Signal Concepts

What is a Signal?

- A signal is a pattern of variation of some form
- Signals are variables that carry information

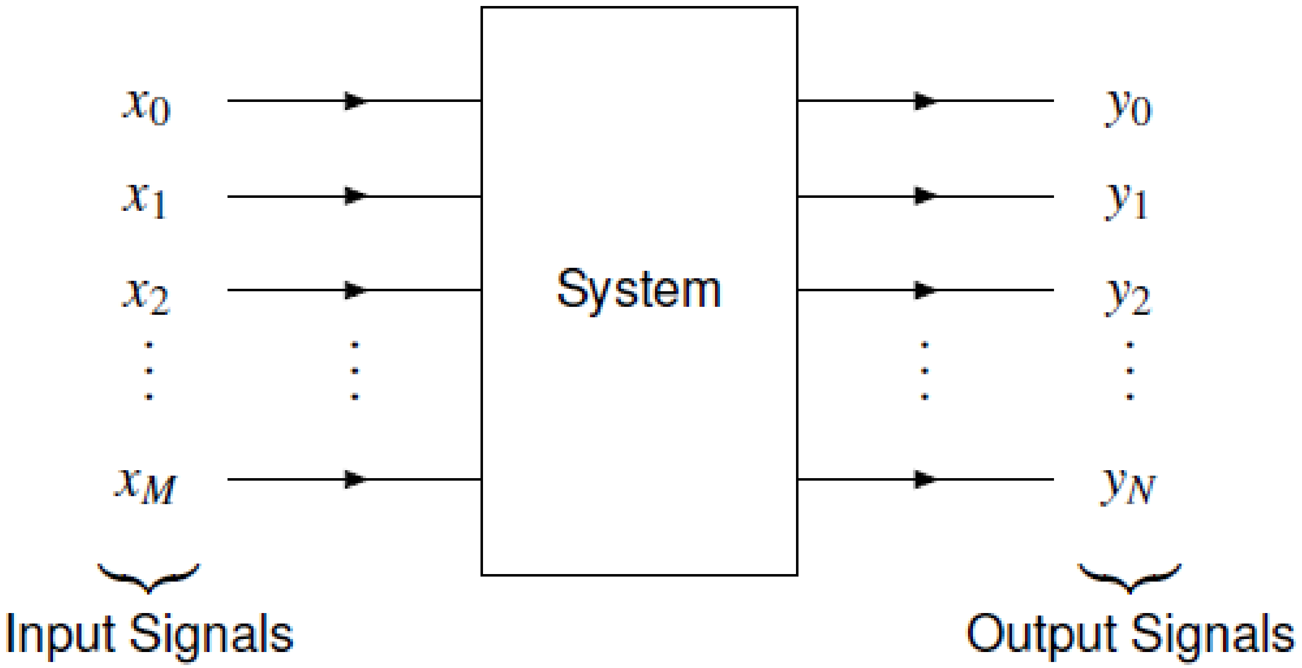
- Examples of signal include:
- Electrical signals
  - Voltages and currents in a circuit
- Acoustic signals
  - Acoustic pressure (sound) over time

- Mechanical signals
  - Velocity of a car over time
- Video signals
  - Intensity level of a pixel in an image (camera, video) over time

- Biological signals --- EEG, ECG etc

Systems

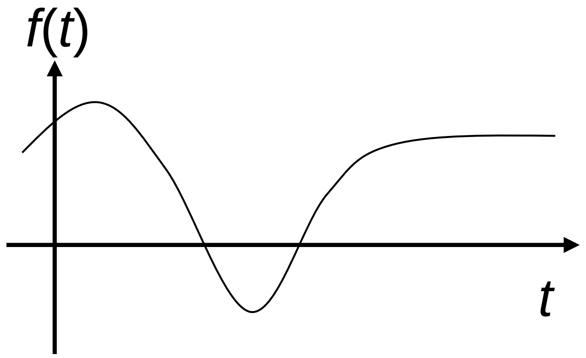
- A system is an entity that processes one or more input signals in order to produce one or more output signals



Why study Signals and Systems ?

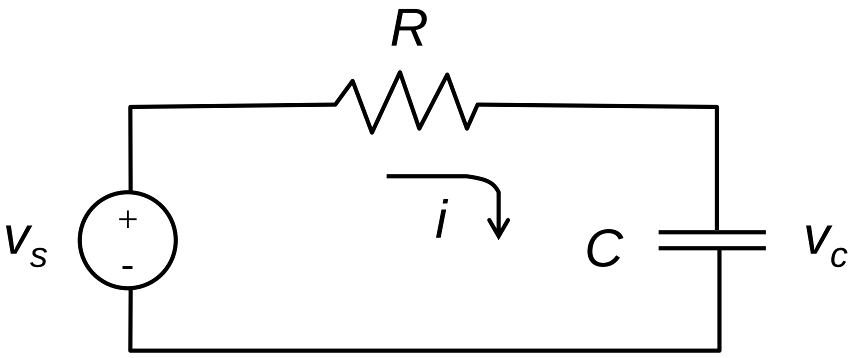
- Engineers build systems that process/manipulate signals.
- We need a formal mathematical framework for the study of such systems.
- Such a framework is necessary in order to ensure that a system will meet the required specifications (e.g., performance and safety).
- If a system fails to meet the required specifications or fails to work altogether, negative consequences usually ensue.
- When a system fails to operate as expected, the consequences can some times be catastrophic

How is a Signal Represented?



- Mathematically, signals are represented as a function of one or more **independent variables**.
- For instance a black & white video signal intensity is dependent on x, y coordinates and time t $f(x,y,t)$
- We shall be exclusively concerned with signals that are a function of a single variable: time

Example: Signals in an Electrical Circuit

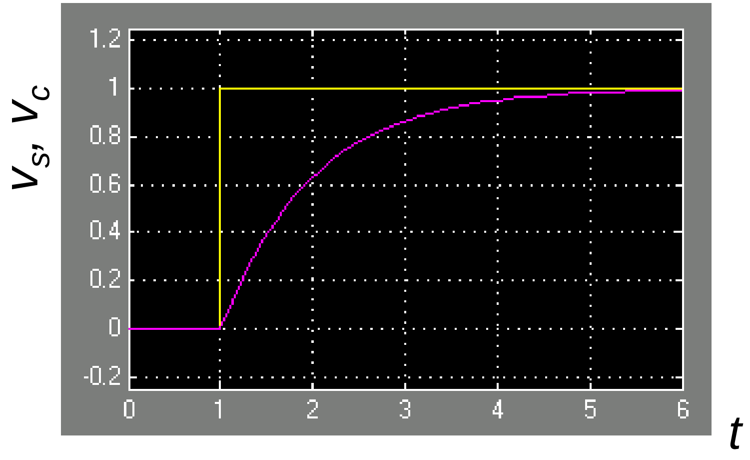


$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

- The signals v_c and v_s are patterns of variation over time



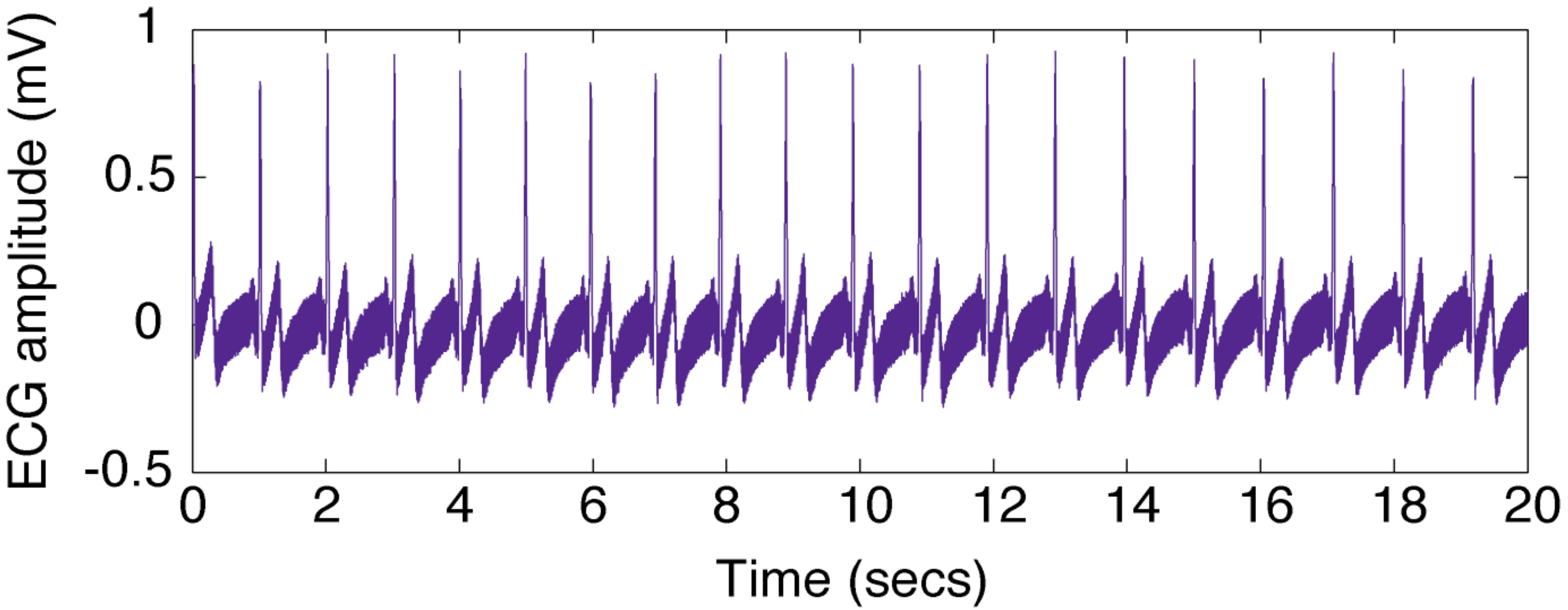
Step (signal) v_s at t=1
RC = 1
First order (exponential) response for v_c

- Note, we could also have considered the voltage across the resistor or the current as signals

Signal Classification

Type of Independent Variable

Time is often the independent variable. Example: the electrical activity of the heart recorded with chest electrodes — the electrocardiogram (ECG or EKG).



The variables can also be spatial

Eg. Cervical MRI

In this example, the signal is the intensity as a function of the spatial variables *x* and *y*.

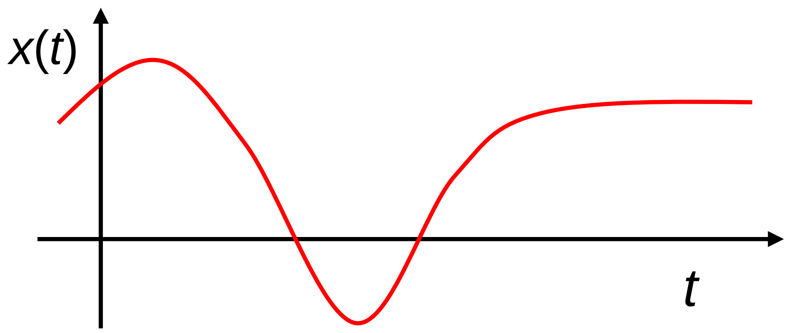
Independent Variable Dimensionality

An independent variable can be 1-D (*t* in the ECG) or 2-D (*x*, *y* in an image).

1. Continuous & Discrete-Time Signals

- **Continuous-Time Signals**
- Most signals in the real world are continuous time, as the scale is infinitesimally fine.

