Lecture: 10

Maxima and Minima for a function of two variables–

A function $f(x,y)$ is said to have a maximum at (a,b) according as $f(a+h, b+k) < f(a,b)$, for all +ve or −ve small values of h and k.

↳ Minimum at (a,b) according as
$$f(a+h, b+k) > f(a,b)$$

$\Delta = f(a+h, b+k) - f(a,b)$ is of the same sign for all small values of h, k and this sign is −ve, then $f(a,b)$ is maximum.

If the sign is +ve, $f(a,b)$ is minimum.

* Maximum or minimum value of a function is called its extreme value.

* A function $f(x,y)$ can have many extreme values. These extreme values are called the local or relative extreme values of the f^n $f(x,y)$.

* If $f(a,b) \geq f(x,y)$ for all x & y, then $f(a,b)$ is called the global or absolute maximum value of $f(x,y)$.

IIIly if $f(a,b) \leq f(x,y)$ for all x & y, then $f(a,b)$ is called the global or absolute minimum of $f(x,y)$.

* The global extreme value of a f^n is unique.

Necessary conditions for a function to attain an extreme value: –

If $f(a,b)$ is an extreme value then

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = 0$$

$$\Delta = f(a+h, b+k) - f(a,b)$$

Using Taylor's series,

$$\Delta = \cancel{f}(a,b) + \left(hf_x + kf_y\right)_{(a,b)} + \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) + \cdots$$
$$- f(\cancel{a},b) \quad \text{——— (1)}$$

$$\text{Sign of } \Delta = \text{Sign of} \left\{ \left[hf_x + kf_y\right]_{(a,b)} + \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) + \cdots \right\}$$

For small values of h and k, the second & higher order terms can be neglected.

$$\text{Sign of } \Delta = \text{sign of } \left[hf_x + kf_y\right]_{(a,b)}$$

Taking $h=0$, RHS changes sign when k changes sign

Hence $f(x,y)$ can not have a max or min at (a,b) unless $f_y(a,b) = 0$

Similarly, taking $k=0$, we find that $f(x,y)$ cannot have a max or min unless $f_x(a,b) = 0$.

Hence the necessary conditions for $f(x,y)$ to have a max or min at (a,b) are that $f_x(a,b) = 0$, $f_y(a,b) = 0$

$f(x,y)$ fn of 2 variables.

Let $g(x) = f(x,b)$, fn of x alone, that attains an extreme value at $x=a$

Then $\left.\dfrac{d}{dx} g(x)\right|_{x=a} = 0$ i.e., $\dfrac{\partial}{\partial x} f(a,b) = 0$

III'y $h(y) = f(a,y)$ attains an extreme value at $y=b$

$\left.\dfrac{d}{dy} h(y)\right|_{y=b} = 0$ i.e., $\dfrac{\partial}{\partial y} f(a,b) = 0$

The conditions $f_x(a,b) = 0$, $f_y(a,b) = 0$ only necessary conditions but not sufficient.

If $f_x(a,b) = f_y(a,b) = 0$ then $f(a,b)$ need not be an extreme value.

For eg: $$f(x,y) = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise.} \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

But $f(0,0)$ is not an extreme value.

Sufficient conditions to attain extreme values –

Let $f(x,y)$ posses continuous 2^{nd} order order partial derivatives in a neighbourhood of a point (a,b) & if $f_x(a,b)=0$, $f_y(a,b)=0$ and

$f_{xx}(a,b) = \underline{\underline{A}}$, $f_{xy}(a,b) = \underline{\underline{B}}$, $f_{yy}(a,b) = \underline{\underline{C}}$ then

i) $f(a,b)$ is a maximum if $AC-B^2>0$ and $A<0$.

ii) $f(a,b)$ is minimum if $AC-B^2>0$ and $A>0$

iii) $f(a,b)$ is not an extreme value if $AC-B^2<0$

iv) The case is doubtful and needs further consideration if $AC-B^2=0$

Pf: If $f_x(a,b)=0$, $f_y(a,b)=0$ then

eqn (1) $\Rightarrow$

$$\text{Sign of } \Delta = \text{Sign of } \frac{1}{2!}\left[h^2A + 2hkB + k^2C\right]$$

$$= \text{Sign of } \frac{1}{2A}\left[\underline{h^2A^2} + \underline{2hkAB} + k^2AC \underset{+\underline{k^2B^2}-k^2B^2}{}\right]$$

$$= \text{sign of } \frac{1}{2A}\left[(hA+kB)^2 + k^2(AC-B^2)\right] \longrightarrow (2)$$

In eqn (2), $(hA+kB)^2$ is always positive and.

$k^2(AC-B^2)$ is +ve if $AC-B^2>0$

Hence, If $AC-B^2 > 0$ then $f(x,y)$ has a maximum at (a,b) when $A<0$

If $AC-B^2 > 0$ then $f(x,y)$ has a minimum at (a,b) when $A>0$.

If $AC-B^2 < 0$ and $A \neq 0$ then Δ will change with h & k, and hence there is no max or min at (a,b)

i.e., it a saddle point

* A critical point at which neither minimum nor maximum called saddle point or minimax.

* Points at which $f_x = 0$ & $f_y = 0$ are called Stationary points / critical points of $f(x,y)$.

If $AC-B^2 = 0$, then there may or may not be extreme value.

Note: Every extreme value is a stationary value but the converse is not true.

① Examine $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.

Ans: Necessary condition for max or min

$f_x = 0$, $f_y = 0$

$4x^3 - 4x + 4y = 0 \rightarrow (1)$ $\qquad$ $4y^3 + 4x - 4y = 0 \rightarrow (2)$

Adding ① & ② $\quad 4x^3 + 4y^3 = 0$

$$x^3 + y^3 = 0$$

$$(x+y)(x^2 - xy + y^2) = 0$$

$\Rightarrow x + y = 0$ $\quad$ or $\quad$ $x^2 - xy + y^2 = 0$

$x = -y$ $\qquad$ $\underset{+ve}{x^2 + y^2} = \underset{+/-}{xy}$

Not possible.

For $x = -y$ in ①, we get

$$-4y^3 + 4y + 4y = 0$$

$$-y^3 + 2y = 0$$

$$y^3 - 2y = 0$$

$$y(y^2 - 2) = 0$$

$y = 0$, $y^2 - 2 = 0$

$y^2 = 2$

$y = \pm\sqrt{2}$

Since $x = -y$

when $y = 0$, $x = 0$

$y = \sqrt{2}$, $x = -\sqrt{2}$

$y = -\sqrt{2}$, $x = \sqrt{2}$

$\therefore$ The stationary points are $(0,0)$, $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$

$A = f_{xx} = 12x^2 - 4$

$B = f_{xy} = 4$

$C = f_{yy} = 12y^2 - 4$

Stationary pts	A	$AC - B^2$	
$(0,0)$	-4	0	
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	$384 > 0$	Min
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	$384 > 0$	Min

$\therefore f(x,y)$ has minima at $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

And Minimum value $f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = \underline{\underline{-8}}$