22-06-21

Laplace Transforms.

Defn: The LT of a real valued function $f(t)$ denoted by $L\{f(t)\}$ is defined as $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, ——— ①.

Note that the integral is a function of the parameter s;

Hence, we write $L\{f(t)\} = F(s)$

Note: i) $L\{c f(t)\} = c L\{f(t)\}$, c is a constant

ii) $L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$

Laplace transforms of elementary functions.

I) $f(t) = e^{at}$.

$$L\{f(t)\} = L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{-0}\right] = \frac{-1}{s-a}[0-1]$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}} \quad , \quad s > a$$

Note: $L\{e^{-at}\} = \frac{1}{s-(-a)} = \frac{1}{s+a}$

Eg: 1) $L\{e^{3t}\}$ Here $a = 3$

$L\{e^{3t}\} = \frac{1}{s-3}$

2) $L\{e^{-t/2}\} = \frac{1}{s+\frac{1}{2}}$

3) $L\{e^{0t}\} = L\{1\} = \frac{1}{s}$

Note: $L\{1\} = \frac{1}{s}$

$L\{a\} = a L\{1\} = \frac{a}{s}$.

II $L\{\sinh at\}$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}\left\{L\{e^{at}\} - L\{e^{-at}\}\right\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{s+a-s+a}{2(s^2-a^2)} = \frac{2a}{2(s^2-a^2)}$$

$$\therefore \boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}} \quad , \quad s > a$$

III $L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\} = \frac{s}{s^2-a^2}$, $s > a$

IV $L\{t^n\}$

By defn, $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$

Put $x = st$

$t = \frac{x}{s}$

$dx = s\,dt$

As $t: 0$ to ∞

$x: 0$ to ∞

$\therefore L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$ ↳ Gamma

$$\boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}} \quad , \quad s > 0$$

When n is a +ve integer, $L\{t^n\} = \frac{n!}{s^{n+1}}$ $\because \Gamma(n+1) = n!$

Eg:

i) $L\{t^2\} = \frac{2}{s^3}$

ii) $L\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$

iii) $L\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} = \underline{\frac{1}{2}}\frac{\sqrt{\pi}}{s^{3/2}}$

V. $L\{\sin at\}$

By defn, $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$

$a = -s$, $b = a$

$\therefore L\{\sin at\} = \frac{e^{-st}}{s^2+a^2}\left[-s \sin at - a\cos at\right]\Big|_0^\infty$

$= \frac{1}{s^2+a^2}\left\{0 - 1(0 - a)\right\}$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

VI $L\{\cos at\}$

$$L\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\therefore \boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

Eg: $L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{\cos t\} = \frac{s}{s^2+1}$

Find LT of the following.

1. $L\{t^2 + 4t - 5\}$

$L\{t^2\} + 4L\{t\} - 5L\{1\}$

$\frac{2}{s^3} + \frac{4}{s^2} - \frac{5}{s}$

2. $L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\}$

$L\left\{t^{3/2} - \frac{1}{t^{3/2}} - 3\cdot\sqrt{t}\times\frac{1}{\sqrt{t}}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\right\}$

$L\{t^{3/2}\} - L\{t^{-3/2}\} - 3L\{t^{1/2}\} + 3L\{t^{-1/2}\}$

$\frac{\Gamma(3/2+1)}{s^{3/2+1}} - \frac{\Gamma -3/2+1}{s^{-3/2+1}} - 3\frac{\Gamma 1/2+1}{s^{1/2+1}} + 3\frac{\Gamma -1/2+1}{s^{-1/2+1}}$

$\frac{3\sqrt{\pi}}{4s^{5/2}} - \frac{(-2\sqrt{\pi})}{s^{-1/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$

$\sqrt{\pi}\left[\frac{3}{4s^{5/2}} + 2\sqrt{s} - \frac{3}{2s^{3/2}} + \frac{3}{\sqrt{s}}\right]$ //

