FOUREIR SERIES REPRESENTATION OF CONTINUOUS TIME SIGNALS

Introduction to System Eigen functions

- Lets imagine what (basis) signals $\phi_k(t)$ have the property that:

$$x(t) = \phi_k(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = \lambda_k \phi_k(t)$$

- i.e. the output signal is the same as the input signal, multiplied by the constant "gain" λ_k (which may be complex)

- For CT LTI systems, we also have that

$$x(t) = \Sigma_k\, a_k \phi_k(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = \Sigma_k\, a_k \lambda_k \phi_k(t)$$

- Therefore, to make use of this theory we need:
  have to decompose $x(t)$ in terms of $\phi_k(t)$ by calculating the coefficients $\{a_k\}$.

Complex Exponentials are Eigen functions of any CT LTI System

- Consider a CT LTI system with impulse response $h(t)$ and input signal $x(t)=\phi(t) = e^{st}$,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

- Assuming that the integral on the right hand side converges to $H(s)$, this becomes

$$y(t) = H(s)e^{st}$$

- Therefore $\phi(t)=e^{st}$ is an eigenfunction, with eigenvalue $\lambda=H(s)$

Time Delay & Imaginary Input

- Consider a CT, LTI system where the input and output are related by a pure time shift: $$y(t) = x(t-3)$$

- Consider a purely imaginary input signal: $x(t) = e^{j2t}$

- Then the response is: $y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$

- e^{j2t} is an eigen function (as we'd expect) and the associated eigenvalue is $H(j2) = e^{-j6}$.
$$H(s) = \int_{-\infty}^{\infty} \delta(\tau - 3)e^{-s\tau}d\tau = e^{-3s}$$

- The eigenvalue could be derived "more generally". The system impulse response is $h(t) = \delta(t-3)$, therefore:

- So $H(j2) = e^{-j6}$!

Example 1: Phase Shift

- Note that the corresponding input e^{-j2t} has eigenvalue e^{j6}, so lets consider an input cosine signal of frequency 2 so that:

$$\cos(2t) = \tfrac{1}{2}\left(e^{j2t} + e^{-j2t}\right)$$

- By the system LTI, eigen function property, the system output is written as:

$$\begin{aligned} y(t) &= \tfrac{1}{2}\left(e^{-j6}e^{j2t} + e^{j6}e^{-j2t}\right) \\ &= \tfrac{1}{2}\left(e^{j(2t-6)} + e^{-j(2t-6)}\right) \\ &= \cos(2(t-3)) \end{aligned}$$

- So because the eigenvalue is **purely imaginary**, this corresponds to a **phase shift** (time delay) in the system's response. If the eigenvalue had a real component, this would correspond to an amplitude variation

Time Delay & Superposition

- Consider the same system (3 time delays) and now consider the input signal $x(t)$ = cos(4t)+cos(7t), a superposition of two sinusoidal signals that are not harmonically related. The response is obviously:

$$y(t) = \cos(4(t-3)) + \cos(7(t-3))$$

- Consider $x(t)$ represented using Euler's formula: $x(t) = \tfrac{1}{2}e^{j4t} + \tfrac{1}{2}e^{-j4t} + \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}e^{-j7t}$

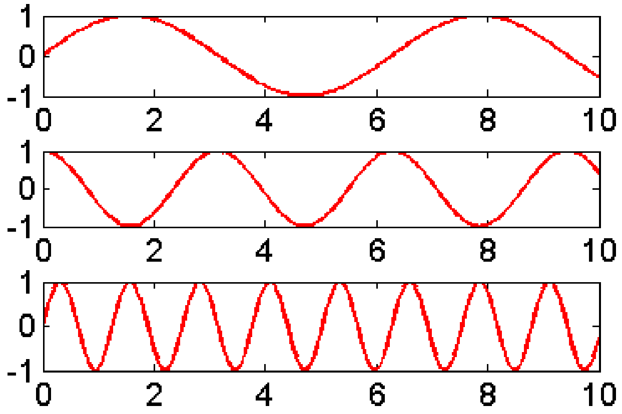
- Then due to the superposition property and $H(s) = e^{-3s}$

$$\begin{aligned} y(t) &= \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} + \tfrac{1}{2}e^{-j21}e^{j7t} + \tfrac{1}{2}e^{j21}e^{-j7t} \\ &= \tfrac{1}{2}e^{j4(t-3)} + \tfrac{1}{2}e^{-j4(t-3)} + \tfrac{1}{2}e^{j7(t-3)} + \tfrac{1}{2}e^{-j7(t-3)} \\ &= \cos(4(t-3)) + \cos(7(t-3)) \end{aligned}$$

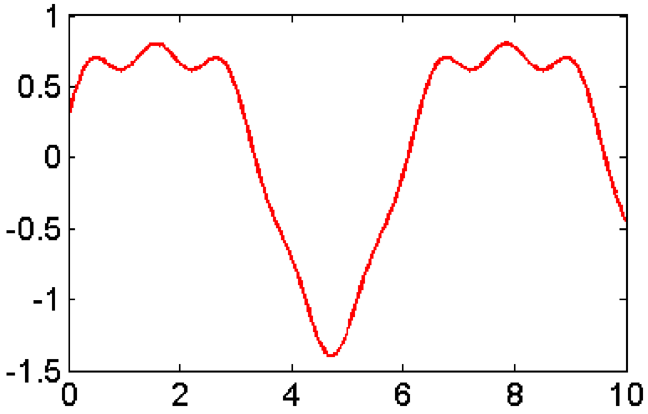
- While the answer for this simple system can be directly spotted, the superposition property allows us to apply the eigen function concept to more complex LTI systems.

Fourier Series and Fourier Basis Functions

- The theory derived for LTI convolution, used the concept that any input signal can represented as a linear combination of shifted impulses (for either DT or CT signals)
- We will now look at how (input) signals can be represented as a linear combination of **Fourier basis functions** (LTI eigen functions) which are purely imaginary exponentials
- These are known as continuous-time **Fourier series**
- The bases are **scaled and shifted sinusoidal signals**, which can be represented as **complex exponentials**



$e^{j\omega t}$



$x(t)$

$x(t) = \sin(t) + 0.2\cos(2t) + 0.1\sin(5t)$

Periodic Signals & Fourier Series

- A periodic signal has the property $x(t) = x(t+T)$, T is the fundamental period, $\omega_0 = 2\pi/T$ is the fundamental frequency. Two periodic signals include:

$$x(t) = \cos(\omega_0 t)$$

$$x(t) = e^{j\omega_0 t}$$

- For each periodic signal, the **Fourier basis** the set of **harmonically related complex exponentials**:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \qquad k = 0, \pm 1, \pm 2, \ldots$$

- Thus the **Fourier series** is of the form: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$

- k=0 is a constant
- k=+/-1 are the fundamental/first harmonic components
- k=+/-N are the N^{th} harmonic components
- For a particular signal, are the values of $\{a_k\}_k$?

Fourier Series Representation of a CT Periodic Signal

- Given that a signal has a Fourier series representation, we have to find $\{a_k\}_k$. Multiplying through by $e^{-jn\omega_0 t}$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

T is the fundamental period of $x(t)$

- Using Euler's formula for the complex exponential integral

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t)dt + j\int_0^T \sin((k-n)\omega_0 t)dt$$

- It can be shown that

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Fourier Series Representation of a CT Periodic Signal

- Therefore

$$a_n = \tfrac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

- which allows us to determine the coefficients. Also note that this result is the same if we integrate over any interval of length *T* (not just [0,*T*]), denoted by $\int_T$
-

$$a_n = \tfrac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt$$

- To summarise, if x(t) has a Fourier series representation, then the pair of equations that defines the Fourier series of a periodic, continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

