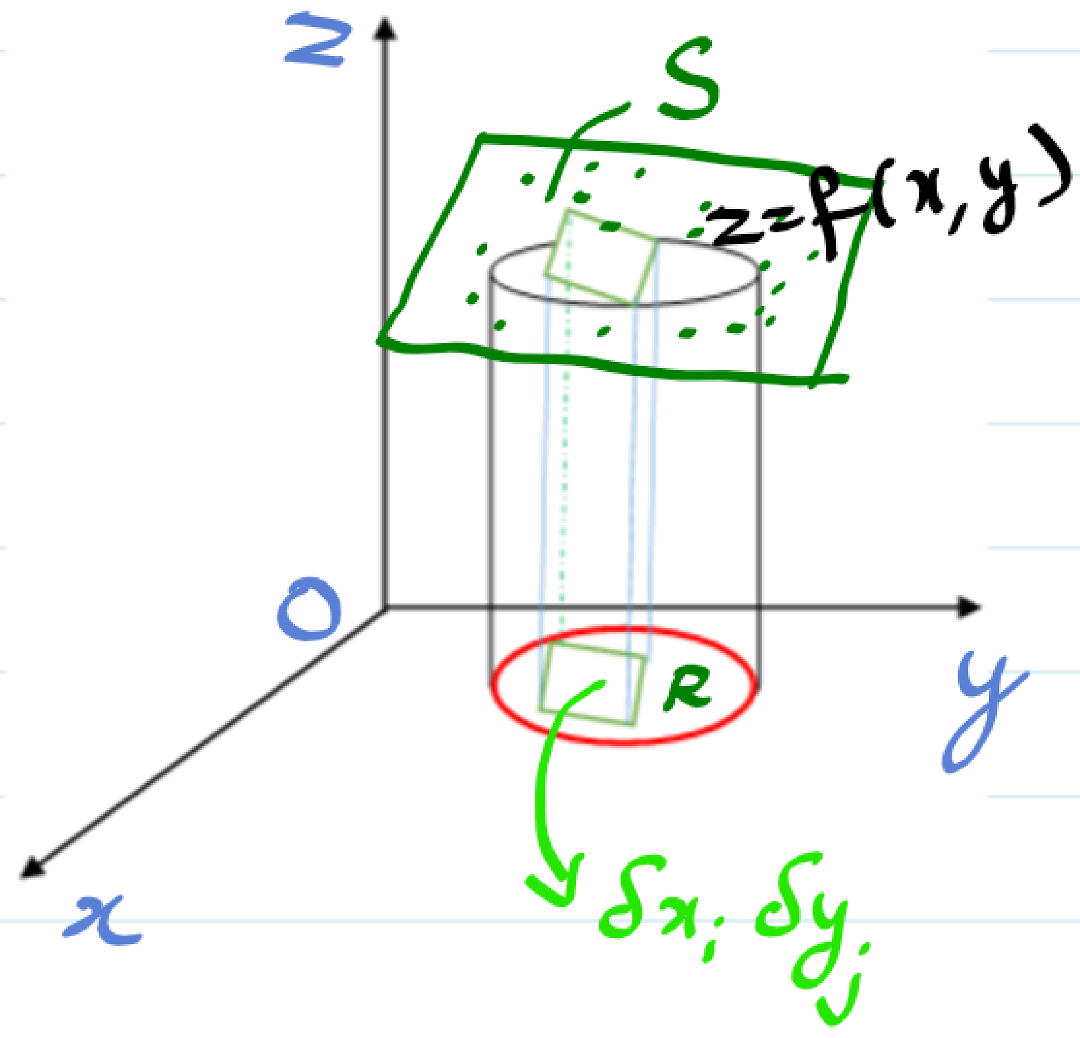
Volume of a region in three dimensions by double integration –

Consider a region enclosed by a cylinder over a plane region R in XOY plane, bounded by a surface S at the top.

If $\delta x_i \, \delta y_j$ is elementary area of R and z_{ij} is the value of z, then the volume of the elementary region above $\delta x_i \delta y_j$ upto the surface S is $z_{ij} \cdot \delta x_i \, \delta y_j$

