State Feedback Controller

- Pole placement method.

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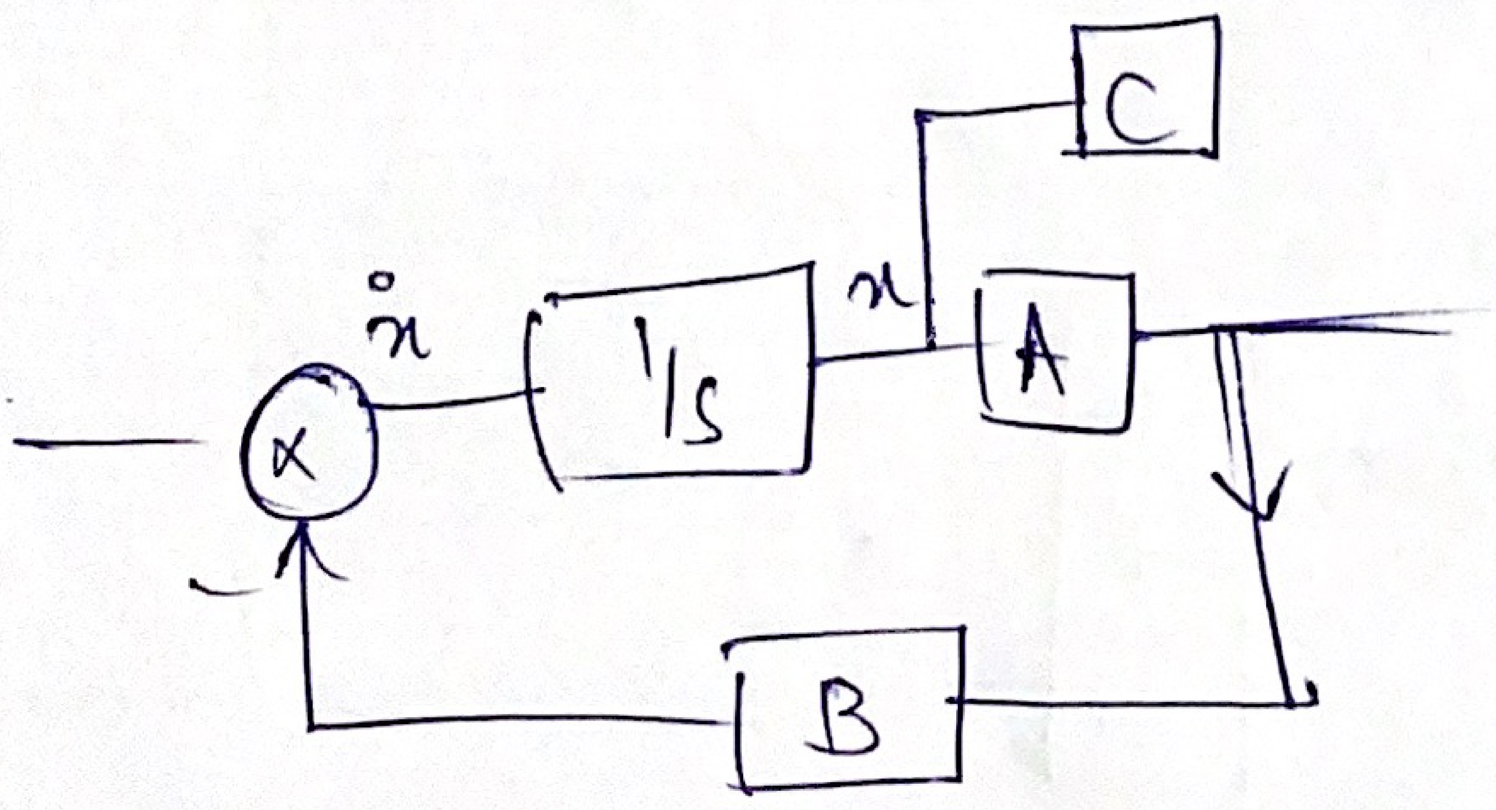
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

we shall choose $u = -kx$.

then $\dot{x} = (A - Bk)x \rightarrow (I)$

$y = (C - Dk)x$.



The solution of (1) is given by

$$x(t) = e^{(A-BK)t} \cdot x(0).$$

$x(0)$ is initial state.

Stability are determined by eigen values of $A-BK$.

Determination of matrix K using transformation matrix T.

Method (1)

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

Step 1:

Check for controllable.

Step 2:

Finding the values of $a_1, a_2, \ldots a_n$ from the char. eq of matrix A.

$$|sI - A| = s^n + a_1 s^{n-1} + \ldots a_{n-1} s + a_n$$

Step 3:

Determine matrix T.

if controllable canonical form, $T = I$

if not then $T = MW$

$M = [B : AB : \ldots : A^{n-1}B] \rightarrow$ controllability matrix.

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Step4:

Determine char eq with desired closed loop poles.

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

Step 5:

$$K = \left[\alpha_n - a_n \,\vdots\, \alpha_{n-1} - a_{n-1} \,\vdots\, \cdots \,\vdots\, \alpha_2 - a_2 \,\vdots\, \alpha_1 - a_1\right]T^{-1}$$

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}_{3\times3}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1}$

Choose desired loop poles at $s = -2+j4$ $s = -2-j4$ $s = -10$

Determine state feedback gain matrix K.

Step 1: Check controllability.

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix}$$

Sol. $|sI - A|$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix}$$

$$= s(s^2+6s+5) + 1(1)$$

$$= s^3 + 6s^2 + 5s + 1$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 1 \\ -6 \\ 31 \end{bmatrix}$$

controllability matrix = $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{vmatrix} = -1$

Completely state controllable.

$|sI - A| = s^3 + 6s^2 + 5s + 1$

$s^3 + a_1 s^2 + a_2 s + a_3 = 0$

$a_1 = 6 \quad a_2 = 5 \quad a_3 = 1$

Here A = controllable canonical form. So, $T = I$.

desired char. polynomial.

$\Rightarrow (s - (-2+j4))(s - (-2-j4))(s - (-10))$

$\Rightarrow (s + 2 - j4)(s + 2 + j4)(s + 10)$

$(s^2 + 2s + \cancel{sj4} + 2s + 4 + \cancel{2j4} - \cancel{sj4} - \cancel{2j4} - 16j^2)(s+10)$

$(s^2 + 4s + 20)(s + 10)$

$s^3 + 4s^2 + 20s + 10s^2 + 40s + 200$

$s^3 + 14s^2 + 60s + 200 = 0.$

$\alpha_1 = 14 \quad \alpha_2 = 60 \quad \alpha_3 = 200$

then

$K = [200 - 1 \; : \; 60 - 5 \; : \; 14 - 6] . T^{-1}$

$= [199 \; : \; 55 \; : \; 8]$

control signal, $u = -Kx = -[199 \quad 55 \quad 8]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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Method ②
Determination of matrix K using direct substitution method:

System order ≤ 3

if $n \geq 3$, then

$$K = [k_1 \quad k_2 \quad k_3]$$

Sub K into desired characteristic polynomial $|SI - A + BK|$ and equate it to $(s-\mu_1)(s-\mu_2)(s-\mu_3)$.

$$|SI - A + BK| = (s-\mu_1)(s-\mu_2)(s-\mu_3)$$

equate the coeff. of like powers of s on both sides

Method ③
Determination of matrix K using Ackermann's formula

Suppose $\dot{x} = Ax + Bu$

Control signal $u = -Kx$.

desired poles are at $s = \mu_1, s = \mu_2, \dots s = \mu_n$.

then, $K = [0 \; 0 \dots 0 \; 1][B : AB : A^2B \dots : A^{n-1}B]^{-1} \phi(A)$

where,

$$\phi(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \dots + \alpha_{n-1} A + \alpha_n I$$

$$|sI - A + BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \underset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}} \right|$$

$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{vmatrix}$$

$$= s\left(s(s+6+k_3) + (5+k_2) \right) + 1\left((1+k_1) \right)$$

$$= s^3 + (6+k_3)s^2 + (5+k_2)s + 1 + k_1$$

$$\Rightarrow \; = s^3 + 14s^2 + 60s + 200$$ desired poles char. polynomial.

Compare

$6 + k_3 = 14 \quad \rightarrow k_3 = 8$

$5 + k_2 = 60 \quad k_2 = 55$

$1 + k_1 = 200 \quad k_1 = 199.$

General eq.

$k_3 = \alpha_1 - a_1$

$k_2 = \alpha_2 - a_2$

$k_1 = \alpha_3 - a_3.$

ex:. Method (3)

$$K = [0 \quad 0 \quad 1][B : AB : A^2B]^{-1} \phi(A)$$

$$\phi(A) = A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3 I$$

$$= \begin{bmatrix} -1 & -5 & -6 \\ 6 & 29 & 31 \\ -31 & -149 & -157 \end{bmatrix} + 14\begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix} + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

$$+ 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$K = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$K = [0 \quad 0 \quad 1]\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$K = [199 \quad 55 \quad 8]$$

when there is an input.

$$\dot{x} = Ax + Bu$$

then $u = -Kx + r$

then, $\dot{x} = Ax + B(-Kx + r)$

$$\dot{x} = (A - BK)x + Br$$

State Observer.

mathematical model of Observer

observer gain.

