MANIPAL INSTITUTE OF TECHNOLOGY
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Modern Control Theory (ICE 3153)

State Space Modeling of Mechanical systems

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- The mechanical system involves two types of motion,

- one is called translation motion
- and other is called a rotational motion.

- Mechanics of translational motion are-

  Mass   Spring   Damper

Mechanics of rotational motion are-

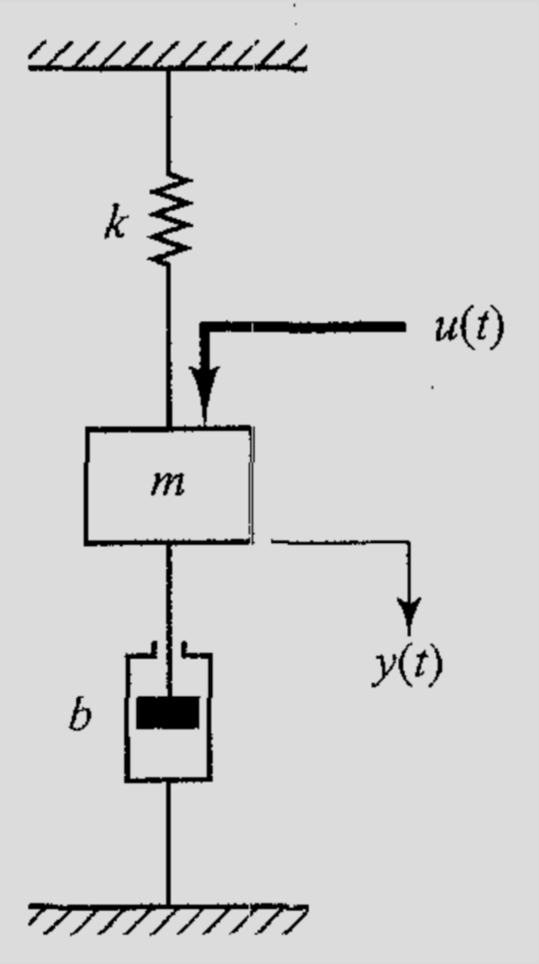
  Inertia   Spring   Damper

Example-1

- Consider the mechanical system shown in Figure. We assume that the system is linear. The external force u(t) is the input to the system, and the displacement y(t) of the mass is the output.

- The displacement y(t) is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

- Obtain the state space model of the system.



<u>Step 1</u>: Understand the physics of the system

It's a mechanical system, governing equation can be written based on Newton's law.

<u>Step 2</u>: Identify the input and output of the system

Input – External force U(t)
Output – Displacement Y(t)

<u>Step 3</u>: Write the differential equation system

$$m\ddot{y} + b\dot{y} + ky = u$$

<u>Step 4</u>: Identify the states of the system

It's a second order system, so two states required to define the system.

Two integrators will be present in the system and the output of the integrators are the states – Position and Velocity.

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Step 5: Write the state equation and Output Equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

