- Fourier Series for function $f(x)$ in interval $\alpha < x < \alpha + 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

$$c = \frac{UL - LL}{2} = \frac{\alpha + 2\pi - \alpha}{2} = \pi$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2\pi} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2\pi} f(x)\sin\frac{n\pi x}{c}\,dx$$

- Some Definite Integrals

1. $$\int_{\alpha}^{\alpha+2\pi} \cos nx = \left[\frac{\sin nx}{n}\right]_{\alpha}^{\alpha+2\pi}$$

2. $$\int_{\alpha}^{\alpha+2\pi} \sin nx = -\left[\frac{\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi}$$

3. $$\int_{\alpha}^{\alpha+2\pi} (\cos mx)(\cos nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{\alpha}^{\alpha+2\pi}$$

4. $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_{\alpha}^{\alpha+2\pi}$$

5. * $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi}$$

6. $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = \left[\frac{\sin^2 nx}{2n}\right]_{\alpha}^{\alpha+2\pi}$$

7. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi}$$

8. $$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{\alpha}^{\alpha+2\pi}$$

* $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

- $\sin n\pi = 0$ $\qquad \sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$

  $\cos n\pi = (-1)^n$ $\qquad \cos\left(n+\frac{1}{2}\right)\pi = 0$

- $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

- Functions having points of discontinuity

$$f(x) = \begin{cases} \phi(x), & \alpha < x < c \\ \psi(x), & c < x < \alpha + 2\pi \end{cases}$$; c is pt of discontinuity

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\,dx + \int_c^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\cos nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\sin nx\,dx + \int_c^{\alpha+2c} \psi(x)\sin nx\,dx\right]$$

- Even & Odd Functions

* $f(x) \rightarrow$ even $\qquad f(-x) = f(x)$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx \qquad b_n = 0$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

* $f(x) \rightarrow$ odd $\qquad f(-x) = -f(x)$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx \qquad a_0 = 0,\ a_n = 0$$

- Half Range Series

* Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \ ; \ b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$ Same as odd fnctn

* Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$ (Same as even fnctn)

* $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$; $\cos x = \dfrac{e^{ix} + e^{ix}}{2i}$

$e^{ix} = \cos(x) + i\sin(x)$

Integration

- Integration as anti-derivative

if $\frac{d}{dx} f(x) = g(x)$

Then $\int g(x)\,dx = f(x)$

- $\int (u_1 + u_2 + \dots + u_n)\,dx = \int u_1\,dx + \int u_2\,dx + \dots + \int u_n\,dx$;

$u_1, u_2, \dots, u_n$ are all functions of x or constants.

- $\int A u\,dx = A \int u\,dx$; A is any constant

- Fundamental Integrals

- $\int x^n\,dx = \frac{x^{n+1}}{n+1}$
- $\int \frac{1}{x}\,dx = \log_e|x|$
- $\int e^{mx}\,dx = \frac{e^{mx}}{m}$
- $\int e^x\,dx = e^x$
- $\int a^x\,dx = \frac{a^x}{\log_e a}$

$$\int \sin x \, dx = -\cos x$$

- $$\int \sin mx \, dx = -\frac{\cos mx}{m}$$

$$\int \cos x \, dx = \sin x$$

- $$\int \cos mx \, dx = \frac{\sin mx}{m}$$

- $$\int \sec^2 x \, dx = \tan x$$

- $$\int \sec^2 mx \, dx = \frac{\tan mx}{m}$$

- $$\int \operatorname{cosec}^2 x \, dx = -\cot x$$

- $$\int \operatorname{cosec}^2 mx \, dx = -\frac{\cot mx}{m}$$

- $$\int \sec x \tan x \, dx = \sec x$$

- $$\int \sec mx \tan mx \, dx = \frac{\sec mx}{m}$$

- $$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$$

- $$\int \operatorname{cosec} mx \cot mx \, dx = -\frac{\operatorname{cosec} mx}{m}$$

- $$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x \text{ (or) } -\cos^{-1} x \quad ; \quad |x| < 1$$

- $$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x \text{ (or) } -\operatorname{cosec}^{-1} x \quad ; \quad |x| > 1$$

- $$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x \text{ or } -\cot^{-1} x$$

★ $$\int_a^b \phi(x)\,dx = \Big[f(x) \Big]_a^b = f(b) - f(a)$$

⊙ ★ Transformations of simple trigonometric functions

- $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
- $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
- $\tan^2 x = \sec^2 x - 1$
- $\cot^2 x = \operatorname{cosec}^2 x - 1$
- $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$
- $\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$
- $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
- $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$
- $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$
- $\sin mx \cos mx = \frac{1}{2}\sin 2mx$
- $\sin^2 mx = \frac{1}{2}(1-\cos 2mx)$
- $\cos^2 mx = \frac{1}{2}(1+\cos 2mx)$

- $$\int \{f(x)\}^n f'(x)\,dx = \frac{1}{n+1}\{f(x)\}^{n+1}$$

★ - $$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$$

$\int \log_e x\,dx = x(\log_e x - 1)$

Some standard Integrals

$\int \tan x = \log_e |\sec x|$

$\int \cot x = \log_e |\sin x|$

$\int \sec x = \log_e |\sec x + \tan x| = \ln \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c$

$\int \operatorname{cosec} x = \log_e |\operatorname{cosec} x - \cot x| = \ln \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c$

Elementary Rules of Integration

$\int (ax+b)^n \, dx$ $\left[\int (ax+b)\,dx, \int \frac{dx}{ax+b}, \int \sqrt{ax+b}\,dx, \int \frac{dx}{\sqrt{ax+b}} \right]$

