Exercise

1. If C is a circle $|z-2|=1$, evaluate $\int_C (\bar{z})^2 dz$

2. Evaluate $\int_C \mathrm{Re}(z)\,dz$ where C is

(i) the path from $1+i$ to $(3+2i)$

(ii) along the straight line from $(1,1)$ to $(3,1)$ and then from $(3,1)$ to $(3,2)$.

3. Evaluate: $\int_C \bar{z}\,dz$ where (i) $C: |z-2|=3$

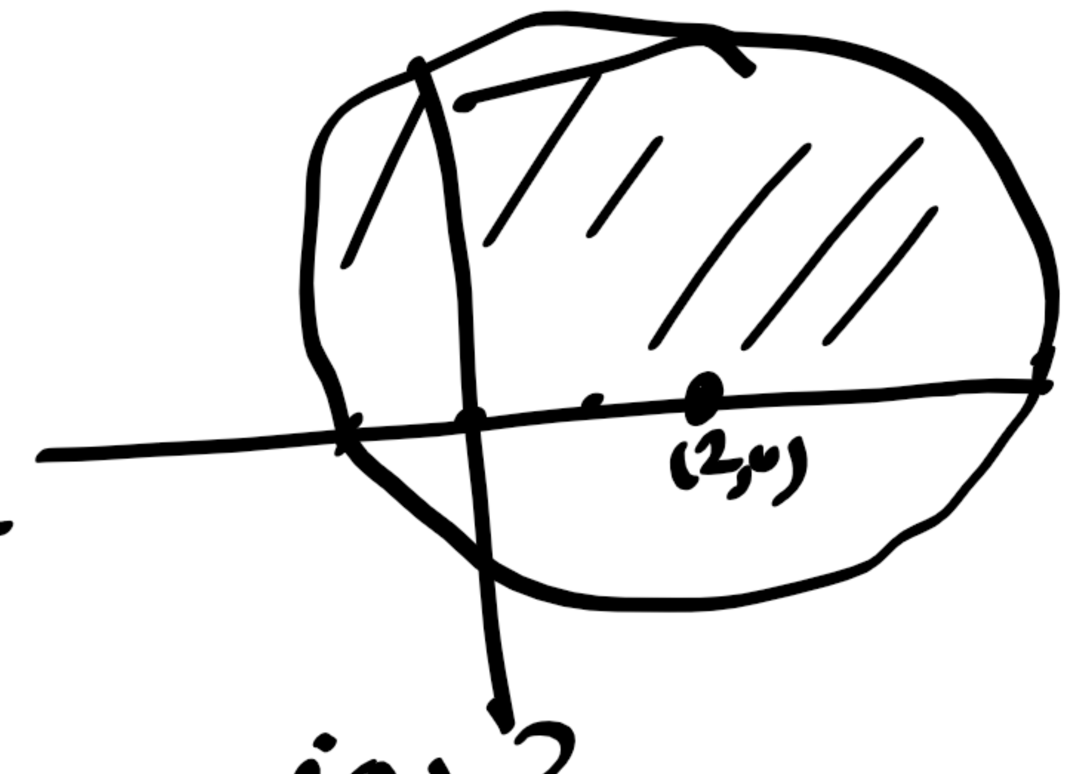
(ii) C is a square with vertices at $z=0, 2, 2i, 2+2i$

If $m=-1$

$$\int_C \frac{1}{(z-z_0)}\,dz = \int_0^{2\pi} \frac{1}{\cancel{\rho e^{i\theta}}}\, i\cancel{\rho e^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta$$

$$= \underline{\underline{2\pi i}}$$

4. Evaluate $\int_C (z-z^2)\,dz$ where C is the upper half of the circle $|z-2|=3$



$z-2 = 3e^{i\theta}$, $0\le\theta\le\pi$

$z = 2+3e^{i\theta}$, $\quad z^2=(2+3e^{i\theta})^2$

$dz = 3ie^{i\theta}\,d\theta$, $\quad = 4+12e^{i\theta}+9e^{2i\theta}$

$$\int_C (z-z^2)\,dz = \int_0^{\pi} \left(2+3e^{i\theta}-4-12e^{i\theta}-9e^{2i\theta}\right)3ie^{i\theta}\,d\theta$$

