Classify the following equations

1) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$

$A = 1, \quad B = 2, \quad C = 4.$

$AC - B^2 = 4 - 4 = 0$

$B^2 - 4AC = 16 - 16 = 0 \quad (\text{Here } B = 4)$

$\therefore$ Eqn: is parabolic

2) $y^2 u_{xx} - 2y u_{xy} - u_y = 8y.$

$A = y^2 \quad B = -2y \quad C = 0$

$B^2 - 4AC = 4y^2 - 0 = 4y^2 > 0$

Eqn: is hyperbolic

3) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

$A = x+1, \quad B = -2(x+2), \quad C = x+3$

$B^2 - 4AC = 4(x+2)^2 - 4(x+1)(x+3)$

$= 4 > 0$

Eqn: is hyperbolic

Finite difference method for solving P.D.E

$$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0 \rightarrow (1)$$

Equation (1) is said to be parabolic if

$$AC - B^2 = 0$$

Ex:- $\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$ → One dimensional heat Eqn:

$A = c, \ B = 0, \ C = 0 \quad \therefore AC - B^2 = 0$

2). Eqn: (1) is said to be hyperbolic if

$$AC - B^2 < 0$$

Ex:- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ → One dimensional wave equation.

$A = c^2, \ B = 0, \ C = -1$

$AC - B^2 = -c^2 < 0$

(3) Eqn: (1) is said to elliptic if $AC - B^2 > 0$.

Ex:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

$A = 1, \ B = 0, \ C = 1$

$AC - B^2 = 1 > 0$

If the General PDE is

$$A u_{xx} + B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0$$

Eqn: is parabolic if $B^2 - 4AC = 0$, hyperbolic if $B^2 - 4AC > 0$, Eliptic if $B^2 - 4AC < 0$

Solve: 2, 3 and 4

$$y_1 = y\left(\tfrac{1}{3}\right) = -0.55$$

$$y_2 = y\left(\tfrac{2}{3}\right) = -0.88$$

$$y_3 = y(1) = -1$$

5. Solve: $y'' + xy' - 2y = 2(x+1)$, $y(0)=0$, $y'(1)=0$, $h = 1/3$

$x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$

$y_0 = 0$, $y_1 =$ $y_2 =$ $y_3' = 0$

$y_3 = ?$

$$y_i'' + x_i y_i' - 2y_i = 2(x_i+1)$$

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - 2y_i = 2(x_i+1)$$

$$9(y_{i+1} - 2y_i + y_{i-1}) + \frac{3x_i}{2}(y_{i+1} - y_{i-1}) - 2y_i = 2(x_i+1)$$

$$(9 + 1.5x_i)y_{i+1} - 20y_i + (9 - 1.5x_i)y_{i-1} = 2(x_i+1) \longrightarrow (1)$$

Put $i=1$ in (1)

$$9.5y_2 - 20y_1 + 8.5y_0 = 2\left(\tfrac{1}{3}+1\right)$$

$$-20y_1 + 9.5y_2 = \frac{8}{3} \longrightarrow (2)$$

Put $i=2$ in (1)

$$10y_3 - 20y_2 + 8y_1 = 2\left(\tfrac{2}{3}+1\right)$$

$$8y_1 - 20y_2 + 10y_3 = \frac{10}{3} \longrightarrow (3)$$

Put $i=3$ in (1)

$$10.5y_4 - 20y_3 + 7.5y_2 = 4 \Rightarrow 18y_2 - 20y_3 = 4 \longrightarrow (4)$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i' = \frac{3}{2}(y_{i+1} - y_{i-1})$$

$$y_3' = \frac{3}{2}(y_4 - y_2)$$

$$y_3' = 0$$

$$\Rightarrow y_4 - y_2 = 0$$

$$\Rightarrow y_4 = y_2.$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = x_i y_i$$

$$4(y_{i+1} - 2y_i + y_{i-1}) - x_i y_i = 0$$

$$4y_{i+1} - (8 + x_i)y_i + 4y_{i-1} = 0 \longrightarrow 1$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i' = y_{i+1} - y_{i-1}$$

$$y_0' = y_1 - y_{-1}$$

$$y_{-1} = y_1 - y_0'$$

$$y_{-1} = y_1 - (y_0 - 1)$$

$$= y_1 - y_0 + 1$$

Put $i=0$ in 1

$$4y_1 - (8 + x_0)y_0 + 4y_{-1} = 0$$

$$4y_1 - 8y_0 + 4(y_1 - y_0 + 1) = 0$$

$$8y_1 - 12y_0 + 4 = 0$$

$$3y_0 - 2y_1 = 1 \longrightarrow 2$$

Put $i=1$ in 1

$$4y_2 - (8 + x_1)y_1 + 4y_0 = 0$$

$$4 \times 1 - 8.5y_1 + 4y_0 = 0$$

$$4y_0 - 8.5y_1 = -4 \longrightarrow 3$$

$$y_0 = y(0) = 0.94 \quad ; \quad y_1 = y(0.5) = 0.91$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} = y_{i+1} - y_{i-1}$$
$$(\because h = 1/2)$$

