Four Different Fourier Representation

Time Domain	*Periodic* (t, n)	*Non periodic* (t, n)	
Continuous (t)	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	*Nonperiodic* (k, ω)
Discrete (n)	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	*Periodic* (k, Ω)
	Discrete (k)	*Continuous* (ω, Ω)	*Frequency Domain*

1

DFT to the rescue!

Could we calculate the **frequency spectrum** of a signal using a **digital computer** with **CTFT/DTFT**?

- Both CTFT and DTFT produce continuous function of frequency → can't calculate an infinite continuum of frequencies using a computer
- Most real-world data is not in the form like $a^n u(n)$

DFT can be used as a FT approximation that can calculate a **finite set of discrete-frequency spectrum values** from a finite set of discrete-time samples of an analog signal

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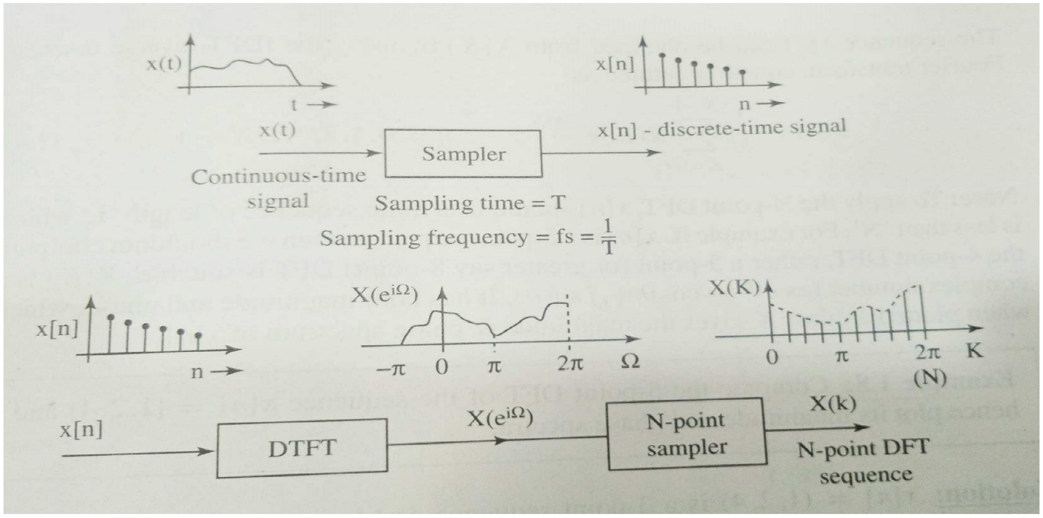
Discrete Fourier Transform (DFT) Realization



3

Building the DFT formula (cont)

Continuous time signal **x(t)**

sample ↓

Discrete time signal **x(n)**

window ↓

Discrete time signal **x_0(n)**
Finite length

DTFT ⟷

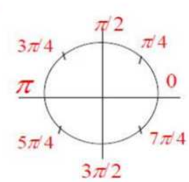
Discrete Time Fourier Transform (DTFT), $X_o(\Omega)$
(periodic over $[0,2\pi)$)

$$\boxed{X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x_0[n]e^{-j\Omega n}}$$

$$\mathbf{X(\Omega)|_{\Omega=\frac{2\pi k}{N}} = X(k)}$$

DFT

Sample at N values around the unit circle ↑

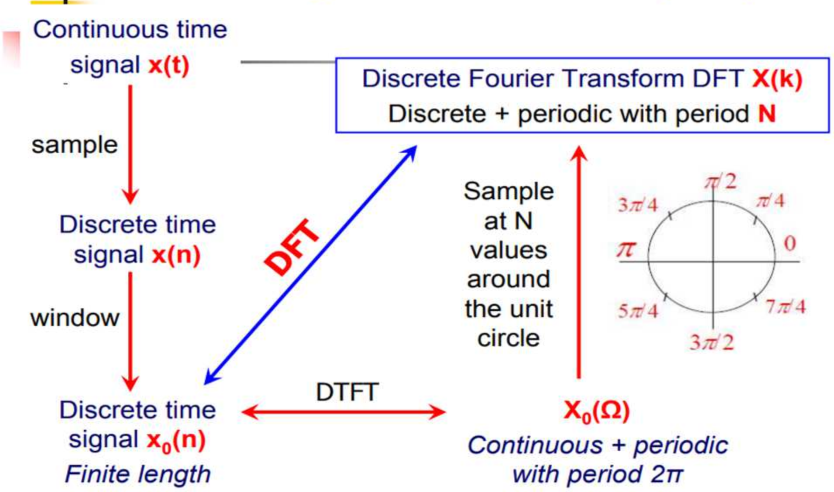


$$\mathbf{X(k) = \sum_{n=0}^{N-1} x_0(n)e^{-j2\pi kn/N}}$$

Discrete Fourier Transform DFT **X(k)**
Discrete + periodic with period **N**

4

Building the DFT formula (cont)



5

Computation of DFT by extracting one period of DFS

To a finite-length sequence $x(n)$:

$X(k)$ $\Leftrightarrow$ $DFS[x((n))_N]$

DFT

Periodical copies

DFS of periodic sequence

Get DFT by extracting one period of DFS

Attention: **DFT is acquired by extracting one period of DFS, it is not a new kind of Fourier Transform.**

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6.4 RELATION BETWEEN DFT AND Z-TRANSFORM

The Z-transform of N-point sequence $x(n)$ is given by

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

Let us evaluate $X(z)$ at N equally spaced points on the unit circle, i.e., at $z = e^{j(2\pi/N)k}$.

$$\left|e^{j(2\pi/N)k}\right| = 1 \text{ and } \underline{\left|e^{j(2\pi/N)k}\right.} = \frac{2\pi}{N}k$$

Hence, when k is varied from 0 to $N-1$, we get N equally spaced points on the unit circle in the z-plane.

$$\therefore \qquad X(z)\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n) z^{-n}\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)kn}$$

By the definition of N-point DFT, we get

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)kn}$$

From the above equations, we get

$$X(k) = X(z)\Big|_{z=e^{j(2\pi/N)k}}$$

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Discrete Fourier Transform DFT

- The formulas for DFT and IDFT may be expressed as (in terms of twiddle factor)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad 0 \le n \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

- The relationship between $x(n)$ and $X(k)$ is denoted as

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

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DFT and IDFT examples

Ex.4. Given $x(n) = \delta(n)$
$N = 4$. Find $X(k)$.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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DFT and IDFT examples

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$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X}(\mathbf{k}) = \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \delta(\mathbf{n}) \mathbf{W}_{\mathbf{N}}^{\mathbf{kn}} \mathbf{X}(\mathbf{k})$$

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DFT and IDFT examples

Ex.4. Given x(n) = δ(n)
N = 4. Find X(k).