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x})$$

$$= (A - K_e C)\tilde{x} + Bu + K_e y$$

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Non-linear

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Common Nonlinear System behaviour

$$\dot{x} = -x + x^2$$

linearised system I $\quad \dot{x} = -x \quad \Rightarrow \quad x(t) = x_0 e^{-t}$

non-linear system II $\quad \dot{x} = -x + x^2 \quad \Rightarrow \quad x(t) = \dfrac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$

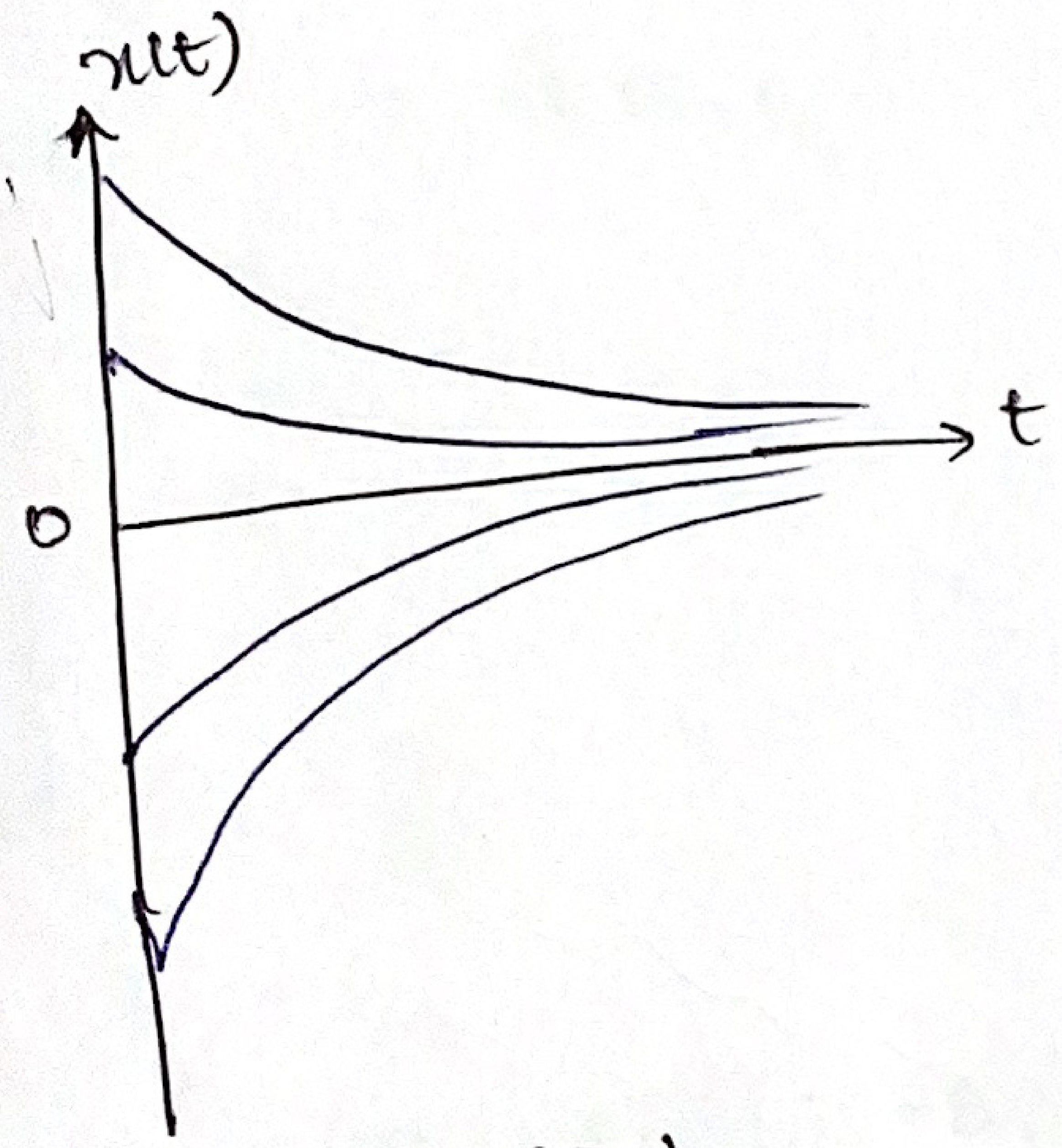
eq. points for I

$-x = 0 \quad \Rightarrow \quad x = 0.$

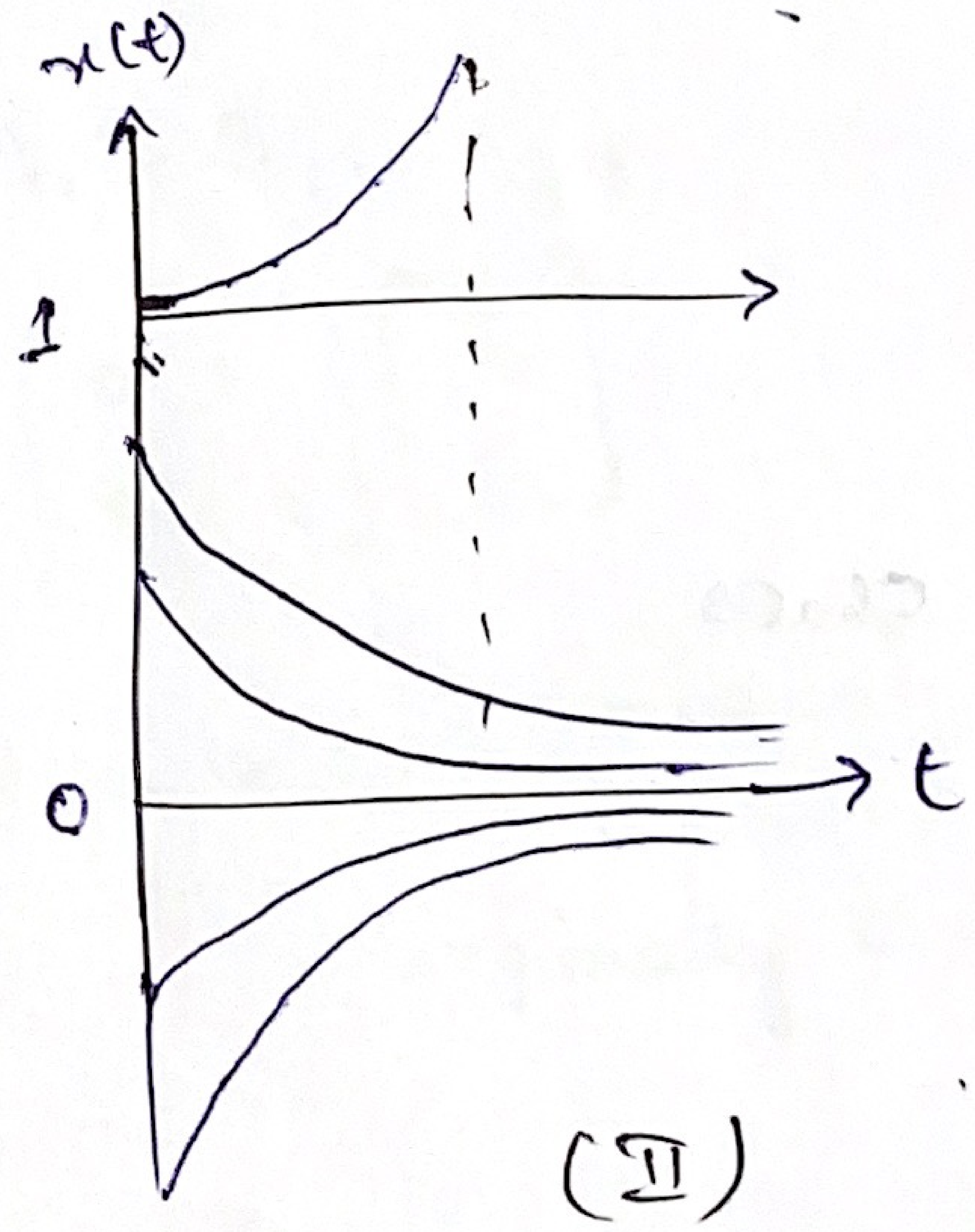
for II

$-x + x^2 = 0$

$\Rightarrow \quad x = 0, 1 \quad \rightarrow$ eq. points.



(I)



(II)

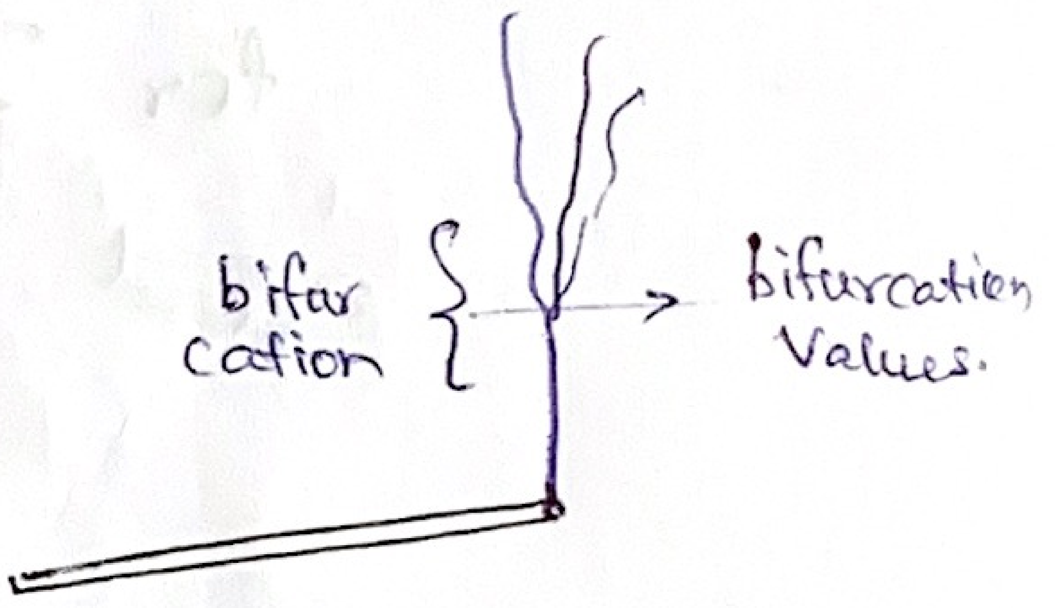
Second order nonlinear diff. eqⁿ

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0 \longrightarrow \text{Vander Pol equation.}$$

Limit cycles

Self excited oscillations.

Bifurcation.



Chaos

Jump response.

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$

$x_1 = \theta$ $\qquad \dot{\theta} = \omega$ $\qquad$ eq. points

$x_2 = \dot{\theta}$ $\qquad \dot{\omega} = -\frac{g}{l}\sin\theta$ $\qquad (0,0)\ (\pi,0)$

for $l = 1m.$

$\dot{\theta} = \omega$
$\dot{\omega} = -9.81\sin\theta.$

At eq, all derivatives are zero

$0 = \omega_d$ $\qquad \omega_d = 0$

$0 = -9.81\sin\theta_d.$ $\qquad \theta_d = \begin{cases} 0 \text{ rad} \\ \pi \text{ rad} \end{cases}$

Consider a small deviation around eq. poind $\theta_d = 0$

$\theta = \theta_d + \xi_1 = \xi_1$

$\omega = \omega_d + \xi_2 = \xi_2.$

for $\theta_d = 0.$

$\dot{\xi}_1 = \xi_2$ $\qquad$ Using Mclauren series.

