EXAMPLE 6.28 Perform the linear convolution of the following sequences using (a) overlap-add method, (b) overlap-save method.

$$x(n) = \{1,\ -2, 2,\ -1, 3,\ -4, 4,\ -3\} \text{ and } h(n) = \{1,\ -1\}$$

EXAMPLE 6.28 Perform the linear convolution of the following sequences using (a) overlap-add method, (b) overlap-save method.

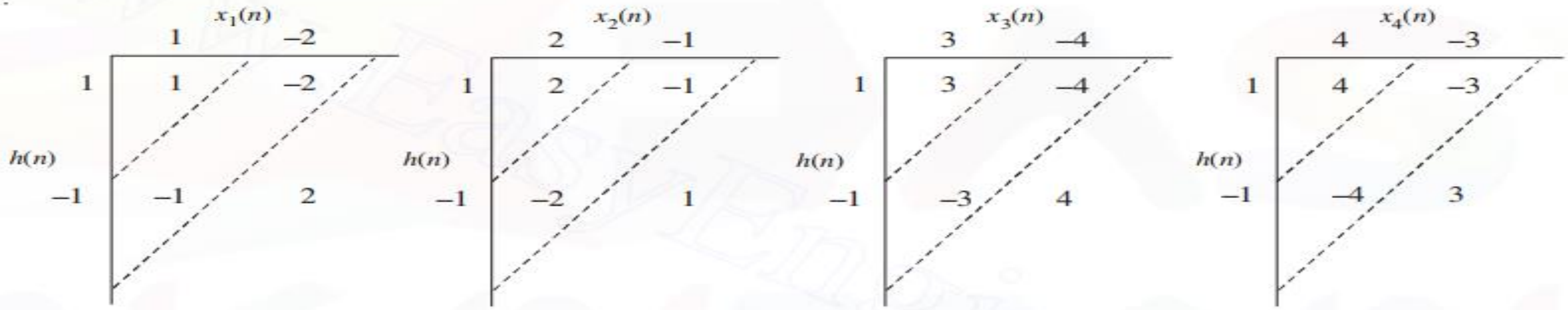
$$x(n) = \{1, \ -2, 2, \ -1, 3, \ -4, 4, \ -3\} \text{ and } h(n) = \{1, \ -1\}$$

Solution: (a) *Overlap-add method*

Here the longer sequence is $x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\}$ of length $L = 8$ and the smaller sequence is $h(n) = \{1, -1\}$ of length $N = 2$. So $x(n)$ is sectioned into 4 blocks $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ each of length 2 samples as shown below.

$$\begin{array}{rl|rl|rl|rl} x_1(n) = & 1;\ n = 0 & x_2(n) = & 2;\ n = 2 & x_3(n) = & 3;\ n = 4 & x_4(n) = & 4;\ n = 6 \\ = & -2;\ n = 1 & = & -1;\ n = 3 & = & -4;\ n = 5 & = & -3;\ n = 7 \end{array}$$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ respectively with $h(n)$.



$y_1(n) = \{1, -3, 2\}$ $y_2(n) = \{2, -3, 1\}$ $y_3(n) = \{3, -7, 4\}$ $y_4(n) = \{4, -7, 3\}$

In overlap-add method, the last $N-1 = 2-1 = 1$ sample in an output sequence overlaps with the first $N-1 = 2-1 = 1$ sample of the next output sequence. The overall output $y(n)$ is obtained by combining the outputs $y_1(n)$, $y_2(n)$, $y_3(n)$, and $y_4(n)$ as shown in Table 6.4. Here the overlapping samples are added.

TABLE 6.4 Combining the output of the convolution of each section

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	1	−3	2						
$y_2(n)$			2	−3	1				
$y_3(n)$					3	−7	4		
$y_4(n)$							4	−7	3
$y(n)$	1	−3	4	−3	4	−7	8	−7	3

EXAMPLE 6.29 Perform the linear convolution of the following sequences by (a) overlap-add method, (b) overlap-save method

$$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\} \text{ and } h(n) = \{1, 2, -1\}$$

Solution: (a) *Overlap-add method*

EXAMPLE 6.29 Perform the linear convolution of the following sequences by (a) overlap-add method, (b) overlap-save method

$$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\} \text{ and } h(n) = \{1, 2, -1\}$$

Solution: (a) *Overlap-add method*

Here the longer sequence is $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$ of length 8 and the smaller sequence is $h(n) = \{1, 2, -1\}$ of length 3. So pad a zero to $x(n)$ and make $x(n)$ of length 9. So $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4, 0\}$.

Section $x(n)$ into 3 blocks $x_1(n)$, $x_2(n)$ and $x_3(n)$, each of length 3 samples as shown below.

$$\begin{array}{ll|ll|ll}
x_1(n) = 1; & n = 0 & x_2(n) = 2; & n = 3 & x_3(n) = 3; & n = 6 \\
\quad = -2; & n = 1 & \quad = -3; & n = 4 & \quad = -4; & n = 7 \\
\quad = 3; & n = 2 & \quad = 4; & n = 5 & \quad = 0; & n = 8
\end{array}$$

$h(n)$ \ $x_1(n)$	1	−2	3
1	1	−2	3
2	2	−4	6
−1	−1	2	−3

$y_1(n) = \{1, 0, -2, 8, -3\}$

$h(n)$ \ $x_2(n)$	2	−3	4
1	2	−3	4
2	4	−6	8
−1	−2	3	−4

$y_2(n) = \{2, 1, -4, 11, -4\}$

$h(n)$ \ $x_3(n)$	3	−4	0
1	3	−4	0
2	6	−8	0
−1	−3	4	0

$y_3(n) = \{3, 2, -11, 4, 0\}$

Since, in overlap-add method, the last $N - 1 = 3 - 1 = 2$ samples in an output sequence overlaps with the first $N - 1 = 3 - 1 = 2$ samples of next output sequence, the overall output $y(n)$ is obtained by combining the outputs $y_1(n)$, $y_2(n)$ and $y_3(n)$ as shown in Table 6.6. Here the overlap samples are added.

TABLE 6.6 Combining the output of the convolution of each section

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	1	0	−2	8	−3						
$y_2(n)$				2	1	−4	11	−4			
$y_3(n)$							3	2	−11	4	0
$y(n)$	1	0	−2	10	−2	−4	14	−2	−11	4	0

The last zero is discarded because $x(n)$ was padded with one zero.

$$\therefore \quad y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$$

(b) *Overlap-save method*

Given $x(n) = \{1,\ -2, 2,\ -1, 3,\ -4, 4,\ -3\}$ $\therefore L = 8$

and $$h(n) = \{1, -1\};\quad N = 2$$

$$M = 2N = 4$$

Add $N - 1 = 2 - 1 = 1$ leading zero to the longer sequence $x(n)$

(b) *Overlap-save method*

Given $x(n) = \{1,\ -2, 2,\ -1, 3,\ -4, 4,\ -3\}$ $\therefore L = 8$

and
$$h(n) = \{1, -1\}; \quad N = 2$$

$$M = 2N = 4$$

Add $N - 1 = 2 - 1 = 1$ leading zero to the longer sequence $x(n)$

$\therefore$
$$x(n) = \{0, 1,\ -2, 2, -1, 3,\ -4, 4, -3\}$$

If we choose $M = 4$, we get three overlapping sections of $x(n)$ (we need to zero pad the last one) described by $x_1(n) = \{0, 1,\ -2, 2\}$, $x_2(n) = \{2, -1, 3, -4\}$, $x_3(n) = \{-4, 4, -3, 0\}$.

$$x_1(n) \oplus h(n) = \{0, 1, -2, 2\} \oplus \{1, -1, 0, 0\} = \{-2, 1, -3, 4\}$$

$$x_2(n) \oplus h(n) = \{2, -1, 3, -4\} \oplus \{1, -1, 0, 0\} = \{6, -3, 4, -7\}$$

$$x_3(n) \oplus h(n) = \{-4, 4, -3, 0\} \oplus \{1, -1, 0, 0\} = \{-4, 8, -7, 3\}$$

$$\begin{bmatrix} 0 & 2 & -2 & 1 \\ 1 & 0 & 2 & -2 \\ -2 & 1 & 0 & 2 \\ 2 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & -4 & 3 & -1 \\ -1 & 2 & -4 & 3 \\ 3 & -1 & 2 & -4 \\ -4 & 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 4 \\ -7 \end{bmatrix}, \quad \begin{bmatrix} -4 & 0 & -3 & 4 \\ 4 & -4 & 0 & -3 \\ -3 & 4 & -4 & 0 \\ 0 & -3 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -7 \\ 3 \end{bmatrix}$$

In overlap-save method, the first $N - 1 = 2 - 1 = 1$ sample of each output overlaps with the last $N - 1 = 2 - 1 = 1$ sample of previous output. So discard the first sample in each section

TABLE 6.5 Combining the output of the convolution of each section

n	-1	0	1	2	3	4	5	6	7	8
$y_1(n)$	-2	1	-3	4						
$y_2(n)$				6	-3	4	-7			
$y_3(n)$							-4	8	-7	3
$y(n)$		1	-3	4	-3	4	-7	8	-7	3

(b) *Overlap-save method*

Given $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$ $\quad \therefore L = 8$

and $h(n) = \{1, 2, -1\}$ $\quad \therefore N = 3$

$$M = 2N = 6$$

Add $N - 1 = 3 - 1 = 2$ leading zeros to the longer sequence $x(n)$

$\therefore$ $x(n) = \{0, 0, 1, -2, 3, 2, -3, 4, 3, -4\}$

(b) *Overlap-save method*

Given $$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\} \quad \therefore L = 8$$

and $$h(n) = \{1, 2, -1\} \quad \therefore N = 3$$

$$M = 2N = 6$$

Add $N - 1 = 3 - 1 = 2$ leading zeros to the longer sequence $x(n)$

$\therefore$ $$x(n) = \{0, 0, 1, -2, 3, 2, -3, 4, 3, -4\}$$

If we choose $M = 6$, we get three overlapping sections of $x(n)$ (we need to zero pad the last one) described by

$$x_1(n) = \{0, 0, 1, -2, 3, 2\},\ x_2(n) = \{3, 2, -3, 4, 3, -4\},\ x_3(n) = \{3, -4, 0, 0, 0, 0\}$$

$$\begin{bmatrix} 0 & 2 & 3 & -2 & 1 & 0 \\ 0 & 0 & 2 & 3 & -2 & 1 \\ 1 & 0 & 0 & 2 & 3 & -2 \\ -2 & 1 & 0 & 0 & 2 & 3 \\ 3 & -2 & 1 & 0 & 0 & 2 \\ 2 & 3 & -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ -2 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 3 & -4 & 3 & 4 & -3 & 2 \\ 2 & 3 & -4 & 3 & 4 & -3 \\ -3 & 2 & 3 & -4 & 3 & 4 \\ 4 & -3 & 2 & 3 & -4 & 3 \\ 3 & 4 & -3 & 2 & 3 & -4 \\ -4 & 3 & 4 & -3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \\ -2 \\ -4 \\ 14 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & -4 \\ -4 & 3 & 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -11 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

n	–2	–1	0	1	2	3	4	5	6	7	8	9	10	11
$y_1(n)$	1	–2	1	0	–2	10								
$y_2(n)$					–8	12	–2	–4	14	–2				
$y_3(n)$									3	2	–11	4	0	0
$y(n)$			1	0	–2	10	–2	–4	14	–2	–11	4		

$\therefore \qquad y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$

The result is same as that obtained earlier by the overlap-add method.

Z-Transform

Department of Instrumentation and Control

Associate Professor Dr. Vikash Singh

Discrete-Time Fourier Transform

A discrete-time signal can be represented in the frequency domain using discrete-time Fourier transform. Therefore, the Fourier transform of a discretetime sequence is called the *discrete-time Fourier transform (DTFT)*.

Mathematically, if $x(n)$ is a discrete-time sequence, then its discrete-time Fourier transform is defined as –

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

The discrete-time Fourier transform X(ω) of a discrete-time sequence $x(n)$ represents the frequency content of the sequence $x(n)$. Therefore, by taking the Fourier transform of the discrete-time sequence, the sequence is decomposed into its frequency components. For this reason, the DTFT X(ω) is also called the **signal spectrum.**

Condition for Existence of Discrete-Time Fourier Transform

The Fourier transform of a discrete-time sequence $x(n)$ exists if and only if the sequence $x(n)$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

The discrete-time Fourier transform (DTFT) of the exponentially growing sequences do not exist, because they are not absolutely summable.

Also, the DTFT method of analysing the systems can be applied only to the asymptotically stable systems and it cannot be applied for the unstable systems, i.e., the DTFT can only be used to analyse the systems whose transfer function has poles inside the unit circle.

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The Fourier transform of a discrete-time sequence $x(n)$ exists if and only if the sequence $x(n)$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

Example 1: $\mathbf{x[n] = a^n\ u(n)}$ where a>1

$$\sum_{\mathbf{n}=-\infty}^{\infty} |\mathbf{x(n)}| = \sum_{\mathbf{n=0}}^{\infty} \mathbf{a^n} \rightarrow \infty$$

DTFT does not exist

The discrete-time Fourier transform (DTFT) of the exponentially growing sequences do not exist, because they are not absolutely summable.

Also, the DTFT method of analysing the systems can be applied only to the asymptotically stable systems and it cannot be applied for the unstable systems, i.e., the DTFT can only be used to analyse the systems whose transfer function has poles inside the unit circle.

