Equations with non-homogeneous coefficients:

Consider a differential equation of the form

$$(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2)\,dy = 0 \quad \text{——— (1)}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants. (real numbers).

Let

$$a_1x+b_1y+c_1=0 \quad \text{— (2)}$$

$$a_2x+b_2y+c_2=0 \quad \text{— (3)}$$

(2) and (3) represent straight lines in plane.

They may be parallel or intersecting.

Case 1: Let (2) and (3) be parallel.

Then we can find an m such that

$$a_2x+b_2y = m(a_1x+b_1y) \qquad \frac{a_1}{a_2}=\frac{b_1}{b_2}$$

Put $a_1x+b_1y=t$

The equation (1) reduces to variable separable and hence can be solved.

Case 2: Let (2) and (3) be intersecting.

Let (h,k) be the point of intersection.

Substitute $x=X+h$, $y=Y+k$.

With this substitution, (1) reduces to a homogeneous differential equation and hence can be solved.

Solve the following differential equations:

(1) $\dfrac{dy}{dx}=\dfrac{2x-6y+7}{x-3y+4}$ $\qquad \dfrac{2}{1}=\dfrac{-6}{-3}$

$2x-6y=2(x-3y)$ $\qquad$ lines are parallel.

Put $x-3y=t$

$$1-3\frac{dy}{dx}=\frac{dt}{dx}$$

$$1-\frac{dt}{dx}=3\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{3}\left(1-\frac{dt}{dx}\right)$$

$$\frac{1}{3}\left(1-\frac{dt}{dx}\right)=\frac{2t+7}{t+4}$$

$$1-\frac{dt}{dx}=\frac{6t+21}{t+4}$$

$$1-\frac{6t+21}{t+4}=\frac{dt}{dx}$$

$$\frac{t+4-6t-21}{t+4}=\frac{dt}{dx}$$

$$\frac{-5t-17}{t+4}=\frac{dt}{dx}$$

$$\frac{5t+20}{5t+17}\,dt=-5\,dx$$

$$\frac{5t+17+3}{5t+17}\,dt=-5\,dx.$$

$$\frac{t+4}{5t+17}\,dt=-dx$$

$$\frac{\frac{1}{5}(5t+17)+\frac{3}{5}}{5t+17}\,dt=-dx$$

$$\left(\frac{1}{5}+\frac{3}{5}\,\frac{1}{5t+17}\right)dt=-dx$$

Integrating

$$\frac{1}{5}t+\frac{3}{5}\,\frac{\log(5t+17)}{5}=-x+\frac{c}{25}$$

$\times$ by 25; $\quad 5t+3\log(5t+17)=-25x+c$

$$5(x-3y)+3\log(5x-15y+17)=-25x+c$$

② $(2x+3y+4)\,dx - (4x+6y+5)\,dy = 0.$

$\frac{a_1}{a_2} = \frac{b_1}{b_2}$

$\frac{2}{4} = \frac{3}{6}$ $\therefore$ parallel.

$$\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$$

$2x+3y = t$

$$2 + 3\frac{dy}{dx} = \frac{dt}{dx}$$

$$3\frac{dy}{dx} = \frac{dt}{dx} - 2$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$$

$4x+6y = 2(2x+3y)$
$= 2t.$

$$\frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t+4}{2t+5}$$

$$\frac{dt}{dx} - 2 = \frac{3t+12}{2t+5}$$

$$\frac{dt}{dx} = \frac{3t+12}{2t+5} + 2$$

$$= \frac{3t+12+4t+10}{2t+5}$$

$$\frac{dt}{dx} = \frac{7t+22}{2t+5}$$

$5 - \frac{44}{7}$

$= \frac{35-44}{7}$

$= -\frac{9}{7}$

$$7t+22 \overline{\big)\, 2t+5\,}\ \frac{2}{7}$$
$2t + \frac{44}{7}$
$(-)\ \ (-)$
$-\frac{9}{7}$

$2t+5 = \frac{2}{7}(7t+22) - \frac{9}{7}$

$$\frac{(2t+5)\,dt}{7t+22} = dx$$

$$\frac{2t+5}{7t+22}$$

$$= \frac{2t}{7t+22} + \frac{5}{7t+22}$$

$$\left(\frac{2}{7} + \frac{-9/7}{7t+22}\right)dt = dx$$

$D_2 \overline{\big)\, D_1\,}\ Q$
R

$$\frac{D_1}{D_2} = Q + \frac{R}{D_2}$$

$$\frac{2}{7}t - \frac{9}{7}\,\frac{\log(7t+22)}{7} = x + \frac{C}{49}$$

x^{ly} by 49

$$14t - 9\log(7t+22) = 49x + C$$

$$14(2x+3y) - 9\log(7(2x+3y)+22) = 49x + C$$

$$-21x + 42y - 9\log(14x+21y+22) = C.$$

③ $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

$$y+x-2=0$$
$$y-x-4=0$$
$$2y-6=0$$
$$y=3, \quad \therefore x=-1.$$

$$-1 = \frac{a_1}{a_2} \neq \frac{b_1}{b_2} = 1$$

point of intersection is $(-1,3)$

Put

$x = X+h, \quad y = Y+k$

$X = x+1 \Leftarrow x = X-1 \qquad y = Y+3 \Rightarrow Y = y-3$

$dx = dX \qquad dy = dY.$

$$\frac{dY}{dX} = \frac{(Y+3)+(X-1)-2}{(Y+3)-(X-1)-4}$$

$\frac{dY}{dX} = \frac{Y+X}{Y-X}$. This is a homogeneous differential equation.