$\dot{\xi}_2 = -9.81\sin\xi_1$ $\qquad \sin\xi_1 \approx \xi_1.$

$\dot{\xi}_1 = \xi_2$
$\dot{\xi}_2 = -9.81\xi_1$

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -9.81 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$$

eigen values $\quad 0+3.13i$
$\qquad 0-3.13i$

marginally stable.

for $\theta_d = \pi$

$\dot{\xi}_1 = \xi_2$
$\dot{\xi}_2 = -9.81\sin(\pi+\xi_1)$
$\dot{\xi}_2 = 9.81\xi_1$

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 9.81 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$$

eigen values $\quad 3.13$
$\qquad -3.13$

Unstable.

Q. $2\ddot{x} + 18\dot{x} + 128000\dfrac{x^2}{x+2} = 0.03f$ find eq. points for $f = 80$.

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Phase plane Analysis

autonomous $\simeq$ LTI system

2nd order autonomous system.

$$\dot{x}_1 = f_1(x_1, x_2)$$

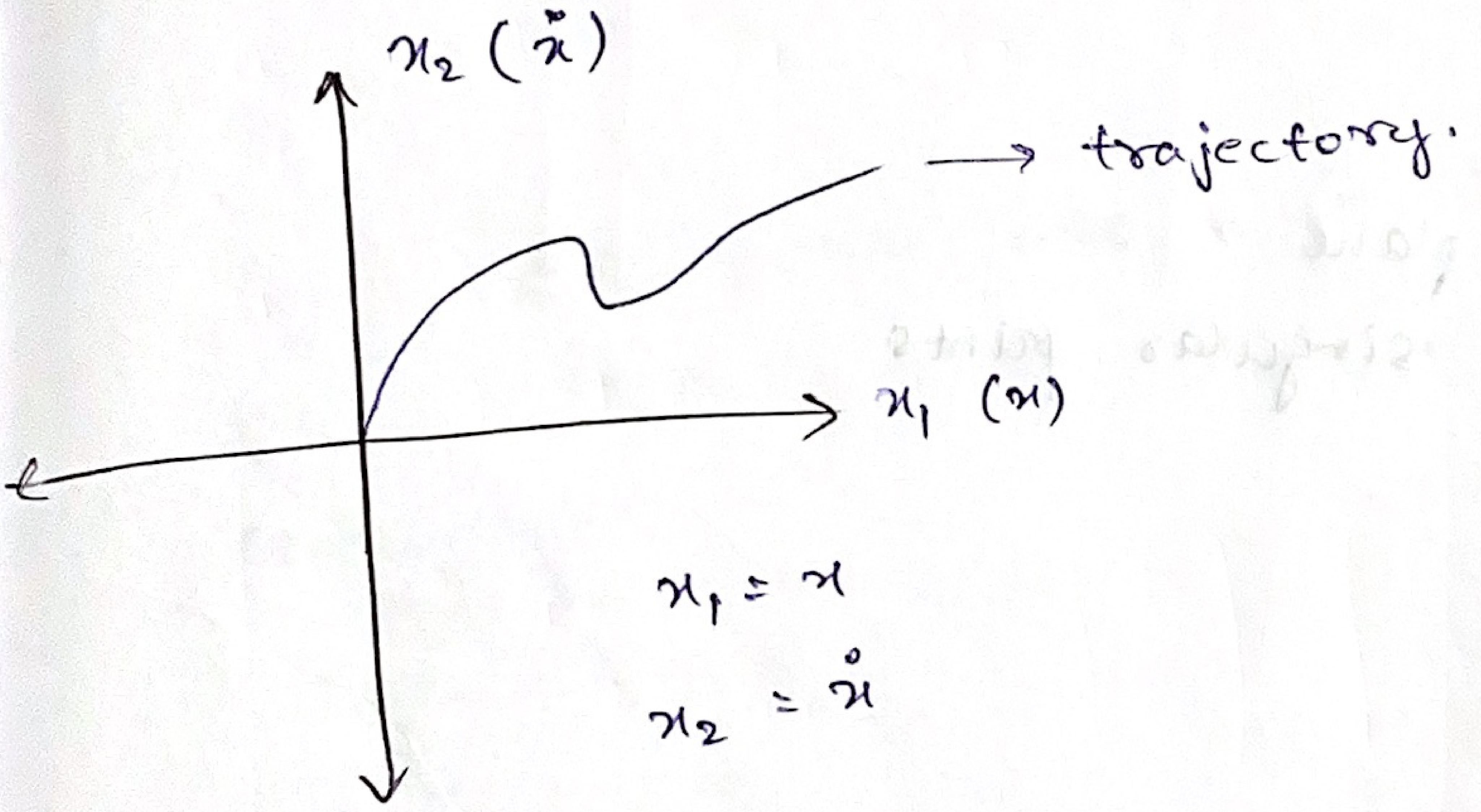
$$\dot{x}_2 = f_2(x_1, x_2)$$

x_1, x_2 are states

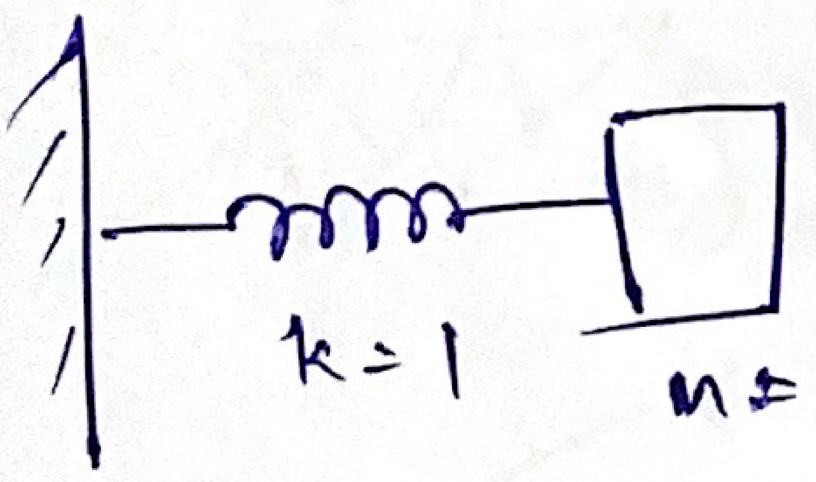
f_1, f_2 are non-linear functions.

State space of this system is a plane having x_1 & x_2 as coordinates.

It is called as phase plane.



Mass spring system.



$\ddot{x} + x = 0.$ $\quad x(0) = x_0$

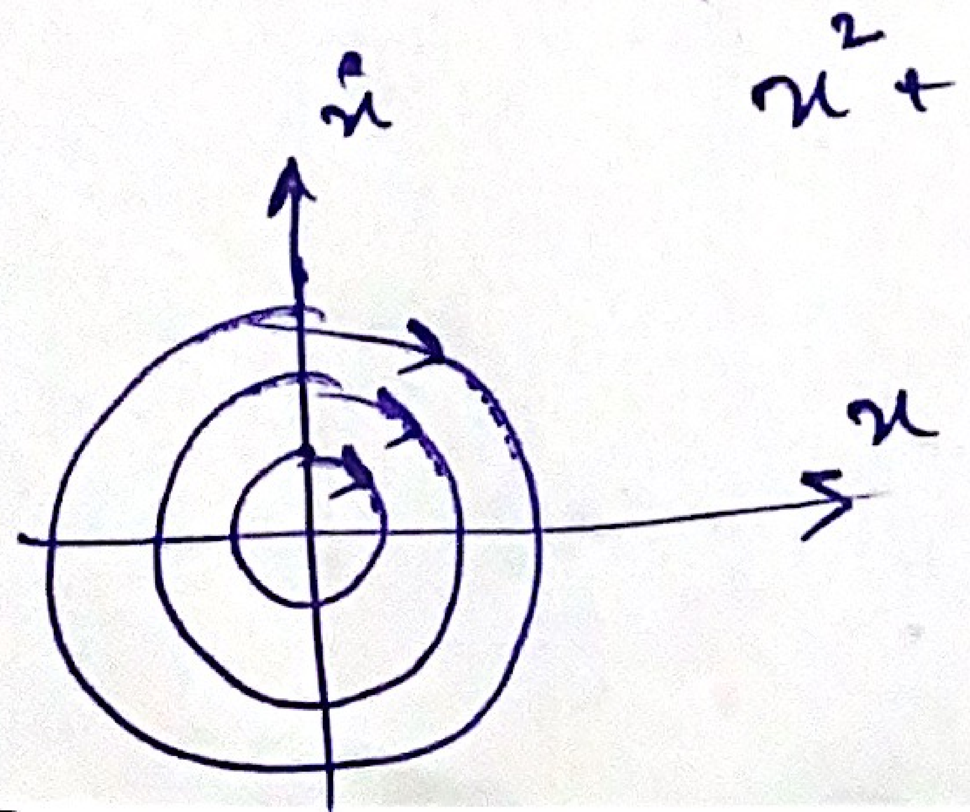
$\dot{x}(0) = 0.$

Solution $x(t) = x_0 \cos t$

$\dot{x}(t) = -x_0 \sin t$

$x^2 + \dot{x}^2 = x_0^2$

This is a circle.



