21/03/22

One dimensional Random variables:

Let 'E' be an experiment x be the sample space associate with 'E' a function 'x' assigning to every element 's'$\in S$, a real number called as a random variable.

Ex: Tossing two coins,

$S = \{HH, TT, HT, TH\}$

X. No. of heads.

$X(HH) = 2 \quad X(TH) = 1 \quad X(HT) = 1 \quad X(TT) = 0.$

R_x = Range space of $X = \{0, 1, 2\}$

$P(X=0) = P(TT)$

$= P(HT \text{ or } TH)$

$= P(HT) + P(TH)$

$= 1/4 + 1/4 = 1/2$

$P(X=2) = P(HH) = 1/4$

Types of Random variables:

① Discreate random variable (DRV):

If the range space of 'x' is finite (or) countabley infinite then 'x' is called as descreate random variable

Probability mass function (pmf): Let 'x' be a discreate random variable then the probability of ~~all the~~ $P(x_i) = P(X = x_i)$ is called the probability mass function pmf of x, if it satisfys the following conditions.

(i) $P(x_i) > 0 \ \forall i$

(ii) $\sum_{i=1}^{n} P(x_i) = 1$

② Countinuous random variabel (CRV):

A random variabel 'x' is said to be countinuous if its range space is an uncountable set. If 'x' is a countinuous random variabel then the (pdf) probability density function of x, is function f(x) satisfy the condition (i) $f(x) \geq 0 \ \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Note: $P(a \leq x \leq b) = \int_a^b f(x)\,dx$

$P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b) = P(a < x < b)$

Cumulative Distribution function: (cdf) [F(x)]:

$F(x) = P(X \leq x)$

If 'x' is DRV

$F(x) = P(X \leq x_i) = \Sigma P(x_i).$

If 'x' is CRV

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(s)\,ds.$

Exam Must question DRV/Σ

Ex: Suppose that the random variabel x assumes the values with {0, 1 & 2} with probabilities {1/3, 1/6 & 1/2} Respectively. find the cdf of x

Ans

$R_x = \{0, 1, 2\}$

$P(x=0) = 1/3$, $P(x=1) = 1/6$, $P(x=2) = 1/2$.

$P(x_i) = ?$

$F(x) = P(x \leq x)$

$x < 0$
$F(x) = 0$

$F(x) = 1/3$

$0 < x < 1$
$F(x) = 1/3$

$x = 1$
$F(x) = 1/3 + \frac{1}{6} = 1/2$

$1 < x < 2$
$F(x) = 1/3 + \frac{1}{6} = \frac{1}{2}$

$x = 2$
$F(x) = 1/3 + 1/6 + 1/2$
$= 1$

$x > 2$
$F(x) = 1.$

$$F(x) = \begin{cases} 0, & x<0 \\ \frac{1}{3}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

Ex: ~~Sp~~suppose 'x' is countinuous random variable with Pdf $f(x) = \begin{cases} 2x, & x \in (0,1). \\ 0, & \text{else where} \end{cases}$

Find Cdf $F(x)$; $x<0$.

$0 \le x < 1$

$x \ge 1$.

$$F(x) = \int_{-\infty}^{x} f(s)\,ds.$$

$$= \int_{-\infty}^{0} f(s)\,ds + \int_{0}^{x} f(s)\,ds$$

$$= 0 + \int_{0}^{x} 2s\,ds = x^2.$$

$$F(x) = \begin{cases} 0, & x<0 \\ x^2, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

22/03/22

~~Uniformly distributed random variable:~~

Properties of (Cdf):

If 'x' is a discreate random variable with cdf $F(x)$ then $P(x_j) = P(x = x_j) = F(x_j) - F(x_{j-1})$. If '$x$' is a countinous random variable,

$$f(x) = \frac{d}{dx} F(x).$$

Uniformly distrubuted random variables

Suppose x is a countinuos random variable assuming all values in interval $[a, b]$ where a, b are finite. If the pdf of x is given by, then we say that 'x' is uniformly distrubuted over the interval a, b.

$x \sim U[a,b]$.