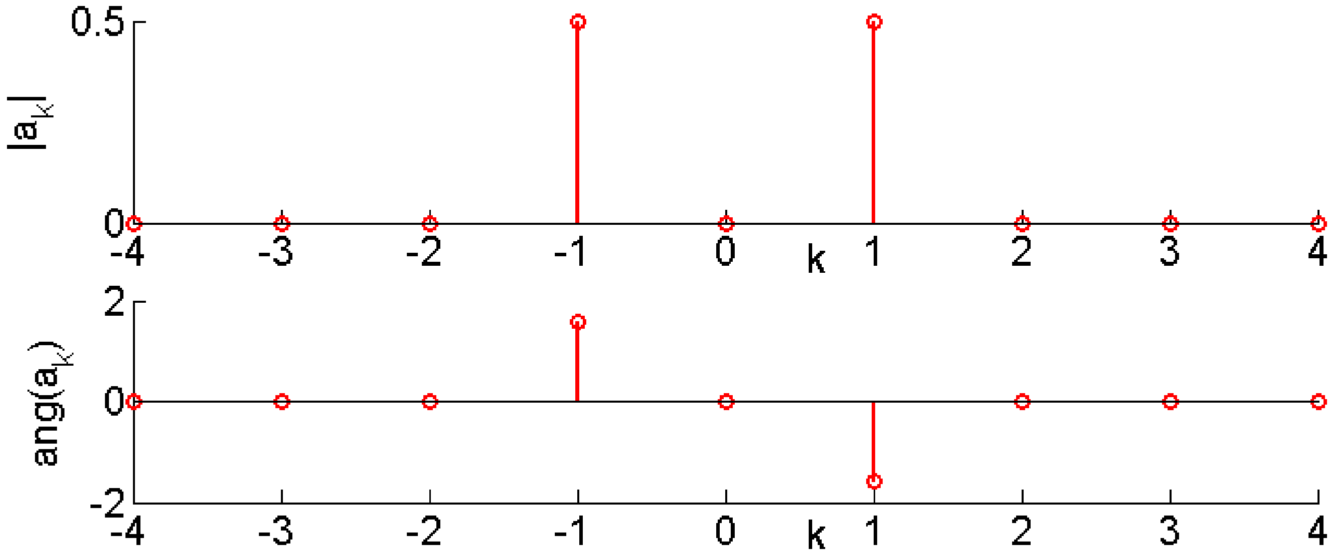
$$a_k = \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \tfrac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt$$

Example : Fourier Series $\sin(\omega_0 t)$

- The fundamental period of $\sin(\omega_0 t)$ is ω_0
- By inspection we can write: $\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

- So $a_1 = 1/2j$, $a_{-1} = -1/2j$ and $a_k = 0$ otherwise

- The magnitude and angle of the Fourier coefficients are:



Example : Fourier Series $\sin(\omega_0 t)$

- The Fourier coefficients can also be explicitly evaluated

$$a_0 = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \sin(\omega_0 t)dt = -\cos(\omega_0 t)\Big|_0^{2\pi/\omega_0} = 1-1=0$$

$$a_k = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \sin(\omega_0 t)e^{-jk\omega_0 t}dt = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\left(\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\right)e^{-jk\omega_0 t}dt$$

$$= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\left(\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\right)e^{-jk\omega_0 t}dt$$

$$= \frac{\omega_0}{4\pi j}\int_0^{2\pi/\omega_0} e^{-j(k-1)\omega_0 t} - e^{-j(k+1)\omega_0 t}dt$$

- When k = +1 or –1, the integrals evaluate to T and $-T$, respectively. Otherwise the coefficients are zero.
- Therefore $a_1 = 1/2j$, $a_{-1} = -1/2j$

Example : Additive Sinusoids

- Consider the additive sinusoidal series which has a fundamental frequency ω_0:

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \tfrac{\pi}{4}\right)$$

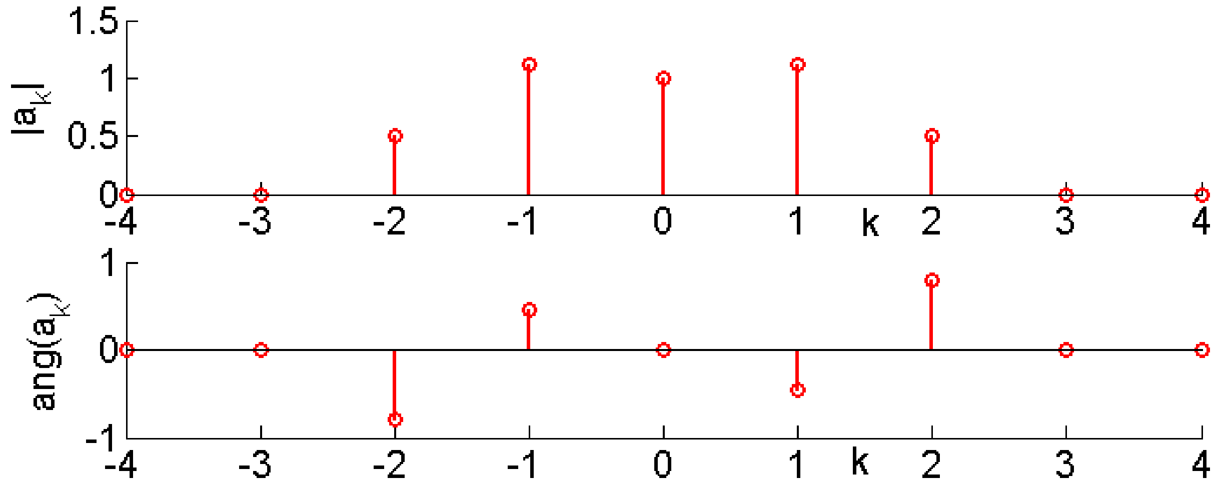
- Again, the signal can be directly written as:

$$x(t) = 1 + \tfrac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) + \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \tfrac{1}{2}\left(e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right)$$

$$= 1 + \left(1 + \tfrac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \tfrac{1}{2j}\right)e^{-j\omega_0 t} + \tfrac{1}{2}e^{j\frac{\pi}{4}}e^{j2\omega_0 t} + \tfrac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2\omega_0 t}$$

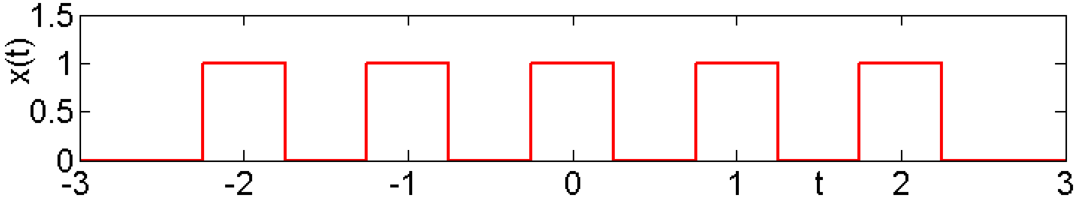
- The Fourier series coefficients can then be visualised as:

$$a_0 = 1 \quad a_1 = \left(1 - \tfrac{1}{2}j\right) \quad a_{-1} = \left(1 + \tfrac{1}{2}j\right) \quad a_2 = \tfrac{1}{2}e^{j\frac{\pi}{4}} \quad a_{-2} = \tfrac{1}{2}e^{-j\frac{\pi}{4}}$$



Example : Periodic Square Signal

- Consider the periodic square wave, illustrated by:



- and is defined over one period as:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

- Fourier coefficients:

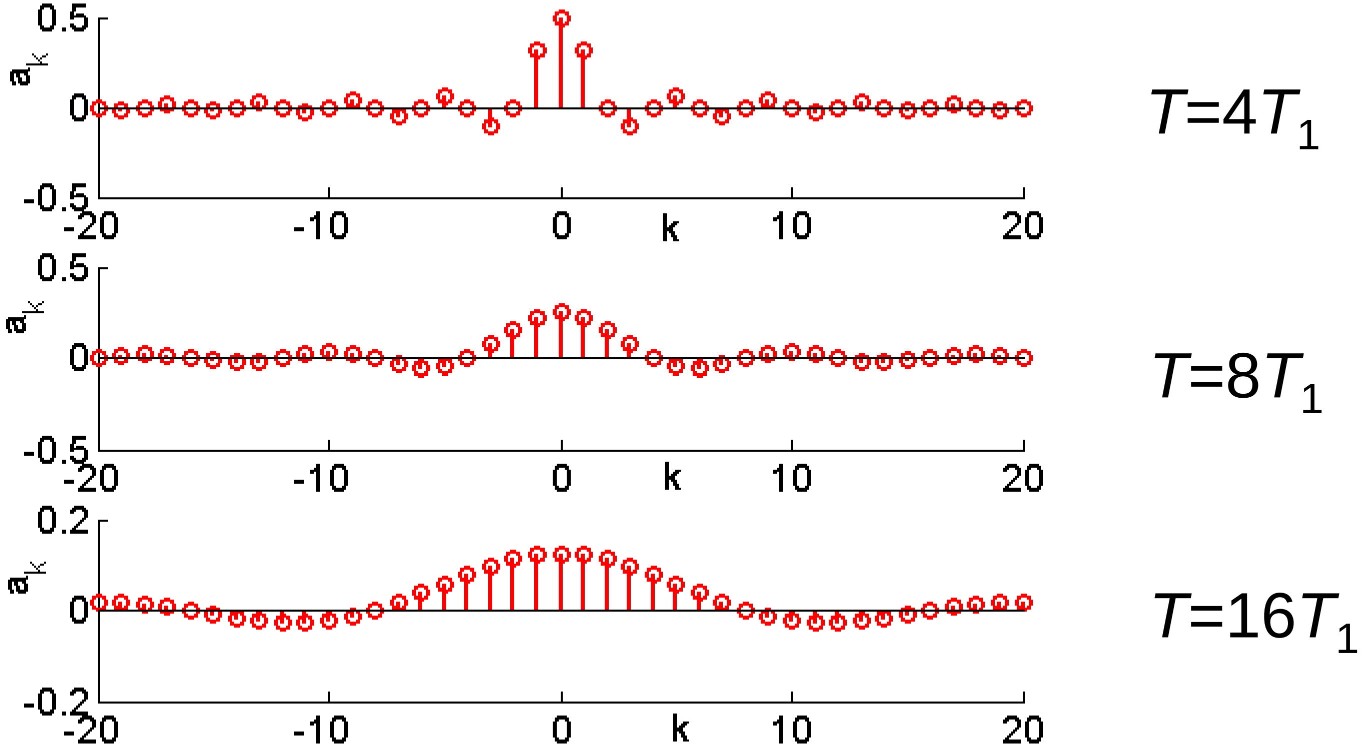
$$a_0 = \tfrac{1}{T}\int_{-T_1}^{T_1} 1 dt = \frac{2T_1}{T}$$