$\Gamma n+1 = n\Gamma n$

$\Gamma 3/2+1 = 3/2\,\Gamma 3/2$

$= 3/2\,\Gamma 1/2+1$

$= 3/2\times 1/2\,\Gamma 1/2$

$= 3/2\times 1/2\times\sqrt{\pi}$

$\therefore \Gamma -1/2 = -2\sqrt{\pi}$

3) $L\{\sin^3 2t\} = L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}$

$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$

$= \frac{6}{4(s^2+4)} - \frac{6}{4(s^2+36)}$ //

5) $L\{\sin 2t \sin 3t\}$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$L\{\sin 2t \sin 3t\} = \frac{1}{2}\{L\{\cos t\} - L\{\cos 5t\}]$

$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$

6) $L\{\cos t \cos 5t\}$

7) $L\{3\cos(4t+7)\}$

8) $L\{\cos^2 t\}$

II Find LT of the following

a) $\phi(t) = \begin{cases} 1, & 0 < t < 2 \\ t, & t > 2 \end{cases}$

$L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)\,dt = \int_0^2 e^{-st}\cdot 1\,dt + \int_2^\infty \underset{II}{e^{-st}}\cdot \underset{I}{t}\,dt$

$= \left.\frac{e^{-st}}{-s}\right|_0^2 + \left\{t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{(-s)^2}\right\}_2^\infty$

$= -\frac{1}{s}\{e^{-2s} - 1\} + \{0 + \frac{2e^{-2s}}{s} - \frac{1}{s^2}(0 - e^{-2s})\}$

$= \frac{1}{s}(1 - e^{-2s}) + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2}$

$= \frac{1}{s}\left(1 + e^{-2s} + \frac{e^{-2s}}{s}\right)$

2) $\psi(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

HW

$L\{\psi(t)\} = \int_0^\pi e^{-st}\sin 2t\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$

Ans: $\frac{2(1 - e^{-\pi s})}{s^2+4}$

Properties of LTs

I. First shifting property.

If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

Proof: $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a)$$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
$= F(s)$

Eg: 1) $L\{e^t t\}$

$a=1$, $f(t) = t$

$L\{f(t)\} = \frac{1}{s^2}$

replace s by $s-a$ $= s-1$

$\therefore L\{e^t t\} = \frac{1}{(s-1)^2}$

2) $L\{e^{-t} \sin t\}$

$L\{\sin t\} = \frac{1}{s^2+1} = F(s)$

$a = -1$

replace s by $s+1$

$L\{e^{-t} \sin t\} = \frac{1}{(s+1)^2+1}$

Note:

a) $L\{e^{at} t^n\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$

b) $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

c) $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$

d) $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2-b^2}$

e) $L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

1) Find $L\{e^{-2t}\cos^2 t\}$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$L\{\cos^2 t\} = L\{\frac{1+\cos 2t}{2}\} = \frac{1}{2} L\{1+\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = F(s)$

replace s by $s+2$

$\therefore L\{e^{-2t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right]$

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2) $L\{e^{4t}\sin 2t\cos t\}$

$L\{\sin 2t\cos t\} = L\{\frac{1}{2}(\sin 3t) + \frac{1}{2}\sin t\}$

Ans : $\frac{2(s-4)^2+6}{[(s-4)^2+9][(s-4)^2+1]}$

3. $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

4) $L\{5t - 7e^{6t} + t^{5/2}\}$

5) $L\{e^{-3t}t^{-1/2}\}$

II : LT of the derivatives.

If $L\{f(t)\} = F(s)$, then $L\{f'(t)\} = sL\{f(t)\} - f(0)$

$= sF(s) - f(0)$

Proof : $L\{f'(t)\} = \int_0^\infty \underset{I}{e^{-st}}\, \underset{II}{f'(t)}\, dt$

$= \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\, dt$

$= \lim_{b\to\infty}\left[e^{-st}f(t)\Big|_0^b - \int_0^b (-s)e^{-st}f(t)\,dt\right]$