$$y = x_1$$

$$\dot{x}_1 = x_2$$

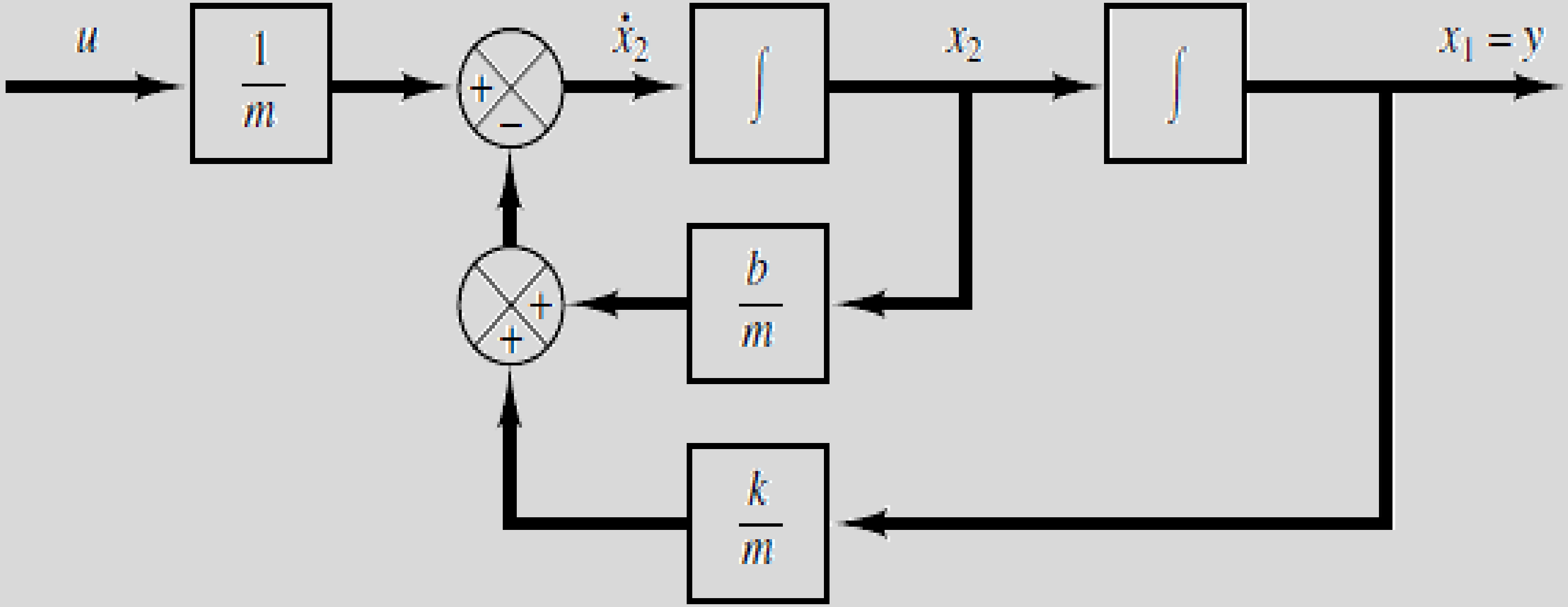
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

Step 6: Write the State space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

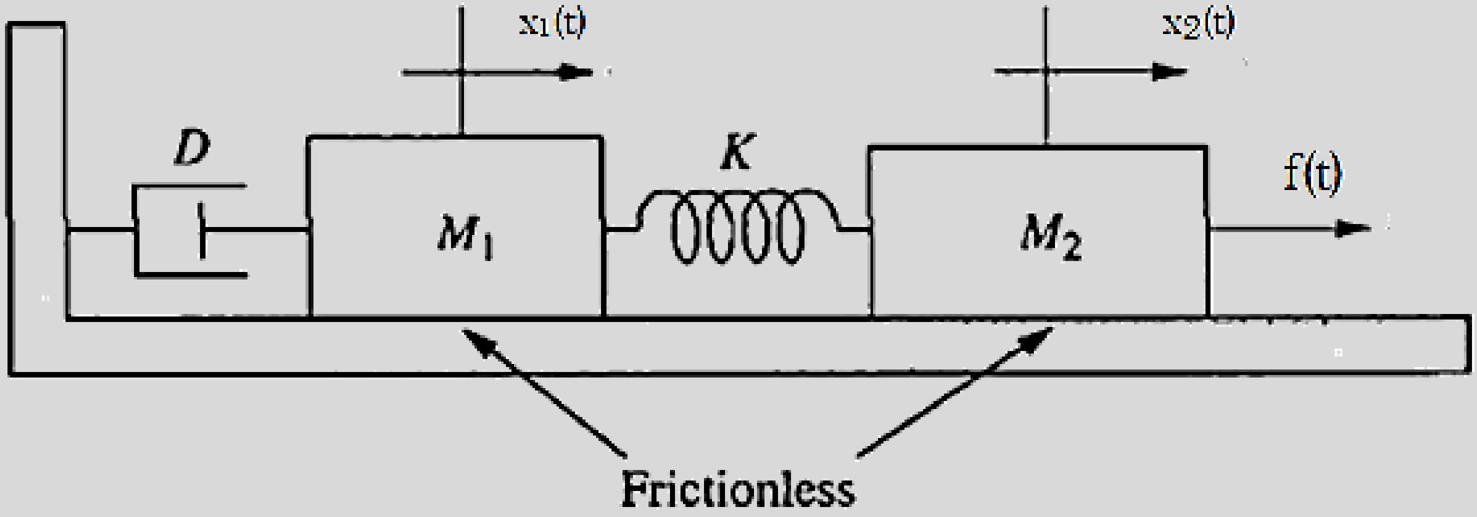
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0], \qquad D = 0$$



Example 2:

Construct the state space model of mechanical system shown in figure.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

Correlation Between Transfer Functions and State-Space Equations

- Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s)$$

- This system may be represented in state space by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

