Inverse Interpolation:

Given a set of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying $y = f(x)$, where explicit nature of $f(x)$ is not known, the process of finding the value of x, for a given a value of $y \in (y_0, y_n)$ is called inverse interpolation. Formula for inverse interpolation can be obtained by interchanging the roles of x & y in Lagrange's interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_1-y_2)\cdots(y_1-y_n)}x_0 + \cdots\cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}x_n$$

1. If $y_1 = 4, y_3 = 12, y_4 = 19$ and $y_x = 7$, find x.

x :	1	3	4
y :	4	12	19

$y = 7$, $x = ?$

$$x = \frac{(y-12)(y-19)}{(4-12)(4-19)} \times 1 + \frac{(y-4)(y-19)}{(12-4)(12-19)} 3 + \frac{(y-4)(y-12)}{(19-4)(19-12)} \cdot 4$$

$$= \frac{(7-12)(7-19)}{(-8)(-15)} + \frac{(7-4)(7-19)}{(8)(-7)} 3 + \frac{(7-4)(7-12)}{(15)(7)} \cdot 4$$

$$= 0.5 + 1.92857 - 0.571428 = 1.85714$$

2. Find a root of the equation $f(x) = 0$ given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, $f(42) = 18$

x :	30	34	38	42
y :	-30	-13	3	18

$f(x) = y = 0$

$x = ?$

$$x = \frac{(y+13)(y-3)(y-18)}{(-30+13)(-30-3)(-30-18)} \times 30 + \frac{(y+30)(y-3)(y-18)}{(-13+30)(-13-3)(-13-18)} 34$$

$$+ \frac{(y+30)(y+13)(y-18)}{(3+30)(3+13)(3-18)} 38 + \frac{(y+30)(y+13)(y-3)}{(18+30)(18+13)(18-3)} \times 42$$

$$= 37.2303781.$$

3. Using Lagrange's interpolation formula, find [redacted] the function y (x) from the following table

X	0	1	3	4
Y	-12	0	12	24

For $f(x)=0$

$x=\alpha$ is a root $\Leftrightarrow$ $x-\alpha$ is factor of $f(x)$

<u>solution</u>: When $x=1$, we have $y=0$.

$\therefore$ $x-1$ is a factor of $y(x)$.

$\therefore$ $y(x) = (x-1)\,z(x)$ —— ①

$z(x) = \dfrac{y(x)}{x-1}$,

$z(0) = \dfrac{y(0)}{0-1} = \dfrac{-12}{-1} = 12$

$z(3) = \dfrac{y(3)}{3-1} = \dfrac{12}{2} = 6$

$z(4) = \dfrac{y(4)}{4-1} = \dfrac{24}{3} = 8$

	x_0	x_1	x_2
x :	0	3	4
$z(x)$:	+12	6	8
	z_0	z_1	z_2

$$z = \frac{(x-3)(x-4)}{(0-3)(0-4)}\cdot(+12) + \frac{(x-0)(x-4)}{(3-0)(3-4)}(6) + \frac{(x-0)(x-3)}{(4-0)(4-3)}(8)$$

$$= (x^2-7x+12)\cdot(+1) + (x^2-4x)(-2) + (x^2-3x)(2)$$

$$= x^2-5x+12$$

$\therefore$ from ①

$$y = (x-1)(x^2-5x+12)$$

<u>Verification</u>:

$y(0) = (-1)(+12) = -12$ ✓

$y(1) = 0$ ✓

$y(3) = (3-1)(9-15+12) = 12$ ✓

$y(4) = (4-1)(16-20+12) = 24$ ✓

① Certain corresponding of x and $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ and $(307, 2.4871)$. Find $\log_{10} 301$

$x_0 = 300$ $\quad y_0 = 2.4771$

$x_1 = 304$ $\quad y_1 = 2.4829$

$x_2 = 305$ $\quad y_2 = 2.4843$

$x_3 = 307$ $\quad y_3 = 2.4871$

$x = 301, \quad y = ?$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \cdots + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}(2.4771) + \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}(2.4829)$$

$$+ \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)} \times (2.4843) + \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} \times 2.4871$$

$$= 1.27393 + 4.96580 - 4.47174 + 0.71057$$

$$\log_{10} 301 = 2.4785$$

Actual value: $\log_{10} 301 = 2.478566$

② Find the Lagrange interpolating polynomial of degree 2 approximating the function $y = \ln x$ defined by the following table of values. Hence determine the value of $\ln 2.7$.

x:	2	2.5	3.0
$\ln x$:	0.69315	0.91629	1.09861

$x_0 = 2, \quad x_1 = 2.5, \quad x_2 = 3.0$

$y_0 = 0.69315, \quad y_1 = 0.91629, \quad y_2 = 1.09861$

$$y = \frac{(x-2.5)(x-3)}{(2-2.5)(2-3)}(0.69315) + \frac{(x-2)(x-3)}{(2.5-2)(2.5-3)}(0.91629) + \frac{(x-2)(x-2.5)}{(3-2)(3-2.5)}(1.09861)$$

$$= \frac{(x^2-5.5x+7.5)(0.69315)}{(-0.5)(-1)} + \frac{(x^2-5x+6)(0.91629)}{(0.5)(-0.5)} + \frac{(x^2-4.5x+5)(1.09861)}{(1)(0.5)}$$

$$= (x^2-5.5x+7.5)(1.3863) + (x^2-5x+6)(-3.66516) + (x^2-4.5x+5)(2.19722)$$

$$y = -0.08164\, x^2 + 0.81366\, x - 0.60761$$

$$\ln 2.7 = (-0.08164)(2.7)^2 + (0.81366)(2.7) - 0.60761$$

$$= 0.994164$$

Actual value $= \ln 2.7 = 0.99325$

Interpolation with unequally spaced points

Lagrange's Interpolation Formula:

Given a set of points $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ satisfying $y = f(x)$, where f is not known explicitly, values of x not necessarily equally spaced, the n^{th} degree $y_n(x)$ such that $y_n(x)$ and $f(x)$ agree at the tabulated values is given by

$$y_n(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n \quad \text{——— } (1)$$

Proof: Since $y_n(x)$ is a polynomial of degree n, we can write

$$y_n(x) = a_0(x-x_1)(x-x_2)\cdots\cdots(x-x_n) + a_1(x-x_0)(x-x_2)\cdots\cdots(x-x_n)$$

$$+\cdots\cdots + a_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \quad \text{——— } (2)$$

$y_n(x_0) = y_0$ gives

$$a_0(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n) + 0 + \cdots + 0 = y_0$$

$$\therefore a_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 \quad \text{——— } (3)$$

$y_n(x_1) = y_1$ gives

$$0 + a_1(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n) + \cdots + 0 = y_1$$

$$\therefore a_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}y_1 \quad \text{——— } (4)$$

$\vdots$

Similarly

$$a_n = \frac{1}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}y_n \quad \text{——— } (*)$$

Substituting (3), (4), $\cdots$ (*) in (2), we get (1).

(1) is called Lagrange's Interpolation formula.