$$\begin{aligned} a_k &= \tfrac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = -\tfrac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} \\ &= \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right) \\ &= 2\sin(k\omega_0 T_1)/k\omega_0 T \\ &= \sin(k\omega_0 T_1)/k\pi \end{aligned}$$

NB, these coefficients are real

Example : Periodic Square Signal

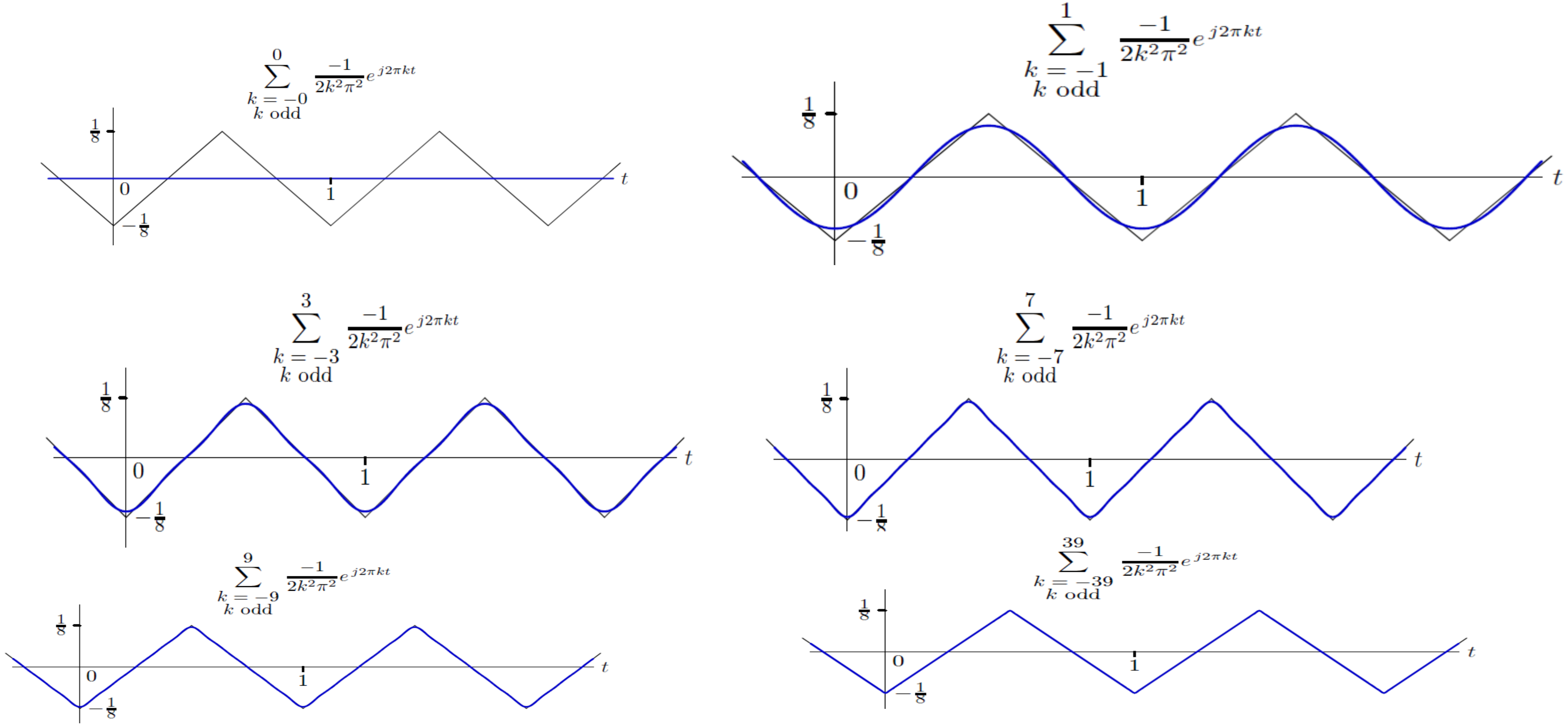
- Instead of plotting both the magnitude and the angle of the complex coefficients, we only need to plot the value of the coefficients.
- Note we have an infinite series of non-zero coefficients



Reconstruction of signal from FS coefficients

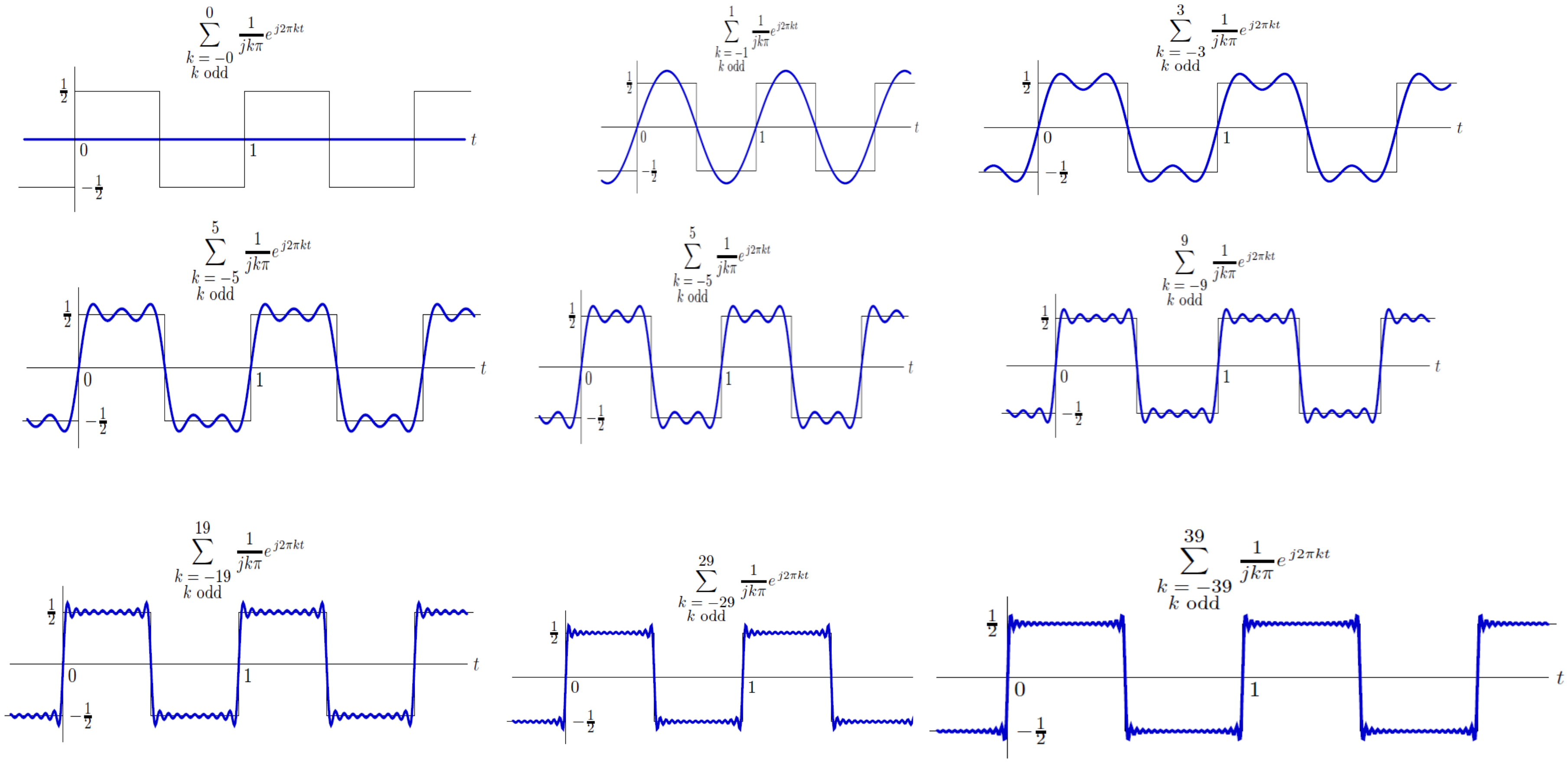
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform



