}$$

$z(n^2) = -z \cdot \frac{d}{dz} z(n)$

$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$

$= -z\left(\frac{(z-1)^2 - z \cdot 2(z-1)}{(z-1)^4}\right)$

$= -z\left(\frac{z-1-2z}{(z-1)^3}\right)$

$z(n^2) = \frac{z+z^2}{(z-1)^3}$

Linearity property

$z(a u_n + b v_n - c w_n) = a z(u_n) + b z(v_n) + c z(w_n)$

Damping rule:

If $z(u_n) = U(z)$

then $z(a^{-n} u_n) = U(az)$

If $z(u_n) = U(z)$

then $z(a^n u_n) = U(z/a)$

$z(n) = \frac{z}{(z-1)^2}$

$z(a^{-n} n) = \frac{az}{(az-1)^2}$

$z(a^n n) = \frac{z/a}{(z/a-1)^2}$

$= \frac{az}{(z-a)^2}$

Find $z(\cos n\theta)$ & $z(\sin n\theta)$

$e^{in\theta} = \cos n\theta + i \sin n\theta$

$z(e^{in\theta}) = z(\cos n\theta + i \sin n\theta)$

$= z(\cos n\theta) + i z(\sin n\theta) \rightarrow (1)$

$z(e^{in\theta}) = z((e^{i\theta})^n \cdot 1)$

$\boxed{\because z(a^n u_n) = U(z/a)}$

$z(1) = \frac{z}{z-a}$

$= \frac{\frac{z}{e^{i\theta}}}{\frac{z}{e^{i\theta}} - 1} = \frac{z}{z - e^{i\theta}}$

$= \frac{z}{z - e^{i\theta}} \frac{(z - e^{-i\theta})}{(z - e^{-i\theta})} = \frac{z(z - (\cos\theta - i\sin\theta))}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}$

$$= \frac{z(z-\cos\theta) + i\, z\sin\theta}{z^2 - 2z\cos\theta + 1} \longrightarrow (2)$$

compare RHS of (1) & (2)

$$z(\cos n\theta) + i\, z(\sin n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + \frac{i\, z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\boxed{\begin{aligned} \therefore\quad z(\cos n\theta) &= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \\ z(\sin n\theta) &= \frac{z\sin\theta}{z^2-2z\cos\theta+1} \end{aligned}}$$

$$z(a^n \cos n\theta) = \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cos\theta + 1} = \frac{z(z-a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

$$z(a^n \sin n\theta) = \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cos\theta + 1} = \frac{\frac{z}{a}\sin\theta}{\frac{z^2 - 2az\cos\theta + a^2}{a^2}}$$

Q. Find the z transform of the following.

(i) $3n - 4\sin\frac{n\pi}{4} + 5a$

$= z\left(3n - 4\sin\frac{n\pi}{4} + 5a\right)$

$3z(n) - 4z\left(\sin\frac{n\pi}{4}\right) + 5a\,z(1)$

$$3\frac{z}{(z-1)^2} - \frac{4\, z\sin\pi/4}{z^2-2z\cos\pi/4+1} + 5a\left(\frac{z}{z-1}\right)$$

$$\frac{3z}{(z-1)^2} - \frac{4z\left(\frac{1}{\sqrt{2}}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1} + \frac{5az}{z-1}$$

(ii) $z\left((n+1)^2\right) = z(n^2+2n+1)$

$= z(n^2) + 2z(n) + z(1)$

$$= \frac{z+z^2}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1}$$

(iii) $\sin(3n+5)$

$$Z(\sin(3n+5)) = Z(\sin 3n \cos 5 + \cos 3n \sin 5)$$

$$= \cos 5\, Z(\sin 3n) + \sin 5\, Z(\cos 3n)$$

$$= \cos 5\left(\frac{z\sin 3}{z^2-2z\cos 3+1}\right) + \sin 5\left(\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right)$$

