23-04-21

Lecture 7

Total Partial derivatives

If $F = f(u, v, w)$ $u = \phi(x, y, z)$, $v = \psi(x, y, z)$, $w = g(x, y, z)$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x}$$

Ill^y we can define $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$.

1. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then find the value of

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$$

Let $u = f(r, s)$ where $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$$s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial r}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial r}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial r}{\partial z} = 0$$

$$\frac{\partial s}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial s}{\partial z} = \frac{1}{z^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} = -\frac{1}{x^2}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$$

Multiply by x^2

$$x^2\frac{\partial u}{\partial x} = -\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) \quad \text{——} \ (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} = \frac{1}{y^2}\frac{\partial f}{\partial r} + 0$$

$$\Rightarrow y^2\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} \quad \text{——} \ (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial z} = 0 + \frac{1}{z^2}\frac{\partial s}{\partial z}$$

$$\Rightarrow z^2\frac{\partial u}{\partial z} = \frac{\partial s}{\partial \cancel{z}s} \quad \text{——} \ (3)$$

Adding (1), (2) and (3) we get

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

2. If $z = f(x, y)$, $x = e^u \sin v$, $y = e^u \cos v$.

P.T $x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v} = (x^2 + y^2) \frac{\partial z}{\partial x}$

$z \longrightarrow (x, y) \longrightarrow (u, v)$

$x = e^u \sin v$

$\frac{\partial x}{\partial u} = e^u \sin v$

$\frac{\partial x}{\partial v} = e^u \cos v$

$y = e^u \cos v$

$\frac{\partial y}{\partial u} = e^u \cos v$, $\frac{\partial y}{\partial v} = -e^u \sin v$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\therefore \frac{\partial z}{\partial u} = e^u \left[\sin v \frac{\partial f}{\partial x} + \cos v \frac{\partial f}{\partial y} \right]$$

$$\Rightarrow x \frac{\partial z}{\partial u} = e^{2u} \sin v \left[\sin v \frac{\partial f}{\partial x} + \cos v \frac{\partial f}{\partial y} \right] \quad \text{— } (1)$$

$|||^{ly}$ $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = e^u \left[\cos v \frac{\partial f}{\partial x} - \sin v \frac{\partial f}{\partial y} \right]$$

$$\therefore y \frac{\partial z}{\partial v} = e^{2u} \cos v \left[\cos v \frac{\partial f}{\partial x} - \sin v \frac{\partial f}{\partial y} \right] \longrightarrow (2)$$

(1) + (2) gives

$$x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v} = e^{2u} \left\{ \sin^2 v + \cos^2 v \right\} \frac{\partial f}{\partial x} = e^{2u} \frac{\partial f}{\partial x}$$

Observe that $x^2 + y^2 = e^{2u} (\sin^2 v + \cos^2 v) = e^{2u}$

$$\Longrightarrow x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v} = (x^2 + y^2) \frac{\partial z}{\partial x}$$

$z = f(x, y)$

HW 3. If $z = f(x, y)$, $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

P.T $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.

4. If $x = r\cos\theta$, $y = r\sin\theta$ then P.T

$$r_{xx} + r_{yy} = \frac{1}{r} \left\{ \left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right\}$$

$(x, y) \longrightarrow (r, \theta)$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}(y/x)$

Differentiation of Implicit functions

If $f(x,y)=c$ is an implicit relation between x and y which defines as a differentiable function of x, then $\frac{df}{dx}=0$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}} \quad \text{or} \quad \frac{dy}{dx}=-\frac{f_x}{f_y}$$

Eg: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$f(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$f_x=\frac{2x}{a^2}, \quad f_y=-\frac{2y}{b^2}$

$\therefore \frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-\cancel{2}x/a^2}{-\cancel{2}y/b^2}=\frac{xb^2}{ya^2}$

1. Find $\frac{dy}{dx}$ for the implicit fns using partial derivatives.

i) $y^x=x$ ii) $a^x+a^y=a^{x+y}$ iii) $xy=e^{x^2+y^2}$

iv) $x^my^n=(x+y)^{m+n}$ v) $x^y=y^x$

v) $x^y=y^x$

$y\ln x=x\ln y$

$f(x,y)=y\ln x-x\ln y=0$

$\therefore \frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-\left[y/x-\ln y\right]}{\ln x-x/y}=\frac{y(y-x\ln y)}{x(x-y\ln x)}$