Relation between Z-transform and DTFT

Taking a look at the equations describing the Z-Transform and the Discrete-Time Fourier Transform:

Discrete-Time Fourier Transform

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x(n)e^{-(j\omega n)}$$

Z-Transform

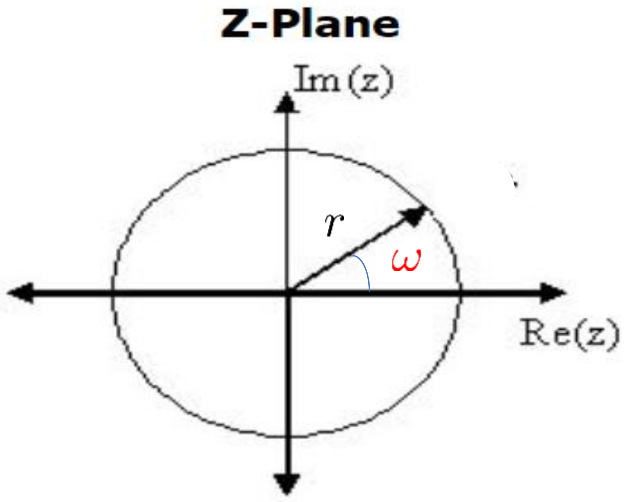
$$\mathbf{X[z]=\sum_{-\infty}^{\infty}x[n]z^{-n}} \quad \textbf{where } \mathbf{z=r\ e^{j\omega}}$$

$$\mathbf{X[z]=\sum_{-\infty}^{\infty}x[n](r\ e^{j\omega})^{-n}} \qquad \textbf{when } \mathbf{r=1} \quad \textbf{DTFT =Z Transform}$$

Z-Transform

$$\mathbf{X[z]} = \sum_{-\infty}^{\infty} \mathbf{x[n]z^{-n}} \qquad \text{where } \mathbf{z = r\ e^{j\omega}}$$

The Z-plane is a complex plane with an imaginary and real axis referring to the complex-valued variable z



Z-Transform and ROC

G. S. $\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

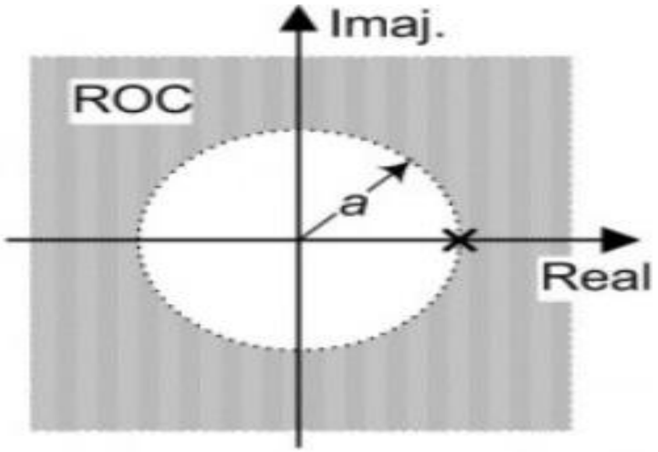
$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{where } z = r\,e^{j\omega}$$

Example 1: $x[n] = a^n\,u(n)$ where a>1

$$X[z] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

when will this G.S. converge ??

when $\frac{a}{|z|} < 1$ or $|z| > a$

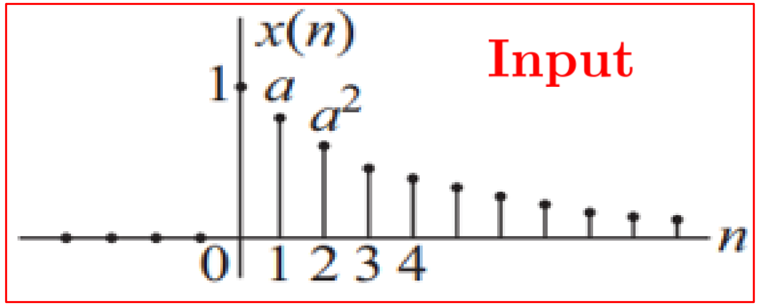


Right-Sided Exponential Sequence

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{because} \quad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



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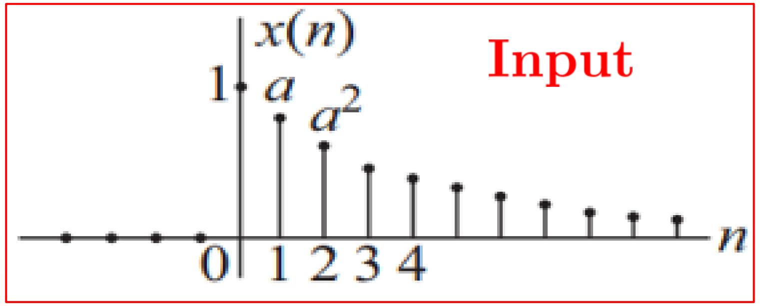

$$\therefore \quad Z[x(n)] = Z[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

Right-Sided Exponential Sequence

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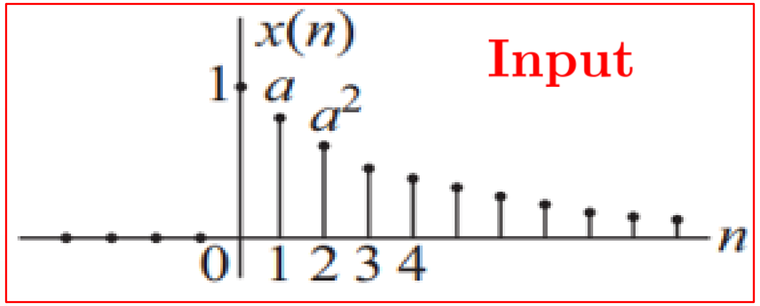

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$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

This is a geometric series of infinite length, and converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$.

$$\therefore \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ ROC}; |z| > |a|$$

which implies that the ROC is exterior to the circle of radius a as shown in Figure 3.1(a)

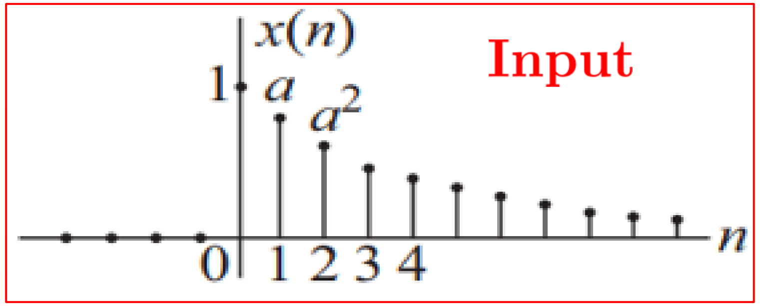
$$\boxed{a^n u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ ROC}; |z| > a}$$

Right-Sided Exponential Sequence

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{because} \quad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



$$\therefore \qquad Z[x(n)] = Z[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n)\, z^{-n}$$

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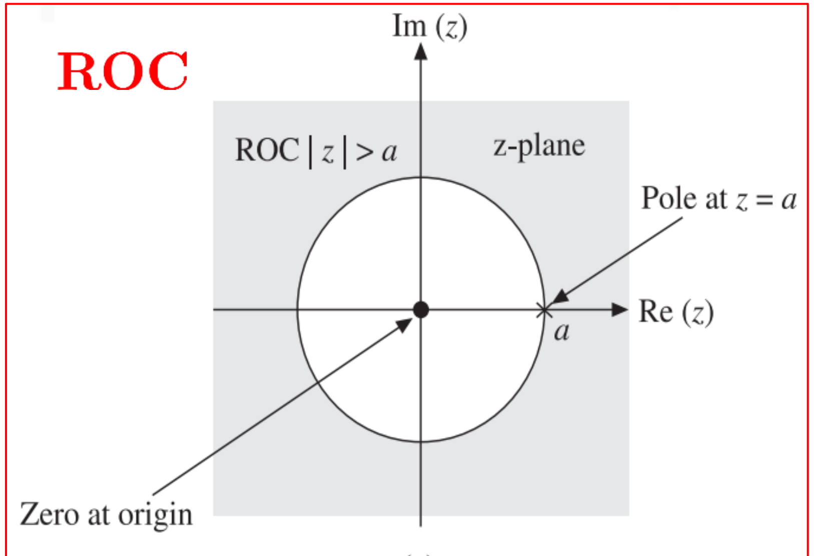


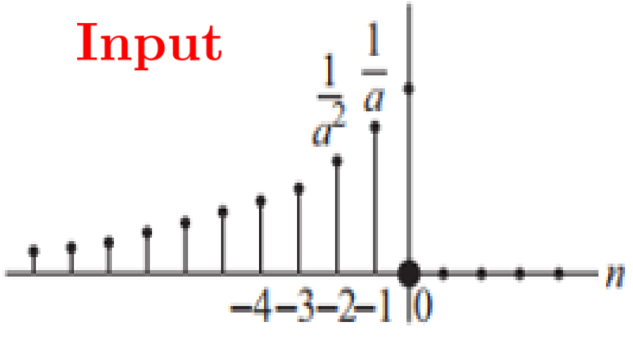
(a)

Left-Sided Exponential Sequence

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

Next consider the *left-sided* discrete time signal $x_2[n] = -a^n u[-n-1]$

$|a| > 1$, a real-valued



$$X_2(z) \equiv -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m \quad \textbf{where } \mathbf{m = -n}$$

$$= \frac{-\left(\frac{z}{a}\right)}{1-\left(\frac{z}{a}\right)} = \left(\frac{z}{z-a}\right) \quad = \frac{\mathbf{1}}{\mathbf{1 - az^{-1}}}$$

Left-Sided Exponential Sequence

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

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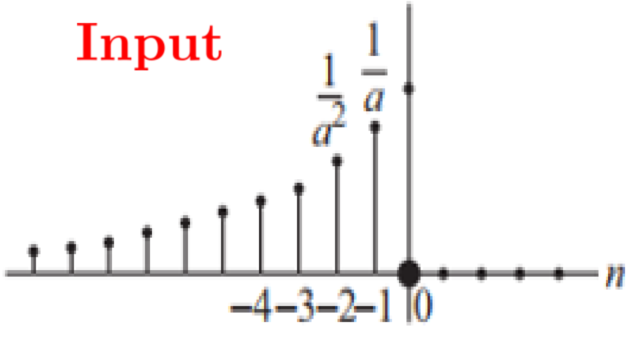


$$X_2(z) \equiv -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m \quad \text{where } m = -n$$

$$= \frac{-\left(\frac{z}{a}\right)}{1-\left(\frac{z}{a}\right)} = \left(\frac{z}{z-a}\right) = \frac{1}{1-az^{-1}}$$

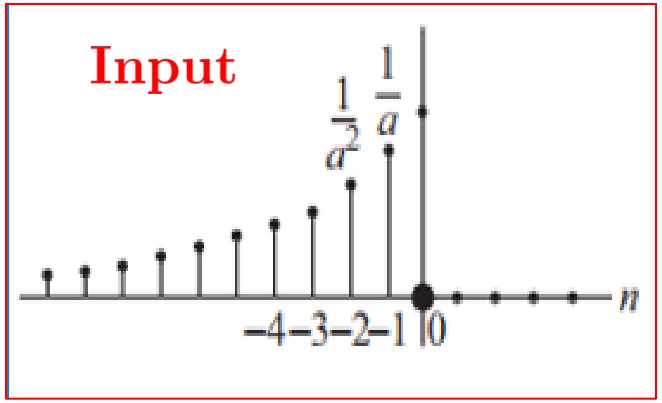
$$\text{for } \left|\frac{z}{a}\right| < 1 \quad \Rightarrow |z| < |a|$$

Left-Sided Exponential Sequence

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

Next consider the *left-sided* discrete time signal $x_2[n] = -a^n u[-n-1]$

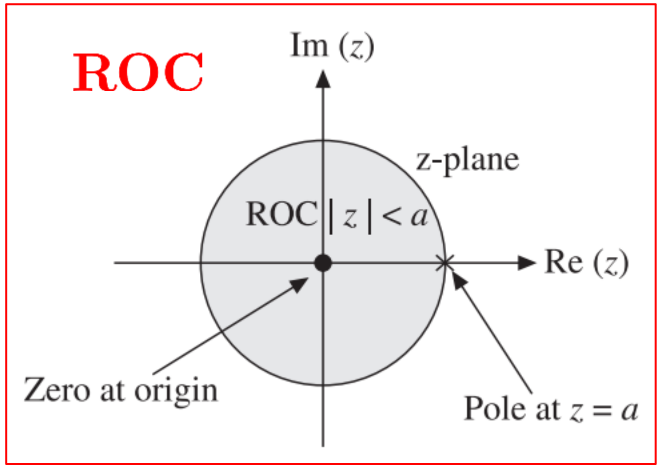
$|a| > 1$, a real-valued



$$X_2(z) \equiv -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m \quad \textbf{where } \mathbf{m = -n}$$

$$= \frac{-\left(\frac{z}{a}\right)}{1-\left(\frac{z}{a}\right)} = \left(\frac{z}{z-a}\right) \quad = \frac{\mathbf{1}}{\mathbf{1 - az^{-1}}}$$

$$\text{for} \quad \left|\frac{z}{a}\right| < 1 \quad \Rightarrow |z| < |a|$$

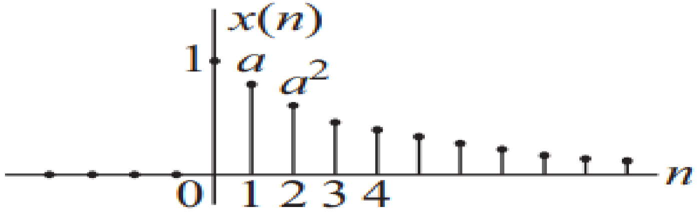


ROC provides the information about the signal in time domain

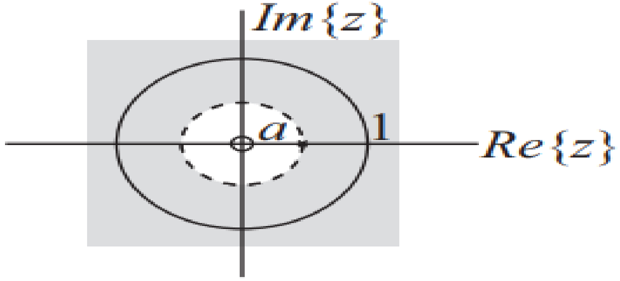
Right-Sided Exponential Sequence

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

The ROC for $X(z)$ is $|z| > |a|$, as shown in the shaded area in Figure.



