Exact Differential Equations:

Partial Differentiation:

Let $z = f(x, y)$ be a function of 2 independent variables x & y.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

if it exists, is called partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

if it exists, is called partial derivative of z with respect to y and is denoted by $\frac{\partial z}{\partial y}$.

Examples: ① $z = x^3 + 3x^2y^2 + 5y^4$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(3x^2y^2) + \frac{\partial}{\partial x}(5y^4)$$

$$= 3x^2 + 3y^2 \frac{\partial}{\partial x}(x^2) + 0$$

$$= 3x^2 + 3y^2 \times 2x \quad = \quad 3x^2 + 6xy^2$$

$$\frac{\partial z}{\partial y} = 0 + 3x^2 \cdot 2y + 5 \cdot 4y^3$$

$$= \quad 6x^2y + 20y^3.$$

② $z = x^2 e^{-3y} + 4\sqrt{y}\,\sin 2x - xy^3$

$$\frac{\partial z}{\partial x} = e^{-3y} \frac{\partial}{\partial x}(x^2) + 4\sqrt{y}\,\frac{\partial}{\partial x}(\sin 2x) - y^3 \frac{\partial}{\partial x}(x)$$

$$= e^{-3y}\, 2x + 4\sqrt{y} \cdot 2\cos 2x - y^3 \times 1$$

$$= 2x e^{-3y} + 8\sqrt{y}\cos 2x - y^3$$

$$\frac{\partial z}{\partial y} = x^2 \frac{\partial}{\partial y}(e^{-3y}) + 4\sin 2x \frac{\partial}{\partial y}(\sqrt{y}) - x \frac{\partial}{\partial y}(y^3)$$

$$= x^2(-3e^{-3y}) + 4\sin 2x \cdot \frac{1}{2\sqrt{y}} - x \cdot 3y^2$$

$$= -3x^2 e^{-3y} + \frac{2\sin 2x}{\sqrt{y}} - 3xy^2.$$

③ $z = \frac{xy}{x^2+y^2}$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(xy) - xy\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot y \cdot 1 - xy\cdot(2x+0)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}, \qquad \text{IIIly} \quad \frac{\partial z}{\partial y} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

A differential equation of the form

$$M\,dx + N\,dy = 0$$

is said to be exact if there exist a function $F(x,y)$ such that the total derivative of F,

$$dF = M\,dx + N\,dy$$

In this case, the solution is given by $F(x,y) = c$, c arbitrary constant.

Note: If F is a function of 2 independent variables, then its total derivative is defined as

$$\boxed{dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy}$$

Example: i) $$d(\underset{F}{\underset{\uparrow}{xy}}) = \frac{\partial}{\partial x}(xy)\,dx + \frac{\partial}{\partial y}(xy)\cdot dy$$

$$= y\,dx + x\,dy.$$

ii) $$d\left(\frac{y}{x}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x}\right)dx + \frac{\partial}{\partial y}\left(\frac{y}{x}\right)dy$$

$$= y\left(\frac{-1}{x^2}\right)dx + \frac{1}{x}\cdot 1\,dy$$

$$= \frac{-y\,dx + x\,dy}{x^2}$$

Theorem: If M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous functions of x and y, then a necessary and sufficient condition for

$$M\,dx + N\,dy = 0$$

to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Note: If $M\,dx + N\,dy = 0$ is exact, then its solution is given by

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C.$$

OR

$$\int_{x\text{ constant}} N\,dy + \int (\text{terms of } M \text{ not containing } y)\,dx = C.$$

Solve the following differential equations:

① $(2x+e^y)\,dx + x\,e^y\,dy = 0.$

$M = 2x+e^y, \quad N = xe^y$

$\frac{\partial M}{\partial y} = 0+e^y, \quad \frac{\partial N}{\partial x} = e^y.1$

$= e^y \qquad\qquad = e^y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The differential equation is exact. Solution is

$\int\limits_{y\text{ const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

$\int (2x+e^y)\,dx + 0 = c$

$2.\frac{x^2}{2} + e^y.x = c$ ie $x^2+xe^y = c$

Grouping the terms;

$(2x+e^y)\,dx + xe^y\,dy = 0$

$2x\,dx + e^y\,dx + xe^y\,dy = 0$

$d(x^2) + d(xe^y) = 0$

$d(x^2+xe^y) = 0$

Integrating

$\underline{x^2+xe^y = c}$

$d(xe^y) = x\,d(e^y) + e^y\,d(x)$

$= x.e^y\,dy + e^y\,dx.$

② $(x^2-4xy-2y^2)\,dx + (y^2-4xy-2x^2)\,dy = 0.$

$M = x^2-4xy-2y^2, \quad N = y^2-4xy-2x^2$

$\frac{\partial M}{\partial y} = 0-4x.1-4y, \quad \frac{\partial N}{\partial x} = 0-4y.1-4x$

$= -4x-4y, \qquad = -4y-4x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad \therefore$ Equation is exact.

