FIR Filter

- An FIR system does not have feedback. Hence $y(n-k)$ term is absent in the system. FIR output is expressed as

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

- If there are M coefficients then

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

- The coefficients are related to unit sample response as

$$h(n) = \begin{cases} b_n & \textit{for} \quad 0 \leq n \leq M-1 \\ 0 & \textit{otherwise} \end{cases}$$

- Expanding the summation

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots b_{(M-1)} x(n-M+1)$$

- Since $h(n) = b_n$ then y(n) is

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

Advantages of the FIR digital filter

- Relatively easy to design and computationally more efficient.
- FIR filters are implemented in hardware or software.
- The phase response is linear. Linear phase property implies that the phase is a linear function of the frequency. FIR filter output is delayed by the same amount of time for all frequencies, thereby eliminating the phase distortion (Group delay).
- FIR filters are always stable i.e. for a finite input, the output is always finite.
- In linear phase, for the filter of length N the number of operations are of the order of N/2.

Disadvantages of the FIR digital filter (compared to IIR filters)

- They require more memory and/or calculation to achieve a given filter response characteristic. Also, certain responses are not practical to implement with FIR filters.
- For a desired frequency response, with tight constraints on the passband, transition band and the stopband, a FIR filter may have large number of coefficients, thereby have more arithmetic operations and hardware components.

An LTI system is causal iff

- Input/output relationship: $y[n]$ depends only on current and past input signal values.
- Impulse response: $h[n] = 0$ for $n < 0$
- System function: number of finite zeros $\leq$ number of finite poles.

FIR Filter

The following are the main advantages of FIR filters over IIR filters:

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

The disadvantages of FIR filters are as follows:

1. The implementation of narrow transition band FIR filters is very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

Disadvantage of IIR Filter

A limit cycle oscillation is a periodic low-level oscillatory disturbance (useless signal) that may exist in an otherwise stable filter. It creeps into the system due to the non-linearities that arise from the inherent quantization in the system.

Interestingly, limit cycle oscillations occur only in recursive systems. That is, it's just the infinite-impulse response (IIR) filters that face this issue. Non-recursive FIR filters don't experience limit cycle oscillations.

What are the types of limit cycle oscillations?

- Zero-limit cycle oscillations
- Overflow limit cycle oscillations

Zero limit cycle oscillations

When a system output enters the limit cycle oscillation zone and continues to show the periodic oscillations even after the input is made 0, it is known as the zero limit cycle oscillations.

Overflow limit cycle oscillations

In the fixed-point addition of two binary numbers, an overflow occurs when the sum exceeds the finite word length of the register used to store the sum.

The overflow, in addition, may lead to oscillations in the output, which we call overflow limit cycles.

Types of FIR Filters

Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric), there are following four possible types of impulse response for linear phase FIR filters.

1. Symmetrical impulse response when N is odd.
2. Symmetrical impulse response when N is even.
3. Antisymmetric impulse response when N is odd.
4. Antisymmetric impulse response when N is even.

9.4 DESIGN TECHNIQUES FOR FIR FILTERS

The well known methods of designing FIR filters are as follows:

1. Fourier series method
2. Window method
3. Frequency sampling method
4. Optimum filter design

Linear-phase filters

The ability to have an exactly linear phase response is the one of the most important of FIR filters

$$H(\omega) = |H(\omega)|\, e^{j\phi(\omega)} \qquad \text{where } \phi(\omega) = -\omega n_0$$

A general FIR filter does not have a linear phase response but this property is satisfied when

$$h(n) = \pm h(M-1-n), \quad n = 0, 1, \ldots, M-1.$$

linear phase filter types

Impulse response	# coefs	Type
$h(n) = h(M-1-n)$	Odd	1
$h(n) = h(M-1-n)$	Even	2
$h(n) = -h(M-1-n)$	Odd	3
$h(n) = -h(M-1-n)$	Even	4

1. Symmetrical impulse response when N is odd.
2. Symmetrical impulse response when N is even.
3. Antisymmetric impulse response when N is odd.
4. Antisymmetric impulse response when N is even.

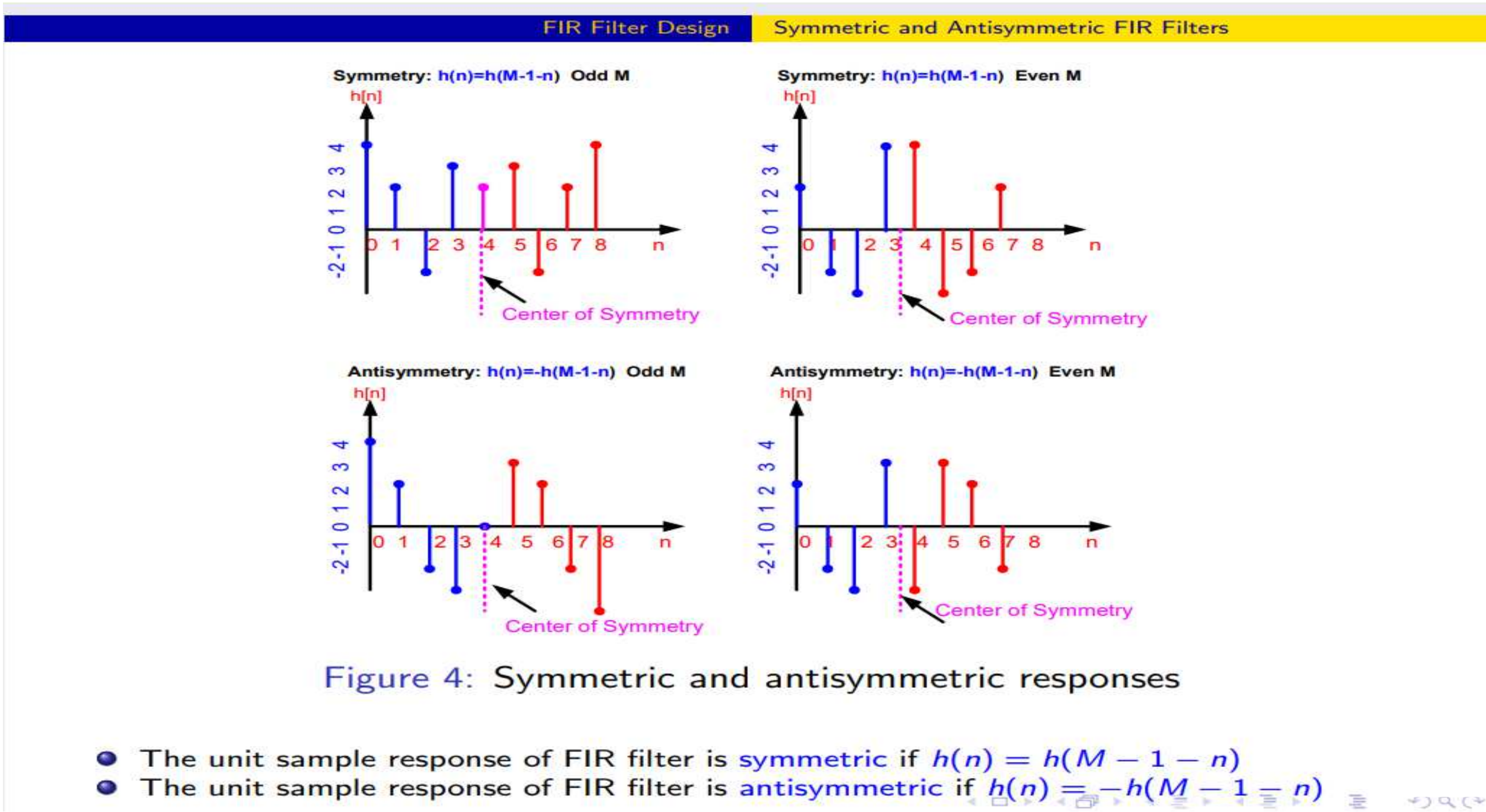
Symmetric and Antisymmetric FIR Filters Linear Phase FIR structure

- Linear phase is a property of a filter, where the phase response of the filter is a linear function of frequency. The result is that all frequency components of the input signal are shifted in time (usually delayed) by the same constant amount, which is referred to as the phase delay. And consequently, there is no phase distortion due to the time delay of frequencies relative to one another.
- Linear-phase filters have a symmetric impulse response.
- The FIR filter has linear phase if its unit sample response satisfies the following condition:

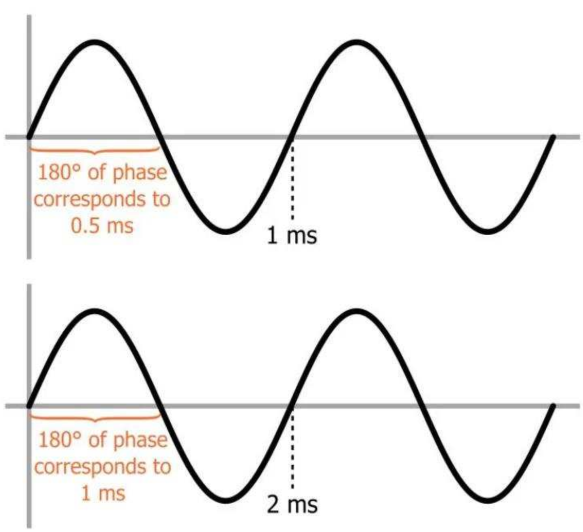
$$h(n) = h(M-1-n) \qquad n = 0, 1, 2, \ldots, N-1$$

- The Z transform of the unit sample response is given as

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$



Linear-Phase

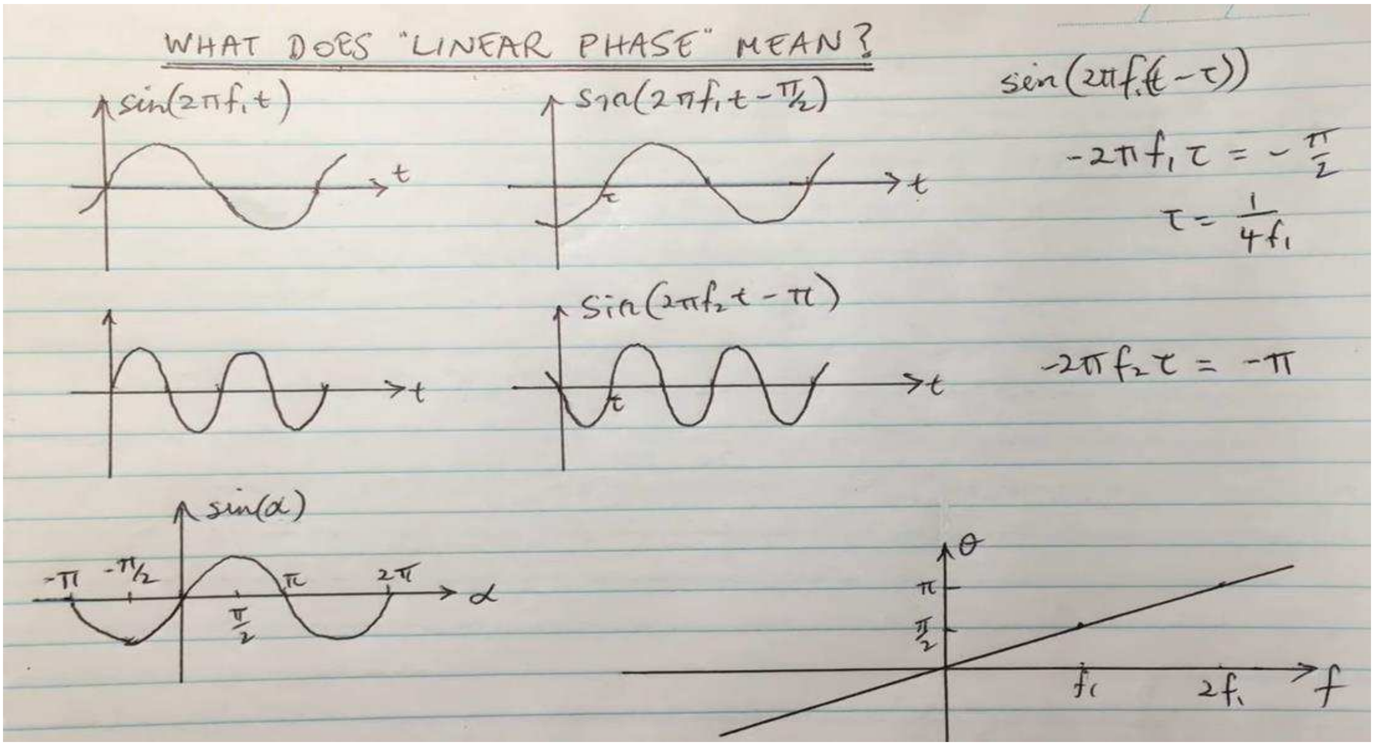


- **Two sine waves, one at 1 kHz (i.e., period = 1 ms) and one at 500 Hz (i.e., period = 2 ms).**

