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Modern Control Theory (ICE 3153)

Construction of Phase Trajectories

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Sketching Phase Trajectories

- The sketching of the phase trajectory is the basis of phase plane analysis.
- Analytical method and graphical method are two main methods for plotting the phase trajectory.
- The analytical method leads to a functional relationship between x and $\dot{x}$ by solving the differential equation, then the phase trajectory can be constructed in the phase plane.
- The graphical method is used to construct the phase trajectory indirectly.

Direction of the phase trajectory

- In the upper half of the phase plane, $\dot{x} > 0$, the phase trajectory moves from left to right along the x axis.

- In the lower half of the phase plane, $\dot{x} < 0$, the arrows on the phase trajectory point to the left. In a word, the phase trajectory moves clockwise.

- The phase trajectory always passes through x axis vertically.

Analytical method

The state equations of a second order system is given by,

$$\dot{x_1} = f_1(x_1, x_2)$$
$$\dot{x_2} = f_2(x_1, x_2)$$

On dividing the above two equations, we get,

$$\frac{\dot{x_2}}{\dot{x_1}} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

We can also write the above equation as

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

The above equation defines the slope of the phase trajectory at every point in the phase plane, except at singular point.

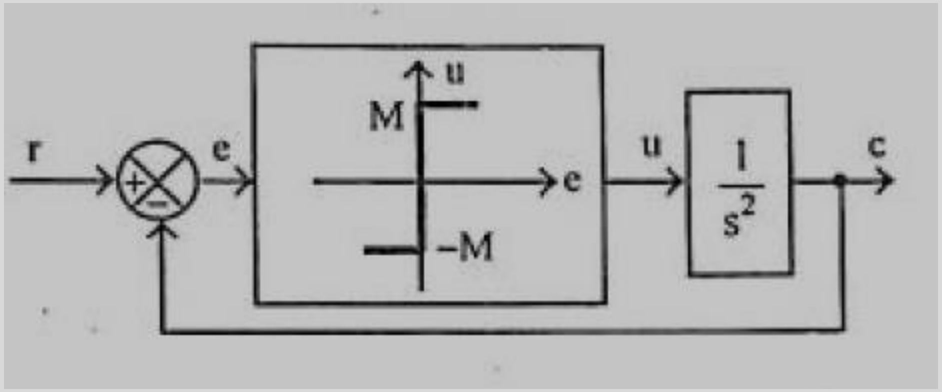
(At singular points, the slope of the phase trajectory is indeterminate)

The above slope equation is integrated and the resulting equation is used to construct the phase trajectory.

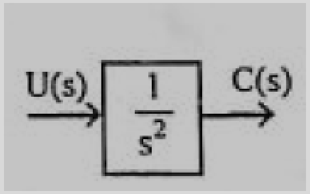
The analytical method is useful if the differential equations describing the systems can be approximated by piecewise linear differential equations. (ie. Equations are linearized for small regions)

Question:1
Example 2.4 , Advanced Control Theory , 2nd Edition Nagoor Kani

Consider a system with an ideal relay as shown in the below figure. Determine the singular points. Construct phase trajectories corresponding to initial conditions, (i) $c(0) = 2, \dot{c}(0) = 1$, (ii) $c(0) = 2, \dot{c}(0) = 1.5$. Consider $r = 2\, Volts\ and\ M = 1.2\, Volts$



Consider the linear part of the system,



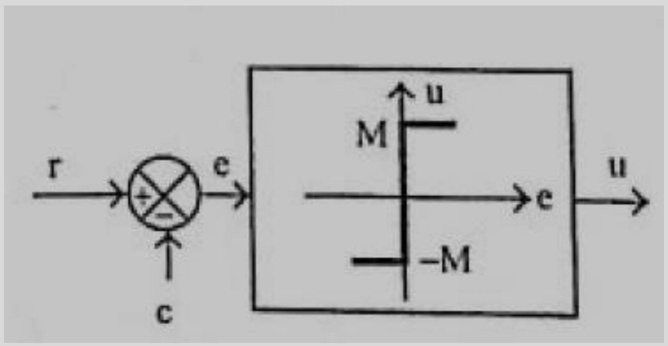
We can write, $C(s) = \frac{1}{s^2} U(s)$
Or $s^2 C(s) = U(s)$
On taking inverse LT, we get

$$\ddot{c}(t) = u(t)$$

Choose state variables as,

$$\therefore x_1 = c(t) \qquad \therefore \dot{x}_1 = x_2$$
$$x_2 = \dot{c}(t) \qquad \dot{x}_2 = u$$

