Formation Of Differential Equations.

An n^{th} order differential equation, has n arbitrary constants in its solution. Hence if a relation between the dependent and independent variables is given that involves n arbitrary constants, then we need to differentiate n times to form a differential equation.

By elimination of these n arbitrary constants we obtain a differential equation which is consistent with the original relation. In other words, we will obtain a differential equation for which the given relation is the general solution.

① Form the DE from $x = A\cos(\omega t + \alpha)$, where A and α are arbitrary constants. ω is a parameter not to be eliminated.

$$x = A\cos(\omega t+\alpha)$$

$$\frac{dx}{dt} = -A\omega\sin(\omega t+\alpha)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t+\alpha)$$

$$= -\omega^2 \underbrace{A\cos(\omega t+\alpha)}$$

$$= -\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\frac{d}{dx}\left(\cos(ax+b)\right) = -a\sin(ax+b)$$

$$\frac{d}{dx}\left(\sin(ax+b)\right) = a\cos(ax+b)$$

② Eliminate the constant a from the equation $(x-a)^2+y^2=a^2$

Method 1:

$$(x-a)^2+y^2=a^2 \quad \text{—— } ①$$

$$2(x-a) + 2y\frac{dy}{dx} = 0$$

$$x-a = -y\frac{dy}{dx} \quad \text{—— } ②$$

$$a = x+y\frac{dy}{dx} \quad \text{—— } ③$$

Substituting ② & ③ in ①

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = \left(x+y\frac{dy}{dx}\right)^2$$

$$\cancel{y^2\left(\frac{dy}{dx}\right)^2} + y^2 = x^2 + \cancel{y^2\left(\frac{dy}{dx}\right)^2} + 2xy\frac{dy}{dx}$$

$$x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

Method 2:

$$x^2+a^2-2ax+y^2=a^2$$

$x^2+y^2 = 2ax$ —— Differentiate this and substitute for a

$\dfrac{x^2+y^2}{x} = 2a$ —— Differentiate this get rid of a.

(3) Form the differential from $y = c_1 e^{-2x} + c_2 e^{3x}$, c_1 & c_2 are arbitrary constants.

$$\frac{d}{dx}\left(e^{ax}\right) = a e^{ax}$$

$$y = c_1 e^{-2x} + c_2 e^{3x} \quad \text{——} (1)$$

$$\frac{dy}{dx} = -2c_1 e^{-2x} + 3c_2 e^{3x} \quad \text{——} (2)$$

$$\frac{d^2y}{dx^2} = 4c_1 e^{-2x} + 9c_2 e^{3x} \quad \text{——} (3)$$

(3) − (2) gives

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6c_1 e^{-2x} + 6c_2 e^{3x}$$
$$= 6\left(c_1 e^{-2x} + c_2 e^{3x}\right)$$
$$= 6y$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

Note: Consider the system

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$
$$a_3 x + b_3 y + c_3 = 0$$

Eliminating x & y from the above system leads to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Eliminating c_1 & c_2 from (1), (2) & (3) gives

$$\begin{vmatrix} y & e^{-2x} & e^{3x} \\ \frac{dy}{dx} & -2e^{-2x} & 3e^{3x} \\ \frac{d^2y}{dx^2} & 4e^{-2x} & 9e^{3x} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & -2 & 3 \\ \frac{d^2y}{dx^2} & 4 & 9 \end{vmatrix} = 0$$

Evaluating the determinant, we get the differential equation as

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

Problems for practice.

1. Form the DE: $x = c_1 \cos \omega t + c_2 \sin \omega t$, c_1 and c_2 are arbitrary constants and ω is a parameter.

2. Form the DE: $y = x^2 + c_1 e^{x} + c_2 e^{-x}$, c_1 and c_2 are arbitrary constants

3. Form the DE: $y = Ae^{2x} + Bxe^{2x}$, A and B are arbitrary constants

4. Find the differential equation of the family of parabolas, having their vertices at the origin and their foci on y-axis.

5. Find the DE. of the family of circles having their centres on y-axis.

6. Find the DE of the family of straight lines with slope and x-intercept equal.

7. Find the DE of the family of Circles with fixed radius r and tangent to the x-axis.