Ex:

① Polar coordinates $\quad x = r\cos\theta \quad , \; y = r\sin\theta .$

$(x,y) \to (r,\theta)$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{J = r}$$

② In cylindrical co-ordinates -

$(x,y,z) \to (\rho,\phi,z)$

$x = \rho\cos\phi \quad , \quad y = \rho\sin\phi \quad , \quad z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$\boxed{J = \rho}$$

③ Sperical coordinates -

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2\sin\theta$$

$$\boxed{J = r^2\sin\theta}$$

Change of variables —

Let $I = \iint_R f(x,y)\, dx\, dy$

Substitute $x = g(u,v)$, $y = h(u,v)$

then, $I = \iint_R f(x,y)\, dx\, dx = \iint_{R^*} f(g(u,v), h(u,v))\, |J|\, du\, dv$,

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ and

R^* is the region in uv-plane corresponding the region R in xy-plane.

Geometrically, when the region R of xy-plane transforms into the R^* of uv-plane, the elementary area $dx\,dy$ transforms to $J\, du\, dv$

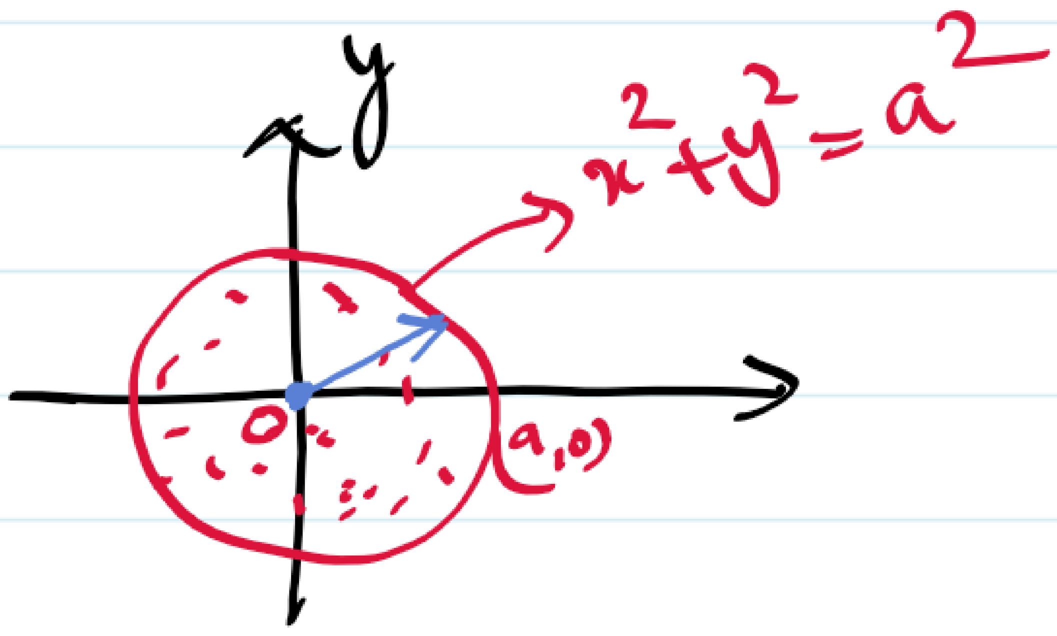
Eg: Given $\iint_R f(x,y)\, dx\, dy$.

changing to polar co-ordinates,

$x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

① Evaluate $\iint\limits_R e^{-(x^2+y^2)}\,dx\,dy$, where $R: x^2+y^2 \le a^2$



Using polar co-ordinates,

$x = r\cos\theta$, $y = r\sin\theta$

$J = r$

$$4\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} e^{-x^2}\cdot e^{-y^2}\,dx\,dy.$$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2$$

$$\iint\limits_R e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} e^{-r^2}\,\frac{r(-2)}{(-2)}\,dr\,d\theta.$$

$$= \int_0^{2\pi} -\frac{1}{2}\,e^{-r^2}\Big]_{r=0}^{a}\,d\theta = \int_0^{2\pi} -\frac{1}{2}\left(e^{-a^2}-1\right)d\theta$$

