Mathematical Modelling of Systems

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Why mathematical modelling?

- To improve understanding of the process
- To train plant operating personnel
- To design the control strategy for a new process
- To select the controller setting
- To optimize process operating conditions

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Modeling Principles

- Conservation law

Within a defined system boundary (control volume)

$$\begin{bmatrix}\text{Rate of}\\ \text{Accumulation}\end{bmatrix} = \begin{bmatrix}\text{Rate of}\\ \text{input}\end{bmatrix} - \begin{bmatrix}\text{Rate of}\\ \text{output}\end{bmatrix} + \begin{bmatrix}\text{Rate of}\\ \text{generation}\end{bmatrix} - \begin{bmatrix}\text{Rate of}\\ \text{disapperance}\end{bmatrix}$$

- Mass balance (overall, components)
- Energy balance
- Momentum or force balance

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Mathematical Modelling of Fluid Systems

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Fluid Systems

- As the most versatile medium for transmitting signals and power, fluid (gas or liquid) have wide usage in industry.
- In engineering terms
  - Hydraulic describes fluid systems that use liquids (e.g., oil or water)
  - Pneumatic applies to those using air or other gases

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Why Model Fluid Systems?

- Hydraulic systems is used in machine tool applications, aircraft control systems, where high power to weight ratio, accuracy and quick response is required.
- Industrial processes often involve systems consisting of liquid-filled tanks connected by pipes having orifices, valves, and other flow restricting devices.
- Therefore, it is important to develop a systematic method to mathematically model different types of fluid systems

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Hydraulic Resistance

- Figure shows liquid flow in a pipe, with a restricting device (a valve) providing a hydraulic resistance (R_h) to the flow.
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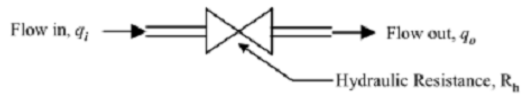

- Note that the walls of the pipe will also provide a small amount of resistance to flow, depending on how rough they are.

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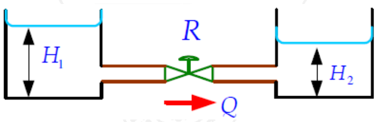
Resistance of Liquid-Level Systems

- Consider the flow through a short pipe connecting two tanks as shown in Figure.



- Where H_1 is the height (or level) of first tank, H_2 is the height of second tank, R is the resistance in flow of liquid and Q is the flow rate.

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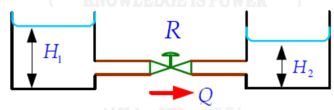
Resistance of Liquid-Level Systems

- The resistance for liquid flow in such a pipe is defined as the change in the level difference necessary to cause a unit change inflow rate.



$$Resistance = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate} = \frac{m}{m^3/s}$$

$$R = \frac{\Delta(H_1 - H_2)}{\Delta Q} = \frac{m}{m^3/s} \qquad R \equiv \frac{Potential}{Flow} = \frac{h}{q_0}$$

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Resistance in Laminar Flow

- For laminar flow, the relationship between the steady-state flow rate and steady state height at the restriction is given by:

$$Q = k_l H$$

- Where Q = steady-state liquid flow in m/s^3
- K_l = constant in m/s^2
- and H = steady-state height in m.
- The resistance R_l is $R_l = \dfrac{dH}{dQ}$

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Resistance in Turbulent Flow

- When turbulent flow occurs from a tank discharging under its own head or pressure, the flow is found by the following equation:

$$q_0 = KA\sqrt{2gh}$$

- The instantaneous rate of change of hydraulic resistance to flow is.

$$R_{hi} = \frac{dh}{dq_0}$$

- Find out R

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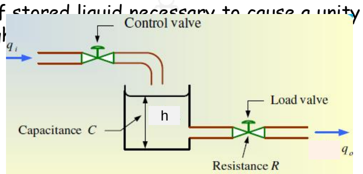
Capacitance of Liquid-Level Systems

- The capacitance of a tank is defined to be the change in quantity of stored liquid necessary to cause a unity change in the heigh



$$Capacitance = \frac{change\ in\ liquid\ stored}{change\ in\ height} = \frac{m^3}{m}\ or\ m^2$$

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Capacitance of Liquid-Level Systems

In a tank being filled with a liquid, the equation for the volume (V) of the liquid in the tank is given by the following equation:

V(t)=A * h(t)

Where
V(t)= the volume of liquid as a function of time
h(t)= height of liquid
A= the surface area of the liquid in the tank
Note that the volume V of the tank and the liquid height or head are a function of time.