2nd order system,

$$\ddot{x} + f(x, \dot{x}) = 0.$$

$x = f_1(x_1, x_2)$

$\dot{x} = f_2(x_1, x_2)$

$$\dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -f(x_1, x_2) = f_2(x_1, x_2)$$

for eq. point. /

$$f_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) = 0.$$

eq. points on phase plane are called singular points ex:. $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$

$x_1 = x$

$x_2 = \dot{x}$

6

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = -x^2 - 3x - 0.6\dot{x}$

$\dot{x}_2 = -x_1^2 - 3x_1 - 0.6x_2$

for eq points

$\dot{x}_1 = 0 \Rightarrow x_2 = 0.$

$\dot{x}_2 = 0 \Rightarrow -x_1^2 - 3x_1 - 0.6x_2 = 0$

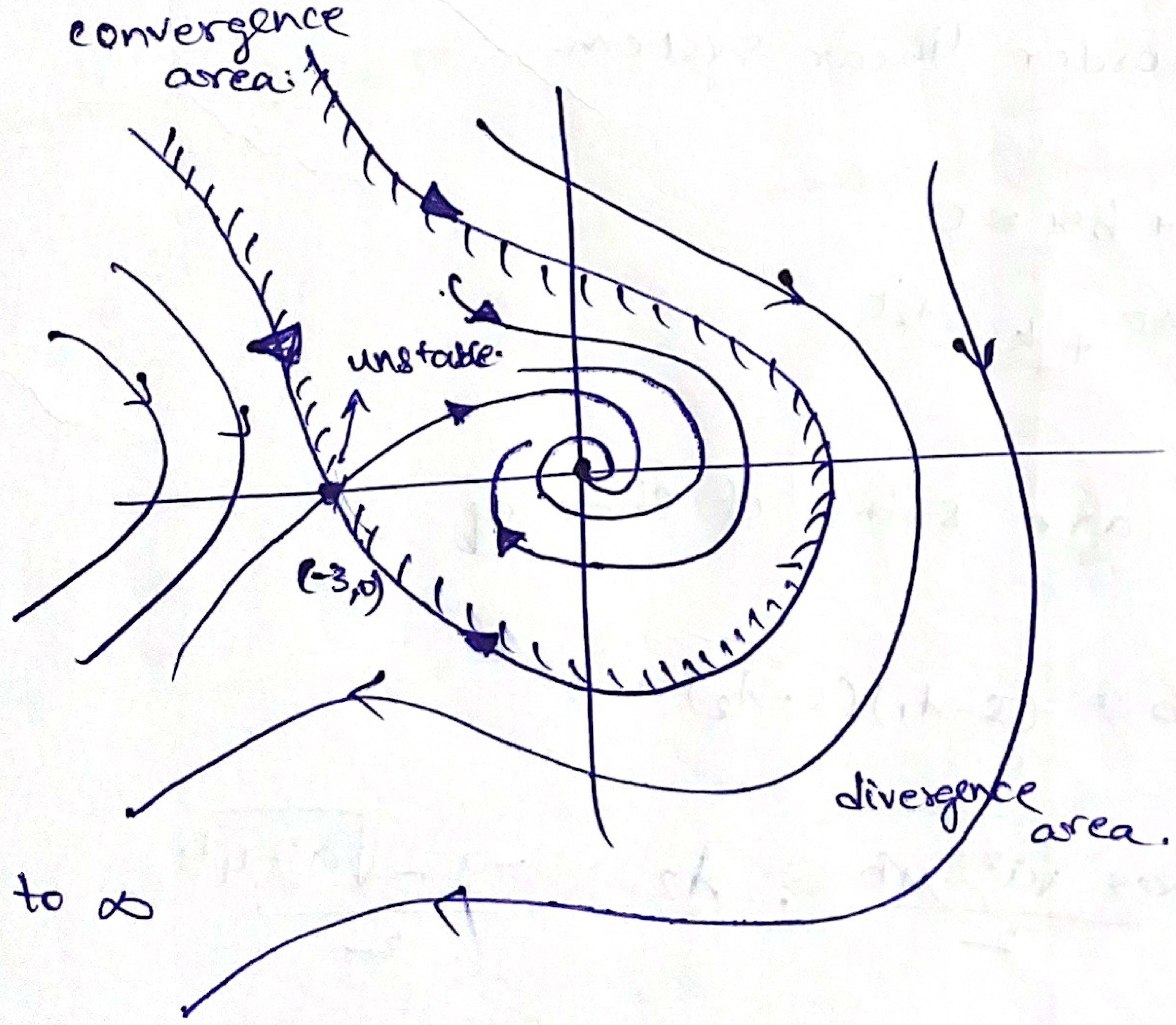
$x_1^2 + 3x_1 = 0$

$x_1(x_1 + 3) = 0$

$\Rightarrow x_1 = 0, -3.$

eq. points are $(0,0)$

& $(-3,0)$



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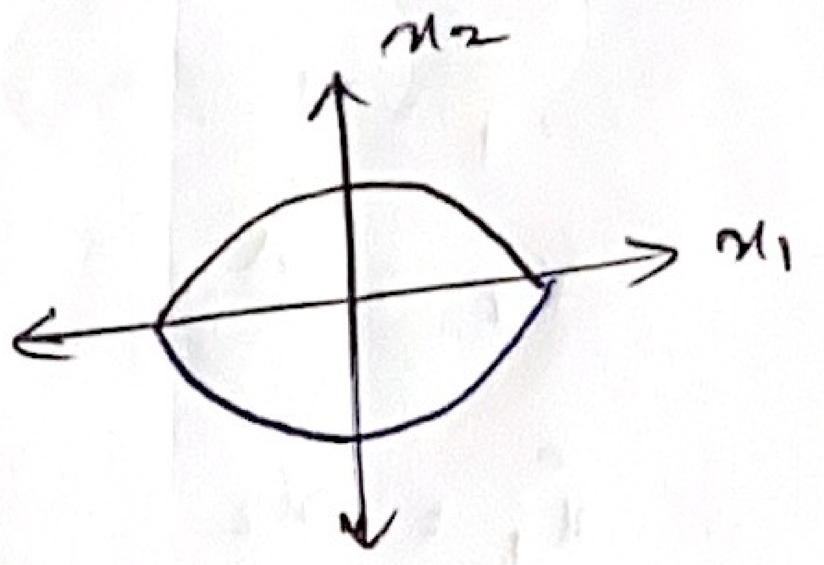
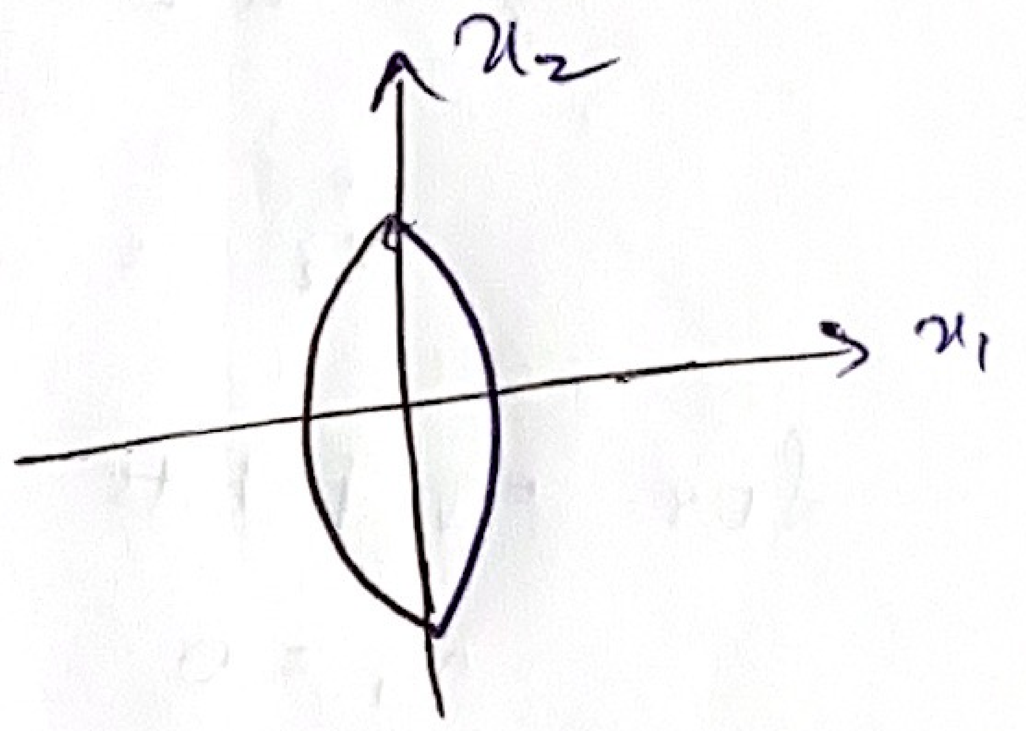
Symmetry in phase plane portraits

Symmetry about the x_1 axis



$f(x_1, x_2) = f(x_1, -x_2)$

Symmetry about the x_2 axis



$f(x_1, x_2) = -f(-x_1, x_2)$

about x_1 & x_2 axes.

$f(x_1, x_2) = f(x_1, -x_2) = -f(-x_1, x_2)$

only one quarter of it has to be explicitly considered.

about origin

$f(x_1, x_2) = -f(-x_1, -x_2)$

Consider 2nd order linear system.

$\ddot{x} + a\dot{x} + bx = 0$

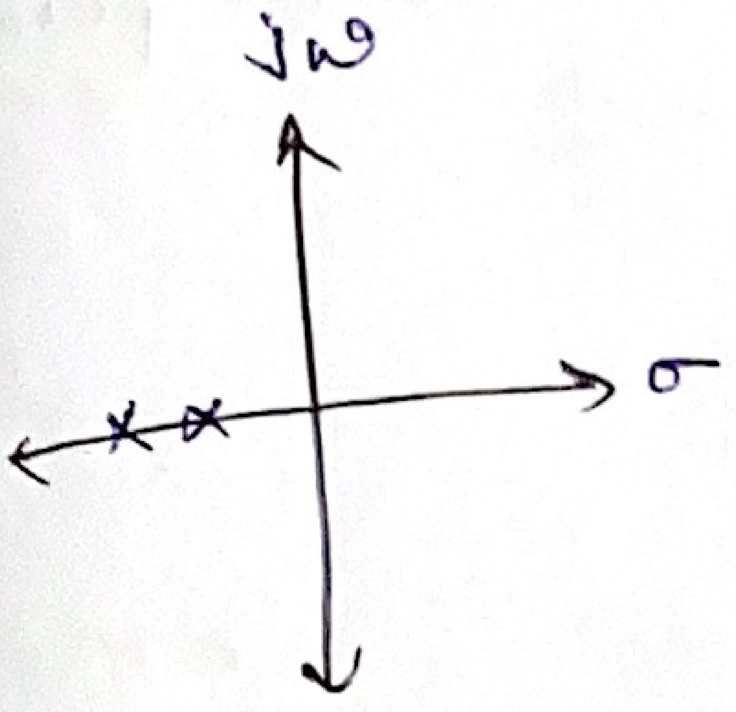
$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$

