Linear Differential Equation:

A differential equation linear in y is of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x alone.

$$IF = e^{\int P\,dx}$$

Solution is

$$y(IF) = \int Q(IF)\,dx + C$$

Equation linear in x is of the form

$$\frac{dx}{dy} + Px = Q$$

where P and Q are functions of y alone.

$$IF = e^{\int P\,dy}$$

solution is

$$x(IF) = \int Q(IF)\,dy + C.$$

Solve the following differential equations:

① $\frac{dy}{dx} + y\sec x = \tan x$

equation is linear in y.

$P = \sec x, \qquad Q = \tan x$

$$IF = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

solution is

$$y(IF) = \int Q(IF)\,dx + C$$

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$$

$$y(\sec x + \tan x) = \int \tan x\sec x\,dx + \int \tan^2 x\,dx + C$$

$\tan^2 x + 1 = \sec^2 x$

$$y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)\,dx + C$$

$$y(\sec x + \tan x) = \sec x + \tan x - x + C.$$

(2) $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

$\div 1+x^2$ $\quad \frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{e^{\tan^{-1}x}}{1+x^2}$

$P = \frac{1}{1+x^2}, \quad Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$IF = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

solution is

$y(IF) = \int Q(IF)\,dx + c$

$e^{\tan^{-1}x}\,e^{\tan^{-1}x} = e^{\tan^{-1}x+\tan^{-1}x}$
$= e^{2\tan^{-1}x}$

$y(e^{\tan^{-1}x}) = \int \frac{e^{\tan^{-1}x}}{1+x^2}e^{\tan^{-1}x}\,dx + c$

$= \int e^{2\tan^{-1}x}\cdot\frac{1}{1+x^2}\,dx + c$

$2\tan^{-1}x = t$

$\frac{1}{1+x^2}dx = \frac{dt}{2}$

$= \int e^t\frac{dt}{2} + c$

you can even substitute $e^{\tan^{-1}x} = t$

$= \frac{e^t}{2} + c$

$y\,e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c$

$$\left[\,y = \frac{e^{\tan^{-1}x}}{2} + c\,e^{-\tan^{-1}x}\right]$$

(3) $(1-x^2)\frac{dy}{dx} - xy = 1$

$\div (1-x^2)$ $\quad \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$

$\frac{dy}{dx} + \frac{x}{x^2-1}y = \frac{1}{1-x^2}$

$x^2-1 = t$

$2x\,dx = dt$

$IF = e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\int\frac{2x\,dx}{x^2-1}} = e^{\frac{1}{2}\int\frac{dt}{t}} = e^{\frac{1}{2}\log t} = e^{\log t^{1/2}} = \sqrt{t}$

$= \sqrt{x^2-1}$

solution is

$y(IF) = \int Q(IF)\,dx + c$

$y\sqrt{x^2-1} = \int \frac{1}{1-x^2}\sqrt{x^2-1}\,dx + c$

$= -\int \frac{1}{x^2-1}\sqrt{x^2-1}\,dx = -\int\frac{1}{\sqrt{x^2-1}} + c$

solution is $y\sqrt{x^2-1} = -\log(x+\sqrt{x^2-1}) + c.$

Note:

$$IF = e^{-\int \frac{x}{1-x^2}dx} = e^{+\frac{1}{2}\log(1-x^2)} = \sqrt{1-x^2}$$

$$y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}\,dx + c$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + c$$

$$y\sqrt{1-x^2} = \sin^{-1}x + c$$

(4) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1}y \qquad \text{Linear in } x.$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$x\left(e^{\tan^{-1}y}\right) = \int e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}\,dy + C$$

$\tan^{-1}y = t$

$\frac{1}{1+y^2}\,dy = dt$

$$= \int e^t\, t\, dt + c$$

$$= te^t - e^t + c$$

$$xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$$

Equations Reducible to Linear Equations:

① Solve $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

$$\frac{dy}{dx} + x \cdot 2\sin y \cos y = x^3 \cos^2 y$$

$\dfrac{1}{\cos^2 y} = \sec^2 y$

$\dfrac{2\sin y \cos y}{\cos^2 y} = 2\dfrac{\sin y}{\cos y} = 2\tan y$

$\div \cos^2 y$

$$\sec^2 y \frac{dy}{dx} + x\, 2\tan y = x^3$$

$\tan y = z$

$\sec^2 y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$$\frac{dz}{dx} + 2xz = x^3$$

linear in z.

$$IF = e^{\int 2x\,dx} = e^{x^2}$$

solution is

$$z(IF) = \int Q(IF)\,dx + \frac{C}{2}$$

$$z(e^{x^2}) = \int x^3 e^{x^2}\,dx + \frac{C}{2}$$

$x^2 = t$

$2x\,dx = dt$

$$= \frac{1}{2}\int x^2 e^{x^2}(2x\,dx) + \frac{C}{2}$$

$$= \frac{1}{2}\int t e^t\,dt + \frac{C}{2}$$

$$2z e^{x^2} = te^t - e^t + C$$

$$2z e^{x^2} = x^2 e^{x^2} - e^{x^2} + C$$

$$2z = x^2 - 1 + C e^{-x^2}$$

$$2\tan y = x^2 - 1 + C e^{-x^2}$$

② $r\sin\theta\,d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)\,dr = 0.$