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn} X(k) = W_4^0 = 1} \quad \mathbf{where\ k = 0 \to N-1}$$

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DFT and IDFT examples

Ex.4. Given x(n) = δ(n)
N = 4. Find X(k).

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn} X(k) = W_4^0 = 1} \quad \mathbf{where\ k = 0 \to N-1}$$

$X[0] = \mathbf{1}$
$X[1] = \mathbf{1}$
$X[2] = \mathbf{1}$
$X[3] = \mathbf{1}$

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$$\mathbf{x(n) = \delta[n - n_0]}$$

$$\boxed{\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n - n_0) W_N^{kn}}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\boxed{\mathbf{X(k) = \sum_{n=0}^{N-1} \delta(n - n_0) W_N^{kn}}}$$

$$\boxed{\mathbf{X(k) = W_N^{kn_0}}}$$

$$\boxed{\mathbf{X(k) = e^{-j\frac{2\pi k n_0}{N}}}}$$

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$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

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$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right) = \frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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$$\cos\left(\frac{2\pi k_0 n}{N}\right) = \frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right] e^{-j\frac{2\pi k n}{N}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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$\cos\left(\frac{2\pi k_0 n}{N}\right) \rightarrow$

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$$= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{-j\frac{2\pi (k-k_0) n}{N}} + e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

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$\cos\left(\frac{2\pi k_0 n}{N}\right) \rightarrow$

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$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} e^{-j\frac{2\pi (k-k_0) n}{N}} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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18

$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

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$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} e^{-j\frac{2\pi (k-k_0) n}{N}} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$= \frac{1}{2}\left[N\delta(k-k_0) + N\delta(k+k_0)\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

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DFT and IDFT examples

Ex.4. Given x(n) = δ(n) + 2δ(n-1) +3δ(n-2) + δ(n-3) and N = 4. Find X(k).

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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$$X[k] = 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + e^{-j3\pi k/2}$$

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$$X[k] = 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + e^{-j3\pi k/2}$$
$$= 1 + 2(-j)^k + 3(-1)^k + (j)^k$$

$$X[0] = 7$$
$$X[1] = 1 - 2j - 3 + j = -2 - j$$
$$X[2] = 1 - 2 + 3 - 1 = 1$$
$$X[3] = 1 + 2j - 3 - j = -2 + j$$

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DFT and IDFT examples

Ex.5. Given X(k) = 2δ(k) + 2δ(k-2) and N = 4. Find x(n).

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]\, W_4^{-kn}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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DFT and IDFT examples

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$$x[n] = \frac{1}{4}2 + \frac{1}{4}2W_4^{-2n} \quad = \frac{1}{2} + \frac{1}{2}W_4^{-2n}$$

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DFT and IDFT examples

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DFT and IDFT examples

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$$\boxed{x[n] = [1 \quad 0 \quad 1 \quad 0]} \qquad \mathbf{n = 0, 1, 2, 3}$$

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EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$\mathbf{X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}}$$

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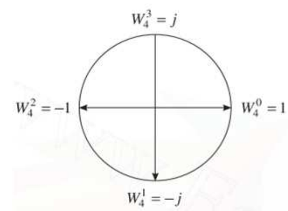

(a) 4-point DFT

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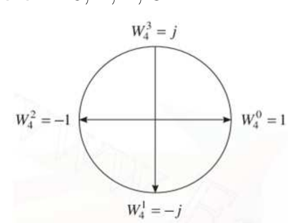
where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$



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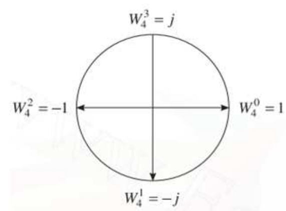

(a) 4-point DFT

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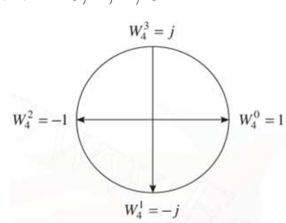
$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$



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$$\mathbf{X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}}$$

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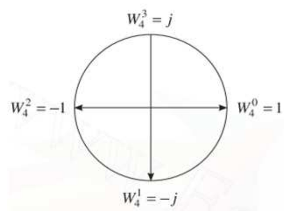
$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

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EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

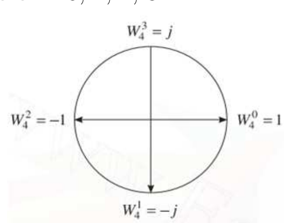
$$\mathbf{X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

$$\mathbf{X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6}$$

36

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

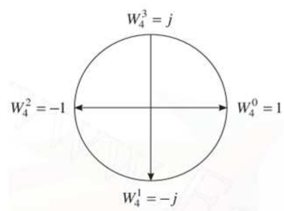
$$\mathbf{X(k) = \sum_{n=0}^{3} x[n]W_4^{nk}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad 0 \le k \le N-1$$

$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

$$\mathbf{X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6}$$

$$\mathbf{X(2) = 1 + 1 + 2 + 2 = 6}$$

37

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

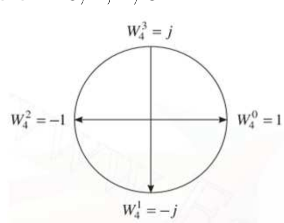
$$\mathbf{X(k) = \sum_{n=0}^{3} x[n]W_4^{nk}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad 0 \le k \le N-1$$