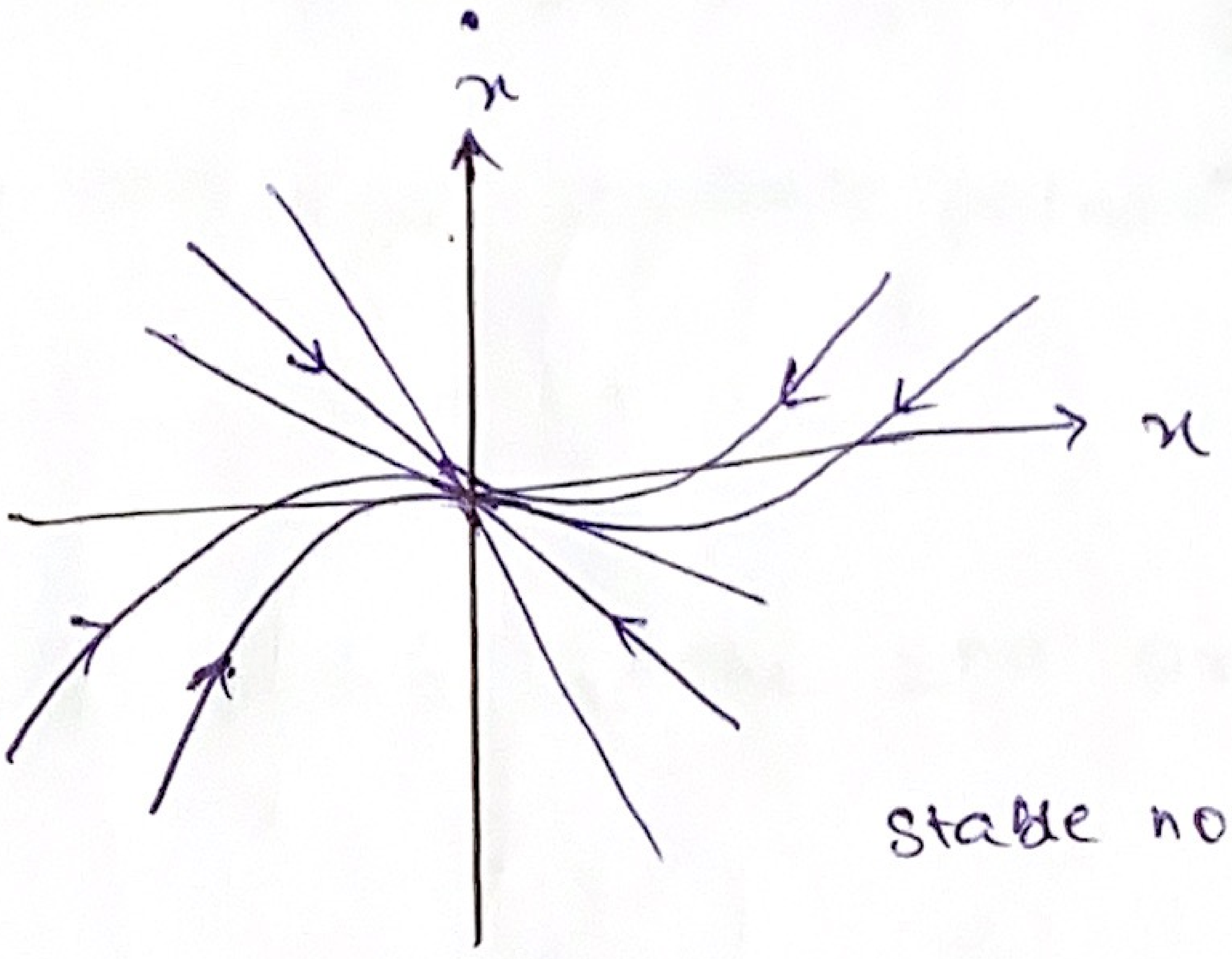
λ_1, λ_2 are soln's of char eq.

$s^2 + as + b = (s-\lambda_1)(s-\lambda_2) = 0$

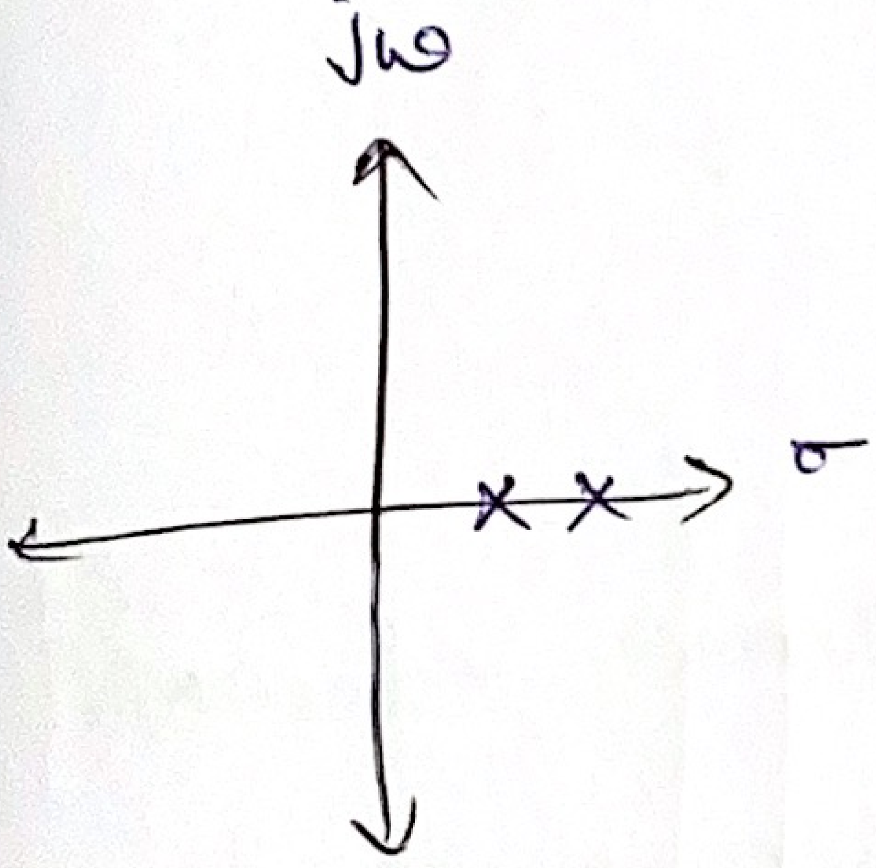
$\Rightarrow \lambda_1 = \dfrac{-a + \sqrt{a^2-4b}}{2}, \quad \lambda_2 = \dfrac{-a - \sqrt{a^2-4b}}{2}$

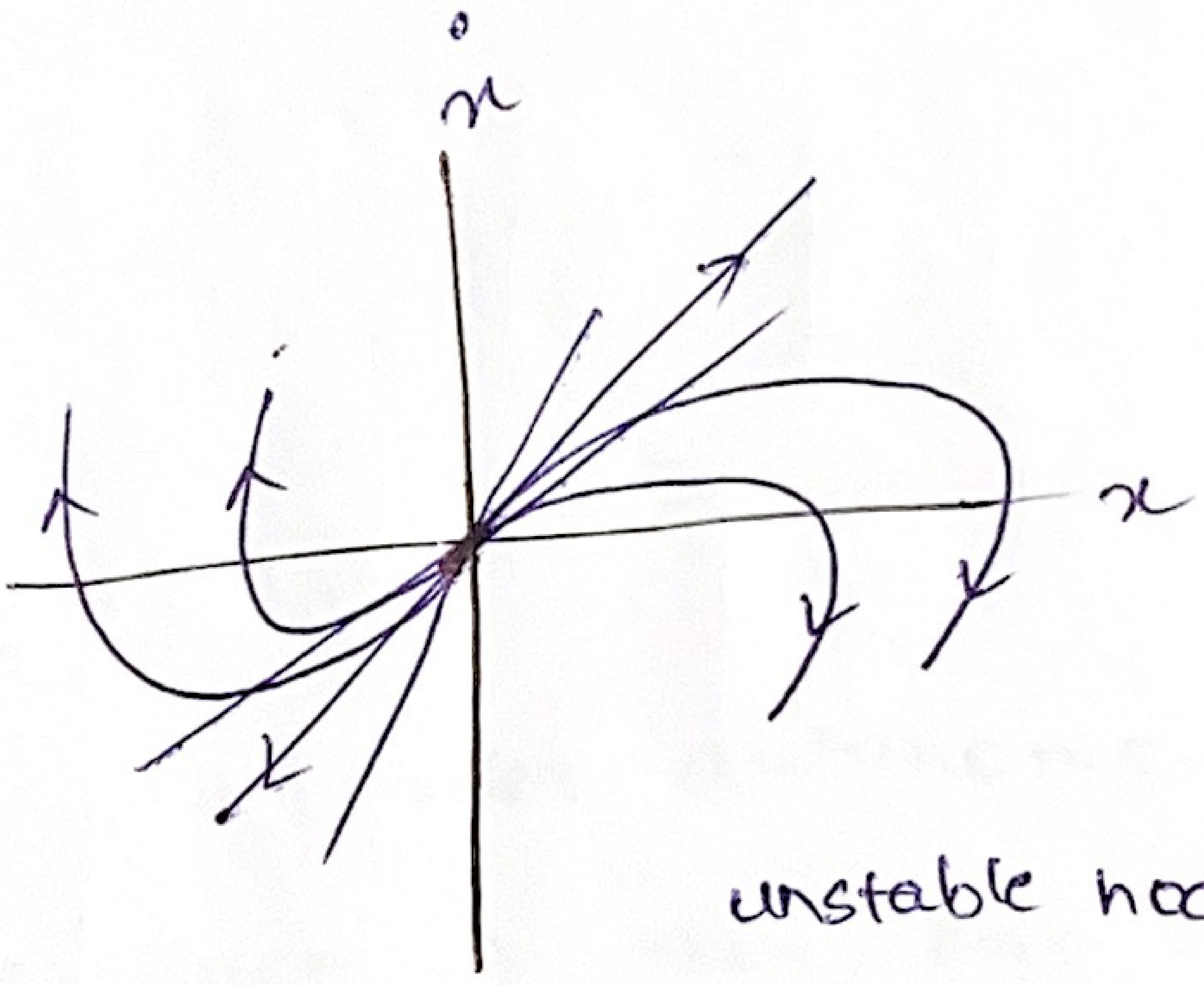
Phase potraits of linear systems.