$\cos\theta = t$

$\sin\theta\,d\theta = dt$

$$-r\,dt + (r^3 - 2r^2 t + t)\,dr = 0$$

$$-r\frac{dt}{dr} + r^3 - 2r^2 t + t = 0$$

$\div -r$

$$\frac{dt}{dr} - r^2 + 2rt - \frac{1}{r}t = 0.$$

$$\frac{dt}{dr} + \left(2r - \frac{1}{r}\right)t = r^2$$

$$IF = e^{\int\left(2r - \frac{1}{r}\right)dr} = e^{r^2 - \log r} = e^{r^2} e^{-\log r} = e^{r^2} e^{\log r^{-1}} = \frac{e^{r^2}}{r}$$

$$t\left(\frac{e^{r^2}}{r}\right) = \int r^2 \frac{e^{r^2}}{r}\,dr + C$$

$$\frac{te^{r^2}}{r} = \frac{1}{2}\int e^{r^2}\, 2r\,dr + C$$

$$\frac{te^{r^2}}{r} = \frac{1}{2}e^{r^2} + C.$$

$$(\cos\theta)e^{r^2} = \frac{re^{r^2}}{2} + Cr$$

Bernoulli's DE:

A differential equation of the form

$$\frac{dy}{dx} + Py = Q y^n \quad (1)$$

where P and Q are functions of x alone, is called a Bernoulli's differential equation.

To solve this equation, we divide (1) by y^n.

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$$

Put $\frac{1}{y^{n-1}} = t$

$$-(n-1)\frac{1}{y^n}\frac{dy}{dx} = dt$$

With this substitution, the equation reduces to linear equation and hence can be solved.

Solve the following differential equations:

(1) $x\frac{dy}{dx} + y = x^3 y^6$.

$\div x$, $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^6$. Bernoulli's D.E.

$\div y^6$ $\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$

$\frac{1}{y^5} = t$

$-5\frac{1}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$

$$-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$\times(-5)$

$$\frac{dt}{dx} - \frac{5}{x}t = -5x^2$$

Linear in t.

$$IF = e^{-\int\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

$$t\left(\frac{1}{x^5}\right) = \int -5x^2\frac{1}{x^5}dx + C$$

$$= -5\int\frac{1}{x^3}dx + C$$

$$\frac{t}{x^5} = -5\left(\frac{-1}{2x^2}\right) + C$$

$$t = \frac{5}{2}x^3 + Cx^5$$

$\frac{1}{y^5} = \frac{5}{2}x^3 + Cx^5$, is the required solution.

(2) $xy(1+xy^2)\frac{dy}{dx} = 1$

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - yx = y^3 x^2, \quad \text{Bernoulli's DE in } x.$$

$\div x^2$

$$\frac{1}{x^2}\frac{dx}{dy} - y\cdot\frac{1}{x} = y^3 \qquad \frac{1}{x} = t$$

$$-\frac{dt}{dy} - y\cdot t = y^3 \qquad -\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$$

$\times^{ly}(-1)$

$$\frac{dt}{dy} + y\cdot t = -y^3 \quad \text{linear in } t$$

$$IF = e^{\int y\,dy} = e^{y^2/2}$$

$$t\,e^{y^2/2} = \int -y^3 e^{y^2/2}\,dy + c \qquad \frac{y^2}{2} = z$$

$$= -\int y^2 e^{y^2/2}\, y\,dy + c \qquad y\,dy = dz$$

$$= -\int 2z\, e^z\,dz + c$$

$$= -2\left[z e^z - e^z\right] + c$$

$$t\,e^{y^2/2} = -2\left[\frac{y^2}{2}e^{y^2/2} - e^{y^2/2}\right] + c$$

$$t\,e^{y^2/2} = -y^2 e^{y^2/2} + 2e^{y^2/2} + c.$$

$$\frac{1}{x}\, e^{y^2/2} = -y^2 e^{y^2/2} + 2e^{y^2/2} + c.$$

(3) $3y^2\frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$

$$y^3 = t$$

$$3y^2\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 2xt = 4xe^{-x^2}$$

linear in t.

$$IF = e^{\int 2x\,dx} = e^{x^2}$$

$$t(e^{x^2}) = \int 4x e^{-x^2} e^{x^2}\,dx + c$$

$$y^3 e^{x^2} = 2x^2 + c.$$

(4) $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$

$\div z$ $\quad \frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2$

$$\frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x}t^2$$

$$\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$$

$$z\frac{dt}{dx} + \frac{z}{x}t = \frac{z}{x}t^2$$

$$\log z = t$$

$$\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$$

Method I Bernoulli in t. $\div t^2$ and proceed.

Method II Equation is variable separable.

$$\frac{dt}{dx} = \frac{t^2-t}{x}$$

$$\frac{dt}{t^2-t} = \frac{dx}{x}$$

$$\frac{1}{t^2-t} = \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$= \frac{-1}{t} + \frac{1}{t-1}$$

$$\left(\frac{1}{t-1} - \frac{1}{t}\right)dt = \frac{1}{x}dx$$

$$\log(t-1) - \log t = \log x + \log C$$

$$\log\left(\frac{t-1}{tx}\right) = c$$

$$(t-1) = cxt$$

$$(\log z - 1) = cx\log z$$