Two dimensional Random variables:

Let 'S' be the sample space associated with a random experiment 'E'. Let $X = X(s)$, $Y = Y(s)$ two functions each assigning a real number to each element $s \in S$. Then we call the pair X, Y as two dimensional random variable.

Ex: Height, weight of randomly selected person.

→ (X, Y) is two dimensional DRV If the range space of (X, Y) is finite or countably infinite.

→ (X, Y) is 2-D CRV. If the range space is uncountable set.

Joint probability distribution:

Let (X, Y) be an 2-D DRV. with each possible outcome (x_i, y_j) be associate a number $P(x_i, y_j)$. representing probability of $P(X = x_i, Y = y_j)$ and satisfying the following condition.

(i) $P(x_i, y_j) \geq 0$

(ii) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(x_i, y_j) = 1.$

Then $P(x_i, y_j)$ is called joint probability mass function of (X, Y).

→ Let (X, Y) be a CRV then its joint. pdf $f(x, y)$ satisfy's the following conditions.

(i) $f(x, y) \geq 0 \ \forall x, y$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1.$

31/3/22

Marginal probability distribution of 'x':

DRV

$$P(X=x_i) = P(x_i) = P(X=x_i, Y=y_1) + P(X=x_i, Y=y_2) + \cdots$$

(joint pmf) $\quad P(x_i) = \sum_{j=1}^{\infty} P(x_i, y_j)$. (marginal pdf of x)

Similarly,

$$P(Y=y_j) = q(y_j) = P(X=x_1, Y=y_j) + P(X=x_2, Y=y_j) + \cdots$$

$$q(y_j) = \sum_{i=1}^{\infty} P(x_i, y_j) \text{ (marginal pdf of y.)}$$

X \ Y	y_1	y_2	y_3	y_4	$P(x_i)$
x_1	P_{11}	P_{12}	P_{13}	P_{14}	$P(x_1)$
x_2	P_{21}	P_{22}	P_{23}	P_{24}	$P(x_2)$
x_3	P_{31}	P_{32}	P_{33}	P_{34}	$P(x_3)$
$q(y_j)$	$q(y_1)$	$q(y_2)$	$q(y_3)$	$q(y_4)$	1

CRV

Let $f(x_i)$ be the joint pdf of two CRV of (x,y) then the marginal pdf of 'x' is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Conditional probability distribution:

DRV

Let (x,y) be a discrete random variable with joint pmf $p(x_i, y_j)$ and marginal PMF $p(x_i)$ and $q(y_j)$,

$$\longrightarrow \quad P(x_i | y_j) = P(X=x_i | Y=y_j) = \frac{P(x_i, y_j)}{q(y_j)}.$$

$$\longrightarrow \quad P(y_j | x_i) = \frac{P(x_i, y_j)}{P(x_i)}$$

CRV

Let (x, y) be a CRV with joint pdf $f(x,y)$ and marginal pdf $h(y)$ and $g(x)$ then

$$\rightarrow \quad P(x|y) = P(X=x | Y=y) = \frac{f(x,y)}{h(y)}$$

$$\rightarrow \quad P(y|x) = P(Y=y | X=x) = \frac{f(x,y)}{g(x)}$$

Q1) A coin is tossed 3times. Let X denotes 0 or 1 according as a tail or a head occurs on the first toss. Let Y denotes the no. of tails. Determine

① Joint Probability distribution of X and Y

② Marginal probability distribution of X and Y

Ans)

X \ Y	0	1	2	3	P(X)
0	0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	0	1/2
q(y)	1/8	3/8	3/8	1/8	1

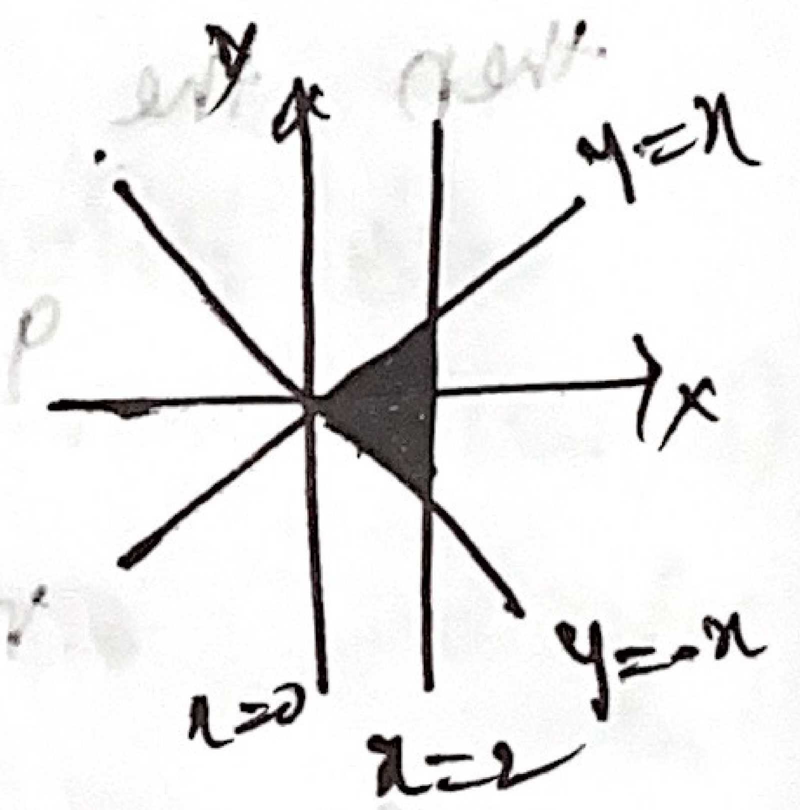
$P(X=0)$

$P(X=1)$

Q2) Suppose that a 2dimensional CRV (X,Y) has Joint PDF $f(x,y) = \begin{cases} kx(x-y), & 0<x<2, -x<y<x \\ 0 & \text{else where.} \end{cases}$

① Find 'k'

② Find marginal pdf of Y & X.



Ans)

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$= k\int_{x=0}^{2}\int_{y=-x}^{x} x(x-y)\,dy\,dx = 1$

$\dfrac{x(x-xy)}{8}$

$= k\int_{x=0}^{2}\int_{y=-x}^{x} (x^2 - \cancel{xy})\,dy\,dx = 1$ → xy will be gone because odd function

$= 2k\int_0^2\int_0^x x^2\,dy = 1$

$= 2k\int_0^2 x^2\,[y]_0^x\,dx = 1$

$= 2k\int_0^2 x^3\,dx = 1$

$= 2k \cdot \dfrac{x^4}{4}\Big|_0^2 = 1$

$\Rightarrow 8k = 1$

$\Rightarrow k = 1/8.$

(2)