Consider the nonlinear part of the system,



$$e = r - c$$
$$\text{But } c = x_1$$
$$\therefore e = r - x_1$$

when $x_1 > r$, r, e is negative, $\therefore u = -M$

when $x_1 < r$, e is positive, $\therefore u = +M$

On substituting the values of u in the state equations we get,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = M \quad ; \quad \text{for } x_1 < r$$
$$= -M \quad ; \quad \text{for } x_1 > r$$

By integrating the below equation we get state of the system at any time t,

$$x_2\, dx_2 = \pm M\, dx_1$$

$$\int_{x_2(0)}^{x_2} x_2 dx_2 = \int_{x_1(0)}^{x_1} \pm M dx_1 \qquad \left[\frac{x_2^2}{2}\right]_{x_2(0)}^{x_2} = \pm M[x_1]_{x_1(0)}^{x_1}$$

$x_2^2 = -2Mx_1 + 2Mx_1(0) + x_2^2(0)$ for $x_1 > r$

$x_2^2 = 2Mx_1 - 2Mx_1(0) + x_2^2(0)$ for $x_1 < r$

For first set of initial conditions, $x_1(0) = 2, x_2(0) = 1$

$$x_2 = \pm\sqrt{-2.4x_1 + 5.8}\ for x_1 > r \qquad x_2 = \pm\sqrt{2.4x_1 - 3.8}\ for x_1 < r$$

For second set of initial conditions, $x_1(0) = 2, x_2(0) = 1.5$

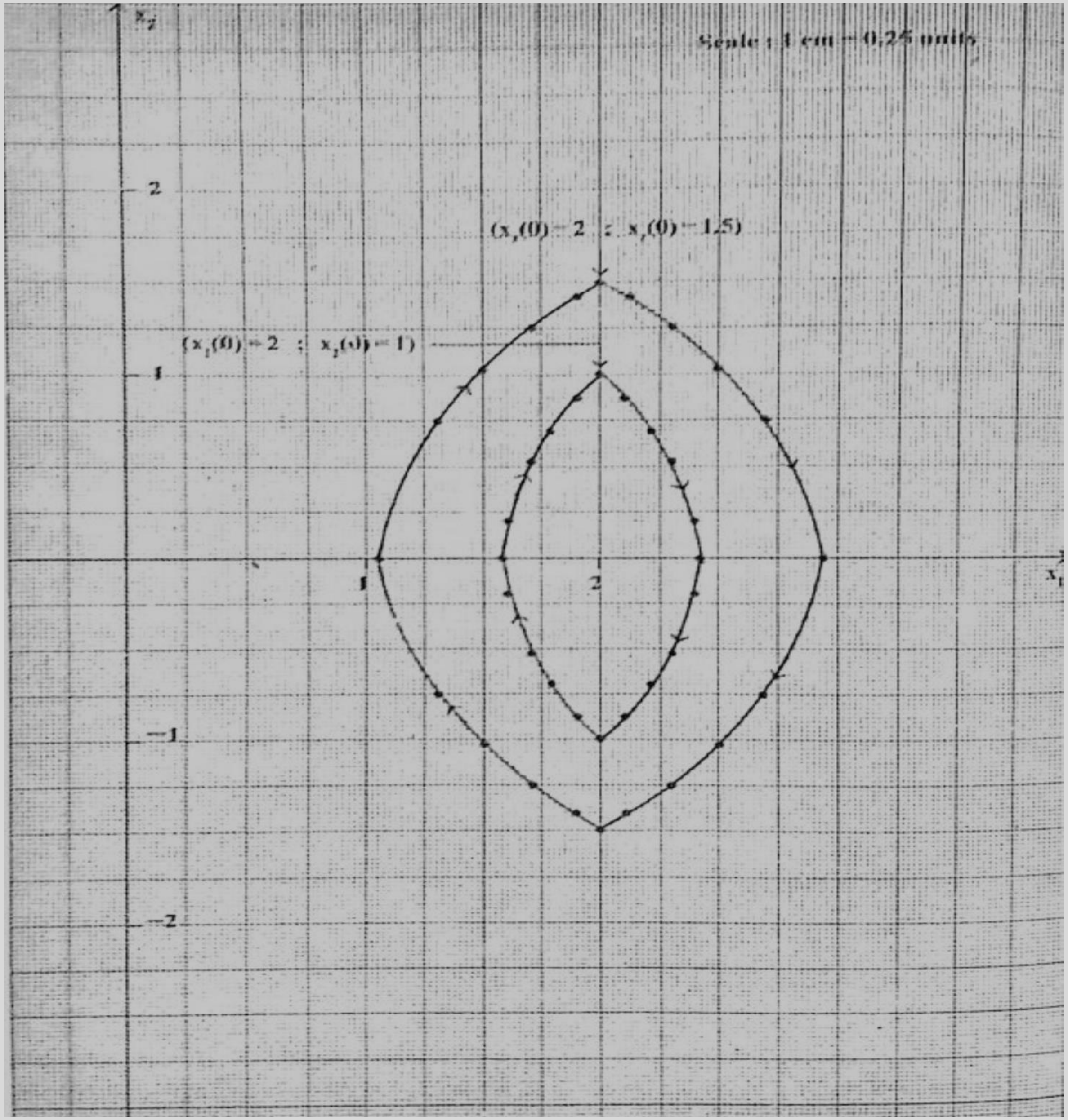
$$x_2 = \pm\sqrt{-2.4x_1 + 7.05}\ for x_1 > r \qquad x_2 = \pm\sqrt{2.4x_1 - 2.55}\ for x_1 < r$$