$$= 3i\int_0^{\pi}\left(-2e^{i\theta}-9e^{2i\theta}-9e^{3i\theta}\right)d\theta$$

$$= 3\cancel{i}\left[-\frac{2e^{i\theta}}{\cancel{i}} - \frac{9e^{2i\theta}}{2\cancel{i}} - \frac{9e^{3i\theta}}{3\cancel{i}}\right]_0^{\pi}$$

$$= 3\left[-2e^{i\pi} - \frac{9}{2}e^{2i\pi} - 3e^{3i\pi} + 2 + \frac{9}{2} + 3\right]$$

$$= 3\left[2-\frac{9}{2}+3+5+\frac{9}{2}\right] = \underline{\underline{30}}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \int_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz$$

$$= 4 + 2i - 4 + 6i = \underline{\underline{8i}}$$

Note:- If $f(z)$ is analytic on a simple closed curve C, then the line integral $\int_C f(z)\,dz$ is independent of the path chosen, it depends only on the end points.

3) Integrate $f(z) = (z - z_0)^m$ where m is an integer, z_0 is constant, around the circle with radius ρ and centre z_0.

$$C : |z - z_0| = \rho$$

$$z - z_0 = \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$z = z_0 + \rho e^{i\theta}$$

$$\therefore dz = \rho i e^{i\theta} d\theta$$

$$\int_C f(z)\,dz = \int_0^{2\pi} (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{im\theta} \rho i e^{i\theta} d\theta$$

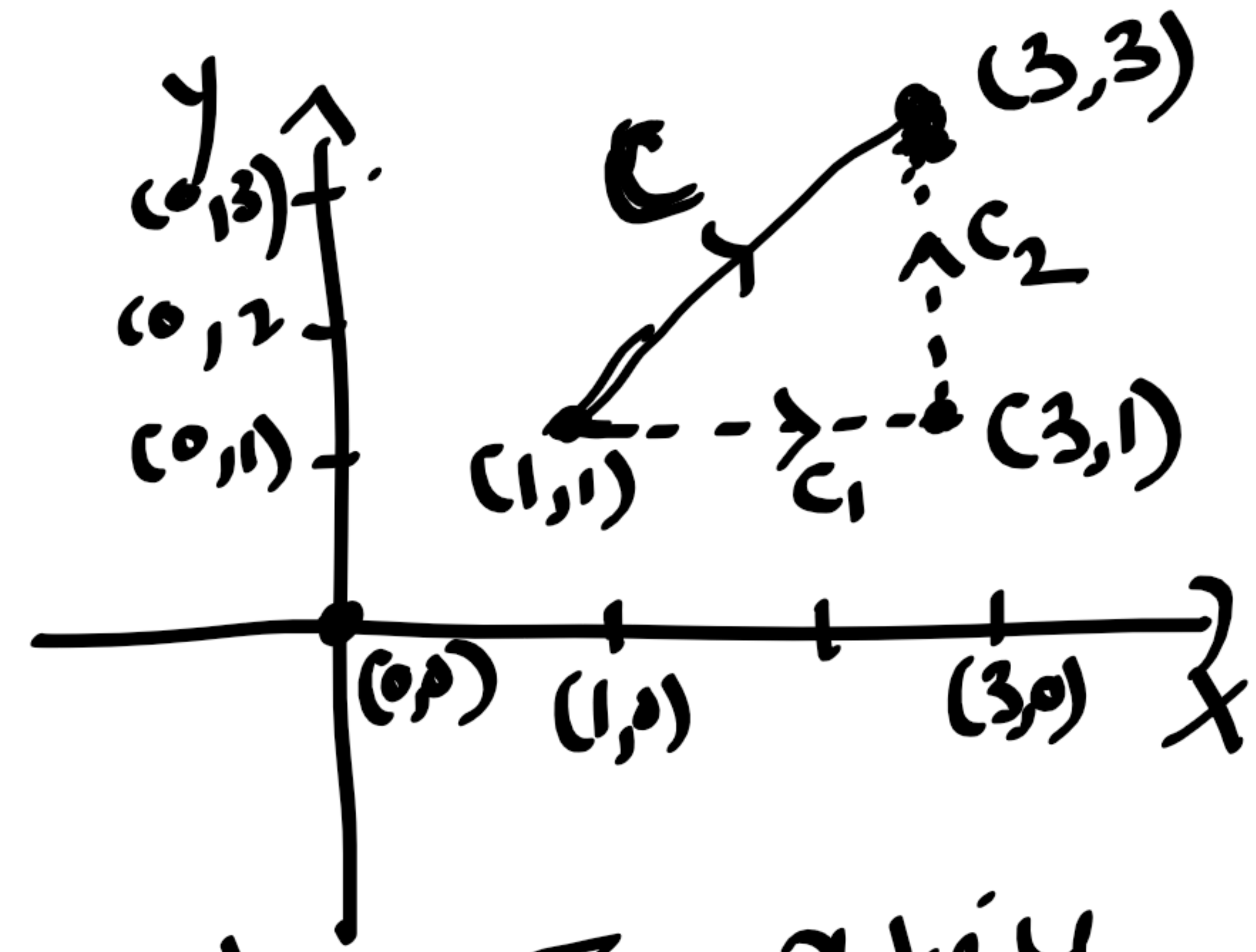
$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta = i\rho^{m+1} \left[\frac{e^{i(m+1)\theta}}{i(m+1)} \right]_0^{2\pi}$$

