Lecture- 2. Maclaurin's series

1. Expand $e^{\tan^{-1}x}$ in powers of x upto x^4!

Solution:

$$y = e^{\tan^{-1}x} \qquad y(0) = e^{\tan^{-1}0} = e^{0} = 1$$

$$y_1 = \frac{e^{\tan^{-1}x}}{1+x^2} \qquad y_1(0) = \frac{e^{\tan^{-1}0}}{1+0} = 1$$

$$y_1 = \frac{y}{1+x^2}$$

$$(1+x^2)y_1 = y$$

$$(1+x^2)y_1 - y = 0$$

$$(1+x^2)y_2 + y_1\,2x - y_1 = 0$$

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \quad (1)$$

$$y_2(1+x^2) = -(2x-1)y_1$$
$$= +y_1$$
$$= +1$$
$$y_2(1+0) = 1$$
$$\underline{y_2 = 1}$$

Differentiating (1), w.r.to x, n times by Leibnitz's theorem

$$D^n(uv) = D^n(u)\,V + nC_1 D^{n-1}(u)\,D(V) + nC_2 D^{n-2}(u)\,D^2(V) + \cdots + nC_n\, u.D^n(v)$$

$$(1) \Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$$

$$\left[(1+x^2)y_{n+2} + n.2x\,y_{n+1} + nC_2.(2)\,y_n\right] + \left\{(2x-1)y_{n+1} + nC_1(2)\,y_n\right\} = 0$$

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + \left[\frac{n(n-1)}{2!}(2) + 2n\right]y_n = 0$$

$$(1+x^2)y_{n+2} + \left[2x(n+1) - 1\right]y_{n+1} + (n^2+n)y_n = 0$$

$x=0$

$$y_{n+2} - y_{n+1} + (n^2+n)y_n = 0$$

$$y_{n+2} = y_{n+1} - (n^2+n)y_n \quad (2)$$

In (2)

$n=0,\quad y_2 = y_1 - 0 = 1 \qquad y_2(0) = y_1(0) = 1$

$n=1,\quad y_3 = y_2 - 2y_1 \qquad y_3(0) = 1 - 2 = -1$

$n=2,\quad y_4 = y_3 - (2^2+2)y_2$

$y_4(0) = -1 - 6(1)$
$= -7$

Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$e^{\tan^{-1} x} = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(-7) + \cdots$$

2) Expand $\log(1+\tan x)$ in powers of x upto 5 terms.

Solution $y = \log(1+\tan x) \qquad y(0) = \log(1+\tan 0) = \log 1 = 0$

$$y_1 = \frac{1}{1+\tan x}\sec^2 x \qquad y_1(0) = \frac{\sec^2(0)}{1+\tan(0)} = 1$$

$$= \frac{\sec^2 x}{1+\tan x}$$

$$(1+\tan x)\, y_1 = \sec^2 x.$$

Diff, $(1+\tan x)\, y_2 + \sec^2 x\,(y_1) = 2\sec x \sec x \tan x$

$\div (1+\tan x)$
$$y_2 + \frac{\sec^2 x}{1+\tan x} y_1 = 2\frac{\sec^2 x \tan x}{1+\tan x}$$

$$y_2 + y_1(y_1) = 2y_1 \tan x$$

$$y_2 + y_1^2 = 2y_1\tan x, \quad y_2(0) = 2y_1(0)\tan(0) - y_1^2(0)$$
$$= 2(1)(0) - 1$$
$$= -1$$

Diff, $y_3 + 2y_1y_2 = 2y_1\sec^2 x + 2\tan x\, y_2$

$$y_3(0) = 2y_1(0)\sec^2(0) + 0 - 2y_1(0)y_2(0)$$
$$= 2 - 2(1)(-1) = 2+2 = 4$$

Diff, $y_4 + 2[y_1y_3 + y_2\cdot y_2] = 2y_2\sec^2 x + 2y_1\cdot 2\sec x\sec x\tan x + 2y_3\tan x + 2\sec^2 x\, y_2$

$$y_4 + 2\left[1(4) + (-1)^2\right] = 2(-1)\sec^2(0) + 0 + 0 + 2(1)(-1)$$

$$y_4 + 10 = -2 - 2$$

$$y_4 = -4 - 10 = \underline{-14}$$

Series $\log(1+\tan x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(4) + \frac{x^4}{4!}(-14) + \cdots$

Expand

3) $\log(1+e^x)$ upto 4 terms

$y = \log_e(1+e^x)$ $\qquad y(0) = \log(1+e^0) = \log 2$

$e^y = 1 + e^x$

Diff, $e^y y_1 = e^x$ $\qquad y_1(0) = \frac{e^x}{e^{y(0)}} = \frac{e^0}{e^{\log 2}} = \underline{\frac{1}{2}}$