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

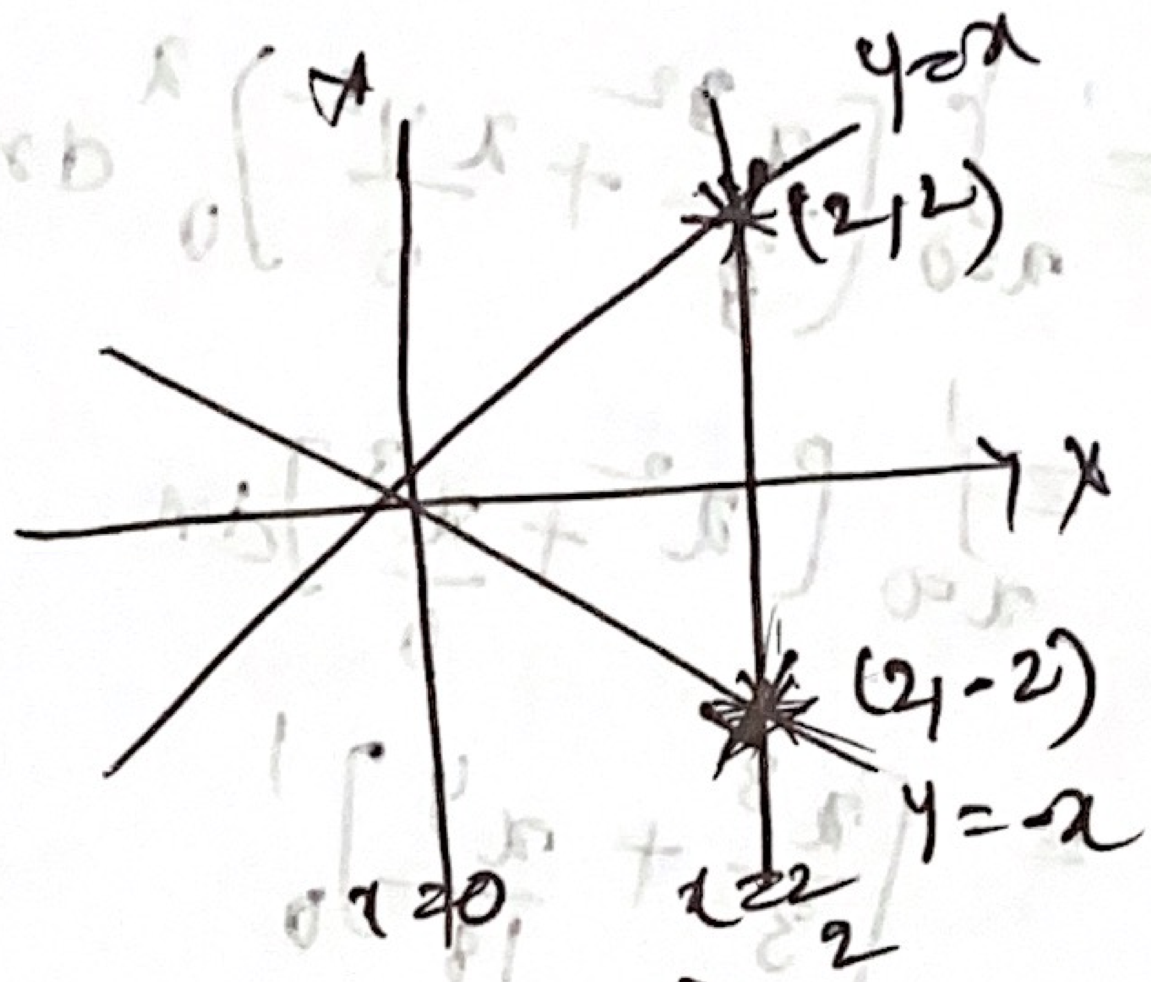
$$= \frac{1}{8}\int_{-x}^{x} (x(x-y))\,dy \quad , 0<x<2$$

$$= \frac{1}{8}\int_{-x}^{x} x^2 - xy \, dy$$

$$= \frac{\cancel{x}}{8y}\left[x^3/y\right]$$

$$= x^3/4$$

$$= \begin{cases} \frac{x^3}{4} , & 0<x<2 \\ 0 , & \text{elsewhere} \end{cases}$$



marginal pdf of y

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$h(y) = \begin{cases} \frac{1}{8}\int_{-y}^{2} x(x-y)\,dx , & -2<y<0 \\ \frac{1}{8}\int_{y}^{2} x(x-y)\,dy , & 0<y<2 \end{cases}$$

(1) (2nd order boundary condition to y')

(2) (laplace, corr, heat, wave

(3) Base theorem, total probability, Random variable) → sessional - 1

$x_1 = 0$, $x_2 = 0.1$, $x_3 = 1$

$y_1 = 1$, $y(0.1) = ?$, $y(1) =$

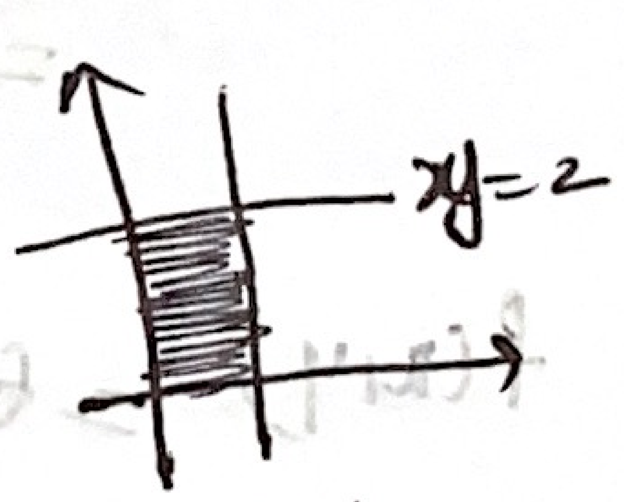
h will be given

11/4/22

Q3) Suppose a joint pdf of a 2-D random variable is given by $f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0<x<1, 0<y<2 \\ 0 , & \text{elsewhere} \end{cases}$

Find: (i) $P(X > \frac{1}{2})$ (ii) $P(Y<X)$

(iii) $P(Y<1/2 \mid X<1/2)$.



(i) $P(X > 1/2) = P(\frac{1}{2} < X < \infty)$

$= P(1/2 < X < 1, 0 < Y < 2)$

$$= \frac{1}{3}\int_{1/2}^{1}\int_{0}^{2} 3x^2 + xy \, dx\,dy$$

$$= \frac{1}{3}\int_{1/2}^{1} (6x^2 + 2x)\,dx$$

$$= \frac{1}{3}\left[\frac{6x^3}{3} + \frac{2x^2}{2}\right]_{1/2}^{1}$$

$$= \frac{1}{3}\left[2+1 - \frac{1}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{3}\left[3 - \frac{2}{4}\right]$$

$$= \frac{10}{12} = \frac{5}{6}$$

$$= \int_{x=1/2}^{1}\int_{y=0}^{2} (x^2 + xy/3)\,dy\,dx$$

$$= \int_{1/2}^{1} \left[x^2 + \frac{xy^2}{6}\right]_0^2 dx$$

$$= \int_{x=1/2}^{1} x + \frac{xy}{63}\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^2}{63}\right]_{1/2}^{1}$$

$$= \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{8} - \frac{1}{12}\right]$$

$= 5/6$

(ii) $P(Y<X) = P(-\infty < Y < X < \infty)$.

$$= \int_{x=0}^{1}\int_{y=0}^{x} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{xy^2}{6}\right]_0^x dx$$

$$= \int_{x=0}^{1} \left[x^2 + \frac{x^3}{6}\right] dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{18}\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{1}{18}\right] = \frac{7}{24}.$$

(iii) $P(Y<1/2 \,/\, X<1/2) = \dfrac{P(X<1/2, Y<1/2)}{P(X<1/2)}$.

$$= \frac{\int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dy\, dx}{\int_0^{1/2}\int_0^{2} \left(x^2 + \frac{xy}{3}\right) dy\, dx}$$

$$= \int_0^{1/2} \left[x^2 + \frac{xy^2}{6}\right]_0^{1/2} dx = \int_0^{1/2} \left[x^2 + \frac{x}{48}\right] dx = \left[\frac{x^3}{3} + \frac{x^2}{96}\right]_0^{1/2} = \left(\frac{1}{24} + \frac{1}{384}\right)$$

$$= \frac{\frac{1}{24} + \frac{1}{384}}{\frac{1}{24} + \frac{1}{12}} = \frac{5}{32}$$

(Q4) $f(x,y) = 6e^{-2x-3y}$; $x, y \geq 0$.

find (i) $P(0 < x < 2, y > 0)$

(ii) Marginal pdf of $g(x)$, $h(y)$.