Put $i=0$

$$y_0' = y_1 - y_{-1}$$
$$y_{-1} = y_1 - y_0' = y_1 - y_0 \quad (\because y_0 = y_0')$$

$$\therefore \quad 5y_1 - 10y_0 + 3(y_1 - y_0) = 0$$
$$8y_1 - 13y_0 = 0 \longrightarrow (2)$$

$i=1$ in (1)

$$(4+x_1)y_2 - 2y_1 + (4-x_1)y_0 = 0$$
$$5.5 \times 5 - 2y_1 + 2.5y_0 = 0$$
$$2.5y_0 - 2y_1 = -27.5 \longrightarrow (3)$$

solve (2) and (3) we get

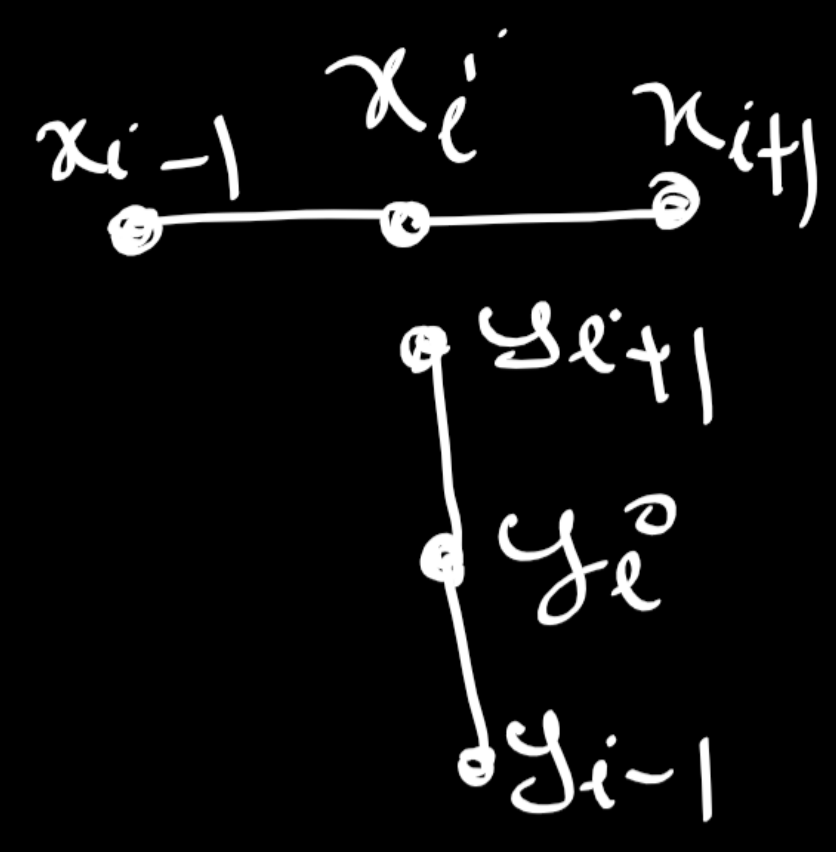
$$y_0 = y(1) = 36.66 \quad ; \quad y_1 = y(1.5) = 59.58$$

4. Solve: $y'' = xy$, $y(0) - y'(0) = 1$, $y(1) = 1$, $h = 1/2$.

$$y_i'' = x_i y_i$$

$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$y_0 - y_0' = 1$	y_1	$y_2 = 1$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$



$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Example 3

Solve $y'' + xy' - 2y = 0, \; y(1) = y'(1), \; y(2) = 5,$

$h = 0.5$

$x_0 = 1 \quad x_1 = 1.5, \; x_2 = 2$

$y_0 = y_0' \quad y_1 \qquad y_2 = 5$

$$y_i'' + x_i y_i' - 2y_i = 0$$

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - 2y_i = 0$$

$$4(y_{i+1} - 2y_i + y_{i-1}) + x_i(y_{i+1} - y_{i-1}) - 2y_i = 0$$

$$(4 + x_i)y_{i+1} - 10y_i + (4 - x_i)y_{i-1} = 0 \longrightarrow (1)$$

$i = 0$ in (1)

$$(4 + x_0)y_1 - 10y_0 + (4 - x_0)y_{-1} = 0$$

$$5y_1 - 10y_0 + 3y_{-1} = 0$$