$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

$$+ \frac{1}{2}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \left(\frac{z-1}{2}\right)^3 + \cdots\right], \quad |z-1|<1$$

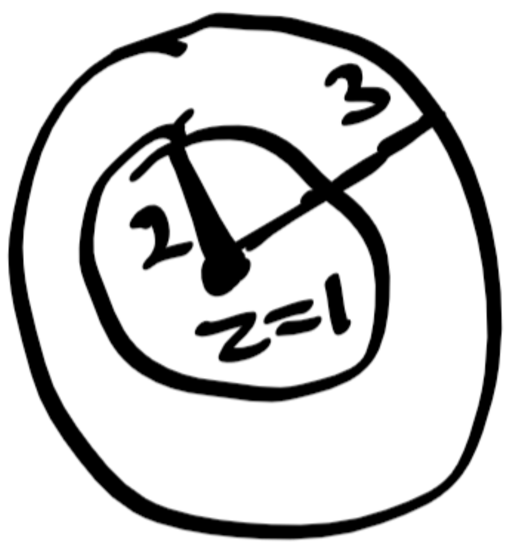
$$\left|\frac{z-1}{2}\right|<1$$

$$\Rightarrow |z-1|<2$$

$$|z-1|<1$$

$$\therefore f(z) = \frac{1}{9}\left\{1 - 2\left(\frac{z-1}{3}\right) + 3\left(\frac{z-1}{3}\right)^2 - 4\left(\frac{z-1}{3}\right)^3 + \cdots\right\}$$

$$- \left\{1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \left(\frac{z-1}{2}\right)^3 + \cdots\right\}, \quad |z-1|<2$$



$$\left|\frac{z-1}{3}\right| < 1 \Rightarrow |z-1| < 3$$

$$\left|\frac{z-1}{2}\right| < 1 \Rightarrow |z-1| < 2$$

2) $f(z) = \dfrac{3}{3z-z^2}$, centre at $z = 1$

$$\frac{3}{3z-z^2} = \frac{3}{z(3-z)} = \frac{A}{z} + \frac{B}{3-z}$$

$$A(3-z) + Bz = 3$$

$$z=0 \Rightarrow 3A = 3 \Rightarrow A = 1$$

$$z=3 \Rightarrow 3B = 3 \Rightarrow B = 1$$

$$\therefore f(z) = \frac{1}{z} + \frac{1}{3-z} = \frac{1}{z} - \frac{1}{z-3}$$

$$= \frac{1}{z-1+1} - \frac{1}{z-1-2}$$

$$= \left[1 + (z-1)\right]^{-1} + \frac{1}{2\left[1 - \frac{(z-1)}{2}\right]}$$

$$= \left[1 + (z-1)\right]^{-1} + \frac{1}{2}\left[1 - \frac{(z-1)}{2}\right]^{-1}$$

Find the Taylor's series expansion of

(1) $f(z) = \dfrac{2z^2+9z+5}{z^3+z^2-8z-12}$ with centre at $z=1$

$\Sigma C_n (z-a)^n$

$$\frac{2z^2+9z+5}{z^3+z^2-8z-12} = \frac{2z^2+9z+5}{(z+2)^2(z-3)} = \frac{A}{(z+2)} + \frac{B}{(z+2)^2} + \frac{C}{z-3}$$

$$A(z+2)(z-3) + B(z-3) + C(z+2)^2 = 2z^2+9z+5$$

$z=3 \Rightarrow 25C = 50 \Rightarrow C=2$

$z=-2 \Rightarrow -5B = -5 \Rightarrow B=1$

$z=0 \Rightarrow -6A-3B+4C = 5 \Rightarrow -6A = 5+3-8 = \underline{\underline{0}}$

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{(z-1+3)^2} + \frac{2}{z-1-2}$$

$$= \frac{1}{9\left[1+\frac{z-1}{3}\right]^2} - \frac{1}{\left[1-\frac{z-1}{2}\right]}$$

$$= \frac{1}{9}\left[1+\frac{z-1}{3}\right]^{-2} - \left[1-\frac{z-1}{2}\right]^{-1}$$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots, \quad |x|<1$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

$(1-x)^{-2} = 1+2x+3x^2+\cdots$

$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3+\cdots$

2) $f(z) = e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z,

3) $f(z) = \cos z$

$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$

$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{z^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{z^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{z^2}{(2n+2)(2n+1)} \right| = 0 < 1$$

series converges for all z. $\therefore R = \infty$

Taylor's Series

A function $f(z)$ which is analytic at all points within a circle C with centre at z_0 and radius R can be represented uniquely as a convergent power series known as Taylor's Series given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

where $$c_n = \frac{f^{(n)}(z_0)}{n!}$$

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$

If $z_0 = 0$ then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ $\longrightarrow$ Maclaurin's Series.

Some standard Maclaurin's Series

1) $f(z) = \frac{1}{1-z}$

$f(0) = 1$ $\quad f'(z) = \frac{1}{(1-z)^2}$, $f'(0) = 1$

$f''(z) = \frac{2}{(1-z)^3}$, $f''(0) = 2$

$$\frac{1}{1-z} = 1 + z + \frac{2}{2!}z^2 + \cdots = 1 + z + z^2 + \cdots,$$

$$= \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

Examples

1) Geometric Series $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$ converges absolutely if $|z| < 1$ and diverges if $|z| > 1$.

$$u_n = z^n \qquad u_{n+1} = z^{n+1}$$

Ratio test

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$$

By Ratio test if $|z| < 1$, series converges.

$$R = 1.$$

2) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\lim_{n \to \infty} \left| \frac{z^{n+1}}{(n+1)!} \times \frac{n!}{z^n} \right| = \lim_{n \to \infty} \left| \frac{z}{n+1} \right| = 0 < 1$$

series converges for all z

$$R = \infty$$

Power series

A power series in powers of $(z-a)$ is

$$\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

where z is a complex variable, $c_0, c_1, c_2 \ldots$ are complex or real constants and a is called the centre of the series.

If $a=0$, we get a power series in powers of z,

$$\sum_{n=0}^{\infty} c_n z^n .$$

Three distinct possibilities exist regarding the region of convergence.

(1) The series converges only at the point $z=a$

(2) The series converges for all z.

(3) The series converges everywhere inside a circular disk $|z-a|<R$ and diverges outside the disk $|z-a|>R$.

Here R is called the radius of convergence.

The circle $|z-a|=R$ is called circle of convergence.

Note:- (i) The series may converge or diverge at the points on the circle of convergence.

(ii) If the series converges for all z then $R=\infty$

(iii) If the series converges only at $z=0$ then $R=0$