$$= \frac{\rho^{m+1}}{m+1} \left[e^{i(m+1)2\pi} - 1 \right], \quad m \ne -1$$

$$= 0, \quad m \ne -1$$

$$f(z) = z = x+iy.$$

(i) C is the straight line joining $(1,1)$ and $(3,3)$



$\therefore C: y = x$

$$\int_C z\,dz = \int_1^3 x(1+i)(1+i)\,dx$$

$$= (1+i)^2 \frac{x^2}{2}\Big|_1^3 = (1+i)^2 \times 4$$

$$= \underline{\underline{4(1+i)^2}} = 4(1+2i-1) = \underline{\underline{8i}}$$

$z = x+iy$
Since $y = x$
$z = (1+i)x$
$dz = (1+i)\,dx$

(ii) Along C_1: $(1,1) \longrightarrow (3,1)$

x varies from 1 to 3 and $y = 1$.

$\therefore z = x+i, \quad dz = dx.$

$$\int_{C_1} z\,dz = \int_1^3 (x+i)\,dx = \frac{x^2}{2} + ix \Big|_1^3$$

$$= \frac{9}{2} + 3i - \frac{1}{2} - i = \underline{4+2i}$$

Along C_2: $(3,1) \longrightarrow (3,3)$, $x = 3$, y varies from 1 to 3

$\therefore z = 3+iy, \quad dz = i\,dy$

$$\int_{C_2} z\,dz = \int_1^3 (3+iy)\,i\,dy = i\left(3y + i\frac{y^2}{2}\right)\Big|_1^3$$

$$= i\left(9 + \frac{9}{2}i - 3 - \frac{1}{2}i\right)$$

$$= i(6+4i) = -4+6i$$

$$\therefore \int_{C_1} (\bar{z})^2 dz = \int_0^2 x^2 dx = \frac{8}{3}.$$

Along C_2, $x = 2$, y varies from 0 to 1

$\therefore z = x + iy = 2 + iy$; $dz = i\,dy$

$(\bar{z})^2 = x^2 - y^2 - i2xy = 4 - y^2 - 4iy.$

$$\int_{C_2} (\bar{z})^2 dz = \int_0^1 (4 - y^2 - 4iy)\, i\,dy$$

$$= i\left[4y - \frac{y^3}{3} - 4i\frac{y^2}{2}\right]_0^1$$

$$= i\left[4 - \frac{1}{3} - 2i\right] = \frac{11}{3}i + 2$$

$$= 2 + \frac{11}{3}i$$

$$\therefore \int_C (\bar{z})^2 dz = \frac{8}{3} + 2 + \frac{11}{3}i = \frac{1}{3}(14 + 11i)$$

