(iii) $|z| > 3 \Rightarrow 1 > \left|\frac{3}{z}\right|$

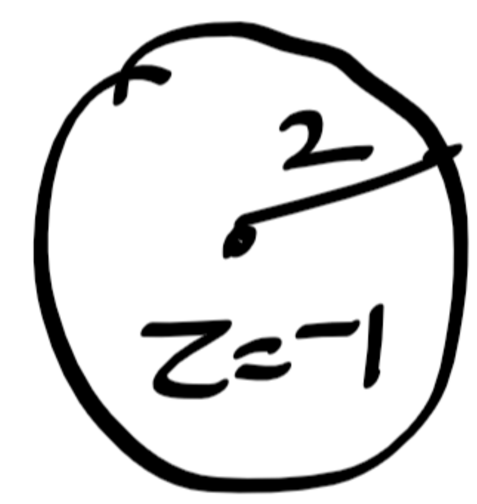
$|z| > 1 \Rightarrow 1 > \left|\frac{1}{z}\right|$

$$f(z) = \frac{1}{2}\left\{\frac{1}{z(1+1/z)} - \frac{1}{z(1+3/z)}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 - \cdots\right) - \frac{1}{z}\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2 - \cdots\right)\right\}$$

$$= \frac{1}{2z}\left\{\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 - \cdots\right) - \left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2 - \cdots\right)\right\}$$

(iv) $|z+1| < 2 \Rightarrow \left|\frac{z+1}{2}\right| < 1$



$$f(z) = \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+3}\right\}$$

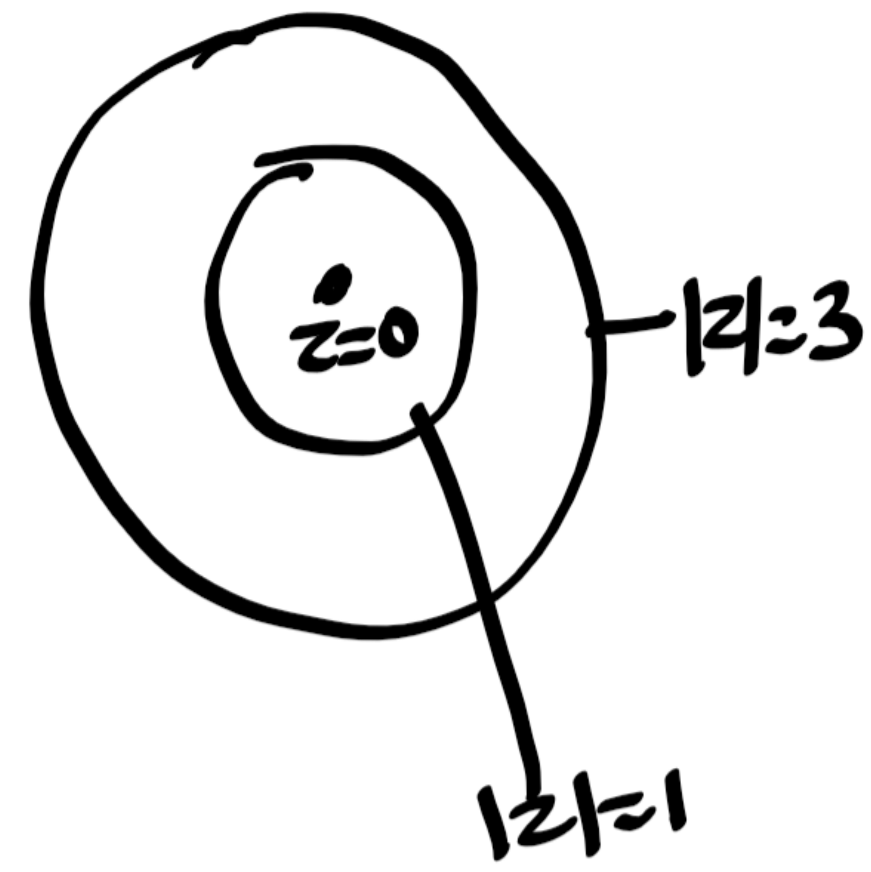
$$= \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+1+2}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{2\left(1+\frac{z+1}{2}\right)}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{2}\left(1-\frac{z+1}{2}+\left(\frac{z+1}{2}\right)^2 - \cdots\right)\right\}$$

2) Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in the region

(i) $|z|<1$, (ii) $1<|z|<3$, (iii) $|z|>3$, (iv) $|z+1|<2$



$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$A(z+3) + B(z+1) = 1$$

$$z=-3 \Rightarrow -2B = 1 \Rightarrow B = -1/2$$

$$z=-1 \Rightarrow 2A = 1 \Rightarrow A = \tfrac{1}{2}$$

$$f(z) = \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+3}\right\}$$

(i) $|z|<1$ then $|z|<3 \Rightarrow |z/3|<1$

$$\therefore f(z) = \frac{1}{2}\left\{(1+z)^{-1} - \frac{1}{3}(1+z/3)^{-1}\right\}$$

$$= \frac{1}{2}\left\{1 - z + z^2 - \cdots - \frac{1}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right)\right\}$$

(ii) $1<|z|<3$

$|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$

$|z|>1 \Rightarrow \frac{1}{|z|}<1$

$$f(z) = \frac{1}{2}\left\{\frac{1}{z(1+1/z)} - \frac{1}{3(1+z/3)}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{z}\left(1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \cdots\right) - \frac{1}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right)\right\}$$

Examples

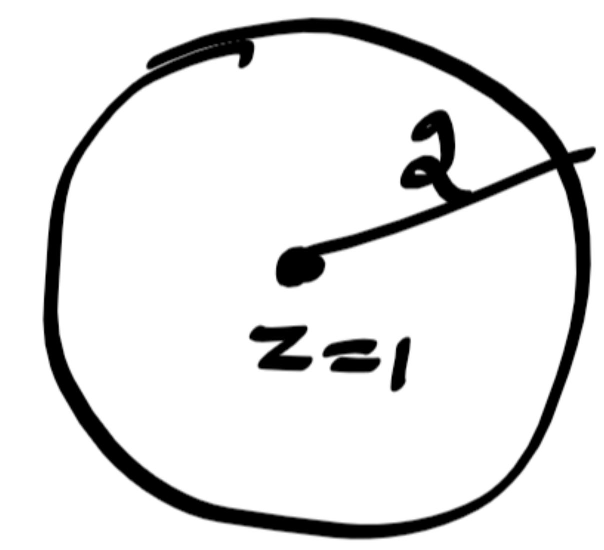
1. Find all Laurent's series expansions of

$$f(z) = \frac{1}{1-z^2} \quad \text{with centre at } z = 1.$$

$$f(z) = \frac{1}{1-z^2} = \frac{1}{(1+z)(1-z)} = \frac{1}{2}\left[\frac{1}{1+z} + \frac{1}{1-z}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z-1}\right]$$

$$\therefore f(z) = \frac{1}{2}\left\{\frac{1}{z-1+2} - \frac{1}{z-1}\right\}$$



$$= \frac{1}{2}\left\{\frac{1}{2\left[1+\frac{z-1}{2}\right]} - \frac{1}{z-1}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left(1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \cdots\right) - \frac{1}{z-1}\right\}, \quad |z-1| < 2$$

$$= \frac{1}{2}\left\{\frac{1}{(z-1)\left[1+\frac{2}{z-1}\right]} - \frac{1}{z-1}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{(z-1)}\left(1 - \frac{2}{z-1} + \left(\frac{2}{z-1}\right)^2 - \cdots\right) - \frac{1}{z-1}\right\}$$

$$|z-1| > 2$$

Laurent's Theorem

If $f(z)$ is analytic in the ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ and with centre at $z=a$, then for all z in R