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{elsewhere} \end{cases}$$

Q4) given $f(x) = \begin{cases} kx^3 ; & x \in [0,1] \\ 0 & , \text{otherwise} \end{cases}$

(1) find 'k'

(2) Find probability of $P(1/4 \le x \le 3/4)$

(3) Find $P(x < 1/2)$

(4) find $P(0 \cancel{x} \ge 0.8)$

(5) calculate cdf $F(x)$.

$\int_{-\infty}^{\infty} f(x)dx = 1$

$P \cup \int_a^b f(x)\,dx$

Ans

(1) $f(x)$ is pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \Rightarrow \int_{-\infty}^{0} = 0 + \int_0^1 + \int_1^{\infty} = 0 = 1$$

$$\Rightarrow \int_0^1 kx^3\,dx = 1 \Rightarrow \boxed{k=4}$$

$$\left(\frac{x^4}{4}\right)_0^1 = 1$$

$$\boxed{k = 4}$$

$$\therefore f x = \begin{cases} 4x^3, & x \in [0,1] \\ 0, & \text{outside} \end{cases}$$

(2) $P\left(\frac{1}{4} \le x \le \frac{3}{4}\right)$ $\qquad \left[\because P(a \le x \le b) = \int_a^b f(x)\,dx\right]$.

$a \le x \le b$

$$P(\because) = \int_a^b f(x)\,dx = \int_{1/4}^{3/4} 4x^3\,dx = \left[x^4\right]_{1/4}^{3/4} = \left(\frac{3}{4}\right)^4 - \left(1/4\right)^4$$

$$= \frac{80}{166}$$

$$= \frac{5}{16}$$

(3) $P(x < 1/2)$

$x \in [-\infty, 1/2]$

$P(x < 1/2) = P(-\infty \le x < 1/2)$.

$$= \int_{-\infty}^{1/2} 4x^3\,dx = \int_{-\infty}^{0} + \int_0^{1/2} 4x^3\,dx = \left[x^4\right]_0^{1/2} = (1/2)^4 = 1/16$$

(4) $P(x > 0.8) = P(0.8 \le x \le \infty)$

$$= \int_{0.8}^{1} 4x^3\,dx + \int_1^{\infty} dx$$

$$= (x^4)_{0.8}^{1} = 1 - (0.8)^4 = 0.5904$$

$$= (0.8)^4 + 1$$

(5) $= \int_{-\infty}^{0} + \int_{0}^{x} 4s^3 dx = x^4, \quad 0 \le x \le 1$

24/3/22

$F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{1} 4s^3 ds = \int_{1}^{x} 0$

Q5) A coin is known to come up heads 3 times as often as tail. This coin is tossed 3 times. Let X be the no. of heads that appear. Find the probability distribution of X and also the cdf

$P(H) = \frac{3}{4}, \quad P(T) = \frac{1}{4}$, $S = \{HHH, TTT, HHT, HTH, THH, THT, TTH, HTT\}$.

$R_x = \{0, 1, 2, 3\}$

x is DRV.

$P(X=0) = P(TTT) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$

$P(X=1) = P(HTT) + P(THT) + P(TTH) = \frac{3}{64} + \frac{3}{64} + \frac{3}{64} = \frac{9}{64}$

$P(X=2) = P(HHT) + P(HTH) + P(THH)$

$= \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$

$P(X=3) = P(HHH) = \frac{27}{64}$.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{64} & 0 \le x < 1 \\ \frac{1}{64} + \frac{9}{64} = \frac{10}{64} & 1 \le x < 2 \\ 37/64 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Q6) A Random variable assumes 4 values with probabilities

$\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}, \frac{1-4x}{4}$. for what values of 'x' this is probability distribution.