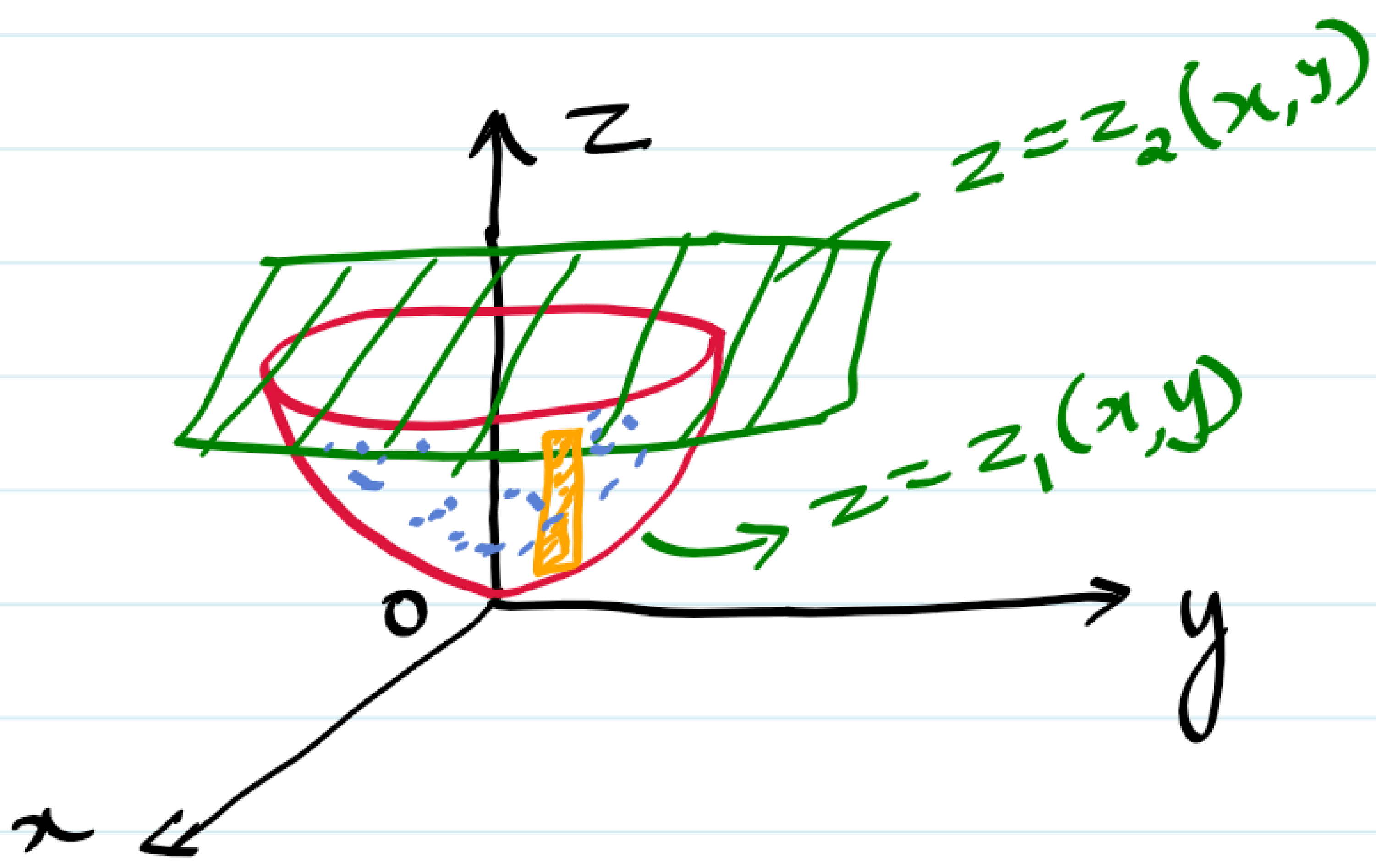


$$\text{Total volume} = \iint_R z \, dx \, dy .$$

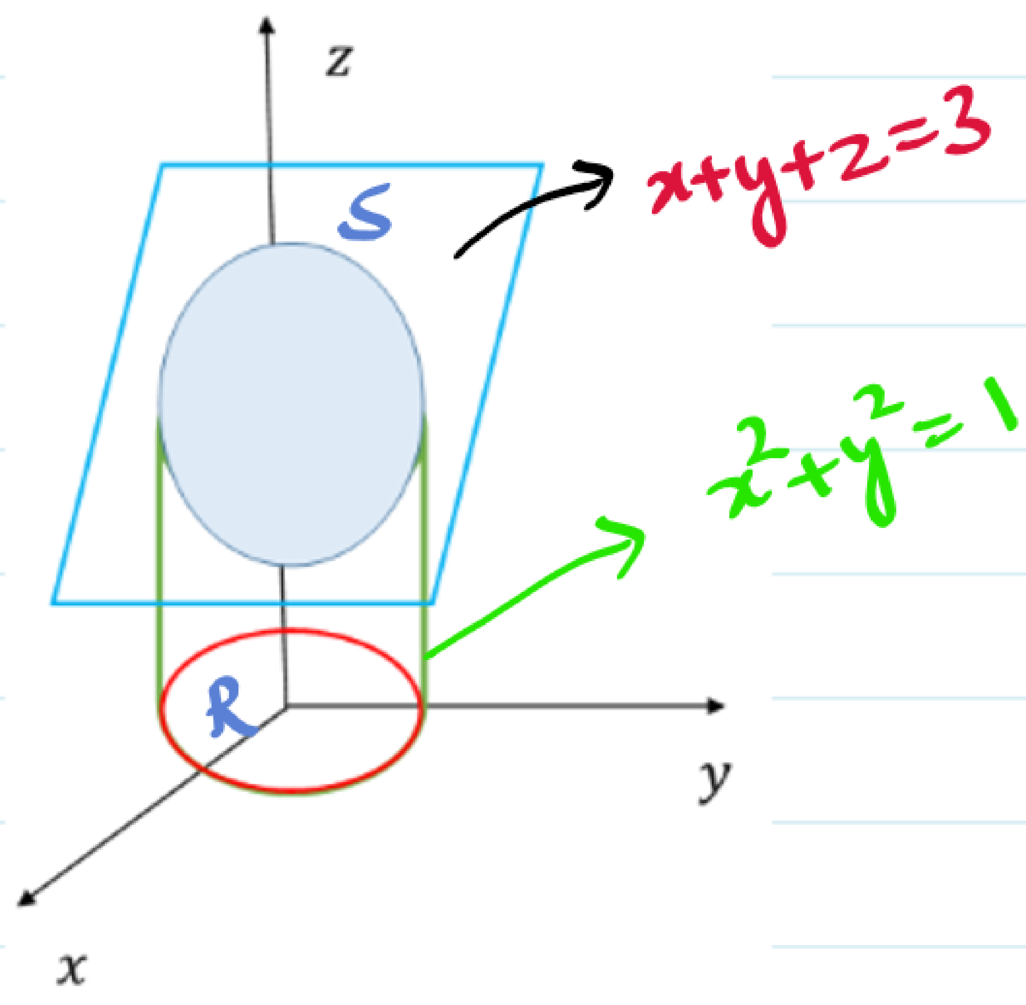
R: Projection of S on xy-plane.

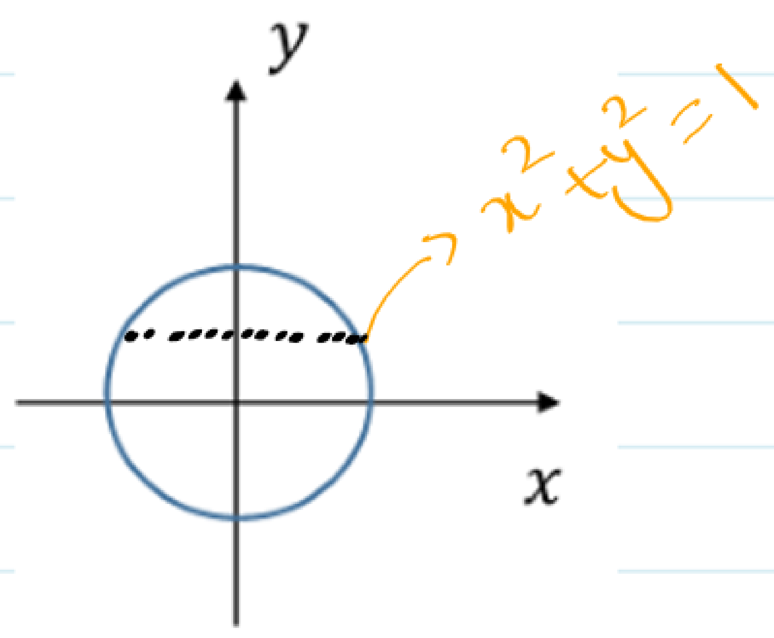
Generally, the volume of the region bounded by two surfaces $z = z_1(x,y)$ and $z = z_2(x,y)$ at the bottom and top, the sides being cylindrical with generators parallel to z-axis, then

$$V = \iint_R |z_2 - z_1| \, dx \, dy ,$$ where R is the projection on the xy plane of both surfaces.



① Using double integrals find the volume of the solid bounded by cylinder $x^2+y^2=1$ and the plane $x+y+z=3$ and $z=0$.





$R: x^2+y^2=1$

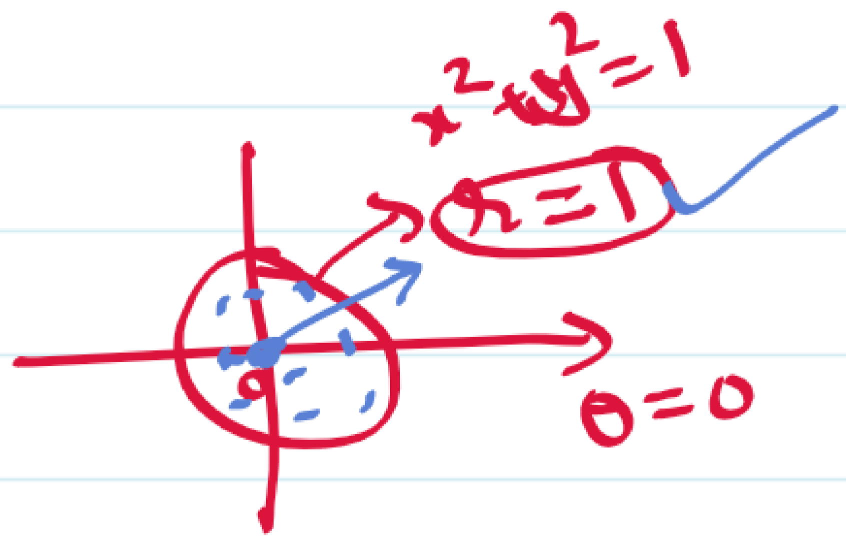
$x+y+z=3$

$z=3-(x+y)$

$$V=\iint_R z\,dx\,dy$$

$$=\iint\left[3-(x+y)\right]dx\,dy$$

$R: x^2+y^2=1$



$$V=\int_{y=-1}^{1}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3-(x+y)\;dx\,dy$$

changing to polar co-ordinates, $x=r\cos\theta$, $y=r\sin\theta$.

$J=r$

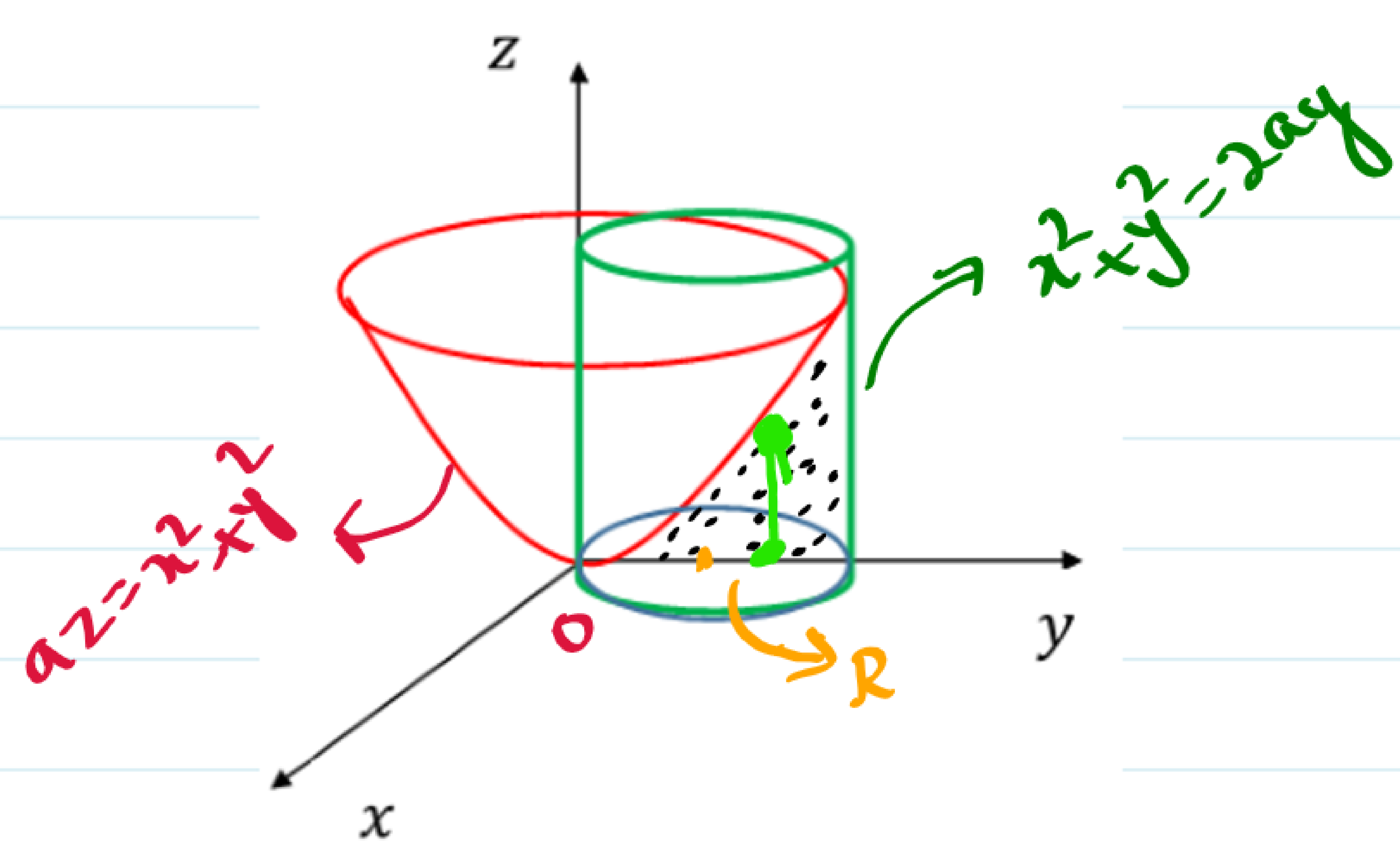
$$V=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left[3-(r\cos\theta+r\sin\theta)\right]r\,dr\,d\theta.$$