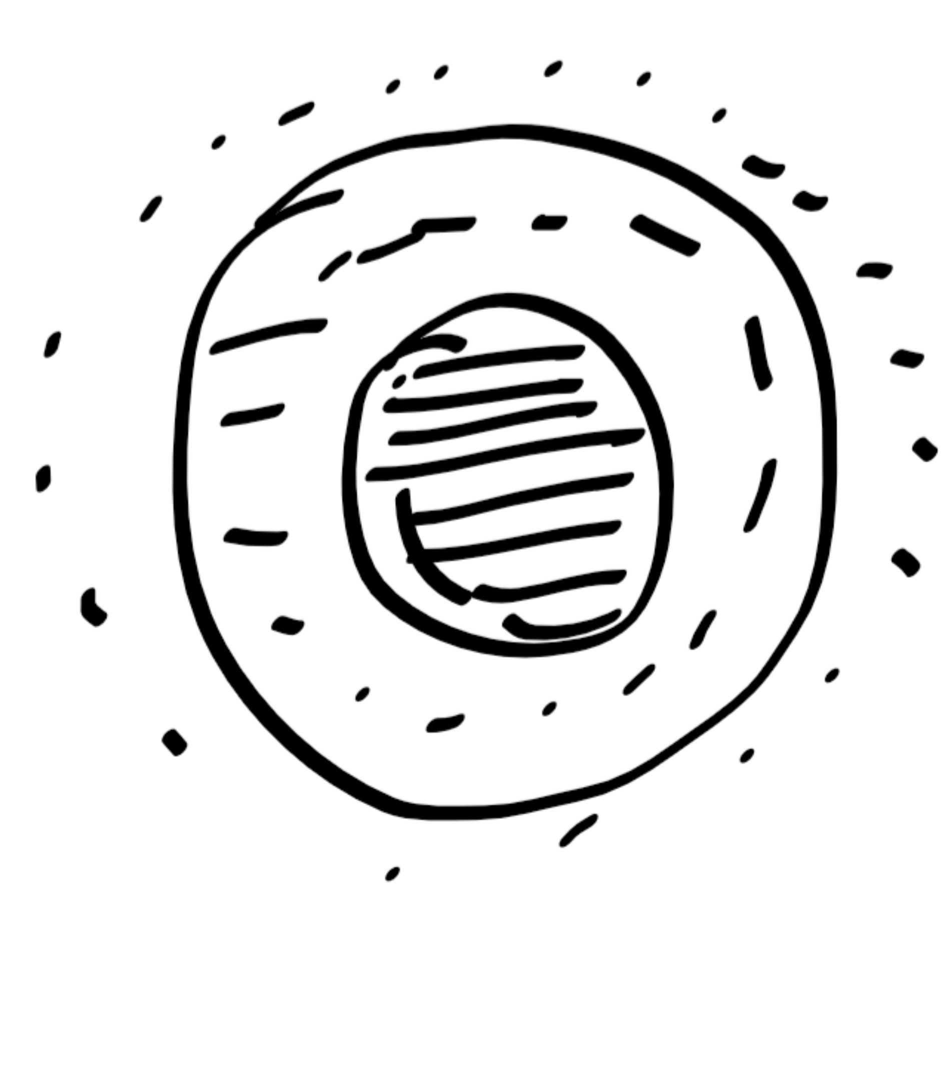
$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ b_1(z-a)^{-1} + b_2(z-a)^{-2} + b_3(z-a)^{-3} + \cdots$$

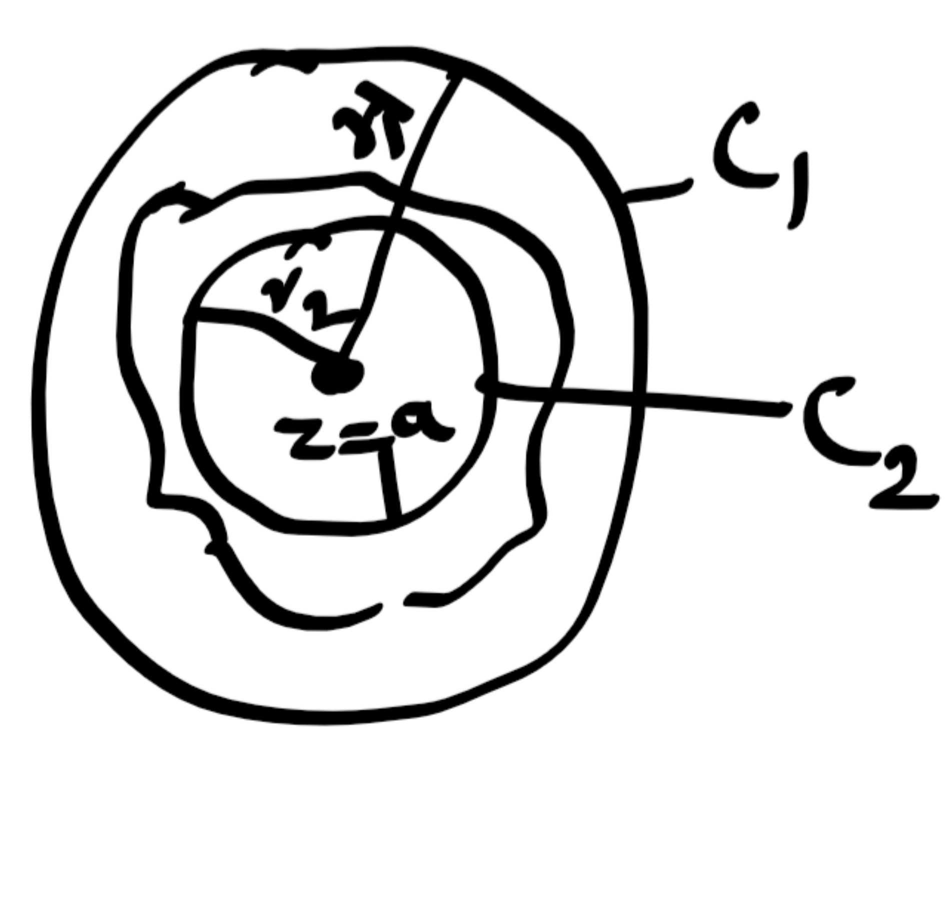
where $a_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\,dz$ and $b_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)\,dz}{(z-a)^{-n+1}}$

Γ is any curve in R encircling C_2.