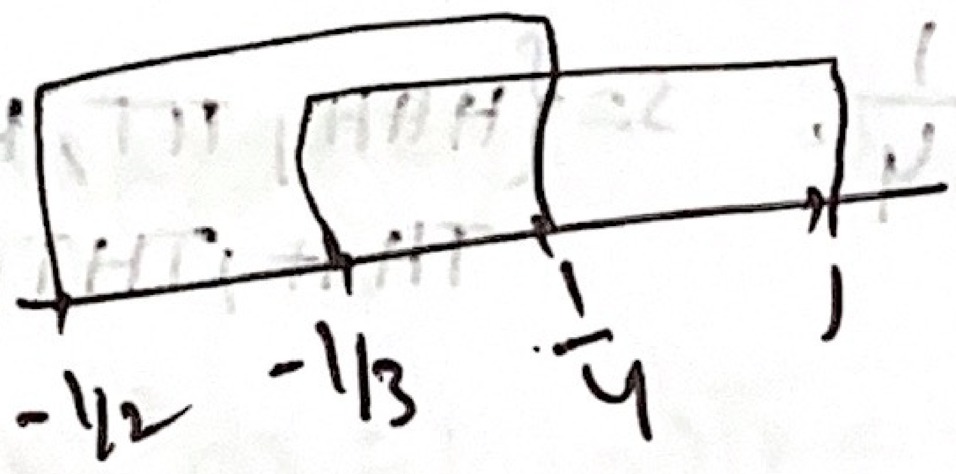
A)

$1-4x \ge 0$

$\frac{1+3x}{4} \ge 0$ $\quad 1 \ge 4x$

$x \ge -1 \quad x \ge -\frac{1}{2} \quad x \le \frac{1}{4} \quad x \le \frac{1}{4}$

$x \ge -\frac{1}{3} \quad x \le -1$



$\{-\frac{1}{3} \le x \le \frac{1}{4}\}$.

Q7) The diameter of electric cable says 'x' is assumed to be a countinuous random variable with pdf $f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

① Check that the above is valid pdf

② Find cdf

③ Find 'b' such that probability of $x<b$ $P(x<b) = 2 P(x>b)$

④ $P(x \le \frac{1}{2} \mid \frac{1}{3} \le x \le \frac{2}{3})$.

Ans

① $\int_0^1 6x(1-x)\,dx = 6\int_0^1 x - x^2 \,dx$

$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 6\left(\frac{3x^2-2x^3}{6}\right)_0^1$

$= 1$

So valid pdf

② $\int_{-\infty}^{\infty} \cancel{[\ldots]}\,dx$

$$f(x) = \begin{cases} 0 & , x<0 \\ 3x^2-2x^3 & , 0 \le x \le 1 \\ 1 & , x \ge 1 \end{cases}$$

$F(x) = \int_{-\infty}^{s} f(t)\,ds$

(CRV)

$= \int_{-\infty}^{0} f(s)\,ds + \int_{0}^{x} f(s)\,dx$

$= 0 + \int_{0}^{x} 6s(1-s)\,ds$

$= 3x^2 - 2x^3$

$F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{1} 6x(1-x)\,dx + \int_{1}^{x} 0$

$= 1$

③ $P(x<b) = 2P(x>b)$.

$= 2[1-P(x<b)]$

$3P(x<b) = 2$

$P(x<b) = \frac{2}{3}$

$\int_{0}^{b} 6x(1-x)\,dx = \frac{2}{3}$

$\int_{-\infty}^{0} f(s)\,ds + \int_{0}^{b} f(s)\,ds = 2/3$

$0 + \int_{0}^{b} 6s(1-s)\,ds = 2/3$

$3b^2 - 2b^3 = 2/3$

$3b^2 - 2b^3 - \frac{2}{3} = 0$

$3b^2 - 2b^3 - \frac{2}{3} = 0$.

$b = 0.613$.

(Bisection) also we can do.

→ mode of distribution:

The value of r which makes $P(X=r)$ largest.

2/8/3/22

Q8) Suppose that a random variable 'x' has possible values 1, 2, 3, ... ∞ and probability

$P(X=r) = K(1-\beta)^{r-1}$, $0<\beta<1$.

(i) find constan k

(ii) find mode of distrubution.

Ans)

(i) Since 'x' is DRV.

$$\sum_{r=1}^{\infty} P(X=r) = 1$$

$$\Rightarrow K \sum_{r=1}^{\infty} (1-\beta)^{r-1} = 1.$$

$$\Rightarrow K\left[1 + (1-\beta) + (1-\beta)^2 + \cdots\right] = 1$$

$$\Rightarrow K\left[\frac{1}{1-(1-\beta)}\right] = 1.$$

$$K = \beta.$$

$$\therefore P(X=r) = \beta(1-\beta)^{r-1}$$

when $r = 1, 2, 3, \ldots$

$$P(X=r) = \beta,\ \beta(1-\beta)^{1},\ \beta(1-\beta)^{2}, \ldots$$

β is the largest probability so, $r=1$ is the mode of distrubution.

