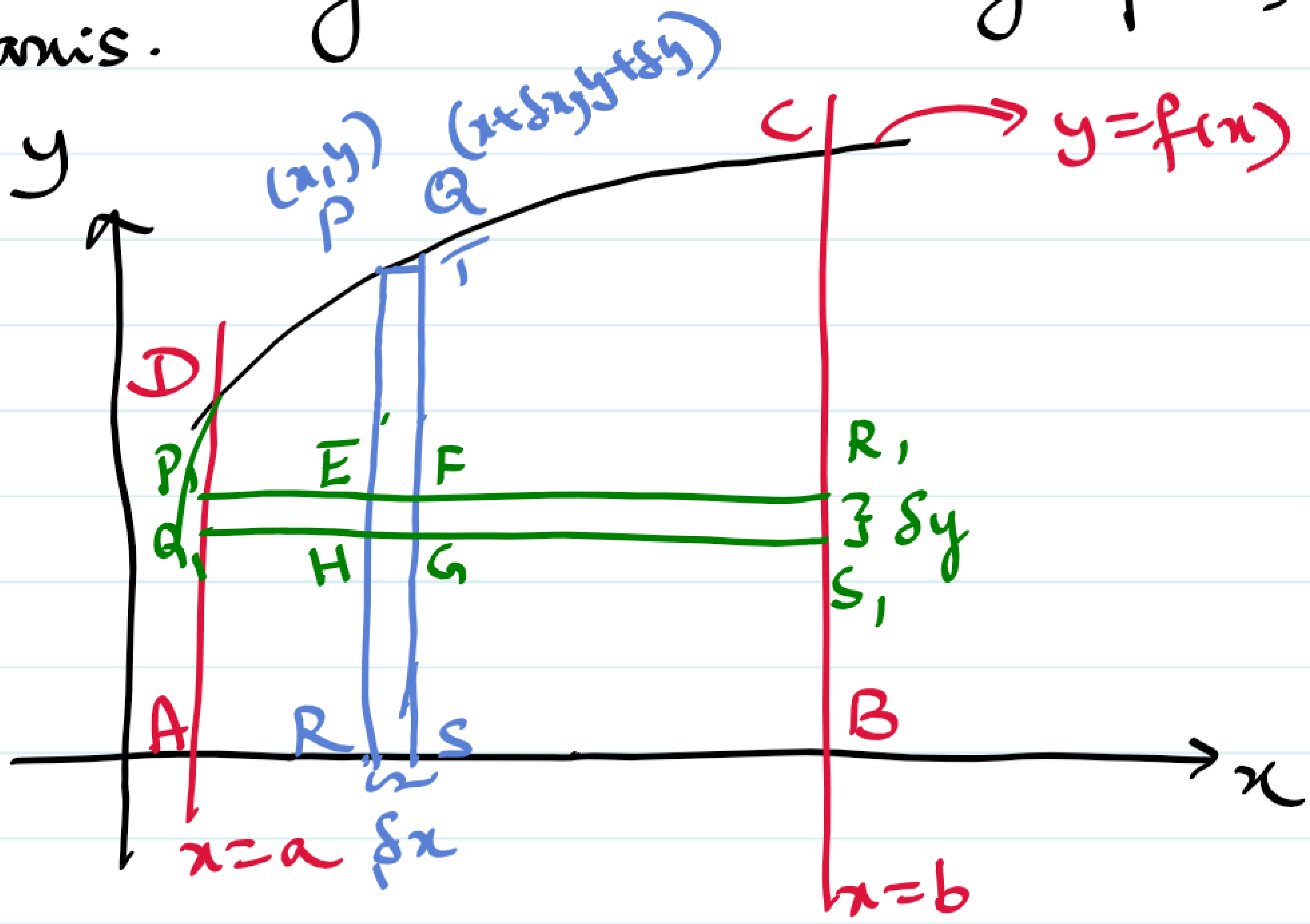
Multiple Integrals

Evaluation of multiple integrals which involve more than one variable:

Double Integration -

Representation of Area as a Double integral:

Consider the region bounded by the curve $y=f(x)$, $x=a$, $x=b$ and the x-axis.



Area of ABCD is,

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{b} \delta x . y$$

Using integral notation

$$A = \int_a^b y\,dx = \int_{x=a}^{b} f(x)\,dx$$

Now, consider the horizontal strip $P_1Q_1R_1S_1$ of width δy.

Area of EFGH, $\delta A = \delta x . \delta y$

First consider the summation in a vertical direction and area PQRS can be considered as the limit of a sum of an infinite number of elements like EFGH situated along the strip PQRS.