  Ex: voltage, velocity, pressure
- Denote by $x(t)$, where the time interval may be bounded (finite) or infinite



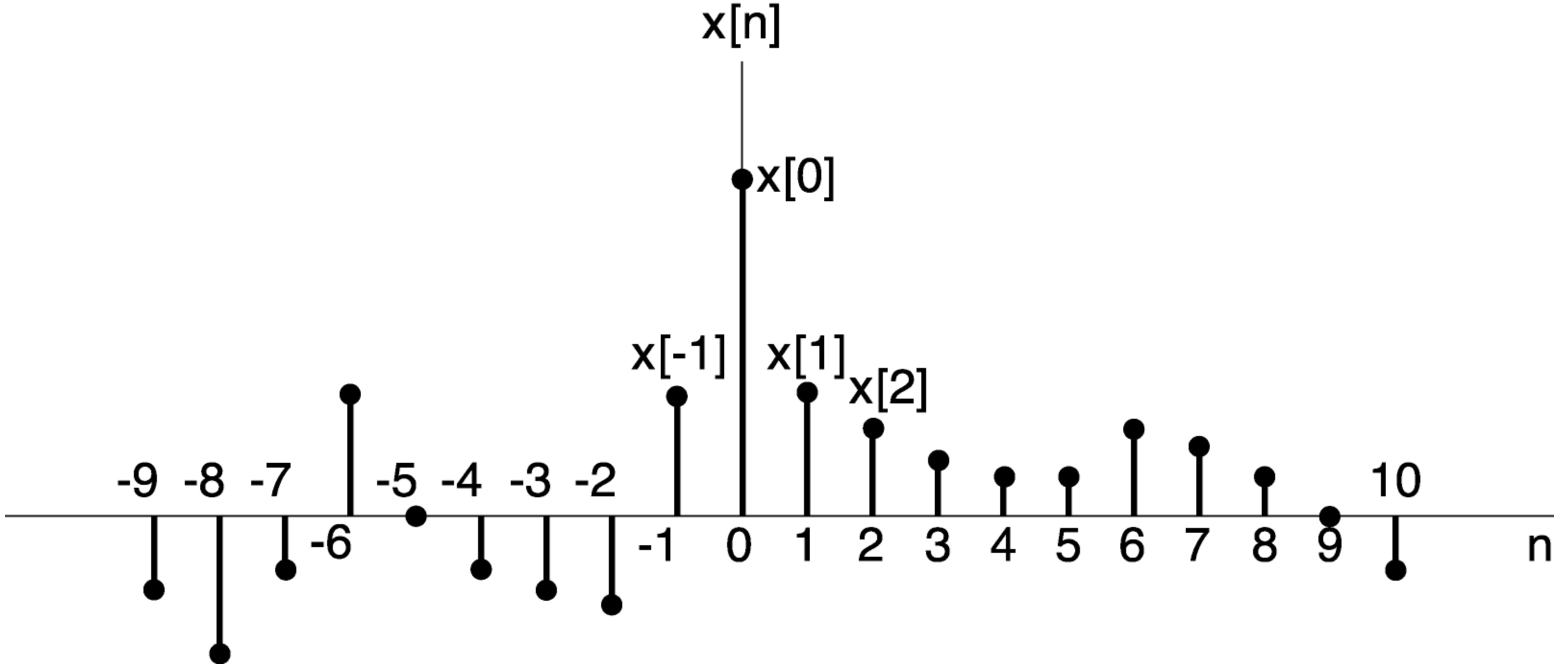
- **Sampled continuous signal** $x[n] = x(nk)$ – k is sample time

Discrete-Time Signals

Some real world and many digital signals are discrete time, as they are sampled

E.g. pixels, daily stock price (anything that a digital computer processes)

Denote by $x[n]$, where n is an integer value that varies discretely



$x[n]$, n — integer, time varies discretely

Examples of DT signals in nature:

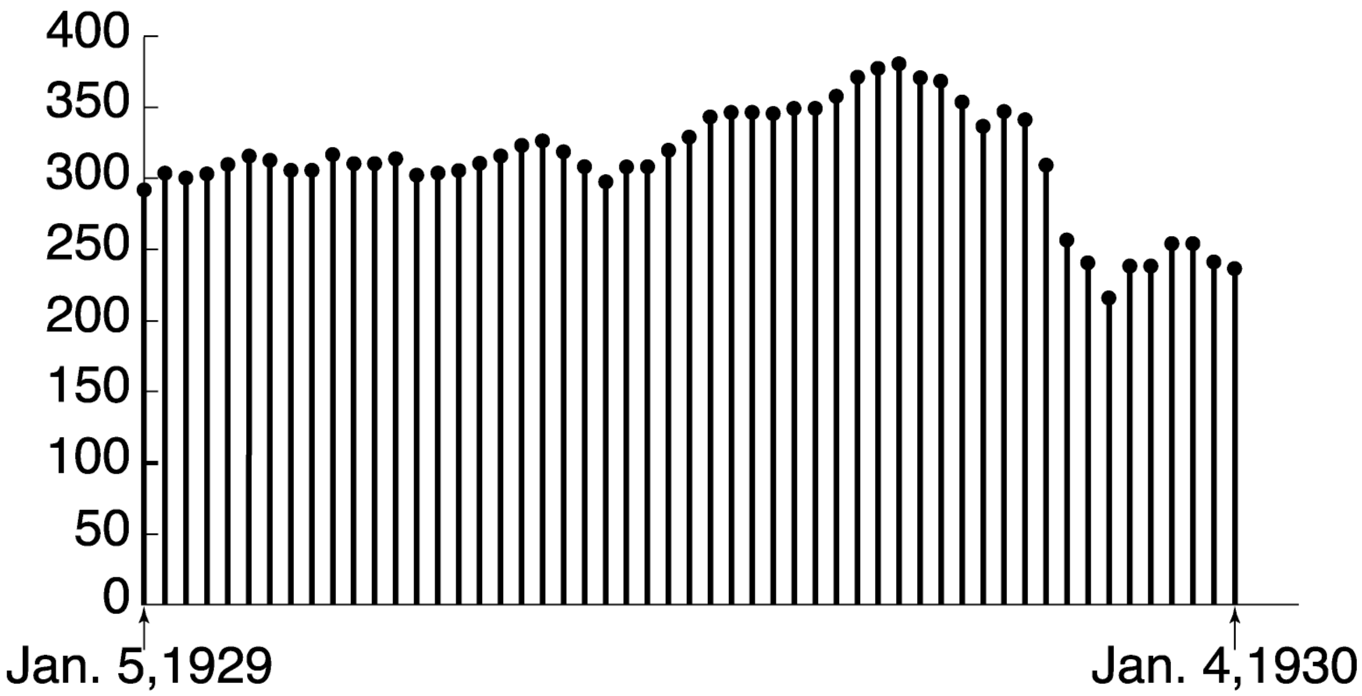
— DNA base sequence

— Population of the *n*th generation of certain species

-- Stock market index

Many human-made Signals are DT

Ex.#1 Weekly Dow-Jones industrial average



Ex.#2 digital image

Why DT? — Can be processed by modern digital computers and digital signal processors (DSPs).

signals

Recap

- A signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.

- For a function f , in the expression f (t1, t2, . . . , tn), each of the {tk} is called an independent variable, while the function value itself is referred to as a dependent variable.

- Some examples of signals include:

a voltage or current in an electronic circuit
the position, velocity, or acceleration of an object
a force or torque in a mechanical system
a flow rate of a liquid or gas in a chemical process
a digital image, digital video, or digital audio

Classification of Signals

- Number of independent variables (i.e., dimensionality):
  A signal with one independent variable is said to be one dimensional (e.g., audio).
  A signal with more than one independent variable is said to be multi-dimensional (e.g., image).

- Continuous or discrete independent variables:
  A signal with continuous independent variables is said to be continuous time (CT) (e.g., voltage).
  A signal with discrete independent variables is said to be discrete time(DT) (e.g., stock market index).

- Continuous or discrete dependent variable:
  A signal with a continuous dependent variable is said to be continuous valued (e.g., voltage).
  A signal with a discrete dependent variable is said to be discrete valued (e.g., digital image).

- A continuous-valued CT signal is said to be analog (e.g., voltage waveform).

- A discrete-valued DT signal is said to be digital (e.g., digital audio).

Transformation of the independent variable

Time Shifting

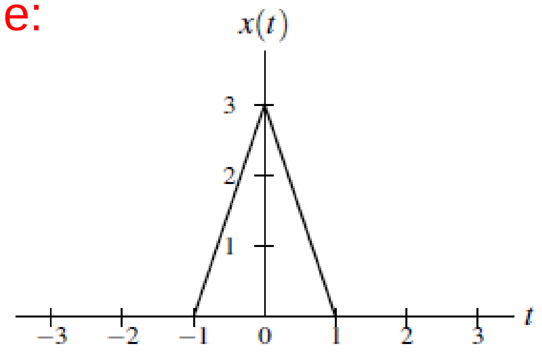
- Time shifting (also called translation) maps the input signal x to the output signal y as given by $y(t) = x(t - b)$,
  where b is a real number.

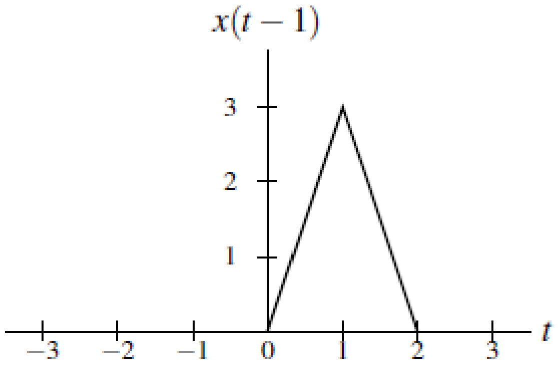
- Such a transformation shifts the signal (to the left or right) along the time axis.

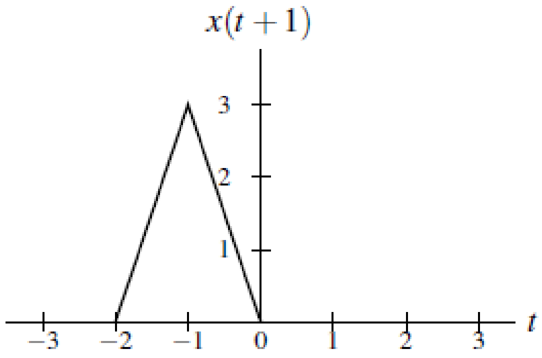
- If $b > 0$, y is shifted to the right by $|b|$, relative to x (i.e., delayed in time).

- If $b < 0$, y is shifted to the left by $|b|$, relative to x (i.e., advanced in time).

Example:



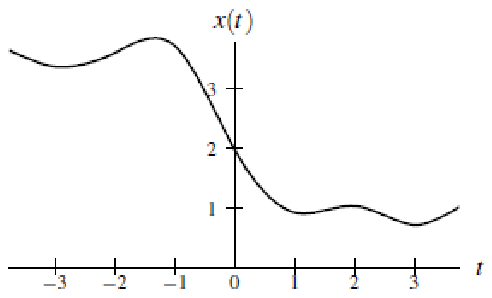


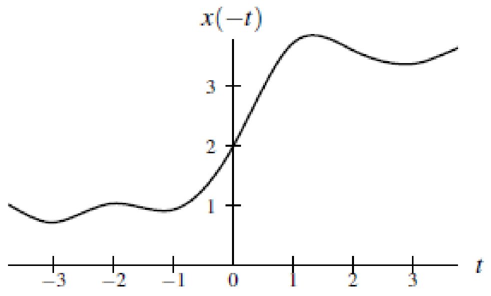


Time reversal (Reflection)

- Time reversal (also known as reflection) maps the input signal x to the output signal y as given by $y(t) = x(-t)$.

- Geometrically, the output signal y is a reflection of the input signal x about the (vertical) line $t = 0$.





Time Compression/Expansion (Dilation)

- **Time compression/expansion** (also called dilation) maps the input signal x to the output signal y as given by

$$y(t) = x(at),$$

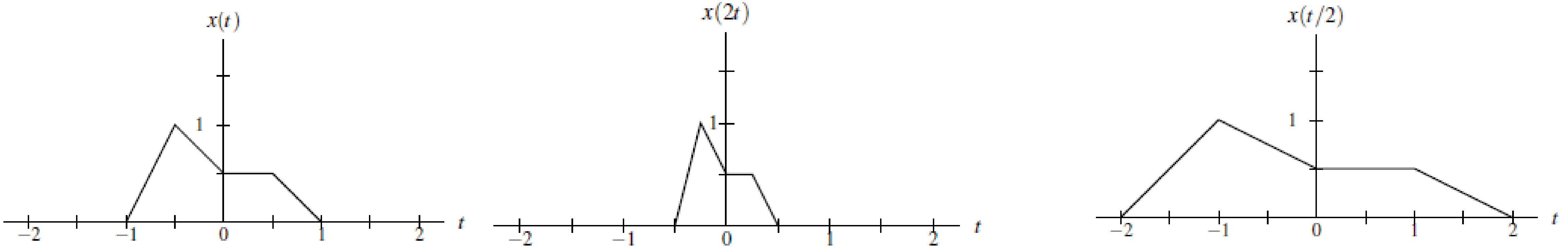
where a is a strictly positive real number.

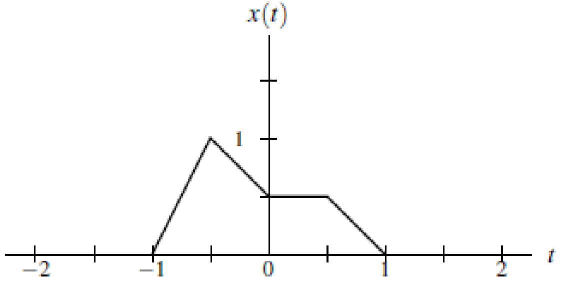
- Such a transformation is associated with a compression/expansion along the time axis.

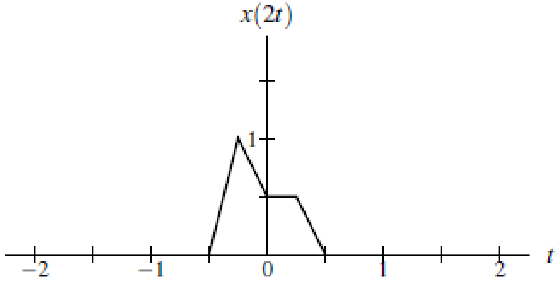
- If $a > 1$, y is compressed along the horizontal axis by a factor of a, relative to x.

