MAT 2258 ENGINEERING MATHEMATICS – IV

PROBABILITY

Addition rule

If A and B are two events of an experiment having sample space S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The conditional probability of an event B, given that the event A already taken place is

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

Baye's Theorem

Let $B_1, B_2, \ldots B_k$ are the partitions of S with $P(B_i) \neq 0, i = 1, 2, \ldots, k$ and A be any event of S, then

$$P(B_i / A) = \frac{P(A/B_i)P(B_i)}{\sum_{i=1}^{k} P(A/B_i)P(B_i)}.$$

The multiplicative rule of probability: $P(A \cap B) = \begin{cases} P(A)P(B|A), & if\ P(A) \neq 0 \\ P(B)P(A|B), & if\ P(B) \neq 0 \end{cases}$

If $P(A \cap B) = P(A)P(B)$, then A and B are independent.

Random Variable: Let S be the sample space of a random experiment. Suppose with each element s of S, a unique real number X is associated according to some rule then X is called random variable. There are two types of random variable, i) Discrete and ii) Continuous.

Discrete Random Variable: A random variable X is said to be discrete, if the number of possible values of X is finite or countably infinite. The probability distribution function (pdf) is named as probability mass function (PMF). Let X be a random variable, hence the range space R_X consists of atmost a countably infinite number of values. The probability mass function is defined as $p(x_i) = \Pr\{X = x_i\}$, satisfying the conditions

i) $p(x_i) \geq 0\ for\ all\ i$

ii) $\sum_{i=1}^{k} p(x_i) = 1$.

Continuous Random Variable: A random variable X is said to be continuous if it can take all possible values between certain limits, here the range space of X is infinite. Therefore the probability distribution function named for such random variable is probability density function (PDF), which is defined as the function $f(x)$ satisfying the following properties

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) $\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$ for any a, b such that $-\infty < a < b < \infty$.

Note:

1. If X is a continuous random variable with pdf f(x), then

$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_a^b f(x)dx.$

2. $P(X = a) = 0$, if X is a continuous random variable.

Cumulative distribution function: Let X be random variable (discrete or continuous), we define F to be the cumulative distribution function of a random variable X given by $F(x) = \Pr\{X \leq x\}$.

Case 1. If X is discrete random variable then $F(t) = \Pr\{X \leq t\} = P(x_1) + P(x_2) + \cdots + P(t)$

Case 2. If x is a continuous random variable then $F(x) = \Pr\{X \leq x\} = \int_{-\infty}^{x} f(x)dx$.

Two dimensional random variable: Let E be an experiment and S be a sample space associated with E. Let X = X(s) and Y = Y(s) be two functions each assigning a real number to each outcome s of S. We call (X, Y) to be two dimensional random variable.

Discrete two dimensional random variable: If the possible values of (X, Y) are finite or countably infinite then (X, Y) is called discrete and it is defined as $P(x_i, y_j)$ satisfying the following condition,

i) $P(x_i, y_j) \geq 0$ and
ii) $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} P(x_i, y_j) = 1$.

The function $P(x_i, y_j)$ defined is called as Joint probability distribution function (Jpdf).

Continuous two dimensional random variable: If (X, Y) is a continuous random variable assuming all values in some region R of the Euclidean plane, then the Joint probability density function $f(x, y)$ is a function satisfying the following conditions

i) $f(x, y) \geq 0$ for all (x, y) $\epsilon\ R$
ii) $\iint f(x, y)dx\, dy = 1$ over the region R.

Marginal Probability distribution: The marginal probability distribution is defined as

Case 1. In the discrete (X, Y), it is defined as $p(x_i) = P\{X = x_i\} = \sum_{j=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of X. Similarly $q(y_j) = P\{Y = y_j\} = \sum_{i=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of Y.

Case 2. In the continuous (X, Y), it is defined as the marginal probability function of X is defined as $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$ and the marginal probability function of Y is defined as $h(y) = \int_{-\infty}^{\infty} f(x, y)dx$.

To calculate the conditional probability:

Case 1. Discrete: Probability of x_i given y_j is defined as $\frac{P(x_i,\ y_j)}{q(y_j)}, q(y_j) > 0$

Probability of y_j given x_i is defined as $= \frac{P(x_i,\ y_j)}{p(x_i)}, p(x_i) > 0$

Case 2. Continuous: The pdf of X for given Y = y is $\frac{f(x,\ y)}{h(y)},\ h(y) > 0$

The pdf off Y for given X = x is $\frac{f(x,\ y)}{g(x)},\ g(x) > 0$.

Independent random variable:

Discrete: If $P(x_i, y_j) = p(x_i).q(y_j)$ for all I and j, then X and Y are independent random variables.

Continuous: If $f(x, y) = g(x).h(y)$ for all x and y, then X and Y are independent random variables.

Mathematical Expectation: If X is a discrete random variable with pmf p(x), then the expectation of X is given by $E(X) = \sum_x xp(x)$, provided the series is absolutely convergent.
If X is continuous with pdf f(x), then the expectation of X is given by $E(X) = \int xf(x)dx$, provided $\int |x|f(x)dx < \infty$.
Variance of X is given by $V(X) = E(X - E(X))^2 = E(X^2) - \left(E(X)\right)^2$.

DISTRIBUTIONS
Binomial distribution : If $X \sim B(n, p)$,

Pdf: $P(x) = {}^nC_k p^k (1-p)^{n-k}, \ \ k = 0,1, 2, ..., n$.

Mean = $E(x) = np$ and Variance = $V(x) = np(1-p)$.
Poisson's distribution: If $X \sim P(\alpha)$,

Pdf: $P(x) = \dfrac{e^{-\alpha}\alpha^k}{k!}, \ k = 0,1,2,...; \alpha > 0$

Mean = $E(x) = \alpha = np$ and Variance = $V(x) = \alpha = np$.

Uniform distribution: If $X \sim U(a,b)$,

Pdf: $f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$

Mean = $E(x) = \dfrac{b+a}{2}$ and Variance = $V(x) = \dfrac{(b-a)^2}{12}$.
Normal distribution: If $X \sim N(\mu, \sigma^2)$,

Pdf: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$.

Mean = $E(x) = \mu$ and Variance = $V(x) = \sigma^2$.
Exponential distribution: If $X \sim E(\lambda)$,

Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$

Mean = $E(x) = \dfrac{1}{\lambda}$ and Variance = $E(x) = \dfrac{1}{\lambda^2}$.
Gamma distribution: If $X \sim G(r, \alpha)$,

Pdf: $f(x) = \begin{cases} \dfrac{x^{r-1}e^{-\alpha x}\alpha^r}{\Gamma(r)}, & x > 0, \ \alpha, r > 0 \\ 0, & elsewhere \end{cases}$

Mean = $E(x) = \dfrac{r}{\alpha}$ and Variance = $V(x) = \dfrac{r}{\alpha^2}$.
Chi-square distribution: If $X \sim \chi^2(n)$,

Pdf: $f(x)=\begin{cases}\dfrac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x>0\\ 0, & elsewhere\end{cases}$

Mean = $E(x)=n$ and Variance = $V(x)=2n$.

Uniform distribution on a two dimensional set: If R is a set in the two-dimensional plane, and R has a finite area, then we may consider the density function equal to the reciprocal of the area of R inside R, and equal to 0 otherwise, i.e., $f(x,y)=\begin{cases}\frac{1}{area\ R}; & if\ (x,y)\in R\\ 0 & Otherwise\end{cases}$.

Chebyshev's inequality:

Let X be random variable with mean μ and variance σ^2 then for any positive real number k (k > 0), $P\{|X-\mu|\geq k\}\leq\frac{\sigma^2}{k^2}$ (Upper bound)

$P\{|X-\mu|<k\}>1-\frac{\sigma^2}{k^2}$ (Lower bound)

Note: Some other forms

1. $P\{|X-\mu|\geq k\sigma\}\leq\frac{1}{k^2}$ and $P\{|X-\mu|<k\sigma\}\geq 1-\frac{1}{k^2}$.
2. $P\{|X-\mu|\geq\in\}\leq\frac{1}{\in^2}E(X-c)^2$ and $P\{|X-\mu|<\in\}\geq 1-\frac{1}{\in^2}E(X-c)^2$.

