(3) Find the interpolating polynomial from the following table

x:	0	1	2	5
y:	2	3	12	147

Also find y when $x = 8$.

	x	y	1st order difference	2nd order difference	3rd order difference
x_0	0	2			
			1		
x_1	1	3		4	
			9		1
x_2	2	12		9	
			45		
	5	147			

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + \cdots$$

$$= 2 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)(1)$$

$$= 2 + x + 4x(x-1) + x(x^2-3x+2)$$

$$= x^3 + x^2 - x + 2$$

when $x = 8$

$$y = 8^3 + 8^2 - 8 + 2$$

$$= 570$$

verification.

$y(0) = 0 + 0 - 0 + 2 = 2$ ✓

$y(1) = 1 + 1 - 1 + 2 = 3$ ✓

$y(2) = 2^3 + 2^2 - 2 + 2 = 12$ ✓

$y(5) = 5^3 + 5^2 - 5 + 2 = 147$ ✓

(4) Given

x:	-3	-1	0	3	5
$f(x)$:	-30	-22	-12	330	3458

Obtain $f(x)$ as a polynomial in x and hence find $f(2.5)$

	x	$f(x)$	1st order Difference	2nd order difference	3rd order difference	4th order difference
x_0	-3	-30				
			4			
x_1	-1	-22		2		
			10		4	
x_2	0	-12		26		5
			114		44	
x_3	3	330		290		
			1564			
x_4	5	3458				

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + \cdots$$

$$= -30 + (x+3)(4) + (x+3)(x+1)(2) + (x+3)(x+1)(x)(4) + (x+3)(x+1)x(x-3)5$$

$$= -30 + 4x + 12 + 2x^2 + 8x + 6 + 4x(x^2+4x+3) + 5(x^2+x)(x^2-9)$$

$$= 5x^4 + 9x^3 - 27x^2 - 21x - 12 \qquad y(2.5) = 102.6875$$

Newton's Divided Difference Formula:

$$y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \cdots$$

1. Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's Divided difference formula.

	x	$f(x)$	1st difference	2nd diff.	3rd diff
x_0	5	150			
			$\frac{392-150}{7-5} = 121$		
x_1	7	392		$\frac{265-121}{11-5} = 24$	
			$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
x_2	11	1452		$\frac{457-265}{13-7} = 32$	
			$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
x_3	13	2366		$\frac{709-457}{17-11} = 42$	
			$\frac{5202-2366}{17-13} = 709$		
x_4	17	5202			

$x = 9$

$$y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \cdots$$

$$= 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1)$$

$$= 810$$

2. Find $f(4)$ from the following table:

x:	0	2	3	6
$f(x)$:	-4	2	14	158

	x	$f(x)$	1st difference	2nd Difference	3rd difference
x_0	0	-4			
			$\frac{2-(-4)}{2-0} = 3$		
x_1	2	2		$\frac{12-3}{3-0} = 3$	
			$\frac{14-2}{3-2} = 12$		$\frac{9-3}{6-0} = 1$
x_2	3	14		$\frac{48-12}{6-2} = 9$	
			$\frac{158-14}{6-3} = 48$		
x_3	6	158			

$x = 4$

$$y = y_0 + (x-x_0)[x_0, x_1] + \cdots$$

$$= -4 + (4-0)(3) + (4-0)(4-2)(3) + (4-0)(4-2)(4-3)(1)$$

$$= 40.$$

Newton's Divided Difference Formula:

Given a set of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying $y = f(x)$, where explicit nature of $f(x)$ is not known, the n^{th} degree polynomial $y_n(x)$ such that $y_n(x)$ & $f(x)$ agree at the tabulated values is given by

$$y_n(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \cdots$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_{n-1})[x_0, x_1, \ldots, x_n]$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_n)[x, x_0, x_1, \ldots, x_n] \quad \text{— (1)}$$

(1) is called Newton's Divided difference formula.
The last term in (1) is called the error term.

Proof:

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

$$y - y_0 = (x-x_0)[x, x_0]$$

$$y = y_0 + (x-x_0)[x, x_0] \quad \text{— (2)}$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

$$[x, x_0] - [x_0, x_1] = (x-x_1)[x, x_0, x_1]$$

$$\therefore [x, x_0] = [x_0, x_1] + (x-x_1)[x, x_0, x_1] \quad \text{— (3)}$$

Substituting (3) in (2), we get

$$y = y_0 + (x-x_0)\{[x_0, x_1] + (x-x_1)[x, x_0, x_1]\}$$

$$= y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x, x_0, x_1] \quad \text{— (4)}$$

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

from which we get

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x-x_2)[x, x_0, x_1, x_2]$$

& substituting this in (4) gives

$$y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2)[x, x_0, x_1, x_2].$$

Continuing this way, we get (1).

Newton's Divided Difference Formula:

Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be the given set of values satisfying $y = f(x)$. The divided differences of order $1, 2, \ldots, n$ are defined by the relations

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

$$\vdots$$

$$[x_0, x_1, \ldots, x_n] = \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, x_2, \ldots, x_{n-1}]}{x_n - x_0}$$

Remark: The divided differences are symmetrical in their arguments i.e.
$[x_0, x_1] = [x_1, x_0]$ and so on.

Construct Divided difference table from the following:

x:	-1	0	3	6	7
$f(x)$:	3	-6	39	522	1611

x	$f(x)$	1st order differences	2nd order Differences	3rd order	4th order
-1	3				
		$\frac{-6-3}{0-(-1)} = -9$			
			$\frac{15-(-9)}{3-(-1)} = 6$		
0	-6			$\frac{24.3-6}{6-(-1)} = 2.614$	
		$\frac{39-(-6)}{3-0} = 15$			
			$\frac{161-15}{6-0} = 24.3$		
3	39				
				$\frac{232-24.3}{7-0} = 29.67$	
		$\frac{522-39}{6-3} = 161$			
			$\frac{1089-161}{7-3} = 232$		
6	522				
		$\frac{1611-522}{7-6} = 1089$			
					$\frac{29.67-2.614}{7-(-1)} = 3.382$
7	1611				