$= \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1}$; if $n \neq -1$

$= \frac{1}{a} \log|ax+b|$; if $n = -1$

Important Standard Integrals

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right|$, $|x| > |a|$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right|$, $|x| < |a|$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \log \left| x + \sqrt{x^2+a^2} \right|$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \log \left| x + \sqrt{x^2-a^2} \right|$, $|x| > |a|$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$, $|x| < |a|$

7. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

8. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

9. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Deriva

- $\frac{d}{dx}(x^n) = nx^{n-1}$

- $\frac{d}{dx}(e^x) = e^x$

- $\frac{d}{dx}(a^x) = a^x \log_e a$

- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

- $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

- $\frac{d}{dx}(\sin x) = \cos x$

- $\frac{d}{dx}(\cos x) = -\sin x$

- $\frac{d}{dx}(\tan x) = \sec^2 x$

- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

- $\frac{d}{dx}(\sec x) = \sec x \tan x$

- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

- $\frac{d}{dx}(c) = 0$

- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}(f(x))$

- $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$

- $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

- $\frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \cancel{\frac{u\frac{d}{dx}(v) \mp v\frac{d}{dx}(u)}{v^2}}$ $\frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

Associate Angles

(1) $\sin(-\theta) = -\sin\theta$

(2) $\cos(-\theta) = \cos\theta$

(3) $\tan(-\theta) = -\tan\theta$

(4) $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

(5) $\sec(-\theta) = \sec\theta$

(6) $\cot(-\theta) = -\cot\theta$

(7) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

(8) $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

(9) $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

(10) $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta$

(11) $\sec\left(\frac{\pi}{2}+\theta\right) = -\operatorname{cosec}\theta$

(12) $\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$

(13) $\sin(\pi-\theta) = \sin\theta$

(14) $\cos(\pi-\theta) = -\cos\theta$

(15) $\tan(\pi-\theta) = -\tan\theta$

(16) $\operatorname{cosec}(\pi-\theta) = \operatorname{cosec}\theta$

(17) $\sec(\pi-\theta) = -\cos\theta$

(18) $\cot(\pi-\theta) = -\cot\theta$

(19) $\sin(\pi+\theta) = -\sin\theta$

(20) $\cos(\pi+\theta) = -\cos\theta$

(21) $\tan(\pi+\theta) = \tan\theta$

(22) $\operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta$

(23) $\sec(\pi+\theta) = -\sec\theta$

(24) $\cot(\pi+\theta) = \cot\theta$

Conclusion :

If angle $(n\times 90 \pm \theta)$ lies in I, II, III or IV quadrant then the sign of the particular trigonometric ratio is determined by "All, sin, tan, cos" rule.

If n is even the trigonometric ratio remains same

If n is odd, sin ⟷ cos, cosec ⟷ sec, tan ⟷ cot.

Compound Angles

(1) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(2) $\cancel{co}\sin(A-B) = \sin A\cos B - \cos A\sin B$

(3) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(4) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(5) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(6) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(7) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(8) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(9) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(10) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(11) $\sin(A+B+C) = \cos A \cos B \cos C\,(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

(12) $\cos(A+B+C) = \cos A \cos B \cos C\,(1 - \tan A \tan B - \tan C \tan A - \tan B \tan C)$

(13) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan C \tan A - \tan B \tan C}$

Sum to Product

$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

$\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

Product to Sum

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

~~2sin~~ $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

Multiple Angles

$\sin 2\theta = 2 \sin\theta \cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(4) $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

(5) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(6) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(7) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(8) $\cot 3\theta = \dfrac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$

Submultiple Angles

(1) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(2) $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

(3) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(4) $\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$

(6) $\tan\theta = \dfrac{3\tan\theta/3 - \tan^3\theta/3}{1 - 3\tan^2\theta/3}$

(5) $\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

General Solutions

(1) $\sin x = 0$, $x = n\pi$

(2) $\cos x = 0$, $x = (2n+1)\frac{\pi}{2}$

(3) $\sin x = \sin y$, $x = n\pi + (-1)^n y$

(4) $\cos x = \cos y$, $x = 2n\pi \pm y$

(5) $\tan x = \tan y$, $x = n\pi + y$

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

Inverse Trigonometry

- Principal value branches

	D	R
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\operatorname{cosec}^{-1}$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	R	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cot^{-1}$	R	$[0, \pi]$

- Properties

1. (i) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x \quad , \quad x \geq 1 \text{ or } x \leq -1$

   (ii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x \quad , \quad x \geq 1 \text{ or } x \leq -1$

   (iii) $\tan^{-1}\frac{1}{x} = \cot^{-1}x \quad , \quad x > 0$

2. (i) $\sin^{-1}(-x) = -\sin^{-1}x \quad , \quad x \in [-1, 1]$

   (ii) $\tan^{-1}(-x) = -\tan^{-1}x \quad , \quad x \in R$

   (iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \quad , \quad |x| \geq 1$

3. (i) $\cos^{-1}(-x) = \pi - \cos^{-1}x \quad , \quad x \in [-1, 1]$

   (ii) $\sec^{-1}(-x) = \pi - \sec^{-1}x \quad , \quad |x| \geq 1$

   (iii) $\cot^{-1}(-x) = \pi - \cot^{-1}x \quad , \quad x \in R$

4. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \quad , \quad x \in [-1, 1]$

   (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \quad , \quad x \in R$

   (iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2} \quad , \quad |x| \geq 1$

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$, $xy < 1$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$, $xy > -1$

(i) $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}$, $|x| \leq 1$

(ii) $2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$, $x \geq 0$

(iii) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$, $-1 < x < 1$

- $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$, $|x|, |y| \leq 1$ & $x+y \geq 0$
- $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$