$$Y = vX$$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{vX+X}{vX-X} = \frac{\cancel{X}(v+1)}{\cancel{X}(v-1)}$$

$$X\frac{dv}{dX} = \frac{v+1}{v-1} - v$$

$$X\frac{dv}{dX} = \frac{v+1-v^2+v}{v-1} = \frac{-v^2+2v+1}{v-1} = \frac{-(v^2-2v-1)}{v-1}$$

$$\frac{v-1}{v^2-2v-1}dv = -\frac{dX}{X}$$

$$\frac{1}{2}\frac{(2v-2)dv}{v^2-2v-1}$$

$$\frac{1}{2}\frac{dz}{z}$$

$v^2-2v-1 = z$

$(2v-2)\,dv = dz$

$(v-1)\,dv = \frac{dz}{2}$

$$\frac{1}{z}\frac{dz}{2} = -\frac{dX}{X}$$

$$\frac{1}{2}\log z = -\log X + \frac{1}{2}\log c$$

$$\log z + 2\log X = \log c$$

$$\log(zX^2) = \log c$$

$\log m^n = n\log m$

$2\log X = \log X^2$

$\log m + \log n = \log(mn)$

$$zX^2 = c$$

$$(v^2-2v-1)X^2 = c$$

$$\left(\frac{Y^2}{X^2} - \frac{2Y}{X} - 1\right)X^2 = c$$

$$\left(\frac{Y^2-2XY-X^2}{X^2}\right)X^2 = c$$

$x = X-1, \quad y = Y+3$

$X = x+1, \quad Y = y-3$

$$Y^2 - 2XY - X^2 = c$$

$$(y-3)^2 - 2(x+1)(y-3) - (x+1)^2 = c$$

④ $\frac{dy}{dx} = \frac{2x-5y+3}{2x+4y-6}$

$2x-5y+3=0$
$2x+4y-6=0$
$\overline{-9y+9=0}$
$\therefore y=1, \quad x=1$

$(1,1)$

$x = X+1, \quad y = Y+1$

$dx = dX, \quad dy = dY.$

$$\frac{dY}{dX} = \frac{2(X+1)-5(Y+1)+3}{2(X+1)+4(Y+1)-6}$$

$$\frac{dY}{dX} = \frac{2X-5Y}{2X+4Y}$$

$Y = vX$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$$v + X\frac{dv}{dX} = \frac{2X-5vX}{2X+4vX}$$

$$v + X\frac{dv}{dX} = \frac{X(2-5v)}{X(2+4v)}$$

$$X\frac{dv}{dX} = \frac{2-5v}{2+4v} - v$$

$$= \frac{2-5v-2v-4v^2}{2+4v} = \frac{-(4v^2+7v-2)}{2+4v}$$

$Nr. = A\frac{d}{dx}(Dr.)+B$

$$\left(\frac{4v+2}{4v^2+7v-2}\right)dv = -\frac{dX}{X}$$

$4v^2+7v-2 = 4v^2+8v-v-2$
$= 4v(v+2)-(v+2)$
$= (4v-1)(v+2)$

$$\left(\frac{4}{3}\frac{1}{4v-1} + \frac{2}{3}\frac{1}{v+2}\right)dv = -\frac{dX}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4} + \frac{2}{3}\log(v+2) = -\log X + \frac{\log c}{3}$$

$\frac{4v+2}{4v^2+7v-2} = \frac{4v+2}{(4v-1)(v+2)}$

$= \frac{A}{4v-1} + \frac{B}{v+2}$

$$\log(4v-1) + \log(v+2)^2 + 3\log X = \log c$$

$4v+2 = A(v+2) + B(4v-1)$

$v=-2, \quad B = 2/3$

$v = +\frac{1}{4}, \quad A = 4/3$

$$(4v-1)(v+2)^2 X^3 = c$$

$$\left(4\frac{Y}{X}-1\right)\left(\frac{Y}{X}+2\right)^2 X^3 = c$$

$$\left(\frac{4Y-X}{X}\right)\left(\frac{Y+2X}{X}\right)^2 X^3 = c$$

$$(4Y-X)(2X+Y)^2 = c$$

$$[4(y-1)-(x-1)][2(x-1)+(y-1)]^2 = c$$

$$(4y-x-3)(2x+y-3)^2 = c.$$

Problems for Practice:

Solve:

1. $(x+3y-4)\,dx + (x+4y-5)\,dy = 0$

2. $\dfrac{dy}{dx} + \dfrac{10x+8y-12}{7x+5y-9} = 0$

3. $\dfrac{dy}{dx} + \dfrac{x-2y+1}{2x-4y+3} = 0$

4. $(4x-6y-1)\,dx - (2x-3y+2)\,dy = 0.$