ICE 3251 DIGITAL SIGNAL PROCESSING

One sided z-Transform	$Z^{+}\{x(n)\} = X^{+}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$ $Z^{+}\{x(n-k)\} = z^{-k}\left[X^{+}(z) + \sum_{n=1}^{k} x[-n]z^{n}\right]$ $Z^{+}\{x(n+k)\} = z^{k}\left[X^{+}(z) - \sum_{n=0}^{k-1} x[n]z^{-n}\right]$	$k > 0$
Linear constant-coefficient difference equation	$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]\ ; a_0 \neq 0$	Discrete LTI system $x[n]: input$ $y[n]: output$
DFT pair	$x(n) = \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1,2 \dots \dots N-1$ $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1,2 \dots \dots N-1$	x(n) : Signal of length L X(k): N-point DFT of x(n) N ≥ L
N-point Circular convolution	$x_3(n) = x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)x_2(n-m)$	$x_1(n), x_2(n)$: signals of length N or less
Pole-zero IIR system	$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$ $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$	

All pole IIR system	$y(n)=\sum_{k=1}^{N} a_k y(n-k)+b_0 x(n)$ $H(z)=\frac{Y(z)}{X(z)}=\frac{b_0}{1-\sum_{k=1}^{N} a_k z^{-k}}$	$x(n)\text{: } input$ $y(n)\text{: } output$ $X(z)$: z-transform of $x(n)$ $Y(z)$: z-transform of y(n) $H(z)$: System function K: Integer part of *(N+1)/2* N ≥ M $C=\frac{b_N}{a_N}$
All zero FIR system	$y(n)=\sum_{k=0}^{M} b_k x(n-k)$ $H(z)=\frac{Y(z)}{X(z)}=\sum_{k=0}^{M} b_k z^{-k}$	
Cascade realization for IIR filters	$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}=\prod_{k=1}^{K} H_k(z)$ $H_k(z)=\frac{b_{k0}+b_{k1}z^{-1}+b_{k2}z^{-2}}{1+a_{k1}z^{-1}+a_{k2}z^{-2}}$	
Parallel realization for IIR filters	$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}=C+\sum_{k=1}^{K} H_k(z)$ $H_k(z)=\frac{b_{k0}+b_{k1}z^{-1}}{1+a_{k1}z^{-1}+a_{k2}z^{-2}}$	
Frequency sampling realization for FIR filters	$H(z)=\frac{1}{M}(1-z^{-M})H_p(z)$ $H_p(z)=\frac{H(0)}{1-z^{-1}}+\sum_{k=1}^{(M-1)/2}\frac{A(k)+B(k)z^{-1}}{1-2cos\left(\frac{2\pi k}{M}\right)z^{-1}+z^{-2}}\quad M\ odd$ $H_p(z)=\frac{H(0)}{1-z^{-1}}+\frac{H\left(\frac{M}{2}\right)}{1+z^{-1}}+\sum_{k=1}^{\frac{M}{2}-1}\frac{A(k)+B(k)z^{-1}}{1-2cos\left(\frac{2\pi k}{M}\right)z^{-1}+z^{-2}}\quad M\ even$ $A(k)=\ H(k)+H(M-k)$	$H(\omega)$: Frequency response $H(k)$: $H(\omega)$ at $\omega=\omega_k=\frac{2\pi k}{M}$ M: Length of the FIR filter