$$= -\frac{1}{2}\left(e^{-a^2}-1\right)\theta\Big]_0^{2\pi}$$

$$= -\pi\left(e^{-a^2}-1\right)$$

$$= \pi\left(1-e^{-a^2}\right)$$

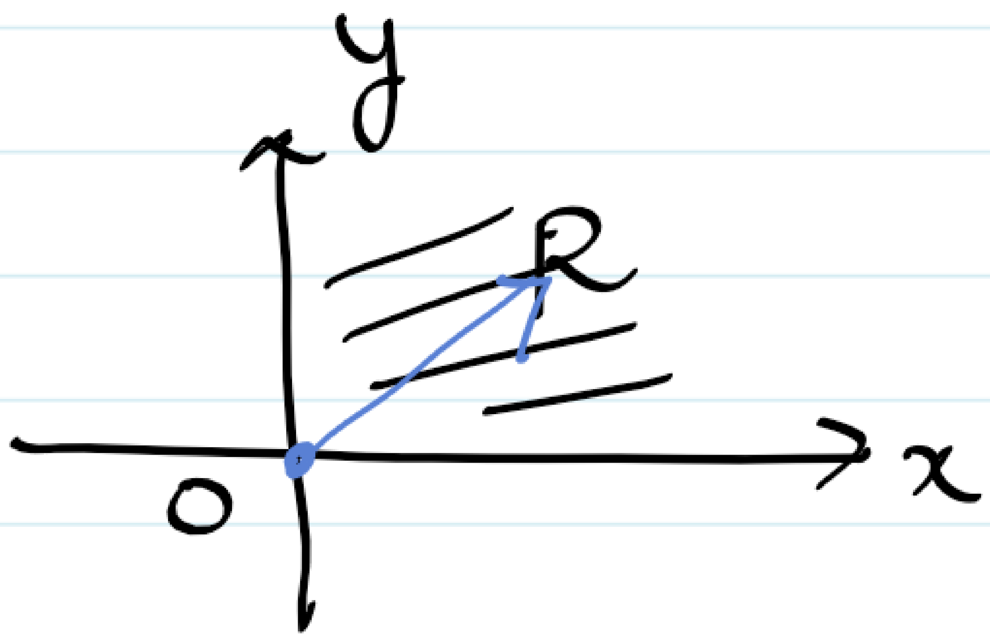
2) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ & hence prove that

$$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

Ans:

Taking polar co-ordinates

$x = r\cos\theta \quad , y = r\sin\theta$



$J = r$

$x^2+y^2 = r^2$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, \frac{r(-2)}{(-2)}\, dr\, d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\left(e^{-r^2}\right)_0^\infty d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}(0-1)\, d\theta = \frac{1}{2}(\theta)_0^{\pi/2} = \frac{\pi}{4}$$

$$I = \int_0^\infty\int_0^\infty e^{-x^2} \cdot e^{-y^2} dx\,dy = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy$$

