2.2 DESCRIBING FUNCTION

Consider the block diagram of the nonlinear system shown in figure 2.12.

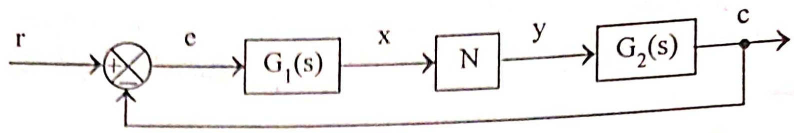

Fig 2.12 : A nonlinear system

In the above system the blocks $G_1(s)$ and $G_2(s)$ represents linear elements and the block N represent nonlinear element.

Let $x = X \sin\omega t$ be the input to nonlinear element. Now the output y of the nonlinear element will be in general a nonsinusoidal periodic function. The fourier series representation of the output y can be expressed as (by assuming that the nonlinearity does not generate subharmonics).

$$y = A_0 + A_1 \sin\omega t + B_1 \cos\omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + \ldots\ldots \qquad \ldots\ldots(2.4)$$

If the nonlinearity is symmetrical the average value of y is zero and hence the output y is given by

$$y = A_1 \sin\omega t + B_1 \cos\omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + \ldots\ldots \qquad \ldots\ldots(2.5)$$

In the absence of an external input (i.e, when $r = 0$)the output y of the nonlinearity N is fedback to its input through the linear elements $G_2(s)$ and $G_1(s)$ in tandem. If $G_1(s)G_2(s)$ has low-pass characteristics, then all the harmonics of y are filtered, so that the input x to the nonlinear element N is mainly contributed by the fundamental component of y and hence x remains sinusoidal. Under such conditions the harmonics of the output are neglected and the fundamental component of y alone considered for the purpose of analysis.

$$\therefore\ y = y_1 = A_1 \sin\omega t + B_1 \cos\omega t = Y_1 \angle\phi_1 = Y_1 \sin(\omega t + \phi_1) \qquad \ldots\ldots(2.6)$$

where, $Y_1 = \sqrt{A_1^2 + B_1^2}$ $\qquad \ldots\ldots(2.7)$

and $\phi_1 = \tan^{-1}\dfrac{B_1}{A_1}$ $\qquad \ldots\ldots(2.8)$

Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The coefficients A_1 and B_1 of the fourier series are given by

$$A_1 = \frac{2}{2\pi}\int_0^{2\pi} y \sin\omega t \; d(\omega t) \qquad .....(2.9)$$

$$B_1 = \frac{2}{2\pi}\int_0^{2\pi} y \cos\omega t \; d(\omega t) \qquad .....(2.10)$$

When the input, x to the nonlinearity is sinusoidal (i.e., $x = X\sin\omega t$) the describing function of the nonlinearity is defined as,

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 \qquad .....(2.11)$$

The nonlinear element N in the system can be replaced by the describing function as shown in figure 2.13.

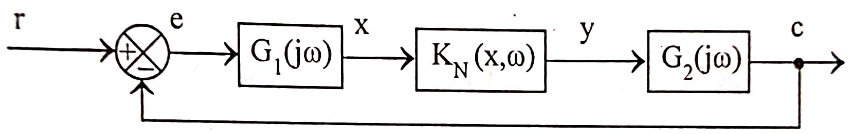

Fig 2.13 : Nonlinear system with nonlinearity replaced by describing function

If the nonlinearity is replaced by a describing function then all linear theory frequency domain techniques can be used for the analysis of the system. The describing functions are used only for stability analysis and it is not directly applied to the optimization of system design. The describing function is a frequency domain approach and no general correlation is possible between time and frequency responses.

2.3 DESCRIBING FUNCTION OF DEAD-ZONE AND SATURATION NONLINEARITY

The input and the output relationship of nonlinearity with dead-zone and saturation is shown in figure 2.14.

The dead-zone region is from $x = -D/2$ to $+D/2$ and in this region the output is zero. The input-output relation is linear for $x = \pm D/2$ to $\pm S$ and when the input, $x > S$, the output reaches a saturated value of $\pm K(S-D/2)$.

The output equation for the linear region can be obtained from the general equation of straight line as shown below.

The equation of straight line is , $y = mx + c$ .....(2.12)

In the linear region, when $x = D/2$, $y = 0$. On substituting this values of x and y in equ(2.12) we get,

$$0 = mD/2 + c \qquad .....(2.13)$$

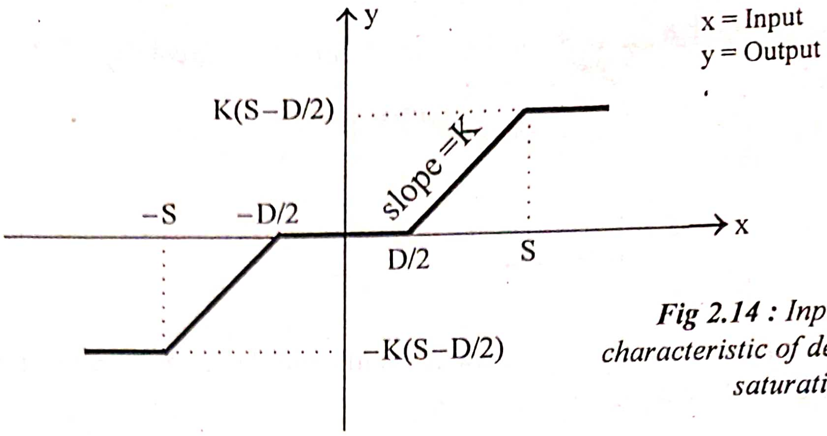

Fig 2.14 : Input-output characteristic of dead -zone and saturation

In the linear region, when x = S, y = K(S−D/2). On substituting this values of x and y in equ(2.12) we get,

$$K(S-D/2) = mS + c \qquad .....(2.14)$$

Equ(2.14) − equ(2.13) yields, $K(S-\frac{D}{2}) = mS + c - m\frac{D}{2} - c$

$$K(S-\frac{D}{2}) = m(S-\frac{D}{2})$$

$$\therefore m = K \qquad .....(2.15)$$

Put m = K in eqn(2.13), $\therefore 0 = K\frac{D}{2} + c$ (or) $c = -K\frac{D}{2}$ .....(2.16)

From equations (2.12), (2.15) and (2.16) the output equation for the linear region can be written as,

$$y = mx + c = Kx - K\frac{D}{2} = K(x - \frac{D}{2}) \qquad .....(2.17)$$

The response or output of the nonlinearity when the input is sinusoidal signal ($x = X \sin\omega t$) is shown in fig 2.15.

The input x is sinusoidal, $\therefore x = X \sin\omega t$ .....(2.18)

where X = Maximum value of input.