Covariance:

$$Cov(X,Y)=E(XY)-E(X)E(Y)$$

Correlation coefficient:

$$\rho_{xy}=\rho=\frac{E(XY)-E(X)E(Y)}{\sqrt{V(X)\,V(Y)}}.$$

<u>Properties:</u>

1. $E(c)=c$, where c is a constant.
2. $V(c)=0$, where c is a constant.
3. If $E(XY)=0$ then X and Y are orthogonal.
4. $V(AX+b)=A^2V(X)$ when AX + B is linear function of X.
5. If $\rho=0$ then X and Y are un correlated.
6. $V(AX+BY)=A^2V(X)+B^2V(Y)+2AB\ COV(X,Y)$, where A and B are constants.

Functions of one dimensional random variables

Let S be a sample space associated with a random experiment E, then it is known that a random variable X on S is a real valued function, i.e., $X:S\rightarrow R$, for each element s $\in$ S, there is a real number associated.

Let X be a random variable defined on S. Let $y = H(x)$ is a real valued function of x. Then $Y = H(X)$ is a random variable on S. i.e., for each element s ∈ S, there is a real number associated, say $y = H\big(X(s)\big)$. Here Y is called a **function of the random variable** X.

Notations:

1. R_X – the set of all possible values of the function X, called the **range space** of the random variable X.
2. R_Y – the set of all possible values of the function $Y = H(X)$, called the **range space** of the random variable Y.

Equivalent Events: Let C be an event associated with the range space R_Y. Let $B \subset R_X$ defined by $B = \{x \in R_X; H(x) \in C\}$, then B and C are called equivalent events.

Distribution function of functions of random variables

Case 1: Let X be a discrete random variable with p.m.f. $p(x_i) = P(X = x_i)$ for $= 1,2,3, \dots$. Let $Y = H(X)$ then Y is also a discrete random variable. If $Y = H(X)$ is a one to one function then the probability distribution of Y is as follows:

For the possible values of $y_i = H(x_i)$ for $i = 1,2,3, \dots$. The p.m.f. of $Y = H(X)$ is $q(y_i) = P(Y = y_i) = P(X = x_i) = p(x_i)$ for $i = 1,2,3, \dots$.

Case 2: Let X be a discrete random variable with p.m.f. $p(x_i) = P(X = x_i)$ for $i = 1,2,3, \dots$. Let $Y = H(X)$ then Y is also a discrete random variable. Suppose that for one value of $Y = y_i$ there corresponds several values of X say $x_{i_1}, x_{i_2}, \dots, x_{i_j}, \dots$ then the p.m.f. of $Y = H(X)$ is

$$q(y_i) = P(Y = y_i) = p\left(x_{i_1}\right) + p\left(x_{i_2}\right) + \cdots + p\left(x_{i_j}\right) + \cdots$$

Case 3: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = H(X)$ be a discrete random variable. Then if the set $\{Y = y_i\}$ is equivalent to an event $B \subseteq R_X$ then the p.m.f. of Y is

$$q(y_i) = P(Y = y_i) = \int_B f(x)\,dx$$

Case 4: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = H(X)$ be a continuous random variable. Then the p.d.f. of Y, say g is obtained by the following procedure:

Step 1: Obtain the c.d.f. of Y, $G(y) = P(Y < y)$, by finding the event $A \subseteq R_X$, which is equivalent to the event $\{Y = y_i\}$.

Step 2: Differentiate $G(y)$ with respect to y to get $g(y)$.

Step 3: Determine those values of y in R_Y for which $g(y) > 0$.

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$ where $f(x) > 0$ for $a < x < b$. Suppose that $Y = H(X)$ is strictly monotonic function on $[a, b]$. Then the p.d.f. of the random variable $Y = H(X)$ is given by $g(y) = f(x)\left|\frac{dx}{dy}\right|$

If $Y = H(X)$ is strictly increasing then $g(y) > 0$ for $H(a) < y < H(b)$.

If $Y = H(X)$ is strictly decreasing then $g(y) > 0$ for $H(b) < y < H(a)$.

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = X^2$ then the p.d.f. of Y is

$$g(y) = \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) + f(-\sqrt{y})\right]$$

Functions of two dimensional random variables

Let (X, Y) be a two dimensional continuous random variable. Let $Z = H(X, Y)$ be a continuous function of X and Y then $Z = H(X, Y)$ is a continuous one dimensional random variable.

To find the p.d.f. of Z, we introduce another suitable random variable say, $W = G(X, Y)$ and obtain the joint p.d.f. of the two dimensional random variable (Z, W), say $k(z, w)$. From this distribution, the p.d.f. of Z can be obtained by integrating k with respect to w.

Theorem: Suppose (X, Y) is a two dimensional continuous random variable with joint p.d.f. $f(x, y)$ defined on a region R of the XY-plane. Let $Z = H_1(X, Y)$ and $W = H_2(X, Y)$. Suppose that H_1 and H_2 satisfies the following conditions;

(i) $z = H_1(x, y)$ and $w = H_2(x, y)$ may be uniquely solved for x, y in terms of z & w say, $x = G_1(z, w)$ and $y = G_2(z, w)$.

(ii) The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous

Then the joint p.d.f. of (Z, W) say $k(z, w)$ is given by,

$$k(z, w) = f[G_1(z, w), G_2(z, w)]|J(z, w)|$$

where $J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$ is called the Jacobian of the transformation $(x, y) \mapsto (z, w)$. Also, $k(z, w) > 0$ for those values of (z, w) corresponding to the values of (x, y) for which $f(x, y) > 0$.

Moment generating function (m.g.f.) of one dimensional random variables

Let X be any one dimensional random variable then the mathematical expectation $E(e^{tX})$ if exists then it is called the moment generating function (m.g.f.) of X, i.e., $M_X(t) = E(e^{tX})$.

In particular, if X is discrete then, $M_X(t) = \sum_{i=1}^{i=\infty} e^{tx_i} P(X = x_i)$.

If X is continuous then, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$.

Properties of m.g.f.

Let X be any one dimensional random variable and $M_X(t)$ be the m.g.f. of X then

1. $M_X^n(0) = E(X^n)$ where $M_X^n(0)$ is the nth derivative of $M_X(t)$ at $t = 0$.
   i.e.; $M_X'(0) = E(X)$
   $M_X''(0) = E(X^2)$
2. $V(X) = M_X''(0) - \left(M_X'(0)\right)^2$
3. Let X be any one dimensional random variable and $M_X(t)$ be the m.g.f. of X. Let $Y = \alpha X + \beta$. Then the m.g.f. of Y is $M_Y(t) = e^{\beta t} M_X(\alpha t)$.

4. Suppose that X and Y are independent random variables. Let $Z = X + Y$. Let $M_X(t), M_Y(t)$ and $M_Z(t)$ be the m.g.f.'s of the random variables X, Y and Z respectively. Then $M_Z(t) = M_X(t)M_Y(t)$
5. Let $X_1, X_2, \ldots, X_n$ be n independent random variables which follows a normal distribution $N(\mu_i, \sigma_i^2)$ for $i = 1,2,3, \ldots, n$. Let $Z = X_1 + X_2 + \cdots + X_n$ then $Z \to N(\mu_1 + \mu_2 + \cdots + \mu_n,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)$.
6. Let $X_1, X_2, \ldots, X_n$ be n independent random variables which follows a Poisson distribution with parameter α_i for $i = 1,2,3, .., n$. Let $Z = X_1 + X_2 + \cdots + X_n$ then Z has a Poisson distribution with parameter $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
7. Let $X_1, X_2, \ldots, X_k$ be k independent random variables which follows a Chi-square distribution with degrees of freedom n_i for $i = 1,2,3, .., k$. Let $Z = X_1 + X_2 + \cdots + X_k$ then Z has a Chi-square distribution with degrees of freedom $n = n_1 + n_2 + \cdots + n_k$.
8. Let $X_1, X_2, \ldots, X_k$ be k independent random variables, each having distribution $N(0,1)$. Then $S = X_1^2 + X_2^2 + \cdots + X_k^2$ has a Chi-square distribution with degrees of freedom k.
9. Let $X_1, X_2, \ldots, X_r$ be r independent random variables, each having exponential distribution with the same parameter α. Let $Z = X_1 + X_2 + \cdots + X_r$ then Z has a Gamma distribution with parameters α and r.
10. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variable with c.d.f.'s $F_1, F_2, \ldots, F_n, \ldots$ and m.g.f.'s $M_1, M_2, \ldots, M_n, \ldots$ Suppose that $\lim_{n\to\infty} M_n(t) = M(t)$, where $M(0) = 1$. Then $M(t)$ is the m.g.f. of the random variable X whose c.d.f is $F = \lim_{n\to\infty} F_n(t)$.