$= (0 - f(0)) + s\int_0^\infty e^{-st}f(t)\,dt$

$= sL\{f(t)\} - f(0) = sF(s) - f(0)$

Eg: $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

Find $L\{\cos at\}$ using the property $L\{f'(t)\} = sF(s) - f(0)$

$f(t) = \sin at \implies F(s) = L\{f(t)\} = L\{\sin at\} = \dfrac{a}{s^2+a^2}$

$f'(t) = a\cos at \qquad f(0) = \sin 0 = 0$

$L\{a\cos at\} = s\cdot\dfrac{a}{s^2+a^2} - 0$

$\cancel{a}\, L\{\cos at\} = \dfrac{\cancel{a}\, s}{s^2+a^2}$

$\therefore L\{\cos at\} = \dfrac{s}{s^2+a^2}$ //

Note: $L\{f''(t)\} = L\{(f'(t))'\}$

$= s\, L\{f'(t)\} - f'(0)$

$= s\{s\, L\{f(t)\} - f(0)\} - f'(0)$

$= s^2 L\{f(t)\} - s f(0) - f'(0).$

In general,

$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0)$

$- - - - - - f^{(n-1)}(0).$

Lecture 2
25-6-21

3. Multiplication by powers of t

If $L\{f(t)\} = F(s)$, Then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$

Proof: By defn $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

diff w.r.t s, $F'(s) = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, f(t)\,dt = \int_0^\infty e^{-st}(-t\, f(t))\,dt = L\{-tf(t)\}$

$\Rightarrow L\{t f(t)\} = -F'(s) = -\frac{d}{ds} F(s)$ — ①

Also $L\{t^2 f(t)\} = L\{t \cdot \underline{t f(t)}\}$

$= -\frac{d}{ds} L\{t f(t)\}$ from ①

$= -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right] = (-1)^n \frac{d^2}{ds^2} F(s)$

$\therefore$ By mathematical induction on n, we obtain

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

eg: 1) $L\{t \sin 3t\}$

$L\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$

$\therefore L\{t f(t)\} = -\frac{d}{ds} F(s)$

$L\{t \sin 3t\} = -\frac{d}{ds} \frac{3}{s^2+9} = -\left[\frac{-3 \times 2s}{(s^2+9)^2}\right] = \frac{6s}{(s^2+9)^2}$

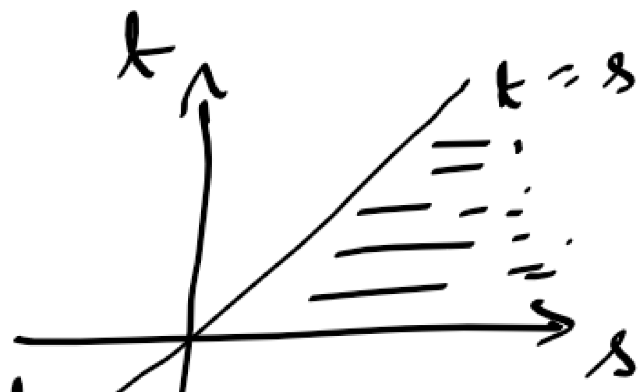
2) $L\{t^{100} e^{-t/2}\}$

$f(t) = t^{100}$ $\quad L\{f(t)\} = F(s) = \frac{100!}{s^{101}}$

$L\{t^{100} e^{-t/2}\} = \frac{100!}{(s+1/2)^{101}}$ replace s by $s+\frac{1}{2}$

4. Division by t

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$



Proof: $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\int_s^\infty F(s)\,ds = \int_{s=s}^\infty \int_{t=0}^\infty e^{-st} f(t)\,dt = \int_{t=0}^\infty f(t)\left(\int_s^\infty e^{-st}\,ds\right)dt$

$$\int_s^{\infty} F(s)\,ds = \int_{t=0}^{\infty} f(t) \int_s^{\infty} e^{-st}\,ds\,dt$$

$$= \int_{t=0}^{\infty} f(t) \left(\frac{e^{-st}}{-t} \Big|_s^{\infty} \right) dt$$

$$= \int_{t=0}^{\infty} f(t) \left\{ -\frac{1}{t} \left(0 - e^{-st}\right) \right\} dt$$

$$= \int_{t=0}^{\infty} e^{-st} \left(\frac{f(t)}{t} \right) dt$$

$$= L\left\{ \frac{f(t)}{t} \right\}$$

$\therefore\ L\left\{ \frac{f(t)}{t} \right\} = \int_s^{\infty} F(s)\,ds$, only if $\lim_{t \to 0} \frac{f(t)}{t}$ exists.

