Exercise

Evaluate

1) $\int_C \frac{\sin 2z}{(z+3)(z+1)^2} dz$ C is the rectangle with vertices at $3 \pm i$, $-2 \pm i$

2) $\int_C \frac{1}{z^4-1} dz$ where (i) $C: |z+1|=1$, (ii) $C: |z+3|=1$
(iii) $C: |z-i|=1$, (iv) $C: |z-1|=1$

3) $\int_C \frac{z^3-2z+1}{(z-i)^2} dz$ where $C: |z|=2$

4) $\int_C \frac{e^z}{(z+1)^4(z-2)} dz$ where $C: |z-1|=3$

5) $\int_C \frac{z^2-1}{z^2+1} dz$ where (i) $C: |z|=\frac{1}{2}$
(ii) $C: |z+i|=1$
(iii) $C: |z-i|=1$
(iv) $C: |z-2i|=2$

6) Evaluate $\int_C \frac{z^3-z}{(z-2)^3}\,dz$ where

(i) $C: |z|=3$ (ii) $C: |z-2|=1$, (iii) $|z|=1$

$f(z) = z^3 - z$

$\frac{z^3-z}{(z-2)^3}$ is not analytic at $z=2$

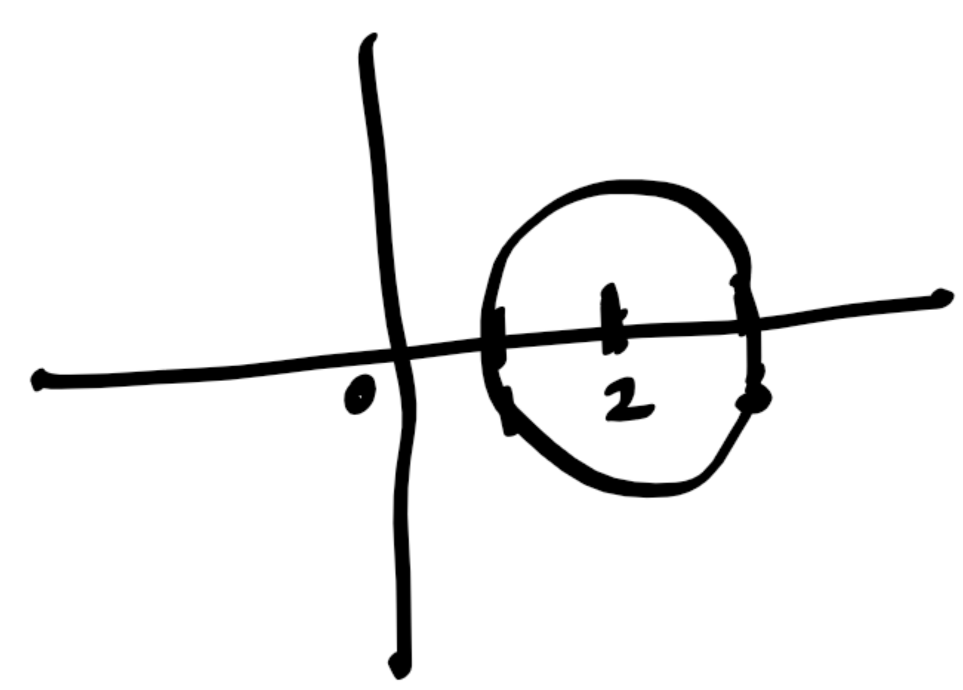
(i) $C: |z|=3$

$z=2$ lies inside the circle $|z|=3$

$$\int_C \frac{z^3-z}{(z-2)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(2) = \pi i\,(6z)_{z=2}$$