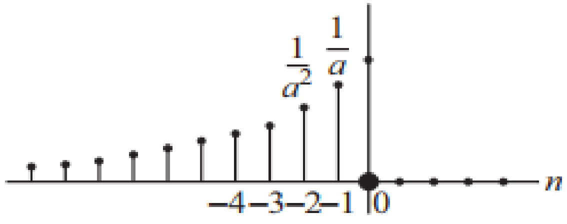
(a)



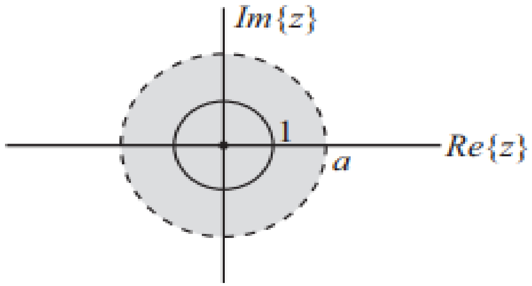
(b)

$$-a^n u(-n-1) \longleftrightarrow \frac{1}{1-az^{-1}}, \qquad |z| < |a|$$

The ROC for $X(z)$ is $|z| < |a|$, as shown in the shaded area in Figure.



(a)



(b)

Left-Sided Exponential Sequence

EXAMPLE 3.6 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, -2, 3, 5, 4\}$$

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We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + x(4)\,z^{-4} + x(5)\,z^{-5}$$

$$\therefore \qquad Z\,[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

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$$\therefore \qquad \mathrm{Z}\,[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 3.8 Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

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$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\ x(-4) = 2,\ x(-3) = 3,\ x(-2) = -1,\ x(-1) = -2,\ x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2) z^2 + x(-1)\, z + x(0)$$

$$\therefore \qquad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

EXAMPLE 3.8 Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\ x(-4) = 2,\ x(-3) = 3,\ x(-2) = -1,\ x(-1) = -2,\ x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2) z^2 + x(-1)\, z + x(0)$$

$$\therefore \qquad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

EXAMPLE 3.9 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$\begin{aligned} X(z) &= x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \\ &= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4} \end{aligned}$$

EXAMPLE 3.9 Find the Z-transform and ROC of the sequence

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We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

Example 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

Example 3.1.1

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$\mathbf{X_1[z] = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}}$$

ROC: entire Z-plane except Z=0

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X_2[z] = z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

ROC: entire Z-plane except Z=0 and Z=∞

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

$\mathbf{X_3[z] = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}}$ **ROC: entire Z-plane except Z=0**

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(d) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$$X_4[z] = 2 + 4z^{-1} + 5z^{-2} + 7z^{-3} + z^{-4}$$

ROC: entire Z-plane except Z=0

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(f) $x_6(n) = \delta(n - k), k > 0$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

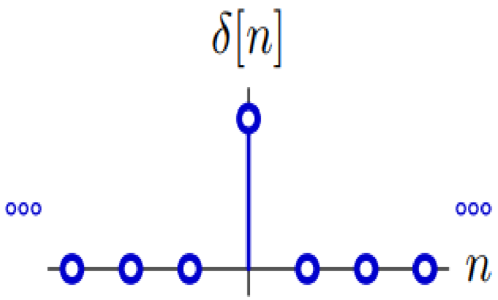
(f) $x_6(n) = \delta(n-k), k > 0$

$$X_6[z] = z^{-k}$$

ROC: entire Z-plane except Z=0

Simple Z transforms

Find the Z transform of the unit-sample signal.



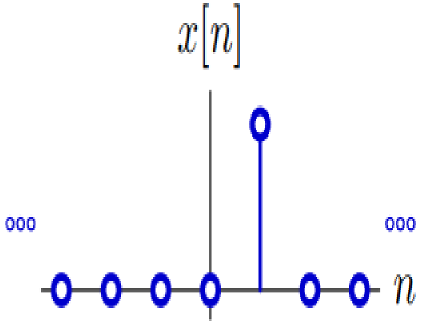
$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[0] z^{0} = 1$$

ROC=?

Simple Z transforms

Find the Z transform of a delayed unit-sample signal.



$$x[n] = \delta[n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[1] z^{-1} = z^{-1}$$

ROC=?

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(g) $x_7(n) = \delta(n+k), k > 0$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

(g) $x_7(n) = \delta(n + k), k > 0$

$\mathbf{X_7[z] = z^k}$

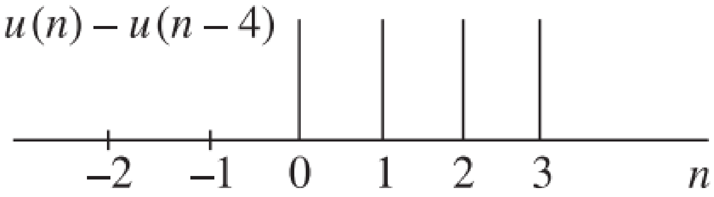
ROC: entire Z-plane except Z=∞

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

EXAMPLE 3.10 Find the Z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$

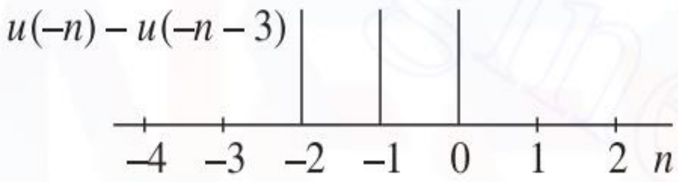
(a)



(a) ANS: $X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$

The ROC is entire z-plane except at $z = 0$.

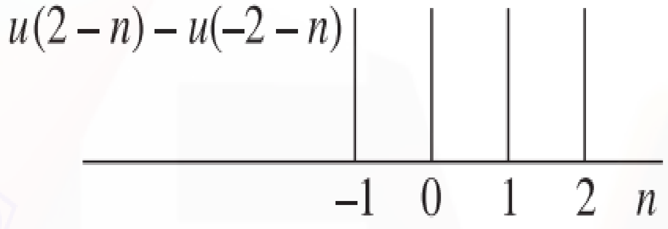
(b)



(b) ANS: $X(z) = 1 + z + z^2$

The ROC is entire z-plane except at $z = \infty$.

(c)



(c) ANS: $X(z) = z + 1 + z^{-1} + z^{-2}$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

Z-Transform of Unit Step Function

The *unit step signal* or *unit step sequence* is defined as –

$$x(n) = u(n) = \{ 1 \quad \text{for } n \geq 0 \; 0 \quad \text{for } n < 0 \}$$

G. S. $\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

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Therefore, the Z-transform of unit step function is given by,

$$Z[x(n)] = X(z) = Z[u(n)]$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

Z-Transform of Unit Step Function

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$$Z[x(n)] = X(z) = Z[u(n)]$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} (1) \cdot z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

Z-Transform of Unit Step Function

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$$x(n) = u(n) = \{1 \quad \text{for } n \geq 0 \; 0 \quad \text{for } n < 0\}$$

Therefore, the Z-transform of unit step function is given by,

$$Z[x(n)] = X(z) = Z[u(n)]$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} (1) \cdot z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

$$\Rightarrow X(z) = \frac{1}{(1 - z^{-1})} = \frac{z}{z - 1}$$

$$\boxed{\text{ROC} \rightarrow |z| > 1}$$

Z-Transform of Unit Ramp Sequence

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}; \quad a < 1$$

The unit ramp sequence is defined as -

$$x(n) = r(n) = \{n \quad \text{for } n \geq 0 \; 0 \quad \text{for } n < 0\}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} r(n)\, z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} n z^{-n} = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots$$

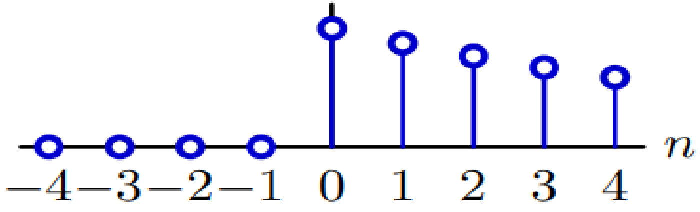
$$\Rightarrow X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}$$

$\text{ROC} \rightarrow |z| > 1$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

What is the Z transform of the following signal.

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$

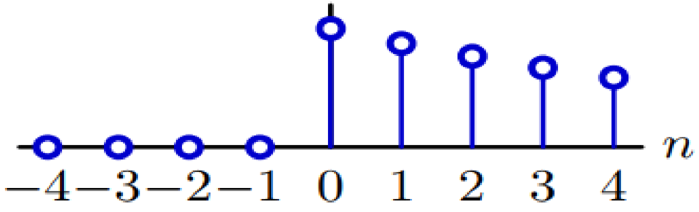


$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8} z^{-1}}$$

$$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

What is the Z transform of the following signal.

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8}z^{-1}}$$

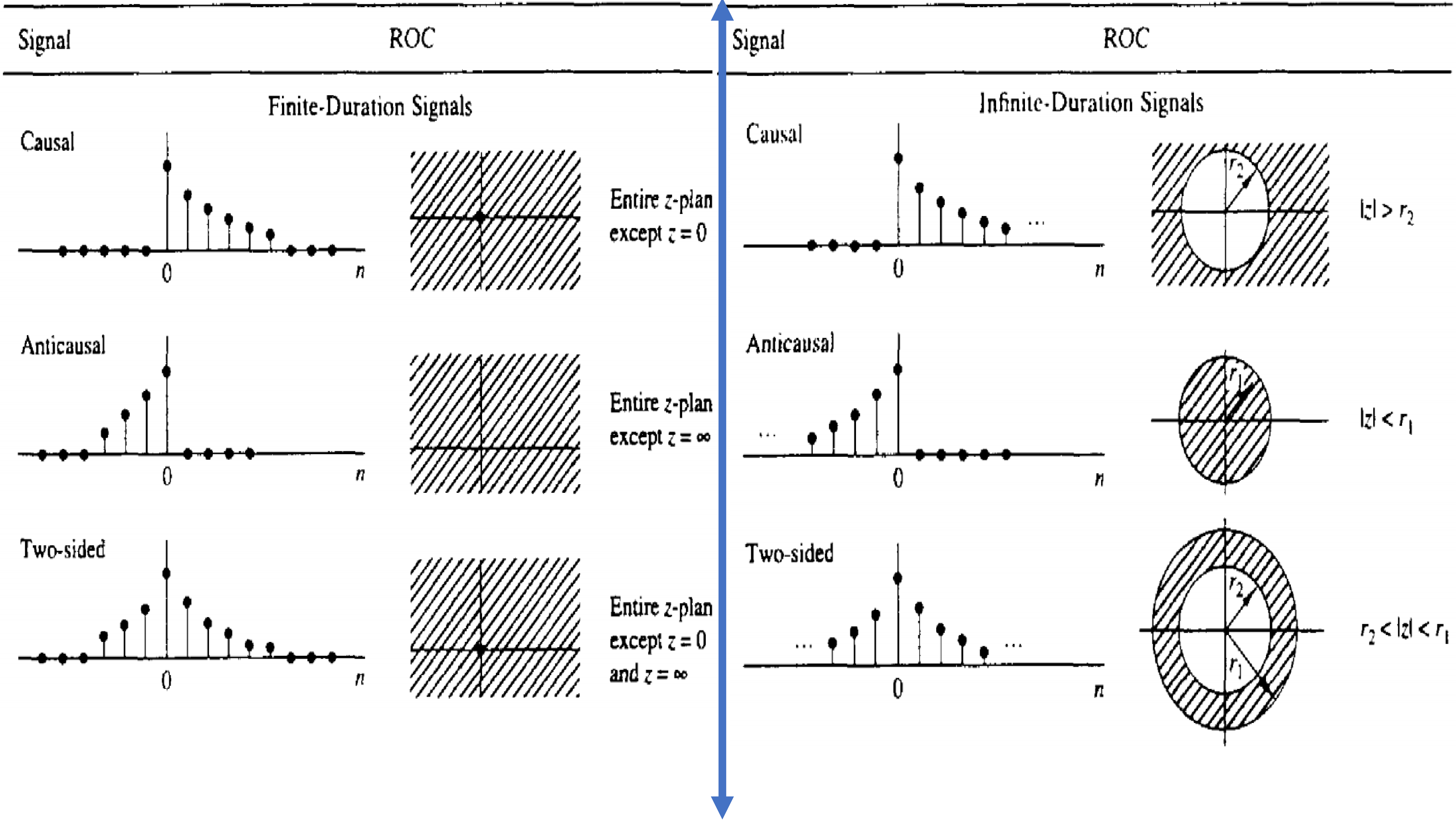
Regions of Convergence

The Z transform $X(z)$ is a function of z defined for all z inside **Region of Convergence (ROC)**.

$$x[n] = \left(\frac{7}{8}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1 - \frac{7}{8}z^{-1}}\,; \quad |z| > \frac{7}{8}$$

ROC: $|z| > \dfrac{7}{8}$

Properties of ROC of Z-Transforms


Signal
ROC
Finite-Duration Signals
Causal
0
n
Entire z-plan except $z = 0$
Anticausal
0
n
Entire z-plan except $z = \infty$
Two-sided
0
n
Entire z-plan except $z = 0$ and $z = \infty$
Signal
ROC
Infinite-Duration Signals
Causal
...
0
n
r_2
$|z| > r_2$
Anticausal
...
0
n
r_1
$|z| < r_1$
Two-sided
...
...
0
n
r_2
r_1
$r_2 < |z| < r_1$

3.4 PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.

   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.