Diff, $e^y y_2 + e^y y_1 \cdot y_1 = e^x$

H.W

4) Expand by Maclaurin's series upto x^4

① $e^{x\sec x}$ ② $(\sin^{-1}x)^2$ ③ $\sqrt{1+\sin x}$

$\sqrt{\sin^2 x/2 + \cos^2 x/2 + 2\sin x/2 \cos x/2}$

5) Using Taylor's theorem prove that

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} \quad \text{for } x > 0$$

Solution Taylor's theorem $\qquad 0<\theta<1$

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^n(a+\theta h)$$

Recalling Maclaurin's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1$$

$$f(x) = f(0) + \sum_{r=1}^{n-1} \frac{x^r}{r!} f^r(0) + \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1 \quad \text{— (1)}$$

$f(x) = \sin x$, $f(0) = \sin 0 = 0$

$$f^r(x) = \frac{d^r}{dx^r}(\sin x) = \sin\left[x + \frac{\pi r}{2}\right] \quad \text{— (2)}$$

(1) $\Rightarrow$

$$\sin x = \sum_{r=1}^{n-1} \frac{x^r}{r!} \sin\left(\frac{r\pi}{2}\right) + \frac{x^n}{n!} \sin\left[\theta x + \frac{n\pi}{2}\right], \quad 0<\theta<1$$

$n=3$,

$$\sin x = \sum_{r=1}^{2} \frac{x^r}{r!} \sin\left(\frac{r\pi}{2}\right) + \frac{x^3}{3!} \sin\left(\theta x + \frac{3\pi}{2}\right)$$

$$= x \sin(\pi/2) + \frac{x^2}{2!} \sin\left(\frac{2\pi}{2}\right) + \frac{x^3}{3!} \sin\left(\theta_1 x + \frac{3\pi}{2}\right), \quad 0<\theta_1<1$$

$$= x + 0 + \frac{x^3}{3!}\left(-\cos(\theta_1 x)\right)$$

$$= x - \frac{x^3}{3!} \cos\theta_1(x), \quad 0<\theta_1<1 \quad \text{— (3)}$$

Since $\cos\theta \leq 1$, for $\theta > 0$ we have for $x>0$

$$x - \frac{x^3}{3!} \leq x - \frac{x^3}{3!} \cos(\theta_1 x) \quad \text{— (4)}$$

$n=5$ in (2)

$$\sin x = \sum_{r=1}^{4} \frac{x^r}{r!} \sin\left(\frac{r\pi}{2}\right) + \frac{x^5}{5!} \sin\left(\theta x + \frac{5\pi}{2}\right)$$

$$= x + 0 + \frac{x^3}{3!}(-1) + \frac{x^5}{5!} \cos(\theta_2 x), \quad 0<\theta_2<1 \quad \text{— (5)}$$

$\cos\theta \leq 1$, for $\theta > 0$.

$\therefore$ we have for $x > 0$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos(\theta_2 x) \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \text{— (6)}$$

In view of (3) and (5)

$$(3) \Rightarrow \sin x = x - \frac{x^3}{3!}\cos(\theta_1 x)$$

$$(5) \Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos(\theta_2 x)$$

(4) and (6)

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$$

Indeterminate forms

While attempting to evaluate certain limits, we often obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0(\infty)$, $\infty - \infty$, 0^0, ∞^0, 1^∞ which doesn't give any real value. Such expressions are called indeterminate forms.

$$\underset{x \to 0}{Lt} \frac{\sin x}{x} = 1, \quad \underset{x \to 0}{Lt} \frac{\tan x}{x} = 1$$

I Form $\left(\frac{0}{0}\right)$

If $f(a) = \phi(a) = 0$, then $\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)}$

Proof By Taylor's series

$$\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots}{\phi(a) + (x-a)\phi'(a) + \frac{(x-a)^2}{2!}\phi''(a) + \cdots}$$

$$= \underset{x \to a}{Lt} \frac{\cancel{(x-a)}\left[f'(a) + \frac{(x-a)}{2!} f''(a) + \cdots\right]}{\cancel{(x-a)}\left[\phi'(a) + \frac{x-a}{2!} \phi''(a) + \cdots\right]}$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)}$$

This is known as L' Hospital's Rule.

In general if $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$

but $f^n(a) \neq 0$

and $\phi(a) = \phi'(a) = \cdots = \phi^{(n-1)}(a) = 0$, but $\phi^n(a) \neq 0$

then from the above, we have

$$\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \frac{f^n(a)}{\phi^n(a)} = \underset{x \to a}{Lt} \frac{f^n(x)}{\phi^n(x)}$$

Problems

1. Find $\underset{x \to 0}{Lim} \frac{\tan x - x}{x^2 \tan x}$

Solution $\underset{x \to 0}{Lt} \frac{\tan x - x}{x^2 \tan x} \left(\frac{0}{0}\right)$