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Capacitance of Liquid-Level Systems

$$A = \frac{V(t)}{h(t)} = \frac{Quantity}{Potential}$$

- Comparing this equation to the equation for electrical capacitance (i.e.,*C=q/V*) clearly shows that liquid capacitance *C* is simply the surface area of the liquid in the tank, or
- C_l= *A*.
- Furthermore, taking the derivative with respect to time yields

$$\frac{dV(t)}{dt} = A\frac{h(t)}{dt}$$

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Liquid capacitance

Capacitance (*C*) is cross sectional area (*A*) of the tank.

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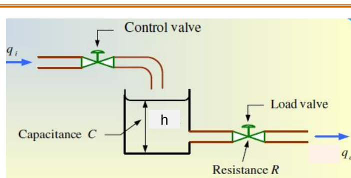
Modelling of Liquid-Level Systems



$$Rate\ of\ change\ of\ fluid\ volume\ in\ the\ tank = flow\ in - flow\ out$$

$$\frac{dV}{dt} = q_i - q_o$$

$$\frac{d(A \times h)}{dt} = q_i - q_o$$

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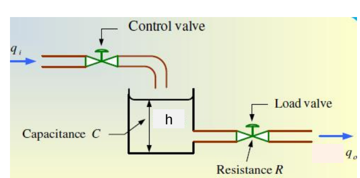
Modelling of Liquid-Level Systems



$$A\frac{dh}{dt} = q_i - q_o$$

$$C\frac{dh}{dt} = q_i - q_o$$

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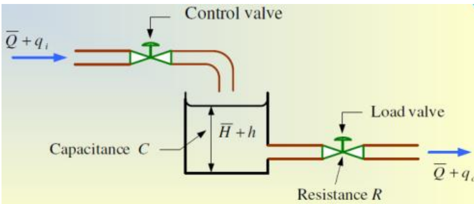
Modelling Example #1



$\bar{H}$ = steady-state head (before any change has occurred), m.
h = small deviation of head from its steady-state value, m.
$\bar{Q}$ = steady-state flow rate (before any change has occurred), m³/s.
q_i = small deviation of inflow rate from its steady-state value, m³/s.
q_o = small deviation of outflow rate from its steady-state value, m³/s.

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Modelling Example#1

- The rate of change in liquid stored in the tank is equal to the flow in minus flow out.

$$C\frac{dh}{dt} = q_i - q_o \qquad (1)$$

- The resistance R may be written as

$$dH = h, \quad dQ = q_o \quad R = \frac{dH}{dQ} = \frac{h}{q_0} \qquad (2)$$

- Rearranging equation (2)

$$q_0 = \frac{h}{R} \qquad (3)$$

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Modelling Example#1

$$C\frac{dh}{dt} = q_i - q_o \qquad (1) \qquad q_0 = \frac{h}{R} \qquad (4)$$

- Substitute q_o in equation (3)

$$C\frac{dh}{dt} = q_i - \frac{h}{R}$$

- After simplifying above equation

$$RC\frac{dh}{dt} + h = Rq_i$$

- Taking Laplace transform considering initial conditions to zero

$$RCsH(s) + H(s) = RQ_i(s)$$

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Modelling Example#1

$$RCsH(s) + H(s) = RQ_i(s)$$

- The transfer function can be obtained as

$$\frac{H(s)}{Q_i(s)} = \frac{R}{(RCs+1)}$$

If output=output flow rate.

