$$= d(x^2 y) + d(xz^3)$$

$$d\varphi = d(x^2 y + xz^3)$$

$$\varphi = x^2 y + xz^3 + k$$

(c) Workdone $= \int_C \vec{F} \cdot dr = \int_{(1,-2,1)}^{(3,1,4)} d\varphi = \varphi(x,y,z)\Big|_{(1,-2,1)}^{(3,1,4)}$

$$= \left(x^2 y + xz^3 + k\right)_{(1,-2,1)}^{(3,1,4)} = 201 - (-1) = \underline{\underline{202}}$$

4. (a) S.T. $\vec{F} = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative force field.

(b) Find the scalar potential.

(c) Find the workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

(a) F is conservative if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} 3xz^2 - \frac{\partial}{\partial z} x^2\right) i$$

$$- \left(\frac{\partial}{\partial x} 3xz^2 - \frac{\partial}{\partial z} (2xy+z^3)\right) j$$

$$+ \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} (2xy+z^3)\right) k$$

$$= 0 - (3z^2 - 3z^2) j + (2x - 2x) k$$

$$= \underline{\underline{0}} \Rightarrow F \text{ is Conservative.}$$

Since $\vec{F}$ is conservative $F = \nabla \varphi$

$$\vec{F} \cdot dr = \nabla \varphi \cdot dr = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = d\varphi$$

$$\therefore d\varphi = \vec{F} \cdot dr = (2xy + z^3) dx + x^2 dy + 3xz^2 dz$$

$$= (2xy\,dx + x^2 dy) + (z^3 dx + 3xz^2 dz)$$

$$= \left(\frac{\partial^2\phi}{\partial y \partial z} - \frac{\partial^2\phi}{\partial y \partial z}\right)i - \left(\frac{\partial^2\phi}{\partial x \partial z} - \frac{\partial^2\phi}{\partial x \partial z}\right)$$

$$+ \left(\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial x \partial y}\right)k = 0.$$

$\Rightarrow \vec{F}$ is irrotational

Note:- If $\nabla \times \vec{F} = 0$ then F is conservative.
i.e. $F = \nabla\phi$.

3. Find the total workdone in moving a particle in the force field given by $\vec{F} = zi + zj + xk$ along the curve C given by $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \pi/2$.

$$\text{workdone} = \int_C \vec{F} \cdot dr = \int_C (z\,dx + z\,dy + x\,dz)$$

$$= \int_0^{\pi/2} \left(t(-\sin t\,dt) + t\cos t\,dt + \cos t\,dt \right)$$

$$= \int_0^{\pi/2} \left[-t\sin t + (t+1)\cos t \right] dt$$

$$= \left[-\{-t\cos t + \sin t\} + (t+1)\sin t + \cos t \right]_0^{\pi/2}$$

$$= \left(t\cos t - \cancel{\sin t} + t\sin t + \cancel{\sin t} + \cos t \right)_0^{\pi/2}$$

$$= \pi/2 - 1$$

4. Suppose $\vec{F}$ is conservative field. Prove that $\vec{F}$ is irrotational.

Since $\vec{F}$ is conservative $\vec{F} = \nabla\phi$.

$$\nabla \times \vec{F} = \nabla \times \nabla\phi = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\int_C \vec{A}\cdot dr = \int_{C_1} \vec{A}\cdot dr + \int_{C_2} \vec{A}\cdot dr + \int_{C_3} \vec{A}\cdot dr$$

$$= 1 + 0 + \frac{20}{3} = \underline{\underline{\frac{23}{3}}}$$

(iii) $$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} \Rightarrow x = y = z$$

$$\int_C \vec{A}\cdot dr = \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

$$= \int_0^1 (3x^2 + 6x - 14x^2 + 20x^3)\,dx$$

$$= \left[-\frac{11x^3}{3} + \frac{6x^2}{2} + \frac{20x^4}{4}\right]_0^1$$

$$= -\frac{11}{3} + 3 + 5 = 8 - \frac{11}{3}$$

$$= \underline{\underline{\frac{13}{3}}}.$$

(i) $\int_c \vec{A}\cdot dr = \int_c \left[(3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz\right]$

$= \int_{t=0}^{1} \left((3t^2+6t^2)dt - 14t^2 t^3\, 2t\,dt + 20\, t\, t^6\, 3t^2 dt\right)$