(iv) $Z(e^{an}) = Z((e^a)^n) = \frac{z}{z-e^a}$

$$Z(n\cdot e^{an}) = Z((e^a)^n \cdot n) = \frac{\frac{z}{e^a}}{\left(\frac{z}{e^a}-1\right)^2} = \frac{ze^a}{(z-e^a)^2}$$

damping rule $Z(n) = \frac{z}{(z-1)^2} \rightarrow$

$$Z(\cos hn\theta) = \frac{1}{2} Z(e^{n\theta}+e^{-n\theta})$$

$$= \frac{1}{2}\left[\left(\frac{z}{z-e^{\theta}}\right)+\left(\frac{z}{z-e^{-\theta}}\right)\right]$$

$$= \frac{1}{2}\left\{\frac{z(z-e^{-\theta}+z-e^{\theta})}{(z-e^{\theta})(z-e^{-\theta})}\right\}$$

$$= \frac{1}{2}\left\{\frac{z(2z-2\cos h\theta)}{z^2-2z\cos h\theta+1}\right\}$$

$$\boxed{Z(\cos h\theta) = \frac{z(z-\cos h\theta)}{z^2-2z\cos h\theta+1}} \quad \boxed{\cos h\theta = \frac{e^{\theta}+e^{-\theta}}{2},\ \sin h\theta = \frac{e^{\theta}-e^{-\theta}}{2}}$$

(v) $Z(\cos(\frac{n\pi}{2}+\pi/4))$

$$= Z\left\{\cos\frac{n\pi}{2}\cdot\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\left(\frac{\pi}{4}\right)\right\}$$

$$= \cos\frac{\pi}{4}\, Z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{4}\, Z\left(\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1} - \frac{1}{\sqrt{2}}\frac{z\sin(\pi/2)}{z^2-2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$= \frac{1}{\sqrt{2}}\frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}}\frac{z}{z^2+1}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right].$$

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Shifting property:

If $z(u_n) = U(z)$

then, $z(u_{n+k}) = z^k\left[U(z) - u_0 - u_1/z - u_2/z^2 - \cdots \frac{u_{k-1}}{z^{k-1}}\right]$.

Pf. $z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)} \cdot z^k = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$.

$n+k=t$.

$z(u_{n+k}) = z^k \sum_{t=k}^{\infty} u_t z^{-t}$

$= z^k \left\{ \sum_{t=0}^{\infty} u_t z^{-t} - \sum_{t=0}^{k-1} u_t z^{-t} \right\}$

$= z^k \left\{ U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)} \right\}$.

Ex: Show that $z\left(\frac{1}{n!}\right) = e^{1/z}$ and hence evaluate $z\left(\frac{1}{(n+1)!}\right)$.

Ans $z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$

$= 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$

$= e^{1/z}$ (Hence proved)

$z\left(\frac{1}{(n+1)!}\right) = z\left\{e^{1/z} - 1\right\}$

$z\left(\frac{1}{(n+2)!}\right) = z^2\left\{e^{1/z} - 1 - z^{-1}\right\}$.

Multiplication by 'n':

If $z(u_n) = U(z)$

$z(nu_n) = -z \frac{d}{dz} U(z)$

$U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

$\frac{d}{dz} U(z) = \sum_{n=0}^{\infty} u_n (-n) z^{-n-1}$

$= -z^{-1} \sum_{n=0}^{\infty} n u_n z^{-n}$

$= -z^{-1} z(nu_n)$

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Q1) Find z transform of $\left(\frac{1}{(n+1)(n+2)}\right)$

A) $= z\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= z\left(\frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+2}\right).$

$\left[\because z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots\right)$

$= -z\left(-\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3}\cdots\right)$

$= -z \log(1 - 1/z).\Big].$

$z\left(\frac{1}{n+2}\right) = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$

$= \frac{1}{2} + \frac{1}{3z} + \frac{1}{4z^2} + \cdots$

$= z^2\left[\frac{1}{2z^2} - \frac{1}{3z^3} + \cdots\right] = -z^2\left[-\frac{1}{z} - \frac{1}{2z^2} + \cdots + \frac{1}{z}\right]$