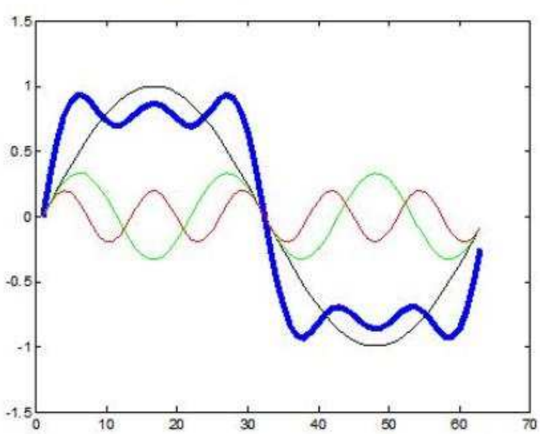
- **180° phase shift as corresponds to a different amount of time for each frequency: a different frequency means a different period, and a phase shift corresponds to a specified proportion of the period.**

- **It follows, then, that maintaining synchronization between the various frequency components of a signal does not mean enforcing a constant phase shift, *because a constant phase shift would result in different temporal delays*.**

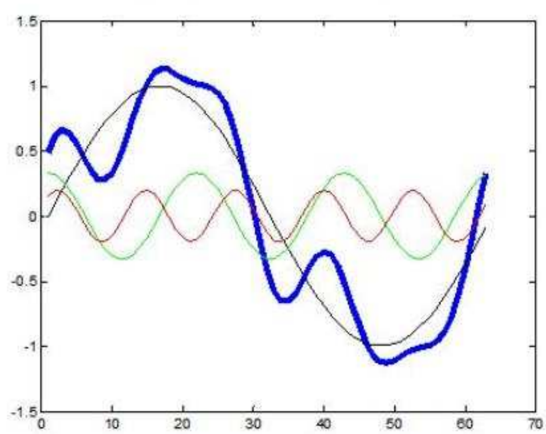
- **An ideal linear-phase filter, then, exhibits phase shift that increases linearly with frequency, and it thereby provides constant temporal delay (this applies primarily to the frequencies within the passband, i.e., the frequencies of interest).**



Linear Phase ☺



Non-Linear Phase ☹



- Low-pass filter is used to eliminate high-frequency fluctuations (eg. noise filtering, demodulation, etc.)
- High-pass filter is used to follow small-amplitude high-frequency perturbations in presence of much larger slowly-varying component (e.g. recording the electrocardiogram in the presence of a strong breathing signal)
- Band-pass is used to select a required modulated carrier signal (e.g. radio)
- Band-stop filter is used to eliminate single-frequency (e.g. mains) interference (also known as notch filtering)

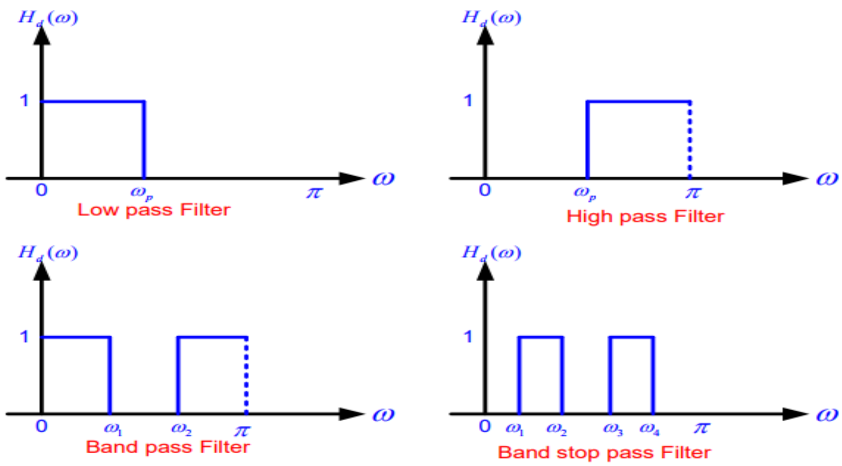


Figure 5: Frequency response characteristic of different types of filters

FIR Filter Design

Design steps for Linear Phase FIR Filter (Fourier Series method)

1. Based on the desired frequency response specification $H_d(e^{j\omega})$ determine the corresponding unit sample response $h_d(n)$.

$$H_d(e^{j\omega}) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n}$$

2. Obtain the impulse response $h_d(n)$ for the desired frequency response $H_d(\omega)$ by evaluating the inverse Fourier transform.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

3. In general the sample response $h_d(n)$ is infinite in duration and must be truncated at some point to get an FIR filter of length M. Truncation is achieved by multiplying $h_d(n)$ by window function.

$$h(n) = h_d(n)w(n)$$

   where $w(n)$ is window function

4. Obtain the $H(z)$ for $h(n)$ by taking z transform
5. Obtain the magnitude response $|H(e^{j\omega})|$ and phase response $\theta(\omega)|$

Back to Our Ideal Low- pass Filter Example

Let us consider for example a simple ideal lowpass filter defined by

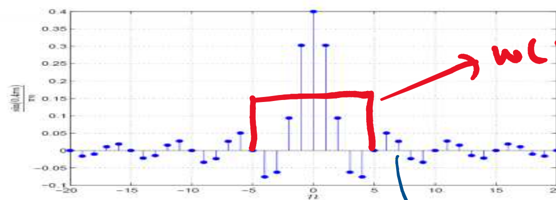
$$H_d(\omega) = \begin{cases} 1 \text{ if } |\omega| \le \omega_c \\ 0 \text{ if } \omega_c < |\omega| < \pi. \end{cases}$$

It can be shown easily that the impulse response is given by

$$h_d(n) = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}.$$

➡ Desired impulse response has a sinc shape which is non-causal and infinite in duration.

clearly cannot be implemented in practice



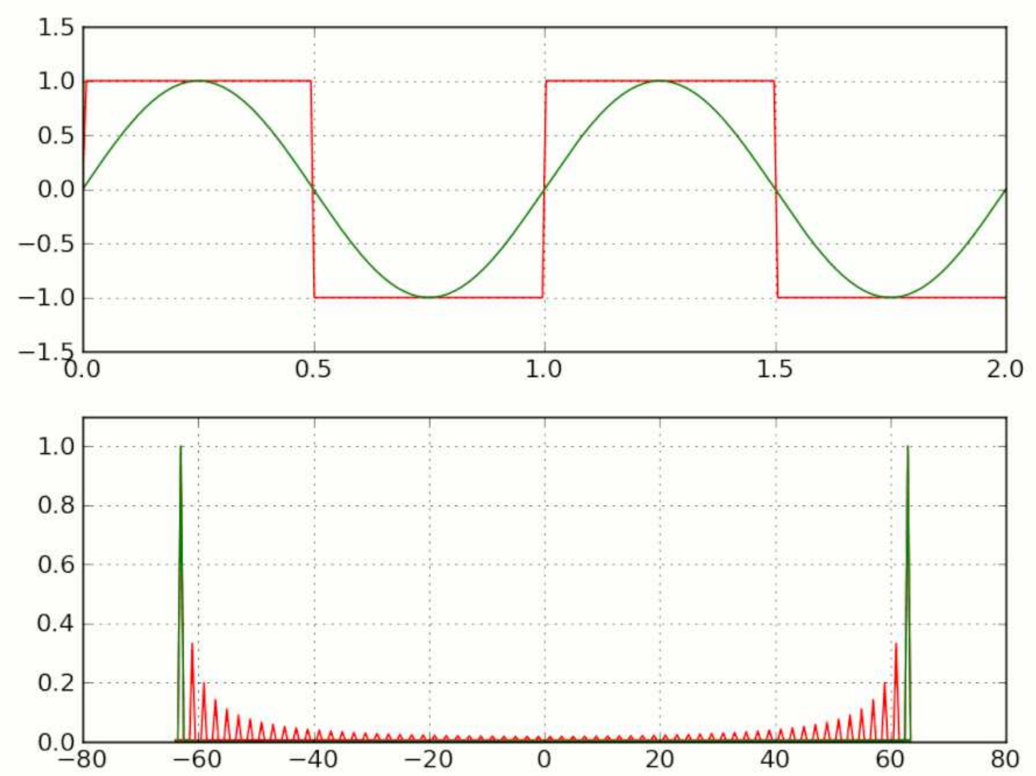
$w(n)$

$h_d(n)$

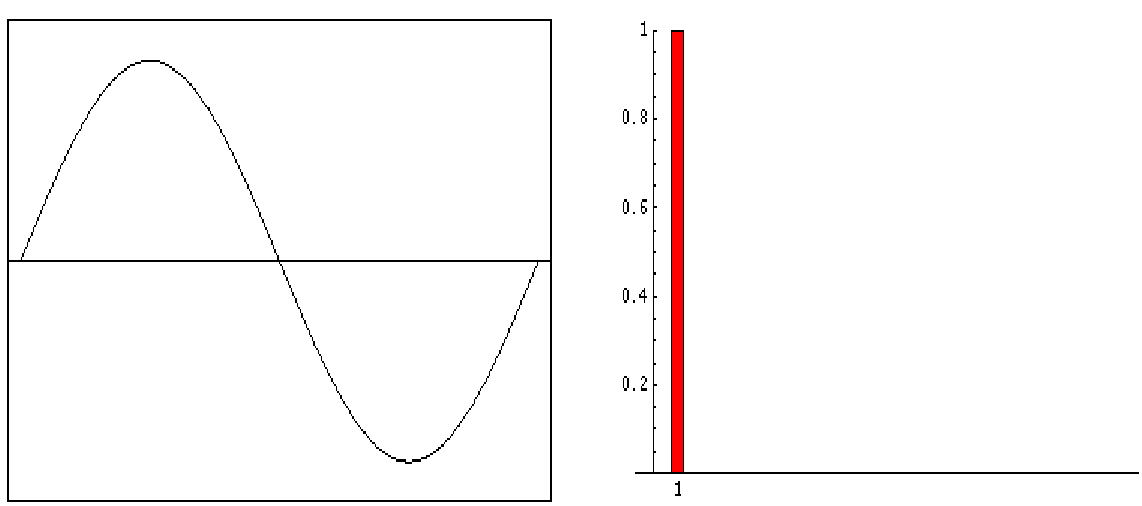
FIR $\quad h(n) = h_d(n) \times w(n)$

Effect of windowing : **Gibbs Phenomenon**

Effect of windowing : Gibbs Phenomenon



Effect of windowing : Gibbs Phenomenon



Rectangular Window

$\sin\theta = \frac{e^{j\theta} - e^{j\theta}}{2j}$

$$\mathbf{w_R(n) = 1 \;; n = -\frac{N-1}{2} \rightarrow \frac{N-1}{2}}$$
$$\mathbf{= 0 \;;} \text{ other } \mathbf{n}$$

DTFT of Rectangular Window

$$\mathbf{W_R(e^{j\omega}) = F\{w_R(n)\} = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega\left(n-\frac{N-1}{2}\right)} = \sum_{n=0}^{N-1} e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)}}$$