Area of strip PQRS, $A_1 = \lim_{\delta y \to 0} (\sum \delta y) . \delta x = \int_{y=0}^{f(x)} dy . \delta x$

We move the strip PQRS along x-axis covering the whole are ABCD,

Area of ABCD, $A = \lim_{\delta x \to 0} \sum_{x=a}^{b} \delta x . \int_{0}^{f(x)} dy$

$$= \int_{x=a}^{b} dx . \int_{y=0}^{f(x)} dy$$

$$\boxed{A = \int_{x=a}^{b} \int_{y=0}^{f(x)} dy\, dx}$$

Inner integration is carried out with respect to y treating x as constant and outer integration is then carried out w.r.to x.

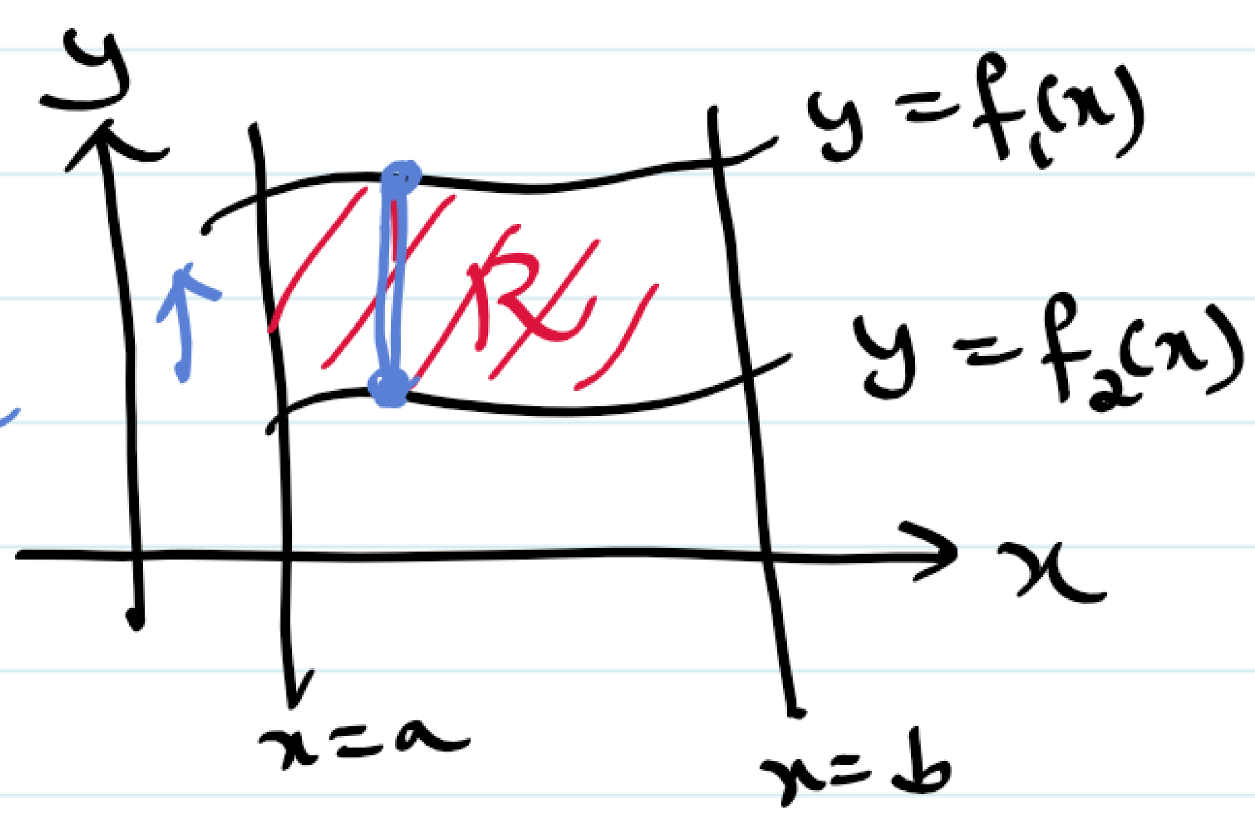
If $f(x,y) \geq 0$, $I = \iint_R f(x,y)\, dx\, dy \rightarrow$ represent volume.

If $f(x,y) = 1$, $I = \iint_R dx\, dy \longrightarrow$ Area.