In fig 2.15, when, $\omega t = \alpha$, $x = D/2$

Hence from equ (2.18) we get

$$D/2 = X \sin\alpha$$

$$\sin\alpha = D/2X$$

$$\therefore \alpha = \sin^{-1}\frac{D}{2X} \qquad .....(2.19)$$

In fig 2.15 when $\omega t = \beta$, $x = S$

Hence from equ (2.18) we get

$$S = X \sin\beta$$

$$\sin\beta = S/X$$

$$\therefore \beta = \sin^{-1}\frac{S}{X} \qquad .....(2.20)$$

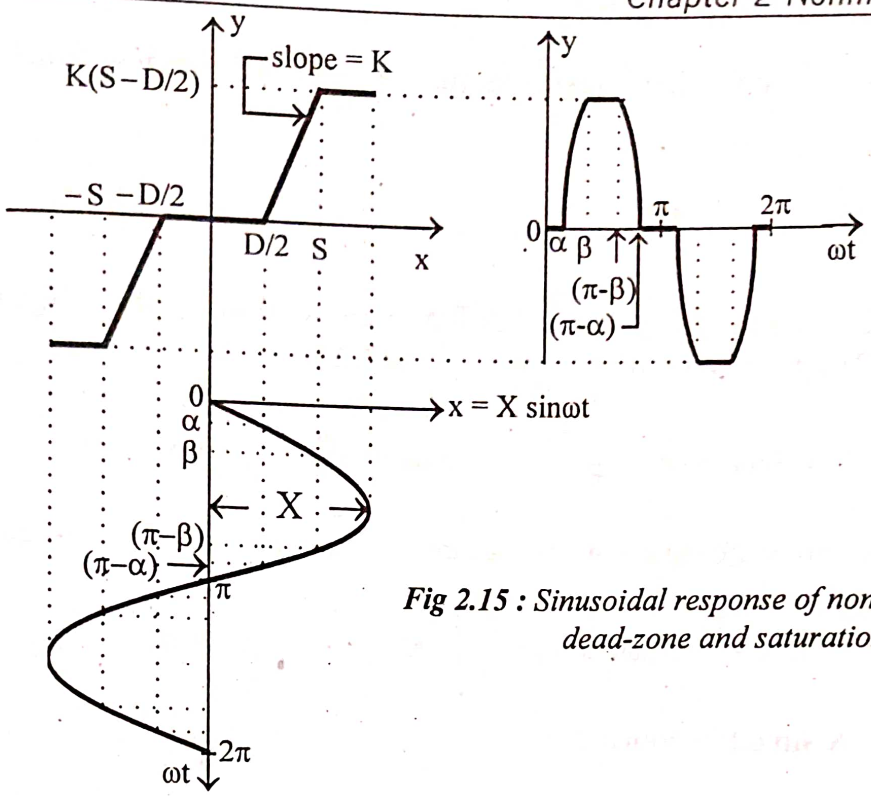

Fig 2.15 : Sinusoidal response of nonlinearity with dead-zone and saturation

The output y of the nonlinearity can be divided into five regions in a period of π and the output equation for the five regions are given below.

$$y = \begin{cases} 0 & ; 0 \le \omega t \le \alpha \\ K(x-\frac{D}{2}) & ; \alpha \le \omega t \le \beta \\ K(S-\frac{D}{2}) & ; \beta \le \omega t \le (\pi-\beta) \\ K(x-\frac{D}{2}) & ; (\pi-\beta) \le \omega t \le (\pi-\alpha) \\ 0 & ; (\pi-\alpha) \le \omega t \le \pi \end{cases} \quad .....(2.21)$$

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by

$$K_N(X,\omega) = \frac{Y_1}{X} \angle\phi_1$$

where, $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}\frac{B_1}{A_1}$

$$A_1 = \frac{2}{2\pi}\int_0^{2\pi} y \sin\omega t \, d(\omega t) \quad \text{and} \quad B_1 = \frac{2}{2\pi}\int_0^{2\pi} y \cos\omega t \, d(\omega t)$$

Here the output has half wave and quarter wave symmetries

$$\therefore \quad B_1 = 0$$

$$\text{and } A_1 = \frac{2}{\pi/2}\int_0^{\pi/2} y \sin\omega t \, d(\omega t) \qquad \text{.....(2.22)}$$

Since the output, y is given by different expressions in the range 0 to $\pi/2$, the equation (2.22) can be written as shown in equ(2.23).

$$A_1 = \frac{4}{\pi}\int_0^{\alpha} y \sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_{\alpha}^{\beta} y \sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} y \sin\omega t \, d(\omega t) \qquad \text{.....(2.23)}$$

On substituting the values of y in the range 0 to $\pi/2$ from equ(2.21) in equ(2.23) we get,

$$A_1 = \frac{4}{\pi}\int_{\alpha}^{\beta} K\left(x - \frac{D}{2}\right) \sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} K\left(S - \frac{D}{2}\right) \sin\omega t \, d(\omega t) \qquad \text{.....(2.24)}$$

Put $x = X \sin\omega t$ in equ(2.24),

$$\therefore A_1 = \frac{4K}{\pi}\left[\int_{\alpha}^{\beta} \left(X \sin\omega t - \frac{D}{2}\right) \sin\omega t \, d(\omega t) + \int_{\beta}^{\frac{\pi}{2}} \left(S - \frac{D}{2}\right) \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\int_{\alpha}^{\beta} X \sin^2\omega t \, d(\omega t) - \int_{\alpha}^{\beta} \frac{D}{2} \sin\omega t \, d(\omega t) + \int_{\beta}^{\frac{\pi}{2}} \left(S - \frac{D}{2}\right) \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\int_{\alpha}^{\beta} (1 - \cos 2\omega t) \, d(\omega t) - \frac{D}{2}\int_{\alpha}^{\beta} \sin\omega t \, d(\omega t) + \left(S - \frac{D}{2}\right)\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_{\alpha}^{\beta} - \frac{D}{2}\left[-\cos\omega t\right]_{\alpha}^{\beta} + \left(S - \frac{D}{2}\right)\left[-\cos\omega t\right]_{\beta}^{\frac{\pi}{2}}\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left(\beta - \sin\frac{2\beta}{2} - \alpha + \frac{\sin 2\alpha}{2}\right) - \frac{D}{2}(-\cos\beta + \cos\alpha)\right.$$

$$\left. + \left(S - \frac{D}{2}\right)\left(-\cos\frac{\pi}{2} + \cos\beta\right)\right] \qquad \left(\because \cos\frac{\pi}{2} = 0\right)$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left(\beta - \alpha - \frac{\sin 2\beta}{2} + \frac{\sin 2\alpha}{2}\right) + \frac{D}{2}\cos\beta - \frac{D}{2}\cos\alpha + S\cos\beta - \frac{D}{2}\cos\beta\right]$$

$$\text{.....(2.25)}$$

$$\text{We know that, } \sin\alpha = \frac{D}{2X} \quad \text{(or)} \quad \frac{D}{2} = X \sin\alpha \qquad \text{.....(2.26)}$$

$$\text{Also, } \sin\beta = \frac{S}{X} \quad \text{(or)} \quad S = X \sin\beta \qquad \text{.....(2.27)}$$

On substituting for D/2 and S from equations (2.26) and (2.27) in equ (2.25) we get,

$$A_1 = \frac{4K}{\pi}\left[\frac{X}{2}\left(\beta - \alpha - \frac{\sin 2\beta}{2} + \frac{\sin 2\alpha}{2}\right) - X\sin\alpha\cos\alpha + X\sin\beta\cos\beta\right]$$

$$= \frac{4KX}{\pi}\left[\frac{\beta}{2} - \frac{\alpha}{2} - \frac{\sin 2\beta}{4} + \frac{\sin 2\alpha}{4} - \frac{\sin 2\alpha}{2} + \frac{\sin 2\beta}{2}\right]$$

$$= \frac{4KX}{\pi}\left[\frac{1}{2}(\beta - \alpha) + \frac{\sin 2\beta}{4} - \frac{\sin 2\alpha}{4}\right]$$

$$= \frac{2KX}{\pi}\left[(\beta - \alpha) + \frac{\sin 2\beta}{2} - \frac{\sin 2\alpha}{2}\right]$$

$$= \frac{KX}{\pi}\left[2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha\right] \qquad .....(2.28)$$

$$Y_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{A_1^2 + 0} = A_1$$

$$\therefore\ Y_1 = A_1 = \frac{KX}{\pi}\left[2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha\right] \qquad .....(2.29)$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}0 = 0 \qquad .....(2.30)$$

The describing function $K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1$ .....(2.31)

On substituting for Y_1 and ϕ_1 from equations (2.29) and (2.30) in equ(2.31) we get

$$\mathbf{K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \frac{K}{\pi}\left[2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha\right]\angle 0^0} \qquad .....(2.32)$$

Depending on the maximum value of input, X, the describing function of equ(2.32) can be written as,

If $X < \frac{D}{2}$, then $\alpha = \beta = \frac{\pi}{2}$ and $\mathbf{K_N(X,\omega) = 0}$ .....(2.33)

If $\frac{D}{2} < X < S$, then $\beta = \frac{\pi}{2}$ and $\mathbf{K_N(X,\ \omega) = K\left[1 - \frac{2}{\pi}(\alpha + \sin\alpha\cos\alpha)\right]}$ .....(2.34)

If $X > S$, $\mathbf{K_N(X,\ \omega) = \frac{K}{\pi}\left[2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha\right]}$ .....(2.35)

2.4 DESCRIBING FUNCTION OF SATURATION NONLINEARITY

The input-output relationship of saturation nonlinearity is shown in fig 2.16.

The input-output relation is linear for $x = 0$ to S. When the input $x > S$, the output reaches a saturated value of KS.