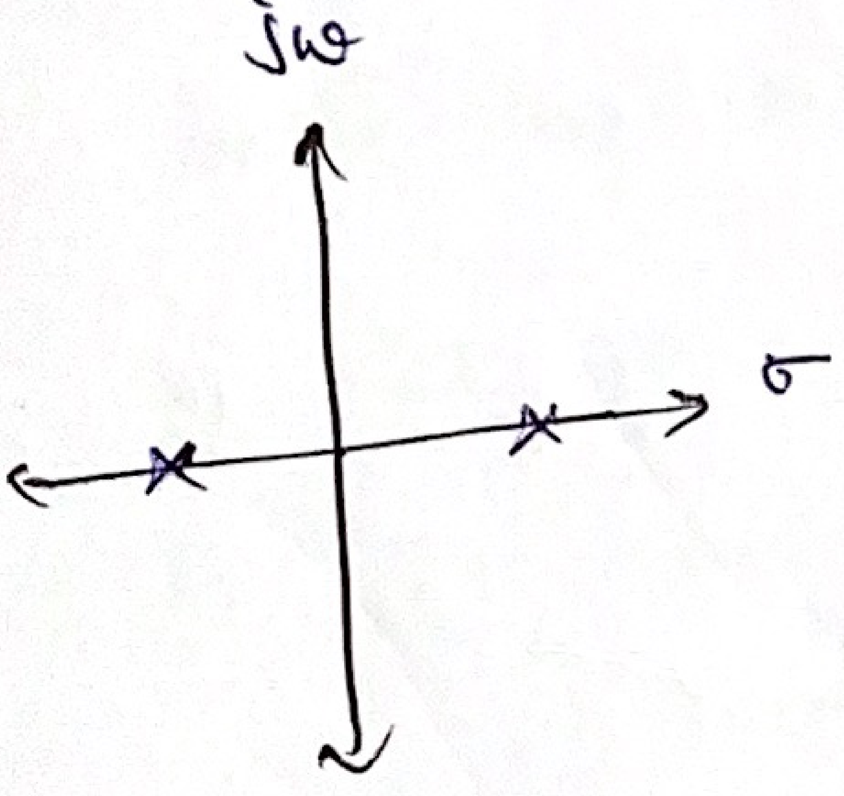


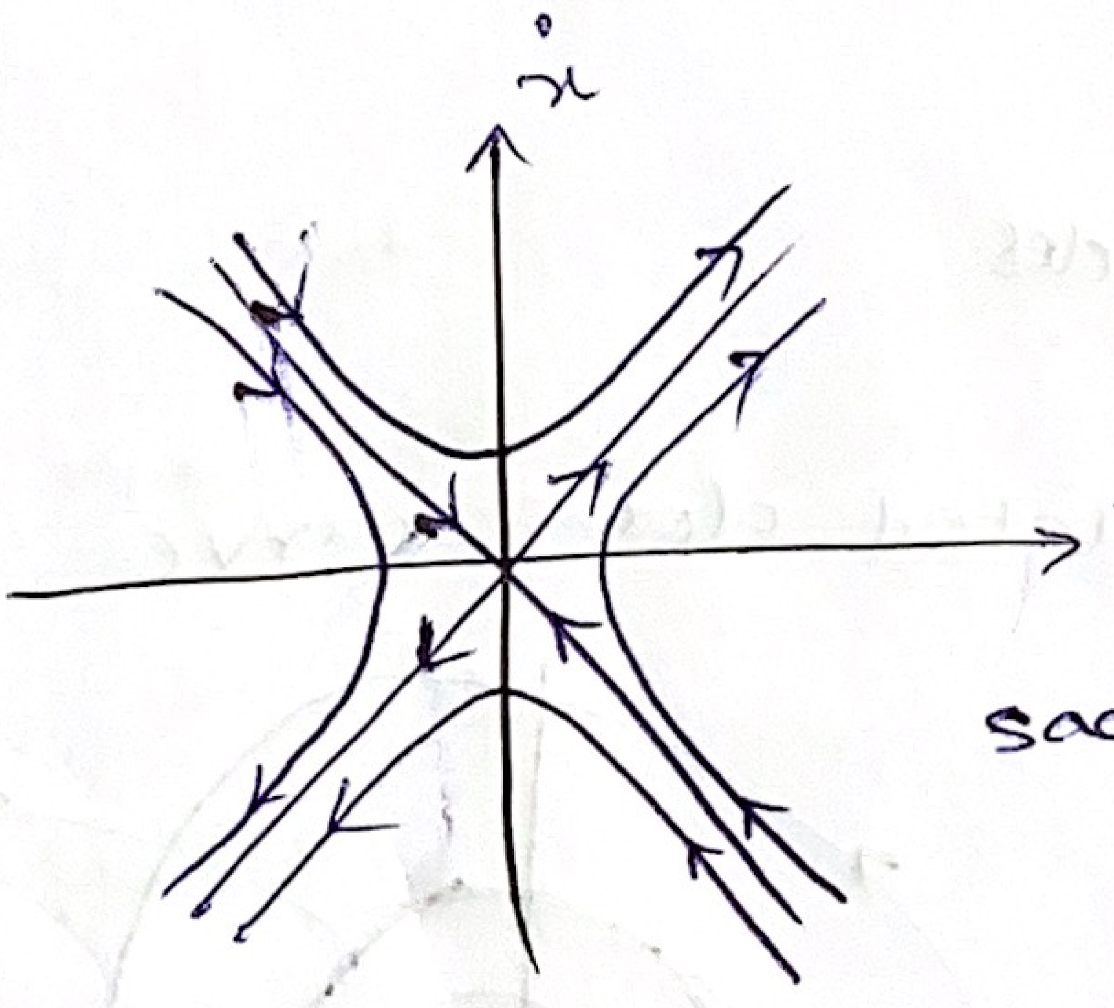
stable node.



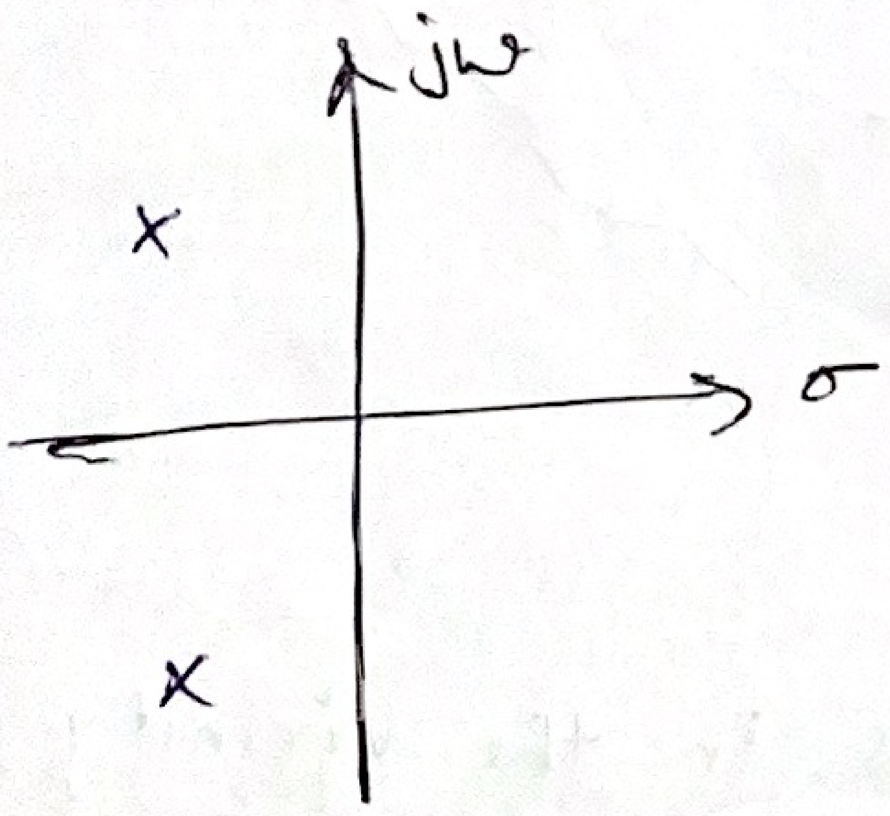


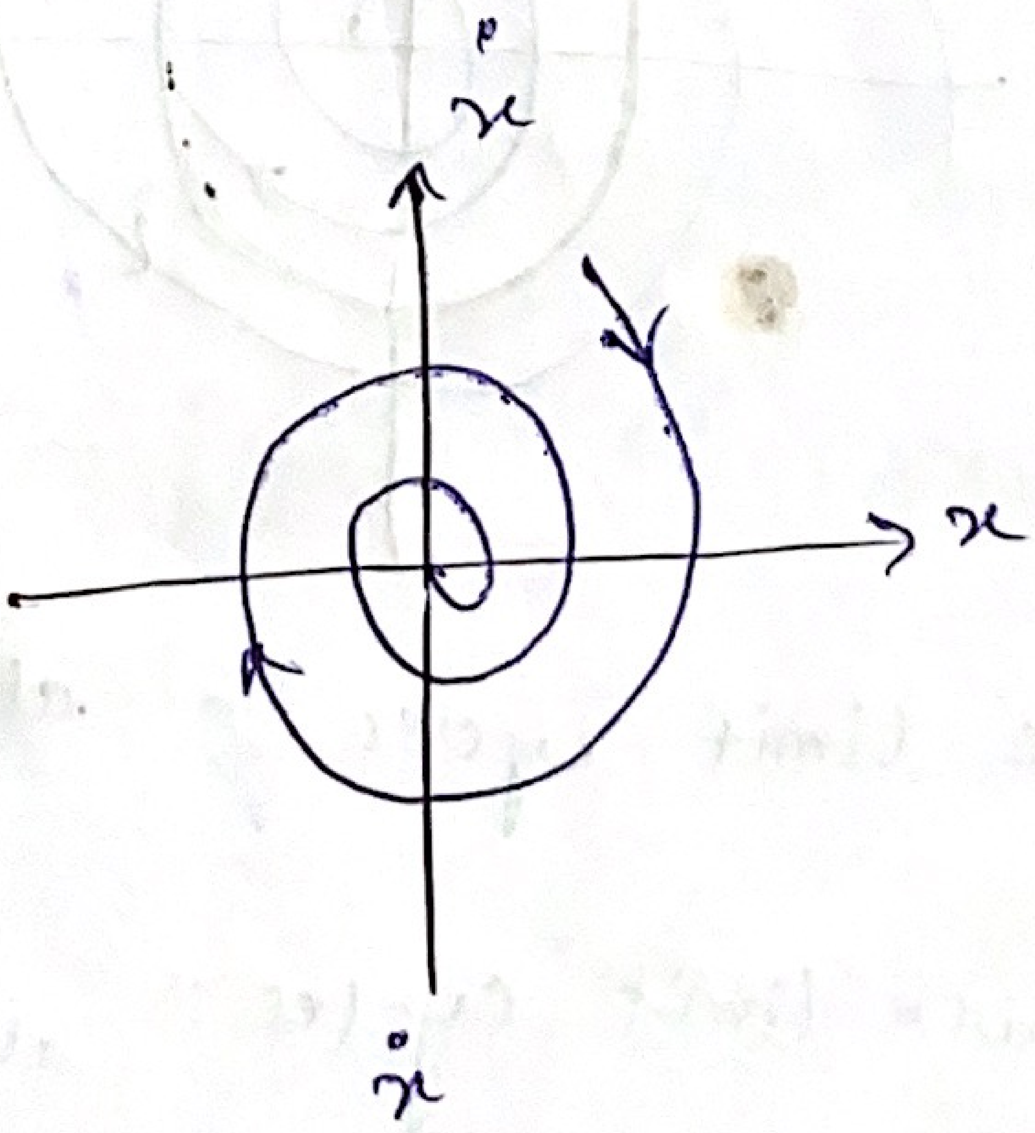
unstable node.