MGF of some standard distributions:

1. Binomial Distributions: $M_X(t) = M_X(t) = \ (pe^t + q)^n$
2. Poisson Distributions: $M_X(t) = e^{\propto(e^t - 1)}$
3. Normal Distributions: $M_X(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$
4. Exponential Distributions: $M_X(t) = \frac{\propto}{\propto - t}$
5. Gamma Distributions: $M_X(t) = \frac{\propto^r}{(\propto - t)^r}$
6. Chi square Distributions: $M_X(t) = (1 - 2t)^{-n/2}$

SAMPLING

In statistical investigation, the characteristics of a large group of individuals (called population) is studied. Sampling is a study of the relationship between a population and samples drawn from it.

The population mean and the population variance are denoted by μ and σ^2 respectively.

Sample mean and sample variance: Let X be the random variable which denotes the population with mean μ and variance σ^2. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n from X. Then,

sample mean, $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ and sample variance, $s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$.

- If $X \to N(\mu, \sigma^2)$ then $\bar{X}$ and s^2 are independent random variables.

- Let X be the random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Let $(X_1, X_2, \dots, X_n)$ be a random sample of size n from X. Then, $E(\bar{X}) = \mu$ and $V(\bar{X}) = \frac{\sigma^2}{n}$.
- Let $X \to N(\mu, \sigma^2)$ then $\bar{X} \to N(\mu, \frac{\sigma^2}{n})$ and $s^2 \to \chi^2(n-1)$.

Central Limit Theorem: Let $X_1, X_2, \dots, X_n$ be n independent random variables all of which have the same distribution. Let $\mu = E(X_i)$ and $\sigma^2 = V(X_i)$ be the common expectation and variance. Let $S = \sum_{i=1}^{n} X_i$ then $E(S) = n\mu$ and $V(S) = n\sigma^2$ then for large values of n, the random variable $T_n = \frac{S - E(S)}{\sqrt{V(S)}}$ has approximately the distribution $N(0,1)$.

NUMERICAL METHODS

Numerical Solution of Boundary Value Problems

Boundary value problems are of great importance in science and engineering. In this section, we have elaborated Numerical methods based on finite difference scheme for the solution of following problems:

1. Boundary value problems in second order ordinary differential equations
2. Boundary value problems governed by linear second order partial differential equations: Laplace equation and Poisson equation.
3. Initial boundary value problems governed by linear second order partial differential equations: One dimensional heat and wave equation.

Boundary value problems governed by second order ordinary differential equations

For our discussion, we shall consider only the linear second order ordinary differential equations

$$y'' + p(x)\, y'(x) + q(x)\, y = r(x), \quad x \in [a, b] \quad \dots \quad (1)$$

Since the ordinary differential equation is of second order, we need to prescribe two conditions to obtain a unique solution of the problem. If the conditions are prescribed at the end points x = a and x = b, then it is called a two point boundary value problem. The two conditions required to solve (1), can be prescribed in one of the following three boundary conditions:

Boundary conditions of the first kind : $y(a) = A, \quad y(b) = B \quad \dots \quad (2)$

Boundary conditions of the second kind : $y'(a) = A, \quad y'(b) = B \quad \dots \quad (3)$

Boundary conditions of the third (or mixed kind) :

$$a_0\, y(a) - a_1\, y'(a) = A, \quad b_0\, y(b) + b_1\, y'(b) = B, \quad \dots \quad (4)$$

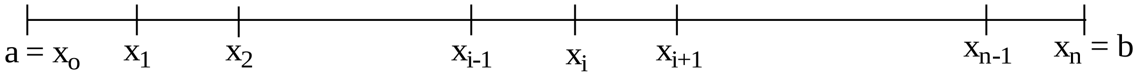
Where a_0, b_0, a_1, b_1, A and B are constants such that

$$a_0\, a_1 \geq 0, \ |a_0| + |a_1| \neq 0 \ ; \ b_0\, b_1 \geq 0, \ |b_0| + |b_1| \neq 0 \ \text{ and } \ |a_0| + |b_0| \neq 0$$

Finite Difference Methods for Ordinary Differential Equation

These are the explicit or implicit relations between the derivatives and function values at the adjacent nodal points. The nodal points on an interval may be defined by

$x_i = x_0 + i\,h$, $i = 0, 1, \ldots, N$,   where $a = x_0$, $b = x_N$ and $h = (b-a)/N$



Finite Difference Approximations to derivatives are given below:

Approximations to $y'(x_i)$ at $x = x_i$ **:**

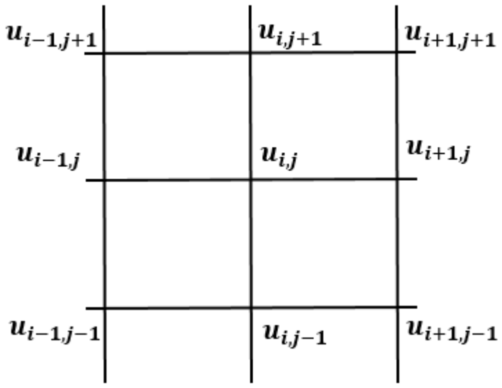
(i) Forward difference approximation of first order or O(*h)* approximation:

$$y'(x_i) = \frac{1}{h}\left[y(x_{i+1}) - y(x_i)\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{h}\left[y_{i+1} - y_i\right] \qquad \ldots \ (5)$$

(ii) Backward difference approximation of first order or O(*h)* approximation:

$$y'(x_i) = \frac{1}{h}\left[y(x_i) - y(x_{i-1})\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{h}\left[y_i - y_{i-1}\right] \qquad \ldots \ (6)$$

(iii) Central difference approximation of second order or O(h^2) approximation:



$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] \quad , \quad \text{or} \quad y'_i = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right] \qquad \ldots \ (7)$$

Approximations to $y''(x_i)$ at $x = x_i$ **:**

Central difference approximation of second order or O(h^2) approximation:

$$y''(x_i)=\frac{1}{h^2}\left[y(x_{i+1}) - 2\,y(x_i)+y(x_{i-1})\,\right] \quad \text{, or} \quad y_i''=\frac{1}{h^2}\left[y_{i+1}-2\,y_i+y_{i-1}\right] \qquad \dots\ (8)$$

The finite difference solution of a boundary value problem (1) is obtained by replacing the differential equation at each nodal point by difference equations (5) to (8) along with given boundary conditions (2) to (4).

Boundary value problems governed by linear second order partial differential equations

Over a two dimensional Cartesian domain, let u be the dependent variable. Then a general second order partial differential equation may be written as,

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,u_x,u_y\right)+G=0 \qquad \text{-------} \qquad (9)$$

where A, B, C are functions of (x, y) and F may be non-linear function, then equation (9) is called a quasi-linear partial differential equation. If F is also a linear function then equation (9) is called a linear partial differential equation. If $G = 0$ then equation (9) is homogeneous otherwise non-homogeneous.

Classification of PDE:

Equation (9) is said to be elliptic, parabolic and hyperbolic depending on $B^2-4AC < 0$, $B^2-4AC = 0$ and $B^2-4AC > 0$ at a point or in a domain.