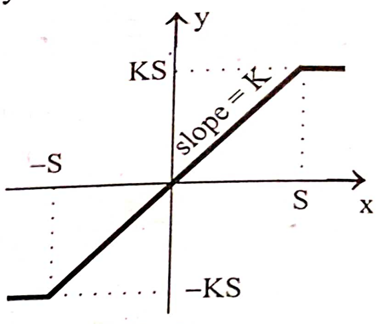

Fig 2.16 : Input-output characteristic of saturation nonlinearity

The response of the nonlinearity when the input is sinusoidal signal ($x = X\sin\omega t$) is shown in fig 2.17.

The input x is sinusoidal,

$$\therefore x = X\sin\omega t \qquad .....(2.36)$$

where X is the maximum value of input.

In fig (2.17), when $\omega t = \beta,\ x = S$.

Hence equ(2.36) can be written as, $S = X\sin\beta$ .....(2.37)

$$\therefore \sin\beta = \frac{S}{X} \quad \text{(or)} \quad \beta = \sin^{-1}\left(\frac{S}{X}\right) \qquad .....(2.38)$$

The output y of the nonlinearity can be divided into three regions in a period of π. The output equation for the three regions are given by equ(2.39).

$$y = \begin{cases} Kx & ;\ 0 \le \omega t \le \beta \\ KS & ;\ \beta \le \omega t \le (\pi - \beta) \\ Kx & ;\ (\pi - \beta) \le \omega t \le \pi \end{cases} \qquad .....(2.39)$$

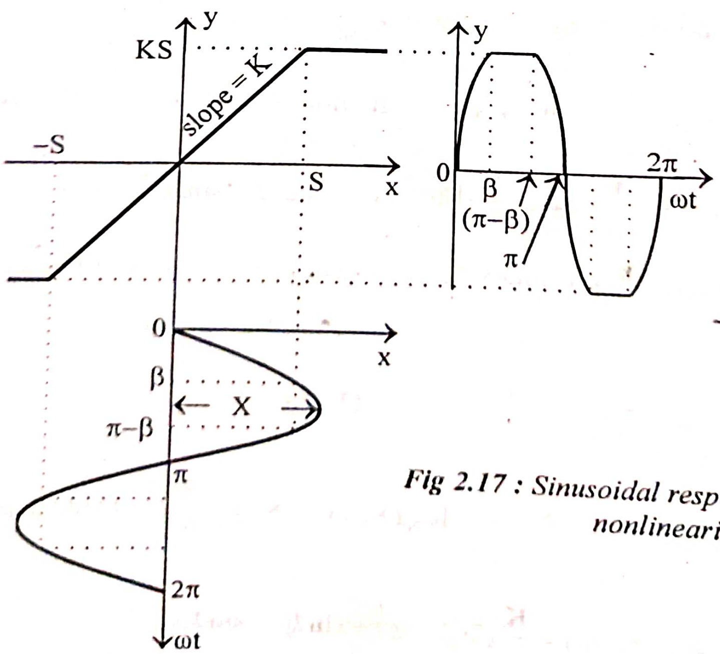

Fig 2.17 : Sinusoidal response of saturation nonlinearity

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by, $K_N(X,\omega) = (Y_1/X)\angle\phi_1$

where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}(B_1/A_1)$

The output y has half wave and quarter wave symmetries

$$\therefore B_1 = 0 \quad \text{and} \quad A_1 = \frac{2}{\pi/2}\int_0^{\frac{\pi}{2}} y \sin\omega t \; d(\omega t) \qquad .....(2.40)$$

The output, y is given by two different expressions in the period 0 to $\pi/2$. Hence equ(2.40) can be written as shown in equ(2.41).

$$A_1 = \frac{4}{\pi}\int_0^{\beta} y \sin\omega t \; d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} y \sin\omega t \; d(\omega t) \qquad .....(2.41)$$

On substituting the values of y from equ(2.39) in equ (2.41) we get,

$$A_1 = \frac{4}{\pi}\int_0^{\beta} Kx \sin\omega t \; d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} KS \; \sin\omega t \; d(\omega t)$$

On substituting $x = X \sin\omega t$, we get,

$$A_1 = \frac{4K}{\pi}\int_0^{\beta} X \sin\omega t \times \sin\omega t \; d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \; d(\omega t)$$

$$= \frac{4KX}{\pi}\int_0^{\beta} \sin^2\omega t \; d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \; d(\omega t)$$

$$= \frac{4KX}{\pi}\int_0^{\beta} \frac{1-\cos 2\omega t}{2} d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \; d(\omega t)$$

$$= \frac{2KX}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^{\beta} + \frac{4KS}{\pi}\left[-\cos\omega t\right]_{\beta}^{\frac{\pi}{2}}$$

$$= \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4KS}{\pi}\left[-\cos\frac{\pi}{2} + \cos\beta\right]$$

$$= \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4KS}{\pi}\cos\beta \qquad .....(2.42)$$

On substituting for S, (i.e, $S = X\sin\beta$) from equ(2.37) in equ(2.42) we get,

$$A_1 = \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4K}{\pi}X\sin\beta\cos\beta$$

$$= \frac{2KX}{\pi}\left[\beta - \frac{2\sin\beta\cos\beta}{2}\right] + \frac{4KX}{\pi}\sin\beta\cos\beta$$

$$= \frac{2KX}{\pi}\left[\beta - \sin\beta\cos\beta + 2\sin\beta\cos\beta\right]$$

$$= \frac{2KX}{\pi}\left[\beta + \sin\beta\cos\beta\right] \qquad .....(2.43)$$

$$Y_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{A_1^2 + 0} = A_1 = \frac{2KX}{\pi}\left[\beta + \sin\beta\cos\beta\right] \qquad .....(2.44)$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}0 = 0 \qquad .....(2.45)$$

The describing function $K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1$ .....(2.46)

Using equations (2.44) and (2.45), the describing function of equ(2.46) can be written as,

$$\mathbf{K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \frac{2K}{\pi}\left[\beta + \sin\beta\cos\beta\right]\angle 0^0} \qquad .....(2.47)$$

Depending on the maximum value of input X, the describing function can be written as,

If $X < S$, then $\beta = \frac{\pi}{2}$, $\mathbf{K_N(X,\omega) = K}$ .....(2.48)

If $X > S$, $\mathbf{K_N(X,\omega) = \frac{2K}{\pi}\left[\beta + \sin\beta\cos\beta\right]}$ .....(2.49)

The equation (2.49) can be expressed in another form as shown below.

From equ(2.37) we get, $S = X\sin\beta$, $\therefore \sin\beta = \frac{S}{X}$ .....(2.50)

On constructing right angle triangle with unity hypotenuse as shown in fig 2.18, $\cos\beta$ can be evaluated. From fig 2.18 we get.

$$\text{adj} = \sqrt{1 - \left(\frac{S}{X}\right)^2} \qquad \therefore \cos\beta = \frac{\text{adj}}{\text{hyp}} = \sqrt{1 - \left(\frac{S}{X}\right)^2} \qquad .....(2.51)$$

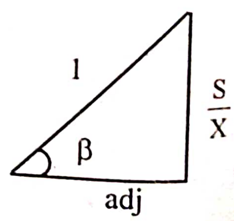

Fig 2.18

In the describing function of equ(2.49), substitute for β, $\sin\beta$ and $\cos\beta$ from equations (2.38), (2.50) and (2.51)

$$\therefore \mathbf{K_N(X,\omega) = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) + \left(\frac{S}{X}\right)\sqrt{1 - \left(\frac{S}{X}\right)^2}\right]} \quad \textbf{for } X > S \qquad .....(2.52)$$

2.5 DESCRIBING FUNCTION OF DEAD-ZONE NONLINEARITY

The input-output relationship of dead-zone nonlinearity is shown in figure 2.19 The output is zero, when the input is less than D/2. The input-output relationship is linear when the input is greater than D/2. The response of the nonlinearity when input is sinusoidal signal ($x = X \sin\omega t$) is shown in fig 2.20.