1. If $u = x\ln(xy)$ where $x^3+y^3+3xy=1$ find $\frac{du}{dx}$.

$\downarrow$ $f(x,y)=c$ $\downarrow$ $\frac{dy}{dx}$

$u \longrightarrow (x,y)$

$u \longrightarrow (x, y(x))$

$\therefore \frac{du}{dx}$

$$f(x,y) = x^3+y^3+3xy-1$$

$$\frac{\partial f}{\partial x} = 3x^2+3y \quad , \quad \frac{\partial f}{\partial y} = 3y^2+3x$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-(3x^2+3y)}{3y^2+3x} = \frac{-(x^2+y)}{y^2+x}$$

$$u = x\ln(xy)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= x\frac{y}{xy} + \ln(xy)\cdot 1 + \frac{x}{xy}\cdot x\left[\frac{-(x^2+y)}{y^2+x}\right]$$

$$= 1+\ln(xy) - \frac{x(x^2+y)}{y^2+x}$$

2. If $z=\sqrt{x^2+y^2}$ and $x^3+y^3 = 3axy$ find $\frac{dz}{dx}$ when $x=-a$ and $y=a$

$$z = \sqrt{x^2+y^2}$$

$$f = x^3+y^3-3axy = 0$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} + \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\cdot\left\{\frac{-(x^2-ay)}{y^2-ax}\right\}$$

$$= \frac{-(3x^2-3ay)}{3y^2-3ax} = \frac{-(x^2-ay)}{y^2-ax}$$

At $x=-a$, $y=a$

$$\left.\frac{dz}{dx}\right|_{(-a,a)} = \frac{-\cancel{a}}{\cancel{a}\sqrt{2}} + \frac{\cancel{a}}{\cancel{a}\sqrt{2}}\left\{0\right\}$$

$$= -\frac{1}{\sqrt{2}}$$

3. Find the total differential of (x^2y) w.r.to x at $(1,1)$ when $x^2+xy+y^2=1$

$u = x^2y$, $f(x,y) = x^2+xy+y^2-1=0$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$$

$$\therefore \frac{du}{dx} = 2xy + x^2\left(-\frac{(2x+y)}{x+2y}\right) \qquad \frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(2x+y)}{(x+2y)}$$

At $(1,1)$

$$\frac{du}{dx} = 2 + \left(-\frac{3}{3}\right) = \underline{\underline{1}}$$

HW 1. If $z = e^{ax+by} f(ax-by)$ then P.T

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

2. If $u = \tan^{-1}\left(\frac{x}{y}\right)$, $x^2+y^2=a^2$, find $\frac{du}{dx}$

Taylor's expansion of function of 2 variables

We have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$f(x+h, y+k) = f(x,y) + \frac{\Delta f}{1!} + \frac{1}{2!}\Delta^2 f + \frac{1}{3!}\Delta^3 f + \cdots$$

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \Rightarrow \Delta f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$\Delta^2 f = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$\Delta^3 f = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f = h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}$$

Note:- $f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f$
$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots$

case (i) :- Put $x = a$ and $y = b$

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\Bigg|_{(a,b)}$$

$$+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y}\right)\Bigg|_{(a,b)} + \cdots$$

case (ii) :- Put $a+h = x$, $b+k = y$

$h = x-a$ $\quad k = y-b$

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right]\Bigg|_{(a,b)}$$

$$+ \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right]_{(a,b)}$$

$$+ \cdots$$

case (iii) :- Put $a = 0$ and $b = 0$

$$\therefore f(x,y) = f(0,0) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\Bigg|_{(0,0)} + \frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}\right.$$

$$\left. + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right]_{(0,0)} + \cdots$$

This series is called Maclaurin's series for x and y variables. This is used to expand $f(x,y)$ in powers of x and y near the origin.

