LECTURE - 1
MEAN VALUE THEOREMS

Syllabus

Max marks

1. Partial derivatives, Limits, Mean Value Theorems — 10

2. Taylor series, Maclaurin's series (in two variables), Max & Min. of a function of two variables, solid geometry } 10

3. Multiple Integrals — 12

4. Laplace Transform — 10

5. Infinite series — 8

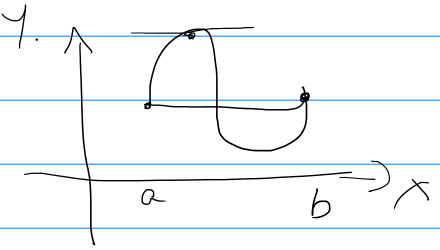
50 Marks

Reference books → 1) B.S. Grewal
2) Differential calculus by Shanti Narayan

Some fundamental Theorems

Rolle's theorem

If the function $f(x)$ is continuous in $[a, b]$ differentiable in (a, b) and if $f(a) = f(b)$, then there exists atleast one point c in (a, b) such that $f'(c) = 0$



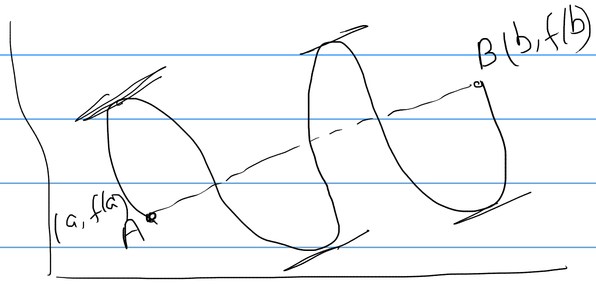
Geometrical meaning

tangents is parallel to x-axis

Langrange's Mean Value Theorem

1) $f(x)$ – continuous, diff., $\exists$ $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



Cauchy's Mean Value Theorem (CMVT)

Theorem

If two functions $f(x)$ and $g(x)$ satisfy the given three conditions

1) Continuous in $[a,b]$
2) Differentiable in (a,b)
3) $g'(x) \neq 0$ in (a,b), then there exists

atleast one point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

[CMVT is a generalisation of LMVT (that means put $g(x)=x$)]

Proof

We define a new function $\phi(x)$ involving $f(x)$ and $g(x)$ s.t $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Let $\phi(x) = f(x) + k\,g(x)$, where k is a constant to be determined

such that $\phi(a) = \phi(b)$

(i.e) $f(a) + k\,g(a) = f(b) + k\,g(b)$

$$k = -\left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] \quad \text{--- (1)}$$

Since $\phi(x)$ satisfies Rolle's theorem, there exists atleast one point $c \in (a,b)$ s.t $\phi'(c) = 0$

ie $\quad f'(c) + kg'(c) = 0$

$$k = -\frac{f'(c)}{g'(c)} \quad \text{--- (2)}$$

From (1) and (2),

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Problems

Verify CMVT and find c in $[a,b]$, a and b being positive

1. $f(x) = e^{x}$, $g(x) = e^{-x}$

Solution

By CMVT, $\quad \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$f(x) = e^{x}$

$f'(x) = e^{x}$

$g(x) = e^{-x}$

$g'(x) = -e^{-x}$

$$\frac{e^{b}-e^{a}}{e^{-b}-e^{-a}} = \frac{e^{c}}{-e^{-c}}$$

$$\frac{e^{b}-e^{a}}{\frac{1}{e^{b}}-\frac{1}{e^{a}}} = -e^{c+c}$$

$$\frac{e^{b}-e^{a}}{\frac{e^{a}-e^{b}}{e^{a}e^{b}}} = -e^{2c}$$

$$\frac{\cancel{(e^{b}-e^{a})}\,e^{a+b}}{-\cancel{(e^{b}-e^{a})}} = -e^{2c}$$

$$e^{a+b} = e^{2c}$$

$$a+b = 2c$$

$$c = \frac{1}{2}(a+b)$$

Thus c lies in (a,b) which verifies CMVT

2. $f(x) = \log_e x \qquad g(x) = \frac{1}{x}$ in $[1,e]$

Solution By CMVT, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ $\left[\begin{array}{l} a=1 \\ b=e \end{array}\right.$

$$\frac{\log b - \log a}{\frac{1}{b} - \frac{1}{a}} = \frac{1/x}{-1/x^2}$$

$$\frac{\log(b/a)}{\frac{a-b}{ab}} = -\frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{1} = -c$$

$$\frac{\log(b/a) \cdot ab}{a-b} = -c$$

$$\frac{\log\left(\frac{e}{1}\right) \cdot (e(1))}{1-e} = -c$$

$$\frac{\log_e (e)}{1-e} = -c$$

$$\frac{e}{-(e-1)} = -c$$

$$c = \frac{e}{e-1} \in [1,e]$$

Thus c lies in $[1,e]$ which verifies CMVT

3. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{1}{\sqrt{x}} \quad \left[\frac{1}{4}, 1\right]$

4. $f(x) = x^3 \quad g(x) = x^2$ in $[1, 2]$

5. $f(x) = x^3, \quad g(x) = 2 - x$ in $[0, 9]$

Taylor's theorem for a function of one variable (Generalised Mean Value Theorem)

Suppose a function $f(x)$ satisfies the following conditions.