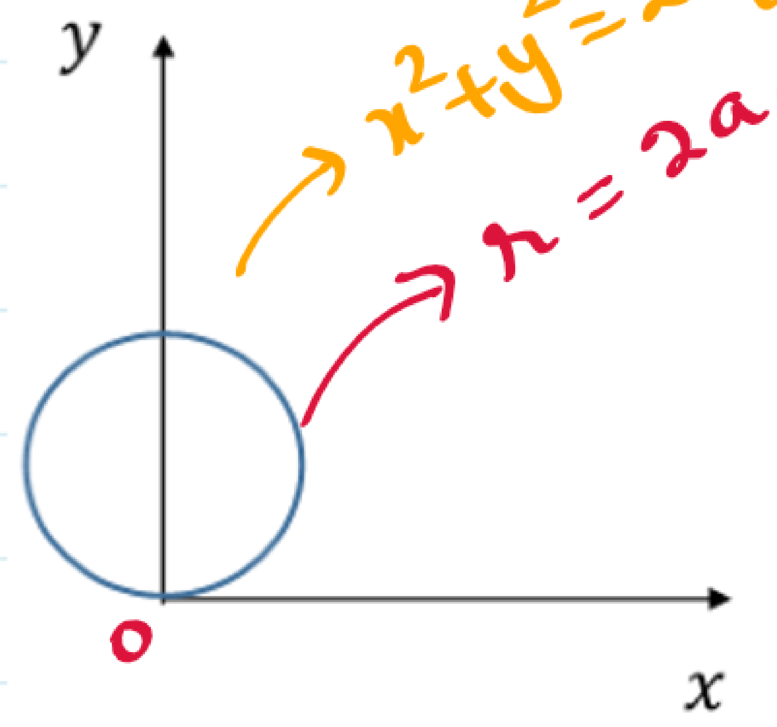
$$=\int_0^{2\pi}\left.\frac{3r^2}{2}-(\cos\theta+\sin\theta)\frac{r^3}{3}\right]_{r=0}^{1}d\theta$$

$$=\int_0^{2\pi}\frac{3}{2}-\frac{1}{3}(\cos\theta+\sin\theta)\,d\theta=\left.\frac{3}{2}\theta-\frac{1}{3}(\sin\theta-\cos\theta)\right|_0^{2\pi}$$

$$=\frac{6\pi}{2}=3\pi$$

2) Find the volume bounded by the paraboloid $az = x^2+y^2$ and the cylinder $x^2+y^2 = 2ay$ and $z=0$.





$x^2+y^2=2ay$

$r = 2a\sin\theta$ $(x^2+y^2=2ay)$

$x^2+(y-a)^2=a^2$

$y=a\pm\sqrt{a^2-x^2}$

$$\int_{x=-a}^{a}\int_{y=a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}}$$

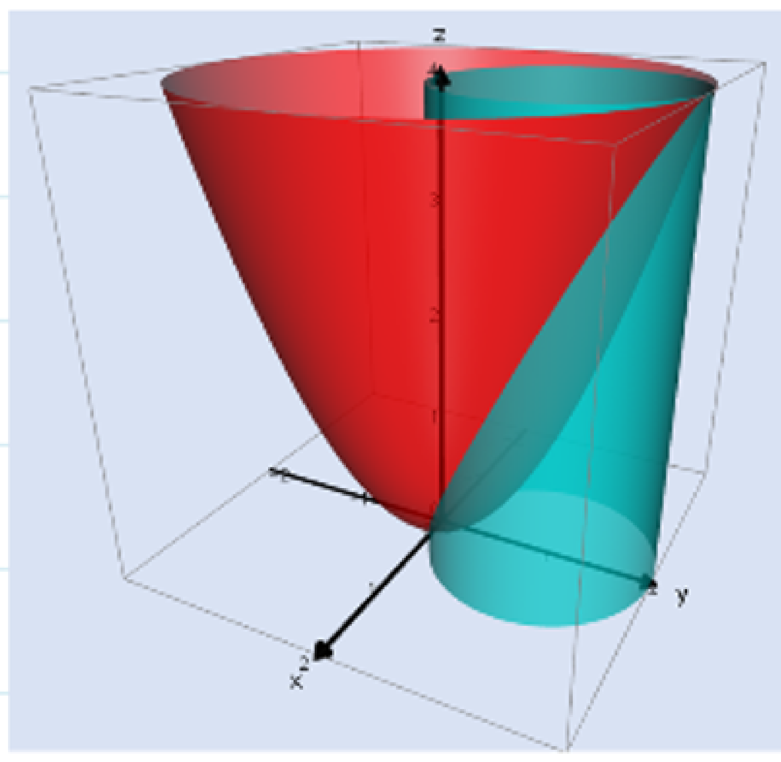
R:

$$V = \iint_R z\,dx\,dy = \iint \frac{x^2+y^2}{a}\,dx\,dy$$

$R: x^2+y^2=2ay$

Using polar co-ordinates, $x = r\cos\theta$, $y = r\sin\theta$, $J = r$.

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{2a\sin\theta} \frac{r^2}{a}\, r\,dr\,d\theta = \int_0^{\pi}\frac{1}{a}\left(\frac{r^4}{4}\right)_0^{2a\sin\theta} d\theta$$



$$= \frac{1}{4a}\int_0^{\pi} 16a^4\sin^4\theta\,d\theta$$

$$= 4a^3\int_0^{\pi}\sin^4\theta\,d\theta.$$

$$= 4a^3\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

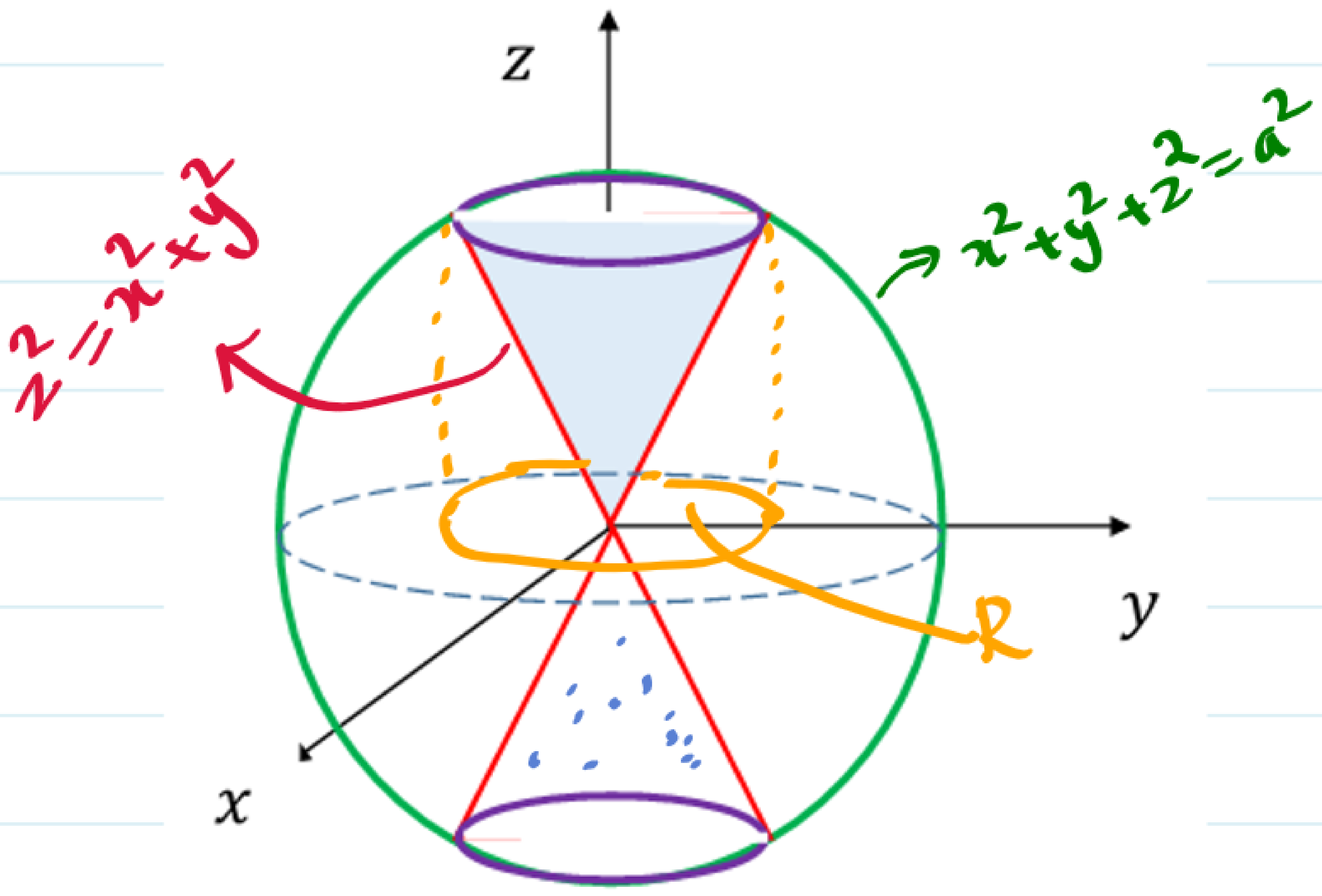
$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x) = -f(x)\end{cases}$$