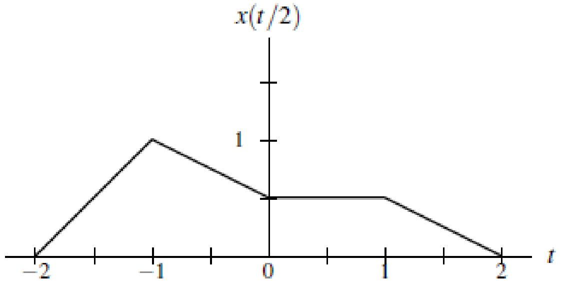
- If $a < 1$, y is expanded (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x

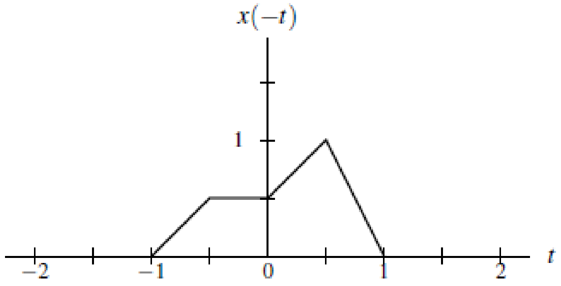
- If |a| = 1, the signal is neither expanded nor compressed.
- If a < 0, the signal is also time reversed.
- Time reversal is a special case of time scaling with a = −1;




x(t)
1
−2
−1
0
1
2
t


x(2t)
1
−2
−1
0
1
2
t


x(t/2)
1
−2
−1
0
1
2
t


x(−t)
1
−2
−1
0
1
2
t

Combination of a Time-scaling and Time-shifting

- Consider a transformation that maps the input signal x to the output signal y as given by

  $$y(t) = x(at - b),$$

  where a and b are real numbers and $a \neq 0$.

- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.

- Since time scaling and time shifting do not commute, we must be particularly careful about the order in which these transformations are applied.

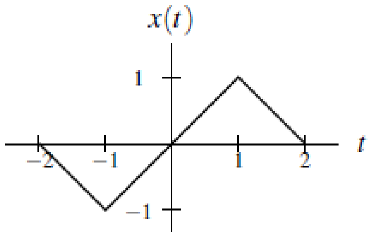
- The above transformation has two distinct but equivalent interpretations:

  first, time shifting x by b, and then time scaling the result by a;

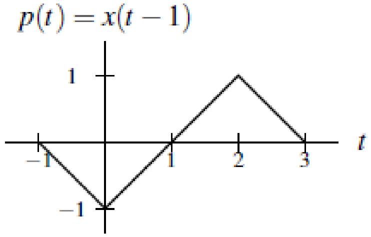
  first, time scaling x by a, and then time shifting the result by b/a.

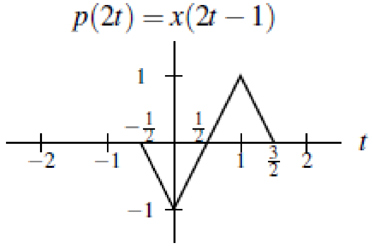
- Note that the time shift is not by the same amount in both cases.

Given $x(t)$ as shown below, find $x(2t-1)$.

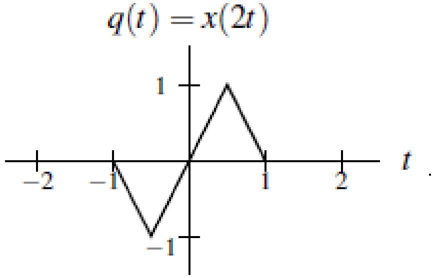


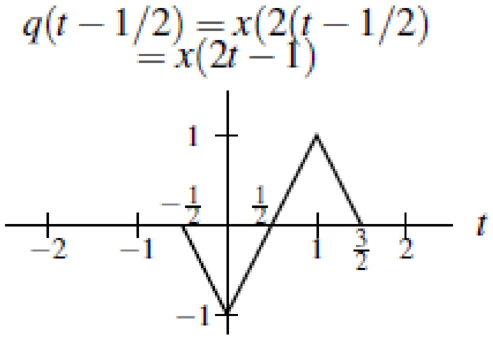
time shift by 1 and then time scale by 2





time scale by 2 and then time shift by $\frac{1}{2}$





Transformation of the dependent variable

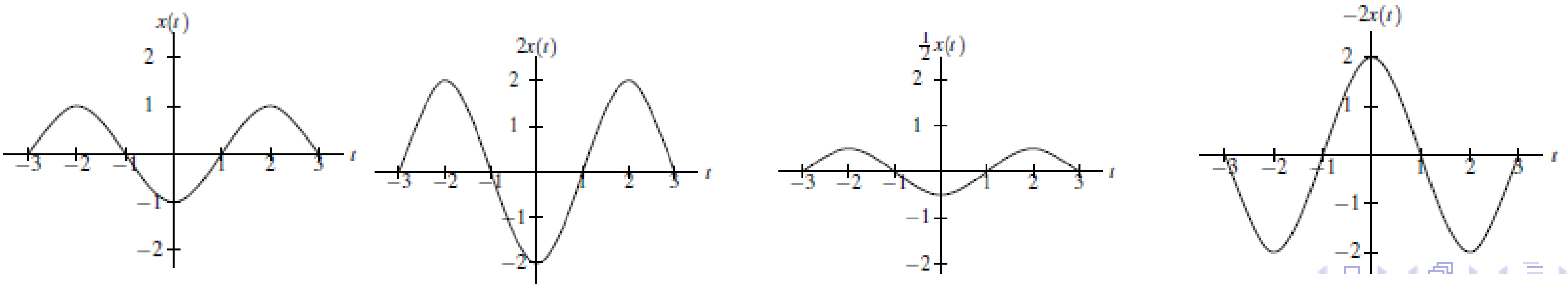
Amplitude Scaling

- Amplitude scaling maps the input signal x to the output signal y as given by

$$y(t) = ax(t),$$

  where a is a real number.

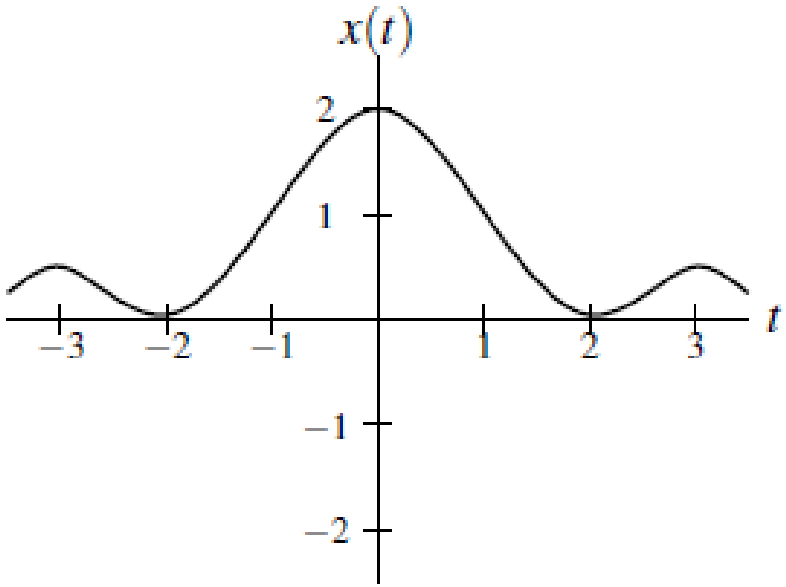
- Geometrically, the output signal y is expanded/compressed in amplitude and/or reflected about the horizontal axis.

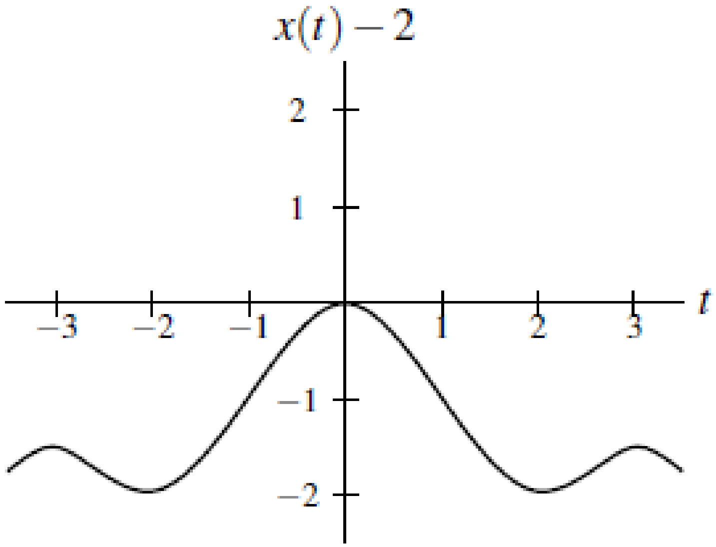


Amplitude shifting

- Amplitude shifting maps the input signal x to the output signal y as given by $y(t) = x(t)+b$,
  where b is a real number.

- Geometrically, amplitude shifting adds a vertical displacement to x.





Combined Amplitude scaling and Amplitude shifting

- We can also combine amplitude scaling and amplitude shifting transformations.

- Consider a transformation that maps the input signal x to the output signal y, as given by

$$y(t) = ax(t)+b,$$

where a and b are real numbers.

- Equivalently, the above transformation can be expressed as

$$y(t) = a[x(t)+ \frac{b}{a}]$$

- .The above transformation is equivalent to:

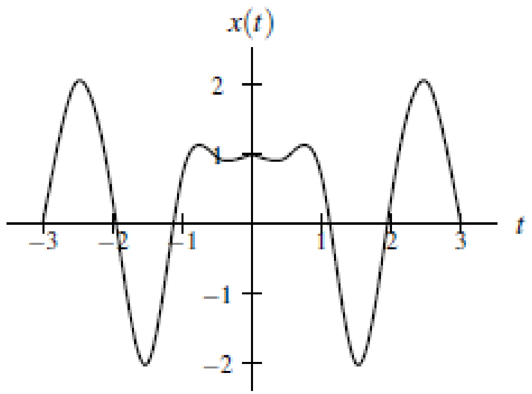
1 first amplitude scaling x by a, and then amplitude shifting the resulting signal by b; or

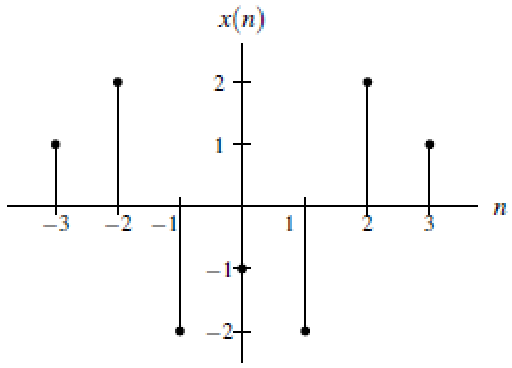
2 first amplitude shifting x by b/a, and then amplitude scaling the resulting signal by a.

Properties of Signals

Even Signals

- A function x is said to be even if it satisfies
  $$x(t) = x(-t) \text{ for all } t.$$
- A sequence x is said to be even if it satisfies
  $$x(n) = x(-n) \text{ for all } n.$$
- Geometrically, the graph of an even signal is symmetric about the origin. Some examples of even signals are shown below.





Odd Signals

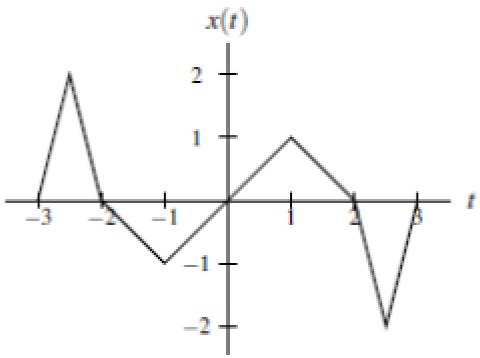
- A function x is said to be odd if it satisfies
  $x(t) = -x(-t)$ for all t.

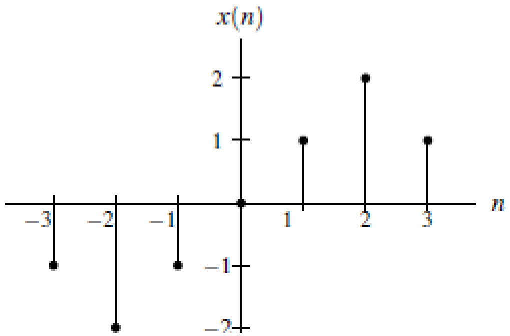
- A sequence x is said to be odd if it satisfies
  $x(n) = -x(-n)$ for all n.

- Geometrically, the graph of an odd signal is antisymmetric about the origin.

- An odd signal x must be such that $x(0) = 0$.

- Some examples of odd signals are shown below





Decomposition of Signals into Even and Odd parts

- Every function x has a unique representation of the form

$$x(t) = x_e(t) + x_o(t),$$

where the functions x_e and x_o are even and odd, respectively.