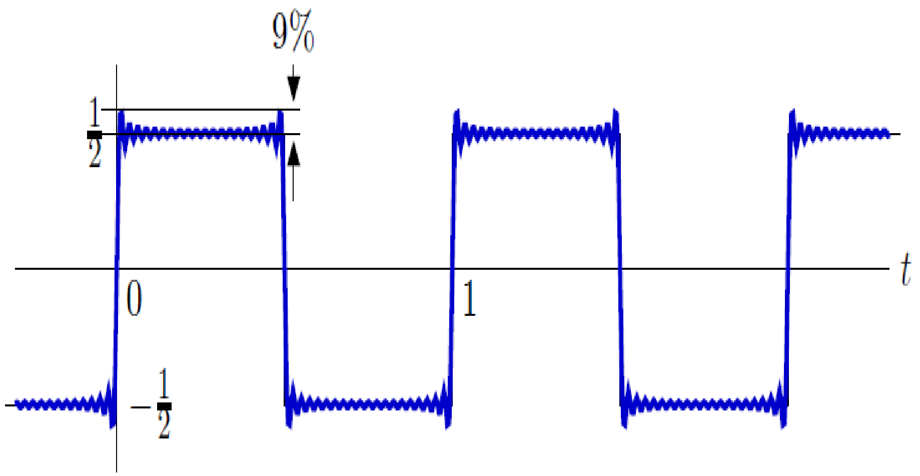
Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.

Example: square wave



Convergence of Fourier Series

Partial sums of Fourier series of discontinuous functions "ring" near discontinuities: Gibb's phenomenon.



Fourier series of a square wave — Gibbs' phenomenon
As harmonics are added to synthesize the square wave, the partial sum of the Fourier series converges to the square wave everywhere except near the discontinuity where the partial sum takes on the value of 1/2. There are oscillations on either side of the discontinuity whose maximum over and undershoot approach 9% of the discontinuity independent of N.

This ringing results because the magnitude of the Fourier coefficients is only decreasing as $\frac{1}{k}$ (while they decreased as $\frac{1}{k^2}$ for the triangle).

You can decrease (and even eliminate the ringing) by decreasing the magnitudes of the Fourier coefficients at higher frequencies.

Convergence of Fourier Series -- Dirichlet Conditions

Condition 1: Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_T |x(t)|dt < \infty ,$$

This guarantees each coefficient a_k will be finite, since

$$|a_k| = \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right|dt = \frac{1}{T}\int_T |x(t)|dt < \infty .$$

A periodic function that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \qquad 0 < t < 1 .$$

Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during a single period of the signal.

An example of a function that meets Condition1 but not Condition 2:

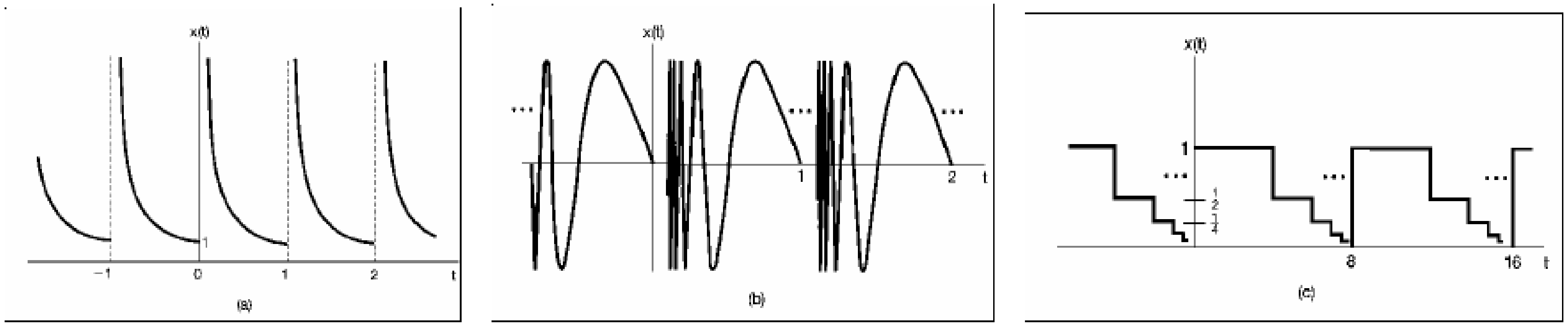
$$x(t) = \sin\left(\frac{2\pi}{t}\right), \qquad 0 < t \le 1, \tag{3.47}$$

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example that violates this condition is a function defined as

$$x(t) = 1,\ 0 \le t < 4,\ x(t) = 1/2,\ 4 \le t < 6,\ x(t) = 1/4,\ 6 \le t < 7,\ x(t) = 1/8,\ 7 \le t < 7.5, \text{ etc.}$$

The above three examples are shown in the figure below.



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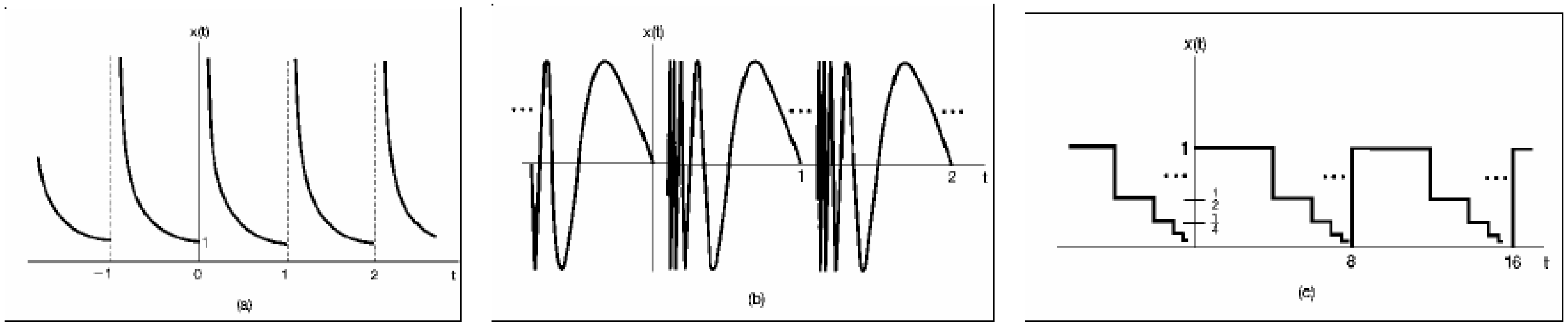


Basic Fourier Series pairs:

Time domain	Frequency domain
$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}$ **Period = T**	$a_k = \frac{1}{T}\int_{t=-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t}dt$ $\omega_0 = \frac{2\pi}{T}$
$x(t) = \begin{cases} 1, \lvert t\rvert \le T_0 \\ 0, T_0 < \lvert t\rvert \le \frac{T}{2} \end{cases}$	$a_k = \frac{sin(k\omega_0 T_0)}{k\pi}$
$x(t) = e^{jp\omega_0 t}$	$a_k = \delta[k-p]$
$x(t) = cos(p\omega_0 t)$	$a_k = \frac{1}{2}\delta[k-p] + \frac{1}{2}\delta[k+p]$
$x(t) = sin(p\omega_0 t)$	$a_k = \frac{1}{2j}\delta[k-p] - \frac{1}{2j}\delta[k+p]$
$x(t) = \sum_{p=-\infty}^{\infty} \delta(t-pT)$	$a_k = \frac{1}{T}$