- The Laplace transforms of the state and o/p equations are given by,

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

- Considering initial conditions to zero for the TF,

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s) \qquad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

- By pre-multiplying **(sI** - **A)^-1** to both sides of this last equation, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

- Substitute the above equation into the o/p equation,

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

- On comparing the above equation with TF defined before,

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

This is the transfer-function expression of the system in terms of **A**, **B**, **C**, and **D**

$$\frac{Y(s)}{U(s)} = G(s) = C\frac{Adj[sI - A]}{\det[sI - A]}B + D$$

$$G(s) = \frac{Q(s)}{|s\mathbf{I} - \mathbf{A}|}$$

where Q(s) is a polynomial in s

$|sI - A|$ is equal to the characteristic polynomial of A. In other words, the eigenvalues of A are identical to the poles of G(s).

Transfer Matrix

- consider a multiple-input-multiple-output system. Assume that there are r inputs m outputs.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

The transfer matrix G(s) relates the output Y(s) to the input U(s), or

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

where G(s) is given by

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Example-3 Consider the mechanical model in Example-1 and it state model.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

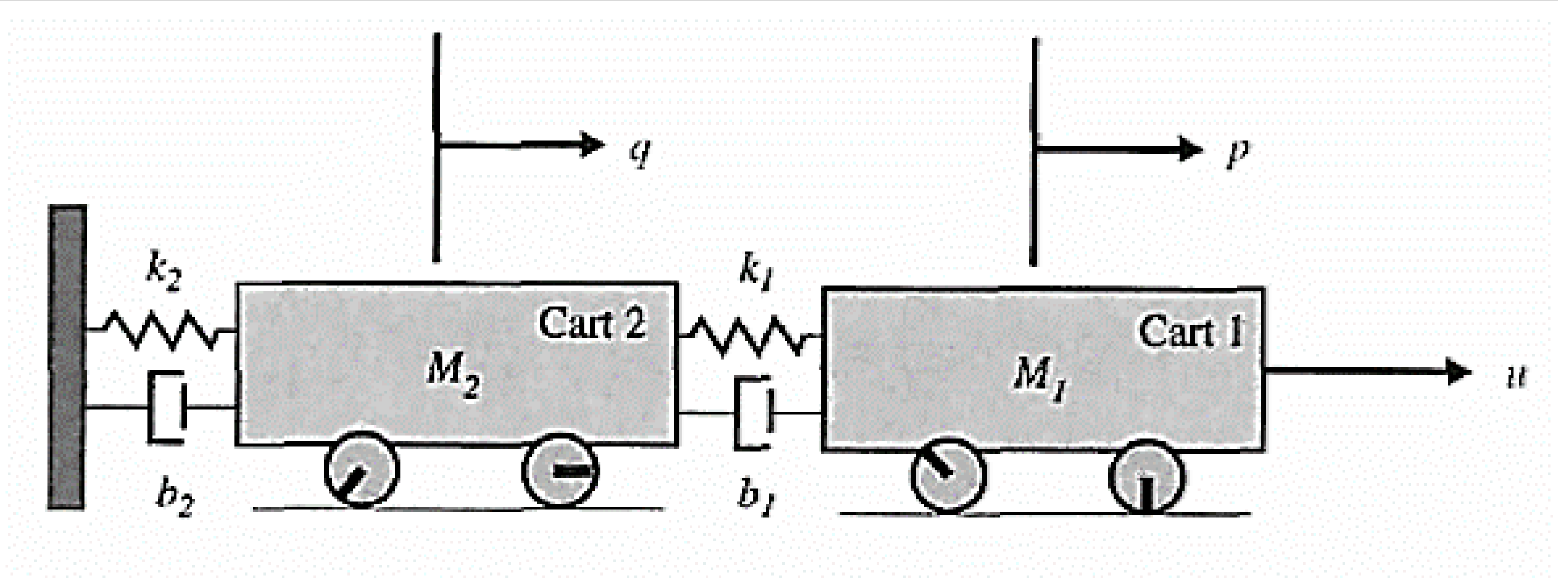
$$G(s) = [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