- In particular, the functions xe and xo are given by

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad \text{and} \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)].$$

- The functions x_e and x_o are called the even part and odd part of x, respectively.

- For convenience, the even and odd parts of x are often denoted as Even{x} and Odd{x}, respectively.

Periodic and Non- periodic Signals

- A function x is said to be periodic with period T (or T-periodic) if, for some strictly-positive real constant T, the following condition holds:
  $x(t) = x(t + T)$ for all t.

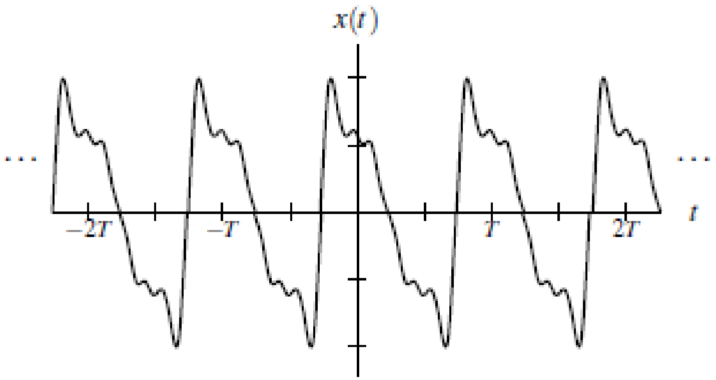
- A T-periodic function x is said to have frequency 1/ T and angular frequency 2π/T .

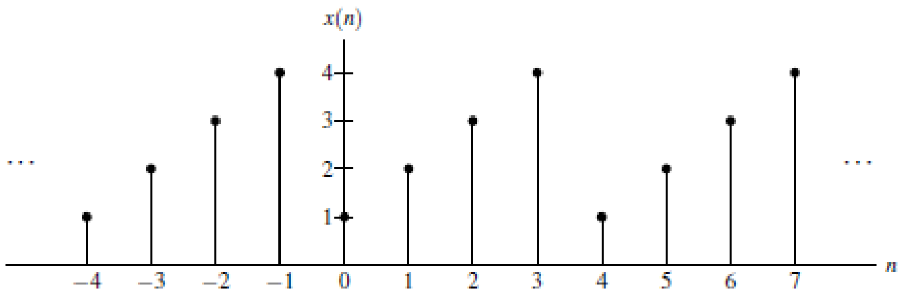
- A sequence x is said to be periodic with period N (or N-periodic) if, for some strictly-positive integer constant N, the following condition holds:
  $x(n) = x(n+N)$ for all n.

- An N-periodic sequence x is said to have frequency 1/N and angular frequency 2π/N .
- A function/sequence that is not periodic is said to be aperiodic

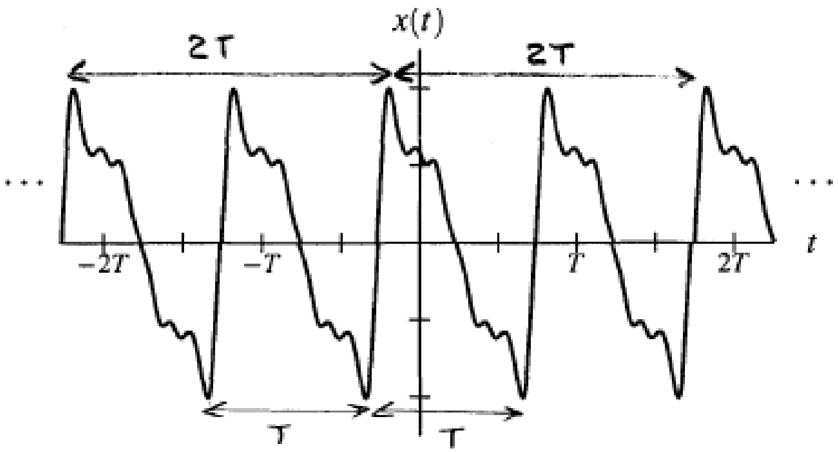
Periodic and Non- periodic Signals

examples





The period of a periodic signal is not unique. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



The smallest period with which a signal is periodic is called the fundamental period and its corresponding frequency is called the fundamental frequency.

Symmetry and Addition/ Multiplication of Signals

- Sums involving even and odd functions have the following properties:
  The sum of two even functions is even.
  The sum of two odd functions is odd.
  The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.

- That is, the sum of functions with the same type of symmetry also has the same type of symmetry.

- Products involving even and odd functions have the following properties:
  The product of two even functions is even.
  The product of two odd functions is even.
  The product of an even function and an odd function is odd.

- That is, the product of functions with the same type of symmetry is even, while the product of functions with opposite types of symmetry is odd

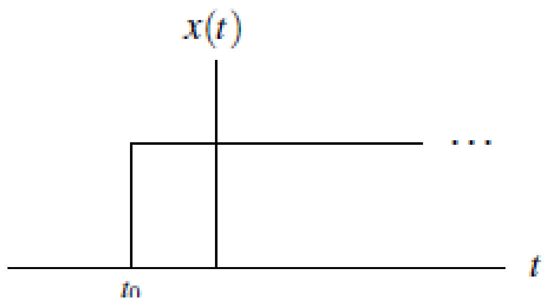
Sum of periodic functions

- **Sum of periodic functions.** Let $x1$ and $x2$ be periodic functions with fundamental periods $T1$ and $T2$, respectively. Then, the sum $y = x1 + x2$ is a periodic function if and only if the ratio $T1/T2$ is a rational number (i.e., the quotient of two integers). Suppose that $T1/T2 = q/r$ where q and r are integers and coprime (i.e., have no common factors), then the fundamental period of y is $rT1$ (or equivalently, $qT2$, since $rT1 = qT2$).
(Note that $rT1$ is simply the least common multiple of $T1$ and $T2$.)

- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N - 1$ times.

Right sided Signal:

- A signal x is said to be right sided if, for some (finite) real t) = 0 for all t < t_0
  (i.e., x is only potentially nonzero to the right of t_0).
- An example of a right-sided signal is shown below.



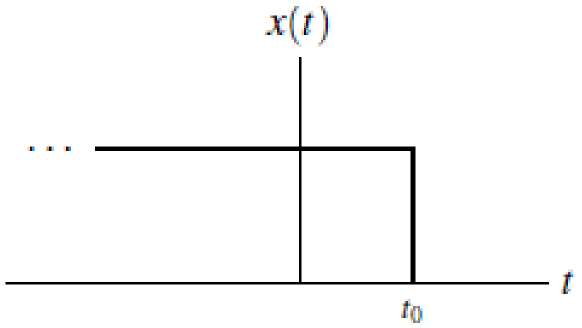
- A signal x is said to be causal if
  $x(t) = 0$ for all $t < 0$.
- A causal signal is a special case of a right-sided signal.
- A causal signal is not to be confused with a causal system. In these two contexts, the word “causal” has very different meanings.

Left Sided Signal:

- A signal x is said to be left sided if, for some (finite) real constant t_0, the following condition holds:

  $x(t) = 0$ for all $t > t_0$

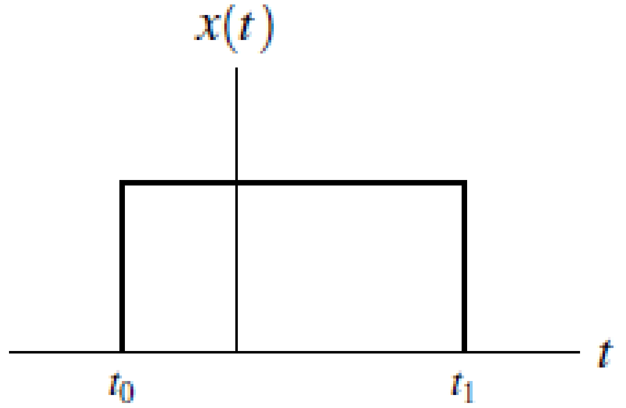
  (i.e., x is only potentially nonzero to the left of t_0).
- An example of a left-sided signal is shown below.



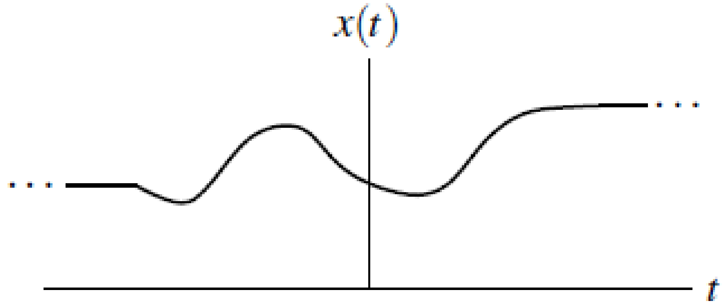
- Similarly, a signal x is said to be anticausal if $x(t) = 0$ for all $t > 0$.
- An anticausal signal is a special case of a left-sided signal.
- An anticausal signal is not to be confused with an anticausal system. In these two contexts, the word “anticausal” has very different meanings.

Finite Duration and Two sided Signals:

- signal that is both left sided and right sided is said to be finite duration (or time limited).
- An example of a finite duration signal is shown below.



- A signal that is neither left sided nor right sided is said to be two sided.
- An example of a two-sided signal is shown below.



Bounded Signals:

- signal x is said to be bounded if there exists some (finite) positive real constant A such that $|x(t)| \leq A$ for all t
  (i.e., $x(t)$ is finite for all t).
- Examples of bounded signals include the sine and cosine functions.
- Examples of unbounded signals include the tan function and any nonconstant polynomial function.

“Electrical” Signal Energy & Power

- It is often useful to characterise signals by measures such as **energy** and **power**
- For example, the **instantaneous power** of a resistor is:

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

- and the **total energy** expanded over the interval $[t_1, t_2]$ is:

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- and the **average energy** is:

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- How are these concepts defined for any continuous or discrete time signal?

Generic Signal Energy and Power

- **Total energy** of a continuous signal $x(t)$ over $[t_1, t_2]$ is:

$$E = \int_{t_1}^{t_2} \left| x(t) \right|^2 dt$$

- where |.| denote the magnitude of the (complex) number.
- Similarly for a discrete time signal $x[n]$ over $[n_1, n_2]$:

$$E = \sum_{n=n_1}^{n_2} \left| x[n] \right|^2$$

- By dividing the quantities by $(t_2 - t_1)$ and $(n_2 - n_1 + 1)$, respectively, gives the **average power**, P

- Note that these are similar to the electrical analogies (voltage), but they are different, both value and dimension.

Energy and Power over Infinite Time

- For many signals, we're interested in examining the power and energy over an infinite time interval (-∞, ∞). These quantities are therefore defined by:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- If the sums or integrals do not converge, the energy of such a signal is infinite

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- Two important (sub)classes of signals
  1. Finite total energy (and therefore zero average power)
  2. Finite average power (and therefore infinite total energy)
- Signal analysis over infinite time, all depends on the "tails" (limiting behaviour)

Energy and Power Signals:

- The energy E contained in the signal x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt. \qquad E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- A signal with finite energy is said to be an energy signal.

- The average power P contained in the signal x is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt. \qquad P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- A signal with (nonzero) finite average power is said to be a power signal.