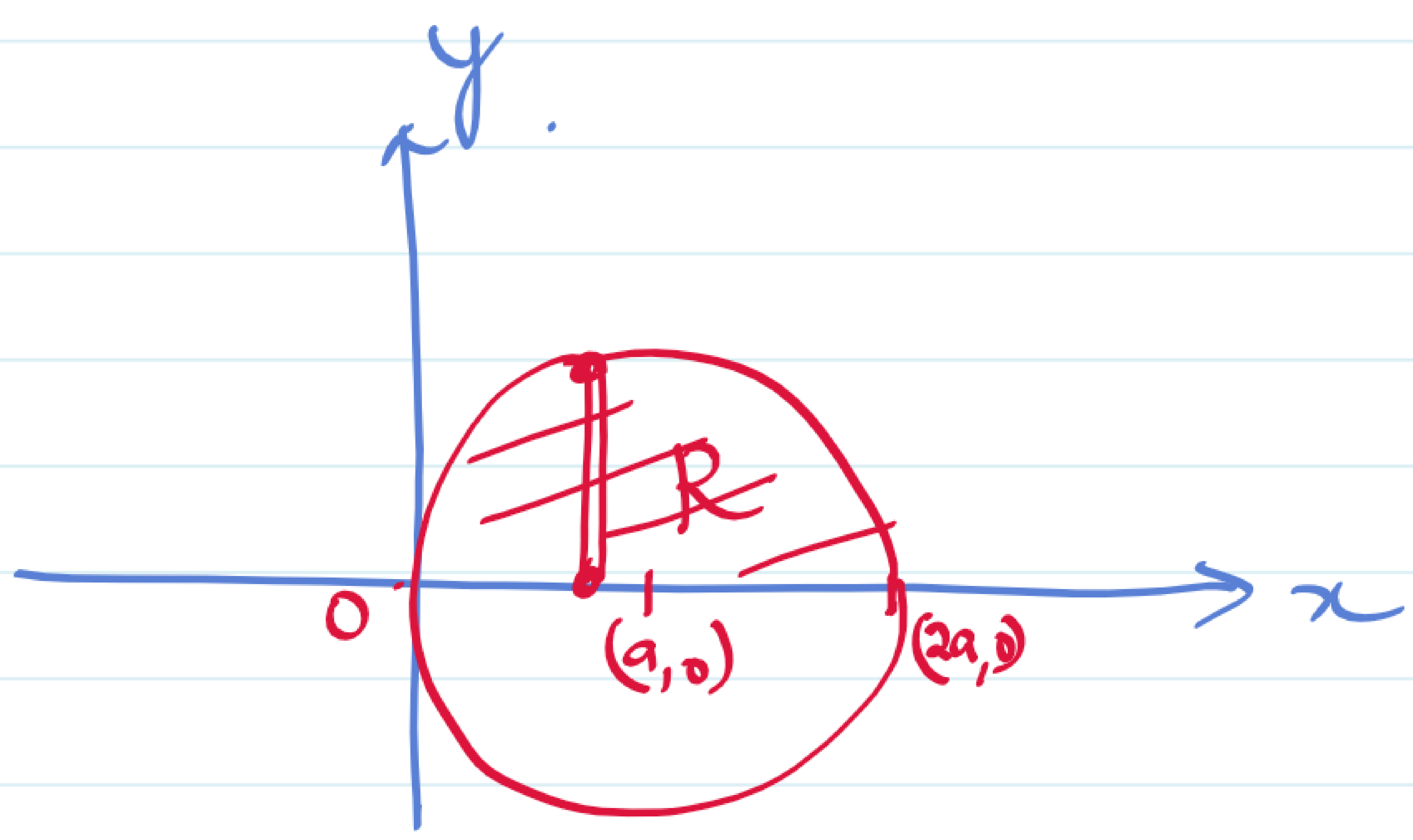
$$= \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-x^2} dx$$

$$\frac{\pi}{4} = \left(\int_0^\infty e^{-x^2} dx\right)^2$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

3) Evaluate $\displaystyle\int_0^{2a}\int_{0}^{\sqrt{2ax-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$

$y=$



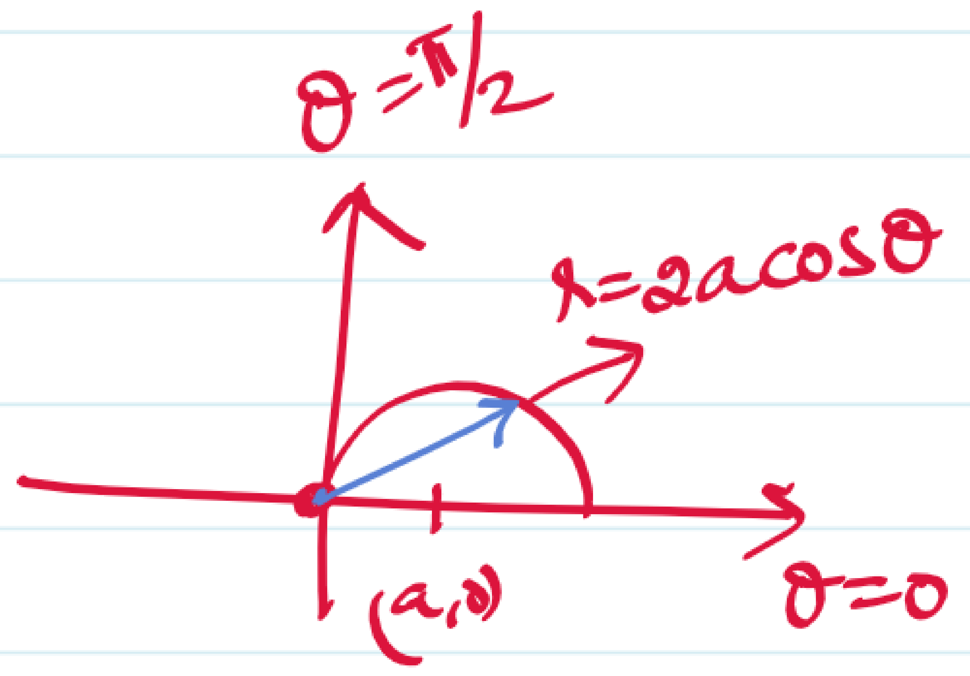
$y=0, \quad y=\sqrt{2ax-x^2}$

$y^2 = 2ax - x^2$

$x^2+y^2-2ax = 0$

$x^2-2ax+\underline{\underline{a^2}}+y^2 = a^2$

$(x-a)^2+y^2 = a^2$



Taking polar co-ordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$J = r. \quad , \quad x^2+y^2 = r^2$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} \frac{r\cos\theta}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