Properties of Fourier Series

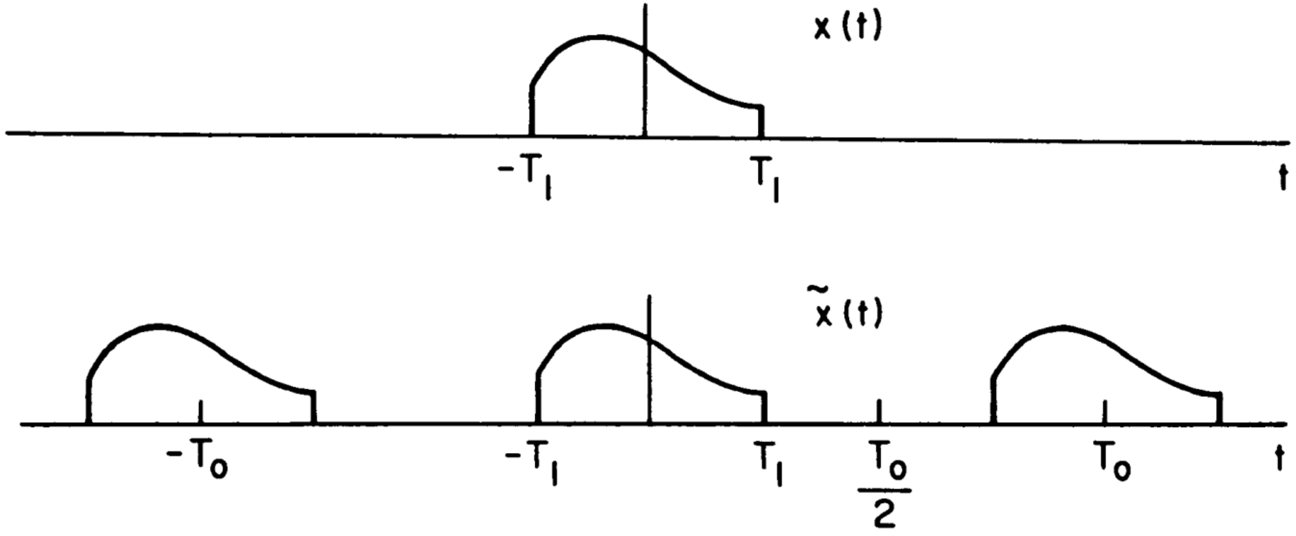
Continuous time signal: Periodic, Period=T, Fundamental frequency $\omega_0 = \frac{2\pi}{T}\ radian/sec$

Property	Continuous time signals	
	Time domain	Frequency domain (FS)
Notation	$x(t)$ $y(t)$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} a_k$
Frequency shift	$e^{jk_0\omega_0 t} x(t)$	a_{k-k_0}
Differentiation in time	$\frac{d}{dt} x(t)$	$jk\omega_0 a_k$
Convolution	$x_1(t) * x_2(t)$	$T a_{k1} a_{k2}$
Multiplication	$x_1(t) x_2(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Symmetry	$x(t)\ \ real$	$a_k^* = a_{-k}$
	$x(t)\ \ imaginary$	$a_k^* = -a_{-k}$
	$x(t)\ \ real\ \&\ even$	$Im\{a_k\} = 0$
	$x(t)\ \ real\ \&\ odd$	$Re\{a_k\} = 0$
Parseval's Theorem	$\frac{1}{T}\int_{t=0}^{T} \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

FOURIER REPRESENTATION OF APERIODIC SIGNALS – FOURIER TRANSFORM

Fourier Series to Fourier Transform

- For periodic signals, we can represent them as linear combinations of harmonically related complex exponentials
- To extend this to non-periodic signals, we need to consider aperiodic signals as periodic signals with infinite period.
- As the period becomes infinite, the corresponding frequency components form a continuum and the Fourier series sum becomes an integral
- Instead of looking at the coefficients a harmonically –related Fourier series, we'll now look at the **Fourier transform** which is a **complex valued function** in the frequency domain
- spectra of aperiodic signals are defined for all real values of the frequency variable ω not just for a discrete set of values



$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_o}{2}$$

$$\text{As } T_o \to \infty \qquad \tilde{x}(t) \to x(t)$$

- use Fourier series to represent $\tilde{x}(t)$
- let $T_o \to \infty$ to represent $x(t)$

$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_o}{2}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad \omega_o = \frac{2\pi}{T_o}$$

$$a_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} \tilde{x}(t)\, e^{-jk\omega_o t}\, dt$$

$$a_k = \frac{1}{T_o} \int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_o t}\, dt$$

$$a_k = \frac{1}{T_o} \int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_o t}\, dt$$

Define: $$X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Then: $$T_o a_k = X(\omega) \Big|_{\omega = k\omega_o}$$

$X(\omega)$ is the <u>envelope</u> of $T_o a_k$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T_o} X(k\omega_o)\, e^{jk\omega_o t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_o)\, e^{jk\omega_o t}\, \omega_o$$

Fourier transform
- analysis

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_o)\, e^{jk\omega_o t}\omega_o$$

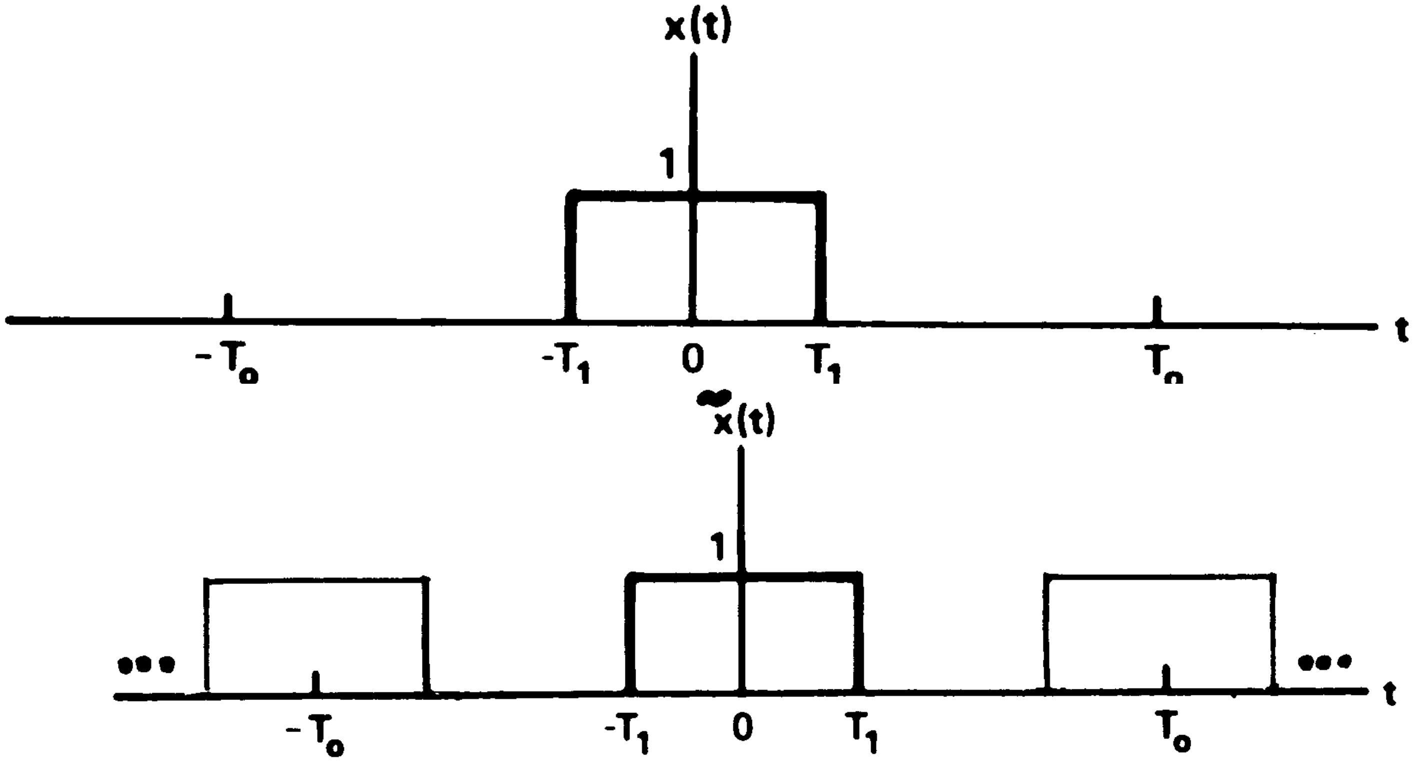
As $T_o \rightarrow \infty$,

$$\omega_o \rightarrow 0,\ \tilde{x}(t) \rightarrow x(t),\ \omega_o \rightarrow d\omega,\ \sum \rightarrow \int$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Inverse
Fourier transform
- synthesis