$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

$$\mathbf{X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6}$$

$$\mathbf{X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9}$$

$$\mathbf{X(2) = 1 + 1 + 2 + 2 = 6}$$

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EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

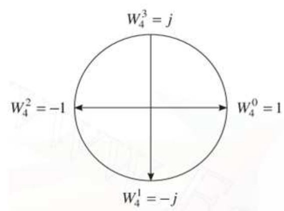
$$X(k) = \sum_{n=0}^{3} x[n] W_4^{nk}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}$$

where k=0, 1, 2, 3

$$W_N = e^{-j\frac{2\pi}{N}}$$



(a) 4-point DFT

$$X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0$$

$$X(0) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3$$

$$X(1) = 1 + j - 2 - 2j = -1 - j$$

$$X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6$$

$$X(2) = 1 + 1 + 2 + 2 = 6$$

$$X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9$$

$$X(3) = 1 - j - 2 + 2j = -1 + j$$

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MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4, \quad k = 0, 1, 2, 3 \;\&\; n = 0, 1, 2, 3}$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4, \quad k = 0, 1, 2, 3 \;\&\; n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right] \end{array}$$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $N = 4, \quad k = 0, 1, 2, 3 \;\&\; n = 0, 1, 2, 3$

$$W_4^{kn} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $N = 4, \quad k = 0, 1, 2, 3 \;\&\; n = 0, 1, 2, 3$

$$W_4^{kn} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ & & & \\ & & & \end{bmatrix} \end{array}$$

MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ & & & \end{bmatrix} \end{array}$$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4, \quad k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

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MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4, \quad k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

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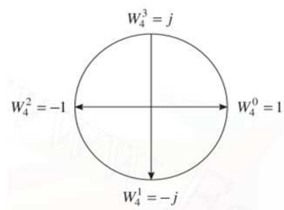
MATRIX FORMULATION OF THE DFT AND IDFT

$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

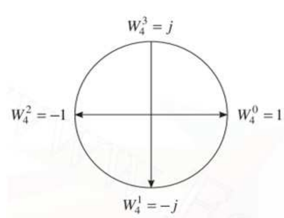
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

MATRIX FORMULATION OF THE DFT AND IDFT

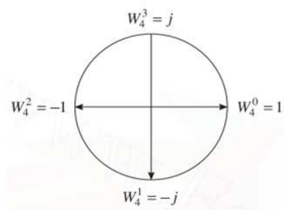
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT N = 4, k = 0, 1, 2, 3 & n = 0, 1, 2, 3

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

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MATRIX FORMULATION OF THE DFT AND IDFT

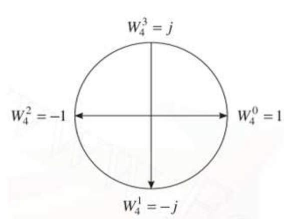
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT N = 4, k = 0, 1, 2, 3 & n = 0, 1, 2, 3

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

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MATRIX FORMULATION OF THE DFT AND IDFT

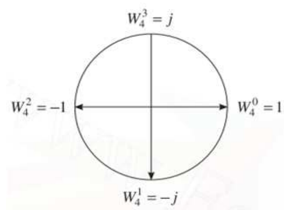
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

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MATRIX FORMULATION OF THE DFT AND IDFT

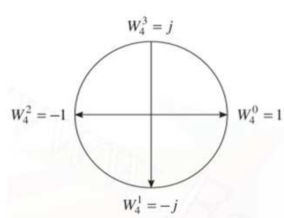
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

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MATRIX FORMULATION OF THE DFT AND IDFT

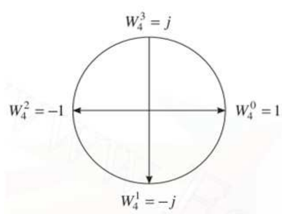
$W_N = e^{-j\frac{2\pi}{N}}$

For 4 point DFT $\mathbf{N = 4}$, $\mathbf{k = 0, 1, 2, 3}$ & $\mathbf{n = 0, 1, 2, 3}$

$$W_4^{kn} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \end{array}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$W_4^{kn} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^0 & w_4^2 \\ w_4^0 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$



(a) 4-point DFT

$$\mathbf{W_4^{kn}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

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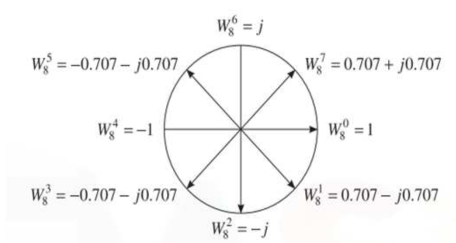
$W_N = e^{-j\frac{2\pi}{N}}$



(b) 8-point DFT

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EXAMPLE 6.8 Find the DFT of the sequence **and also draw the phase and magnitude spectrum**

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

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EXAMPLE 6.8 Find the DFT of the sequence **and also draw the phase and magnitude spectrum**

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$\frac{\pi}{2}$

$\text{Maginitude}|\mathbf{X(k)}| = \sqrt{(\mathbf{im(X(k))^2 + real(X(k)^2}}$

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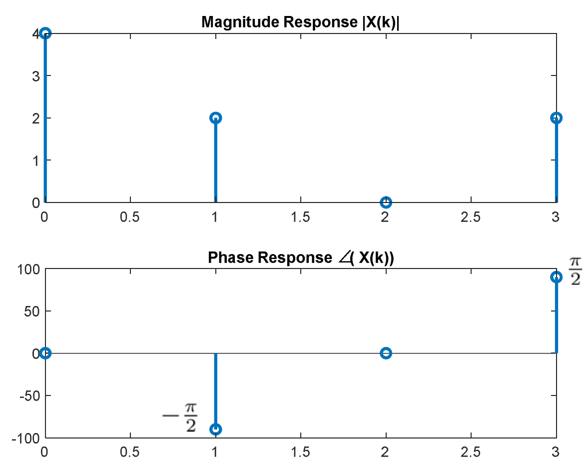