Sub

$$h = Rq_0$$

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Problem No:1

Ex1: Determine the differential equation for flow (q_o) out of the process tank as shown earlier. Find the system time constant if the operating head is 5m, the steady state flow is 0.2m³/s, and the surface area of the liquid is 10m².

$$R_h = \frac{2h}{q_0} = 50\frac{s}{m^2}$$

$$\tau = AR_h = 500s$$

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Problem No:2

- Obtain relationship between q and h for single tank system with a non linear valve at output.

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Modelling Example#2

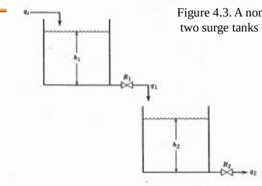

Figure 4.3. A noninteracting system: two surge tanks in series.

Mass Balance: $A_1 \frac{dh_1}{dt} = q_i - q_1$ (4-48)

Valve Relation: $q_1 = \frac{1}{R_1} h_1$ (4-49)

Substituting (4-49) into (4-48) eliminates q_1:

$$A_1 \frac{dh_1}{dt} = q_i - \frac{1}{R_1} h_1 \quad (4\text{-}50)$$

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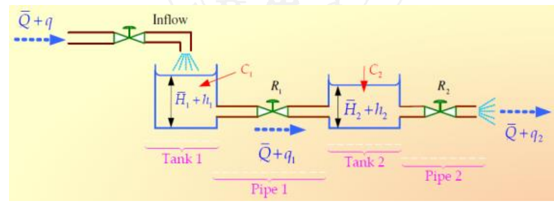
Modelling Example#3

- Consider the liquid level system shown in following Figure. In this system, two tanks interact. Find transfer function *$Q_2(s)/Q(s)$*.



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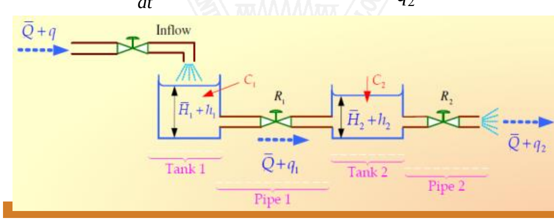
Modelling Example#3

- Tank 1 $C_1 \frac{dh_1}{dt} = q - q_1$ $R_1 = \frac{h_1 - h_2}{q_1}$ Pipe 1
- Tank 2 $C_2 \frac{dh_2}{dt} = q_1 - q_2$ $R_2 = \frac{h_2}{q_2}$ Pipe 2



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Modelling Example#3

Tank 1

$$C_1 \frac{dh_1}{dt} = q - \frac{h_1 - h_2}{R_1}$$

Pipe 1

$$q_1 = \frac{h_1 - h_2}{R_1}$$

- Tank 2

$$C_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

Pipe 2

$$q_2 = \frac{h_2}{R_2}$$

- Re-arranging above equation

$$C_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = q + \frac{h_2}{R_1} \qquad C_2 \frac{dh_2}{dt} + \frac{h_2}{R_1} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

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Modelling Example#3

$$C_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = q + \frac{h_2}{R_1} \qquad C_2 \frac{dh_2}{dt} + \frac{h_2}{R_1} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

- Taking LT of both equations considering initial conditions to zero [i.e. *$h_1(0)=h_2(0)=0$*].

$$\left(C_1 s + \frac{1}{R_1}\right) H_1(s) = Q(s) + \frac{1}{R_1} H_2(s) \quad (1)$$

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1} H_1(s) \quad (2)$$

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Modelling Example#3

$$\left(C_1 s + \frac{1}{R_1}\right) H_1(s) = Q(s) + \frac{1}{R_1} H_2(s) \quad (1) \qquad \left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1} H_1(s) \quad (2)$$

- From Equation (1)

$$H_1(s) = \frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}$$

- Substitute the expression of $H_1(s)$ into Equation (2), we get

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1} \left(\frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}\right)$$

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Modelling Example#3

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1}\left(\frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}\right)$$