   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.
5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

SUMMATION FORMULAS FOR GEOMETRIC SERIES

G. S. $\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

1. $$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

2. $$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}; \quad |a| < 1$$

3. $$\sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}; \quad |a| < 1$$

4. $$\sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1}-a^{n_2+1}}{1-a}; \; n_2 > n_1$$

5. $$\sum_{k=0}^{\infty} ka^k = \frac{a}{(1-a)^2}; \quad a < 1$$

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Z-transform properties

Z-transform properties

1. **Linearity**
2. Time shifting
3. Frequency scaling
4. Multiplication by n
5. Convolution in time
6. Initial value
7. Final value

- **Linearity:** If

$$x_1(n) \longleftrightarrow X_1(z) \quad \text{with ROC} = R_1$$

and

$$x_2(n) \longleftrightarrow X_2(z) \quad \text{with ROC} = R_2$$

then

$$ax_1(n) + bx_2(n) \longleftrightarrow aX_1(z) + bX_2(z) \quad \text{with ROC containing } R_1 \cap R_2$$

Linearity: If

$$x_1(n) \longleftrightarrow X_1(z) \quad \text{with ROC} = R_1$$

and

$$x_2(n) \longleftrightarrow X_2(z) \quad \text{with ROC} = R_2$$

then

$$ax_1(n) + bx_2(n) \longleftrightarrow aX_1(z) + bX_2(z) \quad \text{with ROC containing } R_1 \cap R_2$$

Proof: The z-transform of $ax_1(n) + bx_2(n)$ is given by

$$\begin{aligned} \mathcal{Z}[ax_1(n) + bx_2(n)] &= \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]z^{-n} \\ &= \underbrace{a \sum_{n=-\infty}^{\infty} x_1(n)z^{-n}}_{X_1(z)} + \underbrace{b \sum_{n=-\infty}^{\infty} x_2(n)z^{-n}}_{X_2(z)} \\ &= aX_1(z) + bX_2(z) \end{aligned}$$

Time Reversal Property of Z-Transform

Statement – The time reversal property of Z-transform states that the reversal or reflection of the sequence in time domain corresponds to the inversion in z-domain. Therefore, if

$$x(n) \overset{ZT}{\leftrightarrow} X(z)\,;\ \mathrm{ROC} = R$$

Then,

$$x(-n) \overset{ZT}{\leftrightarrow} X\left(\frac{1}{z}\right) = X\left(z^{-1}\right);\ \mathrm{ROC} = \frac{1}{R}$$

Time Reversal Property of Z-Transform

Proof

From the definition of Z-transform, we have,

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Now, by reversing the sequence in time domain, we get,

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

Substituting $-n = m$ in the above summation, we get,

$$Z[x(-n)] = \sum_{m=\infty}^{-\infty} x(m)\, z^{m}$$

$$\mathrm{Z}[\mathrm{x}(-\mathrm{n})] = \sum_{\mathrm{m}=\infty}^{-\infty} \mathrm{x}(\mathrm{m})(\mathrm{z}^{-1})^{-\mathrm{m}} = \mathrm{X}(\mathrm{z}^{-1})$$

$$\therefore\ Z[x(-n)] = X(Z^{-1}) = Z\left(\frac{1}{z}\right)$$

Time Reversal Property of Z-Transform

$$x(n) = u(-n)$$

Since the Z-transform of the unit step sequence is given by,

$$Z[u(n)] = \frac{z}{z-1}; \quad \text{ROC} \rightarrow |z| > 1$$

Now, by using the time reversal property of the Z-transform $[\text{i. e. , } x(-n) \overset{ZT}{\leftrightarrow} X\left(\frac{1}{z}\right)]$, we get,

$$Z[u(-n)] = \left[\frac{z}{z-1}\right]_{z=(1/z)}$$

$$\Rightarrow Z[u(-n)] = \frac{1/z}{(1/z)-1} = \frac{-1}{z-1}$$

$$\therefore u(-n) \overset{ZT}{\leftrightarrow} \frac{-1}{z-1}; \quad \text{ROC} \rightarrow |z| < 1$$

Time shifting

$$\boxed{x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)}$$

The new ROC is the same as ROC{X(z)} except for z = 0 if $n_0>0$ and z = ∞ if $n_0<0$

Proof:

$$Z\{x[n-n_o]\} = \sum_{n=-\infty}^{\infty} x[n-n_o]z^{-n}$$

$$let \quad l = n - n_o \qquad \sum_{l=-\infty}^{\infty} x[l]z^{-(l+n_o)} = z^{-n_o}\sum_{l=-\infty}^{\infty} x[l]z^{-l} = z^{-n_o}X(z)$$

Delay of k means that the Z-transform is multiplied by z^{-k}

Time shifting

$$x(n) = u(n-3)$$

Since the transform of a unit step sequence is given by,

$$Z[u(n)] = \frac{z}{z-1};\ \ \mathrm{ROC} \rightarrow |z| > 1$$

Therefore, using the time shifting property of Z-transform $\left[\mathrm{i.e.},\ x(n-n_0) \overset{ZT}{\leftrightarrow} z^{-n_0} X(z)\right]$, we get,

$$Z[u(n-3)] = z^{-3} Z[u(n)] = z^{-3}\left(\frac{z}{z-1}\right)$$

$$\therefore Z[u(n-3)] = \frac{1}{z^2(z-1)};\ \ \mathrm{ROC} \rightarrow |z| > 1$$

Time shifting

$$x(n) = \delta(n+5)$$

Solution

The given sequence is,

$$x(n) = \delta(n+5)$$

Since the Z-transform of the impulse sequence is given by,

$$Z[\delta(n)] = 1$$

Now, using the time shifting property of Z-transform $\left[\mathrm{i.e.},\ x(n+n_0) \overset{ZT}{\leftrightarrow} z^{n_0} X(z)\right]$, we get

$$Z[\delta(n+5)] = z^5\,(1) = z^5$$

Frequency scaling

$$\boxed{a^n x[n] \overset{Z}{\leftrightarrow} X\left(\frac{z}{a}\right)}$$

The new ROC is the scaled ROC{X(z)} with factor */a/* (bigger or smaller)

Proof:

$$ZT\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

Multiplication by a^n results in a complex scaling in the z-domain

$$x(n) = (2)^n u(n)$$

$$\boxed{a^n x[n] \overset{Z}{\leftrightarrow} X\left(\frac{z}{a}\right)}$$

Since the Z-transform of unit step function is given by,

$$Z[u(n)] = \frac{z}{z-1};\ \mathrm{ROC} \rightarrow |z| > 1$$

Now, using the scaling in z-domain or exponential property $\left[\mathrm{i.e},\ a^n x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)\right]$ of Z-transform, we get,

$$Z[(2)^n u(n)] = Z[u(n)]_{z=\left(\frac{z}{2}\right)} = \frac{\left(\frac{z}{2}\right)}{\left(\frac{z}{2}\right) - 1}$$

$$\therefore\ Z[(2)^n u(n)] = \frac{z}{z-2};\ \mathrm{ROC} \rightarrow |z| > 2$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Since the Z-transform of the unit step sequence is given by,

$$Z[u(n)] = \frac{z}{z-1};\ \mathrm{ROC} \rightarrow |z| > 1$$

Now, from the property of multiplication by an exponential $\left[\mathrm{i.e,}\ a^n x(n) \overset{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)\right]$,we have,

$$Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = Z[u(n)]_{z=\left[\frac{z}{\frac{1}{2}}\right]=2z} = \frac{2z}{2z-1}$$

$$\therefore Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = \left(\frac{z}{z-\frac{1}{2}}\right);\ \mathrm{ROC} \rightarrow |z| > \frac{1}{2}$$

$$\boxed{a^n x[n] \overset{Z}{\leftrightarrow} X\left(\frac{z}{a}\right)}$$

Multiplication by n

$$\boxed{nx[n] \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}}$$

The new ROC is the same ROC{X(z)}

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \Rightarrow \frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1} = -\frac{1}{z}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$$\Rightarrow ZT\{nx[n]\} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{dX(z)}{dz}$$

Convolution in time

$$\boxed{y[n] = x[n] * h[n] \overset{Z}{\longleftrightarrow} X(z)H(z)}$$

The new ROC is the intersection of ROC{X(z)} and ROC{Y(z)}

If poles cancel zeros then the new ROC is bigger

Proof:

$$y[n] = x[n] * h[n] \overset{Z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x[k]h[n-k]]z^{-n}$$

Switching the order of the summation:

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{n-k=-\infty}^{\infty} h[n-k]z^{-(n-k)}$$

$$= X(z).H(z)$$

$$x(n) = \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)$$

Solution

$$Z[x_1(n)] = X_1(z) = Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$X_1(z) = \frac{z}{\left(z - \frac{1}{3}\right)};\ \mathrm{ROC} \rightarrow |z| > \frac{1}{3}$$

$$Z[x_2(n)] = X_2(z) = Z\left[\left(\frac{1}{5}\right)^n u(n)\right]$$

$$X_2(z) = \frac{z}{\left(z - \frac{1}{5}\right)};\ \mathrm{ROC} \rightarrow |z| > \frac{1}{5}$$

$$\therefore Z\left[\left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)\right] = \frac{z}{\left(z - \frac{1}{3}\right)} \frac{z}{\left(z - \frac{1}{5}\right)}$$

The ROC of the Z-transform of the given sequence is

$$\mathrm{ROC} \rightarrow \left[|z| > \frac{1}{3}\right] \cap \left[|z| > \frac{1}{5}\right] = |z| > \frac{1}{3}$$

$$\therefore \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n) \overset{ZT}{\leftrightarrow} \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{5}\right)};\ \mathrm{ROC} \rightarrow |z| > \frac{1}{3}$$

Initial value theorem

If x[n] is causal then x[0] is the initial value of the function x[n]

$$\boxed{x[0] = \lim_{z\to\infty} X(z)}$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

$$Obviously,\quad as\ z \to \infty,\quad z^{-n} \to 0$$

Initial value theorem

If $x[n] = 0$ with $n < n_0$ then $x[n_0]$ is the initial value, and

$$\boxed{x[n_0] = \lim_{z\to\infty}[z^{n_0} X(z)]}$$

Proof:

$$X(z) = \sum_{n=n_0}^{\infty} x[n]z^{-n} = x[n_0]z^{-n_0} + x[n_0+1]z^{-(n_0+1)} + x[n_0+2]z^{-(n_0+2)} + \ldots$$

$$z^{n_0} X(z) = x[n_0] + x[n_0+1]z^{-1} + x[n_0+2]z^{-2} + \ldots$$

$$As\ z \to \infty, \quad z^{n_0} X(z) = x[n_0]$$

Numerical Example (1)

Using the initial value theorem for Z-transform, find the initial value of signal $x(n)$, i.e., $x(0)$if X(z) is given by,

$$X(z) = \frac{z^2 + z + 1}{(z+2)(z+1)}$$

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Solution

Now, using the initial value theorem of Z-transform $\left[\mathrm{i.e}, x(0) = \lim_{n \to 0} x(n) = \lim_{z \to \infty} X(z)\right]$ .we get,

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[\frac{1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z^2}\right)}{\left(1 + \frac{2}{z}\right)\left(1 + \frac{1}{z}\right)}\right]$$

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$$\Rightarrow x(0) = \frac{1+0+0}{(1+0)(1+0)} = 1$$

Numerical Example (2)

Find the initial value $x(0)$ of a sequence $x(n)$, if X(z) is given by,

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Solution

The given Z-transform of the sequence is,

$$X(z) = \frac{z+4}{(z+2)(z+1)}$$

$$\Rightarrow X(z) = \frac{\mathbf{z}\left(1+\frac{4}{z}\right)}{z^2\left(1+\frac{2}{z}\right)\left(1+\frac{1}{z}\right)} = \frac{\left(1+\frac{4}{z}\right)}{z\left(1+\frac{2}{z}\right)\left(1+\frac{1}{z}\right)}$$

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Using the initial value theorem of Z-transform, we get,

$$x(0) = \lim_{z \to \infty}\left[\frac{\left(1+\frac{4}{z}\right)}{z\left(1+\frac{2}{z}\right)\left(1+\frac{1}{z}\right)}\right] = 0$$

Final value theorem

If this limit exists then x[n] has a final value (steady-state value)

$$\boxed{\lim_{n\to\infty} x[n] = x[\infty] = \lim_{z\to 1}[(z-1)X(z)]}$$

Proof: HW

Numerical Example (1)

