2) Evaluate $\int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$

Ans: Since, limits of inner integral are fn of $x$, integrate w.r.to $\underline{\underline{y}}$ keeping $x$ constant.

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y)\, \underline{dy}\, dx = \int_{x=0}^{1} \left(xy + \frac{y^2}{2}\right)_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} x(1-x) + \frac{(1-x)^2}{2}\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{2}\left(\frac{(1-x)^3}{(-3)}\right)\right]_{x=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{2}\left(\frac{1}{-3}\right)$$

$$= \underline{\underline{1/3}}$$

3) Evaluate $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{1-x^2}\, \sqrt{1-y^2}}$

Ans: Here, we note that limits of both the integrals are constants, and if variables can be separated then the ~~the~~ double integrals is a product of two integrals and can be evaluated separately.

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \times \int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= \left[\sin^{-1} x\right]_0^1 \times \left[\sin^{-1} y\right]_0^1 = \pi/2 \times \frac{\pi}{2} = \frac{\pi^2}{4}$$

4) Evaluate $\int_0^2 \int_0^1 \frac{dx\, dy}{x+y+1}$

<u>Ans</u>: $I = \int_{y=0}^{2} \int_{x=0}^{1} \frac{dx}{x+y+1} \quad dy = \int_{y=0}^{2} \log(x+y+1)\Big|_{x=0}^{1} \, dy$

$$= \int_{y=0}^{2} \left[\log(y+2) - \log(y+1)\right] dy$$

$$\boxed{\int \log x \, dx = x\log x - x}$$

$$= (y+2)\log(y+2) - (y+2) - \left\{(y+1)\log(y+1) - (y+1)\right\}\Big|_{y=0}^{2}$$

$$= (y+2)\log(y+2) - (y+2) - (y+1)\log(y+1) + (y+1)\Big|_{y=0}^{2}$$

$$= 4\log 4 - 4 - 3\log 3 + 3 - 2\log 2 + 2 - 1$$

$$= \log 4^4 - \log 3^3 - \log 2^2$$

$$= \log 4^4 - (\log 27 + \log 4)$$

$$= \log(256) - \log(108)$$

$$= \log\left(\frac{256}{108}\right)$$

5) Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

Ans: $$I = \int_{y=0}^{1} \left( \int_{x=0}^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx}{\sqrt{1-y^2-x^2}} \right) dy$$

Here $1-y^2$ is constant

write $1-y^2 = a^2$

$$= \int_{y=0}^{1} \left( \int_{x=0}^{\frac{1}{\sqrt{2}}a} \frac{dx}{\sqrt{a^2-x^2}} \right) dy$$

$$= \int_{y=0}^{1} \left[ \sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^{a/\sqrt{2}} dy = \int_{y=0}^{1} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) dy$$

$$= \int_{y=0}^{1} \pi/4 \, dy$$

$$= \frac{\pi}{4} \left[ y \right]_0^1 = \frac{\pi}{4}$$

(6) Evaluate $\displaystyle\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} \, x \, dx \, dy$

Ans: $$I = \int_{y=0}^{\infty} \underbrace{\left\{ \int_{x=0}^{\infty} e^{-x^2(1+y^2)} \, x \, dx \right\}}_{I_1} dy \qquad \text{--- (1)}$$

Consider $I_1 = \int_{x=0}^{\infty} e^{-x^2(1+y^2)} x\,dx$,

Here $1+y^2$ constant

write $1+y^2 = k$

$$= \int_{x=0}^{\infty} e^{-x^2 k} x\,dx$$

Let $x^2 k = z$

$2kx\,dx = dz$

$x\,dx = \frac{1}{2k} dz$

$$= \int_{z=0}^{\infty} e^{-z} \frac{1}{2k} dz$$

when $x=0$, $z=0$

$x=\infty$ $z=\infty$

$$= \frac{1}{2k}\left(-e^{-z}\right)_0^{\infty} = \frac{1}{2k}\left(-e^{-\infty} + e^{0}\right)$$

$$\boxed{I_1 = \frac{1}{2k}(0+1) = \frac{1}{2k} = \frac{1}{2(1+y^2)}}$$

Eq ① $\Rightarrow$

$$I = \int_{y=0}^{\infty} \frac{1}{2(1+y^2)} dy = \frac{1}{2}\left(\tan^{-1} y\right)_{y=0}^{\infty}$$