(i)
$$= 6\int_{x=0}^{2}\int_{y=0}^{\infty} e^{-(2x+3y)}\, dy\, dx$$

$$= 6\int_0^2 \left[\frac{e^{-(2x+3y)}}{-3}\right]_0^{\infty} dx$$

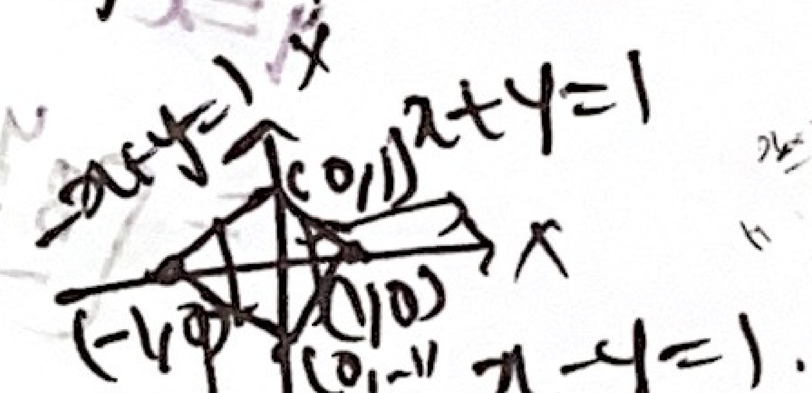
$$= 2\int_0^2 e^{-2x} dx = 1 - e^{-4}.$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= 6\int_{0}^{\infty} e^{-(2x+3y)}\,dy$$
$$= \left[\frac{e^{-(2x+3y)}}{-3}\right]_0^{\infty}$$
$$= 2e^{-2x}, \quad x \geq 0.$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= 6\int_{0}^{\infty} e^{-(2x+3y)}\,dx$$
$$= 6\left(\frac{e^{-2x-3y}}{-2}\right)_0^{\infty}$$
$$= +3e^{-3y}, \quad y \geq 0.$$

★ Q5) Suppose that CRV (2D) (x,y) is uniformly distributed over the square with vertices $(\pm1,0)$ $(0,\pm1)$. Find marginal pdf of x & y. $g(x)$ & $h(y)$.

A) $\text{pdf} = \dfrac{1}{\text{Area of Region}}, \quad x, y \in R.$



$$f(x,y) = \frac{1}{2}, \quad x,y \in \text{Square}$$

$$\frac{(y-y_1)}{y_2-y_1} = \frac{(x-x_1)}{x_2-x_1}$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$
$$= \int_{-1-x}^{1+x} \frac{1}{2}\,dy \quad -1 \leq x \leq 0.$$
$$= 1+x+1+x$$
$$= 2+2x$$

$$g(x) = \int_{x-1}^{1-x} 1/2\,dy,$$
$$= 1-x-x+1$$
$$= 2-2x$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx,$$
$$= \int_{-1-y}^{1+y} \frac{1}{2}\,dx \quad \left(= \int_{-1}^{0} \frac{1}{2}\,dx \quad (-1 \leq y \leq 0)\right).$$

$= \int_{y-1}^{1-y} \frac{1}{2}\,dx, \ 0 \le y \le 1.$

Independent Random Variables:

Let (X,Y) be a 2D DRV then we say that X & Y are independent if $P(x_i, y_j) = P(x_i) \cdot Q(y_j)$

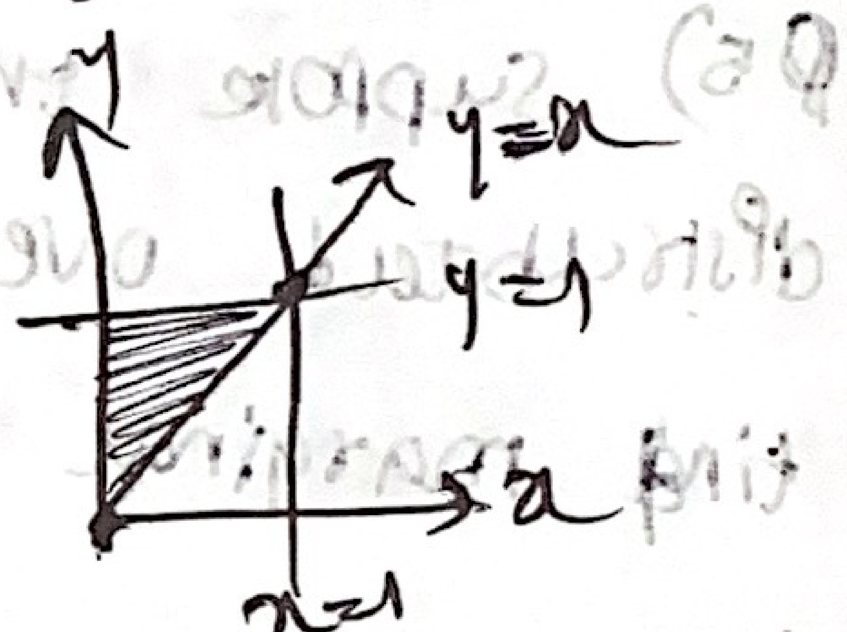
Let (X,Y) be a 2D CRV then we say that X & Y are independent if $f(x,y) = g(x)\,h(y)$.

Q) Ex: 2D RV (X,Y) has joint pdf $f(x,y) = 8xy,\ 0<x<y<1$

(i) find marginal pdf of (x,y)

(ii) are they independent?

$f(x,y) = 8xy$



$g(x) = \int_{y=x}^{1} 8xy\,dy, \ 0<x<1.$

$= \left[\frac{8xy^2}{2}\right]_x^1$

$= [4x - 4x^3], \ 0<x<1$

$h(y) = \int_{x=0}^{y} 8xy\,dx, \ 0<y<1$

$= \left(4x^2y\right)_0^y$

$= 4y^3, \ 0<y<1.$

(ii) $g(x) \cdot h(y) \neq f(x,y)$

X and Y are not independent

12/4/22

Expectation (Mean value):

If x, y is DRV,

$$E(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i\, P(x_i, y_j)$$

$$= \sum_{i=1}^{\infty} x_i\, P(x_i) \quad [\because \text{marginal pmf of } x]$$

$$E(Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y_j\, P(x_i, y_j).$$

$$= \sum_{j=1}^{\infty} y_j\, Q(y_j)$$

If x, y is CRV,

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot g(x)\, dx.$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} y \cdot h(y)\, dx.$$

Properties:

(i) $E(X+Y) = E(X) + E(Y)$

(ii) If x, y is 2D independent RV with joint pdf $f(x,y)$ then $E(XY) = E(X) \cdot E(Y)$.