Find $x(\infty)$ if X(z) is given by,

$$X(z) = \frac{z^2}{(z-1)(z-0.3)}$$

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$$\therefore x(\infty) = \left[\frac{1}{(1-0.3)}\right] = 1.43$$

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Using the final value theorem, calculate $x(\infty)$ if X(z) is given by,

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Solution

The given Z-transform of the sequence is,

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

$$\therefore\ (\mathrm{z}-1)\,X(z) = \frac{z+1}{3(z+0.4)}$$

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Solution

The given Z-transform of the sequence is,

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

$$\therefore\ (z-1)X(z) = \frac{z+1}{3(z+0.4)}$$

As we can see, $(z-1)X(z)$ has no poles on or outside the unit circle. we have,

$$x(\infty) = \lim_{z\to 1}\left[\frac{z+1}{3(z+0.4)}\right] = \left[\frac{1+1}{3(1+0.4)}\right]$$

$$\therefore x(\infty) = \left[\frac{2}{3\times 1.4}\right] = 0.48$$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1-z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1-z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

Solution:

(a) Given $x(n) = u(-n)$

We know that $Z[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1-z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

Solution:

(a) Given $x(n) = u(-n)$

We know that $Z[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

Using the time reversal property,

$$\boxed{Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)}}$$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1 - z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z - 1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

Solution:

(a) Given $x(n) = u(-n)$

We know that $Z[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

Using the time reversal property,

$$Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)} = \boxed{\frac{1/z}{(1/z)-1}}$$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1 - z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z - 1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

Solution:

(a) Given $\qquad x(n) = u(-n)$

We know that $Z[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

Using the time reversal property,

$$Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)} = \frac{1/z}{(1/z)-1} \boxed{= \frac{1}{1-z} = -\frac{1}{z-1}; \text{ ROC; } |z| < 1}$$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1 - z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z - 1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(b) $x(n) = u(-n+1)$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \lvert z \rvert < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a \rvert r_2 < \lvert z \rvert < \lvert a \rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z \rvert < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z \rvert < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(b) $x(n) = u(-n+1)$

step 1. Shift $\mathbf{x[n] = u(n+1)}$

$$\mathbf{X(z) = z^1(z/(z-1)) = z^2/(z-1)} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}, \text{ except } \mathbf{z = \infty}$$

step 2. reverse $\mathbf{x[n] = u(-n+1)}$

$$\mathbf{X(z) = (z^{-1})^2/(z^{-1}-1) = 1/z(1-z) = -1/z(z-1)} \quad \text{ROC} \quad |\mathbf{z}| < \mathbf{1}, \text{ except } \mathbf{z = 0}$$

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1-z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(c) $x(n) = u(-n-2)$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(c) $x(n) = u(-n-2)$

step 1. Shift $\mathbf{x[n] = u(n-2)}$

$$\mathbf{X(z) = z^{-2}(2/(z-1)) = 1/z(z-1)} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

step 2. Reverse $\mathbf{x[n] = u(-n-2)}$

$$\mathbf{X(z) = 1/z^{-1}(z^{-1}-1) = z^2/(1-z) = -z^2/(z-1)} \quad \text{ROC} \quad |\mathbf{z}| < \mathbf{1}$$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\, x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad \mathbf{|z| > 1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(d) $x(n) = 2^n u(n-2)$

step 1. Shift $\mathbf{x[n] = u(n-2)}$

$$\mathbf{X(z) = \frac{z^{-2}}{(z-1)} = \frac{1}{z^2\ (z-1))}} \quad \text{ROC} \quad \mathbf{|z| > 1}$$

step 2. z-scaling $\mathbf{x[n] = 2^n u(n-2)}$

$$\mathbf{X(z) = \frac{1}{\frac{z^2}{2}(\frac{z}{2}-1)} = \frac{4}{z^2(z-2)}} \quad \text{ROC} \quad \mathbf{|z| > 2}$$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except
			$z = 0$ (if $k > 0$)
			and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

Find Z-transform of $\mathbf{x[n] = a^n\ u(-n-1)}$

Step 1. Time Shifting $\mathbf{x[n] = u(n-1)}$

$$\mathrm{X(z)} = \mathbf{z^{-1}\frac{z}{z-1} = \frac{z^{1-1}}{z-1} = \frac{1}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

Step 2. Time Reversal of $\mathbf{x[n] = u(-n-1)}$

$$\mathbf{X(z) = \frac{1}{z^{-1}-1} = \frac{z}{1-z} = -\frac{z}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| < \mathbf{1}$$

Step 3. z-scaling $\mathbf{x[n] = a^n\ u(-n-1)}$

$$\mathbf{X(z) = -\frac{z/a^{-1}}{z/a^{-1}-1} = -\frac{z}{z-a}} \quad \text{ROC} \quad |\mathbf{z}| < \mathbf{a}$$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
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z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$

Sum of Two Exponential Sequences

$$\mathbf{u(n)} \longleftrightarrow \frac{\mathbf{1}}{\mathbf{1 - z^{-1}}} = \frac{\mathbf{z}}{\mathbf{z - 1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

Example 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n - 1)$$

Sum of Two Exponential Sequences

$$u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \quad \text{ROC} \quad |z| > 1$$

Example 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X[z] = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} \qquad \text{ROC } |z| > |a| \cap |z| < |b|$$

Sum of Two Exponential Sequences

$$u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \quad \text{ROC } |z| > 1$$

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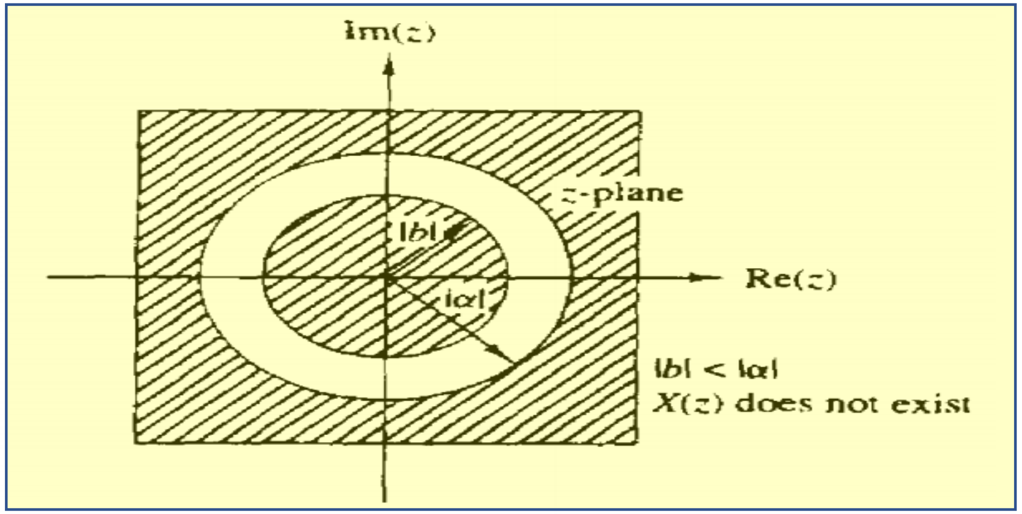

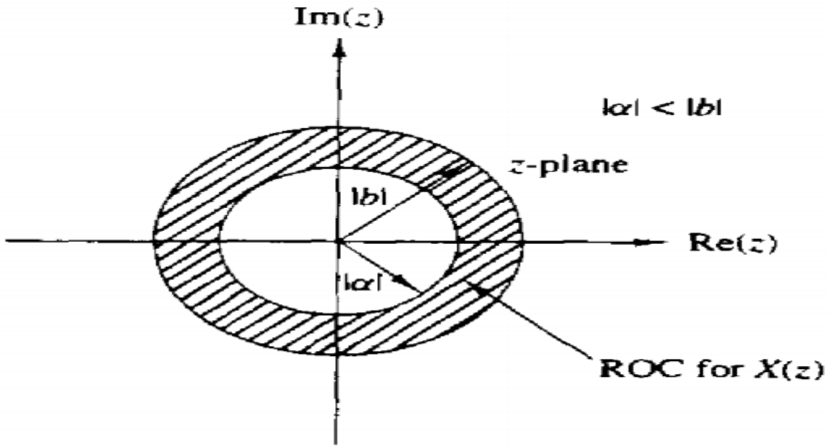

EXAMPLE 3.12 Using properties of Z-transform, find the Z-transform of the sequence

(a) $x(n) = \alpha^{n-2} u(n-2)$ (b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha}; \text{ ROC}; |z| > |\alpha|$$

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Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m} X(z)$$

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Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha}; \text{ ROC; } |z| > |\alpha|$$

Using the time shifting property of Z-transform, we have

$$\text{Z}[x(n-m)] = z^{-m} X(z)$$

In the same way,

$$\text{Z}\,[\alpha^{n-2}\, u(n-2)] = z^{-2}\, \text{Z}[\alpha^n u(n] = z^{-2} \frac{z}{z-\alpha} = \frac{1}{z(z-\alpha)}; \text{ ROC; } |z| > |\alpha|$$

EXAMPLE 3.16 Using Z-transform, find the convolution of the sequences

$$x_1(n) = \{2, 1, 0, -1, 3\};\ x_2(n) = \{1, -3, 2\}$$

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$$x_1(n) = \{2, 1, 0, -1, 3\};\ x_2(n) = \{1, -3, 2\}$$

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$\therefore$ $$X_1(z) = 2 + z^{-1} - z^{-3} + 3z^{-4}$$

and $$x_2(n) = \{1, -3, 2\}$$

$\therefore$ $$X_2(z) = 1 - 3z^{-1} + 2z^{-2}$$

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$\therefore$ $$X_1(z)\, X_2(z) = (2 + z^{-1} - z^{-3} + 3z^{-4})\,(1 - 3z^{-1} + 2z^{-2})$$

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$\therefore$ $$X_1(z)\,X_2(z) = (2 + z^{-1} - z^{-3} + 3z^{-4})(1 - 3z^{-1} + 2z^{-2})$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

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$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

Taking inverse Z-transform on both sides,

$$x(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

Solve the following problems (C.W.)

3. $u[n]$ $\quad \dfrac{z}{z-1} \quad |z| > 1$

4. n

6. a^n $\quad \dfrac{z}{z-a} \quad |z| > |a|$

7. na^n

Solve the following problems (C.W.)

3. $u[n]$	$\dfrac{z}{z-1}$	$\lvert z \rvert > 1$
4. n	$\dfrac{z}{(z-1)^2}$	$\lvert z \rvert > 1$
6. a^n	$\dfrac{z}{z-a}$	$\lvert z \rvert > \lvert a \rvert$
7. na^n	$\dfrac{az}{(z-a)^2}$	$\lvert z \rvert > \lvert a \rvert$

LTI SYSTEM APPLICATIONS

In this section, we illustrate some applications of the z-transform to linear time-invariant (LTI) systems. First, we consider transfer functions, and then certain system properties are investigated.

Transfer Functions

When possible, we model discrete-time systems with linear difference equations with constant coefficients; the model is then linear and time invariant. (See Section 10.4.) From (10.48), the general equation for this model is given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad (11.39)$$

Thus, the z-transform of (11.39), with $M = N$, yields

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{N} b_k z^{-k} X(z),$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-N+1} + b_N z^{-N}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1} + a_N z^{-N}}$$

$$= \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N},$$

Transfer function of a discrete system

$$y[n] - 0.9y[n-1] = 0.1x[n].$$

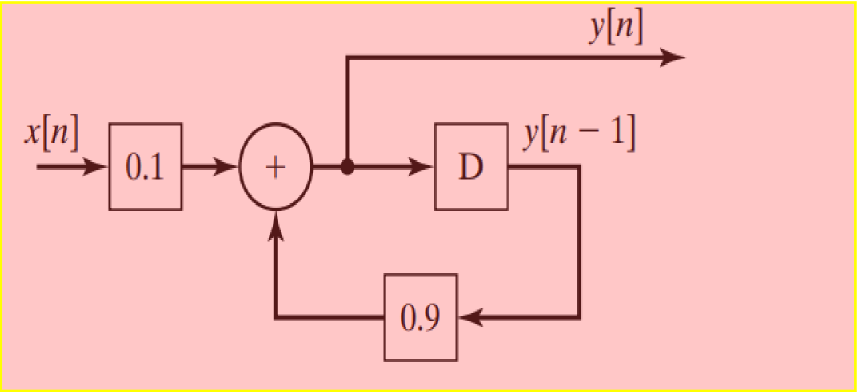
The z-transform of this equation yields

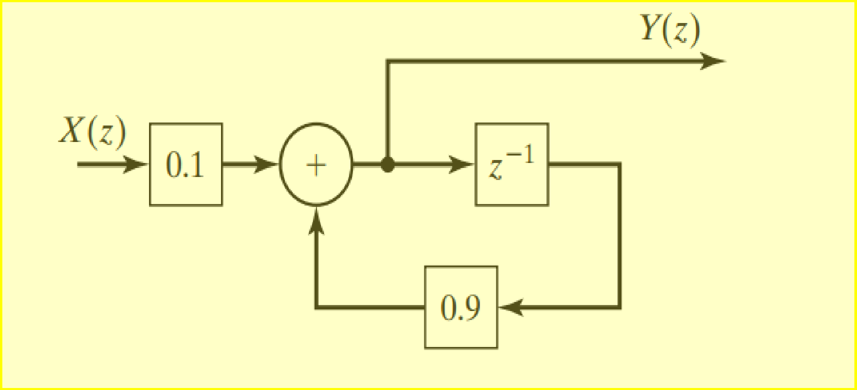
$$(1 - 0.9z^{-1})Y(z) = 0.1X(z),$$

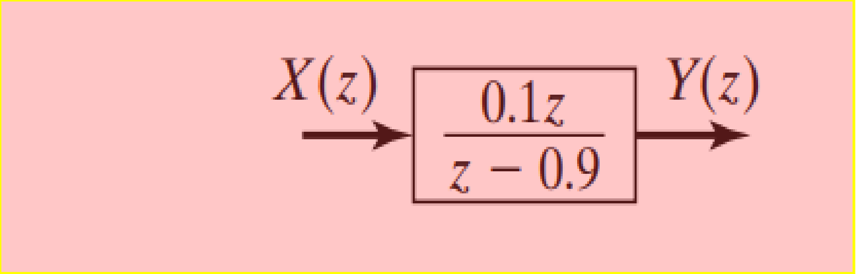
and the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1}{1 - 0.9z^{-1}} = \frac{0.1z}{z - 0.9}.$$

What is h(n)=?