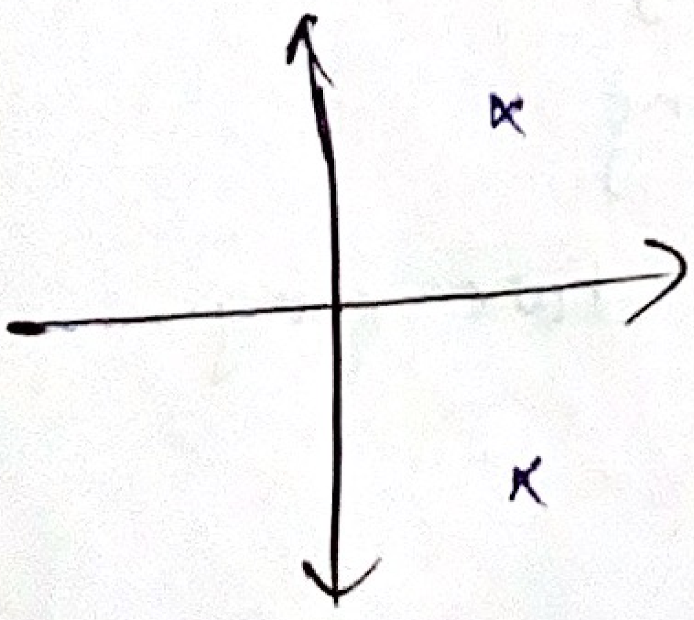


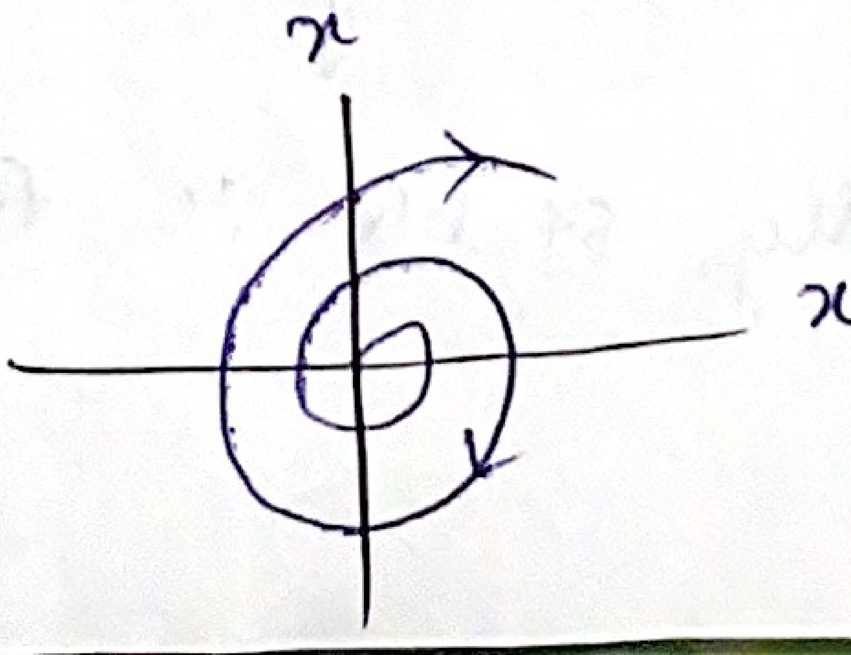
saddle point.



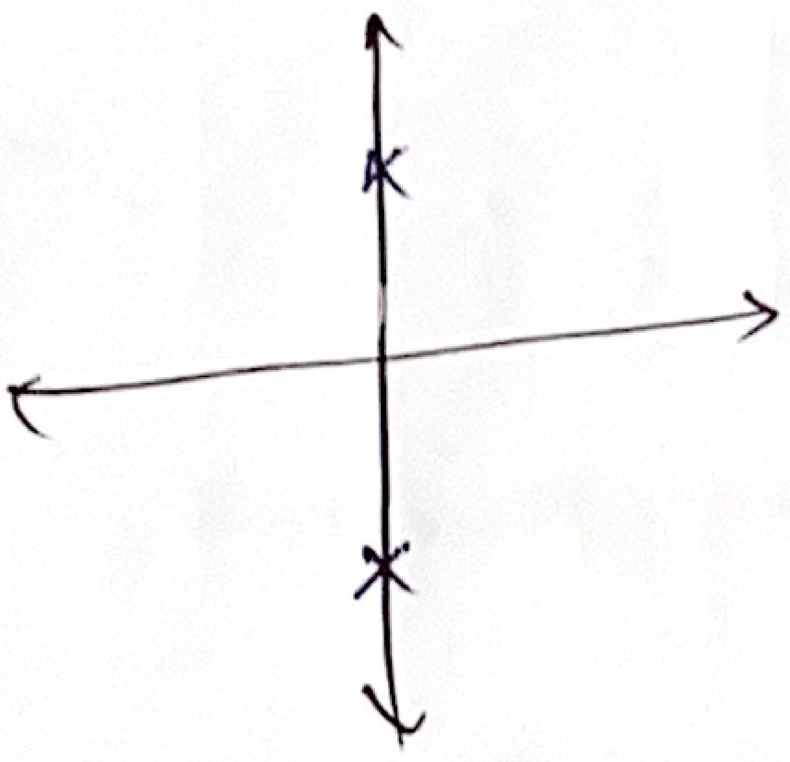


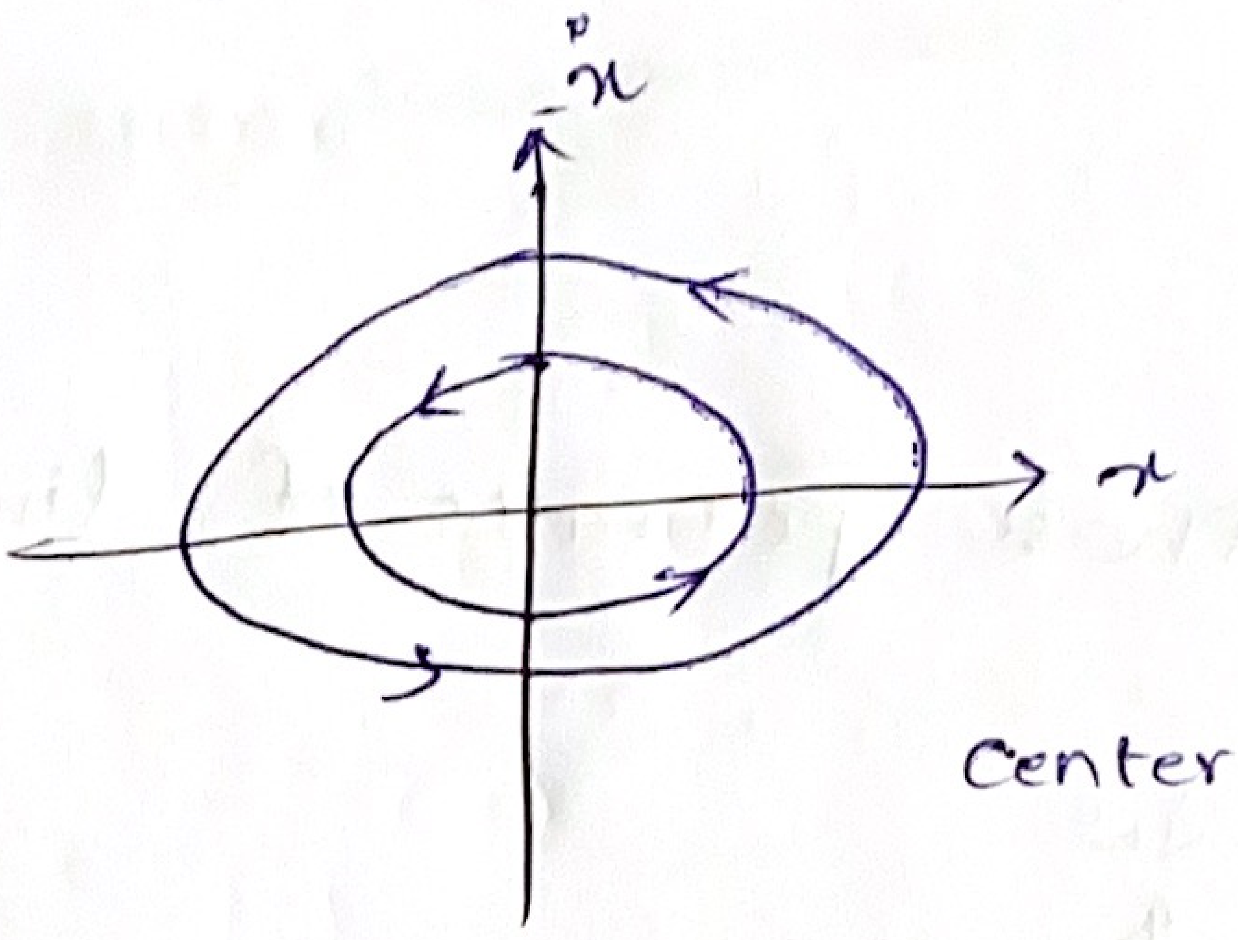
stable focus.





unstable focus.