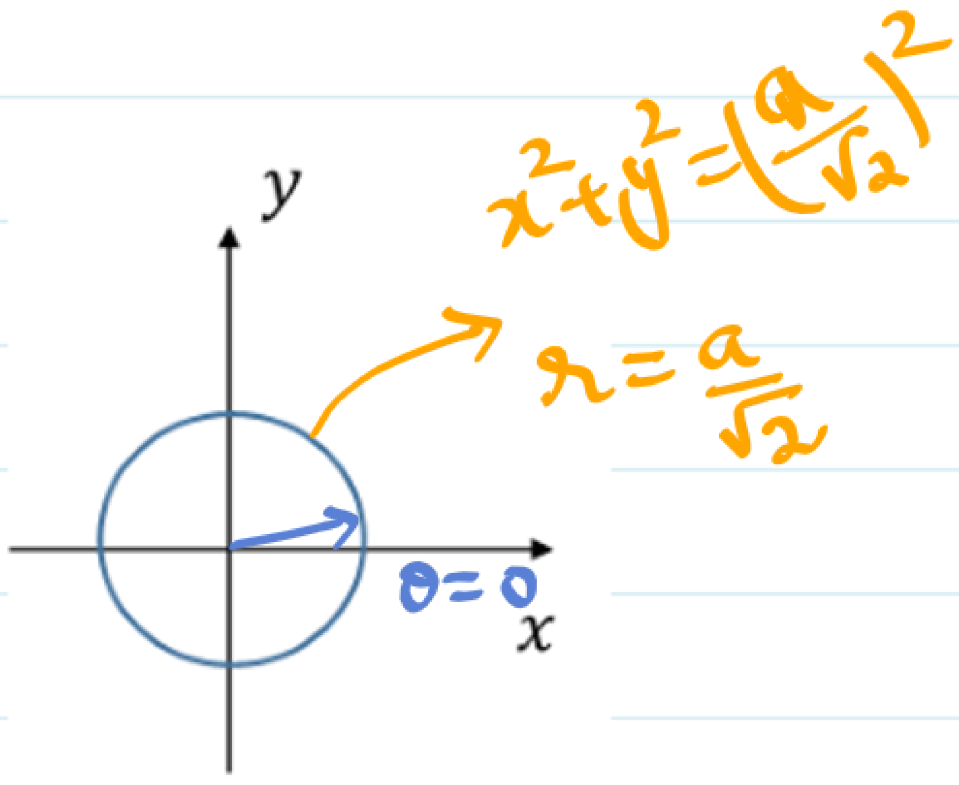
$(\sin(\pi-\theta) = \sin\theta)$

$$= 8a^3\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$V = \frac{3a^3\pi}{2}$$

3) Find the volume inside the cone $x^2+y^2=z^2$ bounded by the sphere $x^2+y^2+z^2=a^2$.





R:

$$x^2+y^2+z^2=a^2 \; ; \; z^2=x^2+y^2$$
$$\downarrow$$
$$x^2+y^2+x^2+y^2=a^2$$
$$2(x^2+y^2)=a^2$$
$$x^2+y^2=\left(\frac{a}{\sqrt{2}}\right)^2$$

$$V = 2\iint_R |z_1-z_2| \, dx\, dy$$

$$= 2\iint \left| \sqrt{a^2-x^2-y^2} - \sqrt{x^2+y^2} \right| dx\, dy$$

R: $x^2+y^2=(a/\sqrt{2})^2$

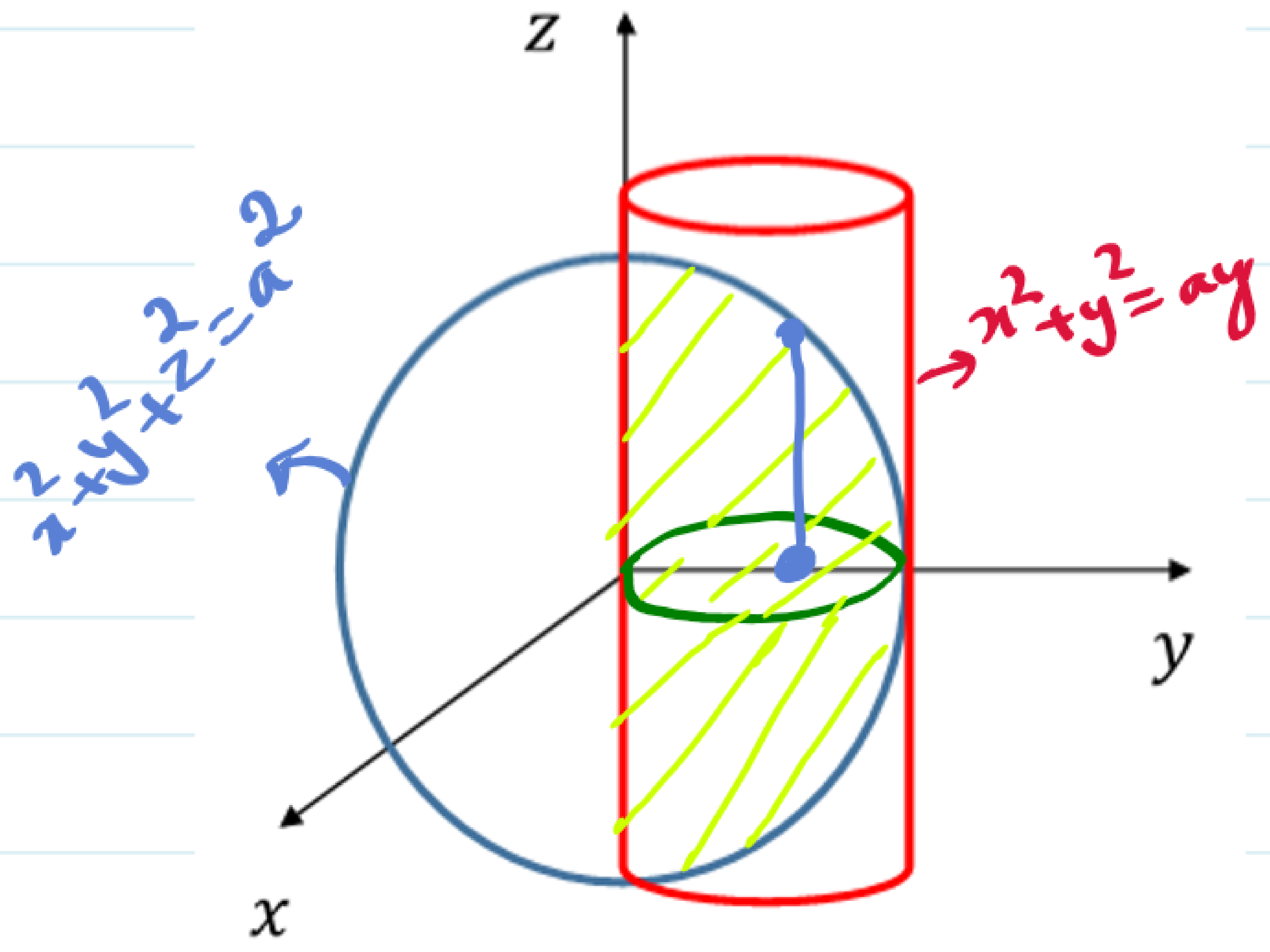
$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{a/\sqrt{2}} \left(\sqrt{a^2-r^2} - r\right) r\, dr\, d\theta$$