Inverse Z-Tranform

3.6 INVERSE Z-TRANSFORM

The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

This is the DTFT of the signal $x(n)\, r^{-n}$. Hence the Inverse Discrete-Time Fourier Transform (IDTFT) of $X(re^{j\omega})$ must be $x(n)\, r^{-n}$. Therefore, we can write

$$x(n) r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

i.e.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^{n}\, d\omega$$

We have

$$z = re^{j\omega}$$

$\therefore$

$$\frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}$$

$\therefore$

$$x(n) = \frac{1}{2\pi j} \oint_{c} X(z)\, z^{n-1}\, dz$$

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We have

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$$\therefore \quad \frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}$$

$$\therefore \quad x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

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This is the DTFT of the signal $x(n)\, r^{-n}$. Hence the Inverse Discrete-Time Fourier Transform (IDTFT) of $X(re^{j\omega})$ must be $x(n)\, r^{-n}$. Therefore, we can write

$$x(n) r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

i.e.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n\, d\omega$$

We have

$$z = re^{j\omega}$$

$\therefore$

$$\boxed{\frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}}$$

$\therefore$

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

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We have

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$\therefore$

$$x(n) = \frac{1}{2\pi j} \oint_{c} X(z)\, z^{n-1}\, dz$$

Inversion of Z-transform

1. Inverse Z-transform formula

2. Using partial fraction expansion to invert the ZT

3. Using power series expansion to invert ZT

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

when (a) ROC: $|z| > \frac{1}{3}$. (b) ROC: $|z| < \frac{1}{3}$, using a power series expansion.

Example: Find the inverse z-transform of

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Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$1 - \tfrac{1}{3}z^{-1} \overline{\big) \; 1 + z^{-1} \qquad\qquad\qquad\qquad\qquad\qquad}$$

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Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$\begin{array}{r|l}
 & 1 + \\
\cline{2-2}
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & 1 - \frac{1}{3}z^{-1} \\
\cline{2-2}
\end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

when (a) ROC: $|z| > \frac{1}{3}$. (b) ROC: $|z| < \frac{1}{3}$, using a power series expansion.

Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$\begin{array}{r|l} & 1 + \\ \hline 1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\ & -1 + \frac{1}{3}z^{-1} \\ \hline & \quad \frac{4}{3}z^{-1} \end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

when (a) ROC: $|z| > \frac{1}{3}$. (b) ROC: $|z| < \frac{1}{3}$, using a power series expansion.

Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$
\begin{array}{r|l}
 & 1 + \frac{4}{3}z^{-1} \\
\hline
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & -1 + \frac{1}{3}z^{-1} \\
\hline
 & \quad \frac{4}{3}z^{-1} \\
 & \qquad \frac{4}{3}z^{-1} - \frac{4}{9}z^{-2} \\
\hline
\end{array}
$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

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$$
\begin{array}{r|l}
 & 1 + \frac{4}{3}z^{-1} \\
\cline{2-2}
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & -1 + \frac{1}{3}z^{-1} \\
\cline{2-2}
 & \frac{4}{3}z^{-1} \\
 & - \quad \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} \\
\cline{2-2}
 & \frac{4}{9}z^{-2}
\end{array}
$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

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Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$\begin{array}{r|l}
 & 1 + \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} \\ \hline
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & -1 + \frac{1}{3}z^{-1} \\ \cline{2-2}
 & \quad \frac{4}{3}z^{-1} \\
 & - \quad \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} \\ \cline{2-2}
 & \qquad\qquad \frac{4}{9}z^{-2} \\
 & \qquad\qquad \frac{4}{9}z^{-2} - \frac{4}{27}z^{-3} \\ \cline{2-2}
\end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

when (a) ROC: $|z| > \frac{1}{3}$. (b) ROC: $|z| < \frac{1}{3}$, using a power series expansion.

Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$\begin{array}{r|l}
 & 1 + \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} \\
\hline
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & -\left(1 - \frac{1}{3}z^{-1}\right) \\
\hline
 & \frac{4}{3}z^{-1} \\
 & -\left(\frac{4}{3}z^{-1} - \frac{4}{9}z^{-2}\right) \\
\hline
 & \frac{4}{9}z^{-2} \\
 & -\left(\frac{4}{9}z^{-2} - \frac{4}{27}z^{-3}\right) \\
\hline
 & \frac{4}{27}z^{-3}
\end{array}$$

Example: Find the inverse z-transform of

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Solution: **(a)** Since the ROC is $|z| > \frac{1}{3}$, we express $X(z)$ as a power series in z^{-1}.

$$\begin{array}{r|l}
 & 1 + \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} + \frac{4}{27}z^{-3} + \cdots \\
\hline
1 - \frac{1}{3}z^{-1} & 1 + z^{-1} \\
 & -\left(1 - \frac{1}{3}z^{-1}\right) \\
\hline
 & \frac{4}{3}z^{-1} \\
 & -\left(\frac{4}{3}z^{-1} - \frac{4}{9}z^{-2}\right) \\
\hline
 & \frac{4}{9}z^{-2} \\
 & -\left(\frac{4}{9}z^{-2} - \frac{4}{27}z^{-3}\right) \\
\hline
 & \frac{4}{27}z^{-3}
\end{array}$$

We can write, therefore,

$$\boxed{X(z) = 1 + \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} + \frac{4}{27}z^{-3} + \cdots}$$

Comparing with

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

we obtain

$$x(0) = 1, \quad x(1) = \frac{4}{3}, \quad x(2) = \frac{4}{9}, \quad x(3) = \frac{4}{27}, \quad \cdots$$

or equivalently, we can express $x(n)$ as

$$x(n) = \left\{ \underset{\uparrow}{\mathbf{1}}, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \cdots \right\}$$

We can write, therefore,

$$\boxed{X(z) = 1 + \frac{4}{3}z^{-1} + \frac{4}{9}z^{-2} + \frac{4}{27}z^{-3} + \cdots}$$

Comparing with $$\mathbf{X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

$$\boxed{X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots}$$

we obtain

$$x(0) = 1, \quad x(1) = \frac{4}{3}, \quad x(2) = \frac{4}{9}, \quad x(3) = \frac{4}{27}, \quad \cdots$$

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or equivalently, we can express $x(n)$ as

$$x(n) = \left\{ \underset{\uparrow}{1}, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \cdots \right\} \qquad \mathbf{x(n) = \delta(n) + 4(1/3)^n u(n-1)}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$-\tfrac{1}{3}z^{-1} + 1 \,\big)\overline{\; z^{-1} + 1 \qquad\qquad\qquad\qquad\qquad\qquad}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$\begin{array}{r|l} & -3 \\ \hline -\frac{1}{3}z^{-1} + 1 & z^{-1} + 1 \\ & z^{-1} - 3 \end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$\begin{array}{r|l} & -3 \\ \hline -\frac{1}{3}z^{-1} + 1 \,) & z^{-1} + 1 \\ & -z^{-1} + 3 \\ \hline & \quad\; 4 \end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$\begin{array}{r|l} & -3 - 12z \\ \hline -\frac{1}{3}z^{-1} + 1 & z^{-1} + 1 \\ & -z^{-1} + 3 \\ \cline{2-2} & \quad 4 \end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$
\begin{array}{r|l}
 & -3 - 12z \\
\hline
-\frac{1}{3}z^{-1} + 1 & z^{-1} + 1 \\
 & - \; z^{-1} + 3 \\
\cline{2-2}
 & \qquad 4 \\
 & \quad - \quad 4 + 12z \\
\cline{2-2}
 & \qquad\qquad 12z
\end{array}
$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$\begin{array}{r|l}
 & -3 - 12z - 36z^2 \\
\hline
-\frac{1}{3}z^{-1} + 1 & z^{-1} + 1 \\
 & -\; z^{-1} + 3 \\
\hline
 & \qquad 4 \\
 & - \qquad 4 + 12z \\
\hline
 & \qquad\qquad 12z \\
 & \qquad\qquad 12z - 36z^2 \\
\hline
 & \qquad\qquad\qquad 36z^2
\end{array}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(b) Since the ROC is $|z| < \frac{1}{3}$, we express $X(z)$ as a power series in z.

$$
\begin{array}{r|l}
 & -3 - 12z - 36z^2 \\
\hline
-\frac{1}{3}z^{-1} + 1 & z^{-1} + 1 \\
 & -\; z^{-1} + 3 \\
\hline
 & \quad 4 \\
 & -\quad 4 + 12z \\
\hline
 & \qquad 12z \\
 & \qquad 12z - 36z^2 \\
\hline
 & \qquad\qquad 36z^2
\end{array}
$$

We can write, therefore,

$$
\begin{aligned}
X(z) &= -3 - 12z - 36z^2 + \cdots \\
&= \cdots + (-36)z^2 + (-12)z + (-3)
\end{aligned}
$$

We can write, therefore,

$$\boxed{\begin{aligned} X(z) &= -3 - 12z - 36z^2 + \cdots \\ &= \cdots + (-36)z^2 + (-12)z + (-3) \end{aligned}}$$

Comparing the above equation with

$$X(z) = \cdots + x(-2)z^2 + x(-1)z + x(0)$$

we obtain

$$\cdots, \quad x(-2) = -36, \quad x(-1) = -12, \quad x(0) = -3$$

or equivalently, we can express $x(n)$ as

$$\boxed{x(n) = \left\{\cdots, -36, -12, \underset{\uparrow}{-3}\right\}}$$

Partial Fraction Expansion Method to Find Inverse Z-Transform

In order to determine the inverse Z-transform of $X(z)$ using partial fraction expansion method, the denominator of $X(z)$ must be in factored form. In this method, we obtained the partial fraction expansion of $\frac{X(z)}{z}$ instead of $X(z)$. This is because the Z-transform of time-domain sequences have Z in their numerators.

The partial fraction expansion method is applied only if $\frac{X(z)}{z}$ is a proper rational function, i.e., the order of its denominator is greater than the order of its numerator.

If $\frac{X(z)}{z}$ is not a proper function, then it should be written in the form of a polynomial and a proper function before applying the partial fraction method.

The disadvantage of the partial fraction method is that, the denominator of $X(z)$ must be in factored form. Once the $\frac{X(z)}{z}$ is obtained as a proper function, then using the standard Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be obtained.

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The disadvantage of the partial fraction method is that, the denominator of $X(z)$ must be in factored form. Once the $\frac{X(z)}{z}$ is obtained as a proper function, then using the standard Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be obtained.