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64

Find IDFT of $X(k) = \{4, -2j, 0, 2j\}$

$$\mathbf{IDFT[X(k)] = \frac{1}{N}(W_N^{nk})^{*}X(k) = x(n)}$$

$$\mathbf{W_4^{kn}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

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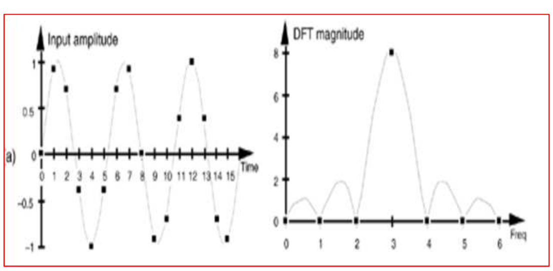
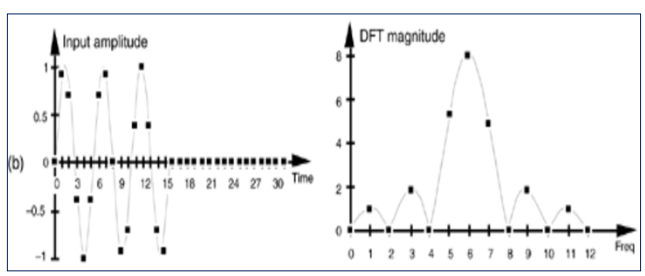
Zero Padding

One popular method used to improve DFT spectral estimation is known as zero padding. This process involves the addition of zero-valued data samples to an original DFT input sequence to increase the total number of input data samples

The more points in our DFT, the better our DFT output approximates the CFT.



(a) 16 input data samples and N = 16



(b) 16 input data samples, 16 padded zeros, and N = 32

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Properties of DFT

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Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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Periodicity

If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

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$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k+N)/N}$$

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But $e^{-j2\pi n} = 1$ for all n (Here n is an integer)

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$$\therefore \qquad X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = X(k)$$

81

Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT }\{x_1(n)\} = X_1(k)$$

and

$$\text{DFT }\{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants a and b,

$$\text{DFT }\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

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Proof:
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84

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$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

85

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$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

$$= aX_1(k) + bX_2(k)$$

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Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

$$x[(-n), \bmod N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\text{DFT}\{x(-n), \bmod N\} = \text{DFT}\{x(N-n)\} = X[(-k), \bmod N] = X(N-k)$$

87

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If DFT $\{x(n)\} = X(k)$, then

$$\text{DFT}\{x(-n), \bmod N\} = \text{DFT}\{x(N-n)\} = X[(-k), \bmod N] = X(N-k)$$

Proof: $$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi nk/N}$$

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Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

$$x[(-n), \bmod N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\begin{aligned}\text{DFT}\,\{x(-n), \bmod N\} &= \text{DFT}\,\{x(N-n)\}\\ &= X[(-k), \bmod N] = X(N-k)\end{aligned}$$

Proof: $\text{DFT}\,\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi nk/N}$

Changing index from n to m, where $m = N-n$, we have $n = N-m$.

$$\mathbf{DFT}\{\mathbf{x[m]}\} = \sum_{\mathbf{m=0}}^{\mathbf{N-1}} \mathbf{x(m)e^{-j2\pi k(N-m)/N}}$$

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$$\mathbf{DFT}\{\mathbf{x[m]}\} = \sum_{\mathbf{m=0}}^{\mathbf{N-1}} \mathbf{x(m)e^{-j2\pi k(N-m)/N}} = \sum_{\mathbf{m=0}}^{\mathbf{N-1}} \mathbf{x(m)e^{-j2\pi kN/N}e^{-j2\pi k(-m)/N}}$$

Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \text{ mod } N]$ and

$$x[(-n), \text{ mod } N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\begin{aligned} \text{DFT}\{x(-n), \text{mod } N\} &= \text{DFT}\{x(N-n)\} \\ &= X[(-k), \text{ mod } N] = X(N-k) \end{aligned}$$

Proof: $\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi nk/N}$

Changing index from n to m, where $m = N-n$, we have $n = N-m$.

$$\text{DFT}\{x[m]\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi kN/N}e^{-j2\pi k(-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k}\, e^{j2\pi km/N}$$

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Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \text{ mod } N]$ and

$$x[(-n), \text{ mod } N] = x(N-n), \qquad 0 \le n \le N-1$$

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Changing index from n to m, where $m = N-n$, we have $n = N-m$.

$$\text{DFT}\{x[m]\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi kN/N}e^{-j2\pi k(-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k}\, e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}$$

92

Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

$$x[(-n), \bmod N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\text{DFT}\{x(-n), \bmod N\} = \text{DFT}\{x(N-n)\} = X[(-k), \bmod N] = X(N-k)$$

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Changing index from n to m, where $m = N-n$, we have $n = N-m$.

$$\text{DFT}\{x[m]\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi kN/N}e^{-j2\pi k(-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k}\, e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}\, e^{-j2\pi km}$$

Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \bmod N]$ and

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Changing index from n to m, where $m = N-n$, we have $n = N-m$.

$$\text{DFT}\{x[m]\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi kN/N}e^{-j2\pi k(-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k}\, e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}\, e^{-j2\pi km}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N} = X(N-k)$$

Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

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Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

Proof

$$\mathbf{DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\ e^{\frac{-j2\pi kn}{N}}}$$

96

Circular Time Shift

$$\mathbf{DFT}\{x(n)\} = X(k), \quad \text{then DFT } \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}$$

Proof

$$\mathbf{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}$$

Let $n - m = p, \quad \therefore n = p + m$

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Circular Time Shift

$$\mathbf{DFT}\{x(n)\} = X(k), \quad \text{then DFT } \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}$$

Proof

$$\mathbf{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}$$

Let $n - m = p, \quad \therefore n = p + m$

$$= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}}$$

98

Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

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$$\mathbf{DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\ e^{\frac{-j2\pi kn}{N}}}$$

Let $\mathbf{n - m = p, \quad \therefore n = p + m}$

$$\mathbf{= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \qquad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi(p+m)}{N}}}$$

99

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$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

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$$\mathbf{DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\ e^{\frac{-j2\pi kn}{N}}}$$