Tutorial -1

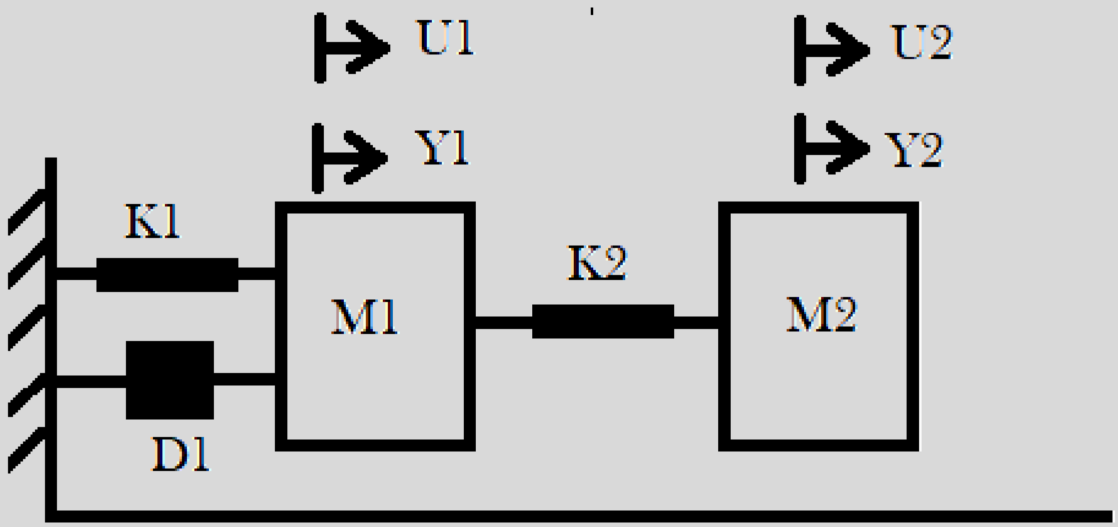
Question 1:

Consider the system shown in figure. The variables of interest are noted on the figure and defined as: M_1 M_2 = mass of carts, p, q = position of carts, u = external force acting on system, k_1, k_2 = spring constants, and b_1 b_2 = damping coefficients. We assume that the carts have negligible rolling friction. We consider any existing rolling friction to be lumped into the damping coefficients, b_1 and b_2.