$= \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right) dt$

$= \underline{\underline{5}}$

$x = t$

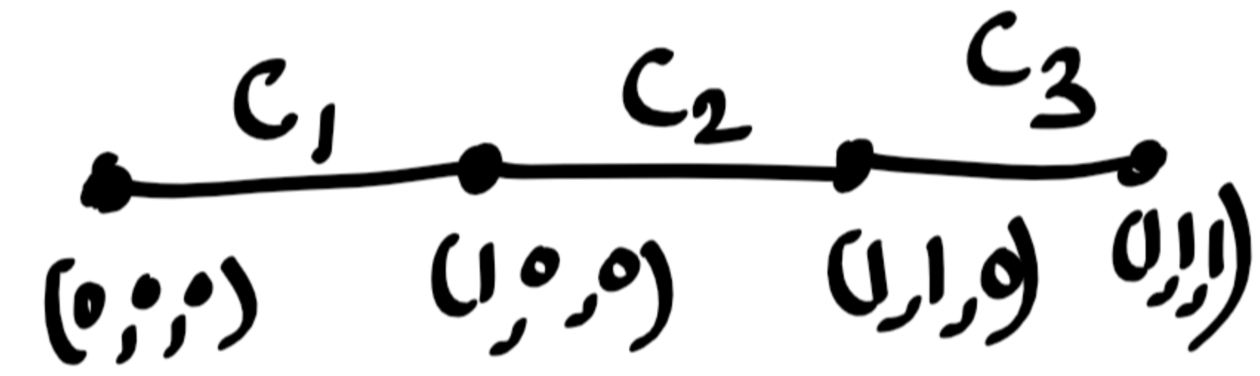
$dx = dt$

$y = t^2$

$dy = 2t\,dt$

$z = t^3, dz = 3t^2 dt$

(ii) C_1: $(0,0,0)$ to $(1,0,0)$



x varies from 0 to 1

$y = z = 0$,

$\int_{C_1} \vec{A}\cdot dr = \int_0^1 3x^2 dx = 1.$

C_2: $(1,0,0)$ to $(1,1,0)$

$x=1$, $z=0$, y varies from 0 to 1.

$\int_{C_2} \vec{A}\cdot dr = 0$

C_3: $(1,1,0)$ to $(1,1,1)$

$x=1$, $y=1$, z varies from 0 to 1.

$\int_{C_3} \vec{A}\cdot dr = 20\int_0^1 z^2 dz = \frac{20}{3}$

$$\left(\nabla\varphi - \vec{F}\right) \cdot \frac{dr}{ds} = 0$$

$$\Rightarrow \nabla\varphi - \vec{F} = 0 \Rightarrow \underline{\underline{\vec{F} = \nabla\varphi}}.$$

<u>Note</u>:- If $\int_{P_1}^{P_2} \vec{F} \cdot dr$ is independent of the path C joining P_1 and P_2, then $\vec{F}$ is called a conservative field and φ is called its scalar potential.

2. Suppose $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\,\hat{j} + 20xz^2\,k$.
Evaluate $\int_C \vec{A} \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

(i) $x = t, \quad y = t^2, \quad z = t^3$

(ii) the straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$ and then to $(1,1,1)$.

(iii) the straight line joining $(0,0,0)$ to $(1,1,1)$.

there exists a function φ such that $F = \nabla\phi$.

(a) Workdone $= \int_C \vec{F}\cdot dr = \int_{P_1}^{P_2} \nabla\phi\cdot dr$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x} i + \frac{\partial\varphi}{\partial y} j + \frac{\partial\varphi}{\partial z} k\right)\cdot(dx\,i + dy\,j + dz\,k)$$

$$= \int_{P_1}^{P_2} \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = \int_{P_1}^{P_2} d\varphi$$

$$= \varphi(x,y,z)\Big|_{P_1}^{P_2} = \varphi(x_2,y_2,z_2) - \varphi(x_1,y_1,z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them.

(b) Let $\vec{F} = F_1 i + F_2 j + F_3 k$. Given $\int_C \vec{F}\cdot dr$ is independent of the path C joining any two points (x_1,y_1,z_1) and (x_2,y_2,z_2).

Let $\varphi(x,y,z) = \int_C \vec{F}\cdot dr = \int_C \vec{F}\cdot\frac{dr}{ds}ds$

$$\frac{d\varphi}{ds} = \vec{F}\cdot\frac{dr}{ds}$$

$$\nabla\varphi\cdot\frac{dr}{ds} = \vec{F}\cdot\frac{dr}{ds}$$

Vector Integration

Line integrals

Any integral that is to be evaluated along a curve is called a line integral.

Let $\vec{A}(x,y,z) = A_1 i + A_2 j + A_3 k$ be a vector function defined along a curve C from P_1 to P_2

$$\int_{P_1}^{P_2} \vec{A}\cdot dr = \int_C \vec{A}\cdot dr = \int_C (A_1 dx + A_2 dy + A_3 dz)$$

Note:- If $\vec{A}$ is the Force on a particle moving along C then $\int_C \vec{A}\cdot dr$ represents workdone by the force.

Examples

1. (a) Suppose $\vec{F} = \nabla\phi$ where ϕ is scalar. Show that the workdone in moving a particle from one point $P_1(x_1, y_1, z_1)$ to another point $P_2(x_2, y_2, z_2)$ is independent of the path joining the two points.

(b) Conversely, suppose $\int_C \vec{F}\cdot dr$ is independent of the path C joining any two points show that