Exercise

1. Show that $f(z) = \cosh z$ is analytic and find $f'(z)$.

2. If $f(z) = u + iv$ is analytic then show that
$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$$

3. If $f(z) = u + iv$ is analytic find $f(z)$ if
$$u - v = (x-y)(x^2+4xy+y^2)$$

4. If $u = r^2\cos 2\theta - r\sin\theta$, find $v$ and hence find $f(z)$

5. If $v = r\sin\theta + \frac{\cos\theta}{r}$, find $u$ and hence find $f(z)$

6. If $v = \left(r - \frac{1}{r}\right)\sin\theta$, find $u$ and hence find $f(z)$

7. If $v = x^2 - y^2 + \frac{x}{x^2+y^2}$, find $u$ and hence find $f(z) = u + iv$.

Integrate w.r.to $\theta$

$$v(r,\theta) = -\frac{\sin 2\theta}{r^2} + g(r)$$

$$\frac{\partial v}{\partial r} = \frac{2\sin 2\theta}{r^3} + g'(r) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \frac{2\sin 2\theta}{r^3} + g'(r) = \frac{2\sin 2\theta}{r^3} \Rightarrow g'(r) = 0$$

$$\therefore g(r) = C$$

$$v(r,\theta) = -\frac{\sin 2\theta}{r^2} + C$$

$$\therefore f(z) = u + iv = \frac{\cos 2\theta}{r^2} + i\left(-\frac{\sin 2\theta}{r^2} + C\right)$$

$$= \frac{1}{z^2} + iC$$

find $f(z) = u + iv$, analytic function if

(1) $u = \frac{1}{r}(\cos\theta - \sin\theta)$

$$\frac{\partial u}{\partial r} = -\frac{1}{r^2}(\cos\theta - \sin\theta)$$

$$\frac{\partial u}{\partial \theta} = -\frac{1}{r}(\sin\theta + \cos\theta)$$

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= e^{-i\theta}\left\{-\frac{1}{r^2}(\cos\theta - \sin\theta) + \frac{i}{r^2}(\sin\theta + \cos\theta)\right\}$$

$$= -\frac{1}{z^2} + \frac{i}{z^2} = \frac{(i-1)}{z^2}$$

$$f(z) = (i-1)\int \frac{1}{z^2}\,dz = \frac{(1-i)}{z} + C$$

2. If $u = \frac{\cos 2\theta}{r^2}$ find $v$ and hence find $f(z)$.

$$\frac{\partial u}{\partial r} = -\frac{2\cos 2\theta}{r^3} \quad ; \quad \frac{\partial u}{\partial \theta} = -\frac{2\sin 2\theta}{r^2}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{2\cos 2\theta}{r^3}$$

$$\therefore \frac{\partial v}{\partial \theta} = -\frac{2\cos 2\theta}{r^2}$$

Laplace's eqn in polar form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \, ,$$

$f(z) = u + iv$ is analytic.

we have $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta}$$

$$-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} = 0$$

# C-R equations in polar form

Let $x = r\cos\theta$, $y = r\sin\theta$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore f(z) = u + iv = f(re^{i\theta}) \longrightarrow (1)$$

Differentiate (1) partially w.r.to $r$.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \longrightarrow (2)$$

Differentiate (1) partially w.r.to $\theta$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta}$$

$$-\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, e^{i\theta} \longrightarrow (3)$$

from (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

These are called C-R eqns in polar form.

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

To express $f(z)$ in terms of $z$ put $r = z$, $\theta = 0$

(Milne Thomson method)

$$= \frac{2(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] - 2(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] - 4uv\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right]}{(u^2+v^2)^2}$$

$$= \underline{\underline{0}}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{\begin{aligned}&(u^2+v^2)\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \\ &- 2u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] - 2v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \\ &- 4uv\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\end{aligned}}{(u^2+v^2)^2}$$

2. S.T. $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0$ if $f(z)$ is analytic

$$f(z) = u + iv$$

$$|f(z)| = \sqrt{u^2+v^2}$$

Let $\varphi = \log|f(z)| = \frac{1}{2}\log(u^2+v^2)$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2}\,\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$= \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{u^2+v^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{(u^2+v^2)\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] - \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)}{(u^2+v^2)^2}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{(u^2+v^2)\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] - \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)}{(u^2+v^2)^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$+ 2v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

Since $f(z)$ is analytic $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Also $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$$\therefore \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 4|f'(z)|^2$$

1: Let $f(z) = u+iv$ be analytic.

S.T. $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

$$f(z) = u+iv$$

Let $\varphi = |f(z)|^2 = (u+iv)(u-iv) = u^2+v^2$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

Let $\varphi = u^2+v^2$ where $\varphi = |f(z)|^2$

$$\frac{\partial \varphi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2$$

Similarly, $$\frac{\partial^2 \varphi}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2$$

Example

Find the analytic function $f(z)$ as a function of $z$ given that the sum of its real and imaginary part is $x^3 + y^3 + 3xy(x-y)$.

Given $u + v = x^3 + y^3 + 3xy(x-y)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \longrightarrow (1) \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 6xy$$

(using C-R eqns)

$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = 3y^2 + 3x^2 - 6xy \longrightarrow (2)$$

$$\frac{1}{2}[(1)+(2)] \Rightarrow \frac{\partial u}{\partial x} = 3x^2$$

$$\frac{1}{2}[(1)-(2)] \Rightarrow \frac{\partial v}{\partial x} = 6xy - 3y^2$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 + i(6xy - 3y^2)$$

$$= 3z^2 \quad \text{(using Milne Thomson method } x = z,\ y = 0\text{)}$$

$$f(z) = \int f'(z)\,dz + c = \underline{\underline{z^3 + c}}$$