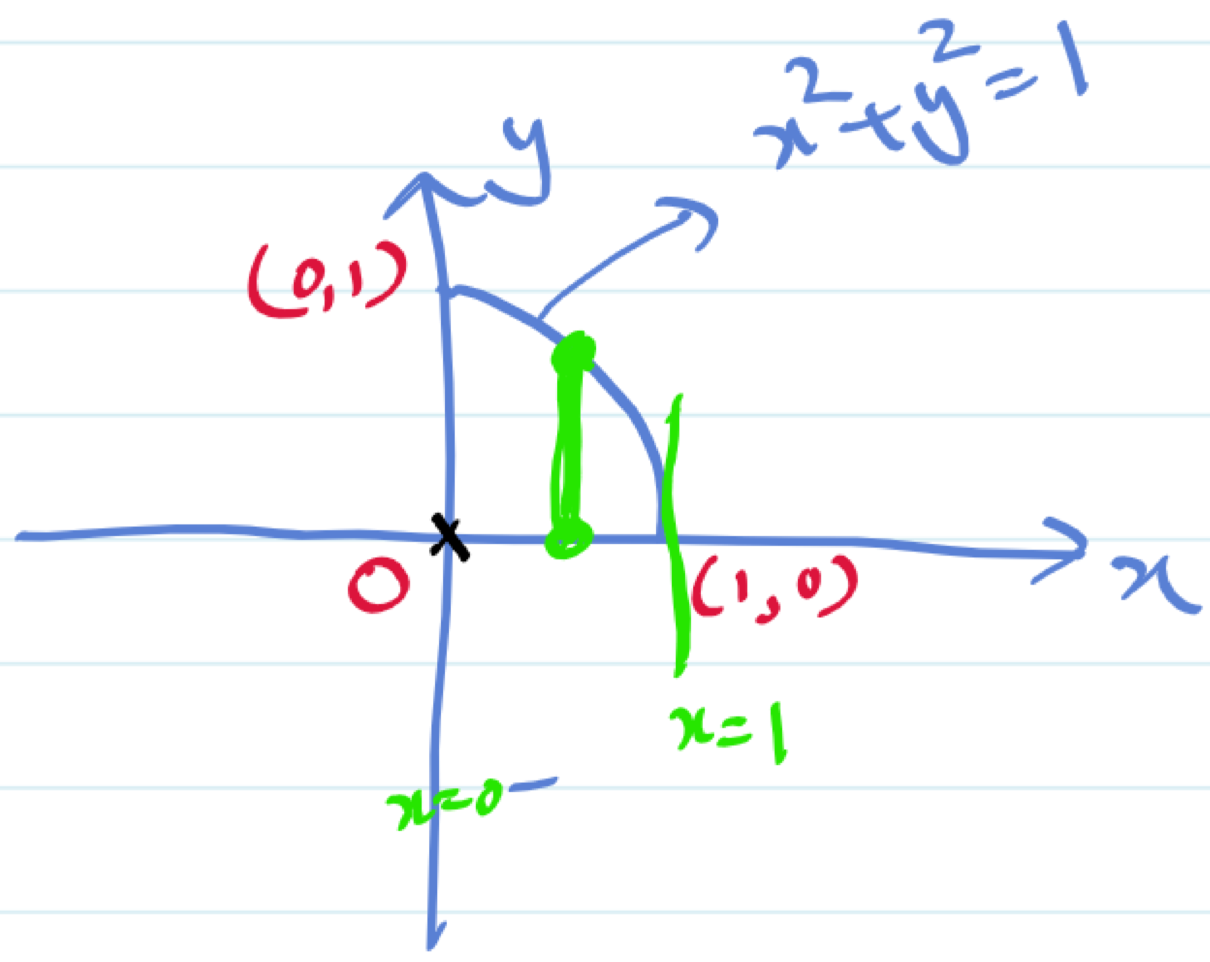
$$= \frac{1}{2}\left(\tan^{-1}\infty - \tan^{-1} 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$$

Problems on double integrals when limits are not provided –

① Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=1$.