Applying L' Hospital's rule, $\underset{x \to 0}{Lt} \frac{\sec^2 x - 1}{x^2 \sec^2 x + 2x \tan x} \left(\frac{0}{0}\right)$

$$\underset{x \to 0}{Lim} \frac{\tan^2 x}{x^2 \sec^2 x + 2x \tan x}$$

$\div$ by x^2

$$\underset{x \to 0}{Lt} \frac{\frac{\tan^2 x}{x^2}}{\cancel{x^2} \cdot \frac{\sec^2 x}{\cancel{x^2}} + 2\cancel{x} \frac{\tan x}{x^{\cancel{2}}}}$$

$$= \underset{x \to 0}{Lt} \frac{(\tan x / x)^2}{\sec^2 x + 2\left(\frac{\tan x}{x}\right)}$$

$$= \frac{1}{1 + 2(1)} = \frac{1}{3}$$

2) $\underset{x \to 0}{Lim} \dfrac{x e^x - \log(1+x)}{x^2}$

Solution $\underset{x \to 0}{Lt} \dfrac{x e^x - \log(1+x)}{x^2} \left(\frac{0}{0}\right)$

$$\underset{x \to 0}{Lt} \frac{x e^x + e^x(1) - \left(\frac{1}{1+x}\right)}{2x} \left(\frac{0}{0}\right)$$

$$\underset{x \to 0}{Lt} \frac{x e^x + e^x + e^x + \left(\frac{1}{1+x}\right)^2}{2}$$

$$= \frac{1+1+1}{2} = 3/2$$

3) $\underset{x \to 0}{Lim} \log_{\tan x} \tan 2x$

Solution By base changing rule

$$\underset{x \to 0}{Lt} \log_{\tan x} \tan 2x = \underset{x \to 0}{Lim} \frac{\log \tan 2x}{\log \tan x} \left(\frac{\infty}{\infty}\right)$$

Applying L'H-rule $\underset{x \to 0}{Lt} \dfrac{\frac{1}{\tan 2x} \cdot \sec^2 2x \, (2)}{\frac{1}{\tan x} \cdot \sec^2 x} \left(\frac{\infty}{\infty}\right)$

$$\underset{x \to 0}{Lt} \frac{2\sec^2 2x \cdot \tan x}{\sec^2 x \cdot \tan 2x} \quad \left(\frac{0}{0}\right)$$

$\div$ Nr and Dr by x, $\underset{x \to 0}{Lt} \dfrac{\sec^2 2x \left(\frac{\tan x}{x}\right)}{\sec^2 x \cdot \frac{\tan 2x}{2x}}$

$$= \frac{1\,(1)}{1\,(1)} = \underline{\underline{1}}$$

4) $\underset{x \to 0}{Lt} \dfrac{a^x - 1 - x\log_e a}{x^2}$

5) $\underset{x \to 1/2}{Lim} \dfrac{\cos^2 \pi x}{e^{2x} - 2e(x)}$

6) $\underset{x \to 0}{Lim} \dfrac{\log(1-x^2)}{\log \cos x}$

7) $\underset{x \to 0}{Lim} \dfrac{a^x - b^x}{x}$

8) Find a and b s.t

$\underset{x \to 0}{Lim} \dfrac{x(1+a\cos x) - b\sin x}{x^3} = 1$

<u>Solution</u> $\underset{x \to 0}{Lim} \dfrac{x(1+a\cos x) - b\sin x}{x^3} \quad \left(\frac{0}{0}\right)$

Applying L-H-Rule

$\underset{x \to 0}{Lt} \dfrac{(1+a\cos x) + xa(-\sin x) - b\cos x}{3x^2}$ (Nr $\neq 0$)

As Dr $= 0$ for $x = 0$, the above will tend to a finite limit iff the Nr $= 0$ for $x = 0$

(ie) $1 + a - b = 0$ — (1)

With this condition assume the above as $\left(\frac{0}{0}\right)$

L.H. Rule

$$\lim_{x \to 0} \frac{-a\sin x - a[x\cos x + \sin x] + b\sin x}{6x}$$

$$\lim_{x \to 0} \frac{\sin x[b - 2a] - ax\cos x}{6x} \quad \left(\frac{0}{0}\right)$$

L-H Rule $\lim_{x \to 0} \frac{(b-2a)\cos x - a[\cos x - x\sin x]}{6} = 1$

$$\Rightarrow \frac{b - 2a - a}{6} = 1$$

$$\Rightarrow b - 3a = 6 \quad — (2)$$

$$(1) \Rightarrow 1 + a - b = 0$$

Solving (1) & (2), $a = -5/2$, $b = -3/2$

9) H.W If the limit of $\frac{\sin 2x + a\sin x}{x^3}$ as $x \to 0$ be finite, find a and the value of the limit.

10) $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

solution $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} \quad \left(\frac{e-e}{0}\right) = \left(\frac{0}{0}\right)$ — (1)

$y = (1+x)^{\frac{1}{x}}$

$\log y = \frac{1}{x} \log(1+x)$

$= \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$

$= 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$

$y = e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)}$

(1) $\Rightarrow \lim_{x \to 0} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)} - e}{x}$

$\left[e^{m+n} = e^m \cdot e^n\right]$

$= \lim_{x \to 0} \frac{e^1 e^{\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)} - e}{x}$

$\left[e^x = 1 + x + \frac{x^2}{2!} + \cdots\right]$

$= \lim_{x \to 0} \frac{e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)\right] - e}{x}$

$\lim_{x \to 0} \frac{\cancel{e} + e\left[\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)^2\right] - \cancel{e}}{x}$

$\lim_{x \to 0} e\left[-\frac{1}{2}\right] + e\left[\frac{x}{3} \cdots\right]$

$= -e/2$