$= -z^2 \log(1 - 1/z) - z$

$= -z \log(1 - 1/z) + z^2 \log(1 - 1/z) + z$

Inverse z-Transform's:

If $z(u_n) = v(z)$ then $z^{-1}(v(z)) = u_n$

$z(a^n) = \frac{z}{z-a}$ $\quad z^{-1}\left(\frac{z}{z-a}\right) = a^n.$

$z(1) = \frac{z}{z-1}$ $\quad z^{-1}\left(\frac{z}{z-1}\right) = 1.$

If $z(u_n) = u(z)$

$z^{-1}(v(az)) = a^{-n} u_n.$

$z^{-1}(v(z/a)) = a^n u_n.$

Partial fraction method to find inverse z-transform.

ex 1) $z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$ (inverse).

$v(z) = \frac{z}{(z-1)(z-2)}$

$\frac{v(z)}{z} = \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}.$

$$U(z) = -\frac{z}{z+1} + \frac{z}{z+2}.$$

$$z^{-1}(U(z)) = -1 + 2^n$$

Q2) $z^{-1}\left(\frac{(1-e^{at})z}{(z-1)(z-e^{-at})}\right)$

$$\frac{U(z)}{z} = \frac{1-e^{at}}{(z-1)(z-e^{-at})} = \frac{A}{z-1} + \frac{B}{z-e^{-at}}$$

$\Rightarrow$ $A(z-e^{-at}) + B(z-1) = 1-e^{at}$

if $z=1$,
$$A = \frac{1-e^{at}}{1-e^{-at}}.$$

equate the coefficients of z, $A+B=0. \Rightarrow B=-A.$

$$B = -\left(\frac{1-e^{at}}{1-e^{-at}}\right).$$

Now, sub (A) & (B)

$$U(z) = \frac{1-e^{at}}{1-e^{-at}}\left\{\frac{z}{z-1} - \frac{z}{z-e^{-at}}\right\}.$$

$$z^{-1}(U(z)) = \frac{1-e^{-at}}{1-e^{-at}}\left\{1-e^{-nat}\right\}.$$

Q3) $z^{-1}\left(\frac{z^2+z}{(z+2)(z-4)}\right)$

$$\frac{U(z)}{z} = \frac{z+1}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}.$$

$$= \frac{-(z+1)}{6(z+2)} + \frac{z+1}{6(z-4)}$$

$A(z-4)+B(z+2) = z+1$

$$= -\frac{z+1}{2z+4}$$

$A = \frac{1}{6}$ $\quad B(6) = z+1$

$B = \frac{5}{6}$ $\quad \cancel{z=4} \quad B = \frac{z+1}{6}$

$A = -\frac{z+1}{2}$

$$= \frac{1}{6z+12} + \frac{5}{6z-24}$$

$$z^{-1}(U(z)) = z^{-1}\left\{\frac{z}{6(z+2)} + \frac{5}{6}\frac{z}{z-4}\right\}$$

$$= \frac{1}{6}(-2)^n + \frac{5}{6}(4)^n$$

Q4) $z^{-1}\left(\frac{8z-z^3}{(4-z)^3}\right)$ $A(4-z)^2+B(4-z)+C=8-z^2$

$\frac{U(z)}{z} = \frac{8-z^2}{(4-z)^3} \Rightarrow \frac{A}{4-z}+\frac{B}{(4-z)^2}+\frac{C}{(4-z)^3}$