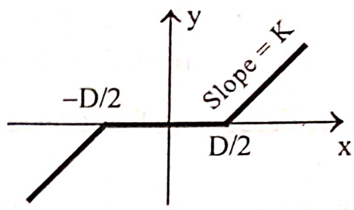

Fig 2.19 : Input-Output characteristic of dead-zone nonlinearity

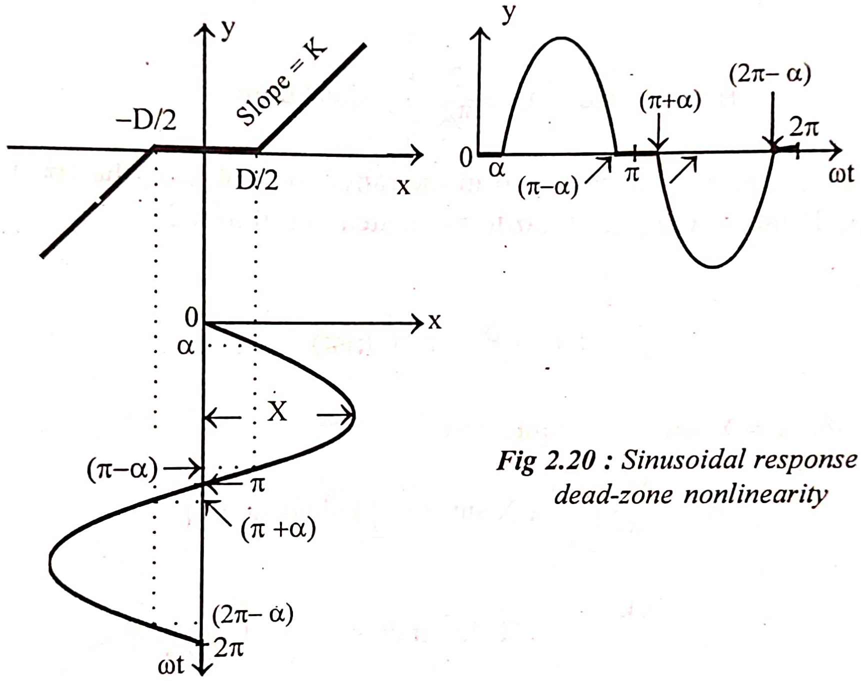

Fig 2.20 : Sinusoidal response of dead-zone nonlinearity

The input x is sinusoidal, $\therefore\ x = X \sin\omega t$ .....(2.53)
where X is the maximum value of input.
In fig 2.20, when $\omega t = \alpha$, $x = D/2$,
Hence when $\omega t = \alpha$, the equ (2.53) can be written as, $D/2 = X \sin\alpha$ .....(2.54)

$$\therefore \sin\alpha = \frac{D}{2X} \qquad .....(2.55)$$

$$\text{and } \alpha = \sin^{-1}\frac{D}{2X} \qquad .....(2.56)$$

The output y can be divided into three regions in a period of π. The output equation for the three regions are given by equ(2.57).

$$y = \begin{cases} 0 & ;\ 0 \le \omega t \le \alpha \\ K(x - \frac{D}{2}) & ;\ \alpha \le \omega t \le (\pi - \alpha) \\ 0 & ;\ (\pi - \alpha) \le \omega t \le \pi \end{cases} \qquad .....(2.57)$$

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by, $K_N(X,\omega) = (Y_1 / X) \angle\phi_1$

where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}(B_1 / A_1)$

The output y has halfwave and quarter wave symmetries.

$$\therefore B_1 = 0 \quad \text{and} \quad A_1 = \frac{2}{\pi/2}\int_0^{\pi/2} y \sin\omega t \; d(\omega t) \qquad .....(2.58)$$

Since the output, y is zero in the range, $0 \le \omega t \le \alpha$, the limits of integration in equ(2.58) can be changed to, α to $\pi/2$ instead of, 0 to $\pi/2$.

$$\therefore A_1 = \frac{4}{\pi}\int_\alpha^{\pi/2} K(x - \tfrac{D}{2}) \sin\omega t \; d(\omega t) \qquad .....(2.59)$$

Put $x = X \sin \omega t$, in equ(2.59)

$$\therefore A_1 = \frac{4K}{\pi}\left[\int_\alpha^{\pi/2} (X \sin\omega t - \frac{D}{2}) \sin\omega t \; d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\int_\alpha^{\pi/2} X \sin^2\omega t \; d(\omega t) - \frac{D}{2}\int_\alpha^{\pi/2} \sin\omega t \; d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\int_\alpha^{\pi/2} (1 - \cos 2\omega t) \; d(\omega t) - \frac{D}{2}\int_\alpha^{\pi/2} \sin\omega t \; d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_\alpha^{\frac{\pi}{2}} - \frac{D}{2}\left[-\cos\omega t\right]_\alpha^{\frac{\pi}{2}}\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left(\frac{\pi}{2} - \frac{\sin\pi}{2} - \alpha + \frac{\sin 2\alpha}{2}\right) - \frac{D}{2}\left(-\cos\frac{\pi}{2} + \cos\alpha\right)\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left(\frac{\pi}{2} - \alpha + \frac{\sin 2\alpha}{2}\right) - \frac{D}{2}(\cos\alpha)\right] \qquad .....(2.60)$$

From equ (2.55) we get, $\sin\alpha = \dfrac{D}{2X}$ $\qquad \therefore D = 2X \sin\alpha \qquad .....(2.61)$

On substituting for D from equ(2.61) in equ(2.60) we get,

$$A_1 = \frac{4K}{\pi}\left[\frac{X}{2}\left(\frac{\pi}{2}-\alpha+\frac{\sin 2\alpha}{2}\right)-X\sin\alpha\cos\alpha\right] = \frac{4KX}{\pi}\left[\frac{\pi}{4}-\frac{\alpha}{2}+\frac{2\sin\alpha\cos\alpha}{4}-\sin\alpha\cos\alpha\right]$$

$$= \frac{4KX}{\pi}\left[\frac{\pi}{4}-\frac{\alpha}{2}-\frac{\sin\alpha\cos\alpha}{2}\right] = KX\left[\frac{4}{\pi}\times\frac{\pi}{4}-\frac{4}{\pi}\times\frac{1}{2}(\alpha+\sin\alpha\cos\alpha)\right]$$

$$= KX\left[1-\frac{2}{\pi}(\alpha+\sin\alpha\cos\alpha)\right] \quad .....(2.62)$$

$$Y_1 = \sqrt{A_1^2+B_1^2} = \sqrt{A_1^2+0} = A_1 = KX\left[1-\frac{2}{\pi}(\alpha+\sin\alpha\cos\alpha)\right] \quad .....(2.63)$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}0 = 0 \quad .....(2.64)$$

The describing function, $K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1$ .....(2.65)

Using equations(2.63) and (2.64) the describing function of equ(2.65) can be written as,

$$\mathbf{K_N(X,\omega) = K\left[1-\frac{2}{\pi}(\alpha+\sin\alpha\cos\alpha)\right]\angle 0^0} \quad .....(2.66)$$

Depending on the maximum value of input X, the describing function can be written as,

If $X < \frac{D}{2}$, $\quad \mathbf{K_N(X,\omega) = 0}$ .....(2.67)

If $X > \frac{D}{2}$, $\quad \mathbf{K_N(X,\omega) = K\left[1-\frac{2}{\pi}(\alpha+\sin\alpha\cos\alpha)\right]}$ .....(2.68)

The equation(2.68) can be expressed in another form as shown below.

From equ(2.55), we get, $\sin\alpha = D/2X$

On constructing right angle triangle with unity hypotenuse as shown in fig 2.21 $\cos\alpha$ can be evaluated.

From fig 2.21, we get,

$$\text{adj} = \sqrt{1-\left(\frac{D}{2X}\right)^2} \qquad \therefore \cos\alpha = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-\left(\frac{D}{DX}\right)^2} \quad .....(2.69)$$

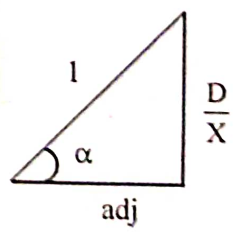

Fig 2.21

In the describing function of equ(2.68), substitute for α, $\sin\alpha$ and $\cos\alpha$ from equations (2.56) (2.55) and (2.69) respectively.

$$\therefore \mathbf{K_N(X,\omega) = K\left[1-\frac{2}{\pi}\left(\sin^{-1}\left(\frac{D}{2X}\right)+\left(\frac{D}{2X}\right)\sqrt{1-\left(\frac{D}{2X}\right)^2}\right)\right]} \quad \textbf{for } X > \frac{D}{2} \quad .....(2.70)$$

2.6 DESCRIBING FUNCTION OF RELAY WITH DEAD-ZONE AND HYSTERESIS

The input and the output relationship of a relay with dead-zone and hysteresis is shown in fig 2.22.