- Using $H_2(s) = R_2 Q_2(s)$ in the above equation

$$[(R_2 C_2 s + 1)(R_1 C_1 s + 1) + R_2 C_1 s] Q_2(s) = Q(s)$$

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{R_2 C_1 R_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1}$$

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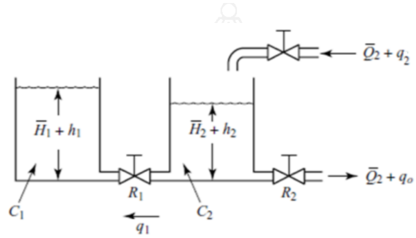
Modelling Example#4



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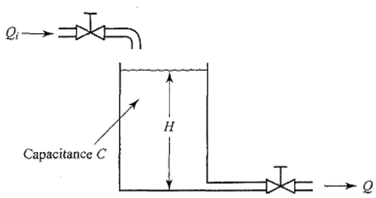
In the liquid-level system of Figure 4–27 assume that the outflow rate Q m^3/sec through the outflow valve is related to the head H m by

$$Q = K\sqrt{H} = 0.01\sqrt{H}$$

Assume also that when the inflow rate Q_i is 0.015 m^3/sec the head stays constant. For $t < 0$ the system is at steady state ($Q_i = 0.015$ m^3/sec). At $t = 0$ the inflow valve is closed and so there is no inflow for $t \geq 0$. Find the time necessary to empty the tank to half the original head. The capacitance C of the tank is 2 m^2.



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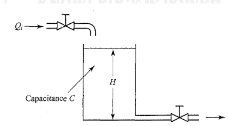

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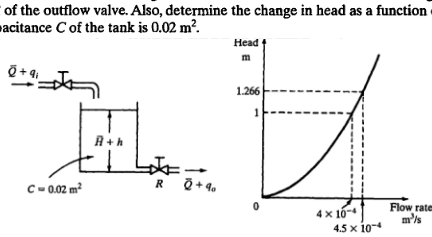
Consider the liquid-level system of Figure 7–17(a). The curve of head versus flow rate is shown in Figure 7–17(b). Assume that at steady state the liquid flow rate is 4×10^{-4} m^3/s and the steady-state head is 1 m. At $t = 0$, the inflow valve is opened further and the inflow rate is changed to 4.5×10^{-4} m^3/s. Determine the average resistance R of the outflow valve. Also, determine the change in head as a function of time. The capacitance C of the tank is 0.02 m^2.



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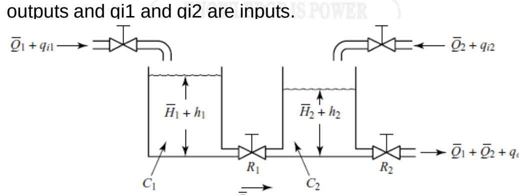
Modelling Example#5

- Write down the system differential equations.Obtain the state space representation of the system when h1 and h2 are outputs and qi1 and qi2 are inputs.



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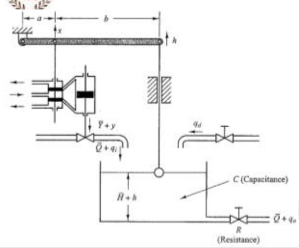
Problem # 7



We assume that the velocity of the power piston (valve) is proportional to pilot valve displacement x, or

$$\frac{dy}{dt} = K_1 x$$

where K_1 is a positive constant. We also assume that the change in the inflow rate q_i is negatively proportional to the change in the valve opening y, or

$$q_i = -K_v y$$

where K_v is a positive constant.

Assuming the following numerical values for the system,

$C = 2\ \text{m}^2$, $R = 0.5\ \text{sec/m}^2$, $K_v = 1\ \text{m}^2/\text{sec}$

$a = 0.25$ m, $b = 0.75$ m, $K_1 = 4\ \text{sec}^{-1}$

obtain the transfer function $H(s)/Q_d(s)$.

- Given: For the feedback lever mechanism, x and h are related by the equation, x = [a/(a+b)]*h.

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