Here, $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the analytic part of $f(z)$ and $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called principal part of $f(z)$.

If $b_n = 0$ then Laurent's series reduces to Taylor's Series





(ii) At $z=1$

$$f(z)=\frac{1}{5}\left\{\frac{1}{z-1-2}-\frac{1}{z-1+3}\right\}$$

$$=\frac{1}{5}\left\{-\frac{1}{2\left(1-\frac{z-1}{2}\right)}-\frac{1}{3\left(1+\frac{z-1}{3}\right)}\right\}$$

$$=-\frac{1}{5}\left\{\frac{1}{2}\left(1+\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\cdots\right)+\frac{1}{3}\left(1-\frac{z-1}{3}+\left(\frac{z-1}{3}\right)^2-\cdots\right)\right\}$$

$|z-1|<2$

$|z-1|<2$

$|z-1|<3$

(2) Expand $f(z) = \dfrac{1}{z^2 - z - 6}$ about the point

(i) $z = -1$

(ii) $z = 1$

$$\frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)}$$

$$= \frac{1}{5}\left[\frac{1}{z-3} - \frac{1}{z+2}\right]$$

$A(z+2) + B(z-3) = 1$

$z = 3 \Rightarrow A = 1/5$

$z = -2 \Rightarrow B = -1/5$

(i) $z = -1$

$$f(z) = \frac{1}{5}\left[\frac{1}{z+1-4} - \frac{1}{z+1+1}\right]$$

$$= \frac{1}{5}\left\{-\frac{1}{4\left(1-\frac{z+1}{4}\right)} - \frac{1}{1+(z+1)}\right\}$$

$$= -\frac{1}{5}\left\{\frac{1}{4}\left[1 + \frac{z+1}{4} + \left(\frac{z+1}{4}\right)^2 + \dots + 1 - (z+1) + (z+1)^2 - \dots\right]\right\}$$

$|z+1| < 1$

$$\left\{\begin{array}{l} \left(1-\frac{z+1}{4}\right)^{-1} \text{ converges for } \left|\frac{z+1}{4}\right| < 1 \Rightarrow |z+1| < 4 \\ (1+(z+1))^{-1} \text{ converges for } |z+1| < 1 \end{array}\right\}$$

Taylor's series expansion of $f(z)$

1) Expand $f(z) = \dfrac{2z^2+15z+34}{(z+4)^2(z-2)}$

$$\frac{2z^2+15z+34}{(z+4)^2(z-2)} = \frac{A}{z+4} + \frac{B}{(z+4)^2} + \frac{C}{z-2}$$

$$A(z+4)(z-2) + B(z-2) + C(z+4)^2 = 2z^2+15z+34$$

$$z=-4 \Rightarrow -6B = 32 - 60 + 34 = 66 - 60 = 6$$

$$\Rightarrow B = -1$$

$$z = 2 \Rightarrow 36C = 8 + 30 + 34 = 72 \Rightarrow C = 2$$

$$z=0 \Rightarrow -8A - 2B + 16C = 34$$

$$\Rightarrow -8A + 2 + 32 = 34 \Rightarrow -8A = 0 \Rightarrow A = 0$$

$$f(z) = \frac{2}{z-2} - \frac{1}{(z+4)^2} = -\frac{1}{(1-z/2)} - \frac{1}{16(1+z/4)^2}$$

$$= -\left\{ (1-z/2)^{-1} + \frac{1}{16}(1+z/4)^{-2} \right\}$$

$$= -\left\{ 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots + \frac{1}{16}\left(1 - 2\left(\frac{z}{4}\right) + 3\left(\frac{z}{4}\right)^2 - \cdots\right) \right\}$$

$|z| < 2$

$(1-z/2)^{-1}$ converges for

$|z/2| < 1 \Rightarrow |z| < 2$

$(1+z/4)^{-2}$ converges

for $|z/4| < 1$

$\Rightarrow |z| < 4$