Frequency Content of the Rectangular Pulse



$$x(t) = \lim_{T \to \infty} x_T(t)$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

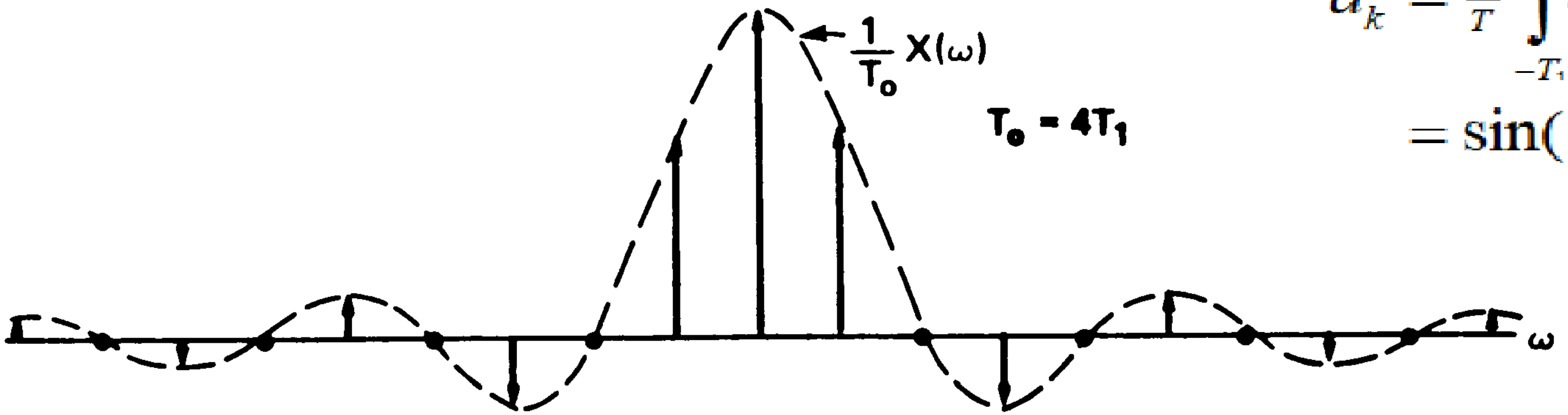
$$= \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right)$$

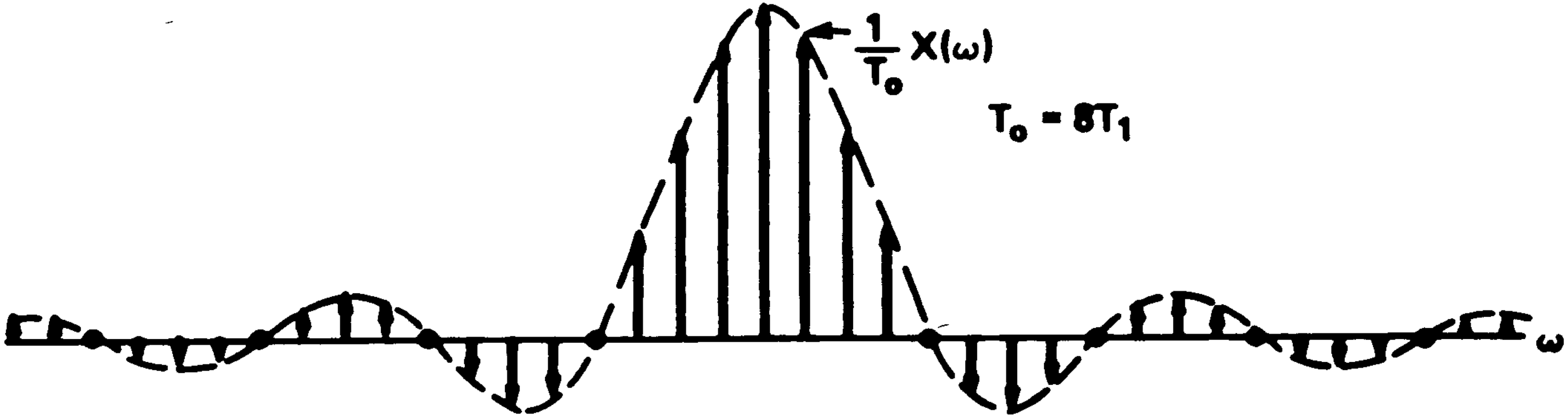
$$= 2\sin(k\omega_0 T_1)/k\omega_0 T$$

$$= \sin(k\omega_0 T_1)/k\pi$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}\,e^{-jk\omega_0 t}$$

$$= \sin(k\omega_0 T_1)\,/\,k\pi$$





Definition of the Fourier Transform

- We will be referring to functions of time and their Fourier transforms. A signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by the **Fourier transform synthesis** and **analysis** equations

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = F\{x(t)\}$$

- and

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = F^{-1}\{X(j\omega)\}$$

- We will refer to $x(t)$ and $X(j\omega)$ as a **Fourier transform pair** with the notation

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

- As previously mentioned, the transform function $X(j\omega)$ can roughly be thought of as a continuum of the previous coefficients

- A similar set of Dirichlet convergence conditions exist for the Fourier transform, as for the Fourier series

Convergence of the Fourier Transform

- The integral does converge if
  1. the signal $x(t)$ is *"well-behaved"*
  2. and $x(t)$ is *absolutely integrable*, namely,

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

- Note: *well behaved* means that the signal has a finite number of discontinuities, maxima, and minima within any finite time interval

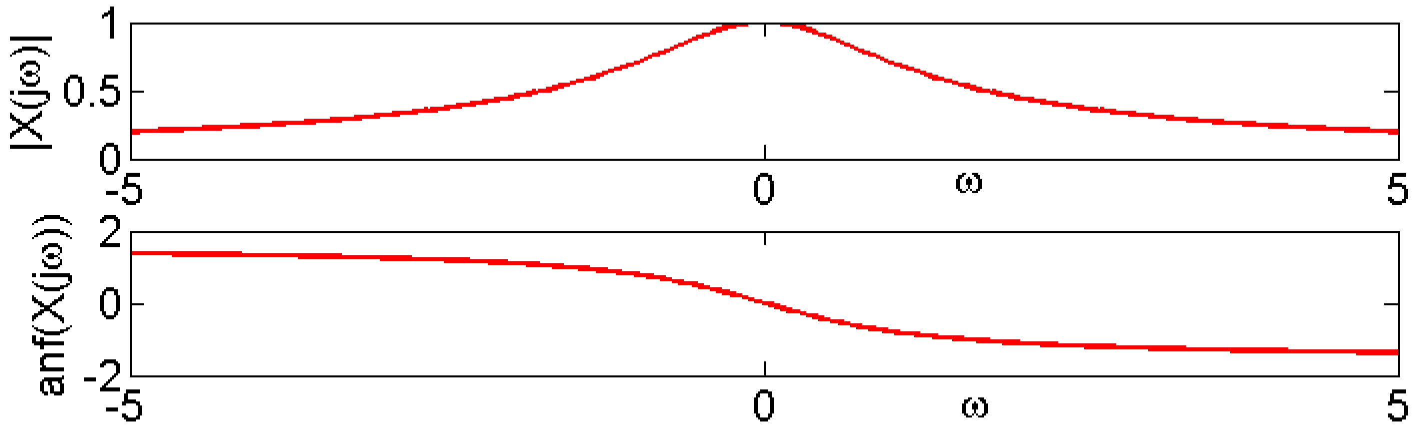
Example: Decaying Exponential

- Consider the (non-periodic) signal $x(t) = e^{-at}u(t) \qquad a > 0$

- Then the Fourier transform is: $$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{(a+j\omega)}$$