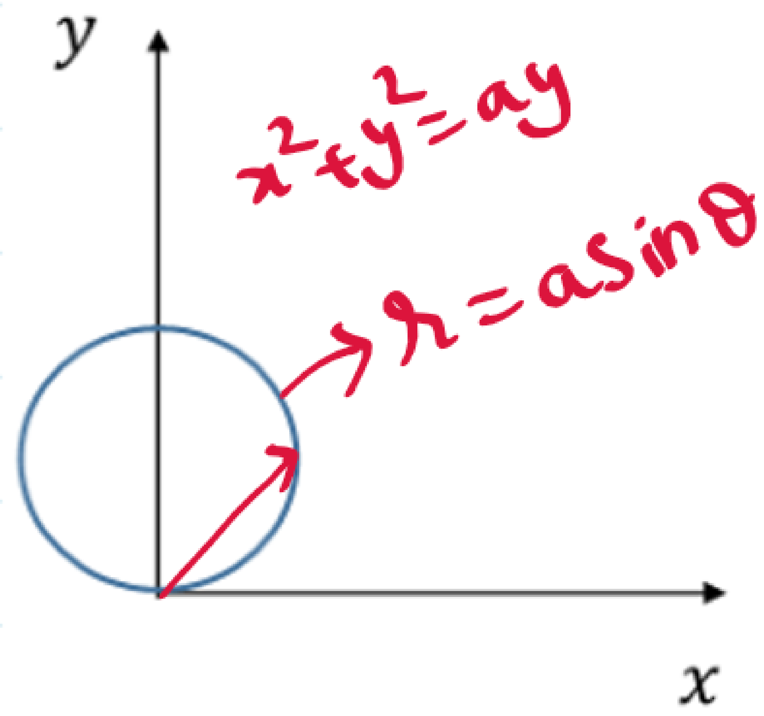
$$= 2\int_0^{2\pi} d\theta \times \int_0^{a/\sqrt{2}} \left(\sqrt{a^2-r^2}\right)r - r^2\, dr$$

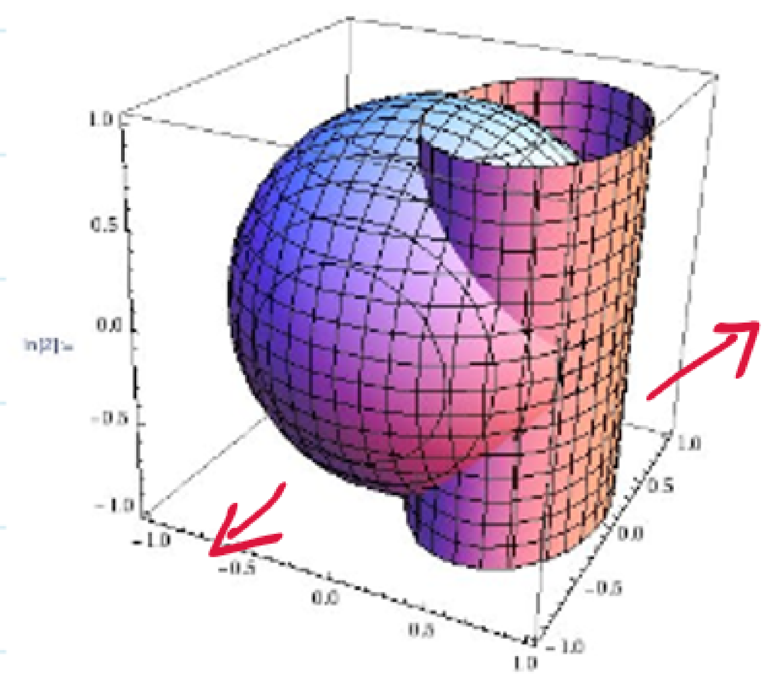
$$= 2\times 2\pi \times \left(-\frac{1}{2}\frac{(a^2-r^2)^{3/2}}{3/2} - \frac{r^3}{3}\right)_0^{a/\sqrt{2}}$$

$$V = \frac{4\pi a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

4) Find the volume bounded by the cylinder $x^2+y^2=ay$ and the sphere $x^2+y^2+z^2=a^2$







$R: x^2+y^2=ay$

$x^2+y^2=ay$

$r^2=ar\sin\theta$

$r=a\sin\theta$

V = 2 volume in upper half of xy-plane.

$$= 2\times \iint_{R:\, x^2+y^2=ay} \sqrt{a^2-(x^2+y^2)}\; dx\,dy$$

$$= 2\times 2 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \sqrt{a^2-r^2}\; \frac{r(-2)}{(-2)}\, dr\, d\theta$$

$$= 4 \int_0^{\pi/2} \left[\frac{(a^2-r^2)^{3/2}}{(-2)\,3/2}\right]_0^{a\sin\theta} d\theta\,.$$

$$= -\frac{4}{3}\int_0^{\pi/2} a^3(1-\sin^2\theta)^{3/2} - a^3 \; d\theta$$

$$= -\frac{4a^3}{3}\int_0^{\pi/2} (\cos^3\theta - 1)\, d\theta$$

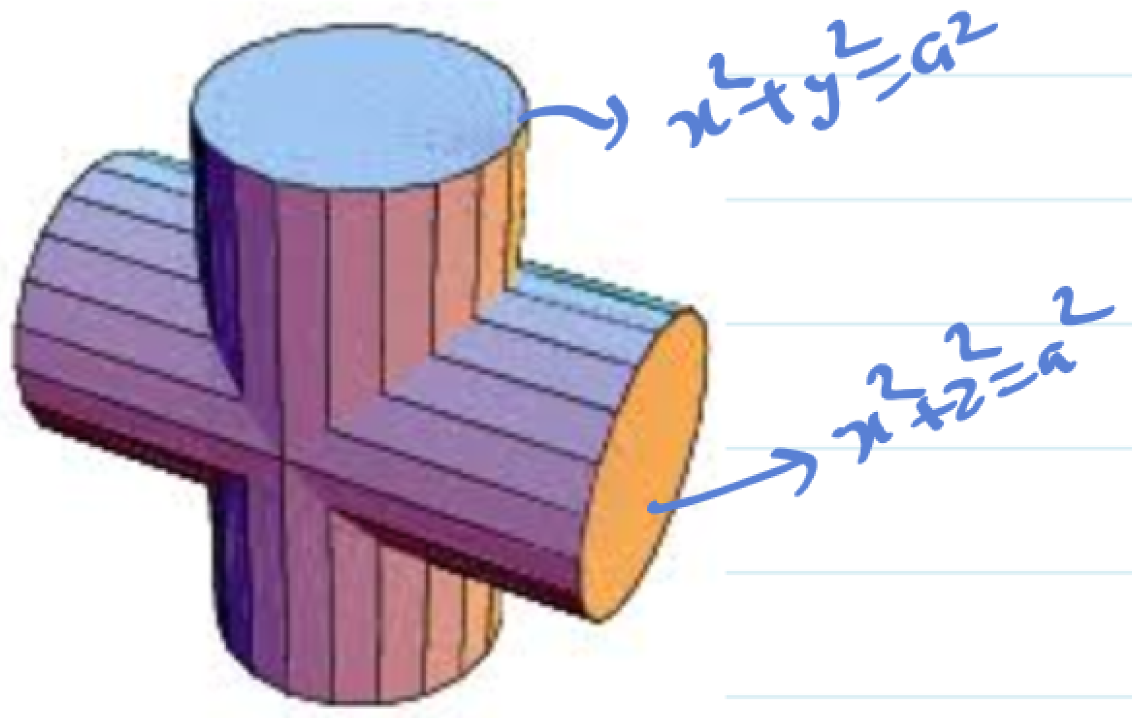
$$= \frac{4a^3}{3}\int_0^{\pi/2} (1-\cos^3\theta)\, d\theta$$