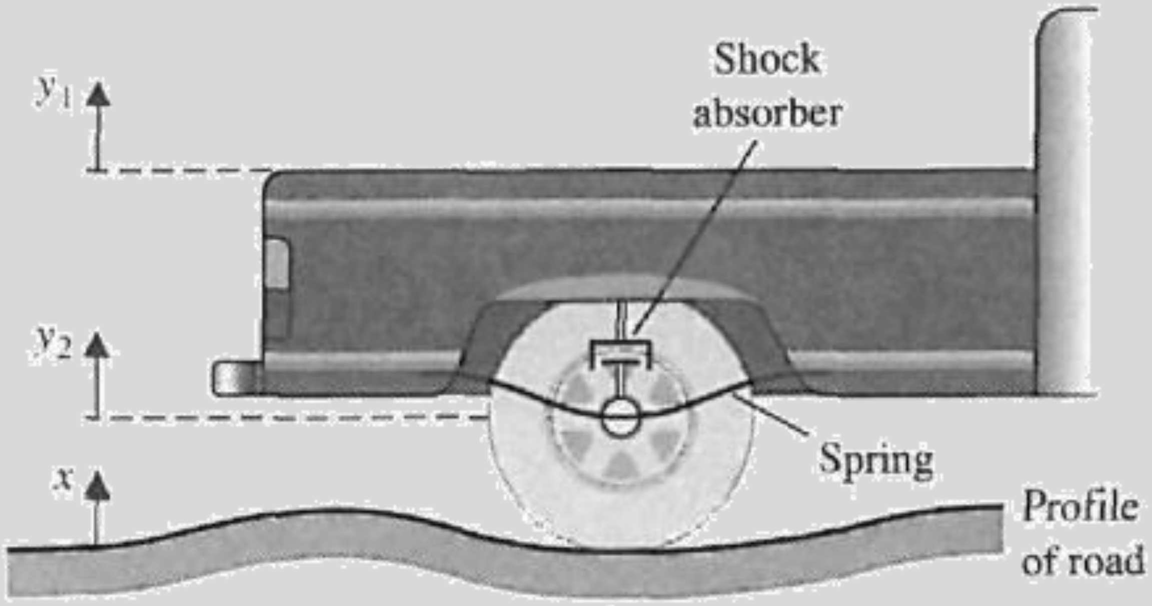


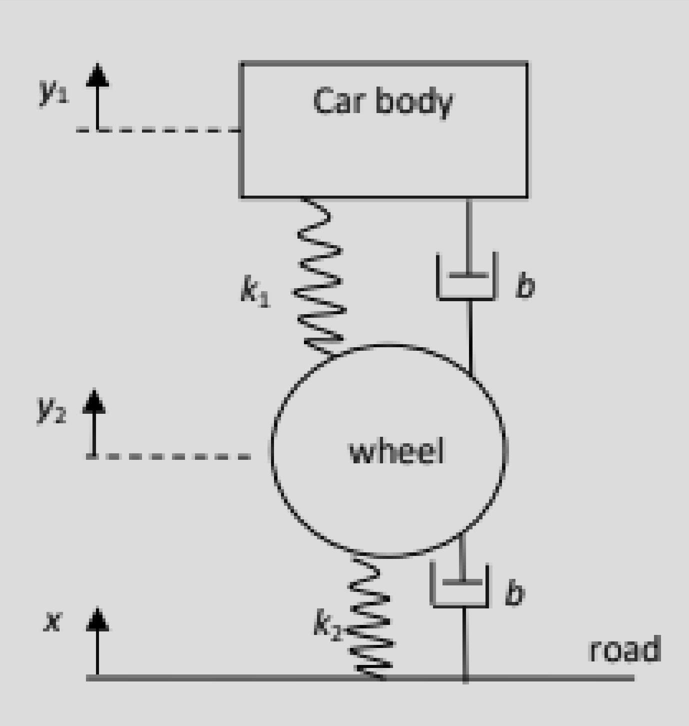
Question 2:

Construct the state space model of mechanical system shown in figure.



Question 3:



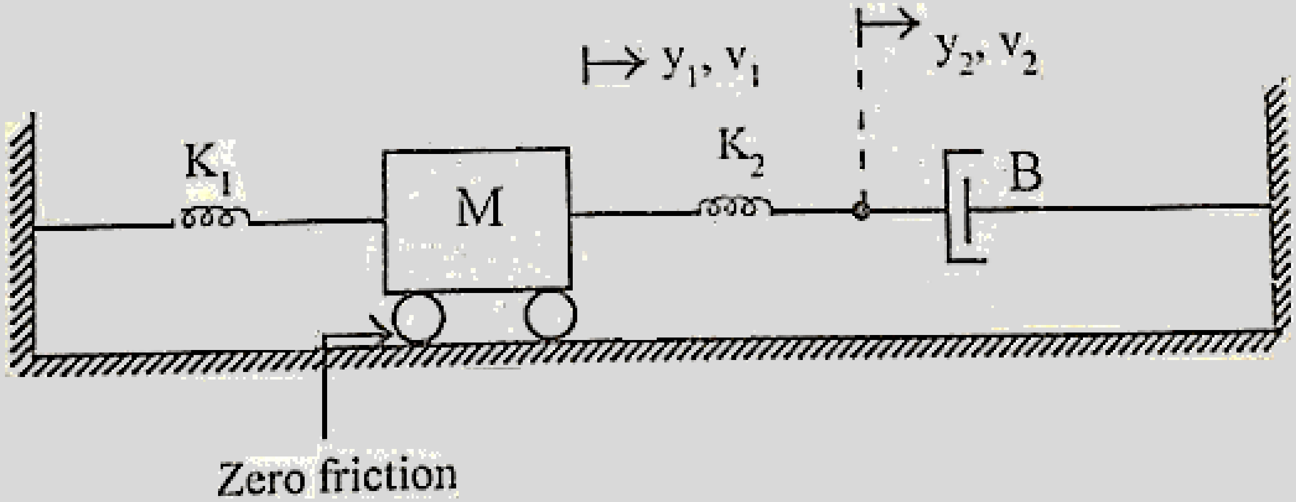


The suspension system for one wheel of a pickup truck is illustrated in Figure. The mass of the vehicle is m1 and the mass of the wheel is m2. The suspension spring has a spring constant k1 and the tire has a spring constant k2. The damping constant of the shock absorber is b. Obtain the state space model of the system considering vehicle body movement as output.

Question 4:

Write the differential equation for the following mechanical systems.

(i)



Question 5:- Obtain the transfer function of the system defined by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)^2(s+2)} \\ 0 & \dfrac{1}{s+1} & \dfrac{1}{(s+1)(s+2)} \\ 0 & 0 & \dfrac{1}{s+2} \end{bmatrix}$$

$$G(s) = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

Question 6:- Obtain the transfer function of the system defined by,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$