Let $\mathbf{n - m = p, \quad \therefore n = p + m}$

$$\mathbf{= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \qquad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \qquad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}} e^{\frac{-j2\pi m}{N}}}$$

$$\mathbf{= \left[\sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}}\right] e^{\frac{-j2\pi m}{N}}}$$

100

Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

Proof

$$\mathbf{DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}}$$

Let $\mathbf{n-m=p,} \quad \mathbf{\therefore n = p+m}$

$$\mathbf{= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \quad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \quad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}}e^{\frac{-j2\pi m}{N}}}$$

$$\mathbf{= \left[\sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}}\right] e^{\frac{-j2\pi m}{N}}} \quad \mathbf{= X(k)e^{\frac{-j2\pi m}{N}}}$$

101

Circular Time Shift

$$\mathbf{DFT\{x(n)\} = X(k), \quad then\ DFT\ \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}}$$

Proof

$$\mathbf{DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}}$$

Let $\mathbf{n-m=p,} \quad \mathbf{\therefore n = p+m}$

$$\mathbf{= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \quad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi(p+m)}{N}}} \quad \mathbf{= \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}}e^{\frac{-j2\pi m}{N}}}$$

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Circular Frequency

If $\qquad \text{DFT}\{x(n)\} = X(k)$

Then, $\qquad \text{DFT}\{x(n)e^{j2\pi ln/N}\} = X\{(k-l), (\text{mod } N)]$

103

Circular Frequency

If $\qquad \text{DFT}\{x(n)\} = X(k)$

Then, $\qquad \text{DFT}\{x(n)e^{j2\pi ln/N}\} = X\{(k-l), (\text{mod } N)]$

Proof: $\qquad \text{DFT}\{x(n)e^{j2\pi l n/N}\} = \sum_{n=0}^{N-1} x(n)e^{j2\pi l n/N}e^{-j2\pi kn/N}$

104

Circular Frequency

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then, $\quad \text{DFT}\{\mathbf{x(n)e^{j2\pi ln/N}}\} = X\{(k-l), (\text{mod } N)]$

Proof: $\quad \text{DFT}\{x(n)e^{j2\pi l n/N}\} = \sum_{n=0}^{N-1} x(n)e^{j2\pi l n/N}e^{-j2\pi kn/N}$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N}$$

105

Circular Frequency

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then, $\quad \text{DFT}\{\mathbf{x(n)e^{j2\pi ln/N}}\} = X\{(k-l), (\text{mod } N)]$

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$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N}$$

$$\text{DFT}\{x(n)e^{j2\pi ln/N}\} = X\{(k-l), (\text{mod } N)]$$

106

Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then $\quad \text{DFT}\ \{x^*(n)\} = X^*(N-k) = X^*[(-k), \bmod N]$

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Complex Conjugate Property

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108

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If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then $\quad \text{DFT}\ \{x^*(n)\} = X^*(N-k) = X^*[(-k), \text{mod } N]$

Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^*$$

109

Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

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Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^*$$

110

Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

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111

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$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

112

Complex Conjugate Property

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then $\quad \text{DFT}\{x^*(n)\} = X^*(N-k) = X^*[(-k), \bmod N]$

Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof: $\quad \text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$

113

Complex Conjugate Property

If $\quad \text{DFT}\{x(n)\} = X(k)$

Then $\quad \text{DFT}\{x^*(n)\} = X^*(N-k) = X^*[(-k), \bmod N]$

Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof: $\quad \text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^*$$

114

Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then $\quad \text{DFT}\ \{x^*(n)\} = X^*(N-k) = X^*[(-k), \text{mod } N]$

Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof: $\quad \text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^* = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}\right]^*$$

115

Complex Conjugate Property

If $\quad \text{DFT}\ \{x(n)\} = X(k)$

Then $\quad \text{DFT}\ \{x^*(n)\} = X^*(N-k) = X^*[(-k), \text{mod } N]$

Proof: $\quad \text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof: $\quad \text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^* = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}\right]^* = x^*(N-n)$$

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Multiplication

If $\quad \mathrm{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \mathrm{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \mathrm{DFT}[x_1(n)\,x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

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Multiplication

If $\quad \mathrm{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \mathrm{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \mathrm{DFT}[x_1(n)\,x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT}\{\mathbf{x_1(n)x_2(n)}\} = \sum_{\mathbf{n=0}}^{\mathbf{N-1}} \mathbf{x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

118

Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right]x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

119

Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right]x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]}$$

120

Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)\,x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}\,e^{\frac{-j2\pi kn}{N}}\right]}$$

Rearranging the exponential terms

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]}$$

121

Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)\,x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}\,e^{\frac{-j2\pi kn}{N}}\right]}$$

Rearranging the exponential terms

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]}$$

Using defination of DFT

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)X_2((k-m))_N}$$

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Multiplication

If $\quad \text{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \text{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\}} = \sum_{n=0}^{N-1} \mathbf{x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n)} = \frac{1}{N}\sum_{m=0}^{N-1} \mathbf{X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} \mathbf{X_1(m)e^{\frac{j2\pi mn}{N}}}\right]\mathbf{x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$= \frac{1}{N}\sum_{m=0}^{N-1} \mathbf{X_1(m)}\left[\sum_{n=0}^{N-1} \mathbf{x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}}\right]$$

Rearranging the exponential terms

$$= \frac{1}{N}\sum_{m=0}^{N-1} \mathbf{X_1(m)}\left[\sum_{n=0}^{N-1} \mathbf{x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}}\right]$$

Using defination of DFT

$$= \frac{1}{N}\sum_{m=0}^{N-1} \mathbf{X_1(m)X_2((k-m))_N}$$

By defination of Circular Convalution

$$\text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$$

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Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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Numerical Problem Based on Properties of DFT

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(a)Circular Convalution (b) Linear Convalution using DFT.

Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \text{mod } N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

Step3: $\mathbf{Y(k)} = \mathbf{X(k)} \times \mathbf{H(k)}$.

Step4: $y(n) = \mathbf{IDFT}(\mathbf{Y(k)})$.