For initial condition (i) the table is,

$x_1 > r$			$x_1 < r$		
x_1	x_2		x_1	x_2	
2	1	−1	1.5834	0	0
2.1	0.87	−0.87	1.6	0.2	−0.2
2.2	0.72	−0.72	1.7	0.53	−0.53
2.3	0.53	−0.53	1.8	0.72	−0.72
2.4	0.2	−0.2	1.9	0.87	−0.87
2.4167	0	0	2	1	−1

For initial condition (ii) the table is,

$x_1 > r$			$x_1 < r$		
x_1	x_2		x_1	x_2	
2	1.5	−1.5	1.0625	[illegible]	0
2.1	1.42	−1.42	1.3	0.75	−0.75
2.3	1.24	−1.24	1.5	1.02	−1.02
2.5	1.02	−1.02	1.7	1.24	−1.24
2.7	0.75	−0.75	1.9	1.42	−1.42
2.9375	0	0	2	1.5	−1.5



x_2
Scale : 1 cm = 0.25 units
2
$(x_1(0) = 2 \; ; \; x_2(0) = 1.5)$
$(x_1(0) = 2 \; ; \; x_2(0) = 1)$
1
1
2
x_1
−1
−2

Isocline method

Let S be nay point in the phase plane,

$$S = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

Let, S_1 = Slope at a point on phase trajectory-1.