Finite Difference Methods for Partial Differential Equation:

Consider a rectangular region R in the XY −plane. Divide the region into rectangular network of sides $\Delta x = h$ and $\Delta y = k$. Writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the finite difference approximations for the first order partial derivatives are as follows:

Forward: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i+1,j}-u_{i,j}\right)+O(h)$

Backward: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i,j}-u_{i-1,j}\right)+O(h)$

Central: $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h}\left(u_{i+1,j}-u_{i-1,j}\right)+O(h^2)$

Similarly,

Forward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j+1}-u_{i,j}\right)+O(k)$

Backward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j}-u_{i,j-1}\right)+O(k)$

Central: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2k}\left(u_{i,j+1}-u_{i,j-1}\right)+O(k^2)$

The finite difference approximations for the second order partial derivatives are as follows.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)+O(h^2)$$

Similar expressions can be written for $\left(\frac{\partial^2 u}{\partial y^2}\right), \left(\frac{\partial^2 u}{\partial x\partial y}\right)$.

Elliptic partial differential equation (Laplace equation & Poisson equation):

Most relevant examples of elliptic PDE are Laplace equation and Poisson equation.

The Poisson equation in Cartesian coordinate system is

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ or $\nabla^2 u=f(x,y),\ \ a\le x\le b,\ \ c\le y\le d.$
subject to boundary condition: u(x, y) = g(x, y) (Dirichlet boundary condition).
The Laplace equation is a special case of Poisson equation with $f(x,y)=0$.

Solution for Laplace Equation (for uniform mesh size h =k)**:**

Standard 5-point formula :

$$u_{i,j}=\frac{1}{4}(u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1})$$

Diagonal 5-point formula **:**

$$u_{i,j}=\frac{1}{4}(u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}+u_{i-1,j-1})$$

Solution to Poisson equation:

$$u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=h^2 f(ih,jh)$$

Parabolic Partial Differential Equation (One – dimensional heat conduction equation)

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2},\qquad 0\le x\le l\ ,\quad t>0$$

In order that the solution of the problem exists and is unique, we need to prescribe the following conditions:

(i) Initial conditions : $u(x,0)=f(x)$, $0\le x\le l$

(ii) Boundary Conditions : $u(0,t)=g(t)$, $u(l,t)=h(t)$, $t>0$

(in this study we restricted to simple boundary condition, i.e., temperature at the ends of the bar is prescribed) where c^2 is the diffusivity of the substance, $u(x,t)$ is a temperature function which is defined for values of x from 0 to l (length of the bar) and for values of time t from 0 to ∞.

Solution of one dimensional heat equation

Explicit Method :

Schmidt Method :

$u_{i,j+1}=\lambda u_{i-1,j}+(1-2\lambda)u_{i,j}+\lambda u_{i+1,j}$ where $\lambda=\frac{kc^2}{h^2}$; $0<\lambda\le\frac{1}{2}$.

Where λ is called the mesh ratio parameter, h mesh length along x-axis and k mesh length along t-axis.

Bender – Schmidt Method (*particular case when* $\lambda=1/2$) **:**

$$u_{i,j+1}=\frac{1}{2}(u_{i-1,j}+u_{i+1,j})$$

Note : For $\lambda=\frac{1}{6}$, the error in Schmidt's formula is least.

Implicit Method:

Crank – Nicolson's method:

$-\lambda u_{i-1,j+1}+2(1+\lambda)u_{i,j+1}-\lambda u_{i+1,j+1}=\lambda u_{i-1,j}+2(1-\lambda)u_{i,j}+\lambda u_{i+1,j}$

where $\lambda=\frac{kc^2}{h^2}$

For $\lambda=1$: $-u_{i-1,j+1}+4u_{i,j+1}-u_{i+1,j+1}=u_{i-1,j}+u_{i+1,j}$ *(particular case)*

Hyperbolic Partial Differential Equation (One dimensional wave equation)

All vibration problems arising in science and engineering are governed by wave equation. Consider the problem of vibrations of an elastic string governed by the one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le l$, $t > 0$

Subject to the initial and boundary conditions:

(i) Initial conditions: $u(x, 0) = f(x)$, $0 \le x \le l$ (initial displacement)

$$\frac{\partial u}{\partial t}(x, 0) = g(x) , \quad 0 \le x \le l \text{ (initial velocity)}$$

(i) Boundary Conditions: $u(0, t) = 0$, $u(l, t) = 0$, $t > 0$

(we consider the case when the ends of the string are fixed)

Where c^2 is a constant and depends on the material properties of the elastic string.

Solution of one dimensional Wave equation:

<u>Explicit Method:</u>

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2k^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$
$$= r^2\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

Where $r = \frac{\mathrm{kc}}{\mathrm{h}}$ is called mesh ratio parameter, h is the mesh length along x-axis and k is the mesh length along t-axis.

The above method is stable when $r \le 1$, higher order method uses the value of $r = 1$

Therefore, *we* have $\boldsymbol{u_{i,j+1} = \left[u_{i+1,j} + u_{i-1,j} - u_{i,j-1}\right]}$, is the explicit finite difference method for wave equation at higher level.

To get the value at first level, we have $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right] + \mathrm{k}g_i$.

If the initial condition is prescribed as $\frac{\partial u}{\partial t}(x, 0) = 0$, that is $g(x) = 0$, then above formula simplifies to

$\boldsymbol{u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]}$ **gives the value of 'u' at first level.**

DIFFERENCE EQUATION

A **difference equation** is a relation between the differences of an unknown function at one or more general values of an argument.

Order of difference equation = $\frac{largest\ argument - smallest\ argument}{unit\ of\ increment}$.

Solution of a difference equation is an expression which satisfies the difference equation.

General solution is a solution in which the number of arbitrary constants is equal to the order of the difference equation.

Particular solution is a solution obtained from general solution by giving particular values to the constants.

A linear difference equation with constant coefficient is of the form

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \cdots + a_r y_n = f(n)$$
…………………………………(i)

where $a_1, a_2, \ldots, a_n$ are constants.

Let $u_1(n),\ u_2(n),\dots,u_r(n)$ be the r independent solutions of $y_{n+r}+a_1y_{n+r-1}+a_2y_{n+r-2}+\cdots+a_ry_n=0$.
$U_n=c_1u_1(n)+c_2u_2(n)+\cdots+c_ru_r(n)$, where $c_1,c_2,\dots,c_r$ are constants, is a solution of the above equation.
If V_n is the particular solution of (i), then the complete solution of (i) is

$y_n=U_n+V_n$, U_n is called complementary function and V_n the particular solution.

Shift Operator: $E^1y_0=y_1$, $E^2y_0=y_2$,…, $E^ny_0=y_n$, $E^ny_r=y_{r+n}$

Rules for finding Complementary Function (CF):
For equation (i), $m^r+a_1m^{r-1}+a_2m^{r-2}+\cdots+a_r=0$, is called auxiliary equation.
Let its r roots be $\lambda_1,\lambda_2,\dots,\lambda_r$.

1. If $\lambda_1,\lambda_2,\dots,\lambda_r$ are real and distinct, $CF=c_1{\lambda_1}^n+c_2{\lambda_2}^n+\cdots+c_r{\lambda_r}^n$,
2. If two roots say λ_1 and λ_2 are real and equal, all other real and distinct, $CF=(c_1+c_2n){\lambda_1}^n+c_3{\lambda_3}^n+\cdots+c_r{\lambda_r}^n$.
   If $\lambda_1=\lambda_2=\lambda_3$, then CF$=(c_1+c_2n+c_3n^2){\lambda_1}^n+c_4{\lambda_4}^n+\cdots+c_r{\lambda_r}^n$.
3. If the two roots are complex, say $\alpha\pm i\beta$,
   then CF = $r^n(c_1\cos n\theta+c_2\sin n\theta)+c_3{\lambda_3}^n+\dots+c_r{\lambda_r}^n$ where $r=\sqrt{\alpha^2+\beta^2}$, $\theta=\tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Rules for finding Particular Solution:
Case 1. Suppose $f(n)=a^n$.
Particular solution = $\frac{1}{f(E)}a^n=\frac{1}{f(a)}a^n$, if $f(a)\neq 0$.
If $f(a)=0$, then (a) for $(E-a)y_n=a^n$, PI = $\frac{1}{E-a}a^n=na^{n-1}$.