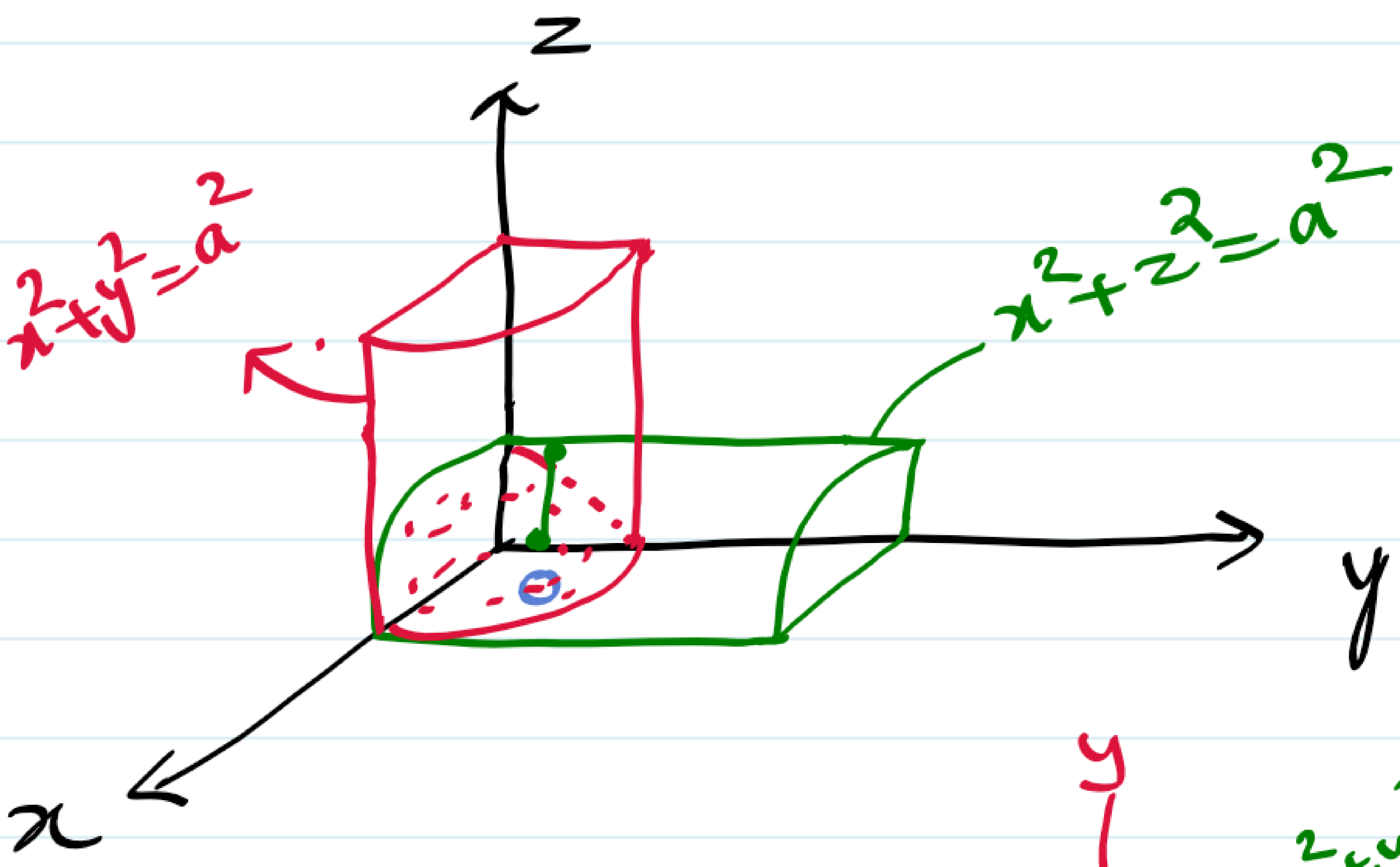
$$= \frac{4a^3}{3}\left\{(\theta)_0^{\pi/2} - \frac{2}{3}\right\}$$

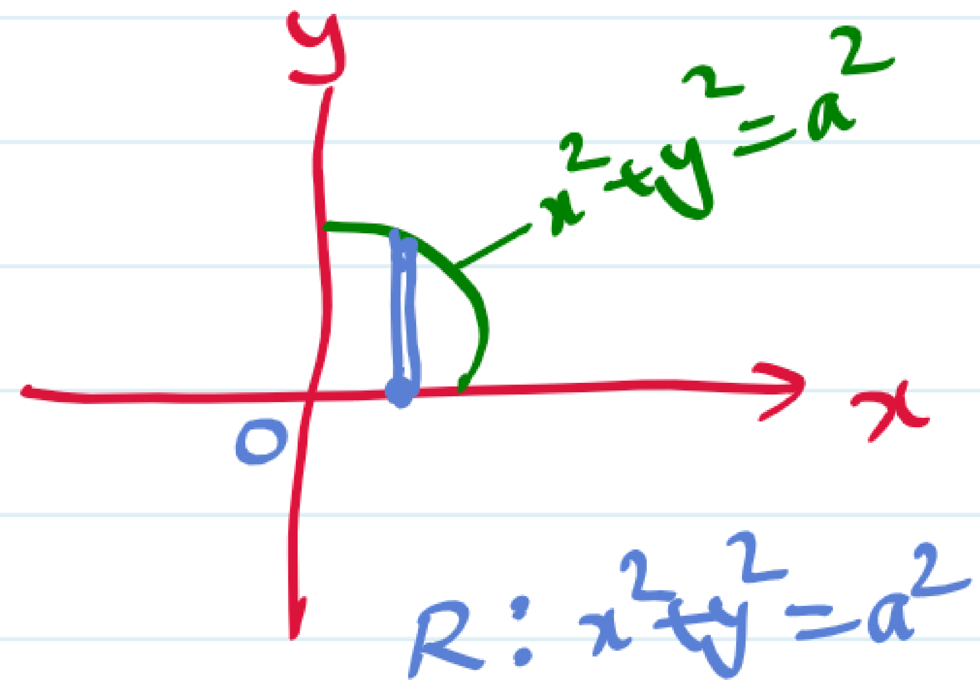
$$= \frac{4a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right) = \frac{4a^3}{18}(3\pi-4)$$

$$= \frac{2a^3}{9}(3\pi-4)$$

5) Find the volume bounded by $x^2+y^2=a^2$ and $x^2+z^2=a^2$.







$V = 8$ volume in 1^{st} octant

$$= 8 \iint z \, dx\,dy$$

$R: x^2+y^2=a^2$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} \; dx\,dy$$

$$8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}}$$

$$\int_{\theta=0}^{\pi/2} \int_{0}^{a} \sqrt{a^2-r^2\cos^2\theta} \; r\,dr\,d\theta$$

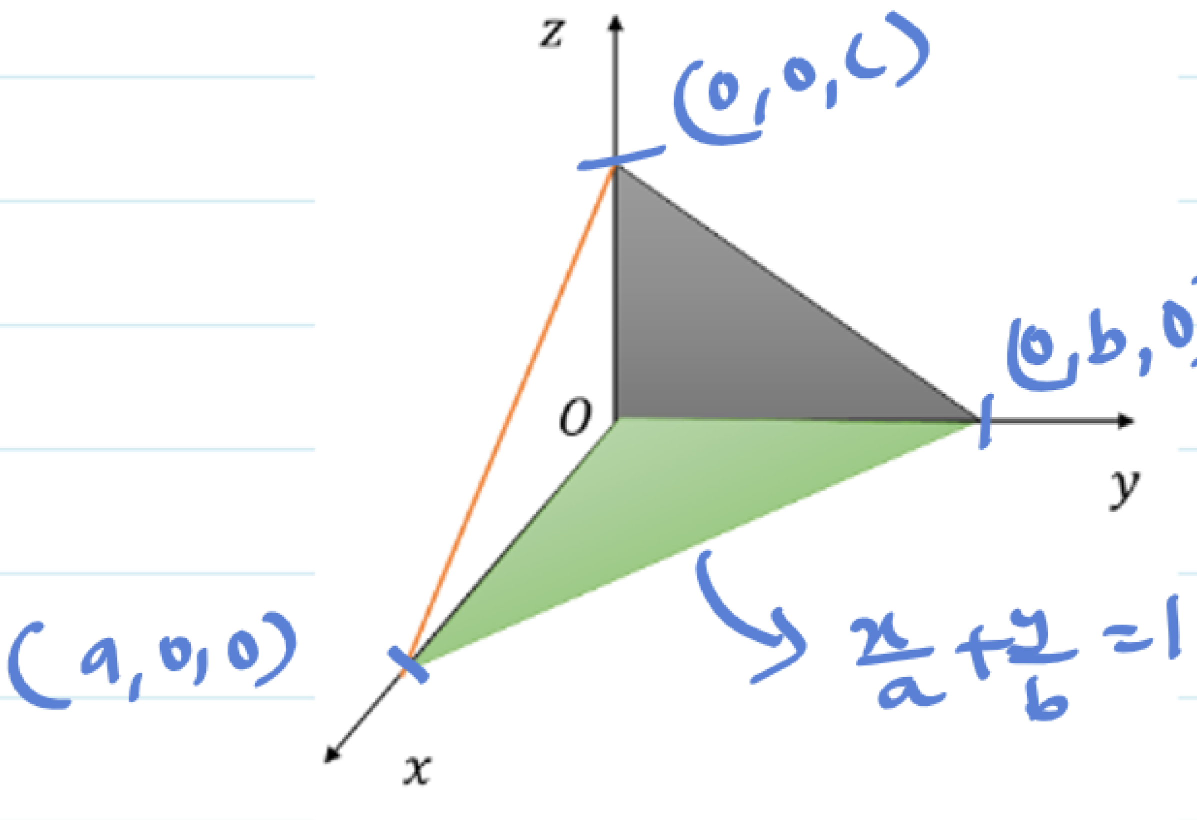
?? ...

$$= 8 \int_0^a \sqrt{a^2-x^2} \Big(y \Big)_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_0^a (a^2-x^2)\,dx = 8 \left(a^2x - \frac{x^3}{3} \right)_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right) = 8\left(\frac{2a^3}{3}\right) = \frac{16a^3}{3}$$