Variance:

If x, y, 2D-RV & x, y are independent then,

$$V(X+Y) = V(X) + V(Y).$$

$$V(X+Y) = E(X+Y)^2 - [E(X+Y)]^2$$

$$= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2$$

$$= E(X^2) + 2\cancel{E(XY)} + E(Y^2) - [E(X)]^2 - 2\cancel{E(X)E(Y)} - [E(Y)^2]$$

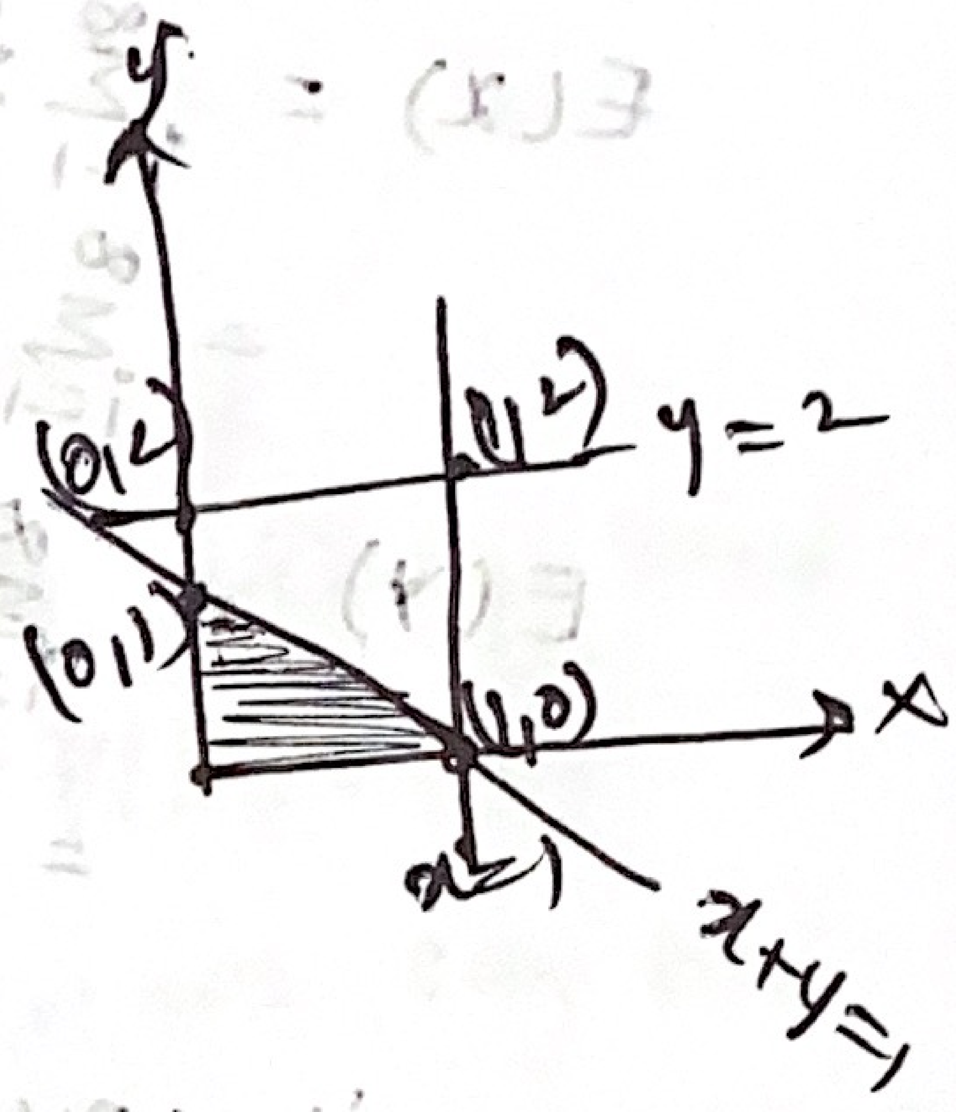
[$\because x, y$ are independent so $E(XY) = E(X) \cdot E(Y)$ cancelled].

$$= V(X) + V(Y)$$

(Q1)

A 2-D RV x,y has joint pdf

$$F(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$



(i) $P(X+Y \ge 1)$ (ii) $E(X)$

(iii) $V(Y)$.

(i) $P(X+Y \ge 1) = 1 - P(X+Y<1)$

$$= \cancel{\int_{x=1}^{2}\int_{y=1}^{2} x^2 + \frac{xy}{3}\,dx\,dy}$$

$$= \cancel{\int_{x=1}^{2}\left[x^2 y + \frac{xy^2}{6}\right]_1^2 dx}$$

$$= 1 - \int_{x=0}^{1}\int_{y=0}^{1-x} x^2 + \frac{xy}{3}\,dy\,dx, \quad (\text{or}) \int_{x=0}^{1}\int_{y=1-x}^{2} \left(x^2 + \frac{xy}{3}\right) dy\,dx$$

$$= 1 - \int_{x=0}^{1} \left[x^2 y + \frac{xy^2}{6}\right]_0^{1-x} dx$$

$$= 1 - \int_{x=0}^{1} x^2(1-x) + \frac{x(1^2 + x^2 - 2x)}{6}\,dx$$

$$= 1 - \frac{1}{6}\int_{x=0}^{1} -6x^3 + 6x^2 + x + x^3 - 2x^2 = 1 - 5x^3 + 4x^2 + x$$

$$= 1 - \frac{1}{6}\cancel{\left[-\frac{6x^4}{4} + \frac{x^3 6}{3} + \frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1}$$

$$= \cancel{1 - \frac{1}{6}\left[-\frac{3}{2} + \frac{2}{1} + \frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right]}$$

$$= \cancel{1 - \frac{x}{6}} = \frac{1}{6}\left[-\frac{5x^4}{4} + \frac{4x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= 1 - \frac{7}{72}$$

$$= \frac{65}{72}$$

(ii) $E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dy\,dx$

$$= \int_{x=0}^{1}\int_{y=0}^{2} x\left(x^2 + xy/3\right) dy\,dx$$

$$= \int_0^2 \left(\frac{x^4}{4} + \frac{x^3 y}{9}\right)_0^1 dy$$

$$= \int_0^2 \left(\frac{1}{4} + \frac{y}{9}\right) dy = \left(y/4 + \frac{y^2}{18}\right)_0^2 = \frac{1}{2} + \frac{4}{18} = \frac{13}{18}$$

(iii) $V(Y) = E(Y^2) - [E(Y)]^2 = \frac{26}{81}$

$E(Y) = \int_0^1 \int_0^2 y\,(x^2 + xy/3)\,dy\,dx = \frac{1}{3}\int_0^1\int_0^2 3x^2y + xy^2\,dy\,dx$

$= \cancel{\int_0^1 \frac{y^2}{2}\left[x^2 + \frac{xy^2}{3}\right]_0^2 dx}$ $= \frac{1}{3}\int_0^1 \left[\frac{3x^2y^2}{2} + \frac{xy^3}{3}\right]_0^2 dx$

$= \cancel{\int_0^1 \left[x^2 + \frac{4x}{2\times 3}\right] dx}$ $= \frac{1}{3}\int_0^1 \left[\frac{3\times 4x^2}{2} + \frac{8x}{3}\right] dx$

$= \cancel{\left[\frac{2x^3}{3} + \frac{2x^2}{4\times 3}\right]_0^1}$ $= \frac{1}{3}\left[\frac{6x^3}{3} + \frac{8x^2}{2\times 3}\right]_0^1$

$= \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$ $= \frac{10}{9}$ $= \frac{1}{3}\left[\frac{6}{3} + \frac{8}{2\times 3}\right]$

$= \frac{1}{3}\left[\frac{10}{3}\right] = \frac{10}{9}$

$E(Y^2) = \int_0^1 \int_0^2 y^2 (x^2 + xy/3)\,dy\,dx$

$= \frac{1}{3}\int_0^1\int_0^2 3y^2x^2 + xy^3\,dy\,dx$

$= \frac{1}{3}\int_0^1 \left[\frac{3y^3x^2}{3} + \frac{xy^4}{4}\right]_0^2 dx$

$= \frac{1}{3}\int_0^1 \left[8x^2 + \frac{16x}{4}\right] dx$

$= \frac{1}{3}\left[\frac{8x^3}{3} + \frac{4x^2}{2}\right]_0^1$

$= \frac{1}{3}\left[\frac{8}{3} + 2\right] = \frac{1}{3}\left[\frac{8+6}{3}\right] = \frac{14}{9}$

$E(Y^2) = \frac{14}{9}$

$V(Y) = \frac{14}{9} - \frac{100}{81} = \frac{26}{81}$.