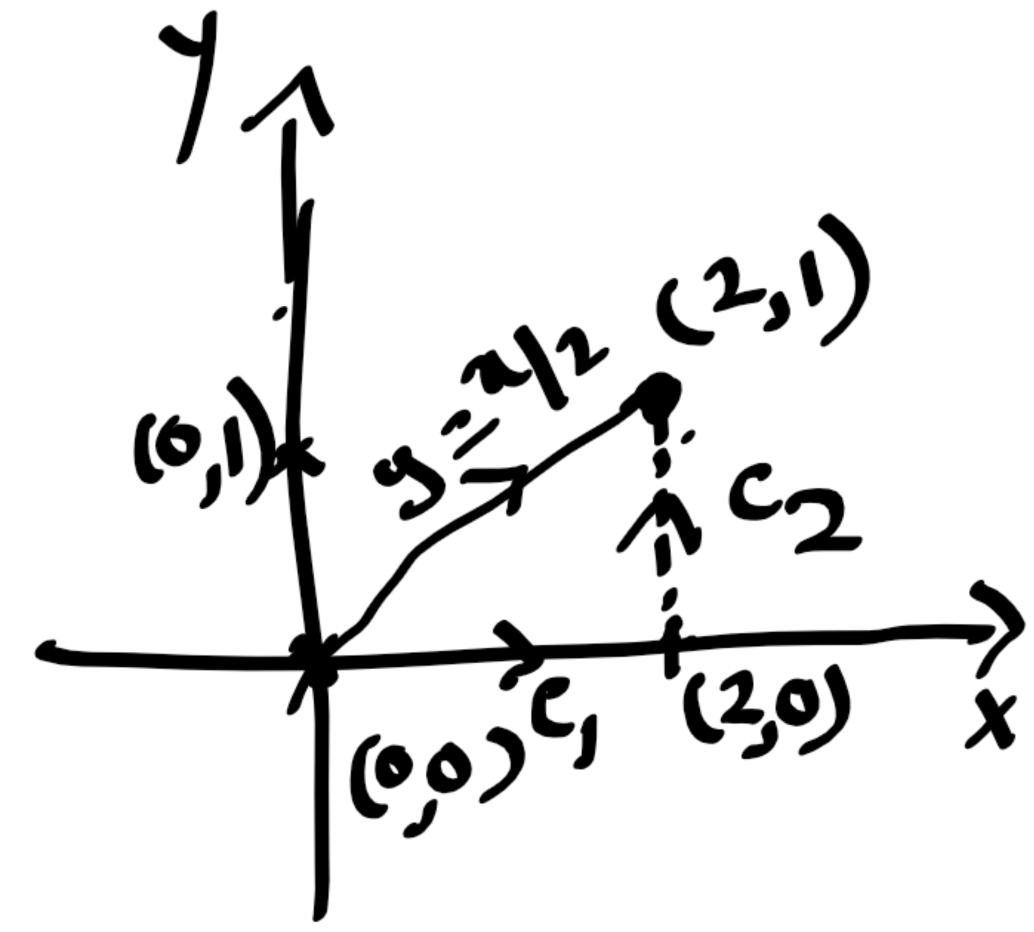
2. Evaluate $\int_C z\,dz$ where C is

(i) the straight line joining the points $1+i$ and $3+3i$.

(ii) the path along the horizontal line from $1+i$ to $3+i$ and then vertically to $3+3i$.

Examples

(1) Evaluate: $\int_0^{2+i} (\bar{z})^2 \, dz$ along



(i) the line $y = x/2$

(ii) the real axis to 2 and then vertically to $(2+i)$

(i) $z = x + iy \quad dz = dx + i\,dy$

$$(\bar{z})^2 = (x - iy)^2 = x^2 - y^2 - i2xy.$$

Along the line $y = x/2$

$$z = x + iy = x + ix/2 = x(1 + i/2)$$

$$\therefore dz = (1 + i/2)\,dx$$

$$(\bar{z})^2 = x^2 - x^2/4 - i2x \cdot \frac{x}{2} = \frac{3}{4}x^2 - ix^2 = \left(\frac{3}{4} - i\right)x^2$$

$$\therefore \int_{(0,0)}^{2+i} (\bar{z})^2 dz = \int_0^2 \left(\frac{3}{4} - i\right)x^2 \left(1 + \frac{i}{2}\right)dx$$

$$= \left(\frac{5}{4} - i\frac{5}{8}\right)\frac{x^3}{3}\Big|_0^2 = \left(\frac{5}{4} - i\frac{5}{8}\right)\frac{8}{3}$$

$$= \underline{\underline{\frac{5}{3}(2 - i)}}$$

(ii) Along C_1 (x-axis), $y = 0$, x varies from 0 to 2

$$\therefore z = x + iy = x, \quad dz = dx$$

$$(\bar{z})^2 = x^2$$

Line integral in the complex plane

The line integral of a function $f(z)$ taken along a curve C is $\int_C f(z)\,dz$