$y^2 = 2ax - x^2$

$x^2+y^2 = 2ax$

$r^2 = 2ar\cos\theta$

$\Rightarrow \underline{\underline{r = 2a\cos\theta}}$

$$= \int_0^{\pi/2} \cos\theta \left(\frac{r^2}{2}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\theta \left(4a^2\cos^2\theta\right) d\theta$$

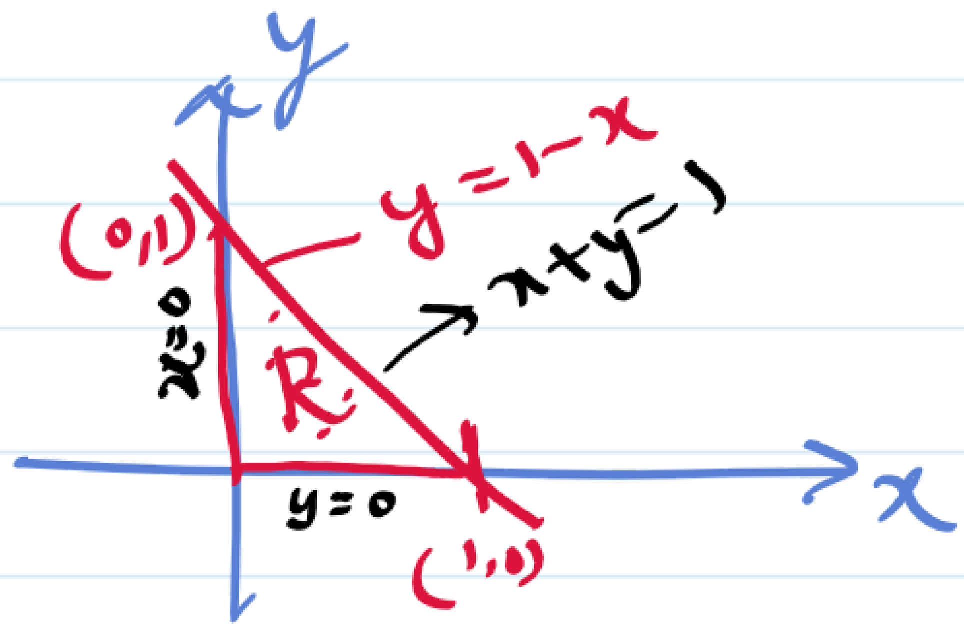
$$= 2a^2 \int_0^{\pi/2} \cos^3\theta\, d\theta = 2a^2 \times \frac{2}{3} = \underline{\underline{\frac{4a^2}{3}}}$$

$\displaystyle\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{(n-3)}{n-2}\cdots\frac{2}{3}$, n odd

$\displaystyle = \frac{n-1}{n}\cdots\frac{1}{2}\frac{\pi}{2}$, n even

4) Evaluate $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\,dy$, using the transformation

$x+y=u, \quad y=uv.$



The given region is bounded by

$y=0, \; y=1-x, \quad x=0, \; x=1$

Here $x+y=u \quad , \quad y=uv$

$x=u-y$

$x=u-uv \quad , \quad y=uv$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=(1-v)u+uv$$

$$=u$$

$J=u$

Taking substitution $x+y=u$, $y=uv$, the boundaries of the region R^* in uv-plane becomes:—

