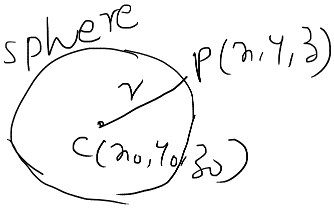
Lecture - 4, Analytical solid Geometry

Sphere

A sphere is the locus of a point in space which moves such that its distance from a fixed point is a constant. The fixed point is called the centre and the constant distance is the radius of the sphere.

Let $C(x_0, y_0, z_0)$ be the centre and 'r' be the radius of the sphere S. Consider a point $P(x, y, z)$ on the sphere. Then the equation of the sphere is



$$l(CP) = r$$

$$[l(CP)]^2 = r^2$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \quad \text{--- (1)}$$

If the centre is the origin, then the equation is

$$x^2 + y^2 + z^2 = r^2$$

General Equation

Expanding (1),

$$x^2 + y^2 + z^2 - 2xx_0 - 2yy_0 - 2zz_0 + x_0^2 + y_0^2 + z_0^2 = r^2$$

This equation is of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{--- (2)}$$

$$u = -x_0, \quad V = -y_0, \quad w = -z_0, \quad d = x_0^2 + y_0^2 + z_0^2 - r^2$$

Since (x_0, y_0, z_0) is the centre and r is the radius of the sphere (2) represents the equation to the sphere whose centre is $(-u, -v, -w)$ and $r = \sqrt{u^2 + v^2 + w^2 - d}$

Thus the general equation of the sphere is such that

1) its of second degree in x, y, z
2) the coefficients of x^2, y^2, z^2 are equal
3) No terms containing xy, yz and zx

Problems

1. Find the eqn of the sphere whose centre is $(3, -1, 4)$ and which passes through $(1, -2, 0)$

Ans → $C(3, -1, 4)$ $\quad P(1, -2, 0)$

$$l(CP) = r$$

$$[l(CP)]^2 = r^2$$

$$(x-3)^2 + (y+1)^2 + (z-4)^2 = \sqrt{(3-1)^2 + (-1+2)^2 + (4-0)^2}$$

$$= \sqrt{4+1+16}$$

$$\underline{(x-3)^2 + (y+1)^2 + (z-4)^2 = 21}$$

2) obtain the equation of the sphere which passes through the points $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ and which has its centre on the plane $x+y+z=6$

Ans → Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ be the required equation of the sphere —— ①

Since $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ lies on (1)

$$1 + 2u + d = 0 \quad \text{—— ②}$$

$$1 + 2v + d = 0 \quad \text{—— ③}$$

$$1 + 2w + d = 0 \quad \text{—— ④}$$

Solving ②, ③, ④

$$u = v = w = \frac{-1}{2}(d+1)$$

Since centre $(-u, -v, -w)$ lies on the plane $x + y + z = 6$,

$$-u - v - w = 6$$

$$+\frac{1}{2}(d+1) + \frac{1}{2}(d+1) + \frac{1}{2}(d+1) = 6$$

$$\frac{3}{2}(d+1) = 6$$

$$d + 1 = 4 \qquad \underline{\underline{d = 3}}$$

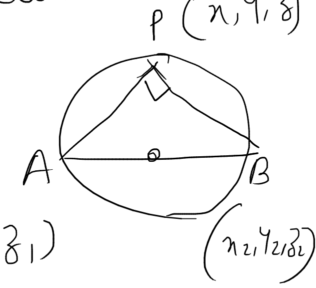
$$u = -\frac{1}{2}(d+1) = -2$$

$$u = v = w = -2$$

Sub in (1)

$$\underline{\underline{x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0}}$$

3) Find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter.



$AP \perp BP$

Direction ratios of AP and BP are

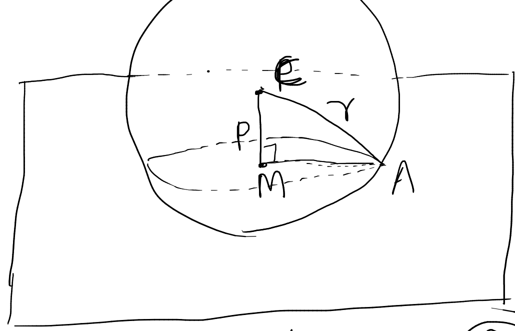
$(x - x_1, y - y_1, z - z_1)$ & $(x - x_2, y - y_2, z - z_2)$

$$\therefore (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

eg - Find the eqn of the sphere which has $(2, 1, -3)$ and $(1, -2, 4)$ as the extremities of a diameter. Also find its centre and radius.