(b) for $(E-a)^2y_n=a^n$, PI = $\frac{1}{(E-a)^2}a^n=\frac{n(n-1)}{2!}a^{n-2}$.

(c) for $(E-a)^3y_n=a^n$, PI = $\frac{1}{(E-a)^3}a^n=\frac{n(n-1)(n-2)}{2!}a^{n-3}$.

Case 2. Suppose $f(n)=\sin kn=\frac{e^{ikn}-e^{-ikn}}{2i}=\frac{1}{2i}[a^n-b^n]$, where $a=e^{ik}$, $b=e^{-ik}$.
If $f(E)=\cos kn=\frac{e^{ikn}+e^{-ikn}}{2}=\frac{1}{2i}[a^n+b^n]$, where $a=e^{ik}$, $b=e^{-ik}$.
Proceed as in case 1.
Case 3. When $f(n)=n^p$,

Particular solution = $\frac{1}{\phi(E)}n^p=\frac{1}{\phi(1+\Delta)}n^p$

1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by binomial theorem as far as the term in Δ^p.
2) Write n^p in the factorial form and operate on it with each term of the expansion.

Note: While expanding in ascending powers, the following formulae are useful.

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots,\qquad \frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\cdots$$

Factorial Notation:
A product of the form x(x - 1)(x - 2)…(x – r + 1) is denoted by $[x]^r$ and is called a factorial.
Thus, $[x]=x$, $[x]^2=x(x-1)$, $[x]^3=x(x-1)(x-2)$ and so on.
The operator Δ is used similar to D, when a polynomial is written in factorial notation.
For example, $\Delta[x]=1$, $\Delta[x]^2=2[x]$, $\Delta^2[x]^2=\Delta 2[x]=2$, $\Delta[x]^3=3[x]^2$, $\Delta^2[x]^3=6[x]$, $\Delta^3[x]^3=6$.

Case 4. Suppose $(n) = a^n n^p$.

Particular solution = $\frac{1}{f(E)}\ a^n n^p = a^n \frac{1}{f(aE)} n^p$, proceed as in case 3.

Z TRANSFORMS

If {un} is a sequence, then its Z-transform is defined as $Z_T\{u_n\} = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, whenever the infinite series converges.

Z-transforms of some standard functions:

u_n	$Z(u_n) = U(z)$
1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$
n^p	$-z\frac{d}{dz}Z(n^{p-1}), \qquad p \in Z_+$
na^n	$\frac{az}{(z-a)^2}$
n^2a^n	$\frac{az^2+za^2}{(z-a)^3}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$a^n\cos n\theta$	$\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
$a^n\, si\ n\ n\theta$	$\frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$
$n^m u_n$	$\left(-z\frac{d}{dz}\right)^m U(z)$

Inverse Z transforms

If $Z_T\{u_n\} = U(z)$, then $\{\ u_n\}$ is called inverse Z transform of U(z) and we denote this by

$$Z^{-1}[U(z)] = \{u_n\}.$$

Inverse Z transforms of some standard functions:

$U(z)$	$u_n = Z^{-1}[U(z)]$
$\frac{1}{z-a}$	a^{n-1}
$\frac{1}{z+a}$	$(-a)^{n-1}$
$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
$\frac{z}{z-a}$	a^n

$\frac{z}{z+a}$	$(-a)^n$
$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$

Properties of Z transforms:

1) **Linearity Property:** If $\{u_n\}$ *and* $\{v_n\}$ are any two sequences then
$$Z_T\{c_1u_n + c_2\, v_n\} = c_1\, Z_T\{u_n\} + c_2\, Z_T\{v_n\}$$
2) **Damping Property:** If $Z_T\{u_n\} = U(\text{z})$, then $Z_T\{k^n u_n\} = U\left(\frac{z}{k}\right)$, $k \neq 0$
3) **Right shifting rule:** If $Z_T\{u_n\} = U(\text{z})$ then $Z_T\{u_{n-k}\} = z^{-k}U(\text{z})$, $k > 0$
4) **Left shifting rule:** If $Z_T\{u_n\} = U(\text{z})$ then $Z_T\{u_{n+k}\} = z^k\, \{U(\text{z}) - \sum_{r=0}^{k-1} u_r\, z^{-r}\}$
5) **Initial Value theorem** If $Z_T\{u_n\} = U(\text{z})$ then $\lim_{z\to\infty} U(\text{z}) = u_0$
6) **Final Value theorem** If $Z_T\{u_n\} = U(\text{z})$ then $\lim_{z\to 1}[\ (z-1)U(\text{z})] = \lim_{n\to\infty} u_n$
7) **Convolution Theorem** : If $Z^{-1}[(U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then
$$Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_m v_{n-m} = u_n * v_n.$$

ICE 2252 INDUSTRIAL INSTRUMENTATION

Bimetallic strip:

$$r = \frac{t[3(1+m)^2 + (1+mn)\left(m^2 + {}^{1}/_{mn}\right)]}{6(\alpha_A - \alpha_B)(T_2 - T_1)(1+m)^2}$$

Where t = total thickness of the strip,

n = ratio of moduli of elasticity = E_B/E_A

m = ratio of thickness = t_B/t_A,

T_2-T_1 = change in temperature,

t_A, t_B = thickness of metal A and B respectively,

r = radius of the arc,

α_A, α_B = thermal co-efficient of expansion of metals A and B respectively.

Resistance temperature detector (RTD):

$$R_T = R_{ref}(1 + \alpha_1\Delta T + \alpha_2\Delta T^2 + \alpha_3\Delta T^3 \ldots)$$

Where R_T = Resistance at given temperature,

R_{ref} = Resistance at reference temperature,

$\alpha_1, \alpha_2, \alpha_3 \ldots \ldots$ = resistance temperature coefficient,

$\Delta T = T - T_{ref}$, T_{ref} = Reference temperature.

Thermistor:

$$R_T = R_{ref} \exp\left(\beta\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)\right)$$

Where β = constant.

Thermocouple:

$$T_{unknown} = T_{reference} + \frac{V}{S}$$

Where V = Voltage,

S = Seebeck Coefficient, µV/°C

Orifice and Venturi flowmeter

$$Q = \frac{C_d\sqrt{2}A_2}{\sqrt{1-\beta^4}}\sqrt{\frac{P_1 - P_2}{\rho}}$$

Where Q = Volumetric Flowrate,

C_d = discharge co-efficient,

A_2 = Area of restriction

P_1 = upstream pressure,

P_2 = downstream pressure,

β = Beta ratio, D/d

D = Diameter of the pipe,

d = diameter of the orifice opening/ Venturi throat

ρ = density of the working medium.

Rotameter:

$$Q = \frac{C_d\sqrt{2}\,A_2}{\sqrt{1-\ \beta^4}}\sqrt{V_f\,\frac{(P_1-P_2)g}{A_f\rho}}$$

Where A_2 = Area of float and tube,

V_f = Volume of the float,

A_f = Area of the float,

g = acceleration due to gravity.

Transit time flowmeters:

$$\Delta t = \frac{2L\hat{u}\,cos\theta}{c^2} \quad \text{and} \quad \Delta f = \frac{2\hat{u}cos\theta}{L}$$

Where Δt = change in time

Δf = Shift in frequency

L = Distance between Transmitter and Receiver

c = Velocity of sound

$\hat{u}$ = Average velocity of fluid at an angle θ to the direction of sound

Doppler Flowmeters:

When observer approaches in the direction of flow:

$$\frac{\Delta f}{f} = \frac{(2vcos\theta)}{c}$$

Where Δf = Doppler frequency shift, f = frequency of incident sound wave

Electromagnetic flowmeter:

$$e = Blv \quad \text{and} \quad Q = k\,\frac{\pi d e}{4B}$$

Where e= motional EMF

B = magnetic flux density

l= length of conductor passing through the magnetic field

v = velocity of conductor

Q = volumetric flow rate

e = Motional EMF

d = diameter of flow tube

k = constant of proportionality

ICE 2253 LINEAR CONTROL THEORY

MATHEMATICAL MODELS OF SYSTEMS:

Electrical Network Transfer Functions

Table: Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = \frac{V(s)}{I(s)}$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls

Note: The following set of symbols and units is used: $v(t)$ - V (volts); $i(t)$ – A (amps); $q(t)$ – Q (coulombs); C – F (farads), R – Ω (ohms); L – h (henries)

Translational mechanical System Transfer functions

Table: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous damper, and mass.