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If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

129

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}} \qquad \mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

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If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT}(\mathbf{x}[\mathbf{n}]) = \mathbf{X}(\mathbf{k}) = \sum_{\mathbf{n=0}}^{\mathbf{1}} \mathbf{x}(\mathbf{n})\mathbf{w_2^{kn}} \qquad \mathbf{X}(\mathbf{k}) = \mathbf{x}(\mathbf{0})\mathbf{w_2^0} + \mathbf{x}(\mathbf{1})\mathbf{w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$\mathbf{X}(\mathbf{0}) = \mathbf{1w_2^0} + \mathbf{2w_2^0} = 1+ 2 =3$

131

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT}(\mathbf{x}[\mathbf{n}]) = \mathbf{X}(\mathbf{k}) = \sum_{\mathbf{n=0}}^{\mathbf{1}} \mathbf{x}(\mathbf{n})\mathbf{w_2^{kn}} \qquad \mathbf{X}(\mathbf{k}) = \mathbf{x}(\mathbf{0})\mathbf{w_2^0} + \mathbf{x}(\mathbf{1})\mathbf{w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$\mathbf{X}(\mathbf{0}) = \mathbf{1w_2^0} + \mathbf{2w_2^0} = 1+ 2 =3$

$\mathbf{X}(\mathbf{1}) = \mathbf{1w_2^0} + \mathbf{2w_2^1} = 1- 2=-1$

132

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}} \qquad \mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0}$ = 1+ 2 =3 $\qquad$ $\mathbf{X(1) = 1w_2^0 + 2w_2^1}$ = 1- 2=-1 $\qquad$ $\mathbf{X(k) = \{3, -1\}}$

133

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}} \qquad \mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0}$ = 1+ 2 =3 $\qquad$ $\mathbf{X(1) = 1w_2^0 + 2w_2^1}$ = 1- 2=-1 $\qquad$ $\mathbf{X(k) = \{3, -1\}}$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k} \qquad \text{where } \mathbf{k = 0, 1}$$

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If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}}$ $\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0} =$ 1+ 2 =3 $\mathbf{X(1) = 1w_2^0 + 2w_2^1} =$ 1- 2=-1 $\mathbf{X(k) = \{3, -1\}}$

$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}}$ $\mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{H(0) = 2w_2^0 + w_2^0} =$ 2+ 1 =3

135

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{1} x(n) w_2^{kn}}$ $\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0} =$ 1+ 2 =3 $\mathbf{X(1) = 1w_2^0 + 2w_2^1} =$ 1- 2=-1 $\mathbf{X(k) = \{3, -1\}}$

$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}}$ $\mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{H(0) = 2w_2^0 + w_2^0} =$ 2+ 1 =3 $\mathbf{H(1) = 2w_2^0 + 1w_2^1} =$ 2- 1=1

136

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step1: Make the length of both signal same.

Step2: Find DFT of $\mathbf{x(n)}$ and $\mathbf{h(n)}$.

$$\mathbf{DFT}(\mathbf{x}[\mathbf{n}]) = \mathbf{X}(\mathbf{k}) = \sum_{\mathbf{n=0}}^{\mathbf{1}} \mathbf{x}(\mathbf{n})\mathbf{w_2^{kn}}$$

$\mathbf{X(k) = x(0)w_2^0 + x(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{X(0) = 1w_2^0 + 2w_2^0}$ = 1+ 2 =3

$\mathbf{X(1) = 1w_2^0 + 2w_2^1}$ = 1- 2=-1

$\mathbf{X(k) = \{3, -1\}}$

$$\mathbf{DFT}(\mathbf{h}[\mathbf{n}]) = \mathbf{H}(\mathbf{k}) = \sum_{\mathbf{n=0}}^{\mathbf{1}} \mathbf{h}(\mathbf{n})\mathbf{w_2^{kn}}$$

$\mathbf{H(k) = h(0)w_2^0 + h(1)w_2^k}$ where $\mathbf{k = 0, 1}$

$\mathbf{H(0) = 2w_2^0 + w_2^0}$ = 2+ 1 =3

$\mathbf{H(1) = 2w_2^0 + 1w_2^1}$ = 2- 1=1

$\mathbf{H(k) = \{3, 1\}}$

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We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

138

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

139

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k) w_2^{-kn}} \qquad \mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)} \qquad \text{where } \mathbf{n = 0, 1}$$

140

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k) w_2^{-kn}} \qquad \mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)} \qquad \text{where } \mathbf{n = 0, 1}$$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9-1)/2 = 4}$$

141

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k) w_2^{-kn}} \qquad \mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)} \qquad \text{where } \mathbf{n = 0, 1}$$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9-1)/2 = 4}$$

$$\mathbf{y(1) = \left(Y(0)w_2^0 + Y(1)w_2^{-1}\right) = (9+1)/2 = 5}$$

142

We have X(k)=3, -1 and H(k)=3, 1

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 2}$

$\mathbf{w_N^{kn} = e^{-j2\pi kn/N} = w_2^1 = e^{-j\pi} = -1}$

(a)Circular Convalution

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1\}}$.

Step4: $\mathbf{y(n) = IDFT(Y(k))}$.

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k)w_2^{-kn}} \qquad \mathbf{y(n) = \frac{1}{2}\left(Y(0)w_2^0 + Y(1)w_2^{-n}\right)} \qquad \text{where } \mathbf{n = 0, 1}$$

$$\mathbf{y(0) = \left(Y(0)w_2^0 + Y(1)w_2^0\right) = (9-1)/2 = 4} \qquad \mathbf{y(1) = \left(Y(0)w_2^0 + Y(1)w_2^{-1}\right) = (9+1)/2 = 5}$$

$$\mathbf{y(n) = \{4, 5\}}$$

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If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

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$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

Step2: By Zero padding make lenght of of $\mathbf{x(n)}$ and $\mathbf{h(n)}$ same as $\mathbf{y(n)}$.

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$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

Step2: By Zero padding make lenght of of $\mathbf{x(n)}$ and $\mathbf{h(n)}$ same as $\mathbf{y(n)}$.

After zero padding $x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

146

$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

If $x(n) = \{1, 2\}$ and h(n)={2, 1}, find

(b) Linear Convalution using DFT.

Step1: Find the length of y(n). where $L = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

Step2: By Zero padding make lenght of of $\mathbf{x(n)}$ and $\mathbf{h(n)}$ same as $\mathbf{y(n)}$.