	$B(k) = H(k)e^{-j\frac{2\pi k}{M}} + H(M-k)e^{j\frac{2\pi k}{M}}$	
Impulse Invariant Transformation	$z = e^{sT}$ $H(s) = \sum_{k=1}^{N} \frac{A_k}{s-s_k} \equiv H(z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{s_kT}z^{-1}}$	N: Order of the analog filter s_k: Poles of $H(s)$ z_k: Zeroes of $H(s)$ T: Sampling interval Ω: Analog filter frequency variable ω: Digital filter frequency variable
Bilinear Transformation	$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$ $\Omega = \frac{2}{T}tan\left(\frac{\omega}{2}\right)$	
Matched z-transform	$H(s) = \frac{\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-s_k)} \equiv H(z) = \frac{\prod_{k=1}^{M} 1-e^{z_kT}z^{-1}}{\prod_{k=1}^{N} 1-e^{s_kT}z^{-1}}$	
Butterworth analog low pass filter response	$\lvert H(\Omega)\rvert^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$ $N = \frac{log\left(\frac{1}{{\delta_2}^2}-1\right)}{2log\left(\frac{\Omega_s}{\Omega_c}\right)}$ $s_k = \Omega_c e^{j\emptyset_k}$	N: order of the filter $\lvert H(\Omega)\rvert^2$: Squared magnitude response Ω_p: Pass band edge frequency Ω_p: Stop band edge frequency Ω_c: 3-dB cut-off frequency $\frac{1}{1+\epsilon^2}$: Pass band edge value of $\lvert H(\Omega)\rvert^2$ ${\delta_2}^2$: Stop band edge value of $\lvert H(\Omega)\rvert^2$ s_k: Poles of H(s) $\emptyset_k = pole\ angle$ $= \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}$, *k= 0, 1, 2, ……N-1*
Chebyshev analog low pass filter response	$\lvert H(\Omega)\rvert^2 = \frac{1}{1+\epsilon^2{T_N}^2\left(\frac{\Omega}{\Omega_p}\right)}$ $T_N(x) = \begin{cases} cos(Ncos^{-1}x), & \lvert x\rvert \le 1 \\ cosh(Ncosh^{-1}x), & \lvert x\rvert > 1 \end{cases}$	

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<td>$$N=\frac{log\left[\frac{\sqrt{1-{\delta_2}^2}+\sqrt{1-{\delta_2}^2(1+\epsilon^2)}}{\varepsilon\delta_2}\right]}{log\left[\left(\frac{\Omega_s}{\Omega_p}\right)+\sqrt{\left(\frac{\Omega_s}{\Omega_p}\right)^2-1}\right]}$$
$s_k=r_2cos(\emptyset_k)+jr_1sin(\emptyset_k)$
$r_1=\Omega_p\frac{\beta^2+1}{2\beta}$; $r_2=\Omega_p\frac{\beta^2-1}{2\beta}$; $\beta=\left[\frac{\sqrt{1+\epsilon^2}+1}{\varepsilon}\right]^{1/N}$</td>
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<tr>
<td>Frequency Transformation for analog filters</td>
<td colspan="2">Prototype Low pass filter has band edge frequency Ω_p
<table>
<tr><th>Type of Transformation</th><th>Transformation</th><th>Band edge frequency of new filter</th></tr>
<tr><td>Low pass</td><td>$s\longrightarrow\frac{\Omega_p}{\Omega_{pn}}s$</td><td>$\Omega_{pn}$</td></tr>
<tr><td>High pass</td><td>$s\longrightarrow\frac{\Omega_p\Omega_{pn}}{s}$</td><td>$\Omega_{pn}$</td></tr>
<tr><td>Band pass</td><td>$s\longrightarrow\Omega_p\frac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$</td><td>$\Omega_l$, Ω_u</td></tr>
<tr><td>Band stop</td><td>$s\longrightarrow\Omega_p\frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_{lp}\Omega_u}$</td><td>$\Omega_l$, Ω_u</td></tr>
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<tr>
<td>Frequency Transformation for digital filters</td>
<td colspan="2">Prototype Low pass filter has band edge frequency ω_p
<table>
<tr><th>Type of Transformation</th><th>Transformation</th><th>Parameters</th></tr>
<tr><td>Low pass</td><td>$z^{-1}\longrightarrow\frac{z^{-1}-a}{1-az^{-1}}$</td><td>$\omega_{pn}=band\ edge\ frequency\ of\ new\ filter$
$a=\frac{sin[(\omega_p-\omega_{pn})/2]}{sin[(\omega_p+\omega_{pn})/2]}$</td></tr>
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	High pass	$z^{-1} \longrightarrow \frac{z^{-1} + a}{1 + az^{-1}}$	$\omega_{pn} = band\ edge\ frequency\ of\ new\ filter$ $a = \frac{cos[(\omega_p - \omega_{pn})/2]}{cos[(\omega_p + \omega_{pn})/2]}$
	Band pass	$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$\omega_l = lower\ band\ edge\ frequency$ $\omega_u = upper\ band\ edge\ frequency$ $a_1 = \frac{-2\alpha K}{(K+1)}$ $a_2 = \frac{(K-1)}{(K+1)}$ $\alpha = \frac{cos[(\omega_u + \omega_l)/2]}{cos[(\omega_u - \omega_l)/2]}$ $K = cot\left(\frac{\omega_u - \omega_l}{2}\right) tan\left(\frac{\omega_p}{2}\right)$
	Band stop	$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$\omega_l = lower\ band\ edge\ frequency$ $\omega_u = upper\ band\ edge\ frequency$ $a_1 = \frac{-2\alpha}{(K+1)}$ $a_2 = \frac{(1-K)}{(1+K)}$ $\alpha = \frac{cos[(\omega_u + \omega_l)/2]}{cos[(\omega_u - \omega_l)/2]}$ $K = tan\left(\frac{\omega_u - \omega_l}{2}\right) tan\left(\frac{\omega_p}{2}\right)$
Linear phase FIR filter frequency response	i) Symmetric impulse response, odd length		M: length of the filter $H(\omega)$: Frequency response