Eg: 1) $L\left\{ \frac{\sin t}{t} \right\}$

$f(t) = \sin t$

$F(s) = L\{f(t)\} = \frac{1}{s^2+1}$

$$L\left\{ \frac{f(t)}{t} \right\} = \int_s^{\infty} F(s)\,ds$$

$$\therefore\ L\left\{ \frac{\sin t}{t} \right\} = \int_s^{\infty} \frac{1}{s^2+1}\,ds = \tan^{-1} s \Big|_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

5. Laplace transforms of integrals.

If $L\{f(t)\} = F(s)$ then $L\left\{ \int_0^t f(u)\,du \right\} = \frac{F(s)}{s}$.

Proof: Let $g(t) = \int_0^t f(u)\,du$

$g'(t) = f(t)$, $g(0) = 0$

$$L\{g'(t)\} = L\{f(t)\}$$

$$s\,L\{g(t)\} - g(0) = L\{f(t)\}$$

$$L\{g(t)\} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s}$$

$$L\left\{ \int_0^t f(u)\,du \right\} = \frac{F(s)}{s}$$

Eg: 1) $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

$L\{\cos t\} = \dfrac{s}{s^2+1}$

$F(s) = L\{e^{-t}\cos t\} = \dfrac{s+1}{(s+1)^2+1}$ $a=-1$, $s \,||\, s+1$

$\therefore L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = \dfrac{1}{s}F(s) = \dfrac{s+1}{s\left((s+1)^2+1\right)}$

2) $L\left\{\int_0^t \dfrac{e^t}{\sqrt{t}}dt\right\}$

$F(s) = L\{e^t t^{-1/2}\} = \dfrac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}\Bigg|_{s\,||\,s-1} = \dfrac{\sqrt{\pi}}{(s-1)^{1/2}}$

$\therefore L\left\{\int_0^t \dfrac{e^t}{\sqrt{t}}dt\right\} = \dfrac{1}{s}F(s) = \dfrac{\sqrt{\pi}}{s\,(s-1)^{1/2}}$

3) $L\left\{\int_0^t t\,e^{-t}\sin 4t\,dt\right\}$

$L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$L\{t\sin 4t\} = -\dfrac{d}{ds}\dfrac{4}{s^2+16} = -\dfrac{4\times(-2s)}{(s^2+16)^2} = \dfrac{8s}{(s^2+16)^2}$

$L\{e^{-t}\,t\sin 4t\} = \dfrac{8(s+1)}{\left((s+1)^2+16\right)^2} = F(s)$

$\therefore L\left\{\int_0^t t\,e^{-t}\sin 4t\,dt\right\} = \dfrac{F(s)}{s} = \dfrac{1}{s}\dfrac{8(s+1)}{\left((s+1)^2+16\right)^2}$

4) $L\left\{\int_0^t t^2\sin at\,dt\right\}$

Find LT of the following.

i) $L\left\{\dfrac{1-\cos t}{t}\right\}$

It is of the form $L\left\{\dfrac{f(t)}{t}\right\}$.

$L\{1-\cos t\} = \dfrac{1}{s} - \dfrac{s}{s^2+1} = F(s)$

$L\left\{\dfrac{1-\cos t}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)ds$

$= \log s - \dfrac{1}{2}\log(s^2+1)\Big|_s^\infty$

$= \dfrac{1}{2}\log\dfrac{s^2}{s^2+1}\Big|_s^\infty$

$$= \frac{1}{2} \ln \frac{s^2}{s^2+1} \Big|_s^\infty = \frac{1}{2} \ln \frac{\cancel{s^2}}{\cancel{s^2}(1+1/s^2)} \Big|_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{1}{1+1/\infty}\right) - \ln \frac{s^2}{s^2+1}\right]$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

2. $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

$f(t) = \cos at - \cos bt$

$F(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty F(s)ds = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \, ds$$

$$= \frac{1}{2}\left(\ln(s^2+a^2) - \ln(s^2+b^2)\right) = \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\Big|_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \ln\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