$16A+4B+C=8$

$A=-1$ $B=8$ $C=-8$

$\Rightarrow \frac{-1}{4-z}+\frac{8}{(4-z)^2}-\frac{8}{(4-z)^3}$

$U(z) = \frac{z}{z-4}+8\frac{z}{(z-4)^2}+8\cdot\frac{z}{(z-4)^3}$

$= \frac{z}{z-4}+2\cdot\frac{4z}{(z-4)^2}+8\frac{z}{(z-4)^3}$

$z^{-1}[U(z)] = (+4)^n+2(n4^n)+8z^{-1}\left(\frac{z}{(z-4)^3}\right)$

Solution of difference eq using z-transform

If
$z(u_n) = U(z)$
then
$z(u_{n+k}) = z^k\left[U(z)-y_0-\frac{y_1}{z}-\frac{y_2}{z^2}+\dots+\frac{y_{k-1}}{z^{(k-1)}}\right]$

solve: $y_{n+2}-5y_{n+1}+6y_n=0$

$= z(y_{n+2})-5z(y_{n+1})+6z(y_n)=0$

$= z^2\left[z(y_n)-y_0-\frac{y_1}{z}\right]-5z(z(y_n)-y_0)+6z(y_n)=0$

$\Rightarrow (z^2-5z+6)\,z(y_n) = z^2y_0+zy_1-5zy_0$

$= (z^2-5z)y_0+zy_1$

$z(y_n) = \frac{z^2-5z}{(z-2)(z-3)}y_0+\frac{z}{(z-2)(z-3)}y_1$

23-5-22 ★★★ One Question will come (pakka)

using z-transform solve this difference equation

① $y_{n+2} - 5y_{n+1} - 6y_n = 2^n$. Given $y_0 = 0, y_1 = 1$.

$z(y_{n+2}) - 5z(y_{n+1}) - 6z(y_n) = z(2^n)$.

$z^2[z(y_n) - y_0 - y_1 z^{-1}] - 5z[z(y_n) - y_0] - 6z(y_n) = \frac{z}{z-2}$

$z(y_n) = \frac{z}{(z+1)(z-6)(z-2)} + \frac{z}{(z+1)(z-6)}$

$y_n = z^{-1}\left\{\frac{z}{(z+1)(z-6)(z-2)}\right\} + z^{-1}\left\{\frac{z}{(z+1)(z-6)}\right\}$.

$z(u_n) = U(z)$

$u_n = z^{-1}(U(z))$.

To find $z^{-1}\left\{\frac{z}{(z+1)(z-6)(z-2)}\right\}$.

$\frac{U(z)}{z} = \frac{1}{(z+1)(z-6)(z-2)}$

[How to find partial fractions in calc search in YF.]

By partial fractions.

$= \frac{A}{(z+1)} + \frac{B}{(z-6)} + \frac{C}{(z-2)}$

$\Rightarrow A(z-6)(z-2) + B(z+1)(z-2) + C(z+1)(z-6) = 1$.

$z=-1$, $A = \frac{1}{21}$. $z=2$, $C = -\frac{1}{12}$. $z=6$, $B = \frac{1}{28}$.

$\frac{U(z)}{z} = \frac{1}{21(z+1)} - \frac{1}{12(z-6)} + \frac{1}{28(z-2)}$

$U(z) = \frac{1}{21} \cdot \frac{z}{(z+1)} - \frac{1}{12} \cdot \frac{z}{(z-6)} + \frac{1}{28} \cdot \frac{z}{(z-2)}$.

$z^{-1}(U(z)) = z^{-1}(U(z))$

$= \frac{1}{21}(-1)^n + \frac{1}{12}(6)^n + \frac{1}{28}(2)^n$.

To find $z^{-1}\left(\frac{z}{(z+1)(z-6)}\right)$

$\frac{U(z)}{z} = \frac{1}{(z+1)(z-6)} = \frac{A}{(z+1)} + \frac{B}{(z-6)}$

$A(z-6) + B(z+1) = 1$

$z=-1$, $z=-6$

$A = \frac{1}{5}$ $B = -\frac{1}{5}$

$$= \frac{1}{5(z+1)} - \frac{1}{5(z-6)}$$

$$z^{-1}(u(z)) = \frac{1}{5}\cdot\frac{z}{z+1} - \frac{1}{5}\,\frac{z}{z-6}$$

$$= \frac{1}{5}(-1)^n - \frac{1}{5}(6)^n.$$

$$y_n = z^{-1}\left(\frac{z}{(z+1)(z-6)(z-2)}\right) + z^{-1}\left(\frac{z}{(z+1)(z-6)}\right)$$

$$= \frac{1}{21}(-1)^n + \frac{1}{5}(-1)^n + \frac{1}{12}(6)^n - \frac{1}{5}(6)^n + \frac{1}{28}(2)^n.$$

$$= \left(\frac{1}{21} + \frac{1}{5}\right)(-1)^n + \left(\frac{1}{12} - \frac{1}{5}\right)(6)^n + \frac{1}{28}(2)^n$$