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$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

ii) Symmetric impulse response, even length

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

iii) Anti-symmetric impulse response, odd length

$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

iv) Anti-symmetric impulse response, even length

$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

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<td>Linear phase FIR filter design using window functions</td>
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Window functions for FIR filter design

<table>
<tr><td>Name of the window</td><td>Window function 0≤n≤M-1</td><td>Main lobe width</td><td>Peak side lobe (dB)</td><td>Normalized transition width#</td><td>Stop band attenuation (dB)</td></tr>
<tr><td>Rectangular</td><td>1</td><td>4π/M</td><td>-13</td><td>0.9/(M-1)</td><td>21</td></tr>
<tr><td>Hanning</td><td>$0.5 - 0.5cos\left(\frac{2\pi n}{M-1}\right)$</td><td>8π/M</td><td>-32</td><td>3.1/(M-1)</td><td>44</td></tr>
<tr><td>Hamming</td><td>$0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$</td><td>8π/M</td><td>-43</td><td>3.3/(M-1)</td><td>53</td></tr>
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<tr><td>Blackman</td><td>$0.42 - 0.5cos\left(\frac{2\pi n}{M-1}\right) + 0.08cos\left(\frac{4\pi n}{M-1}\right)$</td><td>12π/M</td><td>-58</td><td>5.5/(M-1)</td><td>75</td></tr>
<tr><td>Keiser*</td><td>$\frac{I_0\left[\beta\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\beta\left(\frac{M-1}{2}\right)\right]}$</td><td></td><td></td><td></td><td>> 70</td></tr>
</table>
*Keiser window parameters can be controlled by β. $\boldsymbol{I_0}$[.] is modified Bessel function.
#Transition width is normalized to 2π or equivalently to sampling frequency F_s
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<td>Frequency sampling design of FIR filter</td>
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$H_d(\omega) = H_r(\omega)e^{-j\omega(M-1)/2}$

$$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{\frac{(M-1)}{2}} G(k)cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\}, \qquad M\ odd$$

$$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{\frac{M}{2}-1} G(k)cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\}, \qquad M\ even$$
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M: length of the filter
$H_d(\omega)$: Desired frequency response
$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right)$
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<td>Non-parametric power spectrum estimators</td>
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<table>
<tr><th>Estimate</th><th>Quality factor</th><th>Computational requirement</th></tr>
<tr><td>Bartlett</td><td>$Q_B = 1.1N\Delta f$</td><td>$\frac{N}{2}\log_2\frac{0.9}{\Delta f}$</td></tr>
<tr><td>Welch (50% overlap)</td><td>$Q_W = 1.39N\Delta f$</td><td>$N\log_2\frac{5.12}{\Delta f}$</td></tr>
<tr><td>Blackman-Tukey</td><td>$Q_{BT} = 2.34N\Delta f$</td><td>$N\log_2\frac{1.28}{\Delta f}$</td></tr>
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Quality factor
$Q = \frac{\{E[estimate]\}^2}{var[estimate]}$
Δf: *Frequency resolution*
N: Data frame length
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