(Q2) Joint pdf $f(x,y) = \begin{cases} e^{-(x+y)}, & x, y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$ $x \geq 0$, $y \geq 0$

(i) $P(X < Y)$

(ii) $P(X + Y \leq 1)$

(iii) $P(1 < X + Y < 2)$.

$\int_0^\infty \int_0^y e^{-(x+y)}\,dx\,dy$

$= \int_0^y \left[\frac{e^{-x+y}}{-1}\right]_0^\infty dx$

$= e^{-y} \int \left[e^{-x}\right]_0^y$

A) $P(1 < x+y < 2)$ B) $P(X+Y \le 1)$

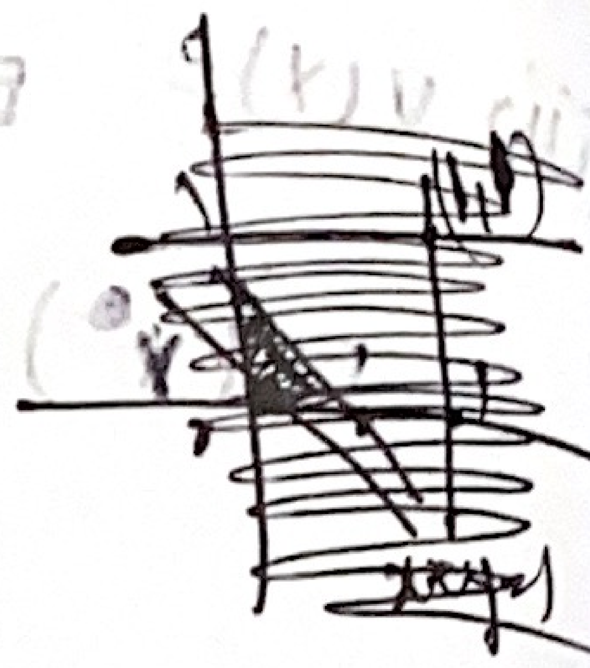
$P(X \le Y) =$

$= \int_0^{2-y}\int_0^{2-x} e^{-(x+y)}\,dy\,dx$ $= \int_0^1\int_0^1 e^{-(x+y)}\,dx\,dy$

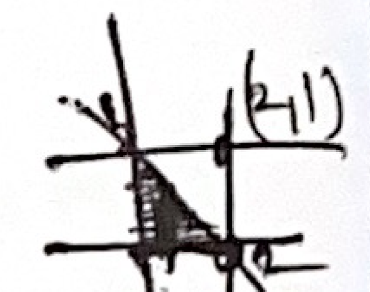
$= \int_0^{2-y} e^{-(x+2-x)}\,dx$ $= \int_0^1 \left[\frac{e^{-x+y}}{1}\right]_0^1 dx$

$= 0\left[\frac{e^{-x+1}}{1}\right]_0^1$

$= [e^{-2}] = 0.13533.$



$1 < x+y < 2$



$x+y=2$

y $0, 2-x$

x $0, 2-y$

14/4/22 (Standard deviation $= \sqrt{V(X)}$) σ.

The correlation coefficient:

Let (X,Y) 2-D RV we define corelation coefficient b/w X & Y

$$\rho_{xy} = \frac{E\{[X-E(X)][Y-E(Y)]\}}{\sqrt{V(X)V(Y)}}.$$

$E\{[X-E(X)][Y-E(Y)]\}$ → covariance of X and Y

σ_{xy} (or) $COV(X,Y)$.

Theorem:

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

$E\{XY - XE(Y) - E(X)Y + E(X)E(Y)\}$

$E\{XY\} - E(X)\cancel{E(Y)} - E(Y)E(X) + E(X)\cancel{E(Y)}$

$E(XY) - E(X)E(Y).$

Note ①. If X and Y are independent $E(XY) = E(X).E(Y)$

$\sigma_{xy} = 0$, $\rho_{xy} = 0$

Note ②. The converse is not true if $\rho = 0$ then x & y need not be independent

Eg: consider RV 'x' define RV $y = x^2$

$f(x) = \frac{1}{2}, \; -1 \leq x \leq 1$

$E(x) = \int_{-1}^{1} x \cdot f(x) dx$

$= \int_{-1}^{1} x \cdot \frac{1}{2} dx$

$= 0.$

$E(x^3) = 0$

Ans

$\sigma_{xy} = E(X^3) - E(X) E(X^2).$

$\sigma_{xy} = 0 - 0 = 0.$

$\sigma_{xy} = 0 \quad \therefore \rho_{xy} = 0$ but X & Y are not independent (note 2)

Note: $\rho = 0$

X and Y are uncorrelated then $\rho = 0$.

Theorem ②:

$-1 \leq \rho \leq 1.$

$$E\left[\frac{X - E(X)}{\sqrt{V(X)}} + \frac{Y - E(Y)}{\sqrt{V(Y)}}\right]^2 \geq 0.$$

$$E\left[\frac{X - E(X)}{V(X)} + \frac{Y - E(Y)}{V(Y)} + \frac{2(X - E(X))(Y - E(Y))}{\sqrt{V(X)V(Y)}}\right] \geq 0.$$

$$= \left[\frac{E(X) - E(X)^2}{V(X)} + \frac{E(Y) - E(Y)^2}{V(Y)} + \text{''}\right] \geq 0$$

$$= \left[\frac{V(X)}{V(X)} + \frac{V(Y)}{V(Y)} + \text{''}\right] \geq 0$$

$= [2 \pm 2\rho \geq 0]$

$= 1 \pm \rho \geq 0. \quad [\therefore -1 \leq \rho \leq 1]$

Theorem ③: If X & Y are linearly related then

$\rho = \pm 1$

since X & Y are linearly related

$Y = a + bX.$

$$\rho = \frac{E(XY) - E(Y)E(X)}{\sqrt{V(X)V(Y)}}$$

$$= \frac{E(aX + bX^2) - E(a + bX) E(X)}{\sqrt{V(X) V(a + bX)}}$$

$$= \frac{aE(X) + bE(X^2) - aE(X) - b[E(X)]^2}{\sqrt{V(X)\, b^2 (V(X))}}$$

$$= \pm \frac{E(X^2) - [E(X)]^2}{V(X)}$$

$$= \pm \frac{V(X)}{V(X)} = \pm 1.$$

***

Q1) If $U = a + bX$, $V = c + dY$ then PT

$$\rho_{UV} = \pm \rho_{XY}$$

Ans)

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sqrt{V(U)V(V)}}$$

$$= \frac{E[(a+bX)(c+dY)] - E(a+bX)E(c+dY)}{\sqrt{V(a+bX)V(c+dY)}}$$

$$= \frac{E[ac + adY + bcX + bXdY] - \cancel{aE(X) - bE(X) \cdot cE(X) - dE(Y)} [a + bE(X)][c + dE(Y)]}{\sqrt{b^2 d^2 V(X) V(Y)}}$$

$$= \frac{ac + adE(Y) + cbE(X) + bdE(XY) - [a + bE(X)][c + dE(Y)]}{\sqrt{b^2 d^2 V(X) V(Y)}}$$

$$= \frac{bd[E(XY) - E(X)E(Y)]}{\pm bd\sqrt{V(X)V(Y)}} = \pm \rho_{xy}$$

Q2) PT $V(aX + bY) = a^2 V(X) + b^2 V(Y) + 2ab\,\sigma_{xy}$.