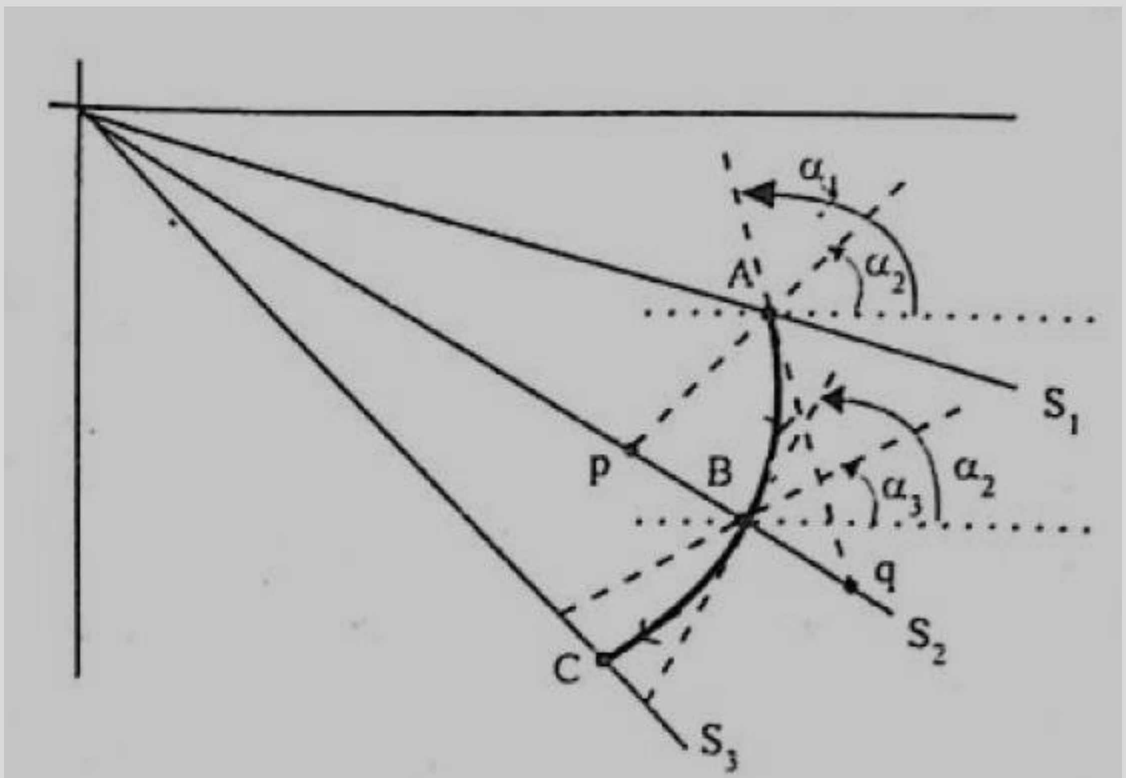
$$f_2(x_1, x_2) = S_1 \times f_1(x_1, x_2)$$

- The above equation defines the locus of all such points in phase plane at which the slope of the trajectory is S1.
- A locus passing through the locus the points of same slope in phase plane is called isocline.
- Slope of any trajectory crossing the isocline will have the slope of the isocline

Let, S_1, S_2, S_3, etc., be the slopes associated with isoclines 1, 2, 3, etc.,

Let, $\alpha_1 = \tan^{-1}(S_1)$; $\alpha_2 = \tan^{-1}(S_2)$; $\alpha_3 = \tan^{-1}(S_3)$; etc.,

Note : If a straight line is drawn at an angle α from a point, then the slope of the line at that point is $\tan \alpha$.



$\alpha_1 = \tan^{-1}(S_1)$
$\alpha_2 = \tan^{-1}(S_2)$
$\alpha_3 = \tan^{-1}(S_3)$

Question:2
Example 2.5 , Advanced Control Theory , 2nd Edition Nagoor Kani

A linear second order servo is described by the equation

$$\ddot{c} + 2\zeta\omega_n\dot{c} + \omega_n^2 c = 0$$

where $\zeta = 0.15$, $\omega_n = 1$ rad/sec, $c(0) = 1.5$ and $\dot{c}(0) = 0$.

Determine the singular point. Construct the phase trajectory, using the method of isoclines.

Let the state variables be, $x_1 = c$, $x_2 = \dot{c}$

The state equations of the system are given by equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

The singular point is (0, 0)

$$S = \frac{dx_2/dt}{dx_1/dt} = \frac{\dot{x}_2}{\dot{x}_1}$$

$$S = -\frac{(\omega_n^2 x_1 + 2\zeta\omega_n x_2)}{x_2}$$

Put $\zeta = 0.15$ and $\omega_n = 1$,

$$S = \frac{-(x_1 + 2 \times 0.15\, x_2)}{x_2} = \frac{-(x_1 + 0.3\, x_2)}{x_2}$$

$$\frac{x_1}{x_2} = -0.3 - S \quad \text{(or)} \quad \frac{x_2}{x_1} = \frac{1}{-0.3 - S}$$

$$\therefore x_2 = \frac{x_1}{-0.3 - S}$$

Let us choose values of S as $-2, -1.0, -0.5, 0, 0.5, 1.0$ and 2.0

S	−2.0		−1.0		−0.5		0		0.5		1.0		2.0	
α	−63°		−45°		−27°		0		27°		45°		63°	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	1.0	0.6	1.0	1.4	0.25	1.25	0.25	−0.8	1.0	−1.25	1.0	−0.77	1.0	−0.43
	2.0	1.2	1.5	2.1	0.5	2.5	0.75	−2.5	2.0	−2.5	2.0	−1.54	2.0	−0.86