In $x+y=u$, $y=uv$, put $x=0$, $y=0$, $x+y=1$

When $x=0$, $\quad 0+y=u \quad , \quad y=uv$

$y=u$

$\Rightarrow u=uv$

$u(1-v)=0 \Rightarrow \boxed{u=0, \; v=1}$ boundaries in R^*

when $y=0$,

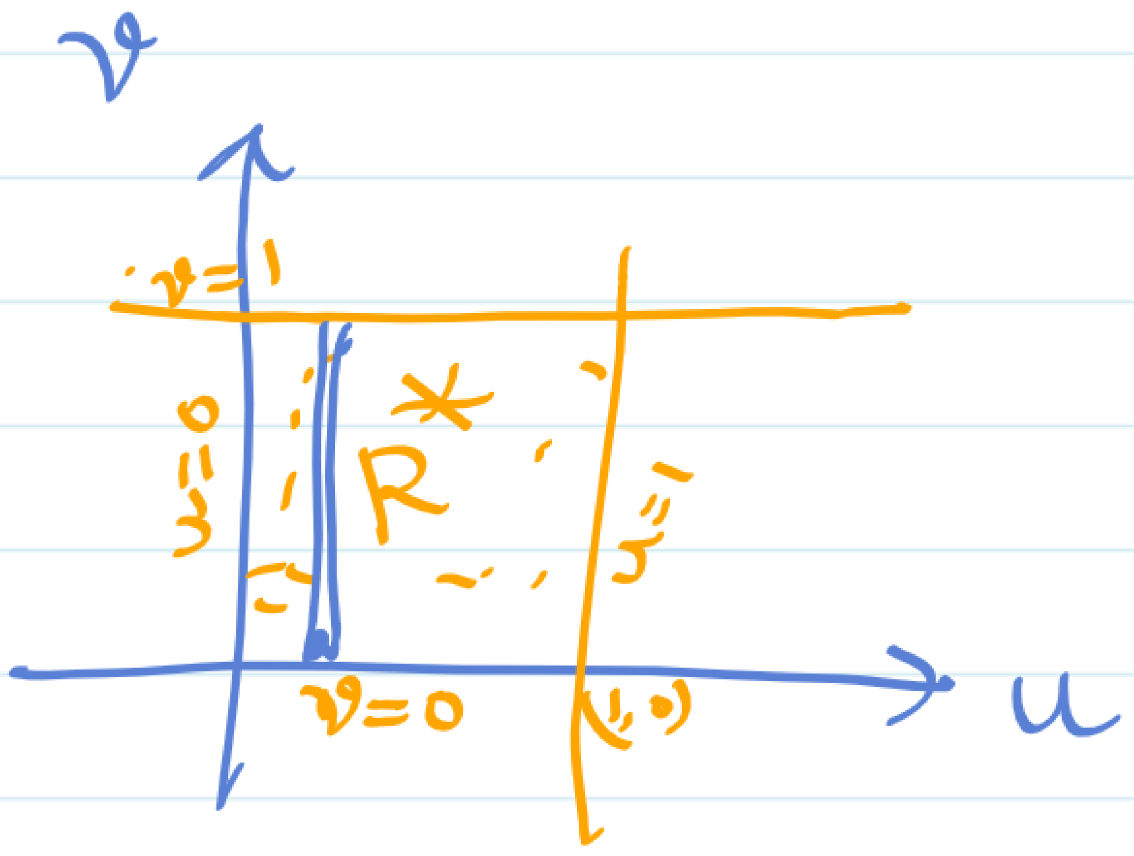
$x=u, \quad uv=0$

$\Rightarrow \boxed{u=0 \; , \; v=0} \longrightarrow$ boundaries in R^* of uv-plane