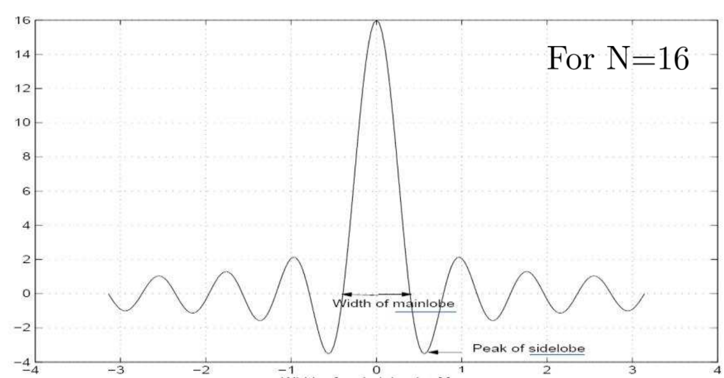
$$\mathbf{= e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = e^{j\omega\left(\frac{N-1}{2}\right)} \frac{e^{\frac{-j\omega N}{2}} e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}} e^{\frac{-j\omega N}{2}}}{e^{\frac{-j\omega}{2}} e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}} e^{\frac{-j\omega}{2}}}}$$

$$\mathbf{= e^{j\omega\frac{N}{2}} \cancel{e^{\frac{-j\omega}{2}}} \frac{e^{-j\omega\frac{N}{2}}\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{\cancel{e^{\frac{-j\omega}{2}}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)} = \frac{\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}} = \boxed{\frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}}$$

Effect of windowing in the frequency domain

$$\mathbf{w_R(n) = 1 \;; n = -\frac{N-1}{2} \rightarrow \frac{N-1}{2}}$$
$$\mathbf{= 0 \;;} \text{ other } \mathbf{n}$$

In frequency domain $\quad \mathbf{W_R(e^{j\omega}) = \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}$



Ideal LPF

$\mathbf{H_d(e^{j\omega}) = 1}$; for $\; -\omega_c \leq \omega \leq \omega_c$

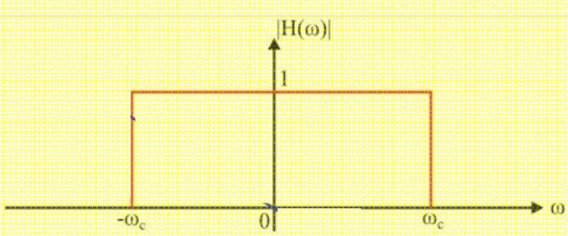
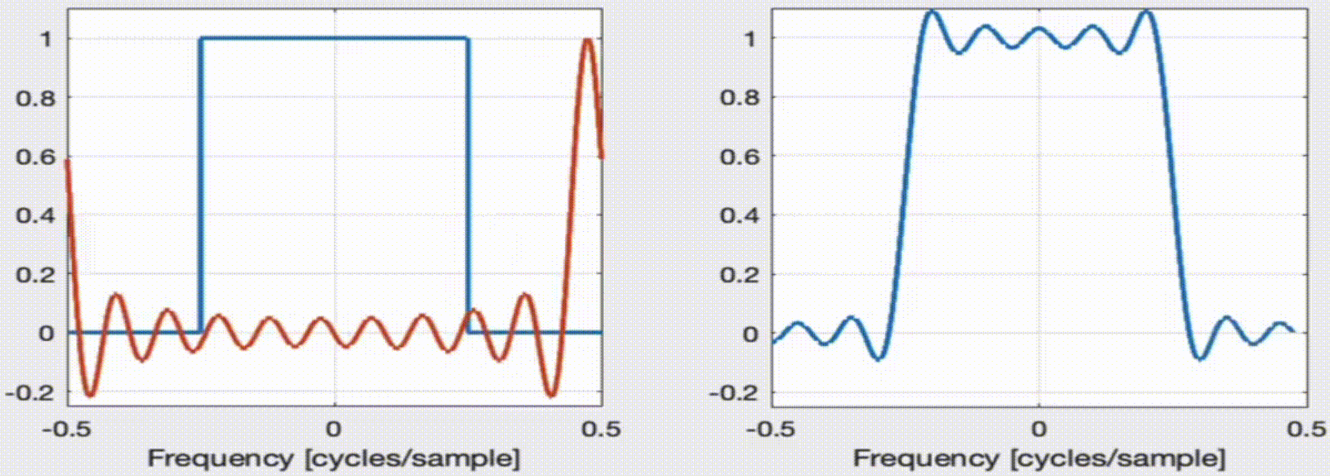


Figure-1

Periodic Convolution in Frequency Domain

multiplication in time — convolution in the frequency

$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$



Conflicting Ideal Requirements

- The width of the transition region between passband and stopband in $H(\omega)$ increases with the width of the main lobe of $W(\omega)$.
- The ripple in the passband and stopband is dependent on the area under the side-lobes.
- For the rectangular window, the width of the main lobe decreases as M increases, but it can be shown that the area under the sidelobes remain constant.

> ⇒ Transition region gets smaller but the ripple remains.
> ⇒ PROBLEM: Sharp discontinuity of rectangular windows!

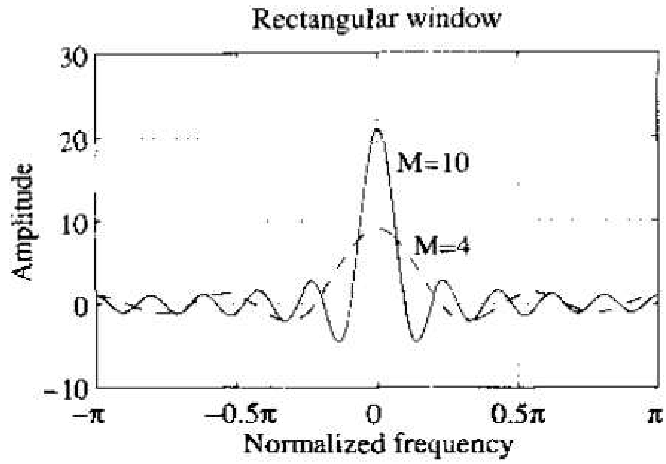


Figure 7.18: Frequency responses of the rectangular window for $M = 4$ and $M = 10$.

$$H(f) = H_i(f) \circledast W(f)$$

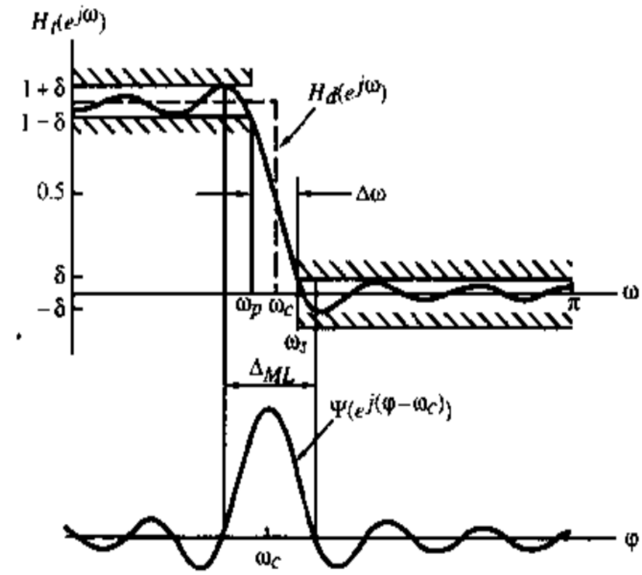
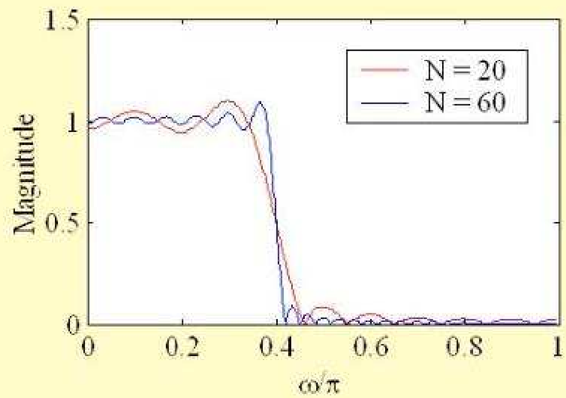


Figure 7.20: Relations among the frequency responses of an ideal lowpass filter, a typical window, and the windowed filter.

Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Solution to Sharp Discontinuity of Rectangular Window

Use windows with no abrupt discontinuity in their time-domain response and consequently low side-lobes in their frequency response.

In this case, the reduced ripple comes at the expense of a wider transition region but this

However, this can be compensated for by increasing the length of the filter.

7.6.4 Fixed Window Functions

Many tapered windows have been proposed by various authors. A discussion of all these suggested windows is beyond the scope of this text. We restrict our discussion to three commonly used tapered windows of length $2M+1$, which are listed below [Sar93]:[6]

$$\textit{Hann:}^{8} \quad w[n] = \frac{1}{2}\left[1+\cos\left(\frac{2\pi n}{2M+1}\right)\right], \qquad -M \le n \le M, \tag{7.74}$$

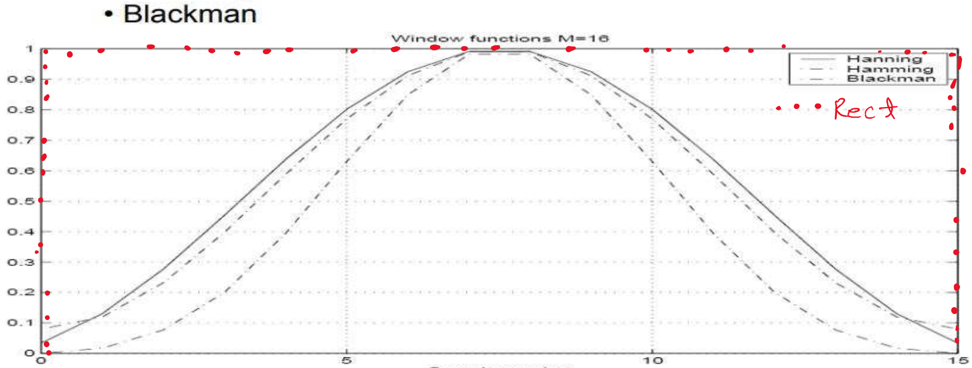
$$\textit{Hamming:} \; w[n] = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M, \tag{7.75}$$

$$\textit{Blackman:} \; w[n] = 0.42 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right) + 0.08\cos\left(\frac{4\pi n}{2M+1}\right), \qquad -M \le n \le M. \tag{7.76}$$

Alternative Windows –Time Domain

Many alternatives have been proposed, e.g.

- Hanning
- Hamming
- Blackman



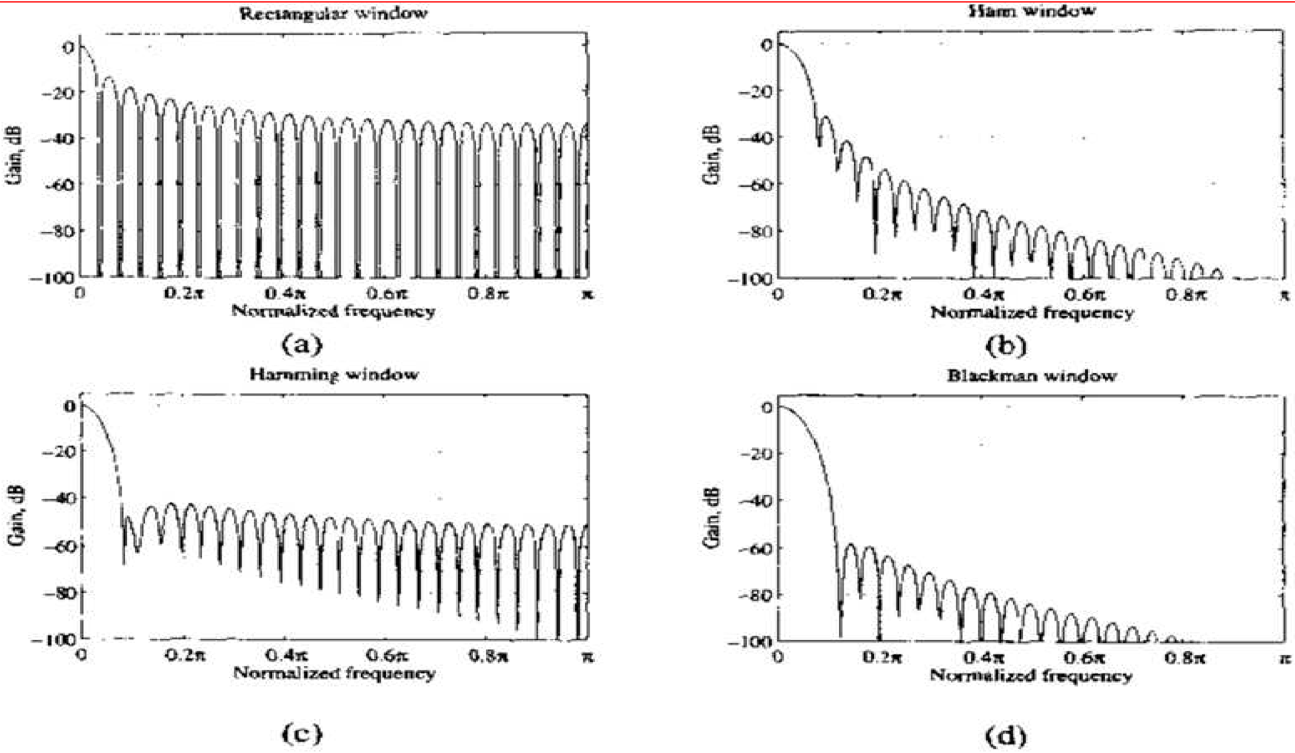


Figure 7.19: Gain response of the fixed window functions.