Intersection of a plane and a sphere

Section of a sphere by a plane is a circle and the section of a sphere through its centre is called great circle

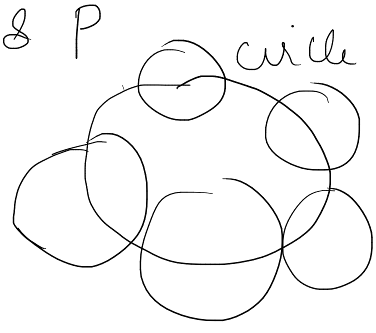


The equations $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — (S)

and $ax+by+cz+d=0$ (plane) taken together represents a circle having centre M and radius $MA = \sqrt{r^2-p^2}$

Family of spheres through a circle of intersection

S & P



The equation of a sphere that passes through the circle of intersection of the sphere and the plane is

$S + kU = 0$, S - sphere, U - plane

Problems

1) Find the centre, radius and area of the circle

$x^2+y^2+z^2-2y-4z=11$, $x+2y+2z=15$

Ans centre of the sphere $(-u,-v,-w)$

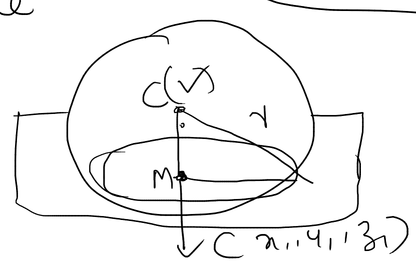
$C\ (0,1,2)$

$r = \sqrt{u^2+v^2+w^2-d} = \sqrt{0+1^2+2^2+11} = 4$

$2ux = 0$
$u = 0$
$2vy = -2y$
$v = -1$
$2wz = -4z$
$w = -2$

Let $M(x_1,y_1,z_1)$ be the centre of the circle

The DR's of CM are



$(x_1-0,\ y_1-1,\ z_1-2)$

CM $\perp$ plane

(line $\perp^r$ to the plane is $\parallel^l$ to normal)

plane eqn is $x+2y+2z=15$

DR's of the plane $(1,2,2)$

Since CM $\perp$ plane the DR's are proportional

$$\frac{x_1}{1} = \frac{y_1-1}{2} = \frac{z_1-2}{2} = t$$

$M(x_1, y_1, z_1) = (t,\ 2t+1,\ 2t+2)$

Since M lies in the given plane $x_1+2y_1+2z_1=15$

$$t+2(2t+1)+2(2t+2)=15$$

$$t=1$$

$M(t,\ 2t+1,\ 2t+2) = (1,\ 3,\ 4)$

$C(0,1,2)$ $\quad M(1,3,4)$ $\quad r = 4$

radius of the circle $= \sqrt{r^2-p^2}$ $\quad p = l(CM)$

$= \sqrt{1+4+4} = 3$

$= \sqrt{4^2-3^2} = \sqrt{7}$

Area of the circle $= \pi r^2$

$= \pi(\sqrt{7})^2$

$= 7\pi$ sq. units

2) Find the eqn of the sphere which passes through the circle $x^2+y^2+z^2-2x-3y+4z-8=0$ (1), $x-2y+z-8=0$ (2) and has its centre on the plane $4x-5y-z-3=0$

Ans Required eqn of the sphere (1) + k(2) = 0

$$x^2+y^2+z^2-2x-3y+4z-8+k(x-2y+z-8)=0 \quad \text{—(A)}$$

Centre of (A) → $x^2+y^2+z^2+x[k-2]+y[-3-2k]+z[4+k]-8k-8=0$

$(-u,-v,-w)$ $= \left(\frac{(2-k)}{2}, \frac{3+2k}{2}, -\left(\frac{4+k}{2}\right)\right)$

Since the centre lies on the plane

$$4x-5y-z-3=0,$$

we get $4\left(1-\frac{k}{2}\right)-5\left(\frac{3}{2}+k\right)+\left(2+\frac{k}{2}\right)-3=0$

$$4[2-k]-5(3+2k)+4+k-6=0$$

$$-13k-9=0 \implies k=\frac{-9}{13}$$

Sub in (A) we get the required sphere

3) Find the eqn of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$ as a great circle.

Ans → Eqn of sphere (required) is $S+kU=0$

$$x^2+y^2+z^2+10y-4z-8+k(x+y+z-3)=0 \quad \text{—(1)}$$

$$x^2+y^2+z^2+kx+(10+k)y-(4-k)z-(8+3k)=0$$

The given circle is a great circle if the centre of this sphere and centre of the circle coincide.