Due to dead-zone the relay will respond only after a definite value of input. Due to hysteresis the output follows a different paths for increasing and decreasing values of input. When the input x is increased from zero, the output follows the path ABCD and when the input is decreased from a maximum value, the output follows the path DCEA.

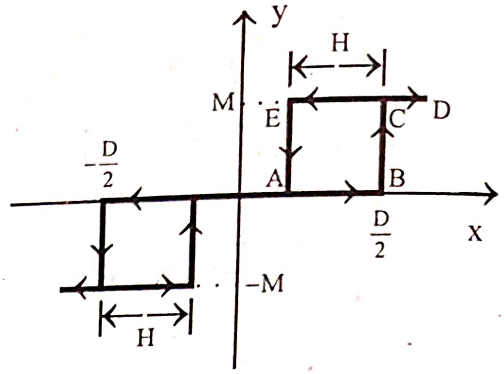

Fig 2.22 : Input Output characteristic of relay with dead-zone and hysteresis.

For increasing values of input, the output is zero when x<(D/2) and the output is M when x>(D/2). For decreasing values of input the output is M when x>(D/2−H) and output is zero when x < (D/2−H).

The response or output of the relay when the input is sinusoidal signal (x = Xsinωt) is shown in fig 2.23.

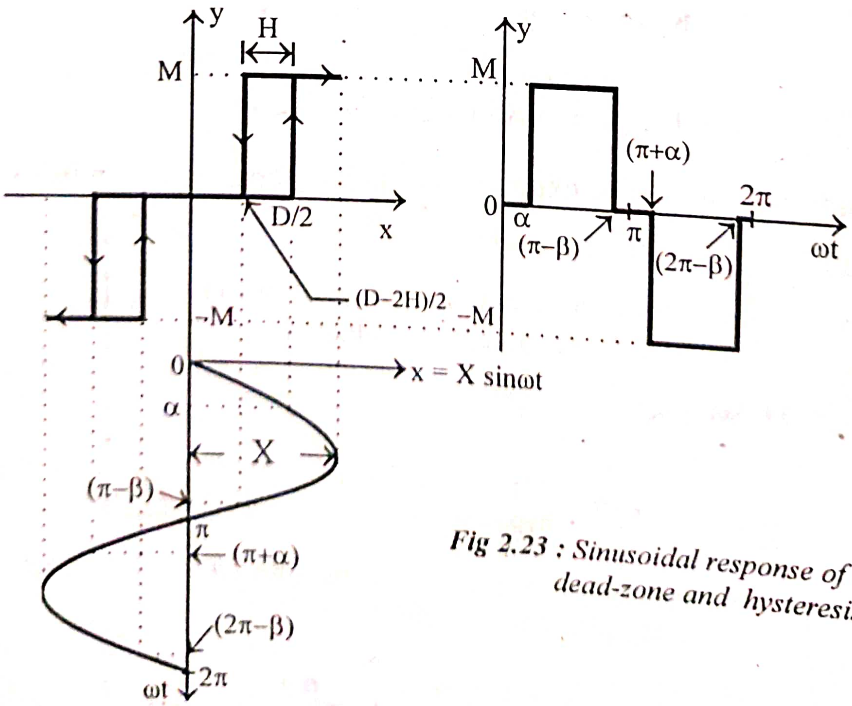

Fig 2.23 : Sinusoidal response of relay with dead-zone and hysteresis

The input x is sinusoidal , $\therefore$ $x = X\sin\omega t$ .....(2.71)

where X = maximum value of input

In fig 2.23, when $\omega t = \alpha$, $x = D/2$

Hence equ(2.71) can be written as

$$D/2 = X\ \sin\alpha$$

$$\therefore \sin\alpha = D/2X \quad .....(2.72)$$

$$\text{and } \alpha = \sin^{-1}\frac{D}{2X} \quad .....(2.73)$$

In fig 2.23, when $\omega t = \pi - \beta$, $x = (D/2) - H$

Hence equ(2.71) can be written as

$$D/2 - H = X\ \sin(\pi - \beta)$$

$$D/2 - H = X\ \sin\beta$$

$$\sin\beta = \frac{1}{X}\left(\frac{D}{2} - H\right) \quad .....(2.74)$$

$$\beta = \sin^{-1}\left(\frac{1}{X}\left(\frac{D}{2} - H\right)\right) \quad .....(2.75)$$

The output can be divided into five regions in a period of 2π and the output equation for the five regions are given by equ(2.76).

$$y = \begin{cases} 0 & ;\ 0 \le \omega t \le \alpha \\ M & ;\ \alpha \le \omega t \le (\pi - \beta) \\ 0 & ;\ (\pi - \beta) \le \omega t \le (\pi + \alpha) \\ -M & ;\ (\pi + \alpha) \le \omega t \le (2\pi - \beta) \\ 0 & ;\ (2\pi - \beta) \le \omega t \le 2\pi \end{cases} \quad .....(2.76)$$

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by, $K_N(X,\omega) = (Y_1/X)\ \angle\phi_1$

where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}(B_1/A_1)$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} y\ \sin\omega t\ d(\omega t) = \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \sin\omega t\ d(\omega t)$$

Note : $\cos(\pi-\beta) = -\cos\beta$

$$= \frac{2M}{\pi}\left[-\cos\omega t\right]_{\alpha}^{\pi-\beta} = \frac{2M}{\pi}\left[-\cos(\pi-\beta)+\cos\alpha\right]$$

$$= \frac{2M}{\pi}(\cos\alpha + \cos\beta) \quad ......(2.77)$$

From equ(2.72) we get, $\sin\alpha = D/2X$

On constructing right angle triangle with unity hypotenuse as shown in fig 2.24, $\cos\alpha$ can be evaluated

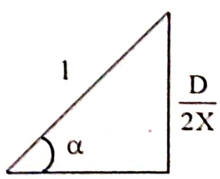

Fig 2.24

$$\text{adj} = \sqrt{1 - (D/2X)^2}$$

$$\therefore \cos\alpha = \frac{\text{adj}}{\text{hyp}} = \sqrt{1 - (D/2X)^2} \quad .....(2.78)$$

From equ(2.74) we get $\sin\beta = \left(\frac{D}{2X} - \frac{H}{X}\right)$.

On constructing right angle triangle with unity hypotenuse as shown in fig(2.25), $\cos\beta$ can be evaluated

$$\text{adj} = \sqrt{1 - \left(\frac{D}{2X} - \frac{H}{X}\right)^2}$$

$$\cos\beta = \frac{\text{adj}}{\text{hyp}} = \sqrt{1 - \left(\frac{D}{2X} - \frac{H}{X}\right)^2} \qquad .....(2.79)$$

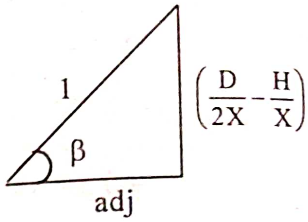

Fig 2.25

On substituting for cos α and cos β from equations (2.78) and (2.79) in equ(2.77) we get,

$$A_1 = \frac{2M}{\pi}\left(\sqrt{1-\left(\frac{D}{2X}\right)^2} + \sqrt{1-\left(\frac{D}{2X} - \frac{H}{X}\right)^2}\right) \qquad .....(2.80)$$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} y \cos\omega t \, d(\omega t) = \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \cos\omega t \, d(\omega t)$$

$\boxed{\textit{Note}: \sin(\pi-\beta) = \sin\beta}$

$$= \frac{2M}{\pi}\left[\sin\omega t\right]_{\alpha}^{\pi-\beta} = \frac{2M}{\pi}\left[\sin(\pi-\beta) - \sin\alpha\right] = \frac{2M}{\pi}(\sin\beta - \sin\alpha)$$

On substituting for sin α and sin β from equ(2.72) and equ(2.74) we get,

$$B_1 = \frac{2M}{\pi}\left[\frac{D}{2X} - \frac{H}{X} - \frac{D}{2X}\right] = \frac{2M}{\pi}\left(\frac{-H}{X}\right) = \frac{2M}{\pi}\left(-\frac{H}{X}\right) \qquad .....(2.81)$$

$$\therefore Y_1 = \sqrt{A_1^2 + B_1^2} = \left[A_1^2 + B_1^2\right]^{\frac{1}{2}}$$

$$Y_1 = \left[\frac{4M^2}{\pi^2}\left\{\sqrt{1-\left(\frac{D}{2X}\right)^2} + \sqrt{1-\left(\frac{D}{2X} - \frac{H}{X}\right)^2}\right\}^2 + \frac{4M^2}{\pi^2}\left(\frac{H^2}{X^2}\right)\right]^{\frac{1}{2}} \qquad .....(2.82)$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}\left(\frac{\frac{2M}{\pi}\left(-\frac{H}{X}\right)}{\frac{2M}{\pi}\left[\sqrt{1-\left(\frac{D}{2X}\right)^2} + \sqrt{1-\left(\frac{D}{2X} - \frac{H}{X}\right)^2}\right]}\right) \qquad .....(2.83)$$

The describing function of the relay with dead-zone and hystersis is given by

$$K_N(X,\omega) = \frac{Y_1}{X} \angle\phi_1 \qquad .....(2.84)$$

Where Y_1 is given by equ(2.82) and ϕ_1 is given by equ(2.83).