7.6.4 Fixed Window Functions

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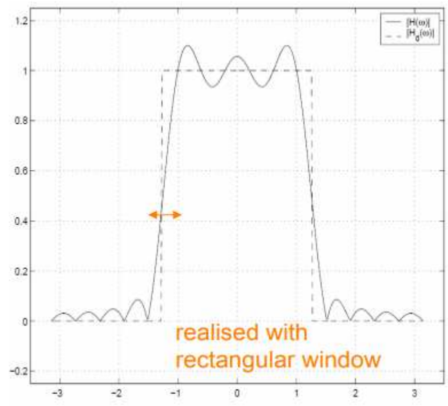
TABLE 9.2 Frequency domain characteristics of some window functions.

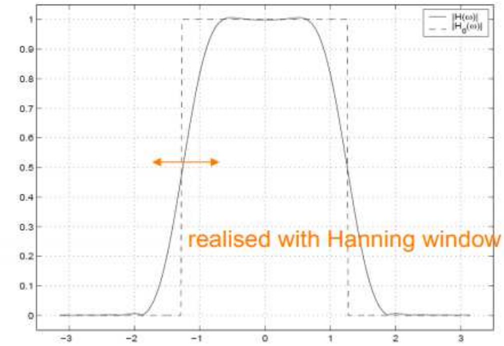
Type of window	*Approximate transition width of main lobe*	*Minimum stop band attenuation* (dB)	*Peak of first side lobe* (dB)
Rectangular	$4\pi/N$	–21	–13
Bartlett	$8\pi/N$	–25	–25
Hanning	$8\pi/N$	– 44	–31
Hamming	$8\pi/N$	–51	– 41
Blackmann	$12\pi/N$	–78	–58

Filter realised with rectangular/Hanning windows



M=16

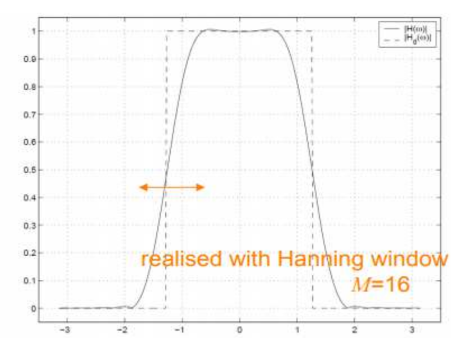
Back to our ideal filter

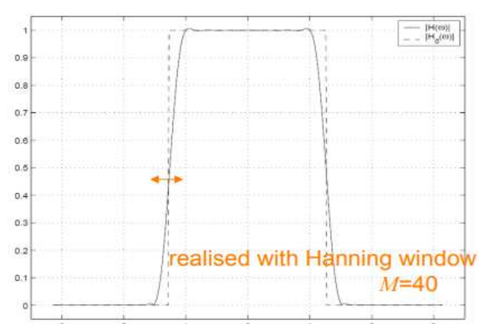


M=16

There are much less ripples for the Hanning window but that the transition width has increased

Filter realised with Hanning windows



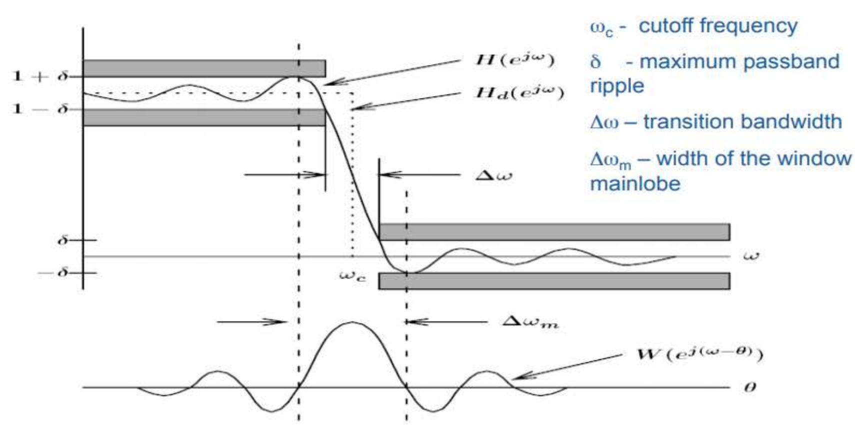


Transition width can be improved by increasing the size of the Hanning window to M = 40

Windows characteristics

- Fundamental trade-off between main-lobe width and side-lobe amplitude
- As window *smoother*, peak *side-lobe decreases*, but the *main-lobe width increases.*
- Need to increase window length to achieve same transition bandwidth.

Specification necessary for Window Design Method



Response must not enter shaded regions

Time-delay in desired response

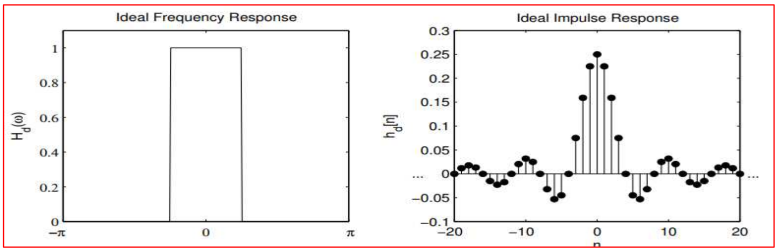
The term $e^{-j\omega(M-1)/2}$ in $\mathcal{W}(\omega)$ above comes from the fact that the rectangular window is not centered around $n = 0$, but rather is time-shifted to be centered around $n = (M-1)/2$. This phase term will cause additional distortion of $\mathcal{H}_d(\omega)$, unless $\mathcal{H}_d(\omega)$ is also phase-shifted to compensate.

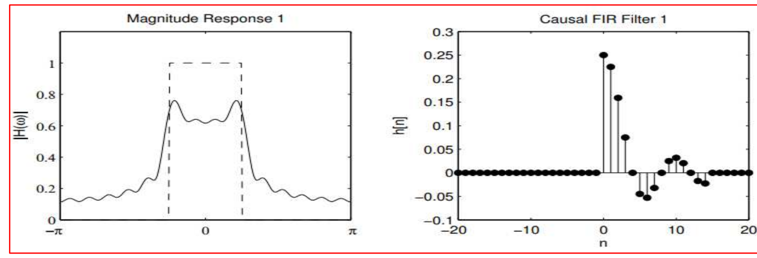
For a lowpass filter with cutoff ω_c, windowed by a length-M window function, the appropriate desired response is:

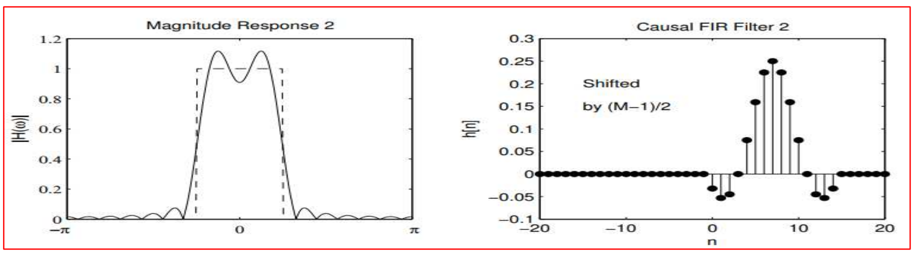
$$\mathcal{H}_d(\omega) = \begin{cases} e^{-j\omega(M-1)/2}, & |\omega| \le \omega_c, \\ 0, & \text{otherwise}, \end{cases} \quad \Longrightarrow \quad |\mathcal{H}_d(\omega)| = \begin{cases} 1, & |\omega| \le \omega_c, \\ 0, & \text{otherwise}, \end{cases}$$

so that

$$h_d[n] = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi}\left[n - \frac{M-1}{2}\right]\right).$$







Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$

where $\alpha = \frac{M-1}{2}$

Type of filter	*Ideal (desired) frequency response*	*Ideal (desired) impulse response*
Low-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\omega_c \le \omega \le \omega_c \\ 0 \quad ; & -\pi \le \omega < -\omega_c \\ 0 \quad ; & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d_\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $= \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$
High-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\pi \le \omega \le -\omega_c \\ e^{-j\alpha\omega}; & \omega_c \le \omega \le \pi \\ 0 \quad ; & -\omega_c < \omega < \omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$ $= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $= \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$

Low-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\omega_c \le \omega \le \omega_c \\ 0 \quad ; & -\pi \le \omega < -\omega_c \\ 0 \quad ; & \omega_c < \omega \le \pi \end{cases}$	

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{-j\alpha w} e^{jwn}\, dw = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jw(n-\alpha)}\, dw$$

$$= \frac{1}{2\pi}\left[\frac{e^{jw(n-\alpha)}}{j(n-\alpha)}\right]_{-w_c}^{w_c}$$

$$= \frac{1}{2j\pi(n-\alpha)}\left(e^{jw_c(n-\alpha)} - e^{-jw_c(n-\alpha)}\right)$$

$$\Rightarrow h_d(n) = \frac{\sin(w_c(n-\alpha))}{\pi(n-\alpha)} \quad \text{when } n \neq \alpha$$

Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$

where $\alpha = \frac{M-1}{2}$

Type of filter	*Ideal (desired) frequency response*	*Ideal (desired) impulse response*
Low-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\omega_c \le \omega \le \omega_c \\ 0 \quad ; & -\pi \le \omega < -\omega_c \\ 0 \quad ; & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d_\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega = \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$
High-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\pi \le \omega \le -\omega_c \\ e^{-j\alpha\omega}; & \omega_c \le \omega \le \pi \\ 0 \quad ; & -\omega_c < \omega < \omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega = \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-w_c} e^{jw(n-\alpha)}\,dw + \frac{1}{2\pi}\int_{w_c}^{\pi} e^{jw(n-\alpha)}\,dw$$

$$= \frac{1}{2\pi}\times\frac{1}{j(n-\alpha)}\left[e^{jw(n-\alpha)}\right]_{-\pi}^{-w_c} + \frac{1}{2\pi}\times\frac{1}{j(n-\alpha)}\left[e^{jw(n-\alpha)}\right]_{w_c}^{\pi}$$

$$= \frac{1}{2j\pi(n-\alpha)}\left(e^{-jw_c(n-\alpha)} - e^{-j\pi(n-\alpha)}\right) + \frac{1}{2\pi j(n-\alpha)}\left(e^{j\pi(n-\alpha)} - e^{jw_c(n-\alpha)}\right)$$

$$= \frac{1}{\pi(n-\alpha)}\left(\frac{e^{-jw_c(n-\alpha)} - e^{jw_c(n-\alpha)}}{2j}\right) + \frac{1}{\pi(n-\alpha)}\left(\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j}\right)$$

$$= \frac{\sin(\pi(n-\alpha)) - \sin(w_c(n-\alpha))}{(n-\alpha)\pi}$$

Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \textit{for } |\omega| \le \omega_c \\ 0 & \textit{Otherwise} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Use rectangular window.

Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \text{for } |\omega| \le \omega_c \\ 0 & \text{Otherwise} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Use rectangular window.