Evaluation of Double integrals –

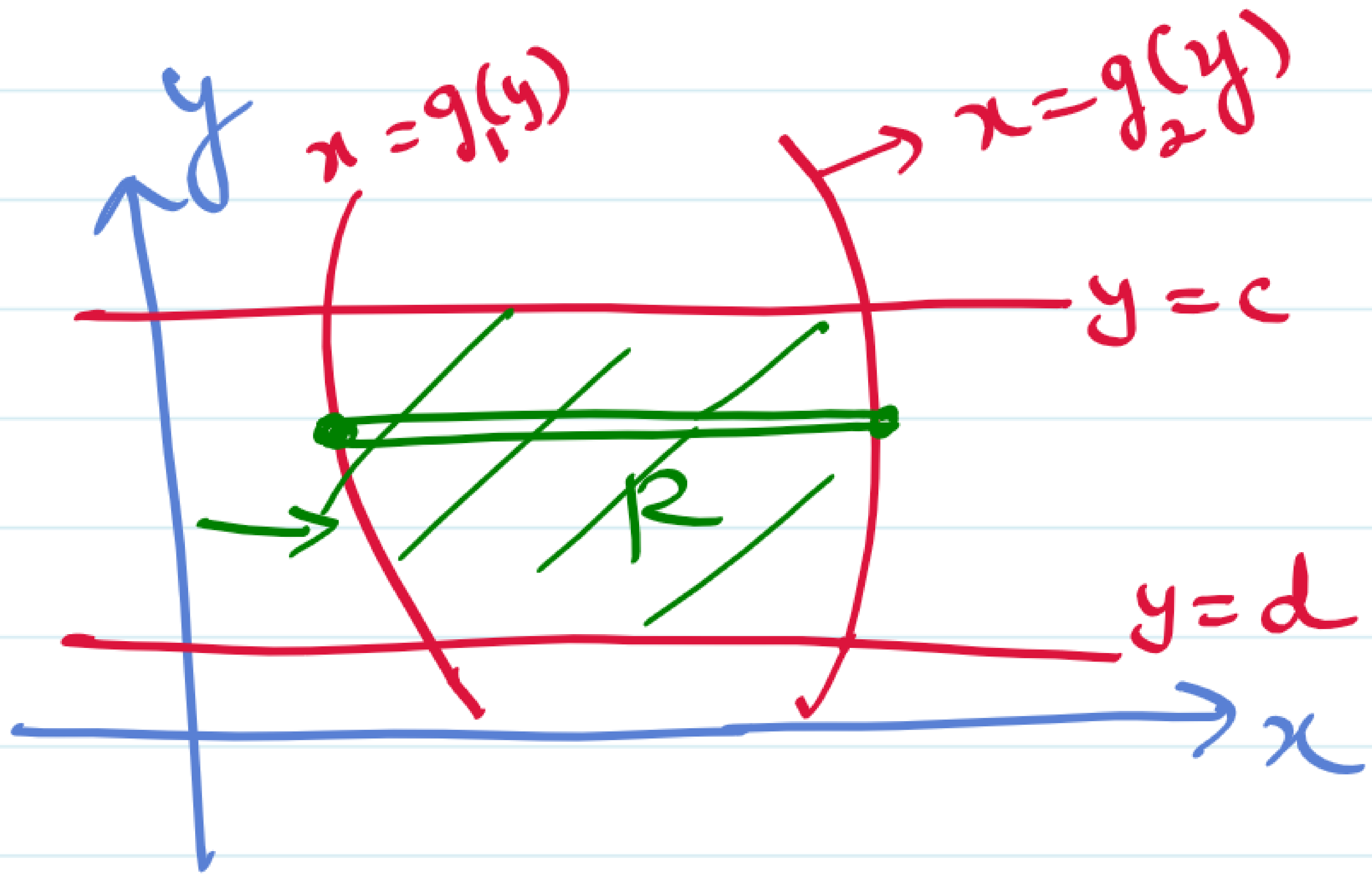
① Suppose that R can be described by $x=a$, $x=b$, $y=f_1(x)$, $y=f_2(x)$ then,

$$I = \iint_R f(x,y)\, dy\, dx = \int_{x=a}^{b} \int_{y=f_2(x)}^{f_1(x)} f(x,y)\, dy\, dx$$