From the equ(2.84), the describing functions of the following three cases of relay can be obtained.

1. *Ideal relay.*
2. *Relay with dead-zone.*
3. *Relay with hysteresis.*

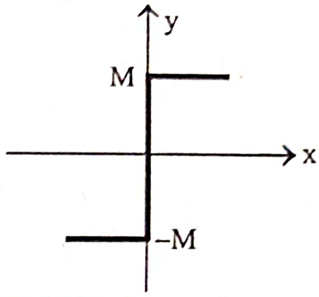

Fig 2.26 : Input - Output characteristics of ideal relay

1. IDEAL RELAY

In this case, $D = H = 0$,

On substituting $D = H = 0$, in equ(2.82) and equ(2.83) we get,

$$Y_1 = \frac{2M}{\pi} \quad \text{and} \quad \phi_1 = 0$$

Hence the describing function of the ideal relay is given by,

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \frac{2M}{\pi X} \qquad .....(2.85)$$

2. RELAY WITH DEAD-ZONE

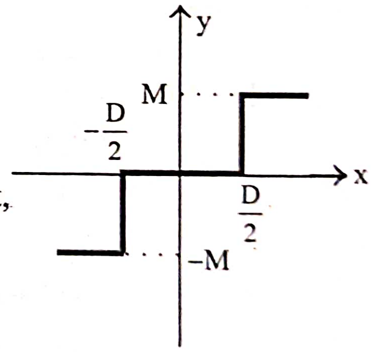

Fig 2.27 : Input - Output characteristics of relay with dead-zone

In this case $H = 0$

On substituting $H = 0$, in equ(2.82) and (2.83) we get,

$$Y_1 = \left[\frac{4M^2}{\pi^2}\left\{2\sqrt{1-\left(\frac{D}{2X}\right)^2}\right\}^2\right]^{\frac{1}{2}}$$

$$= \frac{4M}{\pi}\sqrt{1-\left(\frac{D}{2X}\right)^2}$$

$$\phi_1 = 0$$

Hence the describing function of relay with dead-zone is given by

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \begin{cases} 0 & ; \ X < \frac{D}{2} \\ \frac{4M}{\pi X}\sqrt{1-\left(\frac{D}{2X}\right)^2} & ; \ X > \frac{D}{2} \end{cases} \qquad .....(2.86)$$

3. RELAY WITH HYSTERESIS

In this case D = H

On substituting D = H in equ(2.82) we get,

$$Y_1 = \left[\frac{4M^2}{\pi^2}\left\{\sqrt{1-\left(\frac{H}{2X}\right)^2}+\sqrt{1-\left(\frac{-H}{2X}\right)^2}\right\}^2+\frac{4M^2}{\pi^2}\left(\frac{H^2}{X^2}\right)\right]^{\frac{1}{2}}$$

$$=\frac{2M}{\pi}\left[4\left(1-\frac{H^2}{4X^2}\right)+\left(\frac{H^2}{X^2}\right)\right]^{\frac{1}{2}}$$

$$=\frac{2M}{\pi}\left[4-\frac{H^2}{X^2}+\frac{H^2}{X^2}\right]^{\frac{1}{2}}$$

$$=\frac{4M}{\pi} \qquad .....(2.87)$$

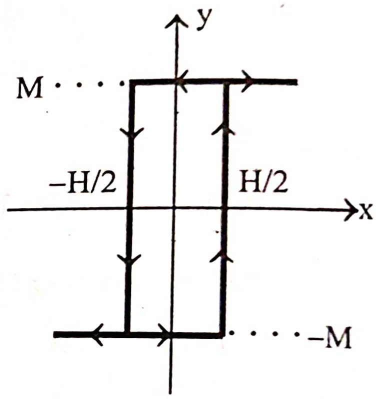

Fig 2.28 : Input - Output characteristics of relay with hysteresis

On substituting D = H in equ(2.83) we get,

$$\phi_1 = \tan^{-1}\frac{\frac{2M}{\pi}\left(-\frac{H}{X}\right)}{\frac{2M}{\pi}\left[\sqrt{1-\left(\frac{H}{2X}\right)^2}+\sqrt{1-\left(\frac{-H}{2X}\right)^2}\right]} = \tan^{-1}\frac{-\frac{H}{X}}{2\sqrt{1-\frac{H^2}{4X^2}}}$$

$$\phi_1 = -\tan^{-1}\frac{\frac{H}{2X}}{\sqrt{1-\frac{H^2}{4X^2}}}$$

$$\therefore -\phi_1 = \tan^{-1}\frac{\frac{H}{2X}}{\sqrt{1-\frac{H^2}{4X^2}}}$$

$$\tan(-\phi_1) = \frac{\frac{H}{2X}}{\sqrt{1-\frac{H^2}{4X^2}}} \qquad .....(2.88)$$

Using the numerator and denominator of equ(2.88) as two sides, we can construct a right angle triangle as shown in fig 2.29.

From fig 2.29 we get, $\sin(-\phi_1) = \frac{H}{2X}$

$$\therefore -\phi_1 = \sin^{-1}\frac{H}{2X}$$

$$\text{(or)}\quad \phi_1 = -\sin^{-1}\frac{H}{2X} \qquad .....(2.89)$$

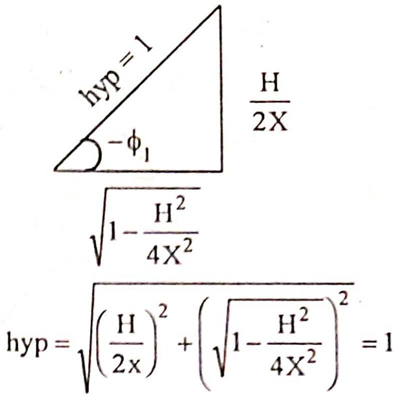

Fig 2.29

Using equations(2.87) and (2.89), the describing function of relay with hysteresis can be written as,

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \begin{cases} 0 & ;\ X < \frac{H}{2} \\ \frac{4M}{\pi X}\ \angle(-\sin^{-1}\frac{H}{2X}) & ;\ X > \frac{H}{2} \end{cases} \qquad .....(2.90)$$

2.7 DESCRIBING FUNCTION OF BACKLASH NONLINEARITY

The input-output relationship of Backlash nonlinearity is shown in fig 2.30.

The response of the nonlinearity when the input is sinusoidal signal ($x = X\sin\omega t$) is shown in fig 2.31.

In fig 2.31, when $\omega t = (\pi - \beta)$, $\quad x = X - b$

On substituting this value of x and ωt in the input signal, $x = X\sin\omega t$, we get,

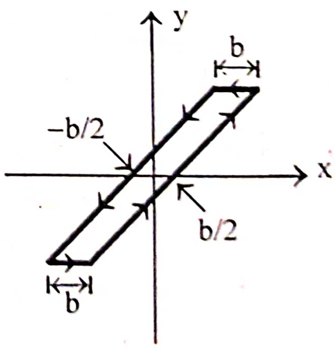

Fig 2.30 : Input-Output characteristic of backlash nonlinearity

$$X - b = X\sin(\pi - \beta)$$

$$X - b = X\sin\beta$$

$$\therefore \sin\beta = \frac{X-b}{X} = 1 - \frac{b}{X} \qquad .....(2.91)$$

$$\text{and } \beta = \sin^{-1}\left(1 - \frac{b}{X}\right) \qquad .....(2.92)$$

The output can be divided into five regions in a period of 2π and the output equation for the five regions are given by equ (2.93).