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} = \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-\alpha)}$$

$$= \frac{1}{2\pi} \times \frac{1}{j(n-\alpha)} \left[e^{j\omega(n-\alpha)} \right]_{-1}^{1} = \frac{1}{\pi(n-\alpha)} \times \frac{1}{2j}\left[e^{j(n-\alpha)} - e^{-j(n-\alpha)} \right]$$

$$h_d(n) = \frac{\sin(n-\alpha)}{\pi(n-\alpha)} \quad \text{when } n \ne \alpha \qquad \text{when } n = \alpha \Rightarrow h_d(n) = \frac{1}{\pi}$$

Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \text{for } |\omega| \le \omega_c \\ 0 & \text{Otherwise} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Use rectangular window.

Sol:- $\alpha = \frac{M-1}{2} = \frac{7-1}{2} = 3$, $n = 0, 1, 2, 3$ $\quad h(n) = h(M-1-n)$

$$\Rightarrow h_d(n) = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} \Rightarrow h_d(0) = \frac{\sin(1(0-3))}{\pi(0-3)} = 0.0150$$

$$h_d(1) = \frac{\sin(1-3)}{\pi(1-3)} = 0.1447 \Rightarrow h_d(2) = \frac{\sin(2-3)}{\pi(2-3)} = 0.2678$$

when $n = \alpha \Rightarrow$ use L' hospital rule

$n = 3 \quad h_d(3) = \frac{1}{\pi} = 0.3183$

$$\begin{aligned} h_d(0) &= h(6) = 0.0150 \\ h_d(1) &= h(5) = 0.1447 \\ h_d(2) &= h(4) = 0.2678 \\ h_d(3) &= 0.3183 \end{aligned}$$

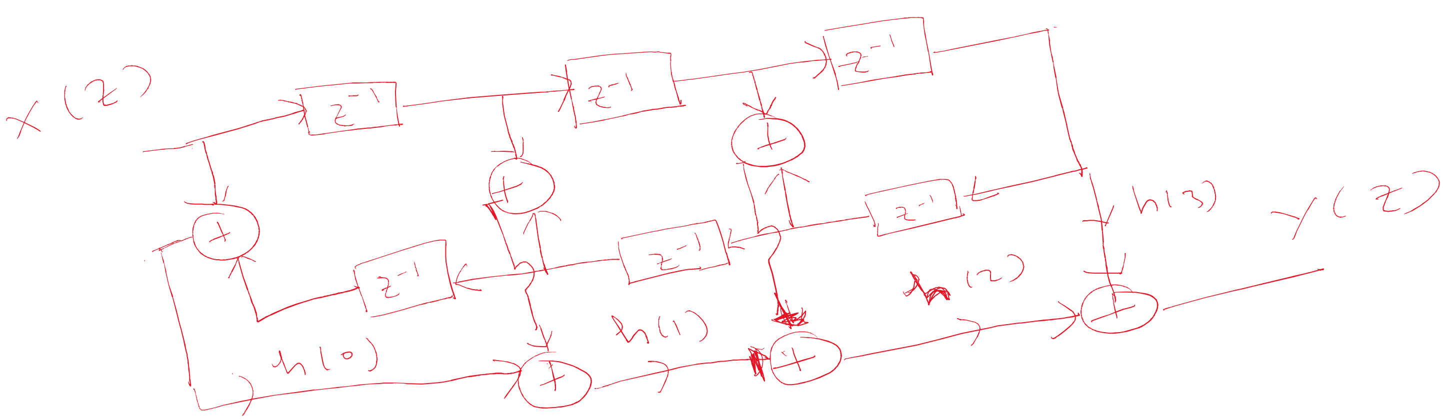
$$\mathbf{H[z] = \sum_{n=0}^{6} h(n)z^{-n}}$$

$$\mathbf{H[z] = \sum_{n=0}^{6} h(n)z^{-n}}$$

$$\mathbf{H[z] = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}}$$

$$\mathbf{H[z] = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}}$$

$$\mathbf{H[z] = h(0)(1 + z^{-6}) + h(1)(z^{-1} + z^{-5}) + h(2)(z^{-2} + z^{-4}) + h(3)z^{-3}}$$



Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & for\ |\omega| \le \omega_c \\ 0 & Otherwise \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Design the FIR filter using Hanning window

Sol:- $\alpha = \frac{M-1}{2} = \frac{7-1}{2} = 3$, $n = 0, 1, 2, 3$ $\quad h(n) = h(M-1-n)$

$\Rightarrow h_d(n) = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}$ $\Rightarrow h_d(0) = \frac{\sin(1(0-3))}{\pi(0-3)} = 0.0150$

$h_d(1) = \frac{\sin(1-3)}{\pi(1-3)} = 0.1447 \Rightarrow h_d(2) = \frac{\sin(2-3)}{\pi(2-3)} = 0.2678$

when $n = \alpha \Rightarrow$ use L' Hospital rule

$n = 3 \quad h_d(3) = \frac{1}{\pi} = 0.3183$

$h_d(0) = h(6) = 0.0150$
$h_d(1) = h(5) = 0.1447$
$h_d(2) = h(4) = 0.2678$
$h_d(3) = 0.3183$

Design the FIR filter using Hanning window

Solution: For M=7

$$\omega(n) = 0.5(1 - cos\frac{2\pi n}{M-1})$$
$$\omega(n) = 0.5(1 - cos\frac{2\pi n}{6})$$

$$\omega(0) = 0.0$$
$$\omega(1) = 0.5(1 - cos\frac{2\pi}{6}) = .25$$
$$\omega(2) = 0.5(1 - cos\frac{4\pi}{6}) = .75$$
$$\omega(3) = 0.5(1 - cos\frac{6\pi}{6}) = 1$$
$$\omega(4) = 0.5(1 - cos\frac{8\pi}{6}) = .75$$
$$\omega(5) = 0.5(1 - cos\frac{10\pi}{6}) = .25$$
$$\omega(6) = 0.5(1 - cos\frac{12\pi}{6}) = 0$$

To calculate the value of h(n)

$$h(n) = h_d(n)w(n)$$

$$h(0) = h_d(0)w(0) = 0.01497 \times 0 = 0$$
$$h(1) = h_d(1)w(1) = 0.014472 \times 0.25 = 0.03618$$
$$h(2) = h_d(2)w(2) = 0.26785 \times 0.75 = 0.20089$$
$$h(3) = h_d(3)w(3) = 0.31831 \times 1 = 0.31831$$
$$h(4) = h_d(4)w(4) = 0.26785 \times 0.75 = 0.20089$$
$$h(5) = h_d(5)w(5) = 0.14472 \times 0.25 = 0.03618$$
$$h(6) = h_d(6)w(6) = 0.014497 \times 0.0 = 0$$

Determine the filter coefficients $h_d(n)$ for the desired frequency response of a low pass filter given by

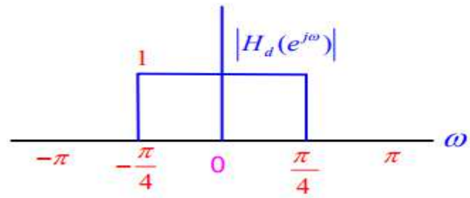
$$H_d(e^{j\omega}) = \begin{cases} e^{-2j\omega} & for \ -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & for \ \frac{\pi}{4} \le |\omega| \le -\pi \end{cases}$$

If we define the new filter coefficients by $h_d(n) = h_d(n)\omega(n)$ where

$$\omega(n) = \begin{cases} 1 & for \ 0 \le n \le 4 \\ 0 & otherwise \end{cases}$$

Solution:

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{-\pi/4}^{\pi/4} \end{aligned}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2j\pi(n-2)}\left[e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}\right] \\ &= \frac{1}{\pi(n-2)}\left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}}{2j}\right] \end{aligned}$$

$n \neq 2$

$$h_d(n) = \frac{sin\frac{\pi(n-2)}{4}}{\pi(n-2)}$$

for n=2 $h_d(n) = \frac{0}{0}$. Using L Hospital's Rule

$$\lim_{n\to 2}\frac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)} = \frac{\pi/4}{\pi} = 0.25$$

The given window function is

$$\omega(n) = \begin{cases} 1 & for \ 0 \le n \le 4 \\ 0 & Otherwise \end{cases}$$

This is rectangular window of length M=5.
In this case $h(n) = h_d(n)$ $for \ 0 \le n \le 4$

n	$h_d(n)$	n	$h_d(n)$
0	0.159091	3	0.224989
1	0.224989	4	0.159091
2	0.25		

Design an FIR filter (lowpass) using rectangular window with passband gain of 0 dB, cutoff frequency of 200 Hz, sampling frequency of 1 kHz. Assume the length of the impulse response as 7.

Solution:

$F_c = 200$ Hz, $F_s = 1000$ Hz,
$f_c = \frac{F_c}{F_s}\frac{200}{1000} = 0.2 cycles/sample$
$\omega_c = 2\pi * f_c = 2\pi \times 0.2 = 0.4\pi rad$
M=7

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ -\omega_c \leq \omega \leq \omega_c \\ 0 & Otherwise \end{cases}$$

$\tau = (M-1)/2 = 7 - 1/2 = 3$
$\omega_c = 0.4\pi$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-0.4\pi}^{0.4\pi} e^{j\omega(n-3)} d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(n-3)}}{j(n-3)}\right]_{-0.4\pi}^{0.4\pi} \end{aligned}$$

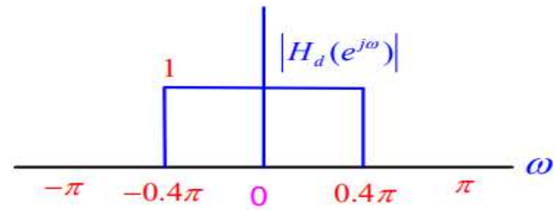


Figure 27: Frequency response of LPF

when $n \neq 3$

$$h_d(n) = \frac{sin[0.4\pi(n-3)]}{\pi(n-3)}$$

when $n = 3$

$$h_d(n) = \frac{0.4\pi}{\pi} = 0.4$$

Determine the value of $h(n)$

Since it is rectangular window $h(n) = w(n) = h_d(n) = h(n)$

For M=7

n	$h(n)$	n	$h(n)$
0	-0.062341	4	-0.062341
1	0.093511	5	0.093511
2	0.302609	6	0.302609
3	0.4		

Using rectangular window design a lowpass filter with passband gain of unity, cutoff frequency of 1000 Hz, sampling frequency of 5 kHz. The length of the impulse response should be 7.
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Solution:
$F_c = 1000$ Hz, $F_s = 5000$ Hz,
$f_c = \frac{F_c}{F_s}\frac{1000}{5000} = 0.2 cycles/sample$
$\omega_c = 2\pi f_c = 2 \times \pi \times 0.2 = 0.4\pi rad$
M=7
The filter specifications (ω_c and M=7) are similar to the previous example. Hence same filter coefficients are obtained.
h(0)=-0.062341, h(1)=0.093511, h(2)=0.302609
h(3)=0.4, h(4)=0.302609, h(5)=0.093511, h(6)=-0.062341

Design a normalized linear phase FIR low pass filter having phase delay of $\tau = 4$ and at least 40 dB attenuation in the stopband. Also obtain the magnitude/frequency response of the filter.

Solution: The linear phase FIR filter is normalized means its cut-off frequency is of $\omega_c = 1 rad/sample$
The length of the filter with given τ is related by

$$\tau = \frac{M-1}{2}$$

For $\tau = 4$ M=9
Desired unit sample response $h_d(n)$ is

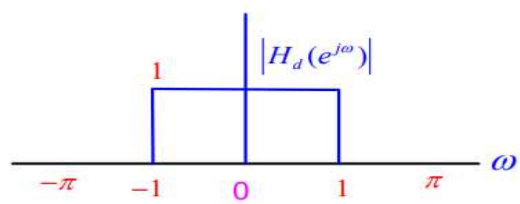


Figure 28: Frequency response of LPF

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-4)} d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(n-4)}}{j(n-4)}\right]_{-1}^{1} \end{aligned}$$

when $n \neq 4$

$$h_d(n) = \frac{sin[(n-4)]}{\pi(n-4)}$$

when $n = 4$

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} 1 d\omega = \frac{\omega}{\pi} = \frac{1}{\pi}$$

Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \text{for } |\omega| \le \omega_c \\ 0 & \text{Otherwise} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Use rectangular window.