Q9) Let $F(x) = 1 - e^{-x^3}$, $x>0$.

(i) Find the pdf $f(x)$.

(ii) Find c if $P(X<c) = 0.5$.

Ans) (i)

$$f(x) = \frac{d}{dx} F(x)$$

pdf cdf

$$f(x) = \frac{d}{dx}(1 - e^{-x^3})$$

$$f(x) = 3x^2 e^{-x^3},\ x>0.$$

(iii) $P(x<c) = 0.5$

$$= \int_0^c 3x^2 e^{-x^3}\,dx = 0.5$$

Put $x^3 = t \mid 3x^2 dx = dt$

$$= \int e^{-t}\,dt = -e^{-t}$$

$$\Rightarrow \left[-e^{-x^3}\right]_0^c = 0.5$$

$$\Rightarrow 1 - e^{-c^3} = 0.5$$

$$\Rightarrow e^{-c^3} = 0.5$$

$$\Rightarrow -c^3 = \log 0.5 \qquad c^3 = 0.6931$$

$$\cancel{c^3 = 0.301} \qquad c = (0.693)^{1/3}$$

$$\cancel{c = \sqrt[3]{0.301}}$$

Q10) If the random variable 'k' is uniformly distributed over interval (0,5). what is probability that roots of equation $\underset{a}{4x^2} + \underset{b}{4xk} + \underset{c}{k+2} = 0$ are real.

$b^2 - 4ac \geq 0$

A)

$$\Rightarrow 16k^2 - 16(k+2) \geq 0$$

$$\Rightarrow 16k^2 - 16k - 32 \geq 0.$$

Pdf of k

$$f(k) = \begin{cases} \frac{1}{5}, & k \in (0,5) \\ 0 \end{cases}$$

$$\Rightarrow k^2 - k - 2 \geq 0$$

$$\Rightarrow (k-2)(k+1) \geq 0$$

$$k \geq 2, \ k \geq -1$$

$$P(\text{Roots are real}) = P(k \geq 2)$$

$$= \int_2^5 f(k)\,dk$$

$$= \int_2^5 \frac{1}{5}\,dk$$

HW

Q11) suppose 'x' is uniformly distributed over the interval $(-a, a)$ $\forall$ $a>0$. whenever possible find 'a' the following conditions are satisfied.

(a) $P(x>1) = 1/3$.

(b) $P(x<\frac{1}{2}) = 0.7$

(c) $P(x<1) = 1/2$

(d) $P(|x|<1) = P(|x|>1)$.

} find 'a' (step① pdf, step② find a).

Ans)

(a)

$P(x>1) = 1/3$

$\frac{1}{b-a} \quad \frac{1}{a+a} = \frac{1}{2a}$

CRV,

so $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_{1}^{a} \frac{1}{2a} dx$

$\Rightarrow \frac{a-1}{2a} = \frac{1}{3}$

$\Rightarrow 3a-3 = 2a$

$\Rightarrow \boxed{a=3}$

b) $\Rightarrow \int_{-a}^{1/2} \frac{1}{2a} dx \Rightarrow \frac{\frac{1}{2}-a}{2a} = 0.7$

$\frac{1}{2}-a = 1.4a$

$\frac{1-2a}{2} = 1.4a$

$1-2a = 2.8a$

$a = \frac{1}{4.8}$

$\boxed{a = 0.208}$

c) $\int_{-a}^{1} \frac{1}{2a} dx$

$\frac{1-a}{2a} = \frac{1}{3}$

$3a-3a = 2a$

$3a+2a = 3$

$5a = 3$

$\boxed{a = \frac{3}{5}}$

d) $P(|x|<1) = P(|x|\geq 1)$

$\int_{1}^{a} \frac{1}{2a} dx = \int_{a}^{1} \frac{1}{2a} dx$

$\frac{1-a}{2a} = \frac{a-1}{2a}$

$2a = 2$

$\boxed{a=1}$

Expected value of random variable: (Mean value)

Let 'x' be a discreate random variable with pmf $P(x_i)$ then the expected value of 'x' is defined as $E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$.