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}; \text{ ROC; } |z| > 1$$

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}};\ \text{ROC};\ |z|>1$$

Solution: Given

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}}=\frac{z}{3z^2-4z+1}$$

$$=\frac{z}{3\,[z^2-(4z/3)+(1/3)]}=\frac{1}{3}\,\frac{z}{(z-1)\,[z-(1/3)]}$$

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}; \text{ ROC; } |z| > 1$$

Solution: Given $X(z) = \dfrac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \dfrac{z}{3z^2 - 4z + 1}$

$$= \frac{z}{3\,[z^2 - (4z/3) + (1/3)]} = \frac{1}{3}\,\frac{z}{(z-1)\,[z - (1/3)]}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{3}\,\frac{1}{(z-1)\,[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}; \text{ ROC; } |z| > 1$$

Solution: Given $$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

$$= \frac{z}{3\,[z^2 - (4z/3) + (1/3)]} = \frac{1}{3}\frac{z}{(z-1)\,[z-(1/3)]}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{3}\frac{1}{(z-1)\,[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\frac{1}{3}\left.\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1} = \frac{1}{3}\frac{1}{1-(1/3)} = \frac{1}{2}$$

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}; \text{ ROC; } |z| > 1$$

Solution: Given

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

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$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{3}\,\frac{1}{(z-1)\,[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\left.\frac{1}{3}\,\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1} = \frac{1}{3}\,\frac{1}{1-(1/3)} = \frac{1}{2}$$

$$B = \left(z - \frac{1}{3}\right)\left.\frac{X(z)}{z}\right|_{z=1/3} = \left(z - \frac{1}{3}\right)\left.\frac{1}{3}\,\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1/3} = \frac{1}{3}\,\frac{1}{(1/3)-1} = -\frac{1}{2}$$

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}};\ \text{ROC; } |z|>1$$

Solution: Given

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}}=\frac{z}{3z^2-4z+1}$$

$$=\frac{z}{3[z^2-(4z/3)+(1/3)]}=\frac{1}{3}\frac{z}{(z-1)[z-(1/3)]}$$

$$\therefore\quad \frac{X(z)}{z}=\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}=\frac{A}{z-1}+\frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A=(z-1)\left.\frac{X(z)}{z}\right|_{z=1}=(z-1)\frac{1}{3}\left.\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1}=\frac{1}{3}\frac{1}{1-(1/3)}=\frac{1}{2}$$

$$B=\left(z-\frac{1}{3}\right)\left.\frac{X(z)}{z}\right|_{z=1/3}=\left(z-\frac{1}{3}\right)\frac{1}{3}\left.\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1/3}=\frac{1}{3}\frac{1}{(1/3)-1}=-\frac{1}{2}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or $$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right]; \text{ ROC; } |z| > 1$$

Since ROC is $|z| > 1$, both the sequences must be causal. Therefore, taking inverse Z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]; \text{ ROC; } |z| > 1$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or

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C. W.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Solve the following problems

a. $a^n u(n)$

b. $a^{n-1} u(n-1)$

c. $a^n u(n-1)$

d. $n u(n)$

e. $n^2 u(n)$

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Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{2-3z^{-1}+z^{-2}}$; ROC $\rightarrow |z| > 1$

$$\Rightarrow X(z) = \frac{z}{2z^2-3z+1} = \frac{z}{2\left[z^2-\left(\frac{3z}{2}\right)+\left(\frac{1}{2}\right)\right]}$$

$$\Rightarrow X(z) = \frac{1}{2}\left\{\frac{z}{(z-1)\left[z-\left(\frac{1}{2}\right)\right]}\right\}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{\left[z-\left(\frac{1}{2}\right)\right]}$$

Where, A and B are determined as follows

$$A = \left[(z-1)\frac{X(z)}{z}\right]_{z=1} = (z-1)\left[\frac{1}{2}\frac{z}{(z-1)\left[z-\left(\frac{1}{2}\right)\right]}\right]_{z=1} = \frac{1}{2}\left[\frac{1}{1-\left(\frac{1}{2}\right)}\right] = 1$$

$$B = \left[\left(z-\frac{1}{2}\right)\frac{X(z)}{z}\right]_{z=\frac{1}{2}} = \left(z-\frac{1}{2}\right)\left[\frac{1}{2}\frac{z}{(z-1)\left[z-\left(\frac{1}{2}\right)\right]}\right]_{z=\frac{1}{2}} = \frac{1}{2}\left[\frac{1}{\left(\frac{1}{2}\right)-1}\right] = -1$$

$$\therefore \frac{X(z)}{z} = \frac{1}{(z-1)} - \frac{1}{\left[z-\left(\frac{1}{2}\right)\right]} \quad \Rightarrow X(z) = \frac{z}{(z-1)} - \frac{z}{\left[z-\left(\frac{1}{2}\right)\right]}; \text{ ROC} \rightarrow |z| > 1$$

Because the region of convergence (ROC) of the given Z-transform is $|z| > 1$, thus both the sequences must be casual. Hence, by taking the inverse Z-transform, we get,

$$Z^{-1}[X(z)] = Z^{-1}\left[\frac{z}{(z-1)} - \frac{z}{\left[z-\left(\frac{1}{2}\right)\right]}\right]$$

$$\therefore x(n) = \left[u(n) - \left(\frac{1}{2}\right)^n u(n)\right]$$

Multiple-order poles. If $X(z)$ has a pole of multiplicity l, that is, it contains in its denominator the factor $(z - p_k)^l$, then the expansion (3.4.15) is no longer true. In this case a different expansion is needed. First, we investigate the case of a double pole (i.e., $l = 2$).

Example 3.4.7

Determine the partial-fraction expansion of

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2} \tag{3.4.23}$$

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$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \tag{3.4.24}$$

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$$A_1 = \frac{X(z)}{z} \times (z+1) \Big|_{z=-1}$$

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$$A_1 = \frac{X(z)}{z} \times (z+1) \Big|_{z=-1}$$

$$= \frac{z^2}{(z-1)^2}\Big|_{z=-1} = \frac{1}{4}$$

$$A_3 = \frac{z^2}{(z+1)\cancel{(z-1)^2}} \times \cancel{(z-1)^2} \Big|_{z=1}$$

$$= \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left(\frac{z^2}{(z+1)\cancel{(z-1)^2}} \times \cancel{(z-1)^2}\right)\Big|_{z=1}$$

$$= \frac{d}{dz}\left(\frac{z^2}{z+1}\right)\Big|_{z=1}$$

$X(z)$ has a simple pole at $p_1 = -1$ and a double pole $p_2 = p_3 = 1$. In such a case the appropriate partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \qquad (3.4.24)$$

The problem is to determine the coefficients A_1, A_2, and A_3.

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}}$$

$$A_1 = \frac{X(z)}{z} \times (z+1) \Big|_{z=-1}$$

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$$A_2 = \frac{d}{dz}\left(\frac{z^2}{(z+1)\cancel{(z-1)^2}} \times \cancel{(z-1)^2}\right) \Big|_{z=1}$$

$$= \frac{d}{dz}\left(\frac{z^2}{z+1}\right)\Big|_{z=1}$$

$$= \frac{(z+1)\cdot 2z - z^2}{(z+1)^2}\Big|_{z=1}$$

$$= \frac{2\times 2 - 1}{2^2} = \frac{3}{4}$$

$X(z)$ has a simple pole at $p_1 = -1$ and a double pole $p_2 = p_3 = 1$. In such a case the appropriate partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \tag{3.4.24}$$

The problem is to determine the coefficients A_1, A_2, and A_3.

$$\frac{X(z)}{z} = \frac{1}{4}\,\frac{1}{(z+1)} + \frac{1}{2}\,\frac{1}{(z-1)} + \frac{3}{4}\,\frac{1}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\,\frac{z}{z+1} + \frac{1}{2}\,\frac{z}{z-1} + \frac{3}{4}\,\frac{z}{(z-1)^2}$$

$X(z)$ has a simple pole at $p_1 = -1$ and a double pole $p_2 = p_3 = 1$. In such a case the appropriate partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \tag{3.4.24}$$

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$$\frac{X(z)}{z} = \frac{1}{4}\frac{1}{(z+1)} + \frac{1}{2}\frac{1}{(z-1)} + \frac{3}{4}\frac{1}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\frac{z}{z+1} + \frac{1}{2}\frac{z}{z-1} + \frac{3}{4}\frac{z}{(z-1)^2}$$

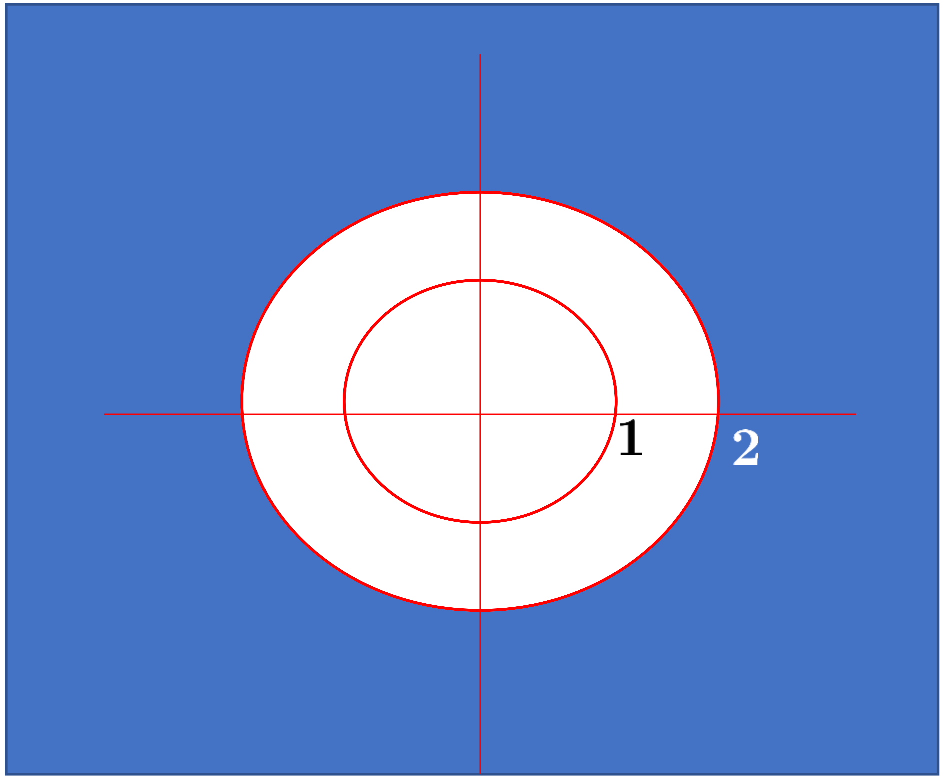
Taking inverse z-transform

$$\Rightarrow x(n) = \frac{1}{4}(-1)^n u(n) + \frac{1}{2}u(n) + \frac{3}{4}\, n\, u(n)$$

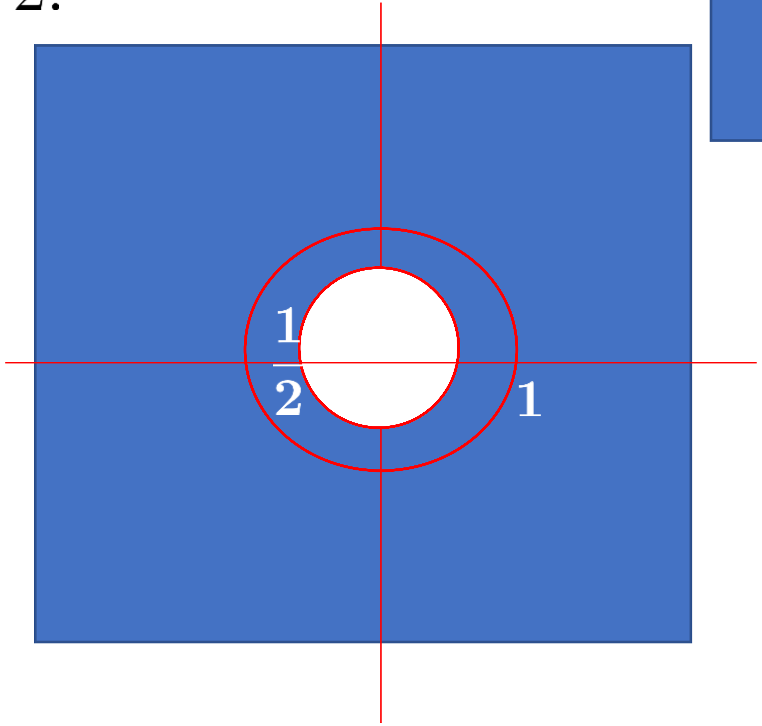
ROC

Select the appropriate ROC

1.



2.



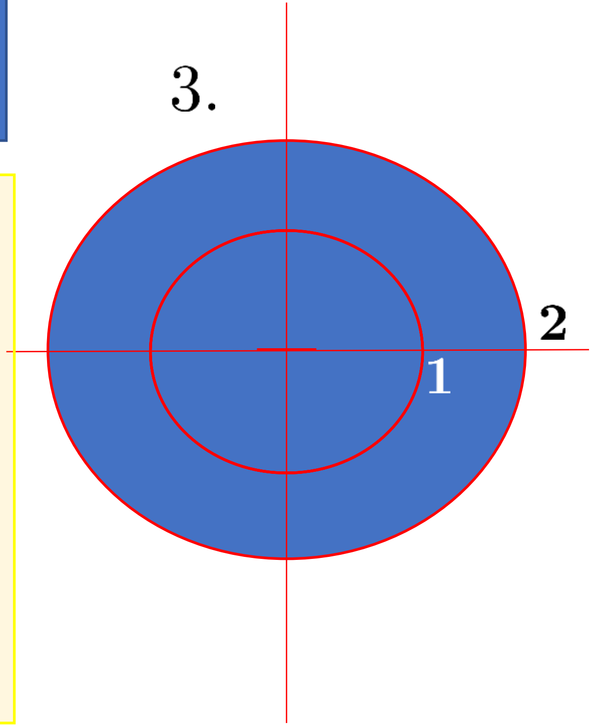
Find Z transfom and Draw the ROC

a. $2^n u(n) \leftrightarrow \frac{z}{z-2}$, ROC $|z| > 2$

b. $(1/2)^n u(n) \leftrightarrow \frac{z}{z-1/2}$, ROC $|z| > 1/2$

c. $-(2)^n u(-n-1) \leftrightarrow \frac{z}{z-2}$, ROC $|z| < 2$

3.