Component	Force-velocity	Force-displacement	Impedance $Z_m(s) = \frac{F(s)}{X(s)}$
Spring	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t))$	K
Viscous damper	$f(t) = Bv(t)$	$f(t) = B\frac{dx(t)}{dt}$	Bs
Mass	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used: $f(t)$ - N (newtons); $x(t)$ – m (meters); $v(t)$ – m/s (meters/second); K – N/m (newton/meter), B – N-s/m (newton-seconds/meter); M – kg (kilograms=newton-second^2/meter)

Rotational mechanical System Transfer functions

Table: Torque-angular velocity, torque angular displacement, and impedance rotational relationships for springs, viscous damper, and inertia.

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_m(s) = \frac{T(s)}{\theta(s)}$
Spring	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t))$	K
Viscous damper	$T(t) = B\omega(t)$	$T(t) = B\frac{d\theta(t)}{dt}$	Bs

Inertia	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used: $T(t)$ – N-m (newton-meters); $\theta(t)$ – rad (radians); $\omega(t)$ – rad/s (radians/second); K – N-m/rad (newton-meter/second), B – N-m-s/rad (newton-meter-second/radians); J – kg-m^2 (kilogram-meters^2=newton-meter-second^2/radian)

Transfer Functions for Systems with Gears

$\boldsymbol{T_1}(t)$- input torque
$\theta_1(t)$ – angle of rotation of input gear
r_1 – radius of input gear
N_1 – number of teeth on input gear
$\boldsymbol{T_2}(t)$- output torque
$\theta_2(t)$ – angle of rotation of output gear
r_2 – radius of output gear
N_2 – number of teeth on output gear

a) $\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$
b) $T_1\theta_1 = T_2\theta_2$
c) $\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$

Note: Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

$$\left[\frac{Number\ of\ teeth\ of\ gear\ on\ destination\ shaft}{Number\ of\ teeth\ of\ gear\ on\ source\ shaft}\right]^2$$

<u>TRANSIENT RESPONSE SPECIFICATIONS:</u>

First order system

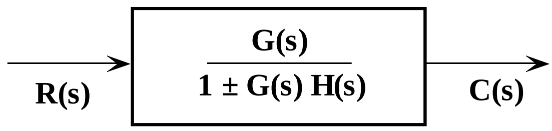
$$\frac{C(s)}{R(s)} = \frac{a}{s+a}$$

a) Time constant $\tau = \frac{1}{a}$
b) Rise time = $\frac{2.2}{a}$
c) Settling time = $\frac{4}{a}$

Second order system

$$\frac{\boldsymbol{C}(\boldsymbol{s})}{\boldsymbol{R}(\boldsymbol{s})} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

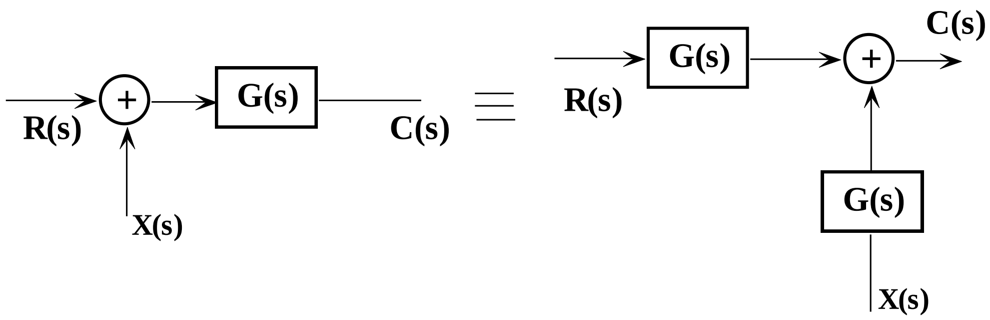
a) Underdamped unit step response $1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$; $\theta = cos^{-1}\xi$;
b) Peak overshoot $M_P = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$
c) $\xi = \frac{-\ln M_P}{\sqrt{\pi^2 + (\ln M_P)^2}}$
d) Rise time $t_r = \frac{\pi-\theta}{\omega_d}$; $\omega_d = \omega_n\sqrt{1-\xi^2}$
e) Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$
f) Settling time $t_s = \frac{4}{\xi\omega_n}$



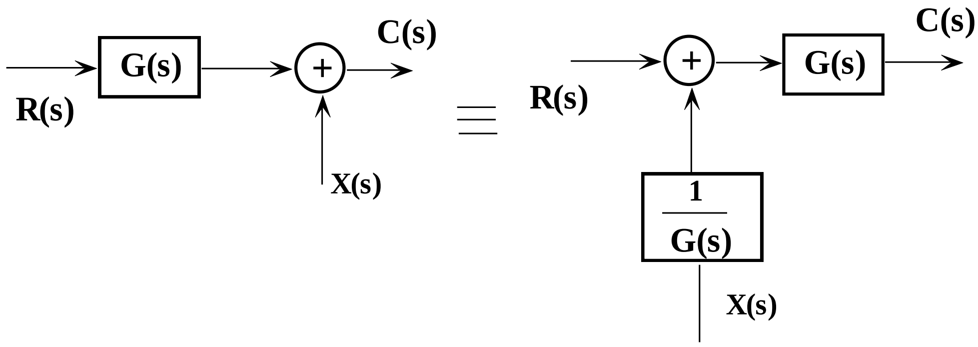
Equivalent transfer function

Moving blocks to create familiar forms:

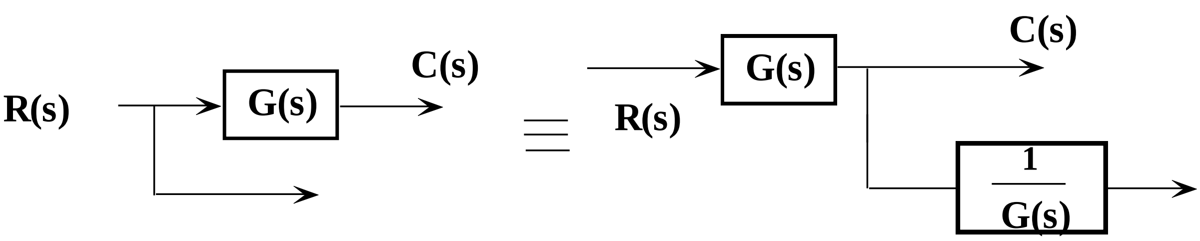
i) Moving a block to the left past a summing junction: (Moving a summing point beyond the block G(s))



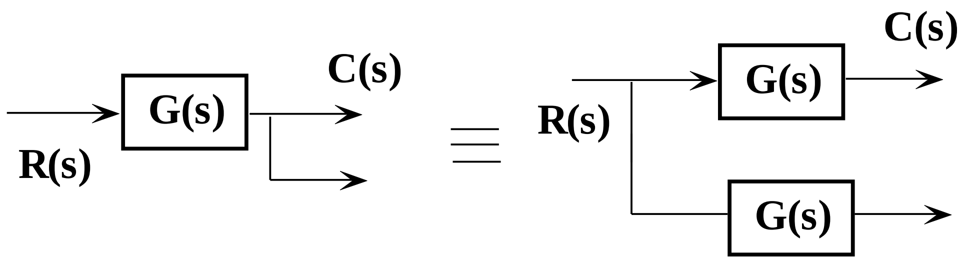
ii) Moving a block to the right past a summing junction: (Moving a summing point ahead of the block G(s))



iii) **Moving a block to the left past a pickoff point: (Moving a Pick-off point beyond the block G(s))**



iv) **Moving a block to the right past a pickoff point: (Moving a pick_off point ahead of the block G(s))**



SIGNAL-FLOW GRAPHS:

Mason's rule:

The transfer function $\frac{C(s)}{R(s)}$ of a system represented by a signal-flow graph is

$$\frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

Where

k= number of forward paths

T_k= the kth forward-path gain

Δ = 1-$\sum$ loop gains + $\sum$ product of non-touching loop gains taken two at a time - $\sum$ product of non-touching loop gains taken three at a time + $\sum$ product of non-touching loop gains taken four at a time …………….