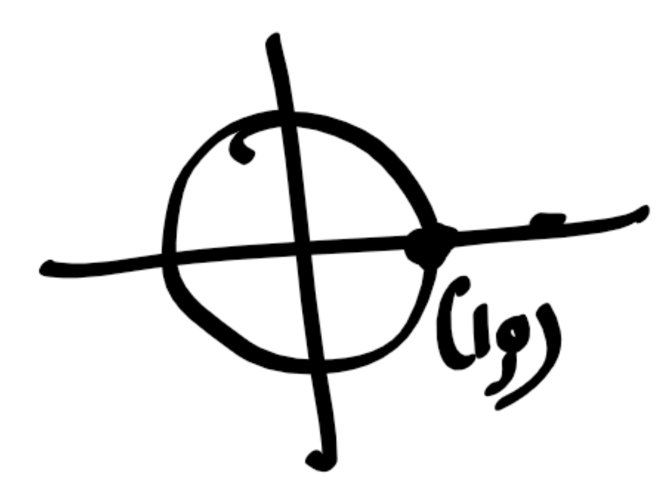
$$= \underline{\underline{12\pi i}}$$

(ii) $C: |z-2|=1$



$$\int_C \frac{z^3-z}{(z-2)^3}\,dz = 12\pi i$$

(iii) $C: |z|=1$

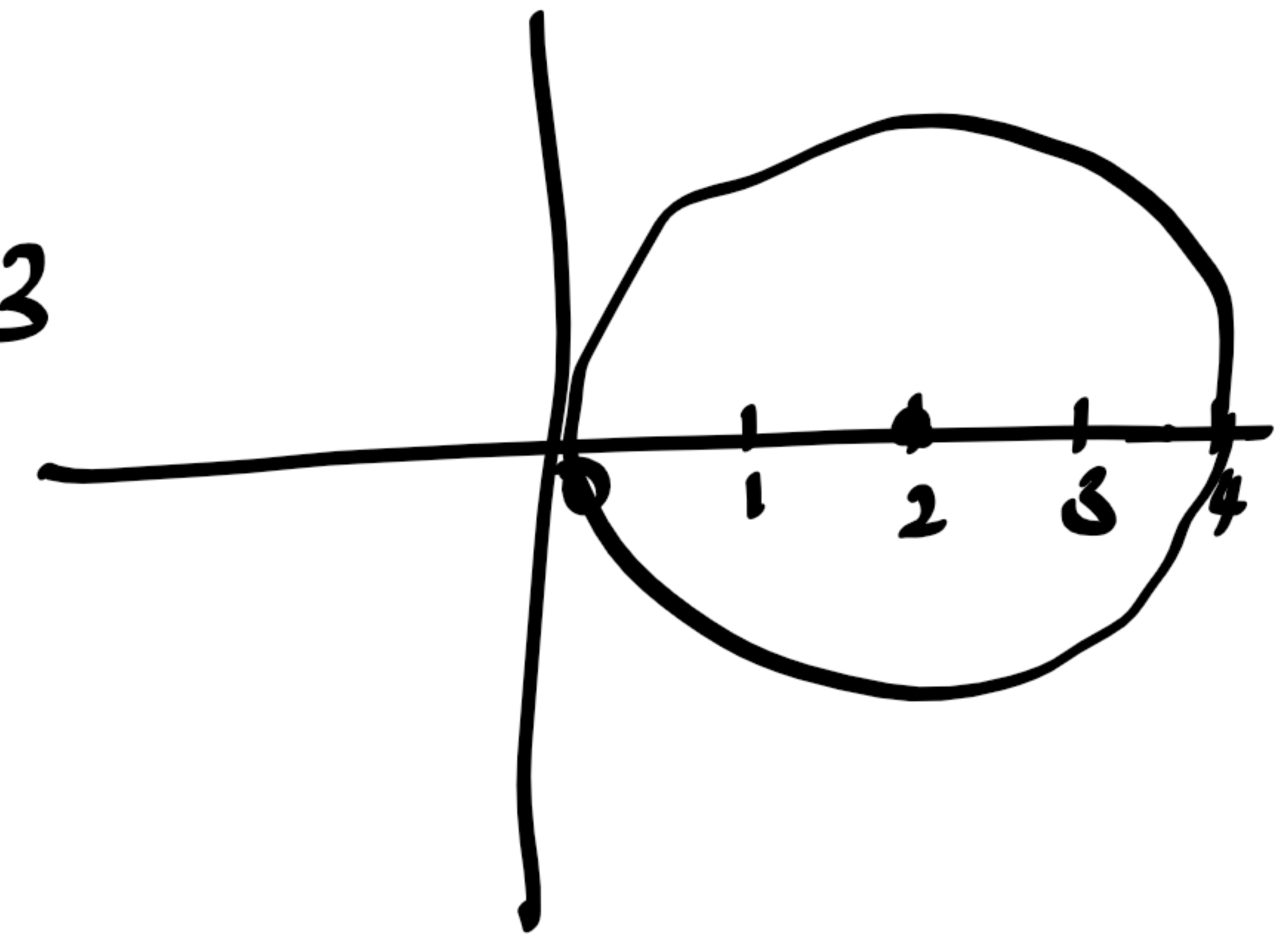


$z=2$ lies outside the circle $|z|=1$

$\therefore$ By CIT

$$\int_C \frac{z^3-z}{(z-2)^3}\,dz = 0$$

(ii) $C: |z-2|=2$



$z = \pm i$ lie outside and $z = \pm 3$ lie inside the circle.

$$\therefore \int_C \frac{z}{(z^2+1)(z^2-9)} dz = -\frac{1}{60} 2\pi i f(-3) + \frac{1}{60} 2\pi i f(3)$$

$$= \frac{1}{10}\pi i + \frac{1}{10}\pi i = \underline{\underline{\frac{\pi i}{5}}}$$

$$\int_C \frac{z}{z+i} dz = 0, \quad \int_C \frac{z}{z-i} dz = 0 \text{ by CIT}$$

5) Evaluate $\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz, \quad C: |z|=1$

$$f(z) = \sin^2 z$$

$$\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} f^{(2)}(\pi/6) = \pi i \left(2\cos 2z\right)_{z=\pi/6}$$