After zero padding $x(n) = \{1, 2, 0\}$ and $h(n) = \{2, 1, 0\}$

Step3: $\mathbf{Y(k) = X(k) \times H(k)}$.

Step4: $y(n) = \mathbf{IDFT(Y(k))}$.

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(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

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(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

Circular convolution	$x_1(n) \oplus x_2(n)$	$\longleftrightarrow$	$X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

149

(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

Circular convolution	$x_1(n) \oplus x_2(n)$	$\longleftrightarrow$	$X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt(3)/2) = j\sqrt{3}}$

150

Circular convolution	$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt{(3)}/2) = j\sqrt{3}}$

$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt{(3)}) = j\sqrt{3}}$

151

Circular convolution	$x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

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(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n) w_3^{kn}} \qquad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} = 1 + 2 = 3$

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt{(3)}/2) = j\sqrt{3}}$

$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt{(3)}) = j\sqrt{3}}$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n) w_2^{kn}} \qquad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k}$$

where $\mathbf{k = 0, 1, 2}$

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(b) Linear Convalution using DFT.

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k} \quad \text{where } \mathbf{k = 0, 1, 2}$$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} =$ 1+ 2 =3

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt{(3)}/2) = j\sqrt{3}}$

$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt{(3)}) = j\sqrt{3}}$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}} \quad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k} \quad \text{where } \mathbf{k = 0, 1, 2}$$

$\mathbf{H(0) = 2w_3^0 + 1w_3^0} =$ 1+ 2 =3

153

(b) Linear Convalution using DFT.

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

$$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}} \quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k} \quad \text{where } \mathbf{k = 0, 1, 2}$$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0} =$ 1+ 2 =3

$\mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt{(3)}/2) = j\sqrt{3}}$

$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt{(3)}) = j\sqrt{3}}$

$$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}} \quad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k} \quad \text{where } \mathbf{k = 0, 1, 2}$$

$\mathbf{H(0) = 2w_3^0 + 1w_3^0} =$ 1+ 2 =3

$\mathbf{H(1) = 2w_3^0 + w_3^1 = 2 - (1/2 - j\sqrt{(3)}/2) = 3/2 + j\sqrt{(3)}/2)}$

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(b) Linear Convalution using DFT.

$x(n) = \{1, 2, 0\}$ and h(n)={2, 1, 0}

Circular convolution	$x_1(n) \oplus x_2(n)$	$\longleftrightarrow$	$X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_N^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

$\mathbf{DFT(x[n]) = X(k) = \sum_{n=0}^{2} x(n)w_3^{kn}}$ $\quad \mathbf{X(k) = x(0)w_3^0 + x(1)w_3^k + x(2)w_3^k}$ $\quad$ where $\mathbf{k = 0, 1, 2}$

$\mathbf{X(0) = 1w_3^0 + 2w_3^0}$ = 1+ 2 =3 $\quad \mathbf{X(1) = w_3^0 + 2w_3^1 = 1 - 2(1/2 - j\sqrt{(3)}/2) = j\sqrt{3}}$

$\mathbf{X(2) = w_3^0 + 2w_3^2 = 1 - (1 - j\sqrt{(3)}) = j\sqrt{3}}$

$\mathbf{DFT(h[n]) = H(k) = \sum_{n=0}^{1} h(n)w_2^{kn}}$ $\quad \mathbf{H(k) = h(0)w_3^0 + h(1)w_3^k + h(2)w_3^k}$ $\quad$ where $\mathbf{k = 0, 1, 2}$

$\mathbf{H(0) = 2w_3^0 + 1w_3^0}$ = 1+ 2 =3 $\quad \mathbf{H(1) = 2w_3^0 + w_3^1 = 2 - (1/2 - j\sqrt{(3)}/2) = 3/2 + j\sqrt{(3)}/2)}$

$\mathbf{H(2) = w_3^0 + 2w_3^2 = 2 - (1/2 - j\sqrt{(3)}/2) = 3/2 + j\sqrt{3}/2}$

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$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$

$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$

Circular convolution	$x_1(n) \oplus x_2(n)$	$\longleftrightarrow$	$X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

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$X(k) = \{3, j\sqrt{3}, j\sqrt(3)\}.$

$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt(3)/2\}.$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

Step3: $\mathbf{Y(k) = X(k) \times H(k) = \{9, -1.5 - j2.598, -1.5 + j2.598\}}.$

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$X(k) = \{3, j\sqrt{3}, j\sqrt(3)\}.$

$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt(3)/2\}.$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt(3)/2}$

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Step4: $\mathbf{y(n) = IDFT(Y(k))}.$

$$\mathbf{IDFT(Y(k)) = y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k) w_3^{-kn}}$$

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$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$

$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

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$$\mathbf{y(0) = \frac{1}{3}\left(Y(0)w_3^0 + Y(1)w_3^{-n} + Y(3)w_3^{-2n}\right) = 6/3 = 2}$$

159

$X(k) = \{3, j\sqrt{3}, j\sqrt{(3)}\}.$

$H(k) = \{3, 3/2 + j\sqrt{3}/2, 3/2 + j\sqrt{(3)}/2\}.$

Circular convolution $x_1(n) \oplus x_2(n) \longleftrightarrow X_1(k)X_2(k)$

$\mathbf{N = 3}$ $\quad \mathbf{w_N^{kn} = e^{-j2\pi kn/N}}$

$\mathbf{w_3^1 = e^{-j2\pi/3} = -1/2 - j\sqrt{(3)}/2}$

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$$\mathbf{y(0) = \frac{1}{3}\left(Y(0)w_3^0 + Y(1)w_3^{-n} + Y(3)w_3^{-2n}\right) = 6/3 = 2}$$

$$\mathbf{y(n) = \{2, 5, 2\}}$$

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Find Linear and Circular Convalution using DFT and IDFT

$$x_1(n) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(n) = \{4, 3, 2, 1\}$$