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy \; dy \; dx$$

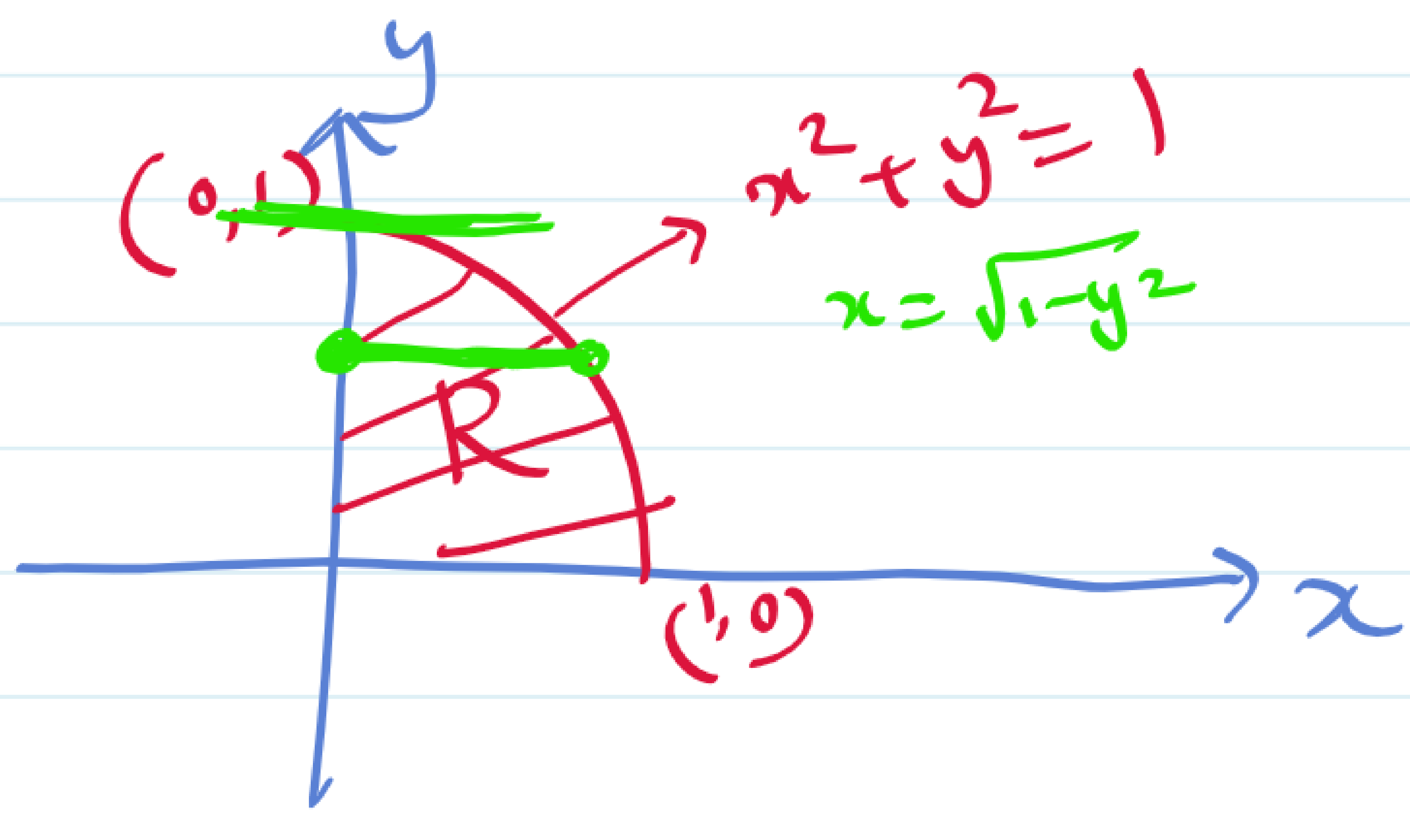



$$= \int_{x=0}^{1} x\left(\frac{y^2}{2}\right)_{y=0}^{\sqrt{1-x^2}} dx$$

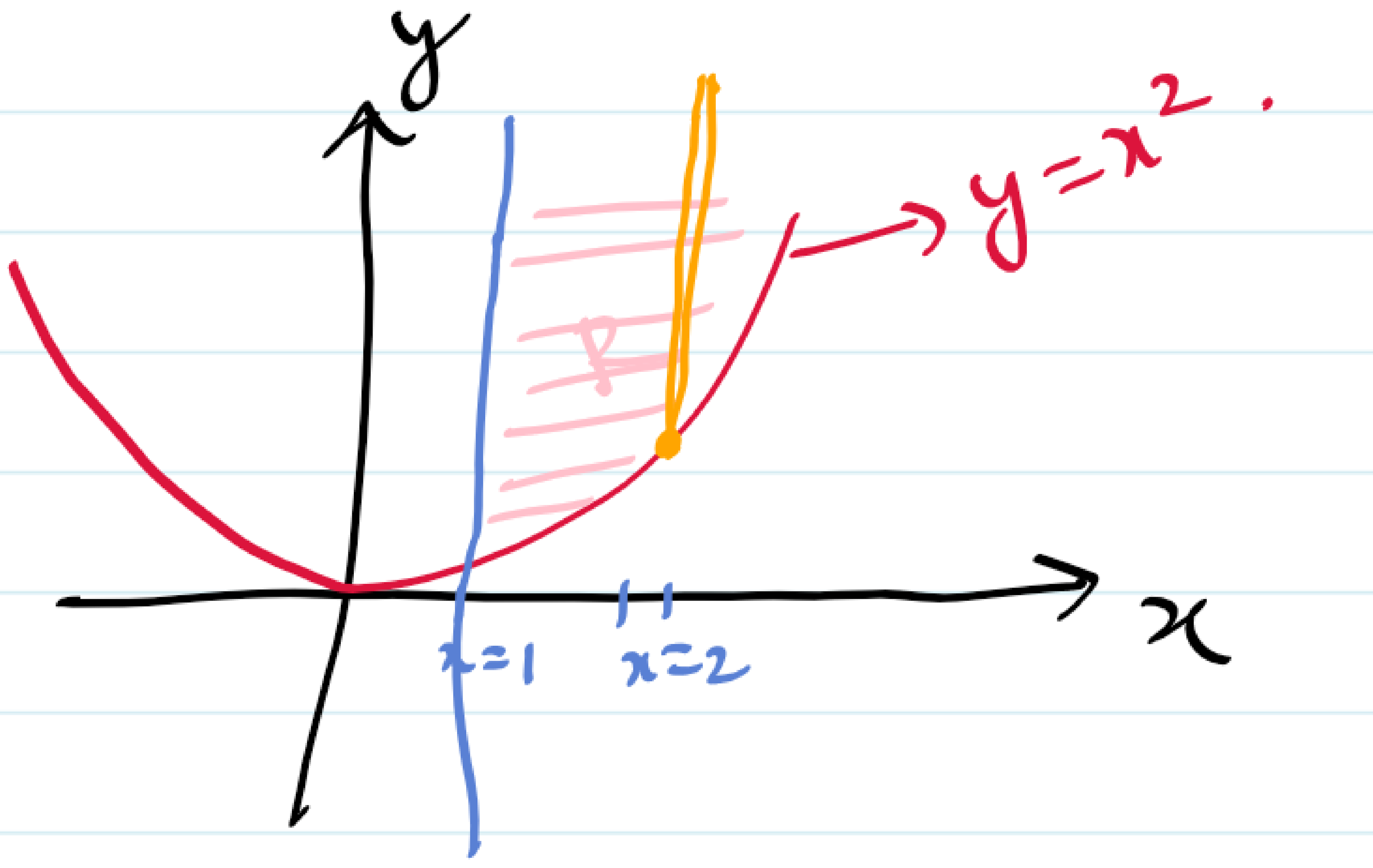
$$= \int_{x=0}^{1} x\left(\frac{1-x^2}{2}\right)dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{8}$$

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} (xy)\; dx\; dy$$



2) Evaluate $\iint \frac{1}{x^4+y^2}\,dx\,dy$ over the region $y \geq x^2$, $x \geq 1$.




Ans:

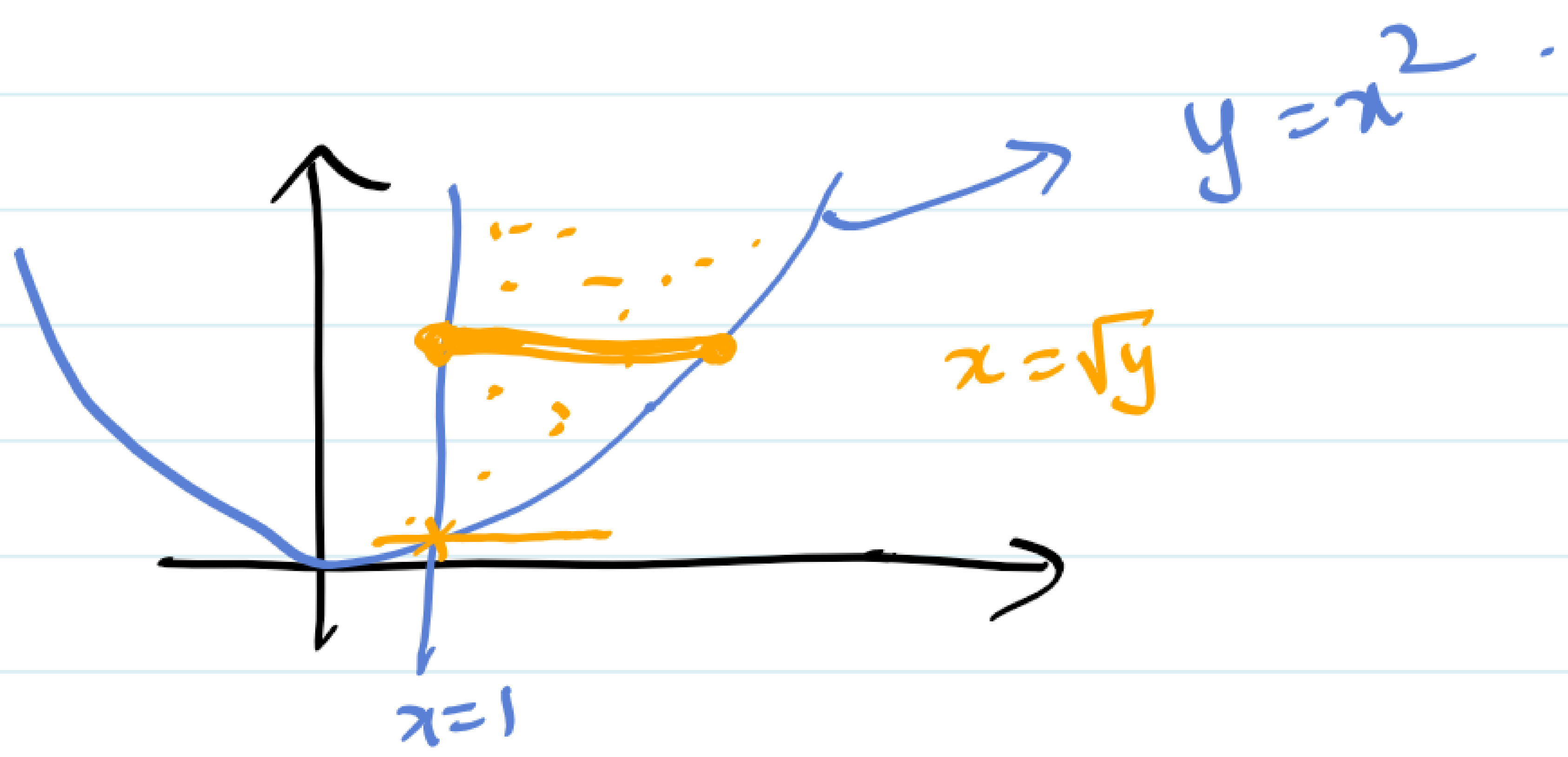
$$I = \int_{x=1}^{\infty}\int_{y=x^2}^{\infty} \frac{1}{(x^2)^2+y^2}\,dy\,dx$$

$$= \int_{x=1}^{\infty} \frac{1}{x^2}\tan^{-1}\left(\frac{y}{x^2}\right)\Bigg|_{y=x^2}^{\infty}\,dx$$