1) $f(x)$ and its first $(n-1)$ derivatives are continuous in $[a, a+h]$
2) $f^{(n)}(x)$ is differentiable in $(a, a+h)$

then there exists atleast one point θ in $(0,1)$ such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h) \quad \text{(A)}$$

Where the last term on the RHS is the remainder $R_n(x)$ which tends to 0 as $n \to \infty$

Take $a + h = x$

then $h = x - a$

Substituting in (A)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

which is called Taylor's series.

Take $a = 0$, $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \cdots$, Maclaurin's series expansion

Problems

1. Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e 1.1$ correct to 4 decimal places.

Solution $\rightarrow$ $f(x) = \log_e x$ $\qquad f(1) = \log_e 1 = 0$ $\qquad \boxed{\begin{aligned} x-a &= x-1 \\ a &= 1 \end{aligned}}$

$f'(x) = \dfrac{1}{x}$ $\qquad f'(1) = \dfrac{1}{1} = 1$

$f''(x) = \dfrac{-1}{x^2}$ $\qquad f''(1) = \dfrac{-1}{(1)^2} = -1$

$f'''(x) = \dfrac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{IV}(x) = \dfrac{-6}{x^4}$ $\qquad f^{IV}(1) = -6$

Taylor's series expansion is

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots$$

$$= 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \dots$$

$$\log_e x = \underline{x-1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

put $x = 1.1$, $x - 1 = 1.1 - 1 = 0.1$

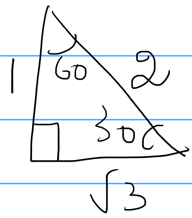
$$\log_e 1.1 = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \dots$$

$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \dots$$

$= 0.0953$

2. Obtain the power series expansion of $f(x) = \cos x$ about $x = \pi/3$. Hence find the approximate value of $\cos 61^\circ$.

Solution: We are required to expand $f(x) = \cos x$ in powers of $x - \pi/3$.



$f(x) = \cos x \qquad f(\pi/3) = \frac{1}{2}$

$f'(x) = -\sin x \qquad f'(\pi/3) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x \qquad f''(\pi/3) = -\frac{1}{2}$

$f'''(x) = \sin x \qquad f'''(\pi/3) = \sqrt{3}/2$

$$\cos x = f(\pi/3) + (x - \pi/3) f'(\pi/3) + \cdots$$

$$= \frac{1}{2} + (x - \pi/3)\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2!}(x - \pi/3)^2\left(-\frac{1}{2}\right) + \cdots$$

$\cos 61^\circ$

To find $\cos 61^\circ$, put $x = 61^\circ$

$$x - \pi/3 = 61^\circ - \pi/3 = 1^\circ = \pi/180^c$$

$$\cos 61^\circ = \frac{1}{2} + \frac{\pi}{180}\left(-\sqrt{3}/2\right) + \frac{1}{2!}\left(\frac{\pi}{180}\right)^2\left(-\frac{1}{2}\right) + \cdots$$

$$= 0.4848$$

3. Expand $\tan x$ in powers of $(x - \pi/4)$ upto four terms

Solution $y = \tan x \qquad y(\pi/4) = 1$

$y_1 = \sec^2 x \qquad y_1(\pi/4) = 2$

$= 1 + y^2$

$y_2 = 2y\,y_1$ $\qquad$ $y_2(\pi/4) = 2\,y(\pi/4)\,y_1(\pi/4)$

$= 2(1)(2) = 4$

$y_3 = 2[y\,y_2 + y_1 y_1]$, $\quad y_3(\pi/4) = 2[1(4) + 2(2)]$

$= 16$

$$\begin{cases} y_1 = \sec^2 x \\ y_2 = 2\sec x \sec x \tan x \\ \quad = 2\sec^2 x \tan x \\ y_3 = 2[2\sec x \sec x \tan x \tan x + \sec^2 x \sec^2 x] \end{cases}$$

$\tan x = f(\pi/4) + (x - \pi/4) f'(\pi/4) + \text{-----}$

$= 1 + (x - \pi/4)2 + \frac{(x-\pi/4)^2}{2!}(4) + \frac{(x-\pi/4)^3}{3!}(16) + \text{---}$

4) Expand $e^{\tan^{-1} x}$ in powers of x upto x^4

5) Expand $\log(1+\tan x)$ ------ upto 5 terms

6) " $\log(1+e^x)$ ------ "