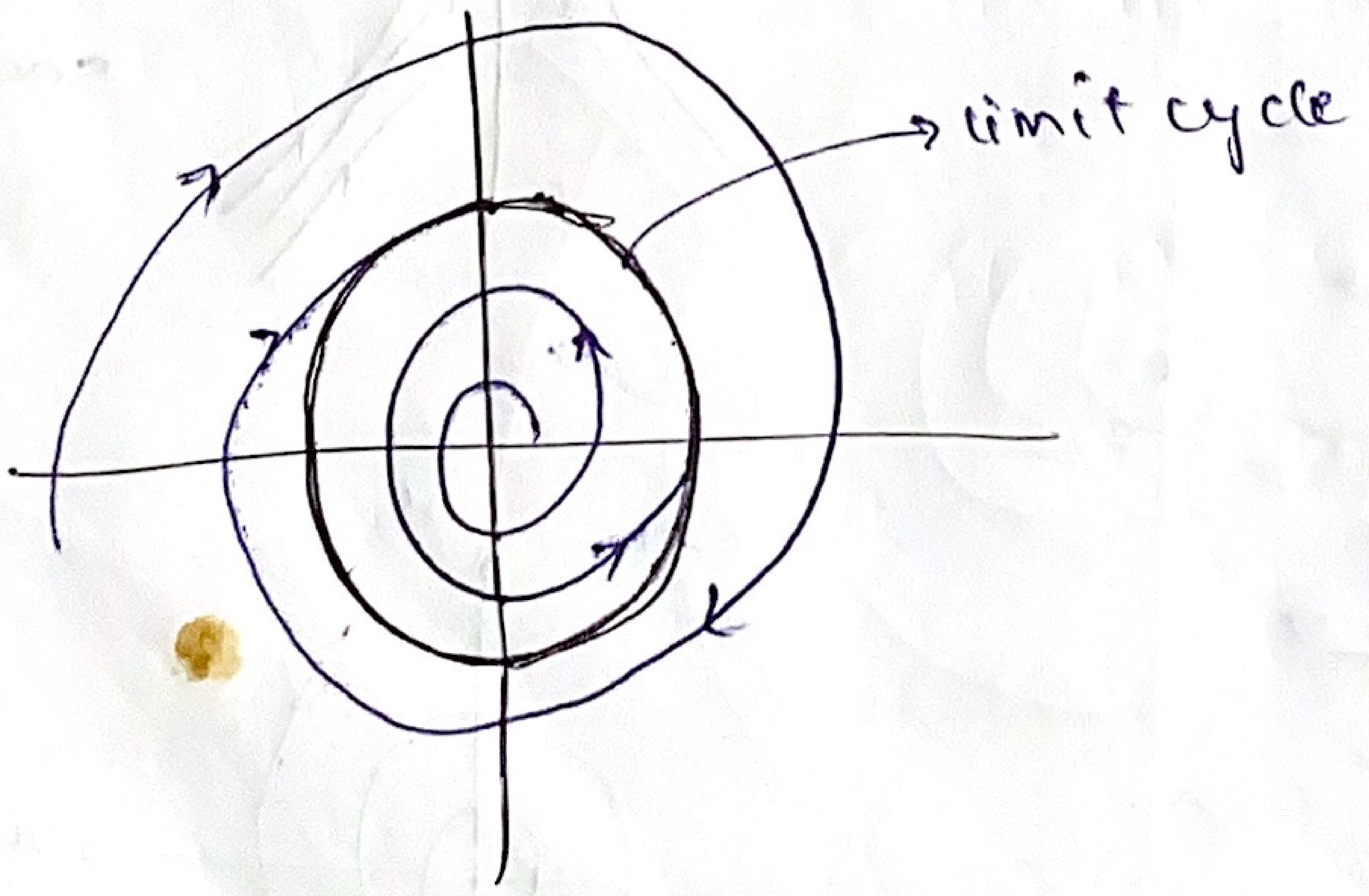




Center point

limit cycles.

Isolated closed curve



Stable limit cycles :- all trajectories in the vicinity are converging

Unstable limit cycles :- all are diverging

Marginally stable :- few converging & diverging.

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Existence of limit cycles

N - no. of nodes, centers, foci
enclosed in a limit cycle.

S - no. of enclosed saddle points.

Poincare: If limit cyle exists in the 2nd order,

then $N = S+1$

Poincare - Bendixson

If a trajectory of 2nd order autonomous system remains in a finite region Q, then one of the following is true

a) It goes to an eq. point.

b) It tends to an asymptotically stable limit cycle

c) The trajectory is itself a limit cycle.

Bendixson

For a nonlinear system, no limit cycle can exist in a region Q in which $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ doesn't vanish and doesn't change sign.

Jacobian linearization.

Consider an autonomous system $\dot{x} = f(x)$ assum $f(x)$ is continuosly differtiable and $f(0) = 0$. then,

$$\dot{x} = \left(\frac{\partial f}{\partial x}\right)_{x=0} x + f_{hot}(x)$$

$f_{h.o.t}$:= higher order terms.

let $A = \left(\frac{\partial f}{\partial x}\right)_{x=0}$

Now, the system is $\dot{x} = Ax$

$$\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Jacobian matrix

for a non-autonomous nonlinar system with control input u, $\dot{x} = f(x, u)$

such that $f(0,0) = 0$

$$\dot{x} = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial u} \cdot u + f_{h.o.t}(x, u)$$

let $A = \left(\frac{\partial f}{\partial x}\right)_{(x=0,\, u=0)}$ $B = \left(\frac{\partial f}{\partial u}\right)_{(x=0,\, u=0)}$

then, $\dot{x} = Ax + Bu$

Tutorial-6

1. For the nonlinear system given below, find the eq. points and determine the type of each isolated eq. point.

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

$-x_1 - x_2 = 0$

$\Rightarrow x_2 = -x_1$

$-x_1 + 2x_1^3 + x_2 = 0$

$-x_1 + 2x_1^3 - x_1 = 0$

$\Rightarrow -x_1 + 2x_1^3 - x_1 = 0$

$\Rightarrow x_1^3 - x_1 = 0$

$x_1 = 1, -1, 0$

Sol. for finding eq. points

$\dot{x}_1 = 0$

$\dot{x}_2 = 0.$

$\cancel{-x_1} + 2x_1^3 + x_2 = \cancel{-x_1} - x_2$

$x_2 = -x_1$

$\cancel{2x_1^3 + 2x_2 = 0}$

$\cancel{x_1^3 + x_2 = 0}$

$\cancel{x_1^3 - x_1 = 0}$

$\cancel{x_1 = 2x_1^3 + x_2}$

$x_2 = -x_1 \quad \cancel{= -2x_1^3 - x_2}$

$2x_2 =$

$x_1 = 1, -1, 0$

$x_2 = -1, 1, 0$

eq. points are $(0,0)$ $(1,-1)$ $(-1,1)$

linearize around the eq. points.

$$\dot{x} = Ax$$

where
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \frac{\partial}{\partial x_1}\left(-x_1 + 2x_1^3 + x_2\right)$$

$$\left.\begin{bmatrix} -1 + 6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}\right|_{(0,0)}$$

$(0,0)$ $(1,-1)$ $(-1,1)$

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$$

$(0,0)$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

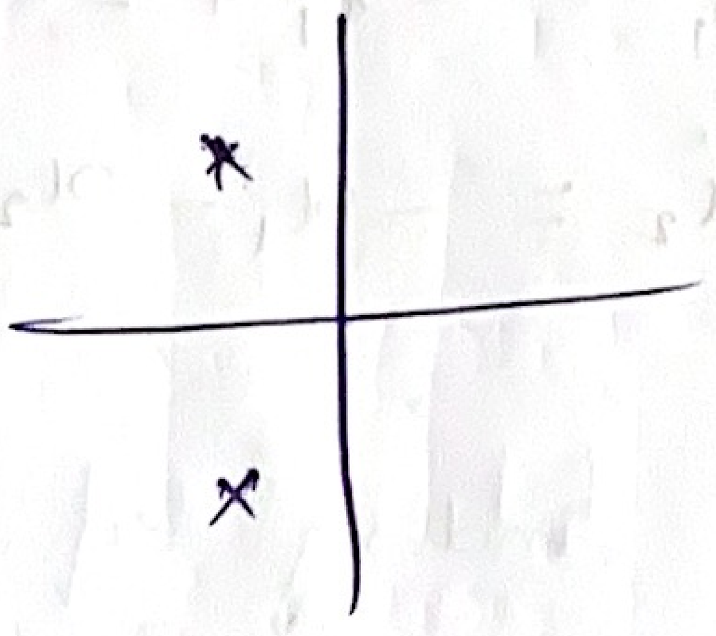
$$\begin{vmatrix} \lambda+1 & -1 \\ 1 & \lambda+1 \end{vmatrix}$$

$(\lambda+1)^2 + 1$

$\lambda^2 + 1 + 2\lambda + 1$

$\lambda^2 + 2\lambda + 2$

$\lambda = -1 + j$
$\quad\;\; -1 - j$



Stable focus.

$(1,-1)$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$$

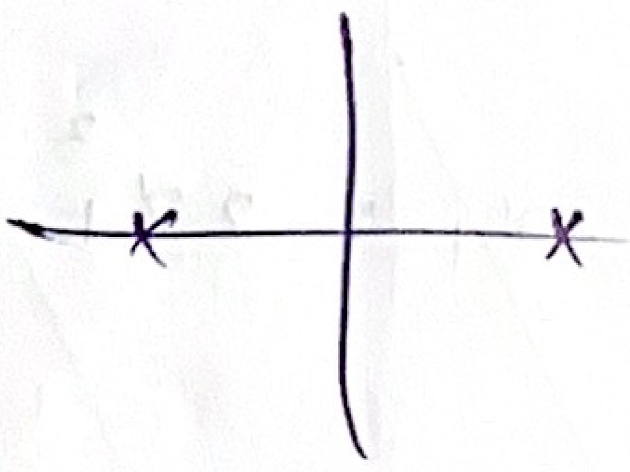
$$\begin{vmatrix} \lambda-5 & -1 \\ 1 & \lambda+1 \end{vmatrix}$$

$= (\lambda-5)(\lambda+1) + 1$

$\lambda^2 - 5\lambda + \lambda - 5 + 1$

$\lambda^2 - 4\lambda - 4 = 0$

$\Rightarrow \lambda = 2+2\sqrt{2},\ 2-2\sqrt{2}$



It is a saddle point.

Q. $\dot{x}_1 = x_1 + x_1 x_2$

$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$

Sol. $\dot{x}_1 = 0$

$\dot{x}_2 = 0$

$\cancel{x_1(1+x_2) = 0}$ $\quad \cancel{x_1} = \cancel{-x_1 x_2}$

$x_2 = -1$

$-x_2 + x_2^2 + x_1 x_2 - x_1^3 = 0$

$-x_2 + x_2^2 + x_1 x_2 - x_1^3 = 0$

$1 + 1 - x_1 - x_1^3 = 0.$

$x_1^3 + x_1 - 2 = 0$

eq. points are

$(0,0)\ (0,1)\ (1,-1)$