$$= \underline{\underline{\pi i}}$$

$$f'(z) = 2\sin z \cos z = \sin 2z$$

$$f''(z) = 2\cos 2z$$

4. Evaluate: $\displaystyle\int_C \frac{z}{(z^2+1)\ (z^2-9)}\, dz$

where (i) $C: |z| = 2$, (ii) $C: |z-2| = 2$.

$f(z) = z.$

$$\frac{1}{(z^2+1)(z^2-9)} = \frac{1}{(z+i)(z-i)(z+3)(z-3)} = \frac{A}{(z+i)} + \frac{B}{z-i} + \frac{C}{z+3} + \frac{D}{z-3}$$

$$A\ (z-i)(z^2-9) + B(z+i)(z^2-9) + C(z^2+1)(z-3) + D(z^2+1)(z+3) = 1$$

$z = i \Rightarrow 2i \times (-10) B = 1 \Rightarrow B = -\frac{1}{20i}$

$z = -i \Rightarrow -2i A (-10) = 1 \Rightarrow A = \frac{1}{20i}$

$z = 3 \Rightarrow 60 D = 1 \Rightarrow D = 1/60$

$z = -3 \Rightarrow -60 C = 1 \Rightarrow C = -1/60$

$$\int_C \frac{z}{(z^2+1)(z^2-9)}\, dz = \frac{1}{20i}\int_C \frac{z}{z+i}\, dz - \frac{1}{20i}\int_C \frac{z}{z-i}\, dz - \frac{1}{60}\int_C \frac{z}{z+3}\, dz + \frac{1}{60}\int_C \frac{z}{z-3}\, dz$$