Sol:- $\alpha = \frac{M-1}{2} = \frac{7-1}{2} = 3$, $n = 0, 1, 2, 3$ $\quad h(n) = h(M-1-n)$

$\Rightarrow h_d(n) = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} \Rightarrow h_d(0) = \frac{\sin(1(0-3))}{\pi(0-3)} = 0.0150$

$h_d(1) = \frac{\sin(1-3)}{\pi(1-3)} = 0.1447 \Rightarrow h_d(2) = \frac{\sin(2-3)}{\pi(2-3)} = 0.2678$

when $n = \alpha \Rightarrow$ use L' Hospital rule

$n = 3 \quad h_d(3) = \frac{1}{\pi} = 0.3183$

$h_d(0) = h(6) = 0.0150$
$h_d(1) = h(5) = 0.1447$
$h_d(2) = h(4) = 0.2678$
$h_d(3) = 0.3183$

Step 1. Select a suitable window function

Choosing a suitable window function can be done with the aid of published data such as

Window's name	Mainlobe	Mainlobe/sidelobe	Peak $20\log_{10}\delta$
Rectangular	$4\pi/\ M$	-13dB	-21dB
Hanning	$8\pi/M$	-32dB	-44dB
Hamming	$8\pi/M$	-43dB	-53dB
Blackman	$12\pi/M$	-58dB	-74dB

The required peak error spec $\delta_2 = 0.01$, i.e. $-20\log_{10}(\delta_s) = -40$ dB $\Rightarrow$ Hanning window

Main-lobe width $\omega_s - \omega_p = 0.3\pi - 0.2\pi = 0.1\pi$, i.e. $0.1\pi = 8\pi / M$ $\Rightarrow$ filter length $M \ge 80$, filter order $N \ge 79$

Type-I filter have even order $\Rightarrow$ $N = 80$

although for Hanning window first and last ones are 0 so only 78 in reality

Step 2 Specify the Ideal Response

Property 1: The band-edge frequency of the ideal response if the midpoint between ω_s and ω_p

$\Rightarrow \omega_c = (\omega_s + \omega_p)/2 = (0.2\pi+0.3\pi)/2 = 0.25\pi$

$$H_d(\omega) = \begin{cases} 1 & \text{if} \quad |\omega| \le 0.25\pi \\ 0 & \text{if} \quad 0.25\pi < |\omega| < \pi \end{cases}$$

our ideal low-pass filter frequency response

Step 3 Compute the coefficients of the ideal filter

- The ideal filter coefficients h_d are given by the Inverse **Discrete time** Fourier transform of $H_d(\omega)$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}. \end{aligned}$$

- Delayed impulse response (to make it causal)

$$\tilde{h}(n) = \hat{h}\left(n - \frac{N}{2}\right)$$

- Coefficients of the ideal filter

$$h(n) = \frac{\sin(0.5\pi(n-40))}{\pi(n-40)}.$$

Step 4 Multiply to obtain the filter coefficients

- Coefficients of the ideal filter

$$h(n) = \frac{\sin(0.5\pi(n-40))}{\pi(n-40)}$$

- Multiplied by a Hamming window function

$$w(n) = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Step 1. Select a suitable window function

Choosing a suitable window function can be done with the aid of published data such as

Window's name	Mainlobe	Mainlobe/sidelobe	Peak $20\log_{10}\delta$
Rectangular	$4\pi/\ M$	-13dB	-21dB
Hanning	$8\pi/M$	-32dB	-44dB
Hamming	$8\pi/M$	-43dB	-53dB
Blackman	$12\pi/M$	-58dB	-74dB

The required peak error spec $\delta_2 = 0.01$, i.e. $-20\log_{10}(\delta_s) = -40$ dB

⇨ Hanning window

Main-lobe width $\omega_s - \omega_p = 0.3\pi - 0.2\pi = 0.1\pi$, i.e. $0.1\pi = 8\pi / M$ ⇨

filter length $M \ge 80$, filter order $N \ge 79$

Type-I filter have even order ⇨ $N = 80$

although for Hanning window first and last ones are 0 so only 78 in reality

Example 1:

Design a low-pass filter with $\omega_p = 0.4\pi$ and $\omega_s = 0.6\pi$ which exhibits a minimum attenuation greater than $50dB$ in the stop-band.

1) Choose the Window Type

Since we need attenuation greater than $50dB$ in the stop-band, we may use either the Hamming or the Blackman from Table I.

2) Approximate the Window Length

As discussed in the previous article, we can find a rough estimation of the window length by equating the transition band of the filter with the main lobe width of the window.

In this example, the transition band is $\omega_s - \omega_p = 0.2\pi$. Since the main lobe width of the Hamming window is

$$\frac{8\pi}{\mathrm{M}} = 0.2\pi \qquad \mathrm{M} = 40$$

$$\boxed{w[n] = \begin{cases} 0.54 - 0.46\cos(\frac{2n\pi}{M}) & 0 \le n \le M \\ 0 & otherwise \end{cases}}$$

EXAMPLE 9.7 A low-pass filter is to be designed with the following desired frequency response:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

Determine the filter coefficients $h(n)$ if the window function is defined as:

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Also determine the frequency response $H(e^{j\omega})$ of the designed filter.

Solution: For the given filter with

$$H_d(\omega) = \begin{cases} e^{-j2\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{-\pi/4}^{\pi/4} = \frac{1}{\pi(n-2)}\left[\frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{2j}\right]$$

$$= \frac{1}{\pi(n-2)}\sin(n-2)\frac{\pi}{4}, \quad n \ne 2.$$

Applying L'Hospital rule

$$h_d(2) = \underset{n\to 2}{\text{Lt}}\ \frac{1}{\pi}\,\frac{\sin(n-2)\frac{\pi}{4}}{(n-2)} = \frac{1}{\pi}\cdot\frac{\pi}{4} = \frac{1}{4}$$

Since it is a linear phase filter, the other filter coefficients are given by

$$h_d(0) = \frac{1}{\pi(0-2)}\sin(0-2)\frac{\pi}{4} = \frac{1}{2\pi}$$

$$h_d(1) = \frac{1}{\pi(1-2)}\sin(1-2)\frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(3) = \frac{1}{\pi(3-2)}\sin(3-2)\frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(4) = \frac{1}{\pi(4-2)}\sin(4-2)\frac{\pi}{4} = \frac{1}{2\pi}$$

The frequency response $H(\omega)$ of the digital filter is given by

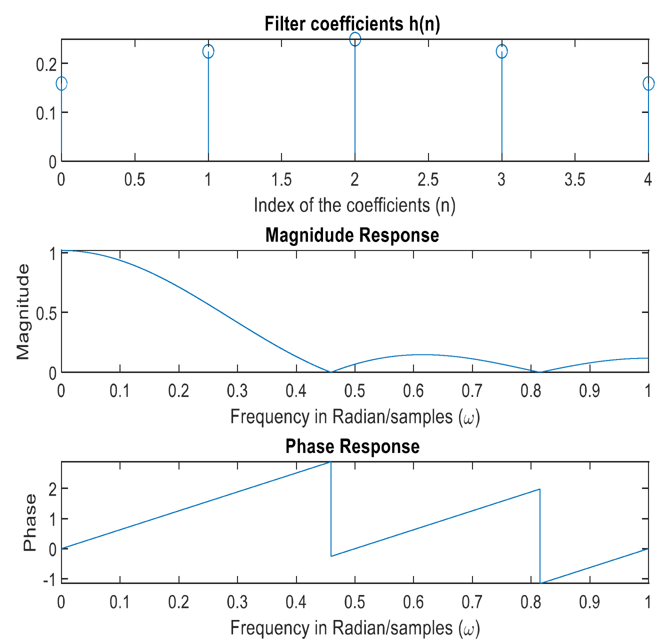
$$H(\omega) = \sum_{n=0}^{4} h(n)e^{-j\omega n}$$

$$= h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + h(3)\,e^{-j3\omega} + h(4)\,e^{-j4\omega}$$

$$= e^{-j2\omega}[h(0)\,e^{j2\omega} + h(1)\,e^{j\omega} + h(2) + h(3)\,e^{-j\omega} + h(4)\,e^{-j2\omega}]$$

$$= e^{-j2\omega}[h(2) + h(1)\,(e^{j\omega} + e^{-j\omega}) + h(0)\,(e^{j2\omega} + e^{-j2\omega})]$$

$$= e^{-j2\omega}\left[\frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\omega + \frac{1}{\pi}\cos 2\omega\right]$$



$$\omega = [\mathbf{0},\ \pi/\mathbf{4},\ \pi/\mathbf{2},\ \mathbf{3}\pi/\mathbf{4},\ \pi]$$

$$\mathbf{H}(\omega) = [\mathbf{1.0185},\ \mathbf{0.5683i},\ \mathbf{0.0683},\ \mathbf{0.0683i},\ \mathbf{0.1182}]$$

$$|\mathbf{H}(\omega)| = \sqrt{\mathbf{re}(\mathbf{H}(\omega))^2 + \mathbf{im}(\mathbf{H}(\omega))^2}$$

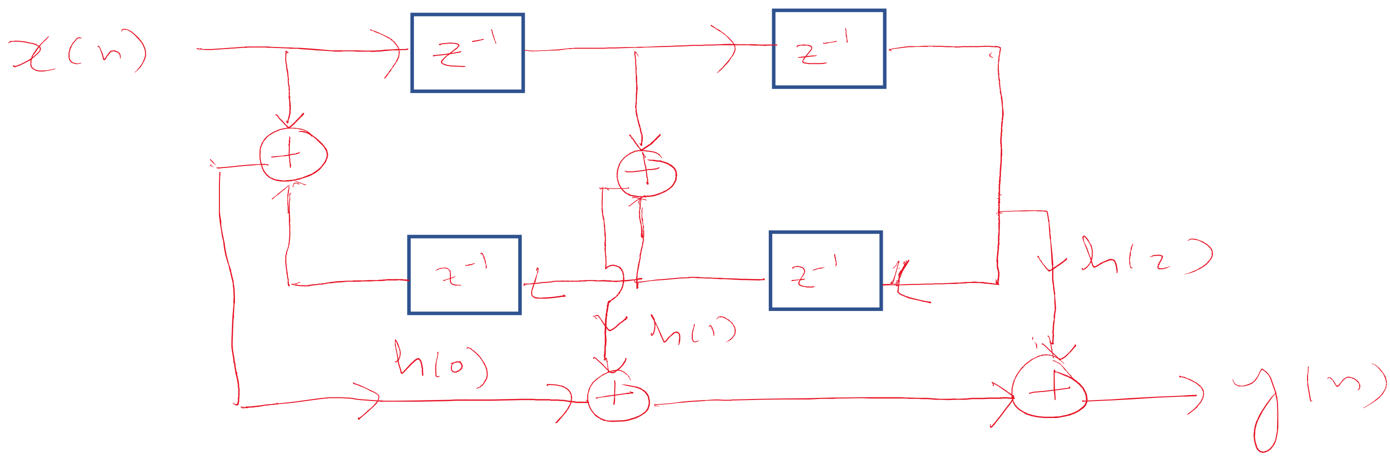
$$\angle\mathbf{H}(\omega) = \mathbf{tan}^{-1}\frac{\mathbf{im}(\mathbf{H}(\omega))}{\mathbf{re}(\mathbf{H}(\omega))}$$

Linear Phase Structure of FIR filter

$$\mathbf{H[z] = \sum_{n=0}^{4} h(n)z^{-n}}$$

$$\mathbf{H[z] = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}}$$

$$\mathbf{H[z] = h(0)(1 + z^{-4})h(1)(z^{-1} + z^{-3}) + h(2)z^{-2}}$$



Design Band-pass and Band-stop filter

Band-pass filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c1} \le \omega < \omega_{c2} \\ 0 \quad ; -\pi \le \omega < -\omega_{c2} \\ 0 \quad ; -\omega_{c1} < \omega < \omega_{c1} \\ 0 \quad ; \omega_{c2} < \omega \le \pi \end{cases}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha}e^{j\omega n}d\omega \\ &= \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$$

Band-stop filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha}; -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c2} < \omega < \pi \\ 0 \quad ; -\omega_{c2} \le \omega \le -\omega_{c1} \\ 0 \quad ; \omega_{c1} < \omega < \omega_{c2} \end{cases}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha}e^{j\omega n}d\omega \\ &\quad + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega \\ &= \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$$

EXAMPLE 9.16 Design an FIR band-stop (band reject or band elimination or notch) filter for the following specifications.