$\Delta_k = \Delta$ -$\sum$ loop gain terms in Δ that touch the k_{th} forward path. Δ_k is formed by eliminating from Δ those loop gains that touch the k_{th} forward path.

STABILITY

Routh-Hurwitz criterion:

Considering the characteristic equation of a 4th order system

$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0$

The subsequent rows of Routh array are constructed as shown below

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_o \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

The system is stable if and only if the first column of the Routh array must be positive. Otherwise system is unstable.

STEADY-STATE RESPONSE SPECIFICATIONS:

Input	Static error constants	Steady state errors	Type zero systems-	Type one systems-	Type two systems-

			Steady state errors	Steady state errors	Steady state errors
Step function	$K_P = \lim_{s \to o} G(s)$	$\frac{1}{1+k_p}$	$\frac{1}{1+k_p}$	0	0
Ramp function (t)	$K_r = \lim_{s \to o} S\, G(s)$	$\frac{1}{k_v}$	∞	$\frac{1}{k_v}$	0
Parabolic function (t^2)	$K_a = \lim_{s \to o} S^2 G(s)$	$\frac{1}{k_a}$	∞	∞	$\frac{1}{k_a}$

FREQUENCY RESPONSE SPECIFICATIONS:

a) Resonant peak $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$; $\xi = 0.707$

b) Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$; $\xi < 0.707$

c) Bandwidth

$$BW = \omega_n\sqrt{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}} = \frac{4}{\xi t_s}\omega_n\sqrt{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}}$$

d) Gain margin $= \frac{1}{|G(j\omega_p)H(j\omega_p)|}$;

Gain margin in $dB = 0 - gain\ in\ dB\ at\ \omega = \omega_p$

ω_p is phase crossover frequency

e) Phase margin=180^o + phase at $\omega = \omega_g$

ω_g is gain crossover frequency

ROOT LOCUS TECHNIQUES

a) Open-loop transfer function $G(s)H(s) = K\ \frac{N(s)}{D(s)}$

b) Magnitude criterion $|G(j\omega)H(j\omega)| = 1$

c) Angle criterion $\angle G(j\omega)H(j\omega) = \pm 180^o$

d) For breakaway and break-in points solve $\frac{dK}{ds} = 0$

e) Centroid of asymptote $\sigma = \frac{\sum open\ loop\ poles - \sum open\ loop\ zeros}{n-m}$

n= number of open loop poles; m= number of open loop zeros

f) Angle of intercept of asymptotes $\beta = \left(\frac{2l+1}{n-m}\right) 180^o$; $l = 0, 1, 2, \dots \dots (n-m-1)$

g) Angle of departure from the complex pole $s = -a + jb$

$\emptyset = 180^o + Angle\ of\ \{(s+a-jb)G(s)H(s)\}$ at $s = -a + jb$

Angle of departure from the complex pole $s = -a - jb$

$\emptyset = 180^o + Angle\ of\ \{(s+a+jb)G(s)H(s)\}$ at $s = -a - jb$

h) Angle of arrival at the complex zero $s = -a + jb$

$\emptyset = 180^o - Angle\ of\ \left\{\frac{G(s)H(s)}{(s+a-jb)}\right\}$ at $s = -a + jb$

Angle of arrival at the complex zero $s = -a - jb$

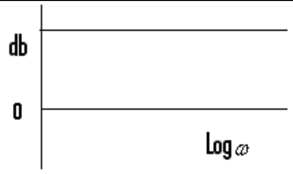
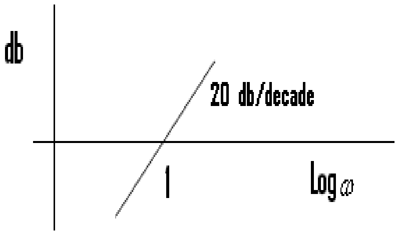
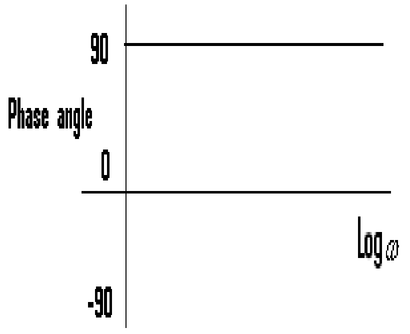
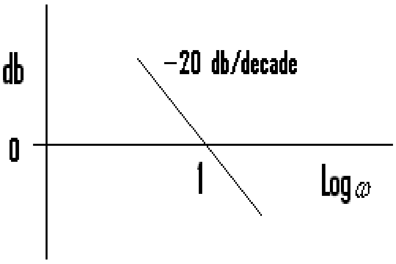
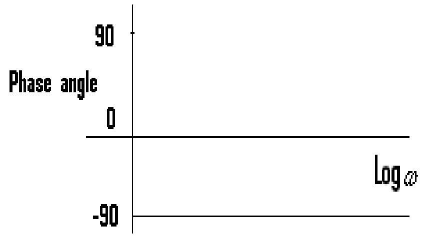
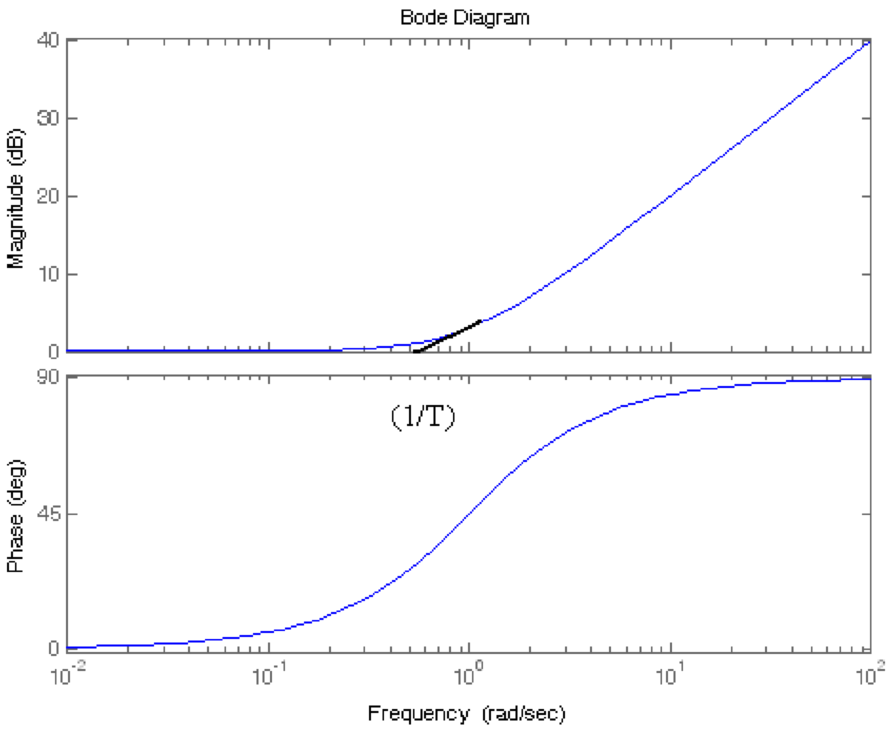
$\emptyset = 180^o - Angle\ of\ \left\{\frac{G(s)H(s)}{(s+a+jb)}\right\}$ at $s = -a - jb$