Ans

$$V(aX+bY) = E[(aX+bY) - E(aX+bY)]^2$$

$$= E[aX + bY - aE(X) - bE(Y)]^2$$

$$= E[a(X - E(X)) + b(Y - E(Y))]^2$$

$$= E[a^2(X - E(X))^2 + b^2(Y - E(Y))^2 + 2ab(X - E(X))(Y - E(Y))].$$

$$= a^2 E(X - E(X))^2 + b^2 E(Y - E(Y))^2 + 2ab\,\sigma_{xy}]$$

$$= a^2 (V(X)) + b^2 (V(Y)) + 2ab\,\sigma_{xy}.$$

Hence proved.

Q3) If X_1, X_2, X_3 are uncorrelated random having the same standard deviation find the correlation coefficient b/w X_1+X_2 and X_2+X_3.

Ans)

$$\rho_{X_1X_2} = \rho_{X_2X_3} = \rho_{X_1X_3} = 0$$

$$V(X_1) = V(X_2) = V(X_3) = \sigma^2$$

let $U = X_1+X_2$, $V = X_2+X_3$

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sqrt{V(U)V(V)}}$$

$$= \frac{E((X_1+X_2)(X_2+X_3)) - E(X_1+X_2)E(X_2+X_3)}{\sqrt{V(X_1+X_2)V(X_2+X_3)}}$$

$$= \frac{E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) - E(X_1)E(X_2) - E(X_1)E(X_3) - [E(X_2)]^2 - E(X_2)E(X_3)}{\sqrt{(V(X_1)+V(X_2))(V(X_2)+V(X_3))}}$$

$$= \frac{E(X_2^2) - E(X_2)^2}{\sqrt{4\sigma^4}} = \frac{E(X_2^2) - E(X_2)^2}{\pm 2\sigma^2} = \frac{V(X_2)}{2\sigma^2}$$

$$= \frac{\sigma^2}{2\sigma^2} = \pm 1/2$$

Q4) If X, Y, Z are uncorrelated RV with standard deviation 5, 12, 9 respectively. If $U = X+Y$, $V = Y+Z$, find ρ_{UV} (correlation coefficient).

A)

$$V(X) = 25 \quad | V(Y) = 144, \ V(Z) = 81$$

$$\rho_{XY} = \rho_{YZ} = \rho_{ZX} = 0.$$

let $U = X+Y$, $V = Y+Z$.

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sqrt{V(U)V(V)}}$$

$$= \frac{E[(X+Y)(Y+Z)] - E(X+Y)\cdot E(Y+Z)}{\sqrt{V(X+Y)V(Y+Z)}}$$

$$= \frac{E[(XY + XZ + Y^2 + YZ)] - E(X)\cancel{E(XY)}\cdot E(Y) - E(X)E(Z) - E(Y^2) - E(Y)E(Z).}{\sqrt{(V(X)+V(Y)) + V(X) + V(Z)}}$$

$$= \frac{E(XY) + E(\cancel{XZ}) + E(Y^2) + E(\cancel{XZ}) - \cancel{E(X)}E(X) - E(X)\cancel{E(Z)} - [E(Y^2)] - E(X)\cancel{E(Z)}.}{\sqrt{25\times144 + 25\times81}}$$

$$= \frac{E(Y^2) - [E(X)]^2}{75}$$

$$= \frac{V(X)}{75} = \frac{144}{75} = \frac{48}{25} = 1.92$$

18/4/22

conditional expectation:

If (X,Y) DRV. we define conditional expectation of x for given y.

$$P(X/Y) = \sum_{i=1}^{\infty} x_i P(x_i/y_j) \qquad E(Y/X) = \sum_{j=1}^{\infty} y_j P(y_j/x_i)$$

$$= \sum x_i \frac{P(x_i, y_j)}{q(y_j)} \qquad = \sum_{j=1}^{\infty} y_j \frac{P(x_i, y_j)}{P(x_i)}$$

If (X,Y) is CRV

$$E(X/Y) = \int_{-\infty}^{\infty} x \frac{f(x,y)}{h(y)}\,dx$$

$$E(Y/X) = \int_{-\infty}^{\infty} y \frac{f(x,y)}{g(x)}\,dy.$$

$\frac{3}{64}$

$\{(x_1y_1)\} = \{(6,4)(4,6),(5,5)\}$,

$\frac{1}{12} =$

1, 2, 3

$x^2 - 4ac$

$b^2 - 4ac$

$-4y^2(4,4)$

$b \geq 0$

123 321 213 312

(1 2 3) (1 3 2) (2 1 3) 2 3 1 3 1 2 3 2 1

$3 = 6. P(A) = \frac{3}{8}$

$P(A) = 1/2$

$P(B) = 2/3$

$P(A \cup B) = \frac{5}{6}$

$P(A) + P(B) + P(A \cap B) = P(A \cup B)$

$P(A \cap B) = \frac{5}{6} - 1/2 - 2/3$

$= 1/3$

$P(C \cap (A \cup B)) = \frac{5}{12}$

$P(A \cap B) = 1/3$

$P(C) + P(A \cup B) + P(A \cap B) + P(C \cap (A \cup B)) = \frac{5}{12}$

$\frac{5}{12} - 1/3 - 1/2 = P(C)$

$\frac{5 - 4 - 6}{12}$

$g(6) = 6$

$y =$

$y(0) = 0$

$\frac{3}{64}$

$b^2 - 4ac < 0$

$\frac{{}^3C_4}{{}^3C_9}$

$x < 0$.

4 vowels

5 cons

6 ser

6^7

$E(x) = \int_{-\infty}^{\infty} x \cdot g(x) dx = \int\int_{-\infty}^{\infty} x f(x,y) dy dx$

$E(Y) = \int_{-\infty}^{\infty} y \cdot h(y) dy = \int\int_{-\infty}^{\infty} y f(x,y) dy dx$.