Cutoff frequencies = 400 Hz and 800 Hz

Sampling frequency = 2000 Hz

$N = 11$

The normalized cutoff frequencies are:

$$\omega_{c1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 400}{2000} = 0.4\pi$$

and

$$\omega_{c2} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 800}{2000} = 0.8\pi$$

The desired frequency response is:

$$H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -\omega_{c2}, -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi \\ 0, & \text{otherwise (i.e } -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2}) \end{cases}$$

$$\therefore \quad H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -0.8\pi, -0.4\pi \le \omega \le 0.4\pi \text{ and } 0.8\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

The desired impulse response of the filter is:

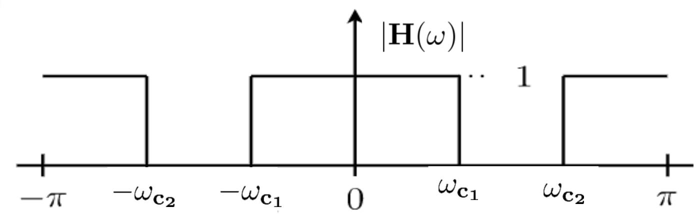
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-0.8\pi} (1)\, e^{j\omega n} d\omega + \int_{-0.4\pi}^{0.4\pi} (1)\, e^{j\omega n} d\omega + \int_{0.8\pi}^{\pi} (1)\, e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-0.8\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{-0.4\pi}^{0.4\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{0.8\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{-j0.8n\pi} - e^{-jn\pi} + e^{j0.4n\pi} - e^{-j0.4n\pi} + e^{jn\pi} - e^{j0.8n\pi}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \frac{e^{j0.8n\pi} - e^{-j0.8n\pi}}{2j} + \frac{e^{j0.4n\pi} - e^{-j0.4n\pi}}{2j}\right]$$



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The desired frequency response is:

$$H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -\omega_{c2}, -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi \\ 0, & \text{otherwise (i.e } -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2}) \end{cases}$$

$$\therefore \quad H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -0.8\pi, -0.4\pi \le \omega \le 0.4\pi \text{ and } 0.8\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

The desired impulse response of the filter is:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-0.8\pi} (1)\, e^{j\omega n} d\omega + \int_{-0.4\pi}^{0.4\pi} (1)\, e^{j\omega n} d\omega + \int_{0.8\pi}^{\pi} (1)\, e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-0.8\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{-0.4\pi}^{0.4\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{0.8\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{-j0.8n\pi} - e^{-jn\pi} + e^{j0.4n\pi} - e^{-j0.4n\pi} + e^{jn\pi} - e^{j0.8n\pi}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \frac{e^{j0.8n\pi} - e^{-j0.8n\pi}}{2j} + \frac{e^{j0.4n\pi} - e^{-j0.4n\pi}}{2j}\right]$$

$$= \frac{1}{n\pi}[\sin n\pi - \sin(0.8n\pi) + \sin(0.4n\pi)]$$

$$= \frac{1}{n\pi}[\sin(0.4n\pi) - \sin(0.8n\pi)]$$

In order to make $h_d(n)$ causal, need to shift it by $\alpha = \frac{N-1}{2}$

$$h_d(n) = \frac{\sin(0.4\pi(n-\alpha))}{(n-\alpha)\pi} - \frac{\sin(0.8\pi(n-\alpha))}{(n-\alpha)\pi}$$

EXAMPLE 9.16 Design an FIR band-stop (band reject or band elimination or notch) filter for the following specifications.

Cutoff frequencies = 400 Hz and 800 Hz

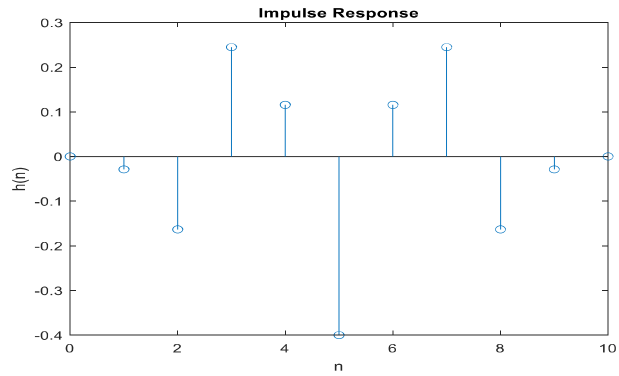
Sampling frequency = 2000 Hz

$N = 11$

$$h_d(n) = \frac{\sin(0.4\pi(n-\alpha))}{(n-\alpha)\pi} - \frac{\sin(0.8\pi(n-\alpha))}{(n-\alpha)\pi}$$

$$\alpha = \frac{11-1}{2} = 5 \qquad h_d(5) = -0.4$$

$$h_d(n) = [0, -0.0289, -0.1633, 0.2449, 0.1156, -0.4]$$



Linear Phase Structure of FIR filter

$$H[z] = \sum_{n=0}^{10} h(n)z^{-n}$$

$$H[z] = h(0)(1+z^{-10}) + h(1)(z^{-1}+z^{-9}) + h(2)(z^{-2}+z^{-8}) + h(3)(z^{-3}+z^{-7} + h(4)(z^{-4}+z^{-6} + h(5)z^{-5}$$

EXAMPLE 9.16 Design an FIR band-stop (band reject or band elimination or notch) filter for the following specifications.

Cutoff frequencies = 400 Hz and 800 Hz

Sampling frequency = 2000 Hz

$N = 11$

$$h_d(n) = \frac{\sin(0.4\pi(n-\alpha))}{(n-\alpha)\pi} - \frac{\sin(0.8\pi(n-\alpha))}{(n-\alpha)\pi}$$

$$\alpha = \frac{11-1}{2} = 5 \qquad h_d(5) = -0.4$$

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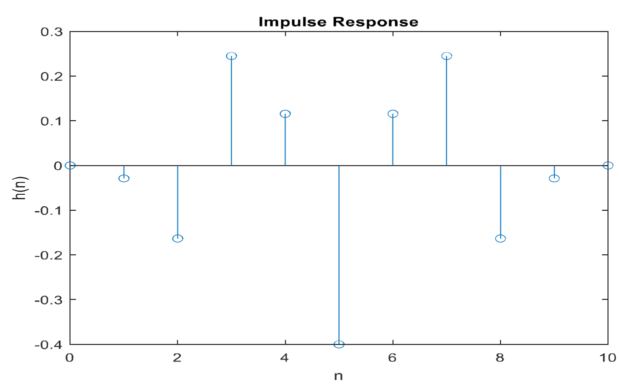
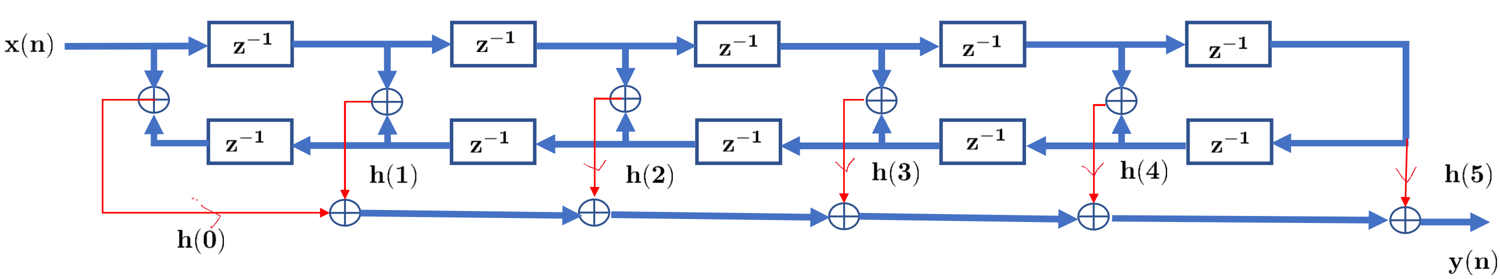


Linear Phase Structure of FIR filter



$$H[z] = h(0)(1+z^{-10}) + h(1)(z^{-1}+z^{-9}) + h(2)(z^{-2}+z^{-8}) + h(3)(z^{-3}+z^{-7} + h(4)(z^{-4}+z^{-6} + h(5)z^{-5}$$

Design Band-pass and Band-stop filter

Band-pass filter

$$H_d(\omega)=\begin{cases} e^{-j\omega\alpha}; -\omega_{c2}\le\omega\le-\omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c1}\le\omega<\omega_{c2} \\ 0 \quad ; -\pi\le\omega<-\omega_{c2} \\ 0 \quad ; -\omega_{c1}<\omega<\omega_{c1} \\ 0 \quad ; \omega_{c2}<\omega\le\pi \end{cases}$$

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\omega)e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}}e^{-j\omega\alpha}e^{j\omega n}d\omega+\frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}}e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$=\frac{\sin\omega_{c2}(n-\alpha)-\sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)}$$

Band-stop filter

$$H_d(\omega)=\begin{cases} e^{-j\omega\alpha}; -\pi\le\omega\le-\omega_{c2} \\ e^{-j\omega\alpha}; -\omega_{c1}\le\omega\le\omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c2}<\omega<\pi \\ 0 \quad ; -\omega_{c2}\le\omega\le-\omega_{c1} \\ 0 \quad ; \omega_{c1}<\omega<\omega_{c2} \end{cases}$$

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\omega)e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}}e^{-j\omega\alpha}e^{j\omega n}d\omega+\frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}}e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$+\frac{1}{2\pi}\int_{\omega_{c2}}^{\pi}e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$=\frac{\sin\omega_{c1}(n-\alpha)+\sin\pi(n-\alpha)-\sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)}$$

EXAMPLE 9.15 Design a band-pass FIR filter for the following specifications:

Cutoff frequencies = 400 Hz and 800 Hz
Sampling frequency = 2000 Hz
N = 11

The normalized cutoff frequencies are:

$$\omega_{c1}=\frac{2\pi f_{c1}}{f_s}=\frac{2\pi\times 400}{2000}=0.4\pi$$

and

$$\omega_{c2}=\frac{2\pi f_{c2}}{f_s}=\frac{2\pi\times 800}{2000}=0.8\pi$$

The desired frequency response is:

$$H_d(\omega)=\begin{cases}1, & -\omega_{c2}\le\omega\le-\omega_{c1} \text{ and } \omega_{c1}\le\omega\le\omega_{c2} \\ 0, & \text{otherwise (i.e. } -\pi\le\omega\le-\omega_{c2}, -\omega_{c1}\le\omega\le\omega_{c1} \text{ and } \omega_{c2}\le\omega\le\pi)\end{cases}$$

i.e.

$$H_d(\omega)=\begin{cases}1, & -0.8\pi\le\omega\le-0.4\pi \text{ and } 0.4\pi\le\omega\le0.8\pi \\ 0, & -\pi\le\omega\le-0.8\pi, -0.4\pi\le\omega\le0.4\pi, 0.8\pi\le\omega\le\pi\end{cases}$$

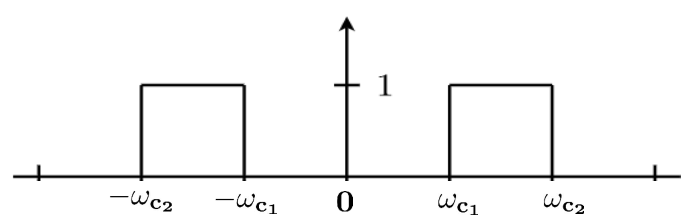
The desired impulse response of the filter is:

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\omega)\,e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-0.8\pi}^{-0.4\pi}(1)\,e^{j\omega n}d\omega+\frac{1}{2\pi}\int_{0.4\pi}^{0.8\pi}(1)\,e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.8\pi}^{-0.4\pi}+\frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.4\pi}^{0.8\pi}$$

$$=\frac{1}{n\pi}\left[\frac{e^{-j0.4\pi n}-e^{-j0.8\pi n}}{2j}+\frac{e^{j0.8\pi n}-e^{j0.4\pi n}}{2j}\right]$$