② Suppose that R canbe described by $y=c$, $y=d$

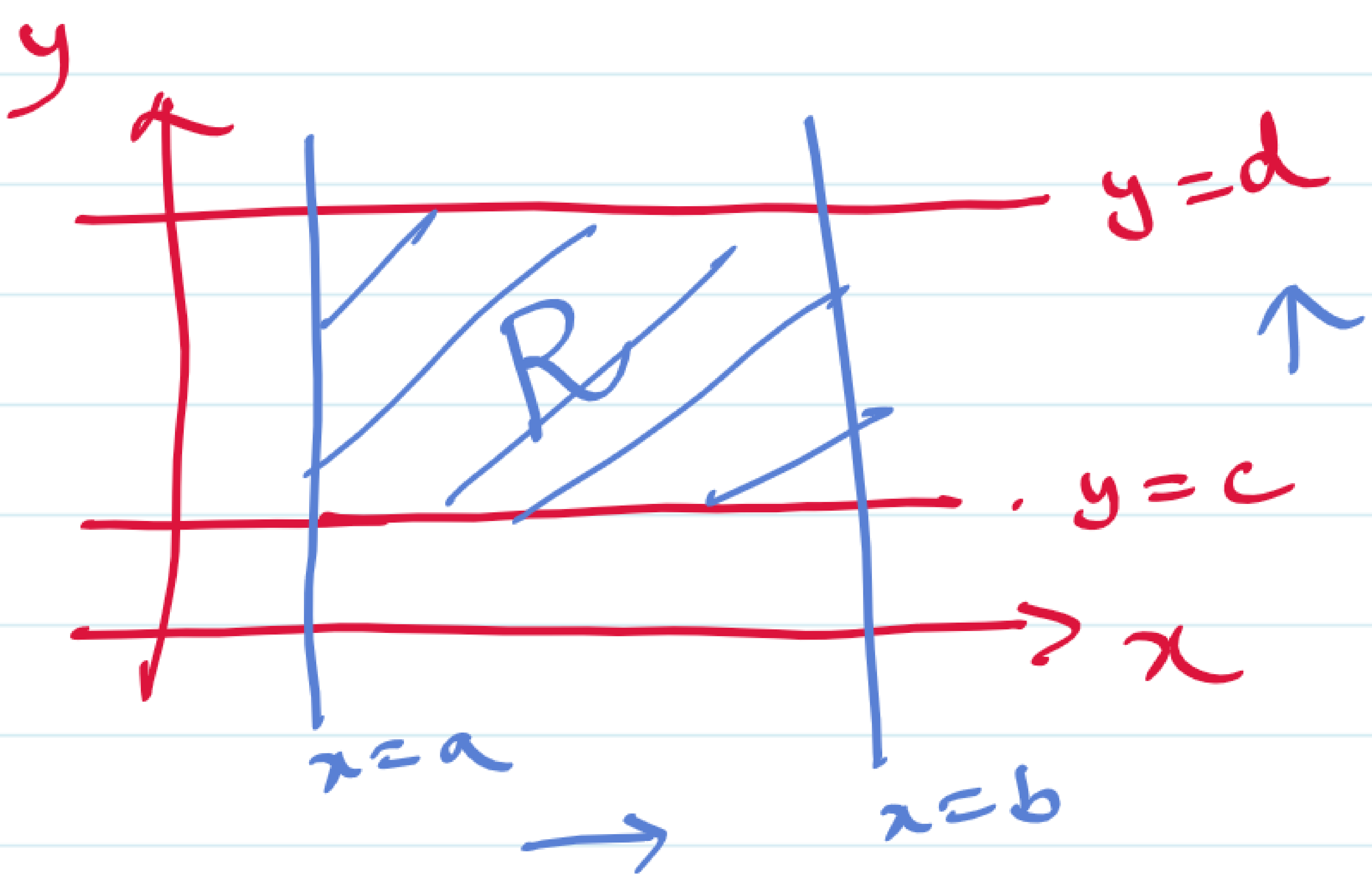
$x = g_1(y)$, $x = g_2(y)$



$$I = \iint_R f(x,y)\,dx\,dy$$

$$= \int_{y=d}^{c} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

③ Suppose that R can be described by $x=a$, $x=b$, $y=c$, $y=d$



$$I = \iint_R f(x,y)\,dx\,dy$$

$$= \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy$$

$$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$$

Evaluate:- $\displaystyle\int_0^1\int_0^y xy\,dx\,dy$

Since, limits of inner integral are functions of y, integrate w. r. to x, keeping y constant.

$$\int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy = \int_{y=0}^{1} y\left(\frac{x^2}{2}\right)_{x=0}^{y} dy$$

$$= \int_{y=0}^{1} y\left(\frac{y^2}{2}-0\right)dy = \int_{y=0}^{1}\frac{y^3}{2}\,dy$$

$$= \frac{1}{2}\left(\frac{y^4}{4}\right)_0^1 = \frac{1}{8}-0$$

$$= \frac{1}{8}$$