Properties of Energy and Power Signals

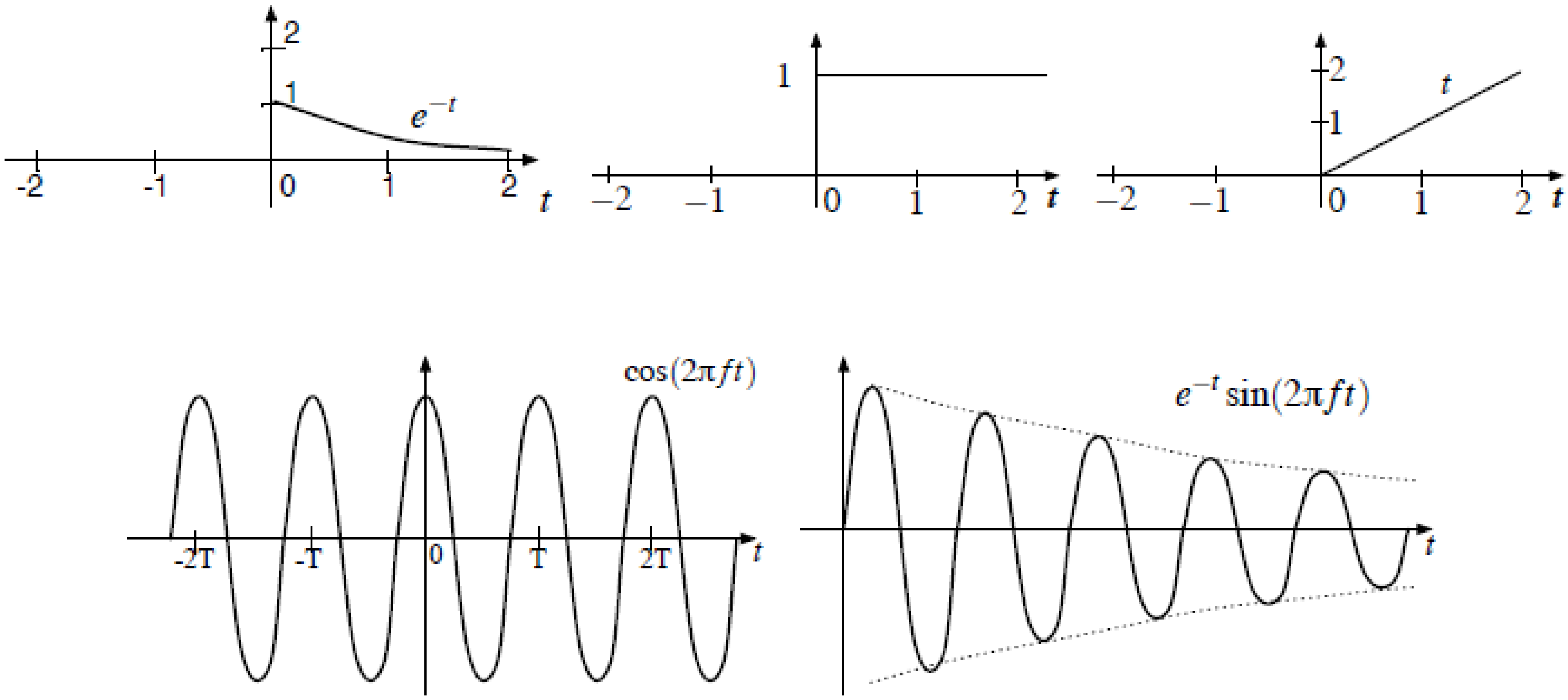
An energy signal $x(t)$ has zero power

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{2T} \underbrace{\int_{-T}^{T} |x(t)|^2 \, dt}_{\to E_x < \infty} \\
&= 0
\end{aligned}
$$

A power signal has infinite energy

$$
\begin{aligned}
E_x &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt \\
&= \lim_{T\to\infty} 2T \underbrace{\frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}_{\to P_x > 0} = \infty.
\end{aligned}
$$

Classify these signals as power or energy signals



A bounded periodic signal.

A bounded finite duration signal.

Elementary Signals

Elementary Signals

Sinusoidal signals
Exponential signals
Complex exponential signals
Unit step and unit ramp
Impulse functions

Sinusoidal Signals

- A sinusoidal signal is of the form

  $$x(t) = \cos(\omega t + \phi):$$

  where the radian frequency is ω, which has the units of radians/s.

- Also very commonly written as

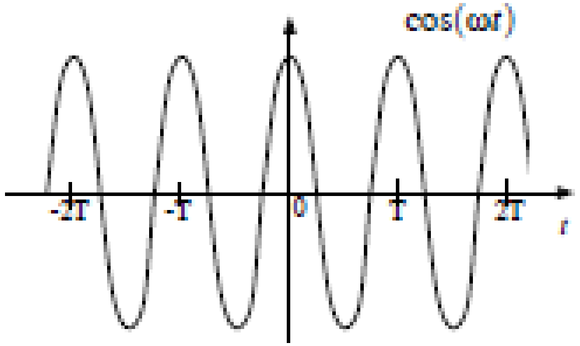
  $$x(t) = A\cos(2\pi ft +\phi):$$

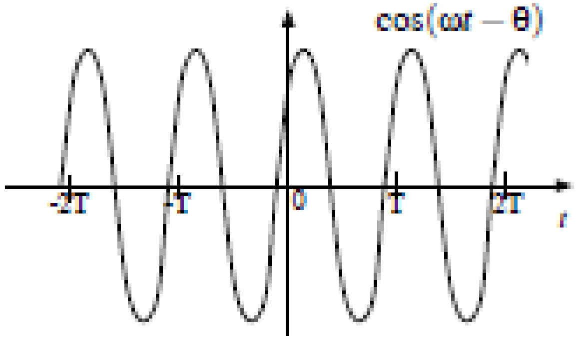
  where f is the frequency in Hertz.

- We will often refer to ω as the frequency, but it must be kept in mind that it is really the radian frequency, and the frequency is actually f .

- The period of the sinusoid is $T = \frac{1}{f} = \frac{2\pi}{\omega}$

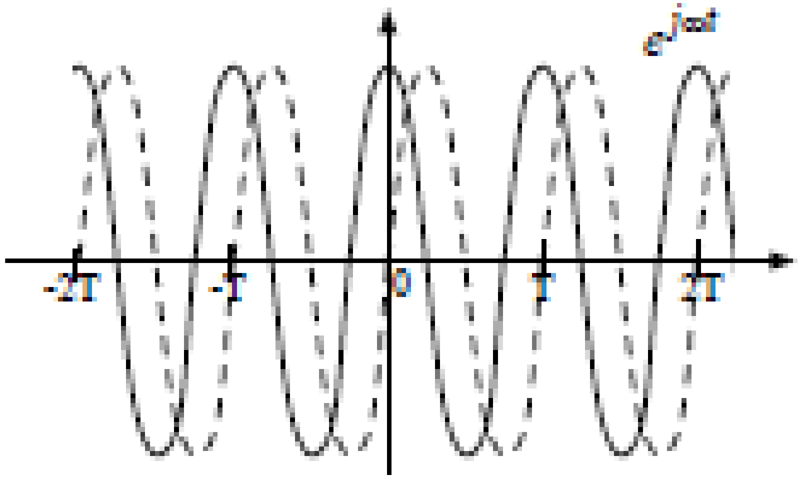
  with the units of seconds.
- The phase or phase angle of the signal is θ, given in radians





Complex Sinusoids

- The Euler relation defines $e^{j\phi} = \cos\phi + j\sin\phi.$
- A complex sinusoid is $Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta).$



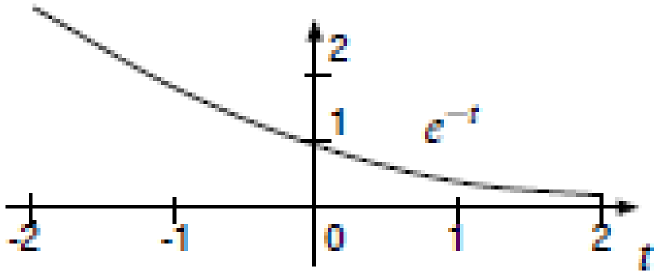
- Real sinusoid can be represented as the real part of a complex sinusoid

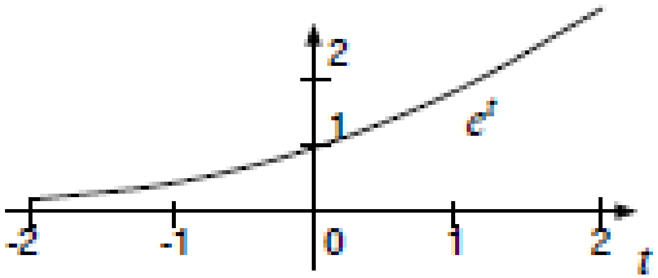
$$\Re\{Ae^{j(\omega t+\theta)}\} = A\cos(\omega t+\theta)$$

Exponential Signals

- An exponential signal is given by $x(t) = e^{\sigma t}$

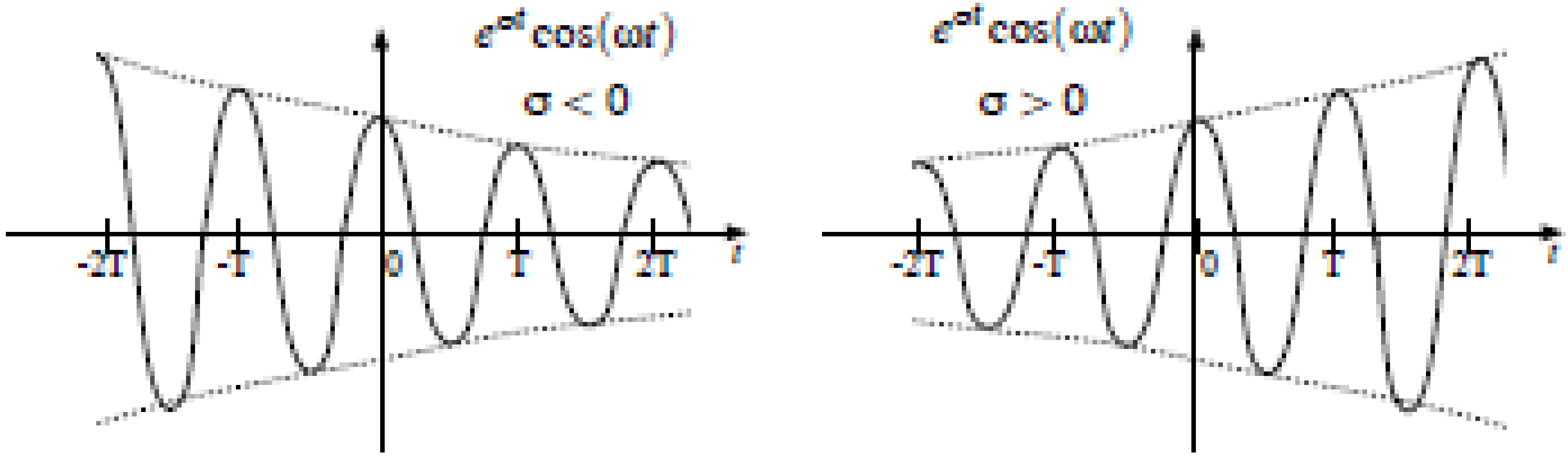
- If $\sigma < 0$ this is exponential decay.
- If $\sigma > 0$ this is exponential growth.





Damped or Growing Sinusoids

- A damped or growing sinusoid is given by $x(t) = e^{\sigma t}\cos(\omega t + \theta)$
- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.



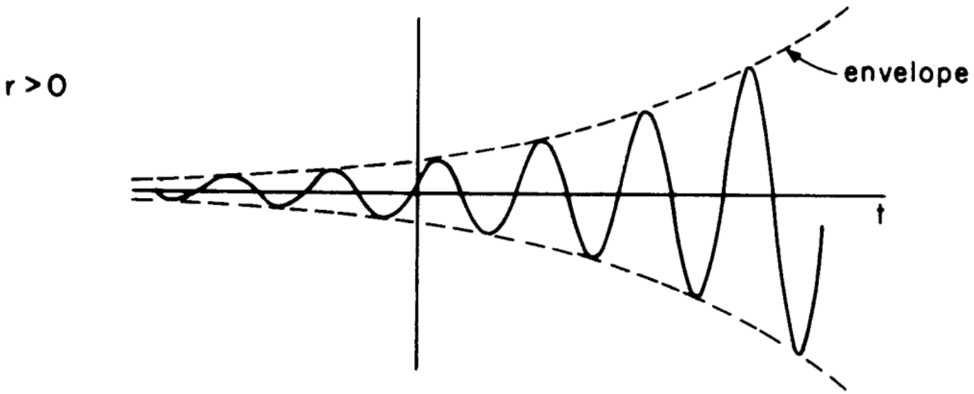
COMPLEX **EXPONENTIAL:** CONTINUOUS TIME

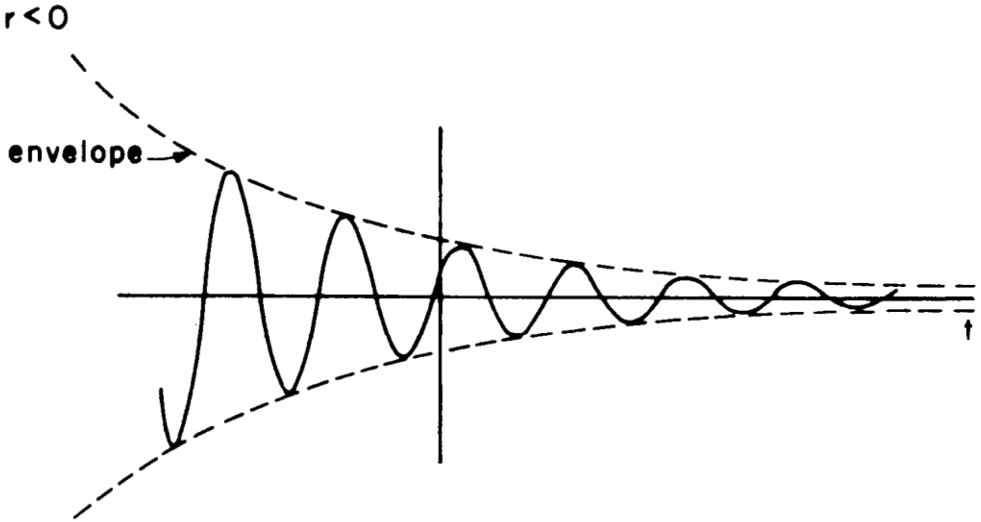
$$x(t) = Ce^{at}$$

C and a are complex numbers

$$x(t) = |C|\, e^{j\theta}\, e^{(r + j\omega_0)t} \qquad = |C|\, e^{rt}\, \underbrace{e^{j(\omega_0 t + \theta)}}$$

$$x(t) = |C|\, e^{rt} \cos(\omega_0 t + \theta) + j\,|C|\, e^{rt} \sin(\omega_0 t + \theta)$$





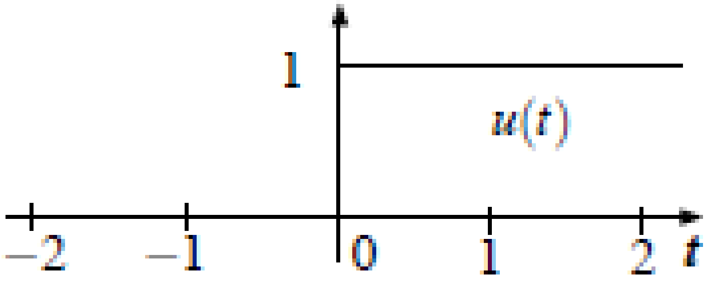
Unit Step Functions

- The unit step function u(t) is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the Heaviside step function.