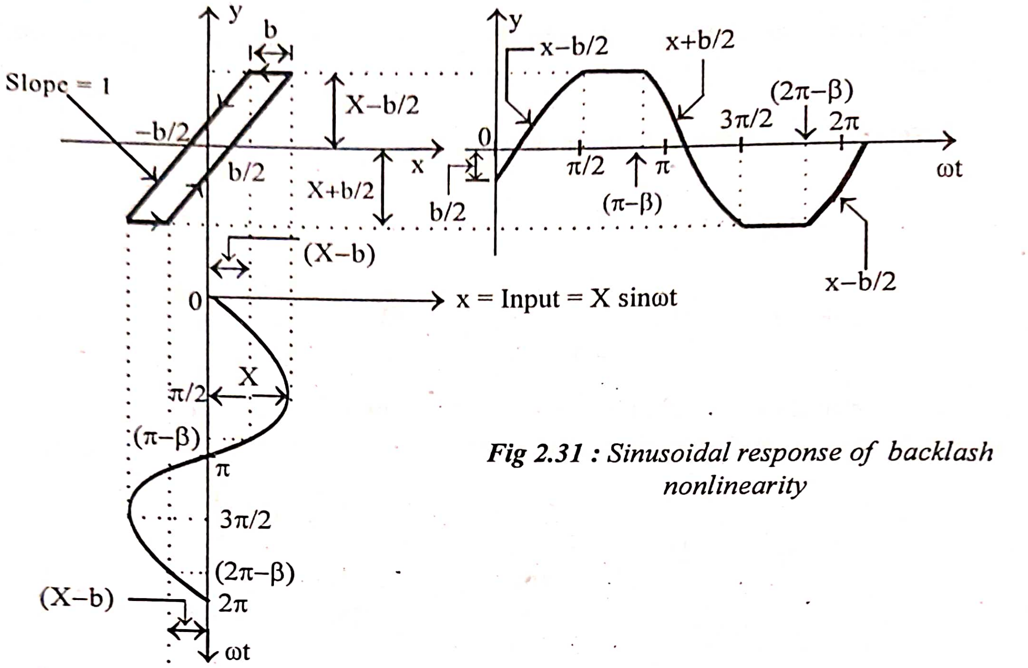

Fig 2.31 : Sinusoidal response of backlash nonlinearity

$$y = \begin{cases} x - b/2 & ; 0 \le \omega t \le \frac{\pi}{2} \\ X - b/2 & ; \frac{\pi}{2} \le \omega t \le (\pi - \beta) \\ x + b/2 & ; (\pi - \beta) \le \omega t \le 3\frac{\pi}{2} \\ -X + b/2 & ; 3\frac{\pi}{2} \le \omega t \le (2\pi - \beta) \\ x - b/2 & ; (2\pi - \beta) \le \omega t \le 2\pi \end{cases} \qquad .....(2.93)$$

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by, $K_N(X,\omega) = (Y_1 / X) \angle\phi_1$

where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}(B_1 / A_1)$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} y \sin\omega t \, d(\omega t) \qquad .....(2.94)$$

The output, y is given by three different equations in range 0 to π, hence equ(2.94) can be written as

$$A_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x - \tfrac{b}{2}) \sin\omega t \, d(\omega t) + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta} (X - \tfrac{b}{2}) \sin\omega t \, d(\omega t) + \frac{2}{\pi}\int_{\pi-\beta}^{\pi} (x + \tfrac{b}{2}) \sin\omega t \, d(\omega t) \qquad .....(2.95)$$

Put $x = X \sin\omega t$ in equ(2.95)

$$A_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(X \sin\omega t - \frac{b}{2}\right)\sin\omega t \, d(\omega t) + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta}\left(X - \frac{b}{2}\right)\sin\omega t \, d(\omega t)$$
$$+ \frac{2}{\pi}\int_{\pi-\beta}^{\pi}\left(X \sin\omega t + \tfrac{b}{2}\right)\sin\omega t \, d(\omega t)$$

$$= \frac{2X}{\pi}\int_0^{\frac{\pi}{2}} \sin^2 \omega t \, d(\omega t) - \frac{b}{\pi}\int_0^{\frac{\pi}{2}} \sin\omega t \, d(\omega t) + \frac{2\left(X - \frac{b}{2}\right)}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta} \sin\omega t \, d(\omega t)$$
$$+ \frac{2X}{\pi}\int_{\pi-\beta}^{\pi} \sin^2\omega t \, d(\omega t) + \frac{b}{\pi}\int_{\pi-\beta}^{\pi} \sin\omega t \, d(\omega t) \qquad \text{......(2.96)}$$

Put $\sin^2\omega t = \dfrac{1-\cos 2\omega t}{2}$ in equ(2.96)

$$\therefore A_1 = \frac{X}{\pi}\int_0^{\frac{\pi}{2}}(1-\cos 2\omega t)\, d(\omega t) - \frac{b}{\pi}\int_0^{\frac{\pi}{2}}\sin\omega t \, d(\omega t) + \frac{2\left(X - \frac{b}{2}\right)}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta}\sin\omega t \, d(\omega t)$$
$$+ \frac{X}{\pi}\int_{\pi-\beta}^{\pi}(1-\cos 2\omega t)\, d(\omega t) + \frac{b}{\pi}\int_{\pi-\beta}^{\pi}\sin\omega t \, d(\omega t)$$

$$= \frac{X}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^{\frac{\pi}{2}} - \frac{b}{\pi}\left[-\cos\omega t\right]_0^{\frac{\pi}{2}} + \frac{2\left(X - \frac{b}{2}\right)}{\pi}\left[-\cos\omega t\right]_{\frac{\pi}{2}}^{\pi-\beta}$$
$$+ \frac{X}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_{\pi-\beta}^{\pi} + \frac{b}{\pi}\left[-\cos\omega t\right]_{\pi-\beta}^{\pi}$$

$$= \frac{X}{\pi}\left(\frac{\pi}{2}\right) - \frac{b}{\pi}(1) + \frac{2\left(X - \frac{b}{2}\right)}{\pi}\left(-\cos(\pi-\beta) + \cos\frac{\pi}{2}\right)$$
$$+ \frac{X}{\pi}\left[\pi - \frac{\sin 2\pi}{2} - (\pi-\beta) + \frac{\sin 2(\pi-\beta)}{2}\right] + \frac{b}{\pi}\left[-\cos\pi + \cos(\pi-\beta)\right]$$

$$= \frac{X}{2} - \frac{b}{\pi} + \frac{2}{\pi}\left(X - \tfrac{b}{2}\right)\cos\beta + \frac{X}{\pi}\left(\beta - \frac{\sin 2\beta}{2}\right) + \frac{b}{\pi}(1-\cos\beta)$$

$$= \frac{X}{2} - \frac{b}{\pi} + \frac{2}{\pi}\left(X - \tfrac{b}{2}\right)\cos\beta + \frac{X\beta}{\pi} - \frac{X}{2\pi}\sin 2\beta + \frac{b}{\pi} - \frac{b}{\pi}\cos\beta$$

$$= \frac{X}{2} + \frac{X\beta}{\pi} + \frac{2}{\pi}\left(X - \tfrac{b}{2} - \tfrac{b}{2}\right)\cos\beta - \frac{X}{2\pi}\sin 2\beta$$

$$= \frac{X}{2} + \frac{X\beta}{\pi} + \frac{2X}{\pi}\left(1 - \tfrac{b}{X}\right)\cos\beta - \frac{X}{2\pi}\sin 2\beta \qquad .....(2.97)$$