(i) $C: |z| = 2$

$z = \pm i$ lie inside the circle $|z|=2$ and $z = \pm 3$ lie outside $|z| = 2$

$$\therefore \int_C \frac{z}{(z^2+1)(z^2-9)}\, dz = \frac{1}{\cancel{20i}\ 10}\ \cancel{2\pi i}\, f(i) - \frac{1}{20i}\, 2\pi i\, f(-i)$$

$$= \frac{\pi i}{10} + \frac{\pi i}{10} = \frac{\pi i}{5}$$

$$\frac{1}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i}$$

$$A(z-2i) + B(z+2i) = 1$$

$$z = 2i \Rightarrow 4i\,B = 1 \Rightarrow B = \frac{1}{4i}$$

$$z = -2i \Rightarrow -4i\,A = 1 \Rightarrow A = -\frac{1}{4i}$$

$$\int_C \frac{z^2}{z^2+4}\,dz = \int_C \frac{z^2}{4i(z-2i)}\,dz - \int_C \frac{z^2}{4i(z+2i)}\,dz$$

$$= \frac{1}{4i}\left[2\pi i f(2i) - 2\pi i f(-2i)\right]$$

$$= \frac{1}{4i}\left[2\pi i \times 4i^2 - 2\pi i\,(-2i)^2\right]$$

$$= \frac{1}{4i}\left[-8\pi i + 8\pi i\right] = \underline{0}$$

2) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where $C: |z| = 3$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A(z-2) + B(z-1) = 1$$

$$z = 1 \Rightarrow -A = 1 \Rightarrow A = -1$$

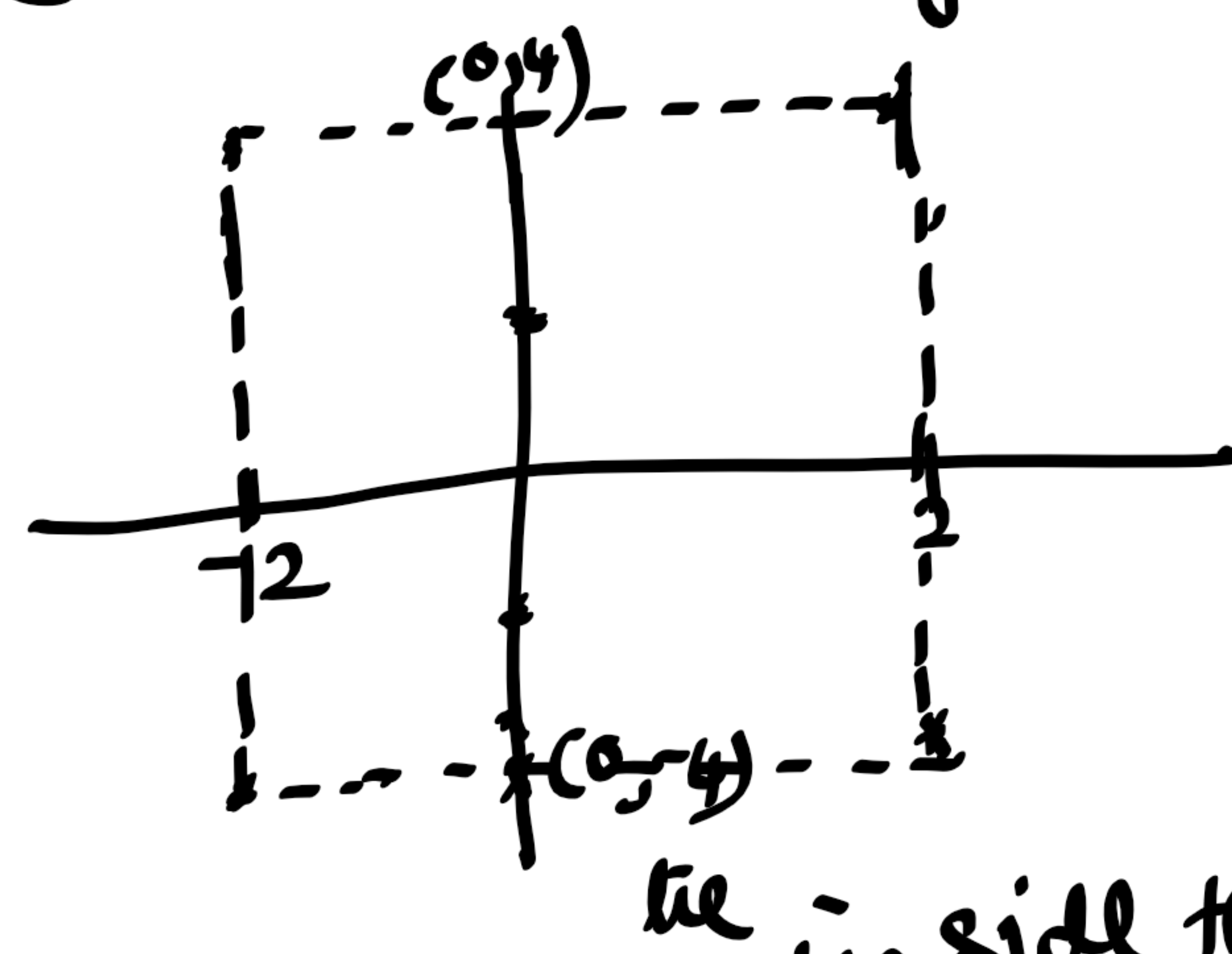
$$z = 2 \Rightarrow B = 1$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i - (-2\pi i) = \underline{\underline{4\pi i}}$$

