Lecture - 3, Indeterminate forms (Contd----)

Some standard Maclaurin's series

1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Proof

$$\begin{aligned}
y &= \sin x & y(0) &= \sin 0 = 0 \\
y_1 &= \cos x & y_1(0) &= \cos 0 = 1 \\
y_2 &= -\sin x & y_2(0) &= -\sin 0 = 0 \\
y_3 &= -\cos x & y_3(0) &= -\cos 0 = -1 \\
y_4 &= \sin x & y_4(0) &= \sin 0 = 0 \\
y_5 &= \cos x & y_5(0) &= \cos 0 = 1
\end{aligned}$$

Maclaurin's series, $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

2) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

4) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

5) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

7) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

8) $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right]$

9) $e^{\sin x} =$

(Sub x as $\sin x$ in (6))

$$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \cdots$$

$$= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \cdots$$

Indeterminate forms (Contd ---)

II $(0 \cdot \infty)$ form

If $\lim_{x \to a} \{f(x) \cdot g(x)\}$ is $0 \cdot \infty$, then write

$f(x) \cdot g(x) = \dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{g(x)}{1/f(x)}$ and then apply L'Hospital's Rule.

Problems

1) $\lim_{x \to 1} (1 - x^2) \tan\left(\frac{\pi x}{2}\right)$

Solution $\lim_{x \to 1} (1 - x^2) \tan\left(\frac{\pi x}{2}\right)$ $[0 \cdot \infty)$

$$\lim_{x \to 1} \frac{1 - x^2}{\cot\left(\frac{\pi x}{2}\right)} \left(\frac{0}{0}\right)$$

L'H Rule $\quad \underset{x \to 1}{Lt} \dfrac{-2x}{-\operatorname{cosec}^2\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}}$

$$\frac{-2 \cdot 2}{-\operatorname{cosec}^2(\pi/2) \cdot \pi} = \frac{4}{\pi}$$

2) $\underset{x \to a}{Lt} \log\left(2 - \frac{x}{a}\right) \cot(x - a)$

Solution $\quad \log(1)(\cot 0) \quad (0 \cdot \infty)$

$$\underset{x \to a}{Lt} \frac{\log\left(2 - \frac{x}{a}\right)}{\tan(x - a)} \left(\frac{0}{0}\right)$$

Applying L'H Rule $\quad \underset{x \to a}{Lt} \dfrac{\frac{1}{2 - \frac{x}{a}} \cdot (-1/a)}{\sec^2(x - a)} = \dfrac{1\,(-1/a)}{1} = \dfrac{-1}{a}$

3) $\underset{x \to \infty}{Lt} \left(a^{1/x} - 1\right) x$

Solution $\underset{x \to \infty}{Lt} \left(a^{\frac{1}{x}} - 1\right) x \quad \left[(a^0 - 1) \cdot \infty\right] \; [0 \cdot \infty]$

$$\underset{x \to \infty}{Lt} \frac{a^{1/x} - 1}{1/x} \left(\frac{0}{0}\right)$$

Put $\frac{1}{x} = t$

$x \to \infty, t \to 0$

$$\underset{t \to 0}{Lt} \frac{a^t - 1}{t} \left(\frac{0}{0}\right)$$

L'H Rule $\underset{t \to 0}{Lt} \frac{a^t \log a - 0}{1}$

$$= a^0 \log a = \underline{\log a}$$

H.W

4) $\underset{x \to 0}{Lt}\; x \log x$

5) $\underset{x \to a}{Lt}\; (a - x) \tan \frac{\pi x}{2a}$

6) $\underset{x \to 0}{Lt} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

Solution $(0/0)$ form

$$\underset{x \to 0}{Lt} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x - x^2}{x^2 + x\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right]}$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{x} + \cancel{x^2} + 1\left(-\frac{x^3}{3!}\right) + \frac{x^2}{2!}(x) + \cdots - \cancel{x} - \cancel{x^2}}{\cancel{x^2} - \cancel{x^2} - \frac{x^3}{2} - \frac{x^4}{3} - \cdots}$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{x^3}{3!} + \frac{x^3}{2!}}{-x^3/2} = \frac{-\cancel{x^3}\left[\frac{1}{6} - \frac{1}{2}\right]}{-\cancel{x^3}/2} = \frac{-2/6}{1/2} = \underline{\frac{-4}{6}}$$

7) $\lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sinh x}{x}\right)$

Solution $\lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sinh x}{x}\right) \left(\frac{0}{0}\right)$

$$\lim_{x \to 0} \frac{\log\left[\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x}\right]}{x^2}$$

$$\lim_{x \to 0} \frac{\log\left[1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots\right]}{x^2} \left(\frac{0}{0}\right)$$