$$\left[\int \frac{dy}{a^2+y^2} = \frac{1}{a}\tan^{-1}(y/a)\right]$$

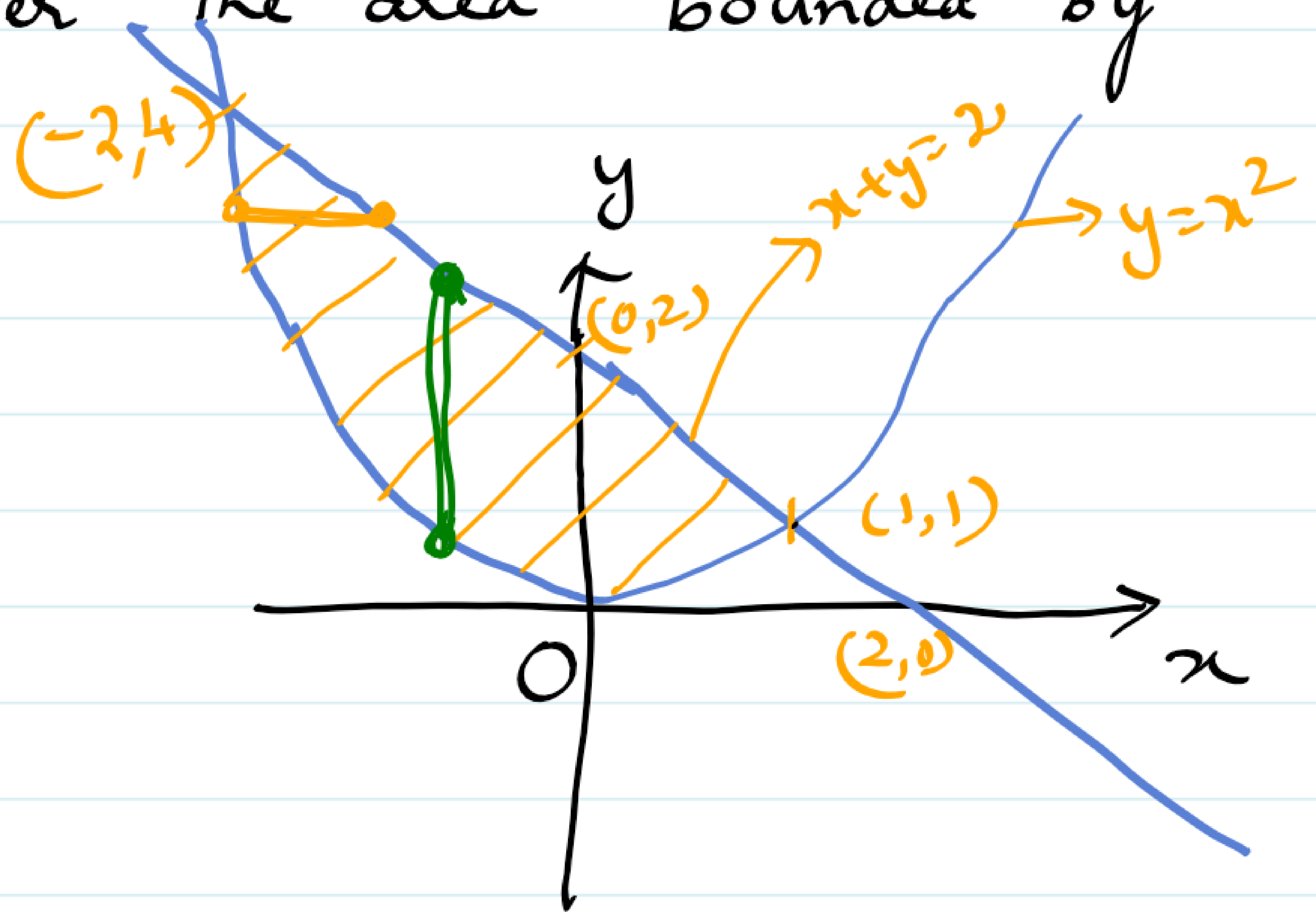
$$= \int_1^{\infty} \frac{1}{x^2}(\tan^{-1}\infty - \tan^{-1}1)\,dx$$

$$= \int_1^{\infty} \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)dx = \frac{\pi}{4}\left(-\frac{1}{x}\right)\Bigg|_1^{\infty} = 0+\frac{\pi}{4} = \pi/4$$




$$I = \int_{y=1}^{\infty}\int_{x=1}^{\sqrt{y}} \frac{1}{(x^2)^2+y^2}\,dx\,dy$$

③ Evaluate $\iint y \, dx\, dy$ over the area bounded by $y = x^2$ and $x + y = 2$.