3. $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$

$L\{\sin t\} = \frac{1}{s^2+1}$, $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1}s \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$L\left\{e^{-t}\frac{\sin t}{t}\right\} = \cot^{-1}(s+1)$

$L\left\{\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \frac{1}{s}F(s) = \frac{\cot^{-1}(s+1)}{s}$

$$\therefore L\left\{t\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = -\frac{d}{ds}\frac{\cot^{-1}(s+1)}{s}$$

$$= -\left[\frac{s\frac{-1}{1+(s+1)^2} - \cot^{-1}(s+1)\cdot 1}{s^2}\right]$$

$$= \frac{s + (s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

4. $L\left\{2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t\right\}$

$L\{2^t\} = L\{e^{\ln 2^t}\} = L\{e^{(\ln 2)t}\} = \frac{1}{s-\ln 2} \longrightarrow (a)$

$L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \ln\left(\frac{s^2+9}{s^2+4}\right)^{1/2} \longrightarrow (b)$

$L\{t\sin t\} = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2} \longrightarrow (c)$

Answer $= (a) + (b) + (c)$

Change of scale property :

6. If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$ Put $at = x$, $a\,dt = dx$, $t : 0$ to ∞, $x : 0$ to ∞

$= \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\,\frac{dx}{a}$

$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\,dx$ since $\int_0^\infty e^{-st} f(t)\,dt = F(s)$

$= \frac{1}{a}F\left(\frac{s}{a}\right)$

Eg: Find $L\{\cos 2t\}$

We have $L\{\cos t\} = \frac{s}{s^2+1}$

$a = 2$ $\therefore L\{\cos 2t\} = \frac{1}{2}\frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^2+1} = \frac{\frac{s}{2}}{\cancel{2}\times\frac{(s^2+4)}{\cancel{4}}} = \frac{s}{s^2+4}$

Evaluation of Integrals by Laplace transform method

1. Evaluate $\int_0^\infty t e^{-3t}\sin t\,dt$ comparing with $\int_0^\infty e^{-st} f(t)\,dt$, we see that

$s = 3$, $f(t) = t\sin t$

$L\{f(t)\} = L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$

$\therefore \int_0^\infty e^{-3t}\, t\sin t\,dt = \left.\frac{2s}{(s^2+1)^2}\right|_{s=3} = \frac{6}{100} = \frac{3}{50}$

2. Evaluate $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt$

Here $s = 0$, $\int_0^\infty e^{-0t}\,\frac{e^{-t} - e^{-3t}}{t}\,dt$

$$L\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = \int_s^\infty \frac{1}{s+1} - \frac{1}{s+3}\, ds$$

$$= \ln(s+1) - \ln(s+3) \Big|_s^\infty$$

$$= \ln\left(\frac{s+1}{s+3}\right)\Big|_s^\infty$$

$$= \ln\left(\frac{\cancel{s}\,(1+\frac{1}{s})}{\cancel{s}\,(1+\frac{3}{s})}\right) - \ln\left(\frac{s+1}{s+3}\right)$$

$$= 0 + \ln\left(\frac{s+3}{s+1}\right)$$

$$\therefore \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\, dt = \ln\left(\frac{s+3}{s+1}\right)\Big|_{s=0} = \underline{\underline{\ln 3}}$$

3. Find $\int_0^\infty e^{-t}\frac{\sin t}{t}\, dt$

Put $s=1$ in $L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}(s)$

$$\cot^{-1} 1 = \pi/4$$

4. Find $\int_0^\infty e^{-2t}\, t \sin^2 t\, dt$

Put $s=2$ in $L\{t \sin^2 t\}$

Answer $\underline{\underline{\frac{1}{8}}}$

Laplace transform of Periodic functions

A function $f(t)$ is called a periodic fn with period T if $f(t) = f(t+T) = f(t+2T) = f(t+3T) = \cdots$

Eg: $f(t) = \sin t$, $T = 2\pi$

$f(t+2\pi) = \sin(t+2\pi) = \sin t$

Also $f(t) = \cos t$ has period 2π.

Theorem: If a function $f(t)$ is periodic with period T, then $L\{f(t)\} =$