- Alternate definitions of value exactly at zero, such as 1/2.

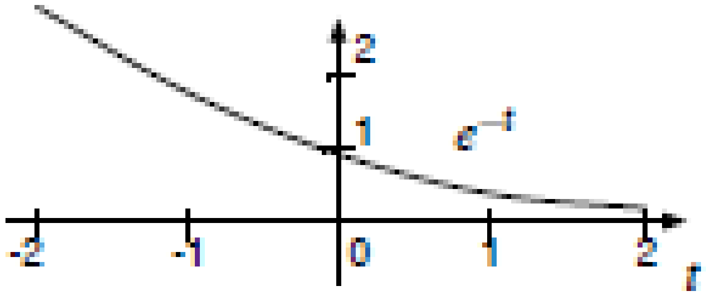


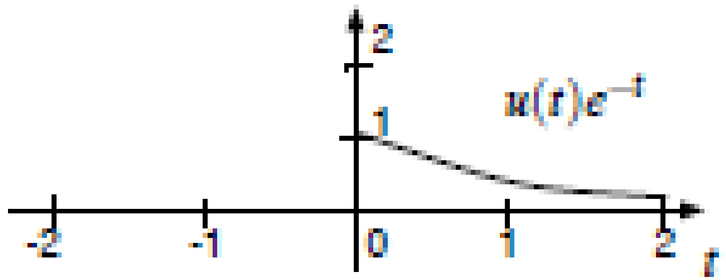
Uses for the unit step:

Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

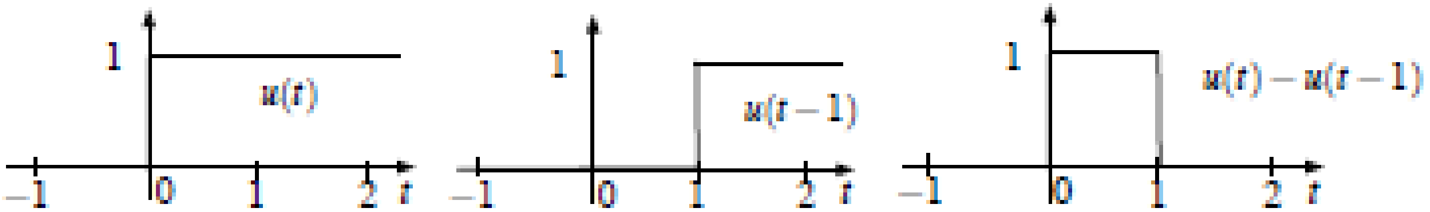




- Combinations of unit steps to create other signals. The rectangular signal

$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

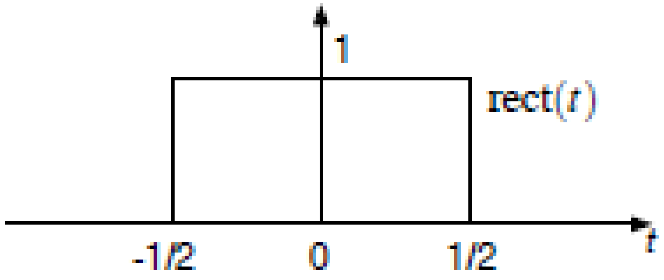
can be written as $x(t) = u(t) - u(t-1)$.



Unit Rectangle

Unit rectangle signal:

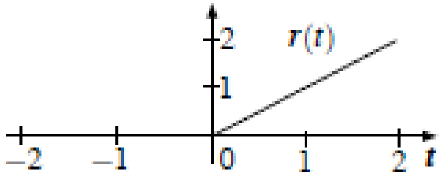
$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



Unit Ramp

The unit ramp is defined as $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

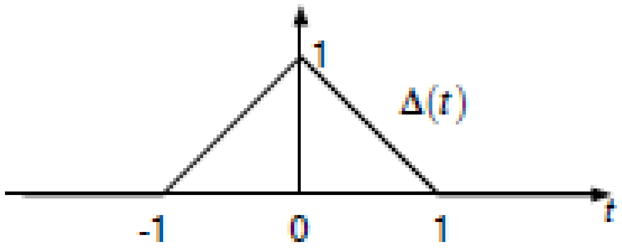
$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$



Unit Triangle

Unit Triangle Signal

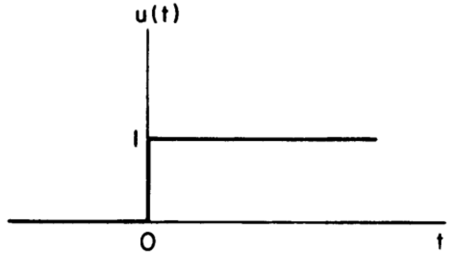
$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



Continuous Time Unit Step Signals

- The continuous **unit step signal** is defined:

$$x(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

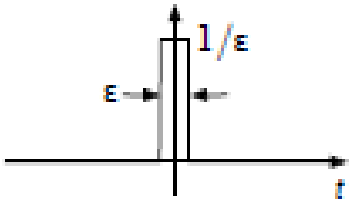


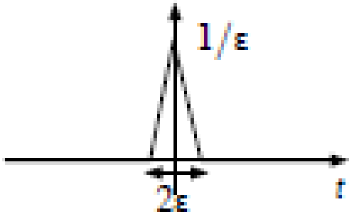
Impulsive signals

(Dirac's) delta function or impulse is an idealization of a signal that

- is very large near t = 0
- is very small away from t = 0
- has integral 1

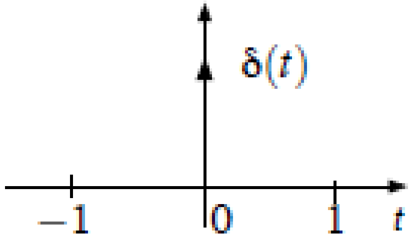
for example:

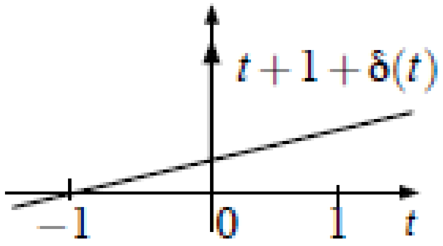




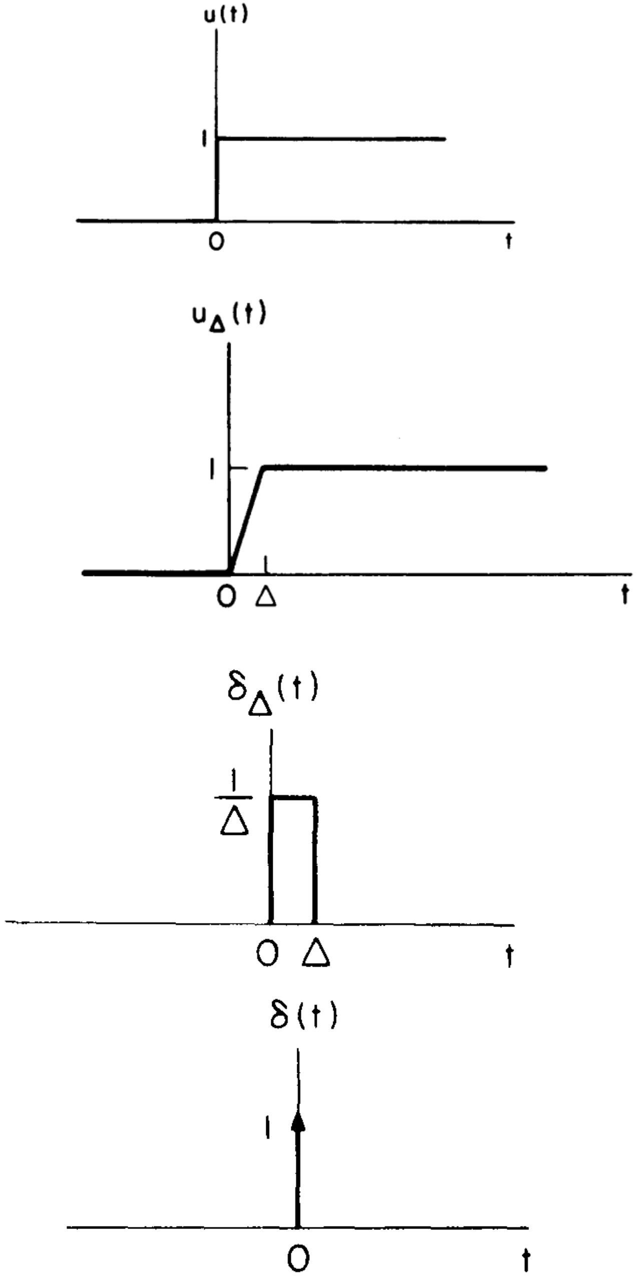
- the exact shape of the function doesn't matter
- ε is small (which depends on context)

On plots is shown as a solid arrow:





Continuous Time Unit Impulse



$u(t) = u_\Delta(t)$ as $\Delta \to 0$

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

area = 1

$\delta(t) = \delta_\Delta(t)$ as $\Delta \to 0$

$$\delta(t) = \frac{du(t)}{dt}$$

height = "∞"

width = "0"

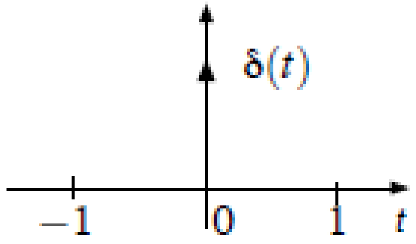
area = 1

The continuous **unit impulse signal** is defined:

$$x(t) = \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Note that it is discontinuous at *t*=0

The arrow is used to denote area, rather than actual value



Formal properties

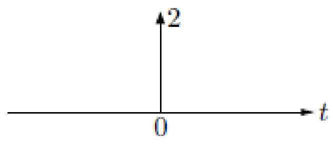
Formally we define δ by the property that $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$

provided f is continuous at t = 0

- δ(t) is not really defined for any t, only its behavior in an integral.
- Conceptually δ(t) = 0 for t ≠ 0, infinite at t = 0, but this doesn't make sense mathematically

Scaled Impulse

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,[\phi(t)\delta(t)]\ dt &= \int_{-\infty}^{\infty} [f(t)\phi(t)]\,\delta(t)\ dt\\ &= f(0)\phi(0)\end{aligned}$$



Multiplication of a Function by an Impulse

Consider a function $\phi(t)$ multiplied by an impulse $\delta(t)$,

If $\phi(t)$ is continuous at $t = 0$, can this be simplified?

Substitute into the formal definition with a continuous δ (t) and evaluate,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,[\phi(t)\delta(t)]\;dt &= \int_{-\infty}^{\infty} [f(t)\phi(t)]\,\delta(t)\;dt \\ &= f(0)\phi(0)\end{aligned}$$

Hence

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

is a scaled impulse, with strength (0).

Sifting property

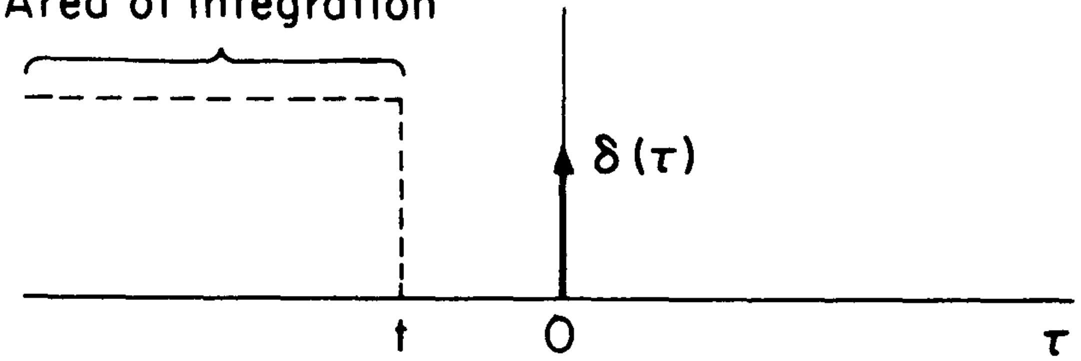
- The signal $x(t) = \delta\ (t - T)$ is an impulse function with impulse at $t = T$.