when $x+y=1$,

$\boxed{u=1} \longrightarrow$ boundari in R^*

$$\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dx\,dy$$

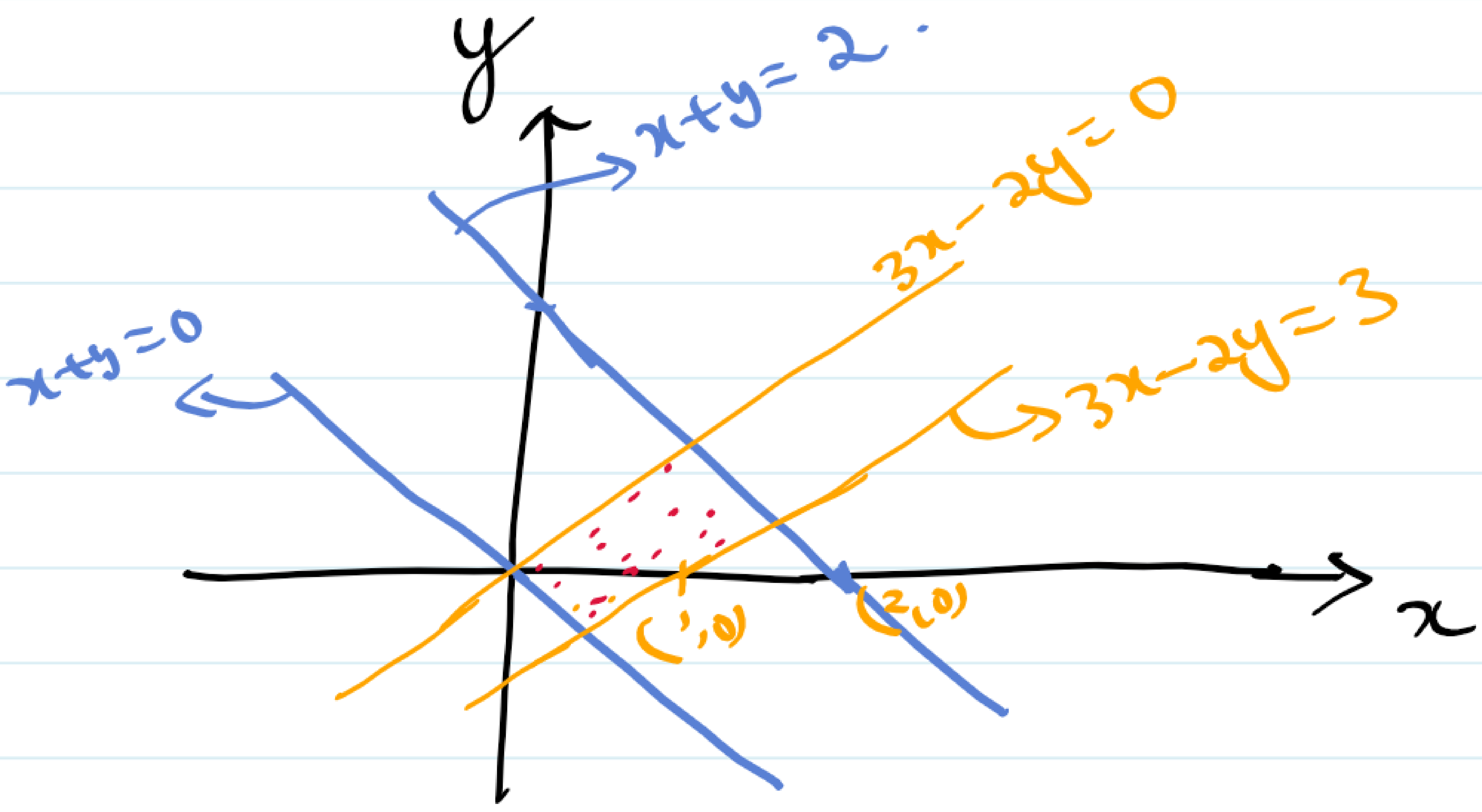


$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{uv/u}\ \underset{\uparrow}{\overset{J}{u}}\ du\,dv$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{v}\, u\ du\,dv$$

$$= \int_{u=0}^{1} u\,du \times \int_{v=0}^{1} e^{v}\,dv \quad = \frac{1}{2}(e-1)$$

(5) Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is the region bounded by the parallelogram $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$



Ans:

Substitute $x+y=u$
$3x-2y=v$

$(x,y)\rightarrow(u,v)$

$$J=\frac{\partial(x,y)}{\partial(u,v)}$$

$$J'=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y\end{vmatrix}=\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix}=-2-3=-5$$

Since $JJ'=1$, we have $J=\frac{1}{J'}=-1/5$

Taking $x+y=u$, $3x-2y=v$, the boundaries of the corresponding region R^* of uv-plane becomes:

In R

$$\begin{cases} x+y=0 \checkmark \\ x+y=2 \\ 3x-2y=0 \\ 3x-2y=3 \end{cases}$$

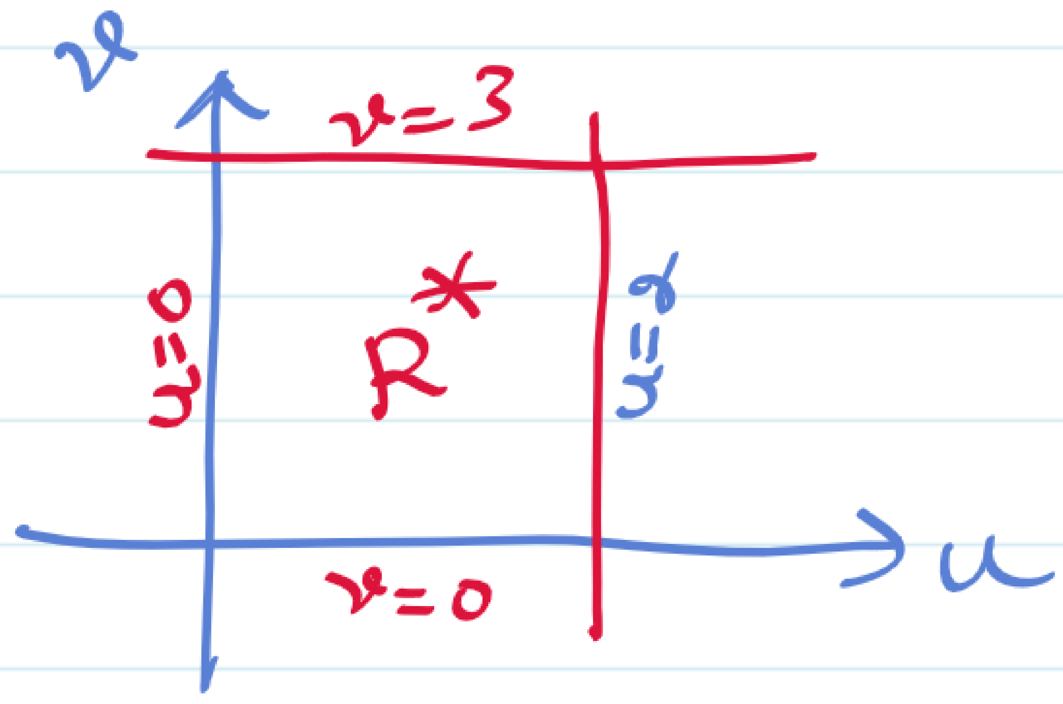
when $x+y=0$, $\quad u=0$

$x+y=2$, $\quad u=2$

$3x-2y=0$, $\quad v=0$

$3x-2y=3$, $\quad v=3$

$\therefore R^*$ is bounded by $u=0$, $u=2$, $v=0$, $v=3$

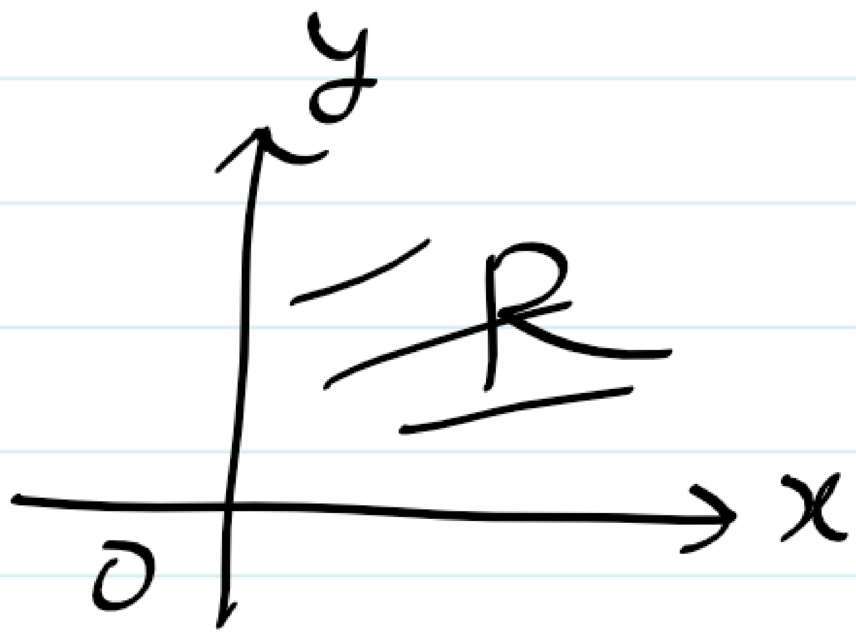


$$I = \iint_R (x+y)^2 \, dx\,dy$$

$$= \int_{u=0}^{2} \int_{v=0}^{3} u^2 \left|\frac{-1}{5}\right| du\,dv$$

$$= \frac{1}{5}\int_0^2 u^2 du \times \int_0^3 dv = \frac{1}{5} \times \left(\frac{u^3}{3}\right)_0^2 \times (v)_0^3 = 8/5$$

(6) Evaluate $\iint_R e^{-(x+y)} \sin\left(\frac{\pi y}{x+y}\right) dx\,dy$, where R is the entire 1^{st} quadrant in xy-plane.