$g(x) = \int_{-\infty}^{\infty} f(x,y) dy$

$h(y) = \int_{-\infty}^{\infty} f(x,y) dx$

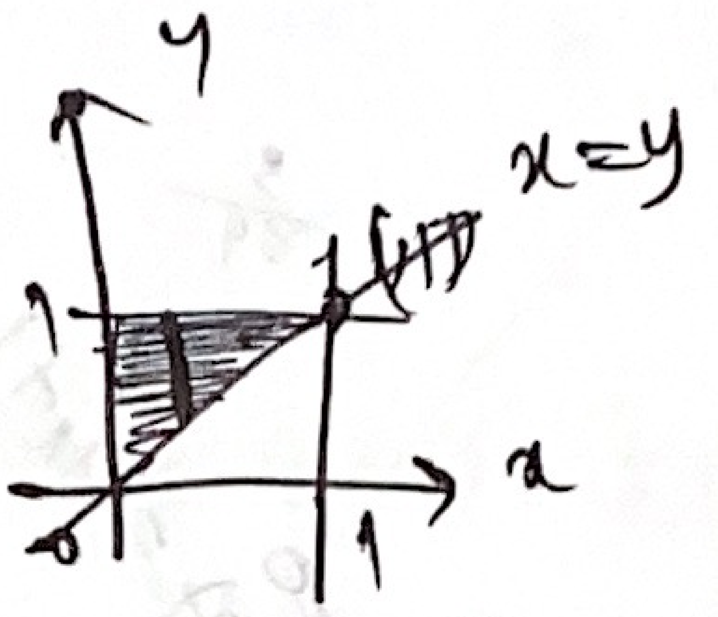
$P(X|Y) = \sum_{i=1}^{\infty} x_i P(x_i, y_j)$

$= \sum_{i=1}^{\infty} x_i \frac{P(x_i, y_j)}{q(y_j)}$

$E(X|Y) = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{h(y)} dx$.

$E(Y|X) = \int_{-\infty}^{\infty} y \cdot \frac{f(x,y)}{g(x)} dy$

Q1) Let $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{elsewhere} \end{cases}$



find, (i) $E(Y/X)$ (ii) $V(Y/X)$

Ans)

(i) $E(Y/X) = ?$

$Y|X$

$$E(Y|X) = \int_{-\infty}^{\infty} y \cdot P(\cancel{X|Y})\, dy$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=x}^{1} 8xy\, dy$$

$$= \left[\frac{8xy^2}{2}\right]_x^1$$

$$= 4x - 4x^3$$

$$g(x) = 4(x - x^3)$$

$$E(Y|X) = \int_x^1 y \cdot \frac{8xy}{4x-4x^3}\, dy$$

$$= \left[\frac{y^2}{2} \cdot \frac{8xy^2}{2(4x-4x^3)}\right]_x^1$$

$$= \frac{2x}{4x-4x^3} - \frac{x^2 \cdot x^3}{4x-4x^3}$$

$$= \frac{2x - x^5}{4x-4x^3} = \frac{x(2-x^4)}{x(4-4x^3)} = \frac{2-x^4}{4(x-x^3)}$$

$$= \frac{2}{3(1-x^2)}(1-x^3)$$

$$= \frac{2}{3(1+x)(1-x)} \cdot (1-x)(1+x+x^2)$$

$$= \frac{2(1+x+x^2)}{3(1+x)}$$

$(0<x<1)$

$$E(Y/X) = \frac{2(1+x+x^2)}{3(1+x)}$$

(ii) $V(Y|X) = E\{(Y|X)^2\} - [E(Y|X)]^2$

$$E(Y^2|X) = \int_x^1 y^2 \cdot \frac{8xy}{4x-4x^3}\, dy$$

$$= \frac{2}{1-x^2} \int_x^1 y^3\, dy$$

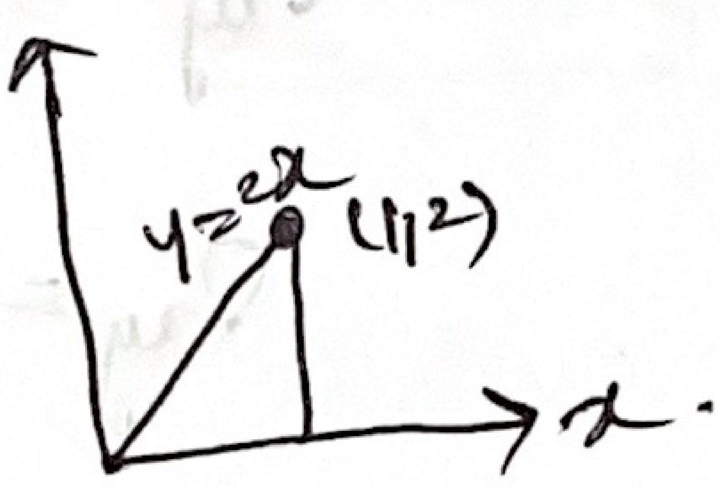
$$= \frac{2}{4(1-x^2)}(1-x^4)$$

$$= \frac{1}{2(1-x^2)}(1+x^2)(1-x^2)$$

$$= \frac{1+x^2}{2}$$

$$V(Y/X) = \frac{1+x^2}{2} - \left[\frac{2(1+x+x^2)}{3(1+x)}\right]^2 //.$$

Q2) suppose that (X,Y) is uniformly distributed over triangular region., then find marginal and conditionaled pdf's of X & Y.



Ans) $f(x,y) = \begin{cases} 1, & (x,y) \in D^{le} \\ 0, & \text{else where} \end{cases}$

$\rightarrow$ Pdf $= \dfrac{1}{\text{area of given region}}$ Pdf $= \dfrac{1}{\frac{1}{2}\times 2\times 1} = 1$

$g(x) = ?$ $h(y) = 2$ $P(X/Y) = ?$ $P(Y/X) = ?$

$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_0^{2x} 1 \cdot dy.$

$= 2x \quad (0<x<1).$

$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_{y/2}^{1} 1\,dx.$ $(0<y<2)$

$= 1 - y/2 = \frac{2-y}{2},$

$P(X/Y) = \dfrac{f(x,y)}{g(x)} = \dfrac{1}{2x}$ $(0<x<1)$

$P(Y/X) = \dfrac{f(x,y)}{h(y)} = \dfrac{1}{\frac{2-y}{2}}$

$= \dfrac{2}{2-y}. \quad (0<y<2)$

① $E(X/Y) = \int_{-\infty}^{\infty} x \cdot P(X/Y)\,dx$

$= \int_{y/2}^{1} \frac{2x}{2-y}\,dx, \; 0<y<2. \quad = \frac{2+y}{4}$

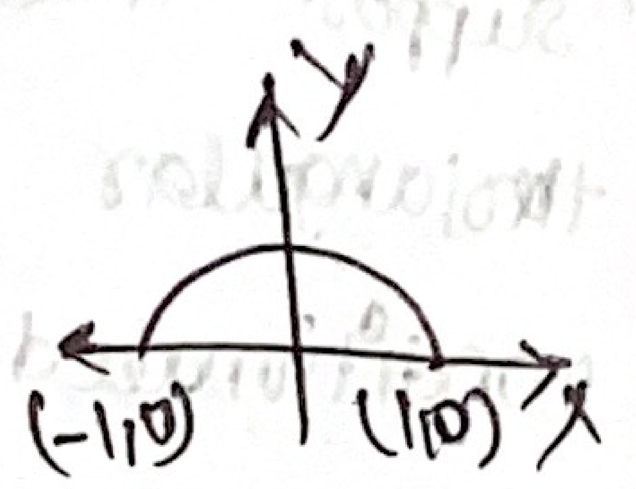
② $E(Y/X) = \int_{-\infty}^{\infty} y \cdot P(Y/X)\,dy$

$= \int_0^{2x} \frac{y}{2x}\,dy, \; 0<x<1. = x.$

$\frac{1}{x}\left(\frac{y^2}{2}\right)^{2x} = \frac{1}{2}\,\frac{4x^2}{2x} = x$

Q3) suppose a 2D RV (X,Y) uniformly distributed R where R is defined by $R:\{(x,y)/x^2+y^2\le 1, y\ge 0\}$. find correlation coefficient