If 'x' is countinuous random variable (CRV),

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Properties:

CRV & DRV
→ if $x = c$
$E(c) = c$.
→ $E(ax+b) = aE(x)+b$.

Variance $V(X)$:

$$V(X) = E[X - E(X)]^2.$$

$$= E[X^2 - 2xE(X) + [E(x)]^2].$$

$$= E(X^2) - 2[E(x)]^2 + [E(x)]^2$$

$$V(x) = E(x^2) - [E(x)]^2.$$

[Cvalue]=0 variation of constant is 0]

Note: $\sqrt{V(X)}$ is called standard deviation.

Properties:

→ if $x = c$
$V(c) = 0$
→ $V(ax+b) = a^2 V(X)$.

29/3/22

Q) Find the mean and variance of random variable x with probability distribution $f(0)=\frac{16}{31}$, $f(1)=\frac{8}{31}$, $f(2)=\frac{4}{31}$, $f(3)=\frac{2}{31}$, $f(4)=\frac{1}{31}$.

Ans)

$$\text{mean}\, E(x) = \sum_{x=0}^{4} x f(x)$$

$$= 0\times\frac{16}{31} + \frac{8}{31} + \frac{4}{31} + \frac{2}{31} + \frac{1}{31}$$

$$E(x) = \frac{8+8+6+4}{31} = \frac{26}{31}$$

$$\text{variance } V(x) = E(x^2) - [E(x)]^2$$

$$= \sum_{x=0}^{4} x^2 f(x) - \left(\frac{26}{31}\right)^2$$

$$= \frac{8+16+18+16}{31}$$

$$= \frac{58}{31} - \frac{676}{961}$$

$$= \frac{58\times 31 - 676}{961}$$

$$= \frac{1122}{961}$$

$$= 1.167$$

26×26 = 156, 52, 676

31×31 = 31, 93, 961

58×31 = 58, 174, 1798, 676, 1122

Q) suppose 'x' is a random variable $E(x)=10$, $V(x)=25$, for what positive values of a & b, $y=ax-b$, has expectation '0' and variance '1'.

Ans)

$E(x)=0$, $V(x)=25$

$E(y)=0$, $V(y)=1$

$$E(y) = E(ax-b)$$

$$= a\cdot E(x) - b = 0.$$

$$= 10a - b = 0. \rightarrow \text{eq}①$$

$V(y) = V(ax-b)$

$= a^2V(X) = 1$

$\Rightarrow 25a^2 = 1$

$\Rightarrow a^2 = \frac{1}{25} \Rightarrow a = \pm\frac{1}{5}$

Take $a = \frac{1}{5}$ ('+ve value)

~~take~~ $a = 1$

$10 \times \frac{1}{5} - b = 0$

$\left(a = \frac{1}{5}, b = 2\right)$

Q) Suppose that an electronic device has a life length 'x' which is considered as a countinuous random variable with PDF, $f(x) = e^{-x}, x > 0$. Suppose that the cost of manufacturing one such item is 2\$. The manufacturer sells item for 5\$ but gurantee is a total refund. If $x \leq 0.9$. What is manufacturer expected profit per item.

Ans) $x \leq 0.9$

$P = -2$ if $x \leq 0.9$

3 if $x > 0.9$

$E(P) = -2P(P=-2) + 3P(P=3)$

$= -2P(x \leq 0.9) + 3P(x > 0.9)$

$= -2 \cdot \int_0^{0.9} e^{-x}dx + 3 \cdot \int_{0.9}^{\infty} e^{-x}dx$

$= -2\left[-e^{-x}\right]_0^{0.9} + 3 \cdot \left[-e^{-x}\right]_{0.9}^{\infty}$

$= 2 \cdot e^{-0.9} - 1 + 3e^{-0.9}$

$= 0.8131 + 1.2197$

$= 2.032 - 1$

$= 1.032$