This is possible only if the centre of the required sphere lies on the plane.

centre $\left(-k/2, -\left(\frac{10+k}{2}\right), \frac{4-k}{2}\right)$

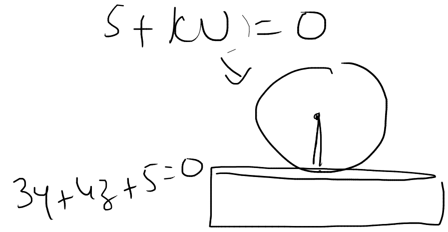
This centre lies on the plane $x+y+z-3=0$

$$-\frac{k}{2} - \frac{(10+k)}{2} + \frac{4-k}{2} - 3 = 0, \quad k = -4$$

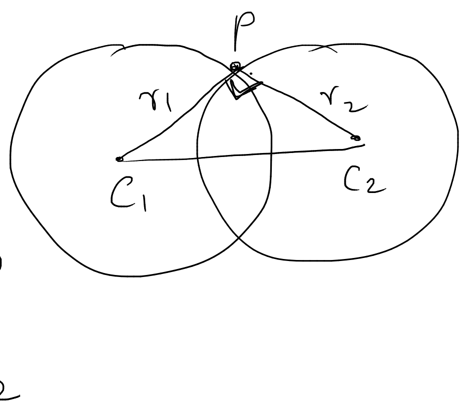
Sub in (1), we get the required sphere

4) S.T $\to$ the plane $x+2y-z=3$ cuts the sphere $x^2+y^2+z^2-x-z-2=0$ in a circle of radius unity. Find also the eqn of the sphere which has the circle as a great circle.

5) Find the sphere passing through the circle $x^2+y^2+z^2-6x-2z+5=0$, $y=0$ and touching the plane $3y+4z+5=0$



Orthogonal spheres



Two spheres are said to be orthogonal to each other if the tangent planes at the point of intersection are at right angles

$$d(C_1C_2)^2 = r_1^2 + r_2^2$$

1) Show that the condition for the spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \&$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

to cut orthogonally is $2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2$.

Proof $C_1 = (-u_1, -v_1, -w_1)$ $\quad C_2 = (-u_2, -v_2, -w_2)$

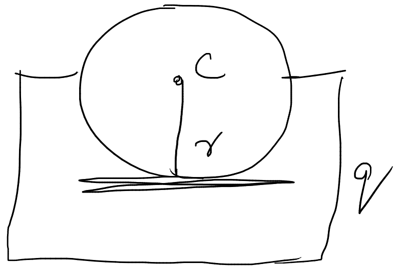
$r_1 = \sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}$ $\quad r_2 = \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}$

they will cut orthogonally if $(C_1C_2)^2 = r_1^2 + r_2^2$

$$(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2$$

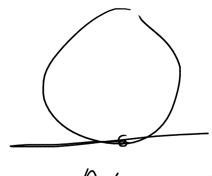
$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

Tangent plane to the sphere



A plane q touches the spheres if the $\perp^r$ distance of the centre C of S from q = radius of the sphere.
Then q is called the tangent plane to S

Eqn of the tangent plane q to S at the point $A(x_1, y_1, z_1)$ is



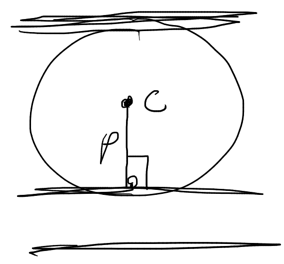
$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

Problems

1) Show that the sphere $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally and find the pt of contact

2) Find the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$

Ans $\rightarrow$ Any plane parallel to a given plane is $2x+2y-z+k=0$ —— (1)



This plane is a tangent plane to the given sphere if the $\perp^r$ distance p of the centre C of S = radius

$C(2,-1,3)$ $\quad r = \sqrt{2^2+(-1)^2+3^2-5} = \sqrt{9} = 3$

$$p = \left| \frac{2(2)+2(-1)+3(-1)+k}{\sqrt{2^2+2^2+(-1)^2}} \right| = 3$$

$$\left| \frac{k-1}{3} \right| = 3 \quad \Rightarrow |k-1| = 9$$

$$\Rightarrow k-1 = \pm 9$$

$$\Rightarrow k = 1 \pm 9$$

$$k = 10, -8$$

Sub $k=10$ & $k=-8$, we get the required planes.