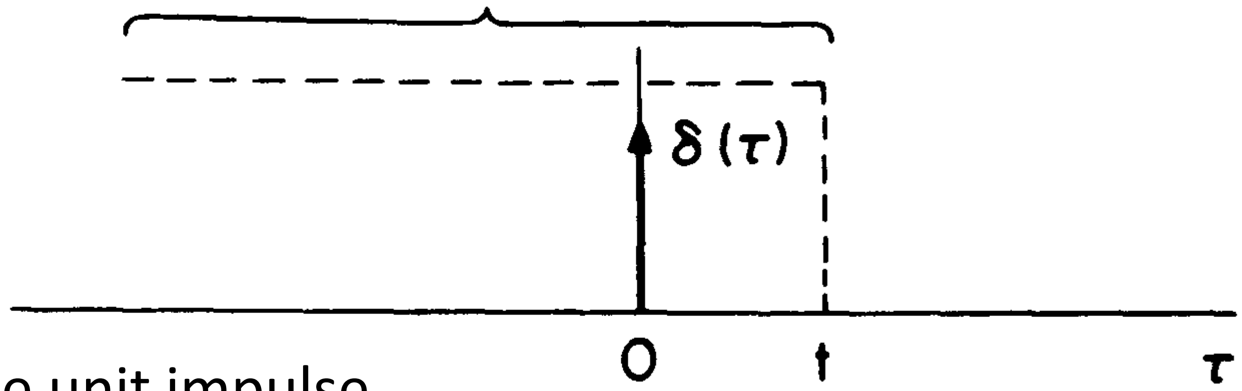
For f continuous at $t = T$,

$$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T)$$

- Multiplying by a function f (t) by an impulse at time T and integrating, extracts the value of f (T).
- This will be important property of the impulse.

$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$





unit step is the running integral of the unit impulse.

Summary of Impulse function

- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \frac{1}{|a|}\delta(t),$$

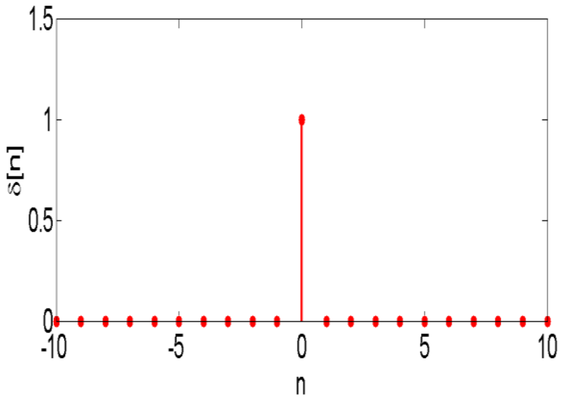
where a is a nonzero real constant.

Discrete Time Unit Impulse and Step Signals

- The discrete **unit impulse signal** is defined:

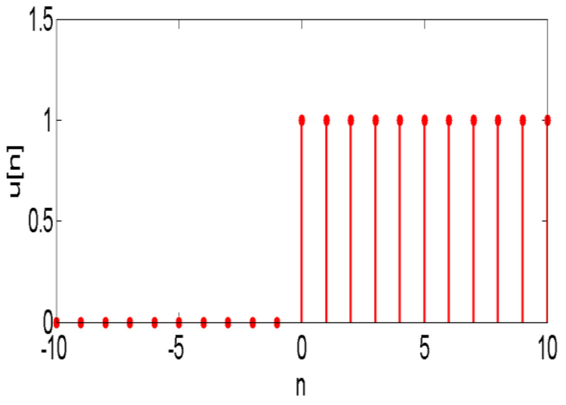
$$x[n] = \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

- Useful as a **basis** for analyzing other signals



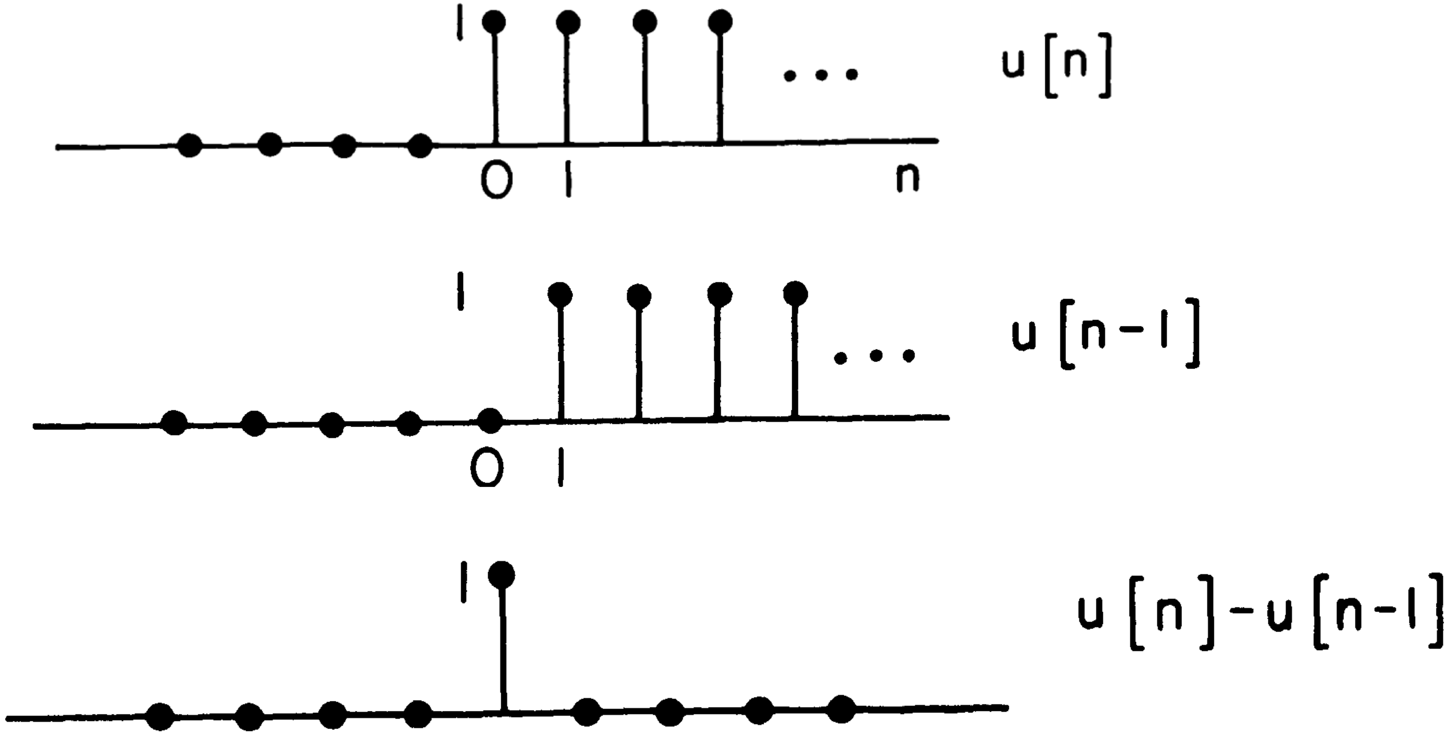
- The discrete **unit step signal** is defined:

$$x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$



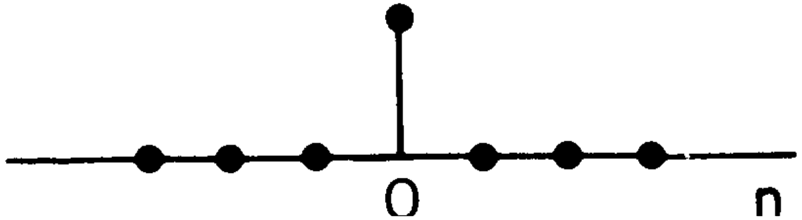
Note that the unit impulse is the first difference of the step signal

$$\delta[n] = u[n] - u[n-1]$$

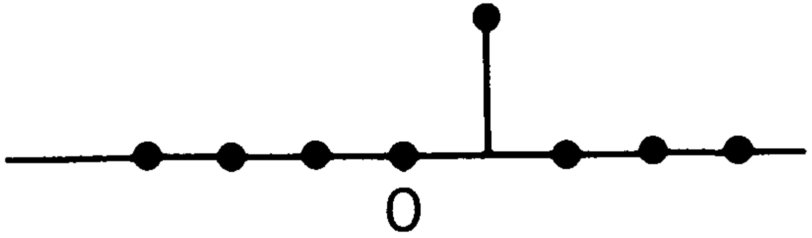


Also unit step can be represented in terms of shifted impulses

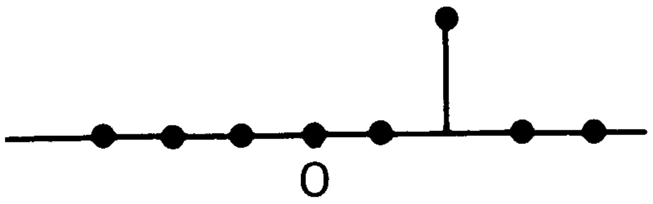
$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



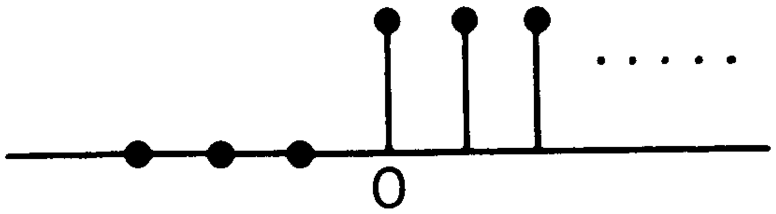
$\delta[n]$



$\delta[n-1]$



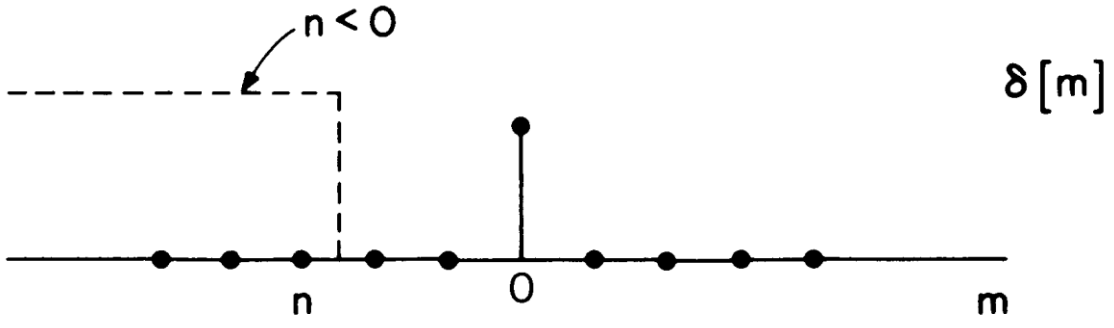
$\delta[n-2]$

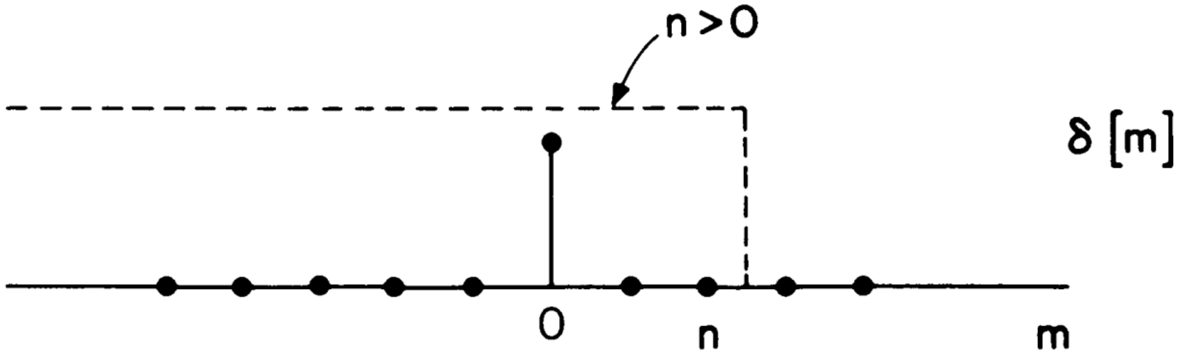


$\delta[n] + \delta[n-1] + \delta[n-2] + \cdots$

Similarly unit step is the running sum of the impulse function.