Example 3.4.8

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

if

(a) ROC: $|z| > 1$
(b) ROC: $|z| < 0.5$
(c) ROC: $0.5 < |z| < 1$

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(c) ROC: $0.5 < |z| < 1$

$$\times \frac{z^2}{z^2} = \frac{z^2}{0.5 - 1.5z + z^2}$$

$$\frac{X(z)}{z} = \frac{z}{0.5 - 1.5z + z^2}$$

Example 3.4.8

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \times \frac{z^2}{z^2} = \frac{z^2}{0.5 - 1.5z + z^2}$$

if

(a) ROC: $|z| > 1$
(b) ROC: $|z| < 0.5$
(c) ROC: $0.5 < |z| < 1$

$$\frac{X(z)}{z} = \frac{z}{0.5 - 1.5z + z^2}$$

Sol':

$$\frac{X(z)}{z} = \frac{z}{0.5 - 0.5z - z + z^2} = \frac{z}{(z-1)(z-0.5)}$$

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Determine the inverse z-transform of

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Sol.:

$$\frac{X(z)}{z} = \frac{z}{0.5 - 0.5z - z + z^2} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

Example 3.4.8

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \times \frac{z^2}{z^2} = \frac{z^2}{0.5 - 1.5z + z^2}$$

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(a) ROC: $|z| > 1$
(b) ROC: $|z| < 0.5$
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$$\frac{X(z)}{z} = \frac{z}{0.5 - 1.5z + z^2}$$

Sol':

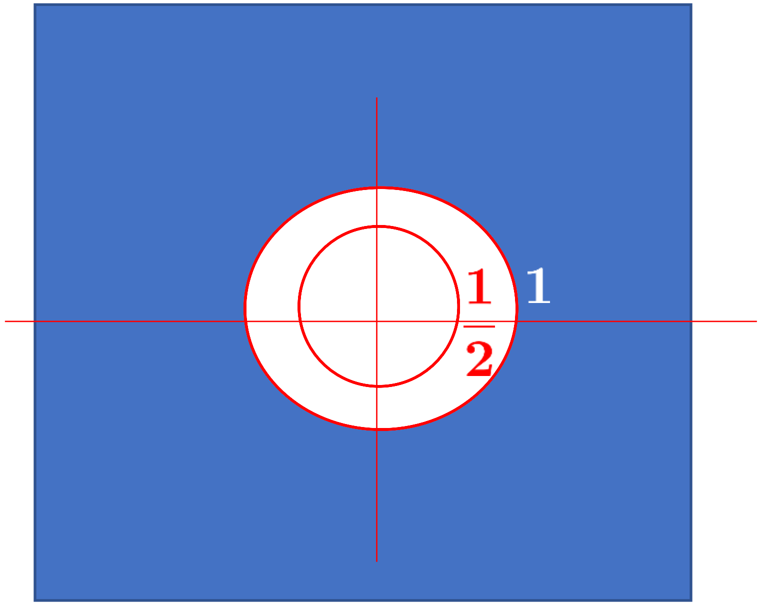
$$\frac{X(z)}{z} = \frac{z}{0.5 - 0.5z - z + z^2} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

$$\boxed{A = \frac{z}{z-0.5}\Big|_{z=1} = 2 \qquad B = \frac{z}{z-1}\Big|_{z=0.5} = -1}$$

(a) ROC: $|z| > 1$

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-1/2} \longrightarrow X(z) = \frac{2z}{z-1} - \frac{z}{z-1/2}$$

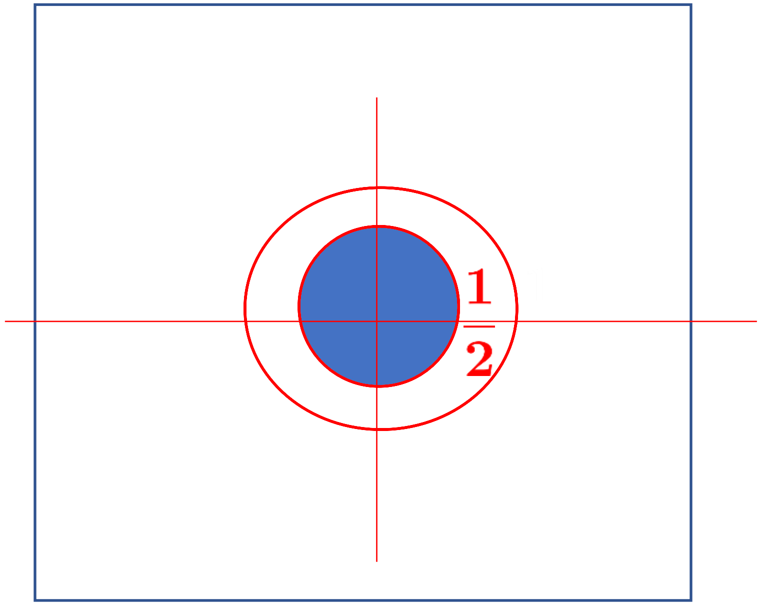


Both the signal with poll Z=1/2 and Z=1 are causal

$$x(n) = 2(1)^n u(n) - (0.5)^n u(n) = (2 - 0.5^n)u(n)$$

(b) ROC: $|z| < 0.5$

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-1/2} \longrightarrow X(z) = \frac{2z}{z-1} - \frac{z}{z-1/2}$$

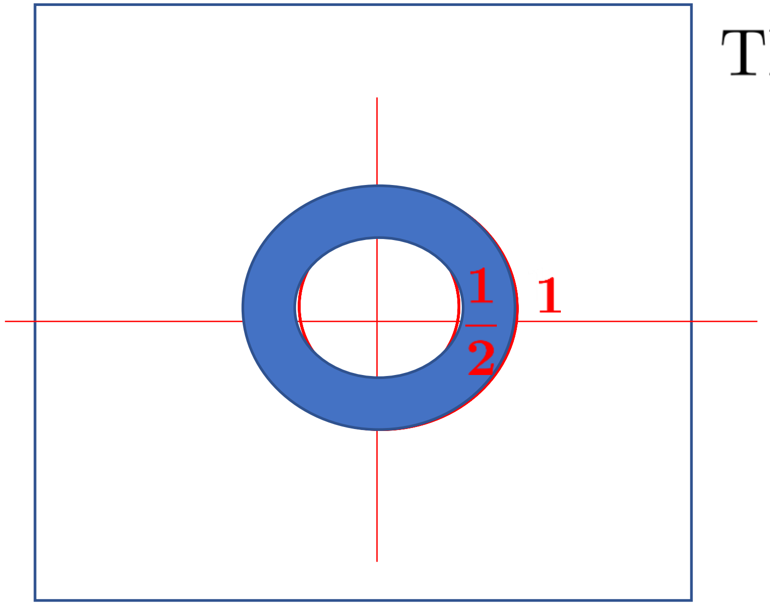


Both the signal with poll Z=1/2 and Z=1 are anticausal

$$x(n) = [-2 + (0.5)^n]u(-n-1)$$

(c) ROC: $0.5 < |z| < 1$

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-1/2} \longrightarrow X(z) = \frac{2z}{z-1} - \frac{z}{z-1/2}$$



The signal with poll Z=1/2 is causal and Z=1 is anticausal

$$x(n) = -2(1)^n u(-n-1) - (0.5)^n u(n)$$

Solution: Given $X(z) = \dfrac{z(z-1)}{(z+1)^3\,(z+2)}$; ROC; $|z| > 2$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{d}{dx}(u) - u\dfrac{d}{dx}(v)}{v^2}$$

Solution: Given $X(z) = \dfrac{z(z-1)}{(z+1)^3\,(z+2)}$; ROC; $|z| > 2$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{z-1}{(z+1)^3\,(z+2)} = \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2} + \frac{C_3}{(z+1)^3} + \frac{C_4}{z+2}$$

Solution: Given $$X(z) = \frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC; } |z| > 2$$

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where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

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where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$C_4 = (z+2)\frac{X(z)}{z}\bigg|_{z=-2} = \frac{z-1}{(z+1)^3}\bigg|_{z=-2} = \frac{-2-1}{(-2+1)^3} = 3$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Solution: Given $X(z) = \dfrac{z(z-1)}{(z+1)^3\,(z+2)}$; ROC; $|z| > 2$

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$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$$

Solution: Given $X(z)=\dfrac{z(z-1)}{(z+1)^3\,(z+2)}$; ROC; $|z|>2$

$$\therefore \qquad \frac{X(z)}{z}=\frac{z-1}{(z+1)^3\,(z+2)}=\frac{C_1}{z+1}+\frac{C_2}{(z+1)^2}+\frac{C_3}{(z+1)^3}+\frac{C_4}{z+2}$$

where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$C_4=(z+2)\,\frac{X(z)}{z}\bigg|_{z=-2}=\frac{z-1}{(z+1)^3}\bigg|_{z=-2}=\frac{-2-1}{(-2+1)^3}=3$$

$$C_3=(z+1)^3\,\frac{X(z)}{z}\bigg|_{z=-1}=\frac{z-1}{(z+2)}\bigg|_{z=-1}=\frac{-1-1}{-1+2}=-2$$

$$C_2=\frac{1}{1!}\frac{d}{dz}\left[(z+1)^3\frac{X(z)}{z}\right]\bigg|_{z=-1}=\frac{d}{dz}\left(\frac{z-1}{z+2}\right)\bigg|_{z=-1}=\frac{(z+2)\,(1)-(z-1)\,(1)}{(z+2)^2}\bigg|_{z=-1}=3$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Solution: Given $$X(z) = \frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC; } |z| > 2$$

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$$C_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z+1)^3\frac{X(z)}{z}\right]\Bigg|_{z=-1} = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z-1}{z+2}\right)\Bigg|_{z=-1}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Solution: Given $X(z) = \dfrac{z(z-1)}{(z+1)^3\,(z+2)}$; ROC; $|z| > 2$

$$\therefore \quad \frac{X(z)}{z} = \frac{z-1}{(z+1)^3\,(z+2)} = \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2} + \frac{C_3}{(z+1)^3} + \frac{C_4}{z+2}$$

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$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{3}{(z+2)^2}\right]\Bigg|_{z=-1} = \frac{1}{2}\left.\frac{-3\times 2(z+2)}{(z+2)^4}\right|_{z=-1} = \frac{-3(-1+2)}{(-1+2)^3} = -3$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$$

$$f(x)=x^n, \text{ then } f'(x)=nx^{n-1}$$

$$\therefore \quad \frac{X(z)}{z}=\frac{-3}{z+1}+\frac{3}{(z+1)^2}-\frac{2}{(z+1)^3}+\frac{3}{z+2}$$

$$\therefore \quad X(z)=\frac{-3z}{z+1}+\frac{3z}{(z+1)^2}-\boxed{\frac{2z}{(z+1)^3}}+\frac{3z}{z+2};\ \text{ROC};\ |z|>2$$

Since ROC is $|z|>2$, all the above sequences must be causal. Taking inverse Z-transform on both sides, we have

$$x(n)=-3(-1)^n u(n)+3n(-1)^n u(n)-\boxed{2(n)(n-1)(-1)^n u(n)}+3(-2)^n u(n)$$

$$=[-3+3n-2n(n-1)](-1)^n u(n)+3(-2)^n u(n)$$

Unknown ROC

- If The Signal Is Known To Be Causal, Always Choose The Right-sided Inverse For Each Term
- If Signal Is Stable, Then Absolutely Summable and Discrete Time Fourier Transform Exists
  - ROC Must Include The Unit Circle
  - Choose Right or Left-sided Inverse To Ensure Unit Circle of z-Plane Is Included

Example 3.6.3

A linear time-invariant system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

(c) The system is anticausal.

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Unknown ROC

Stability

- An LTI system is stable if and only if **ROC** of its system function H(z) **includes the unit circle** $|z| = 1$

- **All the poles** of H(z) should **lie inside the unit circle.**
  - ie, They must have magnitudes smaller than 1.

Example 3.6.3

A linear time-invariant system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

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Solution The system has poles at $z = \frac{1}{2}$ and $z = 3$.

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Solution The system has poles at $z = \frac{1}{2}$ and $z = 3$.

(a) Since the system is stable, its ROC must include the unit circle and hence it is $\frac{1}{2} < |z| < 3$. Consequently, $h(n)$ is noncausal and is given as

$$h(n) = (\tfrac{1}{2})^n u(n) - 2(3)^n u(-n - 1)$$

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(b) Since the system is causal, its ROC is $|z| > 3$. In this case

$$h(n) = (\tfrac{1}{2})^n u(n) + 2(3)^n u(n)$$

This system is unstable.

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(b) Since the system is causal, its ROC is $|z| > 3$. In this case

$$h(n) = (\tfrac{1}{2})^n u(n) + 2(3)^n u(n)$$

This system is unstable.

(c) If the system is anticausal, its ROC is $|z| < 0.5$. Hence

$$h(n) = -[(\tfrac{1}{2})^n + 2(3)^n]u(-n-1)$$

Introduction

- We Now Know How To Analyze Signals Using the z-Transform
- Let's Take a Look At How To Analyze Systems With The z-Transform
- Recall: $y[k] = h[k] * x[k] \stackrel{z}{\longleftrightarrow} Y(z) = H(z)X(z)$

Definition

The input-output behavior of a discrete-time LTI system is determined by the system Transfer Function

$$H(z) = \frac{Y(z)}{X(z)}$$

Example 3.3.4

Determine the system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

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This system has a pole at $z = \frac{1}{2}$ and a zero at the origin. Using Table 3.3 we obtain the inverse transform

$$h(n) = 2(\tfrac{1}{2})^n u(n)$$

This is the unit sample response of the system.

Example 3.6.5

Determine the response of the system

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

to the input signal $x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$.

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This system has two poles, one at $z = \frac{1}{2}$ and the other at $z = \frac{1}{3}$. The z-transform of the input signal is

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$$X(z) = 1 - \tfrac{1}{3}z^{-1}$$

In this case the input signal contains a zero at $z = \frac{1}{3}$ which cancels the pole at $z = \frac{1}{3}$. Consequently,

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$$Y(z) = H(z)X(z)$$

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

and hence the response of the system is

$$y(n) = (\tfrac{1}{2})^n u(n)$$

Clearly, the mode $(\frac{1}{3})^n$ is suppressed from the output as a result of the pole-zero cancellation.

The End