$a = 1$

Example : Single Rectangular Pulse

- Consider the non-periodic rectangular pulse at zero

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| \geq T_1 \end{cases}$$

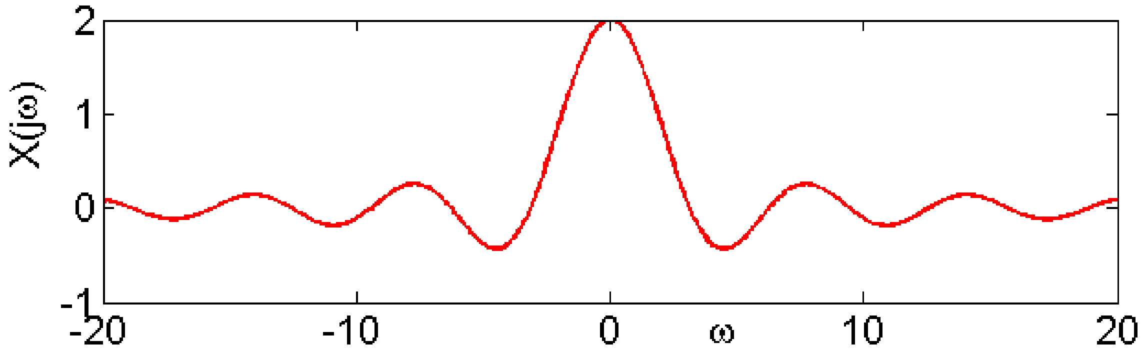
- The Fourier transform is:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

$$= \frac{2\sin(\omega T_1)}{\omega}$$

Note, the values are real



$T_1 = 1$

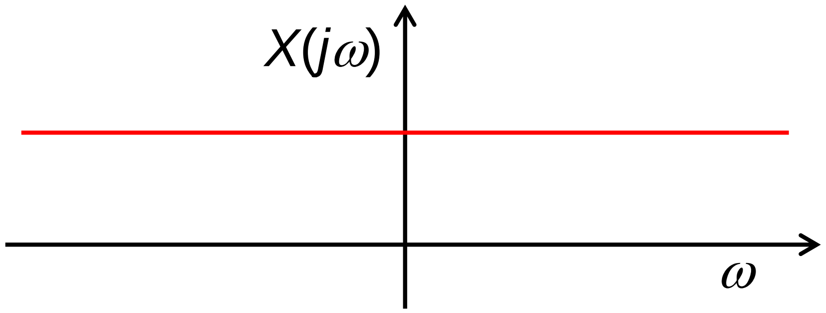
Example : Impulse Signal

- The Fourier transform of the impulse signal can be calculated as follows:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

- Therefore, the Fourier transform of the impulse function has a constant contribution for **all frequencies**



Example: Periodic Signals

- A periodic signal violates condition 1 of the Dirichlet conditions for the Fourier transform to exist
- However, lets consider a Fourier transform which is a single impulse of area 2π at a particular (harmonic) frequency $\omega=\omega_0$.

$$X(j\omega) = 2\pi\delta(\omega-\omega_0)$$

- The corresponding signal can be obtained by: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$
- which is a (complex) sinusoidal signal of frequency ω_0. More generally, when

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

- Then the corresponding (periodic) signal is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- The Fourier transform of a periodic signal is a train of impulses at the harmonic frequencies with amplitude $2\pi a_k$

Properties of Fourier Transform

Reminder: Fourier Transform

- A signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- This is denoted by:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

- For example :

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

- Remember that the Fourier transform is a **density function**, you must integrate it, rather than summing up the discrete Fourier series components

Linearity of the Fourier Transform

- If $$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$
- and $$y(t) \overset{F}{\leftrightarrow} Y(j\omega)$$
- Then $$ax(t) + by(t) \overset{F}{\leftrightarrow} aX(j\omega) + bY(j\omega)$$
- This follows directly from the definition of the Fourier transform (as the integral operator is linear). It is easily extended to a linear combination of an arbitrary number of signals

Time Shifting

- If $$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$
- Then $$x(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega)$$
- Proof $$x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$
- Now replacing *t* by *t*-t_0

$$x(t-t_0) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega$$
$$= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega) \right) e^{j\omega t} d\omega$$

- Recognising this as

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

- A signal which is shifted in time does not have its Fourier transform magnitude altered, only a shift in phase.

Example: Linearity & Time Shift

- Consider the signal (linear sum of two time shifted steps)

$$x(t) = 0.5x_1(t-2.5) + x_2(t-2.5)$$

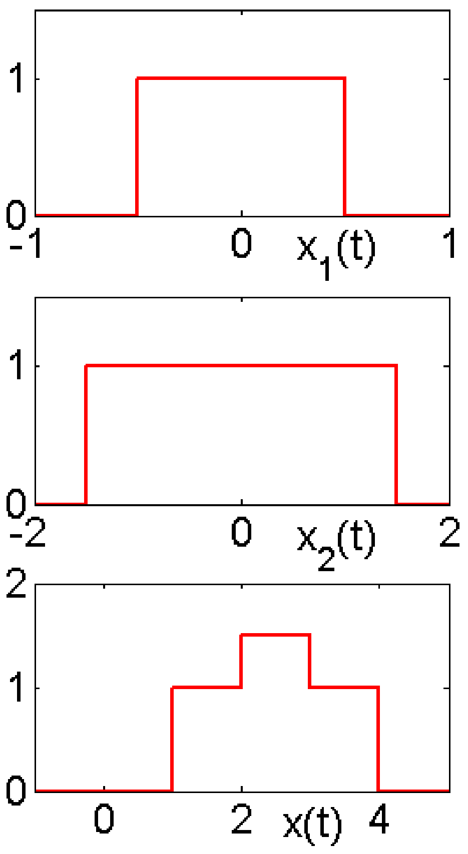
- where $x_1(t)$ is of width 1, $x_2(t)$ is of width 3, centred on zero.
- Using the rectangular pulse example

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

- Then using the **linearity** and **time shift** Fourier transform properties

$$X(j\omega) = e^{-j5\omega/2}\left(\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right)$$



Frequency Shifting

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t} x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$

  where ω_0 is an arbitrary real constant.
- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

Differentiation & Integration

- By differentiating both sides of the Fourier transform synthesis equation:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

- Therefore:

$$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

- This is important, because it replaces **differentiation** in the **time domain** with **multiplication** in the **frequency domain**.
- Integration is similar:

$$\int_{-\infty}^{t} x(\tau)d\tau = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

- The impulse term represents the dc or average value that can result from integration

Example: Fourier Transform of a Step Signal

- Lets calculate the Fourier transform $X(j\omega)$of $x(t) = u(t)$, making use of the knowledge that:

$$g(t) = \delta(t) \overset{F}{\leftrightarrow} G(j\omega) = 1$$

- and noting that:

$$x(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

- Taking Fourier transform of both sides

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

- using the integration property. Since $G(j\omega) = 1$:

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

- We can also apply the differentiation property in reverse

$$\delta(t) = \frac{du(t)}{dt} \overset{F}{\leftrightarrow} j\omega\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) = 1$$

Time Scaling

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

  where a is an arbitrary nonzero real constant.
- This is known as the **dilation (or time/frequency-scaling) property** o the Fourier transform.

Convolution

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

- Therefore, to apply **convolution in the frequency domain**, we just have to multiply the two functions.

- To solve for the differential/convolution equation using Fourier transforms:

1. Calculate Fourier transforms of $x(t)$ and $h(t)$
2. Multiply $H(j\omega)$ by $X(j\omega)$ to obtain $Y(j\omega)$
3. Calculate the inverse Fourier transform of $Y(j\omega)$

- **Multiplication** in the frequency domain corresponds to convolution in the time domain and vice versa.

Proof of Convolution Property

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Taking Fourier transforms gives:

$$Y(j\omega) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt$$

- Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right)d\tau$$

- By the time shift property, the bracketed term is $e^{-j\omega\tau}H(j\omega)$, so

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$

$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= H(j\omega)X(j\omega)$$

Multiplication Property

Since *FT* is highly symmetric,

$$x(t) = \mathfrak{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega, \quad X(j\omega) = \mathfrak{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

thus if $\quad x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

then the other way around is also true $\quad x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