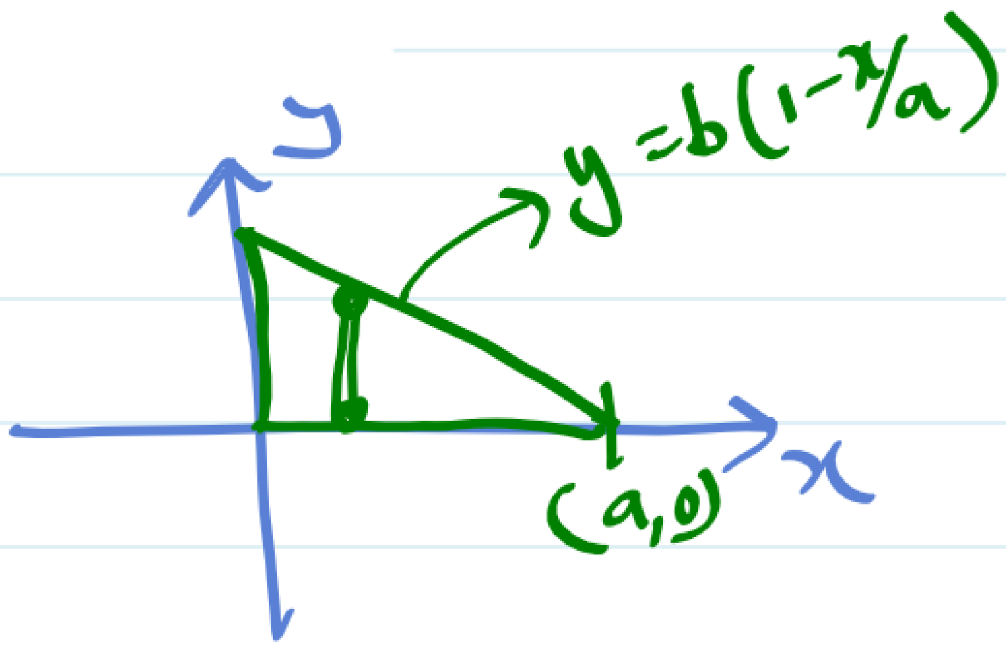
6) Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes.



$$\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y=b\left(1-\frac{x}{a}\right)$$

$$V=\iint_R z \, dx \, dy$$

$$=\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dx \, dy$$



R:

$$=c\int_{x=0}^{a}\left[\left(1-\frac{x}{a}\right)y-\frac{1}{b}\left(\frac{y^2}{2}\right)\right]_{y=0}^{b(1-x/a)} dx$$

$$=c\int_0^a b\left(1-\frac{x}{a}\right)^2-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2 dx = c\times\frac{b}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 dx$$

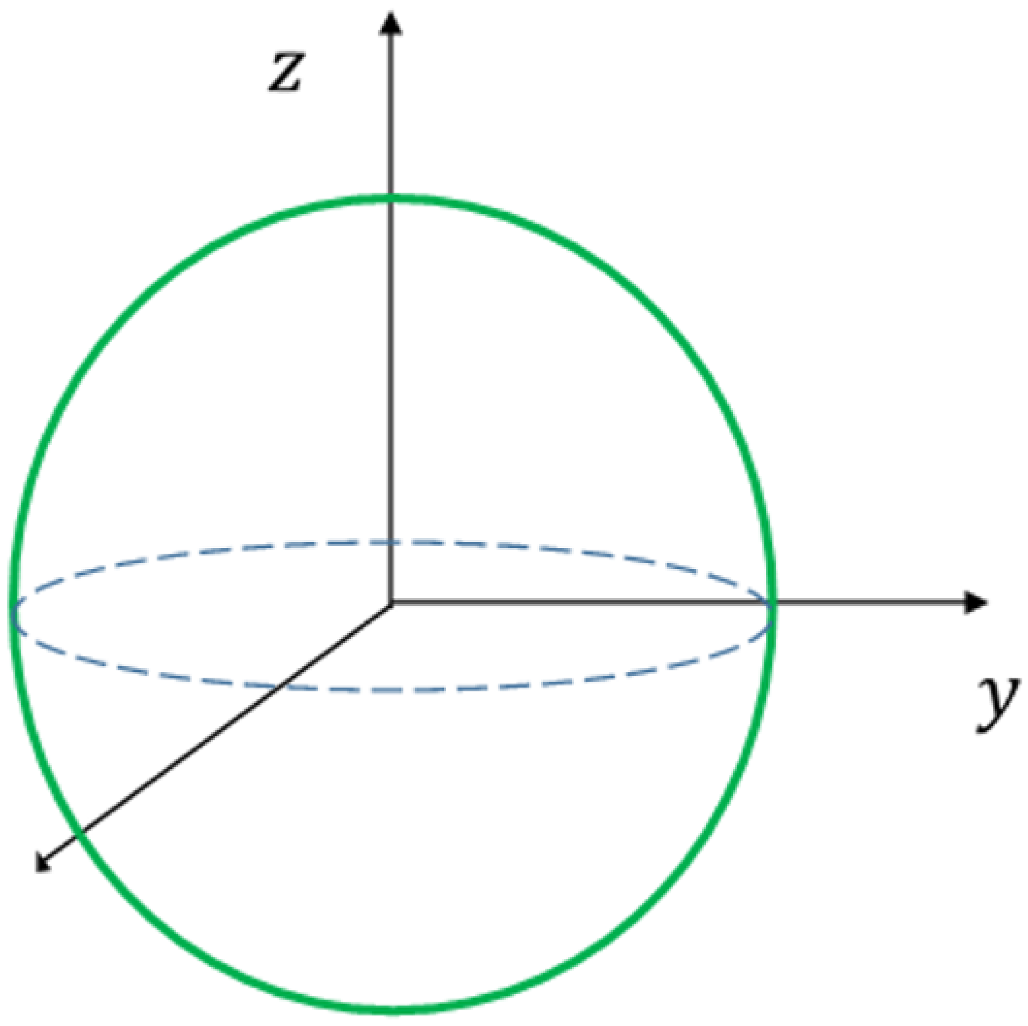
$$=\frac{bc}{2}\left[\frac{\left(1-x/a\right)^3}{3\left(-1/a\right)}\right]_0^a$$

$$=-\frac{abc}{6}(0-1)$$

$$\underline{\underline{V=\frac{abc}{6}}}$$

Practice questions -

① Find by double integration the volume of a sphere $x^2+y^2+z^2=a^2$.



$V = 8 \times$ volume in 1^{st} octant.

Ans: $\frac{4}{3}\pi a^3$

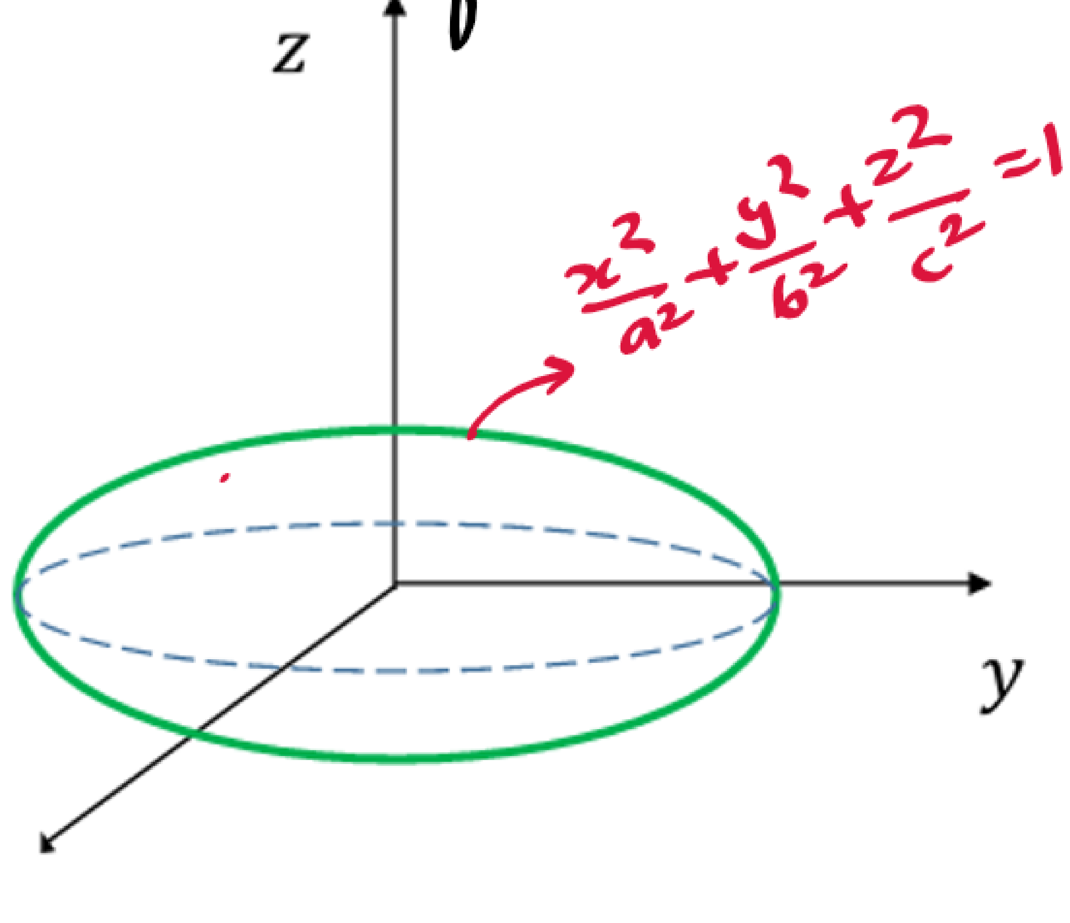
2) Find the volume bounded by $x^2+y^2=az$, $x^2+y^2=a^2$ and $z=0$