On substituting for $(1-b/X)$ from equ(2.91) in equ(2.97) we get,

$$A_1 = \frac{X}{2} + \frac{X\beta}{\pi} + \frac{2X}{\pi}\sin\beta\cos\beta - \frac{X}{2\pi}\sin 2\beta = \frac{X}{2} + \frac{X\beta}{\pi} + \frac{X}{\pi}\sin 2\beta - \frac{X}{2\pi}\sin 2\beta$$

$$= \frac{X}{2} + \frac{X\beta}{\pi} + \frac{X}{2\pi}\sin 2\beta = \frac{X}{\pi}\left(\frac{\pi}{2} + \beta + \tfrac{1}{2}\sin 2\beta\right) \qquad .....(2.98)$$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} y\cos\omega t\, d(\omega t) \qquad .....(2.99)$$

The output, y is given by three different equations in the range 0 to π, hence equ(2.99) can be expressed as,

$$B_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(x - \frac{b}{2}\right)\cos\omega t\, d(\omega t) + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta}\left(X - \frac{b}{2}\right)\cos\omega t\, d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\pi-\beta}^{\pi}\left(x + \frac{b}{2}\right)\cos\omega t\, d(\omega t) \qquad ......(2.100)$$

Put $x = X\sin\omega t$ in equ(2.100)

$$\therefore B_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(X\sin\omega t - \frac{b}{2}\right)\cos\omega t\, d(\omega t) + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\beta}\left(X - \frac{b}{2}\right)\cos\omega t\, d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\pi-\beta}^{\pi}\left(X\sin\omega t + \frac{b}{2}\right)\cos\omega t\, d(\omega t)$$

$$= \frac{X}{\pi}\int_0^{\frac{\pi}{2}} 2\sin\omega t\cos\omega t\, d(\omega t) - \frac{b}{\pi}\int_0^{\frac{\pi}{2}}\cos\omega t\, d(\omega t) + \frac{2}{\pi}\left(X - \tfrac{b}{2}\right)\int_{\frac{\pi}{2}}^{\pi-\beta}\cos\omega t\, d(\omega t)$$

$$+ \frac{X}{\pi}\int_{\pi-\beta}^{\pi} 2\sin\omega t\cos\omega t\, d(\omega t) + \frac{b}{\pi}\int_{\pi-\beta}^{\pi}\cos\omega t\, d(\omega t)$$

$$B_1 = \frac{X}{\pi}\int_0^{\frac{\pi}{2}} \sin 2\omega t\, d(\omega t) - \frac{b}{\pi}\int_0^{\frac{\pi}{2}} \cos\omega t\, d(\omega t) + \frac{2}{\pi}\left(X - \frac{b}{2}\right)\int_{\frac{\pi}{2}}^{\pi-\beta} \cos\omega t\, d(\omega t)$$

$$+\frac{X}{\pi}\int_{\pi-\beta}^{\pi} \sin 2\omega t\, d(\omega t) + \frac{b}{\pi}\int_{\pi-\beta}^{\pi} \cos\omega t\, d(\omega t)$$

$$= \frac{X}{\pi}\left[-\frac{\cos 2\omega t}{2}\right]_0^{\frac{\pi}{2}} - \frac{b}{\pi}\left[\sin\omega t\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left(X - \frac{b}{2}\right)\left[\sin\omega t\right]_{\frac{\pi}{2}}^{\pi-\beta}$$

$$+\frac{X}{\pi}\left[-\frac{\cos 2\omega t}{2}\right]_{\pi-\beta}^{\pi} + \frac{b}{\pi}\left[\sin\omega t\right]_{\pi-\beta}^{\pi}$$

$$= \frac{X}{\pi}\left[-\frac{\cos\pi}{2} + \frac{\cos 0}{2}\right] - \frac{b}{\pi}\left[\sin\frac{\pi}{2} - 0\right] + \frac{2}{\pi}\left(X - \frac{b}{2}\right)\left[\sin(\pi-\beta) - \sin\frac{\pi}{2}\right]$$

$$+\frac{X}{\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 2(\pi-\beta)}{2}\right] + \frac{b}{\pi}\left[\sin\pi - \sin(\pi-\beta)\right]$$

$$= \frac{X}{\pi}\left(\frac{1}{2} + \frac{1}{2}\right) - \frac{b}{\pi}(1-0) + \frac{2}{\pi}\left(X - \frac{b}{2}\right)(\sin\beta - 1) + \frac{X}{\pi}\left(-\frac{1}{2} + \frac{\cos 2\beta}{2}\right) + \frac{b}{\pi}(0 - \sin\beta)$$

$$= \frac{X}{\pi} - \frac{b}{\pi} + \frac{2X}{\pi}\sin\beta - \frac{b}{\pi}\sin\beta - \frac{2X}{\pi} + \frac{b}{\pi} - \frac{X}{2\pi} + \frac{X}{2\pi}\cos 2\beta - \frac{b}{\pi}\sin\beta$$

$$= \frac{X}{\pi}\left(1 - 2 - \frac{1}{2}\right) + \frac{2X}{\pi}\sin\beta - \frac{2b}{\pi}\sin\beta + \frac{X}{2\pi}\cos 2\beta$$

$$= -\frac{3X}{2\pi} + \frac{2X}{\pi}\sin\beta\left(1 - \frac{b}{X}\right) + \frac{X}{2\pi}\cos 2\beta \qquad \text{.....(2.101)}$$

Since $(1-b/X) = \sin\beta$ and $\cos 2\beta = (1-2\sin^2\beta)$, the equ(2.98) can be written as

$$B_1 = -\frac{3X}{2\pi} + \frac{2X}{\pi}\sin\beta\,(\sin\beta) + \frac{X}{2\pi}(1 - 2\sin^2\beta)$$

$$= -\frac{3X}{2\pi} + \frac{2X}{\pi}\sin^2\beta + \frac{X}{2\pi} - \frac{X}{\pi}\sin^2\beta = -\frac{X}{\pi} + \frac{X}{\pi}\sin^2\beta$$

$$= -\frac{X}{\pi} + \frac{X}{\pi}\left(1 - \cos^2\beta\right) = -\frac{X}{\pi} + \frac{X}{\pi} - \frac{X}{\pi}\cos^2\beta$$

$$= -\frac{X}{\pi}\cos^2\beta \qquad \text{.....(2.102)}$$

Note :
$\sin^2\beta + \cos^2\beta = 1$
$\therefore \sin^2\beta = 1 - \cos^2\beta$

$$Y_1 = \sqrt{A_1^2 + B_1^2}$$

$$= \sqrt{\frac{X^2}{\pi^2}\left(\frac{\pi}{2}+\beta+\frac{1}{2}\sin 2\beta\right)^2 + \frac{X^2}{\pi^2}\cos^4\beta}$$

$$\therefore Y_1 = \frac{X}{\pi}\left[\left(\frac{\pi}{2}+\beta+\frac{1}{2}\sin 2\beta\right)^2 + \cos^4\beta\right]^{\frac{1}{2}} \quad .....(2.103)$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}\frac{-\frac{X}{\pi}\cos^2\beta}{\frac{X}{\pi}\left(\frac{\pi}{2}+\beta+\frac{1}{2}\sin 2\beta\right)} \quad .....(2.104)$$

The describing function of backlash nonlinearity is given by,

$$K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 \quad .....(2.105)$$

where Y_1 is given by equ(2.103) and ϕ_1 is given by equ(2.104).

2.8 DESCRIBING FUNCTION ANALYSIS OF NONLINEAR SYSTEMS

The describing functions of nonlinear elements can be used for stability analysis of nonlinear control systems. Also it is used to predict the sustained oscillations or limit cycles in the output of the system.

Consider a unity feedback system shown in fig 2.32 in which the nonlinearity is represented by its describing function, $K_N(X, \omega)$ or K_N.

Note : The describing function, $K_N(X, \omega)$ can be denoted by K_N

Let $C(j\omega)/R(j\omega)$ be the closed loop sinusoidal transfer function of the system shown in fig 2.32.

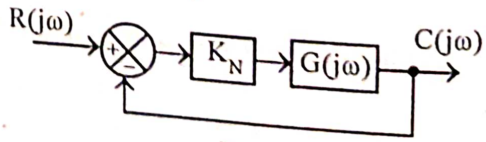

Fig 2.32

$$\therefore \frac{C(j\omega)}{R(j\omega)} = \frac{K_N\, G(j\omega)}{1+K_N\, G(j\omega)} \quad .....(2.106)$$

The characteristic equation of the system is obtained by equating the denominator of equ (2.106) to zero.

Hence the characteristic equation is given by,

$$1+K_N\, G(j\omega) = 0 \quad .....(2.107)$$