A)



$$\rho_{xy} = \frac{E\{x-E(x))(y-E(y))\}}{\sqrt{V(x)V(y)}}$$

$$\rho_{xy} = \frac{E(XY)-E(X)E(X)}{\sqrt{V(X)V(Y)}}.$$

Pdf = $\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

$f(x,y) = \begin{cases} \frac{2}{\pi}, (x,y)\in R \\ 0, \text{elsewhere} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot f(x,y)\, dy\, dx.$$

$$= \int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} x\cdot\frac{2}{\pi}\, dy\, dx$$

$E(X) = 0$

$$\int_0^{\pi/2}\sin^n\theta\, d\theta = \int_0^{\pi/2}\cos^n\theta\, d\theta = \begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & n \text{ is even}\end{cases}$$

$$E(X^2) = \frac{2}{\pi}\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2\, dx\, dy$$

$$= \frac{4}{\pi}\int_0^1\int_0^{\sqrt{1-y^2}} x^2\, dx\, dy$$

$$= \frac{4}{\pi}\int_0^1\left(\frac{1}{3}\right)(1-y^2)^{3/2}\, dy.$$

$y = \sin\theta$

$dy = \cos\theta\, d\theta$

$$= \frac{4}{3\pi}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{4}{3\pi}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{1}{4}.$$

$E(Y) = ?$

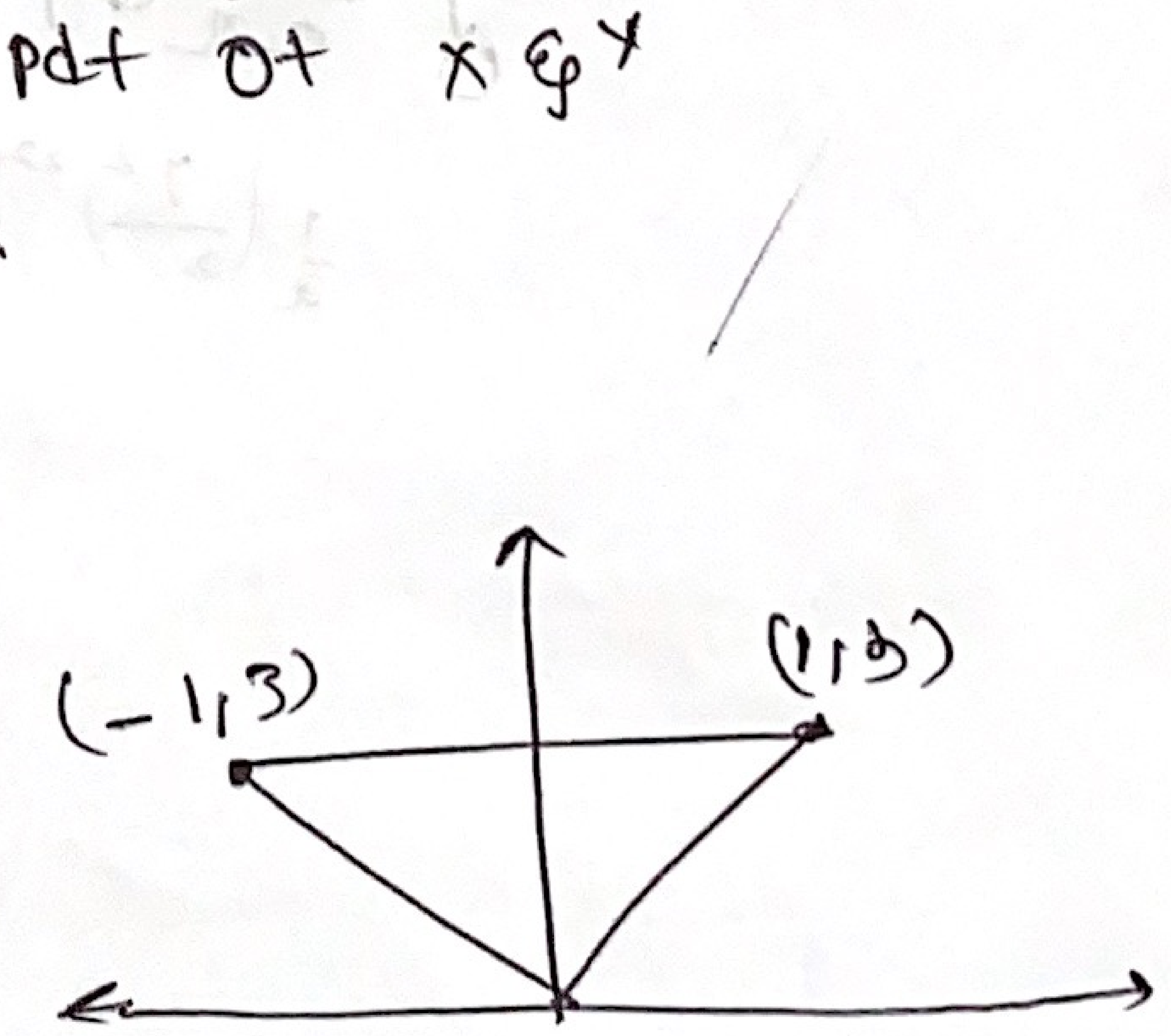
$E(Y^2) = ?$

$E(XY) = ?$

19/4/22

Q4) Suppose that X, Y is uniformly distributed over the triangle (1) obtain the marginal pdf of X & Y

(2) evaluate $V(X), V(Y)$.



Q5) If X & Y are independent Random variables then find $E(X/Y)$ and $E(Y/X)$.

Q6) suppose that X, Y has pdf $f(x,y) = \begin{cases} k e^{-y}, & 0<x<y<1 \\ 0, & \text{elsewhere} \end{cases}$

find the corelation coefficient. $\boxed{k=?} \leftarrow \boxed{\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1}$

$(\rho_{xy} = ?)$.

Difference Equation:

A difference equation is a relation b/w the difference of an unknown function at one or more general values of the argument.

$$\Delta y_{n+1} + y_n = 2 \longrightarrow ①$$

$$\Delta y_{n+1} + \Delta^2 y_{n-1} = 1 \longrightarrow ②$$

equation's 1 & 2 can be rewritten as

$$① \Rightarrow y_{n+2} - y_{n+1} + y_n = 2 \rightarrow ③$$

$$② \Rightarrow y_{n+2} - \cancel{y_{n+1}} + \cancel{y_{n+1}} - y_n - y_n + y_{n-1} = 1.$$

$$② \Rightarrow y_{n+2} + y_{n-1} - 2y_n = 1 \rightarrow ④.$$

Formation of Difference equation:

$y = ax + bx^2$

$\Delta y = a\Delta x + b\Delta(x^2)$

$= a(x+1-x) + b[(x+1)^2 - x^2]$.

$= a + b(2x+1)$.

$\Delta^2 y = 2b$.

$$\begin{vmatrix} y & x & x^2 \\ \Delta y & 1 & (2x+1) \\ \Delta^2 y & 0 & 2 \end{vmatrix} = 0.$$

$= y(2-0) - x(2\Delta y - \Delta^2 y(2x+1)) + x^2(-\Delta^2 y) = 0$

$2y - 2x\Delta y + (2x^2 + x - x^2)\Delta^2 y = 0$

$(x^2 + x)\Delta^2 y - 2x\Delta y + 2y = 0$