5. Find an analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y) \longrightarrow (2)$$

Put $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in (2)

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = e^x(-\sin y - \cos y)$$

$$\frac{\partial u}{\partial x} + \partial v/\partial x = e^x(\sin y + \cos y) \longrightarrow (3)$$

solve eqns: (1) and (3)

$(1)+(3) \Rightarrow \quad \frac{\partial u}{\partial x} = e^x \cos y$

$(1)-(3) \Rightarrow \quad \partial v/\partial x = e^x \sin y$

$$\therefore f'(z) = \partial u/\partial x + i\,\partial v/\partial x$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^z \quad (\text{by taking } x = z, y = 0)$$

$$\therefore f(z) = e^z + c$$

4. If $u = e^x \cos y$ find v and hence find $f(z)$

$$\frac{\partial u}{\partial x} = e^x \cos y \;, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = e^x \sin y$$

Integrate w.r.~~to~~ x

$$v(x,y) = e^x \sin y + g(y)$$

Differentiate w.r.~~to~~ y

$$\partial v/\partial y = e^x \cos y + g'(y) = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\Rightarrow g'(y) = 0 \quad \therefore g(y) = C$$

$$\therefore v(x,y) = e^x \sin y + C$$

$$f(z) = u + iv = e^x \cos y + i(e^x \sin y + C)$$

$$= \underline{\underline{e^z + iC}} = e^z + C_1$$

where $\underline{\underline{C_1 = iC}}$

3. If $u = xy$ find $v(x,y)$ and hence find $f(z)$

$$\frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = y$$

Integrate w.r.to y,

$$v(x,y) = y^2/2 + g(x)$$

differentiate w.r.to x

$$\frac{\partial v}{\partial x} = g'(x) = -\frac{\partial u}{\partial y} = -x$$

$$\therefore g(x) = \frac{-x^2}{2} + C$$

$$\therefore v(x,y) = y^2/2 - x^2/2 + C$$

$$\therefore f(z) = u + iv = xy + i\left(y^2/2 - x^2/2 + c\right)$$

Put $x = z$ $y = 0$

$$\underline{\underline{f(z) = i\left(-z^2/2 + c\right)}}$$

2. Find the analytic function $f(z) = u + iv$

Where $u(x,y) = e^{x}(x\cos y - y\sin y) + 2\sin x \sinh y + x^3 - 3xy^2 + y$.

$$\frac{\partial u}{\partial x} = e^{x}(\cos y - y\sin y + x\cos y) + 2\cos x \sinh y + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = e^{x}(-x\sin y - \sin y - y\cos y) + 2\sin x \cosh y - 6xy + 1$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left(\text{using C-R eqn } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$$

$$= e^{z}(1+z) + 3z^2 - i(2\sin z + 1)$$

$$f(z) = \int \left[e^{z}(1+z) + 3z^2 - i(2\sin z + 1)\right] dz$$

$$= (1+z)e^{z} - e^{z} + z^3 - i(-2\cos z + z) + C$$

$$= ze^{z} + z^3 + i(2\cos z - z) + C$$

$$u(x,y) = \cos x \cosh y + g(x)$$

Differentiate $u(x,y)$ w.r.to x

$$\frac{\partial u}{\partial x} = -\sin x \cosh y + g'(x) = \frac{\partial v}{\partial y}$$

$$\therefore -\sin x \cosh y = -\sin x \cosh y + g'(x)$$

$$\Longrightarrow g'(x) = 0 \Rightarrow g(x) = c$$

$$\therefore u(x,y) = \cos x \cosh y + c$$

$$\left\{ \begin{aligned} f(z) = u + iv &= \cos x \cosh y + c - i \sin x \sinh y \\ &= \underline{\underline{\cos z + c}} \end{aligned} \right\}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0+0=0$$

$\therefore \implies u$ and v are harmonic

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\therefore f(z)$ is not analytic.

<u>Examples</u>

1. S.T. $v(x,y) = -\sin x \sinh y$ is harmonic. Find its harmonic conjugate $u(x,y)$.

$$\frac{\partial v}{\partial x} = -\cos x \sinh y \qquad \frac{\partial v}{\partial y} = -\sin x \cosh y$$

$$\frac{\partial^2 v}{\partial x^2} = \sin x \sinh y \qquad \frac{\partial^2 v}{\partial y^2} = -\sin x \sinh y$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \implies v(x,y) \text{ is harmonic.}$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y = -\frac{\partial u}{\partial y}.$$

$$\therefore \partial u/\partial y = \cos x \sinh y$$

integrate w.r.to y

Harmonic function

A function $f(x,y)$ is said to be harmonic if it satisfies the Laplace's eqn: $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Theorem:- If $f(z)=u+iv$ is analytic then u and v are harmonic.

Pf:- Since $f(z)$ is analytic C-R eqn are satisfied.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow u(x,y) \text{ is harmonic}$$

Similarly $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \Rightarrow v(x,y)$ is harmonic

$u(x,y)$ is called the conjugate harmonic function of $v(x,y)$.

Note:- u and v are harmonic $\not\Rightarrow$ $f(z)=u+iv$ is analytic.

Ex:- $u=\frac{1}{2}\log(x^2+y^2)$, $v=2xy$.

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2} \quad ; \quad \frac{\partial v}{\partial x}=2y, \quad \frac{\partial v}{\partial y}=2x$$

If $f(z) = u + iv$ is analytic, P.T. the curve $u(x,y) = c_1$ is orthogonal to the curve $v(x,y) = c_2$ $\forall$ arbitrary constants c_1 and c_2.

Differentiate $u(x,y) = c_1$ w.r. to x.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \, dy/dx = 0$$

$$\therefore dy/dx = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1 \quad \longrightarrow (1)$$

Differentiate $v(x,y) = c_2$ w.r. to x

$$\partial v/\partial x + \partial v/\partial y \, dy/dx = 0$$

$$dy/dx = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2 \quad \longrightarrow (2)$$

$$m_1 \times m_2 = -\frac{\partial u/\partial x}{\partial u/\partial y} \times -\frac{\partial v/\partial x}{\partial v/\partial y}$$

Since $f(z) = u + iv$ is analytic we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore m_1 \times m_2 = \frac{\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial u/\partial y}{\partial u/\partial x} = \underline{\underline{-1}}$$

Hence the two curves are orthogonal to each other.

Example

S.T. $f(z) = \sin z$ is analytic everywhere and find $f'(z)$.

$$f(z) = \sin z = \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y$$

$\therefore u = \sin x \cosh y$; $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$; $\frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y$; $\frac{\partial v}{\partial y} = \cos x \cosh y$

C-R eqns are satisfied.

therefore $f(z) = \sin z$ is analytic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cos x \cosh y - i \sin x \sinh y$$

Put $x = z$, $y = 0$

$$\underline{f'(z) =} \cos z$$

C-R eqns:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos iy = \frac{e^{-y} + e^{y}}{2}$

$= \cosh y$.

$\sin iy = \frac{e^{-y} - e^{y}}{2i}$

$= i\left(\frac{e^{y} - e^{-y}}{2}\right)$

$= i \sinh y$.

Note To express $f(z)$ in terms of z, put $x=z, y=0$

This method is known as Milne-Thomson's method.