1. Expand $f(x,y) = \sin xy$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ upto the second degree terms.

$x - a = x - 1$, $y - b = y - \frac{\pi}{2}$

$\therefore (a,b) = \left(1, \frac{\pi}{2}\right)$

$f(x,y) = \sin xy$, $f\left(1, \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f_x = \cos xy \cdot y$, $f_x\left(1, \frac{\pi}{2}\right) = 0$

$f_{xx} = y^2[-\sin xy]$, $f_{xx}\left(1, \frac{\pi}{2}\right) = -\frac{\pi^2}{4}$

$f_y = x\cos xy$, $f_y\left(1, \frac{\pi}{2}\right) = 0$

$f_{yy} = -x^2\sin xy$, $f_{yy}\left(1, \frac{\pi}{2}\right) = -1$

$$\therefore f(x,y) = f(a,b) + (x-a)f_x + (y-k)f_y \Big|_{(a,b)}$$

$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right\}\Big|_{(a,b)}$$

$$+ \cdots$$

$$f_{xy} = \frac{\partial}{\partial y}\left(y \cdot \cos(xy)\right\} = y(-x\sin xy) + \cos xy \cdot 1$$

$$f_{xy}\left(1, \frac{\pi}{2}\right) = -\frac{\pi}{2}$$

$$\therefore \sin xy = 1 + (x-1)(0) + \left(y - \frac{\pi}{2}\right) \times 0$$

$$+ \frac{1}{2!}\left\{(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)\right.$$

$$\left. + \left(y - \frac{\pi}{2}\right)^2 (-1)\right\}$$

$$\sin xy = 1 - \frac{1}{2!}\left\{\frac{\pi^2}{4}(x-1)^2 + \pi(x-1)\left(y-\frac{\pi}{2}\right)\right.$$

$$\left. + \left(y - \frac{\pi}{2}\right)^2\right\}$$

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2. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in the neighbourhood of $(1, 1)$ upto 2nd deg terms

$a = 1$, $b = 1$

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right), \quad f(1,1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f_x = \frac{1}{1+\left(y/x\right)^2} \times \left(-\frac{y}{x^2}\right)$$

$$f_x = \frac{-y}{x^2+y^2} \qquad \therefore f_x(1,1) = -\frac{1}{2}$$

$$f_y = \frac{1}{1+\left(y/x\right)^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2} \qquad \therefore f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore f_{xx}(1,1) = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x \times 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{yy}(1,1) = -\frac{1}{2}$$

$$\therefore f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\Big|_{(a,b)} + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right\} + \cdots$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \frac{1}{2}\left[-(x-1)+(y-1)\right] + \frac{1}{4}\left[(x-1)^2-(y-1)^2\right] + \cdots$$

$$= \frac{\pi}{4} + \frac{1}{2}\left[y-x\right] + \frac{1}{4}\left[(x-1)^2-(y-1)^2\right] + \cdots$$

3. Expand $e^x \sin y$ in powers of x and y as far as terms of 3rd degree

$a = 0, \quad b = 0$

$f(x,y) = e^x \sin y$	$f(0,0) = 0$
$f_x = e^x \sin y$	$f_x(0,0) = 0$
$f_y = e^x \cos y$	$f_x(0,0) = 1$
$f_{xx} = e^x \sin y$	$f_{xx}(0,0) = 0$
$f_{xy} = e^x \cos y$	$f_{xy}(a,0) = 1$
$f_{yy} = -e^x \sin y$	$f_{yy}(0,0) = 0$
$f_{xxx} = e^x \sin y$	$f_{xxx}(0,0) = 0$
$f_{xxy} = e^x \cos y$	$f_{xxy}(0,0) = 1$
$f_{xyy} = -e^x \sin y$	$f_{xyy}(0,0) = 0$
$f_{yyy} = -e^x \cos y$	$f_{yyy}(0,0) = -1$

Thus $f(x,y) = f(0,0) + x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}\Big|_{(0,0)}$

$$+ \frac{1}{2!}\left\{x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right\}_{(0,0)}$$

$$+ \frac{1}{3!}\left\{x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right\} + \cdots$$

$$e^x \sin y = 0 + y + \frac{1}{2!}\left\{2xy\right\} + \frac{1}{3!}\left\{3x^2y - y^3\right\} + \cdots$$

$$= y + xy + \frac{1}{6}\left(x^2y - y^3\right) + \cdots$$

HW a) $f(x,y) = e^x \ln(1+y)$ as powers of x and y upto 3rd degree terms.

b) Expand $f(x,y) = e^{xy}$ at $(1,1)$ upto 3rd degree terms

c) Expand $f(x,y) = \cos x \cos y$ in powers of x and y upto 3rd degree terms.

d) $f(x,y) = (1+x+y^2)^{1/2}$ at $(1,0)$ upto 2nd deg terms

e) Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem. $(\because a = 1,\ b = -2)$