Taking $x+y=u$ $\qquad y=v$

$x=u-y$ $\qquad y=v.$

$=u-v$

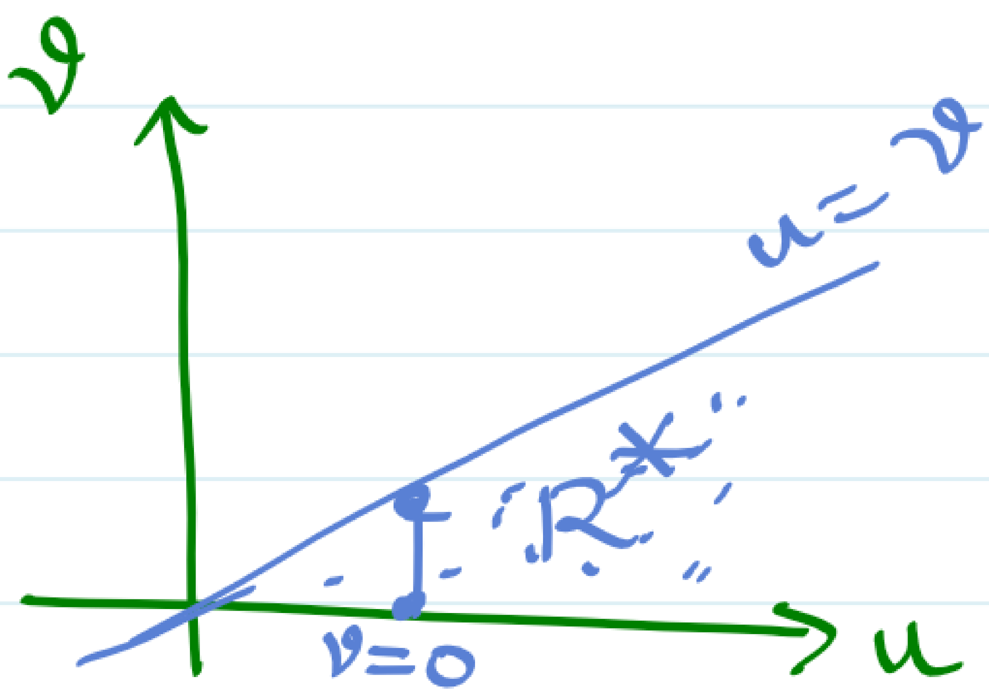
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

To find the boundaries in uv-plane

Put $x=0,\ y=0$ in $x = u-v$, $y = v$

when $x=0$, $u-v=0 \Rightarrow \underline{\underline{u=v}}$

when $y=0$, $\underline{\underline{v=0}}$



$$I = \iint_R e^{-(x+y)} \sin\left(\frac{\pi y}{x+y}\right) dx\,dy$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{u} e^{-u} \sin\left(\frac{\pi v}{u}\right) \overset{|J|}{\overset{\uparrow}{1}} .\ du\,dv$$

$$= \int_0^{\infty} e^{-u}\left(-\cos\left(\frac{\pi v}{u}\right) \times \frac{u}{\pi}\right)_{v=0}^{u} du$$

$$= \int_0^{\infty} \frac{u e^{-u}}{\pi}(1+1)\,du = \frac{2}{\pi}\left\{u(-e^{-u}) - 1(e^{-u})\right\}_0^{\infty}$$

$$= \underline{\underline{2/\pi}}$$

Practice questions –

① Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\,dy$, changing to polar co-ordinates.

(Ans: $a^4/4$)

② Evaluate $\iint_R \sqrt{x^2+y^2}\, dx\,dy$ where R is the region bounded between $x^2+y^2=4$ and $x^2+y^2=9$, by changing to polar coordinates.

(Ans: $\frac{38\pi}{3}$)

③ Evaluate $\iint xy\, dx\,dy$ over the area bounded by $y^2=4x$, $y^2=8x$, $x^2=4y$, $x^2=8y$.

(Ans: 192)

(Hint: You can take the transformation $\frac{y^2}{x}=u$, $\frac{x^2}{y}=v$)

④ Evaluate $\iint_R (x+y)^2\, dx\,dy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $x+y=u$, $x-2y=v$

(Ans: 21)

⑤ Evaluate $\iint_R xy(\sqrt{1-x-y})\, dx\,dy$, where R is the region bounded by $x=0$, $y=0$, $x+y=1$, using the transformation $x+y=u$, $y=uv$.

(Ans: $2/945$)

* Must try this ↓.

⑥ Changing to polar coordinates, evaluate: $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\,dy$

Ans: $8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$