EXAMPLE 9.15 Design a band-pass FIR filter for the following specifications:

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and

$$\omega_{c2} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 800}{2000} = 0.8\pi$$

The desired frequency response is:

$$H_d(\omega) = \begin{cases} 1, & -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2} \\ 0, & \text{otherwise (i.e. } -\pi \le \omega \le -\omega_{c2}, -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi) \end{cases}$$

i.e.

$$H_d(\omega) = \begin{cases} 1, & -0.8\pi \le \omega \le -0.4\pi \text{ and } 0.4\pi \le \omega \le 0.8\pi \\ 0, & -\pi \le \omega \le -0.8\pi, -0.4\pi \le \omega \le 0.4\pi, 0.8\pi \le \omega \le \pi \end{cases}$$

The desired impulse response of the filter is:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-0.8\pi}^{-0.4\pi} (1)\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{0.4\pi}^{0.8\pi} (1)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.8\pi}^{-0.4\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.4\pi}^{0.8\pi}$$

$$= \frac{1}{n\pi}\left[\frac{e^{-j0.4\pi n} - e^{-j0.8\pi n}}{2j} + \frac{e^{j0.8\pi n} - e^{j0.4\pi n}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{j0.8n\pi} - e^{-j0.8n\pi}}{2j} - \frac{e^{j0.4\pi n} - e^{-j0.4\pi n}}{2j}\right]$$

$$= \frac{\sin(0.8n\pi)}{n\pi} - \frac{\sin(0.4n\pi)}{n\pi}$$

In order to make $h_d(n)$ causal, need to shift it by $\alpha = \frac{N-1}{2}$

$$h_d(n) = \frac{\sin(0.8\pi(n-\alpha))}{(n-\alpha)\pi} - \frac{\sin(0.4\pi(n-\alpha))}{(n-\alpha)\pi}$$

EXAMPLE 9.15 Design a band-pass FIR filter for the following specifications:

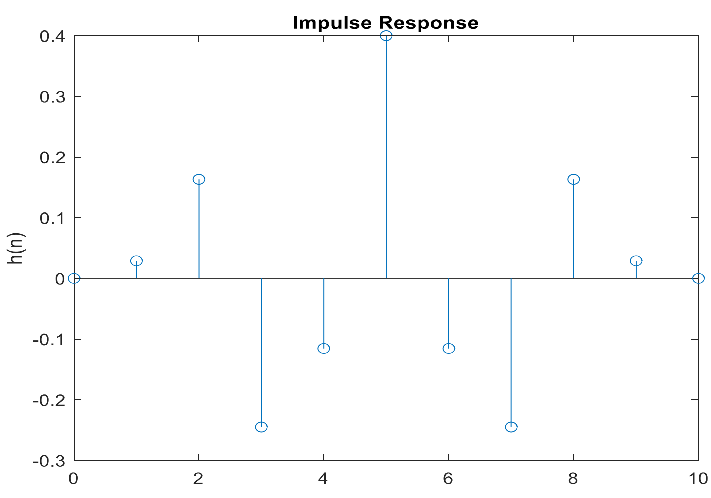
Cutoff frequencies = 400 Hz and 800 Hz

Sampling frequency = 2000 Hz

$N = 11$

$$h_d(n) = \frac{\sin(0.8\pi(n-\alpha))}{(n-\alpha)\pi} - \frac{\sin(0.4\pi(n-\alpha))}{(n-\alpha)\pi}$$

$$h_d(n) = [0, 0.0289, 0.1633, -0.2449, -0.1156, 0.4000, -0.1156, -0.2449, 0.1633, 0.0289, 0]$$



Design Band-pass and Band-stop filter

Band-pass filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c1} \le \omega < \omega_{c2} \\ 0 \quad ; -\pi \le \omega < -\omega_{c2} \\ 0 \quad ; -\omega_{c1} < \omega < \omega_{c1} \\ 0 \quad ; \omega_{c2} < \omega \le \pi \end{cases}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$$

Band-stop filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha}; -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c2} < \omega < \pi \\ 0 \quad ; -\omega_{c2} \le \omega \le -\omega_{c1} \\ 0 \quad ; \omega_{c1} < \omega < \omega_{c2} \end{cases}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &\quad + \frac{1}{2\pi} \int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$$

- An FIR system does not have feedback. Hence $y(n-k)$ term is absent in the system. FIR output is expressed as

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

- If there are M coefficients then

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

- The coefficients are related to unit sample response as

$$h(n) = \begin{cases} b_n & \text{for} \quad 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

- Expanding the summation

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots b_{(M-1)} x(n-M+1)$$

- Since $h(n) = b_n$ then y(n) is

$$\boxed{y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)}$$

Find the frequency response of FIR filter and draw the magnitude and phase response.

$$y(n) = \frac{1}{2\pi}x(n) + \frac{1}{\sqrt{2}\pi}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{\sqrt{2}\pi}x(n-3) + \frac{1}{2\pi}x(n-4)$$

Sol:

Take z-trasform

$$Y(z) = \frac{1}{2\pi}X(z) + \frac{1}{\sqrt{2}\pi}X(z)z^{-1} + \frac{1}{4}X(z)z^{-2} + \frac{1}{\sqrt{2}\pi}X(z)z^{-3} + \frac{1}{2\pi}X(z)z^{-4}$$

$$Y(z) = X(z)\left(\frac{1}{2\pi} + \frac{1}{\sqrt{2}\pi}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{\sqrt{2}\pi}z^{-3} + \frac{1}{2\pi}z^{-4}\right)$$

$$H(z) = \left(\frac{1}{2\pi} + \frac{1}{\sqrt{2}\pi}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{\sqrt{2}\pi}z^{-3} + \frac{1}{2\pi}z^{-4}\right)$$

Put $z = e^{j\omega}$

$$H\left(e^{j\omega}\right) = \left(\frac{1}{2\pi} + \frac{1}{\sqrt{2}\pi}e^{-j\omega} + \frac{1}{4}e^{-2j\omega} + \frac{1}{\sqrt{2}\pi}e^{-3j\omega} + \frac{1}{2\pi}e^{-4j\omega}\right)$$

$$H\left(e^{j\omega}\right) = e^{-2j\omega}\left(\frac{1}{2\pi}e^{2j\omega} + \frac{1}{\sqrt{2}\pi}e^{j\omega} + \frac{1}{4} + \frac{1}{\sqrt{2}\pi}e^{-j\omega} + \frac{1}{2\pi}e^{-2j\omega}\right)$$

$$H\left(e^{j\omega}\right) = e^{-2j\omega}\left(\frac{1}{2\pi}\left(e^{2j\omega} + e^{-2j\omega}\right) + \frac{1}{\sqrt{2}\pi}\left(e^{j\omega} + e^{-j\omega}\right) + \frac{1}{4}\right) = e^{-2j\omega}\left(\frac{1}{\pi}\cos(2\omega) + \frac{\sqrt{2}}{\pi}\cos(\omega) + \frac{1}{4}\right)$$

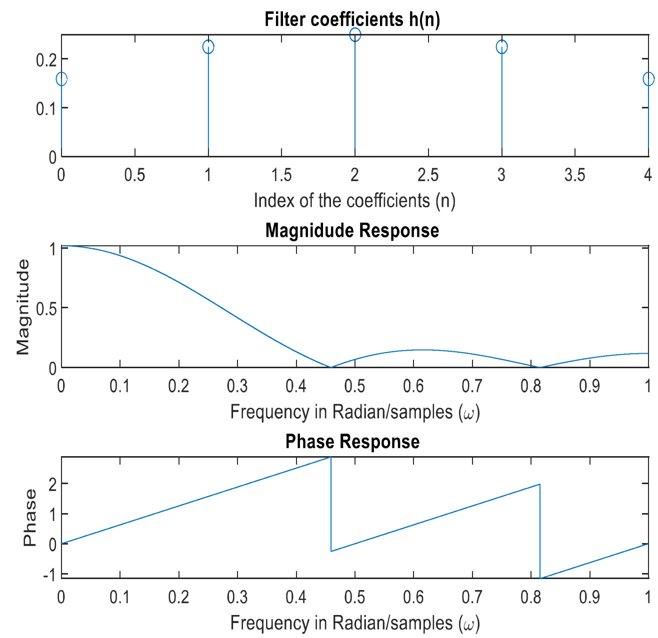
$$\omega = [0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi]$$

$$H\left(e^{j\omega}\right) = e^{-2j\omega}\left(\frac{1}{\pi}\cos(2\omega) + \frac{\sqrt{2}}{\pi}\cos(\omega) + \frac{1}{4}\right)$$

$$H(\omega) = [1.0185,\ 0.5683i,\ 0.0683,\ 0.0683i,\ 0.1182]$$

$$|H(\omega)| = \sqrt{re(H(\omega))^2 + im(H(\omega))^2}$$

$$\angle H(\omega) = \tan^{-1}\frac{im(H(\omega))}{re(H(\omega))}$$



$$\mathbf{h(n)} = \left[\frac{\mathbf{1}}{\mathbf{2}\pi}, \frac{\mathbf{1}}{\sqrt{\mathbf{2}}\pi}, \frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{1}}{\sqrt{\mathbf{2}}\pi}, \frac{\mathbf{1}}{\mathbf{2}\pi}\right]$$

The frequency response $H(\omega)$ of the digital filter is given by

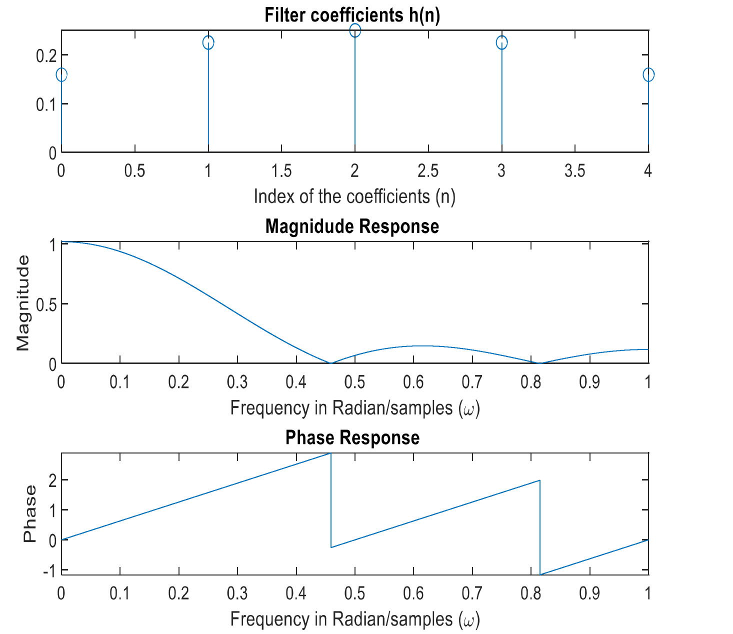
$$H(\omega) = \sum_{n=0}^{4} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$

$$= e^{-j2\omega}[h(0)\, e^{j2\omega} + h(1)\, e^{j\omega} + h(2) + h(3)\, e^{-j\omega} + h(4)\, e^{-j2\omega}]$$

$$= e^{-j2\omega}[h(2) + h(1)\,(e^{j\omega} + e^{-j\omega}) + h(0)\,(e^{j2\omega} + e^{-j2\omega})]$$

$$= e^{-j2\omega}\left[\frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\omega + \frac{1}{\pi}\cos 2\omega\right]$$



$$\omega = [\mathbf{0},\ \pi/\mathbf{4},\ \pi/\mathbf{2},\ \mathbf{3}\pi/\mathbf{4},\ \pi]$$

$$\mathbf{H}(\omega) = [\mathbf{1.0185}, \mathbf{0.5683i}, \mathbf{0.0683}, \mathbf{0.0683i}, \mathbf{0.1182}]$$

$$|\mathbf{H}(\omega)| = \sqrt{\mathbf{re}(\mathbf{H}(\omega))^{\mathbf{2}} + \mathbf{im}(\mathbf{H}(\omega))^{\mathbf{2}}}$$

$$\angle\mathbf{H}(\omega) = \mathbf{tan}^{-1}\frac{\mathbf{im}(\mathbf{H}(\omega))}{\mathbf{re}(\mathbf{H}(\omega))}$$