3) Evaluate: $\int_C \frac{z^2}{z^2+4} dz$ where C is the rectangle with vertices $\pm 2 \pm 4i$.



$$f(z) = z^2$$

$\frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$ is not analytic at $z = \pm 2i$ lie inside the rectangle

Examples

1) Evaluate $\int_C \frac{z^2+1}{z(2z+1)}\,dz$ where $C:|z|=1$.

$\frac{z^2+1}{z(2z+1)}$ is not analytic at $z=0$ and $-1/2$ which lie inside the circle $|z|=1$

$$f(z)=z^2+1$$

$$\frac{1}{z(2z+1)} = \frac{A}{z}+\frac{B}{2z+1}$$

$$A(2z+1)+Bz=1$$

$$z=0 \Rightarrow A=1$$

$$z=-\frac{1}{2} \Rightarrow -\frac{1}{2}B=1 \Rightarrow B=-2$$

$$\frac{1}{z(2z+1)} = \frac{1}{z}-\frac{2}{2z+1} = \frac{1}{z}-\frac{1}{z+1/2}.$$

$$\therefore \int_C \frac{z^2+1}{z(2z+1)}\,dz = \int_C \frac{z^2+1}{z}\,dz - \int_C \frac{z^2+1}{z+1/2}\,dz$$

$$= 2\pi i\, f(0) - 2\pi i\, f\left(-\tfrac{1}{2}\right) \Big\}\text{ By using CIF}$$

$$= 2\pi i\times 1 - 2\pi i \times 5/4$$

$$= 2\pi i - \frac{5\pi i}{2} = \frac{-\pi i}{2}$$

$$\therefore \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

Derivative of analytic functions

If a function $f(z)$ is analytic in a simply connected domain D, then its derivative is also analytic at any point $z=a$ of D and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$\therefore \int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

In general, $$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Cauchy's integral formula

Let $f(z)$ be analytic in a simply connected domain D. Let C be any simple closed curve in D enclosing any point z_0 in D

Then $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

Pf:- Consider the function $\frac{f(z)}{z-z_0}$ which is analytic within C at all points except at $z=z_0$. With z_0 as centre and radius r draw a small circle C_1 lying entirely within C.

Now $\frac{f(z)}{z-z_0}$ being analytic in the region enclosed by C and C_1 we have by Cauchy's integral theorem for multiply connected domains

$C_1: |z-z_0|=r$

$z-z_0=re^{i\theta}$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta.$$

As $r \to 0$ $\int_{C_1} \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0)\,d\theta = 2\pi i f(z_0)$