③ Given that

4

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=1.2$ & $x=2.0$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ
1.0	2.7183						
		0.6018					
$x_0=$ 1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.008		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
$x_n=$ 2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Tabulation is for the function $y=e^x$.

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \dots\right]$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{0.1627}{2} + \frac{0.0361}{3} - \frac{0.0080}{4} + \frac{0.0014}{5}\right] = 3.32031$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{(0.2)^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.008) - \frac{5}{6}(0.0014)\right] = 3.3192$$

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots\right]$$

$$= \frac{1}{0.2}\left[1.3395 + \frac{0.2429}{2} + \frac{0.0441}{3} + \frac{0.0080}{4} + \frac{0.0013}{5}\right]$$

$$= 7.3895$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.2429 + 0.0441 + \frac{11}{12}(0.0080) + \frac{5}{6}(0.0013)\right]$$

$$= 7.38541$$

② Given $\sin 0^\circ = 0$, $\sin 10^\circ = 0.1736$, $\sin 20^\circ = 0.3420$, $\sin 30^\circ = 0.5$, $\sin 40^\circ = 0.6428$.

(i) Find $\sin 23^\circ$ (ii) Find the numerical value of $\cos x$ at $x = 10^\circ$.

3

x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4
0	0				
		0.1736			
$x_0 = 10$	0.1736		−0.0052		
		0.1684		−0.0052	
20	0.3420		−0.0104		0.0004
		0.1580		−0.0048	
30	0.5000		−0.0152		
		0.1428			
40	0.6428				

To find $\sin 23^\circ$: $x = 23^\circ$, $p = ?$

To find $\cos 10^\circ$:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \cdots\right]$$

$$= \frac{1}{10}\left[0.1684 - \left(\frac{-0.0104}{2}\right) + \left(\frac{-0.0048}{3}\right)\right] = 0.0172$$

Actual value $= \cos 10^\circ = 0.9848$

Lot of deviation!

Reason: formula is valid only when angle is used in radians not in degrees.

$$h = 10^\circ = 10 \times \frac{\pi}{180} = \frac{\pi}{18}$$

$$\therefore \frac{dy}{dx} = \frac{18}{\pi}\left[0.1684 - \left(\frac{-0.0104}{2}\right) + \left(\frac{-0.0048}{3}\right)\right]$$

$$= 0.98548$$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\cdots\right]$$

2

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\cdots\right]$$

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\cdots\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\cdots\right]$$

1. Given that

x:	50	51	52	53	54	55	56
y:	3.684	3.7084	3.7325	3.7563	3.7798	3.803	3.8259

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=50$ and $x=56$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.684			
		0.0244		
51	3.7084		−0.0003	
		0.0241		0
52	3.7325		−0.0003	
		0.0238		0
53	3.7563		−0.0003	
		0.0235		0
54	3.7798		−0.0003	
		0.0232		0
55	3.8030		−0.0003	
		0.0229		
56	3.8259			

3^{rd} order differences are zero. ∴ The data represents a polynomial of degree 2.

at $x=50$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2}\right\} = \frac{1}{1}\left[0.0244 - \left(\frac{-0.0003}{2}\right)\right] = 0.02455$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0\right] = \frac{1}{1}\left[-(0.0003)\right] = -0.0003$$

at $x=56$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2}\right] = \left[0.0229 + \left(\frac{-0.0003}{2}\right)\right] = 0.02275$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\nabla^2 y_n = 1 \times (-0.0003) = -0.0003$$

Numerical Differentiation:

Given a set of points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, satisfying $y = f(x)$ where $f(x)$ is not known explicitly, the process of finding $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$ for an $x \in (x_0, x_n)$ is called Numerical Differentiation.

Suppose that the values of x_i's are equally spaced.
i.e. $x_i = x_0 + ih$, $i = 1, 2, \dots, n$, $h > 0$.

We have Newton's forward difference formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$y = y_0 + p\Delta y_0 + (p^2 - p)\frac{\Delta^2 y_0}{2!} + (p^3 - 3p^2 + 2p)\frac{\Delta^3 y_0}{3!} + (p^4 - 6p^3 + 11p^2 - 6p)\frac{\Delta^4 y_0}{4!} + \dots$$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} \text{ (chain Rule)} \qquad x = x_0 + ph, \quad \frac{dx}{dp} = h \quad \therefore \frac{dp}{dx} = \frac{1}{h}$$

$$= \frac{1}{h}\left[0 + \Delta y_0 \cdot 1 + \frac{\Delta^2 y_0}{2!}(2p-1) + \frac{\Delta^3 y_0}{3!}(3p^2 - 6p + 2) + \frac{\Delta^4 y_0}{4!}(4p^3 - 18p^2 + 22p - 6) + \dots\right]$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{\Delta^2 y_0}{2!}(2p-1) + \frac{\Delta^3 y_0}{3!}(3p^2 - 6p + 2) + \dots\right] \text{ --- } (1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx}$$

$$= \frac{1}{h}\left[0 + \frac{\Delta^2 y_0}{2!}(2) + \frac{\Delta^3 y_0}{3!}(6p - 6) + \frac{\Delta^4 y_0}{4!}(12p^2 - 36p + 22) + \dots\right] \cdot \frac{1}{h}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{\Delta^3 y_0}{3!}(6p - 6) + \frac{\Delta^4 y_0}{4!}(12p^2 - 36p + 22) + \dots\right] \text{ --- } (2)$$

In a similar way, we can obtain $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ and so on.

(1) and (2) take simpler forms, ~~for~~ tabulated points.

For example, when $x = x_0$, we get $p = 0$. $\therefore$ (1) & (2) give

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots\right] \text{ --- } (3)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\right] \text{ --- } (4)$$

If we use backward difference formula, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2!}(2p+1) + \frac{\nabla^3 y_n}{3!}(3p^2 + 6p + 2) + \dots\right] \text{ --- } (5)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{\nabla^3 y_n}{3!}(6p + 6) + \frac{\nabla^4 y_n}{4!}(12p^2 + 36p + 22) + \dots\right] \text{ --- } (6)$$

When $x = x_n$, (5) and (6) give

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \dots\right] \text{ --- } (7)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \dots\right] \text{ --- } (8)$$