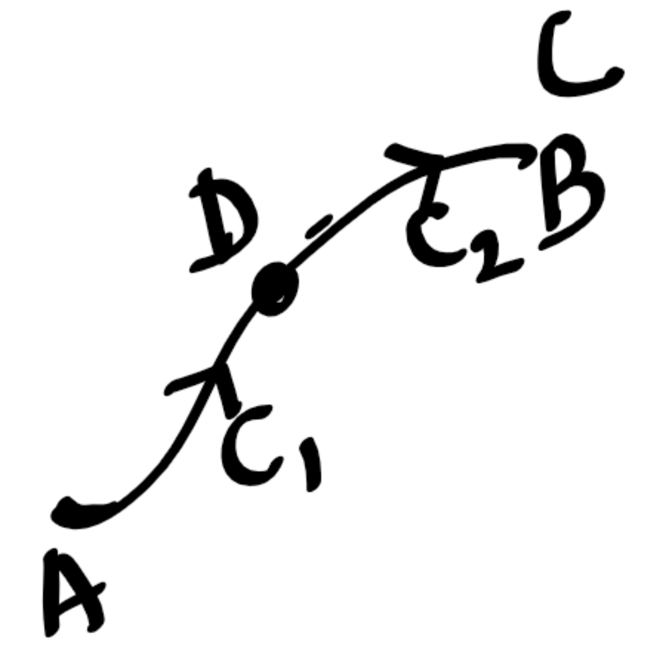
$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (u\,dx - v\,dy) + i\int (u\,dy + v\,dx)$$

Properties

1. $$\int_C [k_1 f(z) + k_2 g(z)]\,dz = k_1 \int_C f(z)\,dz + k_2 \int_C g(z)\,dx$$

2. $$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

3. $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



Complex integration.

$\int_C f(z)\,dz$

Curve:- Let $x(t)$, $y(t)$ be continuous functions of real variable t. Then the equation $z = z(t) = x(t) + i\,y(t)$, $a \le t \le b$ represents a curve in the complex plane.

Ex:- $z(t) = r\cos t + i r \sin t = re^{it}$, $0 \le t \le 2\pi$ represents a circle in the complex plane.

Simple curve :- A curve C is said to be simple if it doesnot intersect itself.

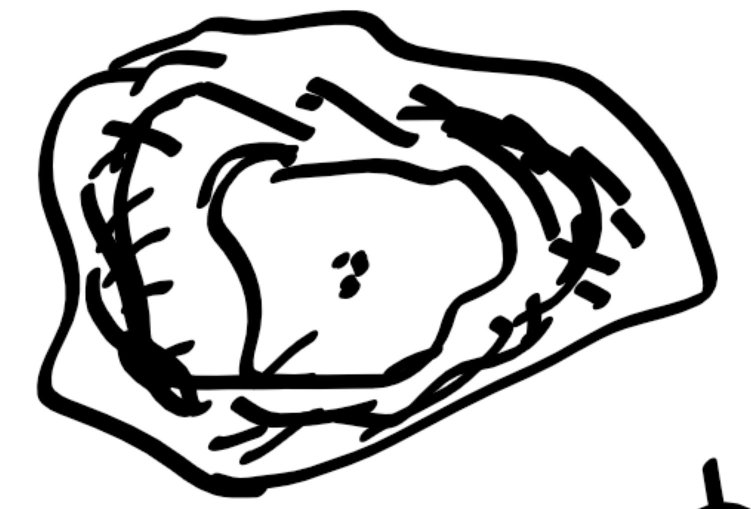
Ex:- Semicircle above x-axis

Simple closed curve :- A curve is said to be a simple closed curve (Jordan curve) if it is a simple curve and its endpoints coincide

Ex:- Circle.

Simply connected region:- It is a region R such that every simple closed curve in R contains only the points of R.

A region which is not simply connected is called multiply connected.



Ex:- The region between two concentric circles