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Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \text{mod } N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\, X(k) = e^{-j\pi k}\, X(k)$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
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(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\, X(k) = e^{-j\pi k}\, X(k)$$

$$= \{X(0)\,e^0, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

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(b) DFT of $x(-n)$
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(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\, X(k) = e^{-j\pi k}\, X(k)$$

$$= \{X(0)\,e^0, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\}$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k} X(k) = e^{-j\pi k} X(k)$$

$$= \{X(0)e^0, X(1)e^{-j\pi}, X(2)e^{-j2\pi}, X(3)e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
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(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k} X(k) = e^{-j\pi k} X(k)$$

$$= \{X(0)e^0, X(1)e^{-j\pi}, X(2)e^{-j2\pi}, X(3)e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

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Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \text{mod } N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

169

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\,X(k) = e^{-j\pi k}\,X(k)$$

$$= \{X(0)e^0, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

(b) Using the flipping (time reversal) property of DFT, we have

$$\text{DFT}\ \{x(-n)\} = X(-k) = X^*(k) =$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\, X(k) = e^{-j\pi k}\, X(k)$$

$$= \{X(0)\,e^{0}, X(1)\,e^{-j\pi}, X(2)\,e^{-j2\pi}, X(3)\,e^{-j3\pi}\}$$

$$= \{4(1),\ -j2(-1),\ 0(1),\ j2(-1)\} = \{4, j2, 0, -j2\}$$

(b) Using the flipping (time reversal) property of DFT, we have

$$\text{DFT}\ \{x(-n)\} = X(-k) = X^*(k) = \{4,\ -j2, 0, j2\}^* = \{4, j2, 0,\ -j2\}$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT}\ \{x^*(n)\} = X^*(-k)$$

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Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
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Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT }\{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT } \{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

Since DFT $\{x^*(n)\} = \text{DFT } \{x(n)\}$, we can say that $x(n)$ is real valued.

(d) Using the property of convolution of product of two signals, we have

$$\text{DFT } \{x(n)\,x(n)\} = \frac{1}{N}[X(k) \oplus X(k)]$$

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Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
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Parseval's theorem	$\sum_{n=0}^{N-1} x(n)\,y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(c) Using the conjugation property of DFT, we have

$$\text{DFT }\{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

Since DFT $\{x^*(n)\} = \text{DFT }\{x(n)\}$, we can say that $x(n)$ is real valued.

(d) Using the property of convolution of product of two signals, we have

$$\text{DFT }\{x(n)\,x(n)\} = \frac{1}{N}[X(k) \oplus X(k)] = \frac{1}{4}[(4, -j2, 0, j2) \oplus (4, -j2, 0, j2)]$$

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$$= \{6, -j4, 0, j4\}$$

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(f) Signal energy

Solution:

(e) Using the circular convolution property of DFT, we have

$$\text{DFT}\,\{x(n) \oplus x(n)\} = [X(k)\,X(k)]$$

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(f) Signal energy

Solution:

(e) Using the circular convolution property of DFT, we have

$$\text{DFT}\{x(n) \oplus x(n)\} = [X(k)X(k)] = \{4, -j2, 0, j2\}\{4, -j2, 0, j2\}$$

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(f) Signal energy

Solution:

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$$\text{DFT}\{x(n) \oplus x(n)\} = [X(k)X(k)] = \{4, -j2, 0, j2\}\{4, -j2, 0, j2\}$$

$$= \{16, -4, 0, -4\}$$

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(f) Using Parseval's theorem, we have

$$\text{Signal energy} = \frac{1}{4}\sum |X(k)|^2$$

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(f) Using Parseval's theorem, we have

$$\text{Signal energy} = \frac{1}{4}\sum |X(k)|^2 = \frac{1}{4}\sum |\{4, -j2, 0, j2\}|^2$$
$$= \frac{1}{4}[16+4+0+4] = 6$$

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EXAMPLE 6.23 If IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$, using properties of DFT, find

(a) IDFT $\{X(k-1)\}$
(b) IDFT $\{X(k) \oplus X(k)\}$
(c) IDFT $\{X(k)X(k)\}$
(d) Signal energy

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(c) IDFT $\{X(k)X(k)\}$
(d) Signal energy

Solution: Given IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$

(a) Using modulation property, we have

$$\text{IDFT } \{X(k-1)\} = x(n)\, e^{j2\pi n/4} = x(n)\, e^{j\pi \frac{n}{2}}$$

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$$= \left\{ x(0)\, e^{0},\ x(1)\, e^{\frac{j\pi}{2}},\ x(2)\, e^{j\pi},\ x(3)\, e^{j\frac{3\pi}{2}} \right\}$$

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$$= 1(1),\ 2(j),\ 1(-1),\ 0(-j) = \{1, j2,\ -1, 0\}$$

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(b) Using periodic convolution property, we have

$$\text{IDFT } \{X(k) \oplus X(k)\} = Nx^2(n)$$

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(d) Signal energy

Solution: Given IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$

(a) Using modulation property, we have

$$\text{IDFT } \{X(k-1)\} = x(n)\, e^{j2\pi n/4} = x(n)\, e^{j\pi \frac{n}{2}}$$

$$= \left\{ x(0)e^{0},\, x(1)\, e^{\frac{j\pi}{2}},\, x(2)\, e^{j\pi},\, x(3)\, e^{j\frac{3\pi}{2}} \right\}$$

$$= 1(1),\, 2(j),\, 1(-1),\, 0(-j) = \{1, j2, -1, 0\}$$

(b) Using periodic convolution property, we have

$$\text{IDFT } \{X(k) \oplus X(k)\} = Nx^2(n) = 4\{1, 2, 1, 0\}^2 = \{4, 16, 4, 0\}$$

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(a) IDFT $\{X(k-1)\}$
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(c) IDFT $\{X(k)X(k)\}$
(d) Signal energy

Solution: Given IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$

(c) Using the convolution in time domain property, we have

$$\text{IDFT } \{X(k)\, X(k)\} = \{1, 2, 1, 0\} \oplus \{1, 2, 1, 0\}$$

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(c) IDFT $\{X(k)X(k)\}$
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(d) Signal energy $= \sum_{n=0}^{N-1} |x(n)|^2 = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$

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$$= (1)^2 + (2)^2 + (1)^2 + (0)^2 = 6$$

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