L'H Rule $$\lim_{x \to 0} \frac{1}{1 + \frac{x^2}{6} + \frac{x^4}{120}} \frac{\left[\frac{2x}{6} + \frac{4x^3}{120} + \cdots\right]}{2x}$$

$$\lim_{x \to 0} \left(\frac{1}{1 + \frac{x^2}{6} + \cdots}\right) \frac{1}{2}\left[\frac{1}{3} + \frac{4x^2}{120} + \cdots\right]$$

$$1\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$$

III $(\infty - \infty)$ form

If $\lim_{x \to a} [f(x) - g(x)]$ is $(\infty - \infty)$ form, then write

$$\lim_{x \to a} (f(x) - g(x)] = \lim_{x \to a} \left\{\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{g(x) f(x)}}\right\}$$

Problems

1) $\lim_{x \to 0} \left\{\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right\}$

Solution $\lim_{x \to 0} \left(\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right] \; [\infty - \infty]$

$$\lim_{x \to 0} \left[\frac{a}{x} - \frac{1}{\tan x/a}\right]$$

$$\lim_{x \to 0} \left[\frac{a \tan(x/a) - x}{x \tan(x/a)}\right] \quad \left[\frac{0}{0}\right]$$

L'H Rule $\lim_{x \to 0} \left[\frac{\cancel{a} \cdot \sec^2(x/a) \cdot \frac{1}{\cancel{a}} - 1}{x \sec^2(x/a) \cdot \frac{1}{a} + \tan\left(\frac{x}{a}\right)}\right]$ $\quad \left\{ \sec^2\left(\frac{x}{a}\right) - 1 = \tan^2\left(\frac{x}{a}\right) \right.$

$$\lim_{x \to 0} \frac{\tan^2(x/a)}{\frac{x}{a} \sec^2(x/a) + \tan(x/a)} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 0} \frac{2 \tan(x/a) \sec^2(x/a)(1/a)}{\left(\frac{x}{a}\right) 2 \sec x/a \sec x/a \tan(x/a) \cdot \frac{1}{a} + \sec^2(x/a) \cdot \frac{1}{a} + \sec^2(x/a) \cdot \frac{1}{a}}$$

$$\frac{0}{0 + \frac{1}{a} + \frac{1}{a}} = \frac{0}{2/a} = \underline{\underline{0}}$$

2) $\lim_{x \to \pi/2} (\sec x - \tan x)$

<u>Solution</u> $\lim_{x \to \pi/2} [\sec x - \tan x] \quad (\infty - \infty)$

$$\lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} \quad \left(\frac{0}{0}\right)$$

<u>L'H Rule</u> $\lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{1} = \underline{\underline{0}}$

3) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

<u>Solution</u> $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \quad (\infty - \infty)$

$$\lim_{x \to 0} \left(\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right)$$

$$\lim_{x\to 0} \frac{\cancel{x^2}\left[\left(\frac{\sin x}{x}\right)^2 - 1\right]}{\cancel{x^2}\sin^2 x}$$

$$\lim_{x\to 0} \frac{\left(\frac{\sin x}{x}\right)^2 - 1}{1-\cos^2 x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{\left(\frac{\sin x}{x}+1\right)\left(\frac{\sin x}{x}-1\right)}{(1+\cos x)(1-\cos x)}$$

$$\lim_{x\to 0} \frac{\frac{\sin x}{x}-1}{1-\cos x} = \frac{\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}}{x}-1}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{1}-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots-\cancel{1}}{x^2/2!-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}$$

$$\lim_{x\to 0} \frac{-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}$$

$$\lim_{x\to 0} \frac{\cancel{x^2}\left[-\frac{1}{6}+\frac{x^2}{5!}+\cdots\right]}{\cancel{x^2}\left[\frac{1}{2!}-\frac{x^2}{4!}+\cdots\right]}$$

$$= \frac{-1/6}{1/2} = \boxed{-\frac{1}{3}}$$

$$\lim_{x\to 0} \frac{\frac{\sin x}{x}+1}{1+\cos x} = \frac{1+1}{1+1} = 1$$

$$\lim_{x\to 0} \frac{\left(\frac{\sin x}{x}+1\right)\left(\frac{\sin x}{x}-1\right)}{(1-\cos x)(1+\cos x)} = -\frac{1}{3}(1) = -\frac{1}{3}$$

