Exercise

1) Find all possible expansion of $f(z) = \dfrac{7z^2+9z-18}{z^3-9z}$ about the points $z=0, -3, 3$.

2) Expand $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$ in the region $1<|z+1|<3$

3) expand $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$ in the region (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

4) find the nature of the singularities

(i) $(z+1)\sin\left(\dfrac{1}{z-2}\right)$ about $z=2$ (ii) $\dfrac{1-\sin z}{z^5}$ about $z=0$

(iii) $\dfrac{1-\cos z}{z^4}$ about $z=0$ (iv) $\dfrac{z^4}{(1-\cos z)^2}$ about $z=0$ (v) $\dfrac{1}{z(1-e^{z^2})}$ about $z=0$

3) $f(z) = \frac{\sin z}{z^2}$

$z = 0$ is a pole.

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

$\therefore$ Residue of $\frac{\sin z}{z^2}$ is coefft of $\frac{1}{z} = 1$.

4) $f(z) = \frac{1 - e^{2z}}{z^4}$

$z = 0$ is a pole.

$$\frac{1 - e^{2z}}{z^4} = \frac{1}{z^4}\left\{1 - \left(1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots\right)\right\}$$

$$= -\left(\frac{2}{z^3} + \frac{2}{z^2} + \frac{8}{6z} + \frac{16}{24} + \cdots\right)$$

$\therefore$ Residue of $f(z)$ at $z = 0$ is coefft of $\frac{1}{z} = -\frac{8}{6} = \underline{\underline{-\frac{4}{3}}}$

Examples

1) Residue of $f(z) = \dfrac{1-e^z}{z\cos z}$ at $z=0$ is 0

$$\frac{1-\left(1+z+\frac{z^2}{2!}+\cdots\right)}{z\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)} = \frac{-z-\frac{z^2}{2!}-\frac{z^3}{3!}-\cdots}{z\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)}$$

$$= -\left(1+\frac{z}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1+\left(\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots\right)\right)$$

$$=$$

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \cancel{z}\, \frac{(1-e^z)}{\cancel{z}\cos z} = \underline{\underline{0}}$$

2) Determine the poles of the function

$f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole.

Poles are $z = 1, -2$

$$\operatorname*{Res}_{z=1} f(z) = \left\{\frac{d}{dz}(z-1)^2 f(z)\right\}_{z=1} = \left\{\frac{d}{dz}\frac{z^2}{z+2}\right\}_{z=1} = \underline{\underline{5/9}}$$

$z=1$ is double pole. and $z=-2$ is a simple pole.

$$\operatorname*{Res}_{z=-2} f(z) = \lim_{z\to -2} \cancel{(z+2)}\frac{z^2}{(z-1)^2\cancel{(z+2)}} = \underline{\underline{4/9}}$$

(formula 1)

$$\operatorname*{Res}_{z=-2} f(z) = \frac{P(-2)}{Q'(-2)} = \frac{4}{\left[(z-1)^2+(z+2)\cdot 2(z-1)\right]_{z=-2}}$$

$= 4/9$ (formula 2)

Formula 2 suppose $f(z) = \frac{P(z)}{Q(z)}$ has a simple pole at $z=a$ such that $P(a) \neq 0$ and $Q(z)$ has a simple zero at $z=a$ (ie, $Q(a)=0$ and $Q'(a) \neq 0$)

then $\underset{z=a}{\text{Res}} f(z) = \underset{z \to a}{\text{Res}} \frac{P(z)}{Q'(z)} = \frac{P(a)}{Q'(a)}$

Calculation of Residue at multiple pole.

Suppose $f(z)$ has a pole of order n, say $z=a$

then $\underset{z=a}{\text{Res}} f(z) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] \right\}_{z=a}$

For example, $z=a$ is a double pole

then $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2}$.

Multiply throughout by $(z-a)^2$

$$(z-a)^2 f(z) = a_0(z-a)^2 + a_1(z-a)^3 + \cdots + b_1(z-a) + b_2$$

Diff w.r.to z,

$$\frac{d}{dz}\left((z-a)^2 f(z)\right) = 2a_0(z-a) + 3a_1(z-a)^2 + \cdots + b_1$$

$$\lim_{z \to a} \frac{d}{dz}\left[(z-a)^2 f(z)\right] = b_1$$

Residues

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{-n+1}}$$

$$\therefore \operatorname*{Res}_{z=a} f(z) = b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$\therefore \int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=a} f(z) = 2\pi i \operatorname{Res} f(a)$$

Calculation of residue at simple pole

Formula 1 $\quad b_1 = \lim_{z\to a} (z-a) f(z)$ if $z=a$ is a simple pole.

Laurent's series expansion of $f(z)$ about $z=a$ is

$$f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \dots + b_1 (z-a)^{-1}$$

Multiply throughout by $z-a$

$$f(z)(z-a) = a_0 (z-a) + a_1 (z-a)^2 + \dots + b_1$$

$$\lim_{z\to a} (z-a) f(z) = b_1$$

3) $f(z) = \dfrac{z^3}{z - \sin z}$ about $z=0$

$$\frac{z^3}{z-\sin z} = \frac{z^3}{\frac{z^3}{3!}\left[1 - \frac{3!\,z^2}{5!} + \frac{3!\,z^4}{7!} \cdots\right]} - 1$$

$$= 6\left[1 - \left(\frac{3!\,z^2}{5!} + \frac{3!\,z^4}{7!} - \cdots\right)\right]$$

$z=0$ is a removable singularity.

4) $f(z) = z^2 \cos 1/z$, $z=0$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$z^2 \cos 1/z = z^2\left(1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \cdots\right)$$

$$= z^2 - \frac{1}{2!} + \frac{1}{4!\,z^2} - \frac{1}{6!\,z^4} + \cdots$$

$z=0$ is an essential singularity.

Singular Points

1) $\dfrac{1}{z\sin z}$ about the point $z=0$

$$\frac{1}{z\sin z} = \frac{1}{z\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]}$$

$$= \frac{1}{z^2\left[1-\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right]}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots\right)^2+\cdots\right]$$

$$= \frac{1}{z^2}+\frac{1}{3!}-\frac{z^2}{5!}+\cdots$$

$z=0$ is a pole of order 2.

2) $f(z)=\dfrac{z^2}{z-\sin z}$ about the point $z=0$.

$$\frac{z^2}{z-\sin z} = \frac{z^2}{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)}$$

$$= \frac{6z^2}{z^3\left[1-\frac{3!z^2}{5!}+\cdots\right]} = \frac{6}{z}\left(1-\frac{3!z^2}{5!}+\cdots\right)^{-1}$$

$$= \frac{6}{z}\left[1+\frac{3!z^2}{5!}+\cdots\right]$$

$\Rightarrow z=0$ is a simple pole.