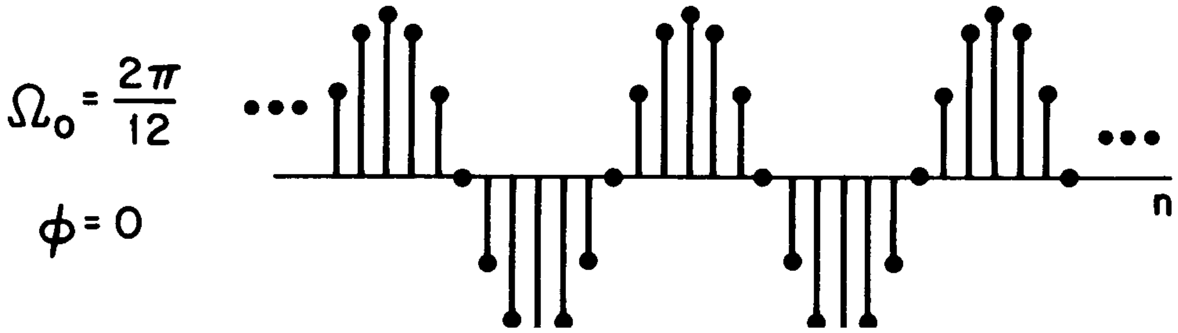
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

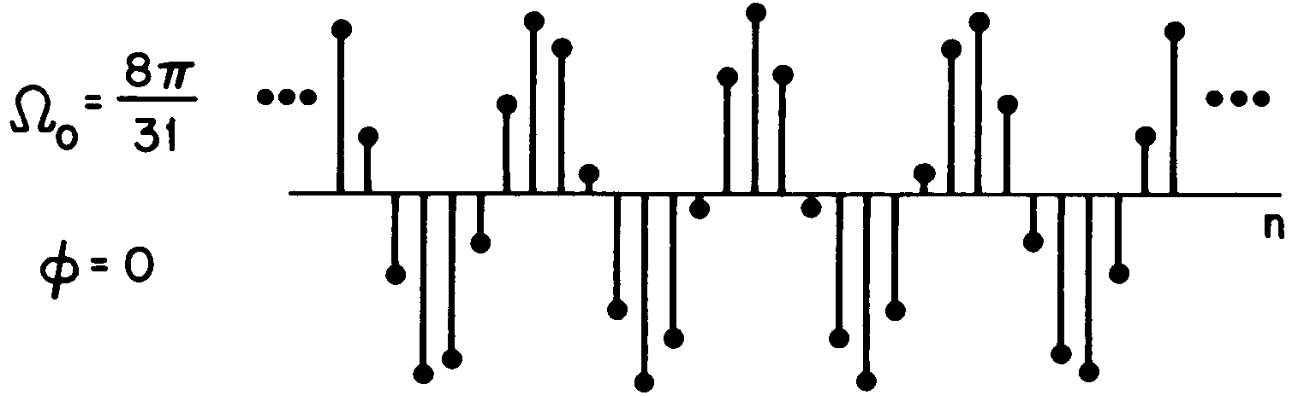




DISCRETE-TIME SINUSOIDAL SIGNAL

$$x[n] = A\cos(\Omega_o n + \phi)$$





Time Shift => Phase Change

$$A\cos[\Omega_o(n + n_o)] = A\cos[\Omega_o n + \Omega_o n_o]$$

Periodicity:

$$x[n] = A\cos(\Omega_o n + \phi)$$

$x[n] = x[n + N]$ smallest integer $N \triangleq$ period

$$A\cos[\Omega_o(n + N) + \phi] = A\cos[\Omega_o n + \underbrace{\Omega_o N}_{\text{integer multiple of } 2\pi\ ?} + \phi]$$

Periodic => $\Omega_o N = 2\pi m$

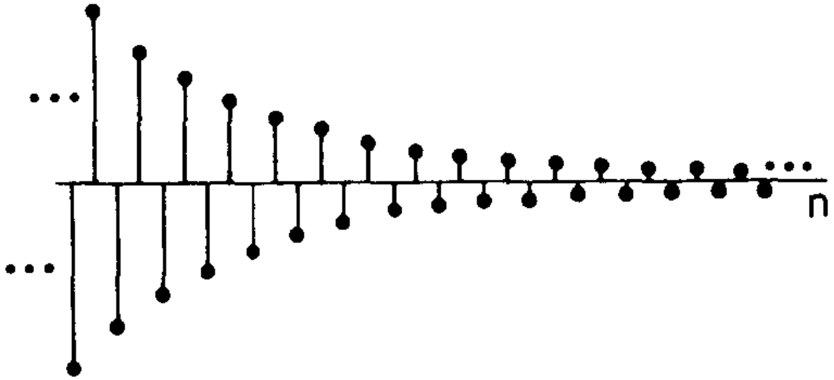
$N = \dfrac{2\pi m}{\Omega_o}$ N,m must be integers

smallest N (if any) = period

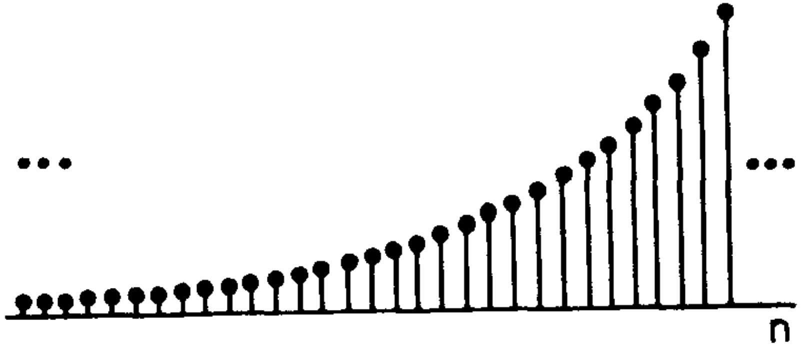
REAL EXPONENTIAL: DISCRETE-TIME

$$x[n] = Ce^{\beta n} = C\alpha^n$$

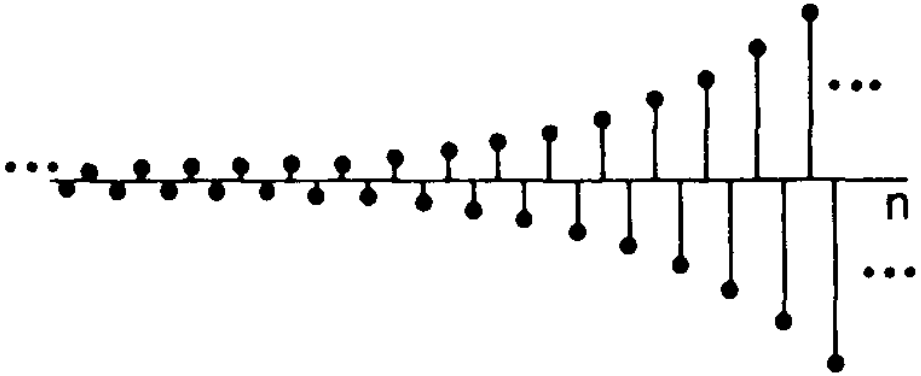
C, α are real numbers



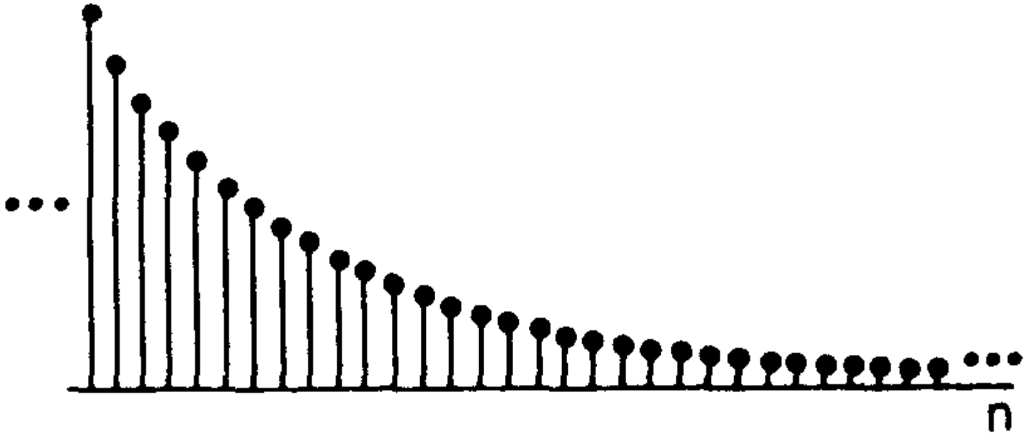
$\alpha < 0$
$|\alpha| < 1$



$\alpha > 0$
$|\alpha| > 1$



$\alpha < 0$
$|\alpha| >$



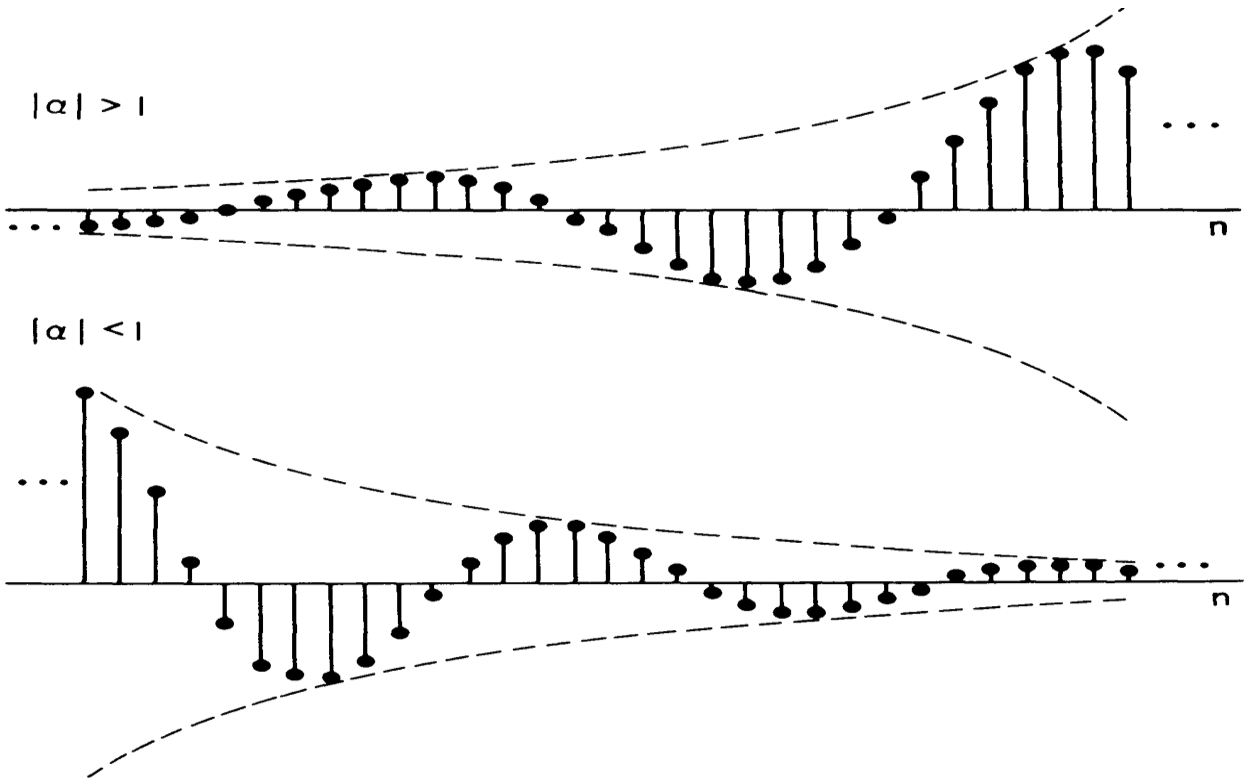
$\alpha > 0$
$|\alpha| < 1$

COMPLEX **EXPONENTIAL:** DISCRETE-TIME

$x[n] = C\alpha^n$ C and α are complex numbers

$$x[n] = |C|\, e^{j\theta} \left(|\alpha|\, e^{j\Omega_o}\right)^n = |C|\, |\alpha|^n \underbrace{e^{j(\Omega_o n + \theta)}}$$

$$x[n] = |C|\, |\alpha|^n \cos(\Omega_o n + \theta) + j\, |C|\, |\alpha|^n \sin(\Omega_o n + \theta)$$



Periodicity Properties of DT Complex Exponential Signals

Although there are many similarities between CT and DT signals, there are also important differences. One of these is the different properties of complex exponential signals $x(t) = e^{j\omega_0 t}$ and $x[n] = e^{j\Omega_0 n}$.

$x(t) = e^{j\omega_0 t}$ previously and we can identify two important properties of it :

- it is periodic for any value of ω_0 and its fundamental period is $T_0 = \dfrac{2\pi}{\omega_0}$
- the larger the magnitude of ω_0, the higher the rate of oscillation (i.e. frequency) in the signals.

Both of the above properties are different for $x[n] = e^{j\Omega_0 n}$:

- $x[n] = e^{j\omega_0 n}$ is periodic only if Ω_0 can be written in the form $\Omega_0 = 2\pi\frac{m}{N}$ for some integers $N > 0$, and m.

- $x[n] = e^{j\Omega_0 n}$ does not have a continually increasing rate of oscillation as we increase the magnitude of Ω_0. In particular, $x_1[n] = e^{j\Omega_0 n}$ is equal to $x_2[n] = e^{j(\Omega_0 + k2\pi)n}$, $k \in \mathbb{Z}$.

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