$$\frac{[\sin^2 x - x^2]}{x^2 x^2 \frac{\sin^2 x}{x^2}}$$

$$\text{Lt} \frac{\sin^2 x - x^2}{x^4} \quad \left(\frac{0}{0}\right)$$

$$\text{Lt} \frac{\sin 2x - 2x}{4x^3} \quad \left(\frac{0}{0}\right)$$

$$\text{〃} \frac{2\cos 2x - 2}{12x^2} \quad \left(\frac{0}{0}\right)$$

$$\text{Lt}_{x\to 0} \left(\frac{\cos 2x - 1}{6x^2}\right)$$

$$\text{Lt}_{x\to 0} \frac{-2\sin 2x}{12x}$$

$$\text{Lt}_{x\to 0} \frac{-2}{6}\left(\frac{\sin 2x}{2x}\right) = \boxed{\frac{-1}{3}}$$

H.W

4) $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

Ans $= (2/3)$

(5) $\lim_{x \to 0} \frac{\frac{1}{x} - \cot x}{x}$

(Ans $= 1/3$)

IV $1^{\infty}, \infty^{0}, 0^{0}$

If $\lim_{x \to a} f(x)^{g(x)} = 0^{0}$ or 1^{∞} or ∞^{0}, then put $y = f^{g}$ and then take logarithm on both sides

Problems

1) $\lim_{x \to a} (x-a)^{(x-a)}$

Solution: $\lim_{x \to a} (x-a)^{(x-a)} \quad (0^{0})$

Let $y = \lim_{x \to a} (x-a)^{(x-a)}$

Taking log,

$$\log y = \lim_{x \to a} (x-a) \log (x-a) \quad (0 \cdot \infty)$$

$$= \lim_{x \to a} \frac{\log(x-a)}{(1/x-a)} \quad \left(\frac{\infty}{\infty}\right)$$

L'H Rule

$$= \lim_{x \to a} \frac{\frac{1}{x-a}}{\frac{-1}{(x-a)^2}} = \frac{-(x-a)^2}{(x-a)} = -(x-a)$$

$$= \lim_{x \to a} -(x-a) = 0$$

$$\log y = 0$$

$$y = e^{0} = 1$$

2) $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$

Solution (1^∞)

$$\log y = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right) \log\left(2-\frac{x}{a}\right) \quad [\infty\cdot 0]$$

$$= \lim_{x\to a} \frac{\log\left(2-x/a\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left(\frac{0}{0}\right)$$

L-H-Rule

$$= \lim_{x\to a} \frac{\frac{1}{2-x/a}\left(-1/a\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}$$

$$\frac{1\left(-1/a\right)}{-1\left(\pi/2a\right)} = \frac{2}{\pi}$$

$$\log_e y = \frac{2}{\pi}$$

$$y = e^{2/\pi} = 1.89$$

3) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Solution $(1^\infty$ form$)$

$$\lim_{x\to 0}\left(\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots}{x}\right)^{\frac{1}{x^2}}$$

$$\lim_{x\to 0}\left\{1+\frac{x^2}{3}+\frac{2}{15}x^4+\cdots\right)^{\frac{1}{x^2}}$$

$$\lim_{x\to 0}\left(1+tx^2\right)^{\frac{1}{x^2}} \quad \left[t=\frac{1}{3}+\frac{2}{15}x^2+\cdots\right]$$

$$\lim_{x\to 0}\left\{\left(1+tx^2\right)^{\frac{1}{tx^2}}\right\}^t \quad \left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e\right\}$$

$$\lim_{x\to 0} e^t = \lim_{x\to 0} e^{\frac{1}{3}+\frac{2}{15}x^2+\cdots} = e^{1/3}$$

4) $\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}}$

Solution (1^∞)

$\log y = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{a^x + b^x}{2} \right) \quad \left(\frac{0}{0}\right)$

L-H Rule $= \lim_{x \to 0} \frac{1}{\frac{a^x + b^x}{2}} \left(\frac{a^x \log a + b^x \log b}{2} \right)$

$= \frac{\log a + \log b}{2}$

$= \frac{1}{2} \log ab$

$\log y = \log (ab)^{\frac{1}{2}}$

$y = (ab)^{\frac{1}{2}}$ or $\sqrt{ab}$

H.W

5) $\operatorname{Lt}_{x \to 0} \cot x^{1/\log x}$ (Ans $= \frac{1}{e}$)

6) $\lim_{x \to 0} \left(\frac{1}{x} \right)^{\frac{1}{x}}$

Ans (1)

7 $\lim_{x \to \pi/2} (\sin x)^{\tan x} \to$ (Ans 1)

8 $\lim_{x \to \infty} \left(\pi/2 - \tan^{-1} x \right)^{1/x} \longrightarrow$ (Ans 1)