Ans: $\frac{\pi a^3}{2}$

3) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Ans: 16π

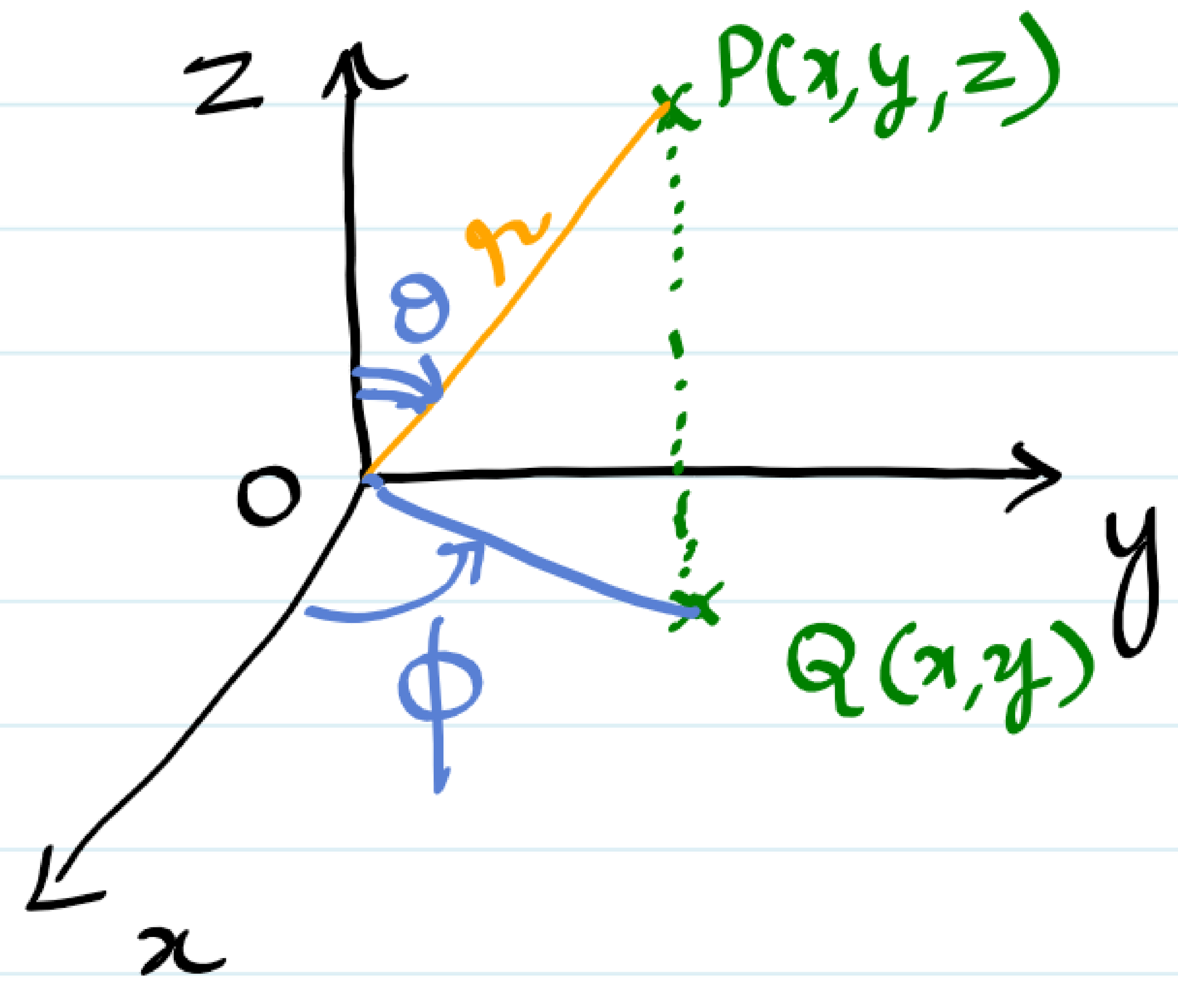
4) Find by double integration the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.



Ans: $\frac{4\pi}{3}abc$

Spherical polar co-ordinates –

Let $P(x, y, z)$ be any point whose projection on the xy-plane is $Q(x, y)$. Then the spherical polar co-ordinates of P are (r, θ, ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.



The spherical polar co-ordinates are,

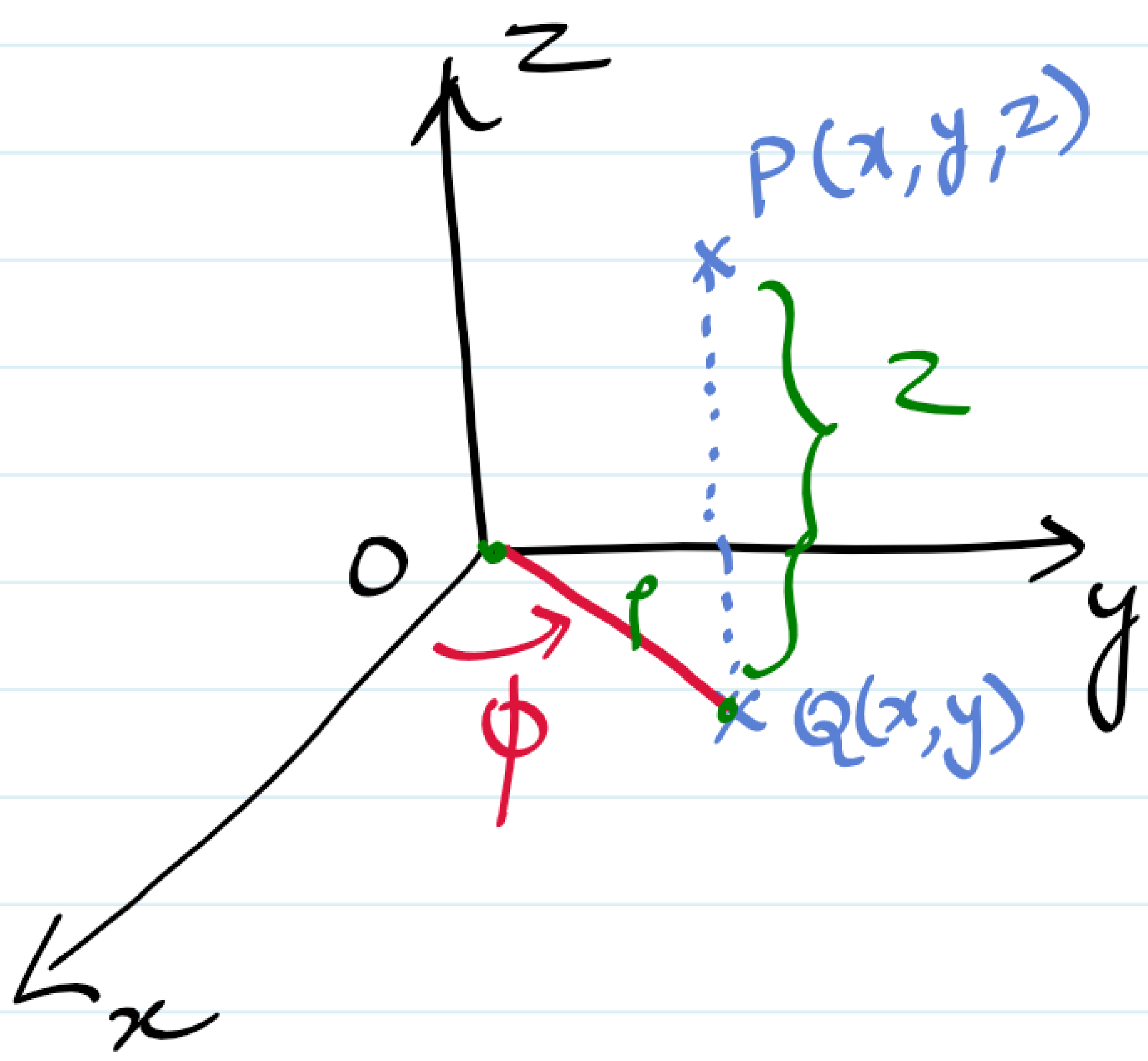
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

$$J = r^2\sin\theta$$

Cylindrical co-ordinates –

Any point $P(x, y, z)$ whose projection on the xy-plane is $Q(x, y)$ has the cylindrical co-ordinates (ρ, ϕ, z), where

$\rho = OQ$, $\phi = \angle XOQ$ and $z = QP$.



The cylindrical co-ordinates are,

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$J = \rho$$