(2) $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0 = y_1 = 0$.

$$z^2\left(z(y_n) - y_0 - y_1 z^{-1}\right) + 6z\left(z(y_n) - y_0\right) + 9\left(z(y_n)\right) = z(2^n)$$

$$(z+3)^2\, z(y_n) = \frac{z}{z-2}$$

$$z(y_n) = \frac{z}{(z-2)(z+3)^2}$$

$$y_n = z^{-1}\left\{\frac{z}{(z-2)(z+3)^2}\right\}$$

$$\frac{u(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}.$$

$$A(z+3)^2 + B(z-2) = 1.$$

$z = 2$, $z = -3$

$A = \frac{1}{25}$, $B = -\frac{1}{5}$

$$\frac{1}{25(z-2)} - \frac{1}{5}\,\frac{1}{(z+3)^2}$$

$$A(z+3)^2 + B(z-2)(z+3) + C(z-2) = 1.$$

$z = -3$, $z = 2$, $z = -3$,

$A = \frac{1}{25}$, $B = -\frac{1}{25}$, $C = -\frac{1}{5}$

$$u(z) = \frac{z}{25(z-2)} - \frac{1}{25}\,\frac{z}{(z+3)} - \frac{1}{5}\,\frac{z}{(z+3)^2}.$$

$$z^{-1}(u(z)) = z^{-1}(y_n) = \frac{1}{25}z^{-1}\left(\frac{z}{z-2}\right) - \frac{1}{25}z^{-1}\left(\frac{z}{z+3}\right) - \frac{1}{5}z^{-1}\left(\frac{z}{(z+3)^2}\right)$$

$$\left[\because Z(na^n) = \frac{az}{(z-a)^2}\right]$$

$$= \frac{1}{25}(-2)^n - \frac{1}{25}(3)^n + \frac{1}{15} z^{-1}\left(\frac{-3z}{(z+3)^2}\right) \quad \left\{\begin{matrix}\text{Multipy and divide} \\ \text{by } -3\end{matrix}\right\}$$

$$= \frac{1}{25}(-2)^n - \frac{1}{25}(3)^n + \frac{1}{15} n(-3)^n$$

HW

1. $y_{n+2} + 2y_{n+1} + y_n = n \;,\; y_0 = y_1 = 0.$
2. $y_{n+1} - 2y_n = 1 ,\; y_0 = 0.$

Probability Distributions:

$n \to$ finite $\leftarrow$ Binomial – pmf
$n \to \infty \leftarrow$ poission } DRV

Normal

$x = 0, 1, 2, \dots n$

Poission distrubution in Biomial not their Proof not be asked

Binomial Distrubution:

Let x be a DRV the assumtions used to define a Binomial Distrubution are:
① There are only 2 possible outcomes of value success and faile
② The probability of succes in each trail is same
③ There are 'n' trails.

Probabilty of (success) = P
Probability of (failure) = $q = 1 - p$.

Let 'x' be a Binomial r.v with parameters n and p if its pmf is given by $n_{C_k} p^k q^{n-k}$.

$$P(X=k) = n_{C_k} \cdot p^k \cdot q^{n-k} \quad , \; k = 0, 1, 2, 3 \dots k$$

n = no of trails , p is p(success) and q(failure).

$X \sim B(n, p) \to$ means x has binomial distrubution.

Mean (or) Expectation:

$$E(X) = \sum_{k=0}^{n} k P(X=k)$$

$$= \sum_{k=0}^{n} k \cdot n_{C_k} \cdot p^k \cdot q^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$$