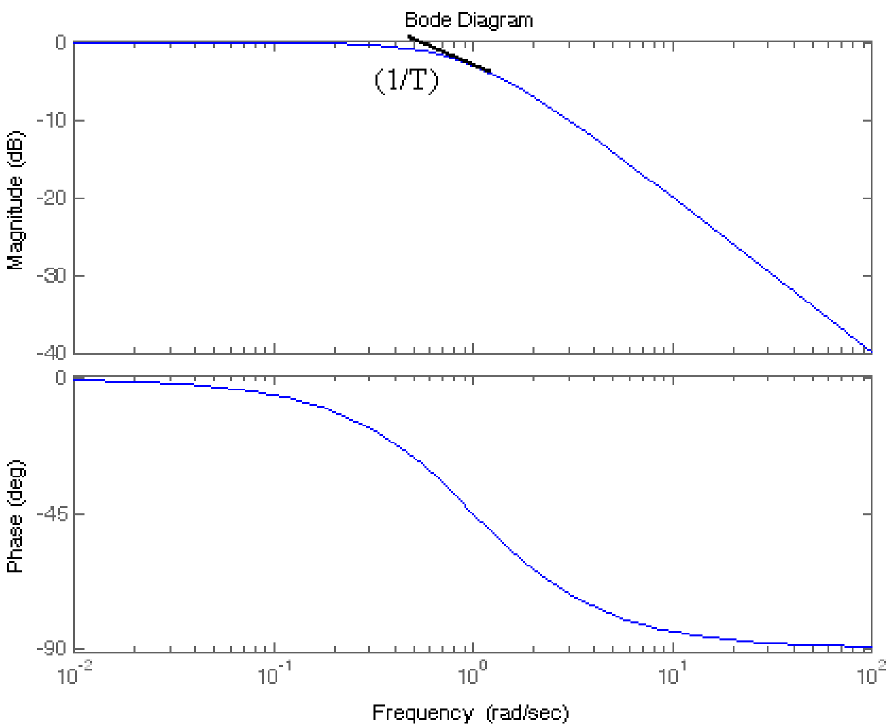
i) Value of K at the point $s=s_o$ on the root locus $K = \left|\frac{D(s_o)}{N(s_o)}\right|$

j) To find the jω axis crossing, use Routh Hurwitz criterion.

Magnitude and phase plot of basic factors.

Bode Plots

Factors	Magnitude plot	Phase plot
Gain (K)		No phase angle contribution
Differential Factor (s) slope 20db/decade		
Integral factor (1/s slope -20db/decade		
First order factors *of the form* *(1+Ts)* *slope 20db/decade* *Corner frequency* *(1/T)*		

First order factors of the form $\frac{1}{1+Ts}$ *Slope - 20db/decad* Corner frequency (1/T)	

STATE-SPACE REPRESENTATION

a) State Equation: $\dot{X} = AX + Br$
b) Output equation: $y = CX + Dr$
c) Eigen Values: Solve $|\lambda I - A| = 0$
d) Transfer function: $T(s) = \frac{Y(s)}{R(s)} = C\,[sI - A]^{-1}\,B + D$
e) State transition matrix: $e^{At} = L^{-1}\,[sI - A]^{-1}$
f) Zero input response: $Y_{ZIR}(s) = C\,[sI - A]^{-1}\,X(0)$
g) Zero state response: $Y_{ZSR}(s) = C\,[sI - A]^{-1}\,B\,R(s)$

Gain margin = 20 log $\frac{1}{|\alpha|}$; where α is the magnitude of the function at $\varphi = -180^0$.
Phase Margin = $\angle GH(j\omega_1) + 180^0$; Where ω_1 is the gain cross over frequency.
Lag compensator $G_c(s)$ = Kc $\frac{s+1/T}{s+1/\alpha T}$; $\alpha > 1$
Lead compensator $G_c(s)$ =Kc $\frac{s+1/T}{s+1/\alpha T}$; $\alpha < 1$
PID Controller $G_c(s)$ $= K_p + \frac{Ki}{s} + K_d s$

Laplace Transforms:

Time domain	**Laplace Domain**
$\delta(t) =$ unit Impulse	1
u(t) = unit step	$\frac{1}{s}$
t = ramp	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$

te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
sin(bt)	$\frac{b}{s^2+b^2}$
cos(bt)	$\frac{s}{s^2+b^2}$
$e^{-at}\sin(bt)$ ċ	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
Sinh(bt)	$\frac{b}{s^2-b^2}$
Cosh(bt)	$\frac{s}{s^2-b^2}$
$\delta(t-kT)$	e^{-kTs}
t^2	$\frac{2}{s^3}$
Initial value Theorem	$\lim_{\vec{t}\to 0} f(t) = \lim_{s\to\infty} sf(s)$
Final value Theorem	$\lim_{\vec{t}\to\infty} f(t) = \lim_{s\to 0} sf(s)$
First differentiation f\`(t)	$sF(s) - f(0)$
Second differential f"(t)	$s^2F(s) - sf(0) - f^1(0)$

ICE 2254 LINEAR INTEGRATED CIRCUITS (E&I)

Normalized Butterworth polynomials

Order of Filter n	Factors of Polynomial
1	$(s+1)$
2	$(s^2+1.414s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.518s+1)(s^2+1.414s+1)(s^2+1.932s+1)$
7	$(s+1)(s^2+0.445s+1)(s^2+1.247s+1)(s^2+1.802s+1)$
8	$(s^2+0.39s+1)(s^2+1.111s+1)(s^2+1.663s+1)(s^2+1.962s+1)$

Where s = Normalized frequency = $j\left(\frac{\omega}{\omega_c}\right)$

ω_c = Cutoff frequency

CMRR = $\rho = \left|\frac{A_d}{A_c}\right|$

A_d = Differential mode gain

A_c = Common mode gain

Expression of the output of an Op-Amp in terms of common mode signals and difference mode signals:

$$v_o = A_d v_d\left(1+\frac{1}{\rho}\frac{v_c}{v_d}\right)$$

where v_d = differential voltage v_c = common mode voltage

Total output offset voltage of an Op-Amp is

$$\Delta V_{\text{ooT}} = E_v = \left(1+\frac{R_F}{R_1}\right)\times\left(\frac{\Delta V_{\text{io}}}{\Delta T}\right)\times\Delta T + R_F\times\left(\frac{\Delta I_{\text{io}}}{\Delta T}\right)\times\Delta T$$

where V_{io} = Input offset voltage

Time period of Op-Amp Astable Multivibrator

$$T = 2\text{RC}\ln\frac{1+\beta}{1-\beta} \quad \text{and approximately, } T = 2.2\text{RC}$$

T = Time period of output voltage

β = Feedback voltage at non-inverting terminal

Pulse width of Op-Amp Monostable Multivibrator

$$t_P = \text{RCln}\left(\frac{1+\frac{V_D}{V_{SAT}}}{1-\beta}\right) \text{ and approximately } t_P = 0.69\text{RC}$$

Pulse width of 555 Timer IC Monostable Multivibrator.

$$t_P = 1.1\text{RC}$$

Time Intervals of 555 Timer IC Astable Multivibrator with duty cycle greater than 50% (Without a Diode across R_2)

$$T_{\text{ON}} = 0.69(R_1 + R_2)C$$
$$T_{\text{OFF}} = 0.69R_2C$$
$$T = T_{\text{ON}} + T_{\text{OFF}} = 0.69(R_1 + 2R)C$$

Time Intervals of 555 Timer IC Astable Multivibrator with any duty cycle (With a Diode across R_2)

$$T_{\text{ON}} = 0.69R_1C$$
$$T_{\text{OFF}} = 0.69R_2C$$
$$T = T_{\text{ON}} + T_{\text{OFF}} = 0.69(R_1 + R_2)C$$

Free running frequency of VCO IC 566

$$f_0 = \frac{2(+V - V_C)}{R_1C_1(+V)}$$

f_0 = Output frequency

+V = Supply voltage

V_C = Modulating input voltage

Free running frequency of PLL IC 565 $f_{\text{OUT}} = \frac{1.2}{4R_1C_1}$ Hz

Lock Range of PLL IC 565 $f_L = \pm\frac{8f_{\text{OUT}}}{V}$ Hz

Capture range of PLL IC 565 $f_C = \pm\left[\frac{f_L}{2\pi\times C_2\times 3.6\times 10^3}\right]^{\frac{1}{2}}$Hz

Inverting all pass filter $\Phi = -2tan^{-1}(2\pi fRC)$

Φ = Phase difference of output voltage w.r.t input voltage