Solution is given by

$\int\limits_{y\text{ const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

$\int\limits_{y} (x^2-4xy-2y^2)\,dx + \int y^2\,dy = \frac{c}{3}$

$\frac{x^3}{3} - 4y.\frac{x^2}{2} - 2y^2.x + \frac{y^3}{3} = \frac{c}{3}$

$x^3 - 6x^2y - 6y^2x + y^3 = c.$

Grouping terms.

$x^2dx - \underline{4xy\,dx} - \underline{2y^2dx} + y^2dy - \underline{4xy\,dy} - \underline{2x^2dy} = 0$

$d\left(\frac{x^3}{3}\right) - (4xy\,dx + 2x^2dy) - (2y^2dx + 4xy\,dy) + d\left(\frac{y^3}{3}\right) = 0$

$d\left(\frac{x^3}{3}\right) - 2(y\cdot 2x\,dx + x^2\cdot dy) - 2(y^2dx + x\cdot 2y\,dy) + d\left(\frac{y^3}{3}\right) = 0$

$d\left(\frac{x^3}{3}\right) - 2\,d(x^2y) - 2\,d(xy^2) + d\left(\frac{y^3}{3}\right) = 0$

$d\left(\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3}\right) = 0$

③ $3re^{3\theta}\,d\theta + e^{3\theta}\,dr = 0$

$M = 3re^{3\theta}, \quad N = e^{3\theta}$

$\frac{\partial M}{\partial r} = 3e^{3\theta}, \quad \frac{\partial N}{\partial \theta} = 3e^{3\theta}$

$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta} \quad \therefore$ Equation is exact.

Solution:

$\int\limits_{r\text{ const}} 3re^{3\theta}\,d\theta + \int 0 = c \qquad \frac{3r.e^{3\theta}}{3} = c$ ie $re^{3\theta} = c.$

Grouping:

$3re^{3\theta}\,d\theta + e^{3\theta}\,dr = 0$

$r(3e^{3\theta}\,d\theta) + e^{3\theta}(dr) = 0$

$d(re^{3\theta}) = 0$

$\Rightarrow re^{3\theta} = c.$

Equations Reducible to Exact DE:

An equation of the form

$$M\,dx + N\,dy = 0$$

which is not exact, can be made exact by multiplying by a suitable function of x & y. Such a multiplier is called an integrating factor (IF) of the differential equation.

I. IF found by inspection:

(1) $x\,dy + y\,dx = d(xy)$ (2) $\dfrac{x\,dy - y\,dx}{x^2} = d(y/x)$

(3) $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$ (4) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\left(y/x\right)\right)$

Solve the following:

① $y\left[2xy + e^x\right]dx = e^x\,dy$

$$2xy^2\,dx + ye^x\,dx = e^x\,dy$$

$$(2x\,dx)y^2 + y\,e^x\,dx - e^x\,dy = 0. \qquad \left(IF = \frac{1}{y^2}\right)$$

$\div y^2$ $$2x\,dx + \frac{y\,e^x\,dx - e^x\,dy}{y^2} = 0 \qquad d\left(\frac{e^x}{y}\right) = \frac{y\,d(e^x) - e^x\,d(y)}{y^2}$$

$$d(x^2) + d\left(\frac{e^x}{y}\right) = 0.$$

$$d\left(x^2 + \frac{e^x}{y}\right) = 0$$

$\therefore$ Solution: $x^2 + \dfrac{e^x}{y} = c.$

② $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

$$x^4\,dy + x^3y\,dx + \operatorname{cosec}(xy)\,dx = 0$$

$$x^3(x\,dy + y\,dx) + \operatorname{cosec}(xy)\,dx = 0$$

$$x^3\,d(xy) + \operatorname{cosec}(xy)\,dx = 0. \qquad \div \operatorname{cosec}(xy)\,x^3$$

$$\frac{d(xy)}{\operatorname{cosec}(xy)} + \frac{dx}{x^3} = 0$$

$$\sin(xy)\,d(xy) + \frac{dx}{x^3} = 0$$

$$-\cos(xy) - \frac{1}{2x^2} = -c$$

$$\cos(xy) + \frac{1}{2x^2} = c.$$

③ $3x^2y\,dx + (y^4 - x^3)\,dy = 0$

$$3x^2y\,dx + y^4\,dy - x^3\,dy = 0.$$

$$y(3x^2dx) - x^3\,dy + y^4\,dy = 0$$

$\div y^2$ $\left(IF = \dfrac{1}{y^2}\right)$ $$\frac{y(3x^2dx) - x^3dy}{y^2} + y^2\,dy = 0$$

$$d\left(\frac{x^3}{y}\right) + d\left(\frac{y^3}{3}\right) = 0,$$ Solution is $\dfrac{x^3}{y} + \dfrac{y^3}{3} = c$