— Definition of convolution in ω

— A consequence of *Duality*

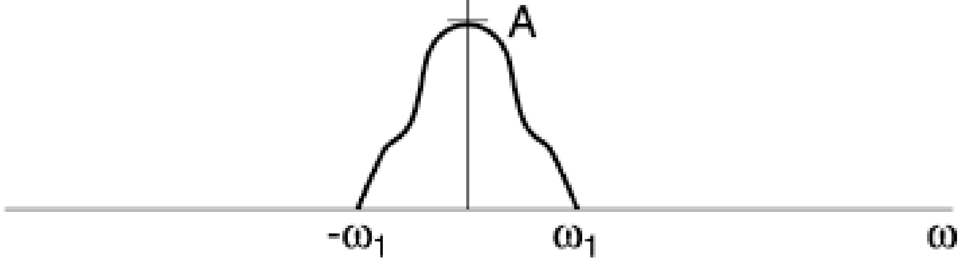
Examples of the Multiplication Property

$$r(t) = s(t) \cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$$

$$\text{For} \quad p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

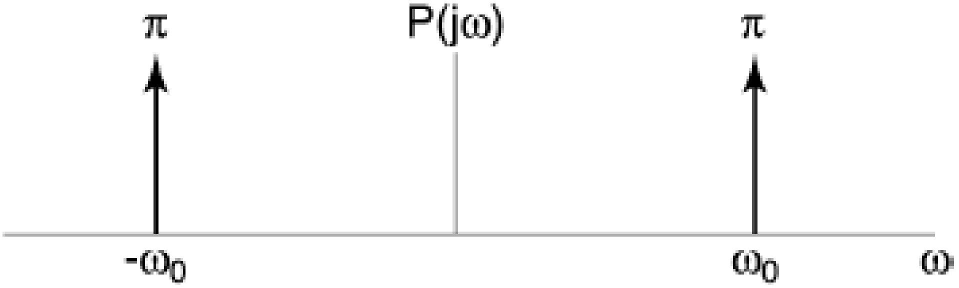
$$\Downarrow$$

$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$



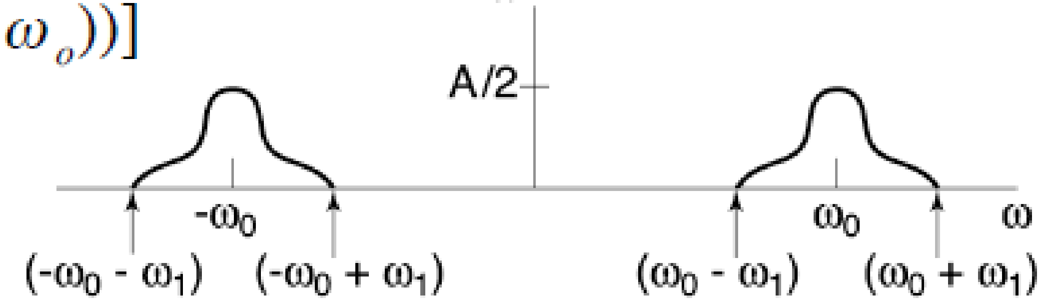
$r(t) = s(t) \cdot \cos(\omega_o t)$

Amplitude modulation (*AM*)



$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_o)) + S(j(\omega + \omega_o))]$$

$$R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$



Drawn assume

$\omega_o - \omega_1 > 0$

i.e. $\omega_o > \omega_1$

Parseval's Relation

$$\underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 \, dt}_{\text{Total energy in the time-domain}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega}_{\text{Total energy in the frequency-domain}}$$

$\frac{1}{2\pi}|X(j\omega)|^2$ — spectral density

Conjugation Property

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then
$$x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-\omega).$$
- This is known as the **conjugation property** of the Fourier transform.

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Two differences:

- minus sign: flips time axis (or equivalently, frequency axis)
- divide by 2π (or multiply in the other direction)

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \to t$ $\quad$ $t \to \omega$; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$

Duality

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

$x(t)$ real and even $\qquad x(t) = x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) \text{ — Real \& even}$$

$x(t)$ real and odd $\qquad x(t) = -x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) \text{ — Purely imaginary \& odd}$$

$$X(j\omega) = \text{Re}\{X(j\omega)\} + j\text{Im}\{X(j\omega)\}$$

For real $\quad x(t) = Ev\{x(t)\} + Od\{x(t)\}$

Basic Fourier Transform pairs:

Time domain	Frequency domain
$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$x(t) = \begin{cases} 1, \lvert t\rvert \le T_0 \\ 0, \lvert t\rvert > T_0 \end{cases}$	$X(\omega) = \frac{2sin(\omega T_0)}{\omega}$
$x(t) = \frac{1}{\pi t}sin(Wt)$	$X(\omega) = \begin{cases} 1, \lvert\omega\rvert \le \mathrm{W} \\ 0, \mathrm{otherwise} \end{cases}$
$x(t) = \delta(t)$	$X(\omega) = 1$
$x(t) = 1$	$X(\omega) = 2\pi\delta(\omega)$
$x(t) = u(t)$	$X(\omega) = \frac{1}{j\omega}\pi\delta(\omega)$
$x(t) = e^{-at}u(t) \quad Re\{a\} > 0$	$X(\omega) = \frac{1}{a + j\omega}$
$x(t) = te^{-at}u(t), \quad Re\{a\} > 0$	$X(\omega) = \frac{1}{(a + j\omega)^2}$
$x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$X(\omega) = \frac{2a}{a^2 + \omega^2}$
$x(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$	$X(\omega) = e^{-\omega^2/2}$
$x(t) = cos(\omega_0 t)$	$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$x(t) = sin(\omega_0 t)$	$X(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$
$x(t) = e^{j\omega_0 t}$	$X(j\omega) = 2\pi\delta(\omega - \omega_0)$
$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$	$X(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$

Properties of Fourier Transform

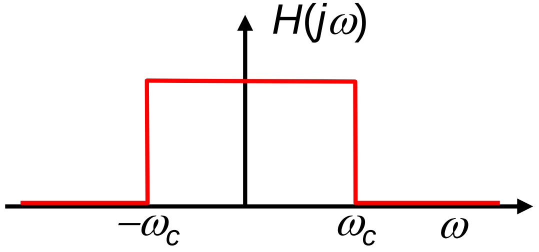
Property	Continuous time signals	
	Time domain	Frequency domain (FT)
Notation	$x(t)$ $x_i(t)$	$X(\omega)$ $X_i(\omega)$
Linearity	$\sum_{i=1}^{N} a_i x_i(t)$	$\sum_{i=1}^{N} a_i\, X_i(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shift	$e^{j\gamma t}x(t)$	$X\big((\omega-\gamma)\big)$
Time reversal	$x(-t)$	$X(-\omega)$
Correlation	$r_{x_1x_2}(\tau)=x_1(\tau)*x_2(-\tau)$	$X_1(\omega)X_2(-\omega)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Differentiation in frequency	$-jtx(t)$	$\frac{d}{d\omega}X(\omega)$
Integration /summation	$\int_{\tau=-\infty}^{t} x(\tau)d\tau$	$\frac{X(j\omega)}{j\omega}+\pi X(j0)\delta(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}\int_{\vartheta=-\infty}^{\infty} X_1(\vartheta)X_2(\omega-\ \vartheta)d\vartheta$
Symmetry	$x(t)\ \ real$	$X^*(\omega)=X(-\omega)$
	$x(t)\ \ imaginary$	$X^*(\omega)=-X(-\omega)$
	$x(t)\ \ real\ \&\ even$	$Im\{X(\omega)\}=0$
	$x(t)\ \ real\ \&\ odd$	$Re\{X(\omega)\}=0$
Parseval's Theorem	$\int_{t=-\infty}^{\infty} \lvert x(t)\rvert^2 dt=\frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\Omega$	
Duality	$X(t)\xleftrightarrow{FT}2\pi x(-\omega)$	

FT representation for a continuous-time periodic signal *g(t)*:

$$g(t) \xleftrightarrow{FT} G(\omega) = 2\pi \sum_{k=-\infty}^{\infty} g_k \, \delta(\omega - k\omega_0)$$

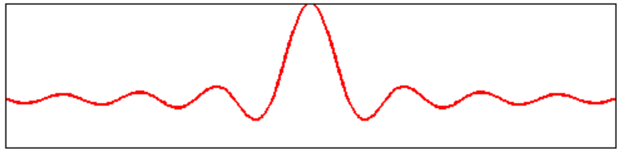
Where g_k are the FS coefficients and ω_0 is the fundamental frequency.

Designing a Low Pass Filter



- Lets design a low pass filter: $$H(j\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & |\omega|>\omega_c\end{cases}$$
- The impulse response of this filter is the inverse Fourier transform

$$h(t)=\tfrac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega=\frac{\sin(\omega_c t)}{\pi t}$$



- which is an ideal low pass filter
  - Non-causal (how to build)
  - The time-domain oscillations may be undesirable
- How to approximate the frequency selection characteristics?
- Consider the system with impulse response:

$$e^{-at}u(t)\overset{F}{\leftrightarrow}\frac{1}{a+j\omega}$$

- Causal and non-oscillatory time domain response and performs a degree of low pass filtering

References:

Signals and Systems,, Allan V Opprnheim , Allan S Willsky with Hamid Nawab

https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-003-signals-and-systems-fall-2011/lecture-videos/MIT6_003F11_lec06.pdf

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