The pt of intersection of parabola and st. line

$$x + y = 2$$

$$x + x^2 - 2 = 0$$

$$x^2 + x - 2 = 0$$

$$x = -2, 1$$

$y = x^2$

when $x = -2$, $y = 4$ (-2, 4)

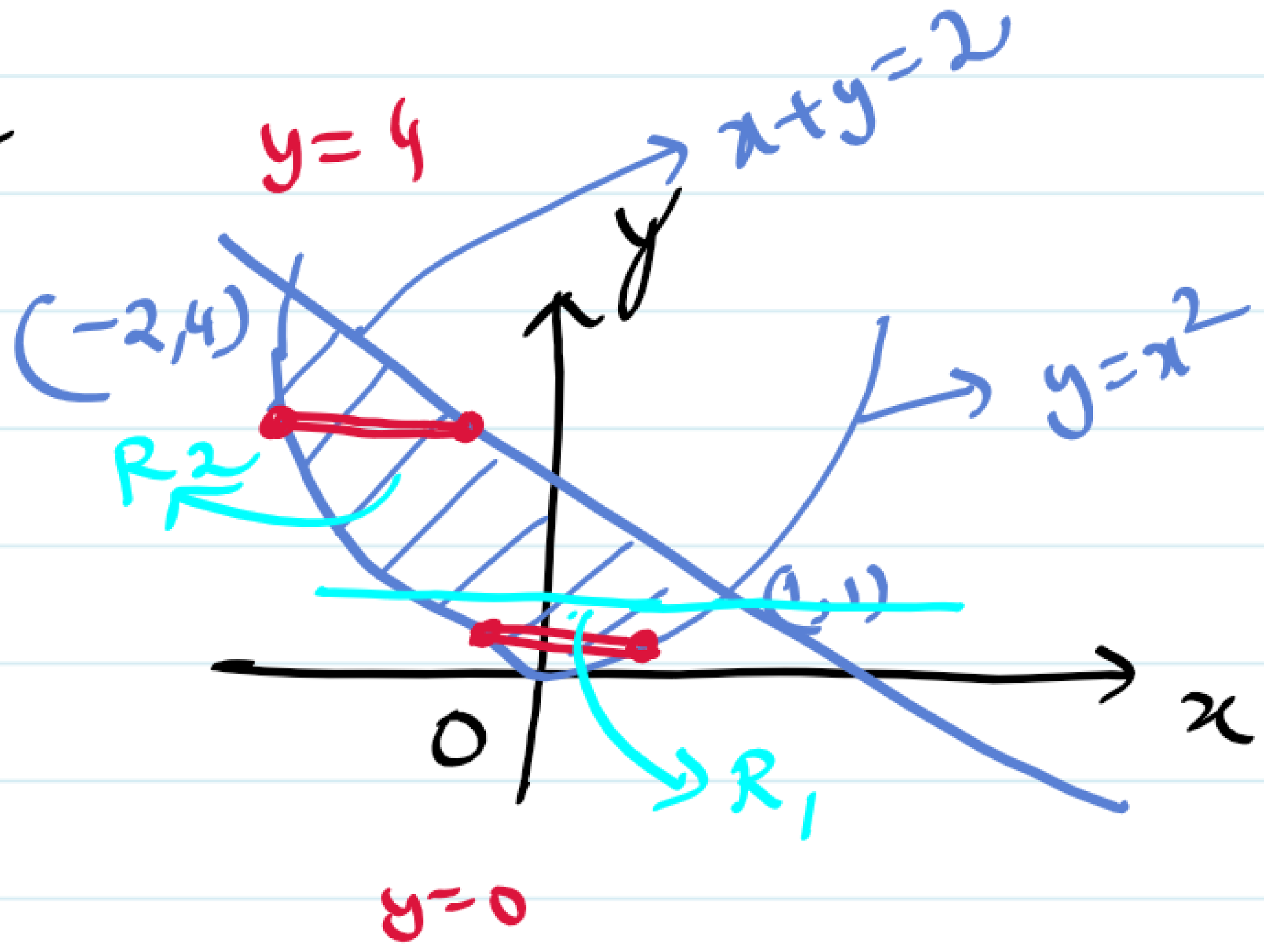
$x = 1$ $y = 1$ (1, 1)

$$I = \int_{x=-2}^{1} \int_{y=x^2}^{2-x} y \, dy \, dx = \int_{x=-2}^{1} \left(\frac{y^2}{2}\right)_{y=x^2}^{2-x} dx$$

$$= \int_{x=-2}^{1} \frac{1}{2}\left((2-x)^2 - (x^2)^2\right) dx$$

$$= \frac{1}{2}\left(4x - 4\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5}\right)_{-2}^{1}$$

$$= \frac{36}{5}$$




$x = \pm\sqrt{y}$

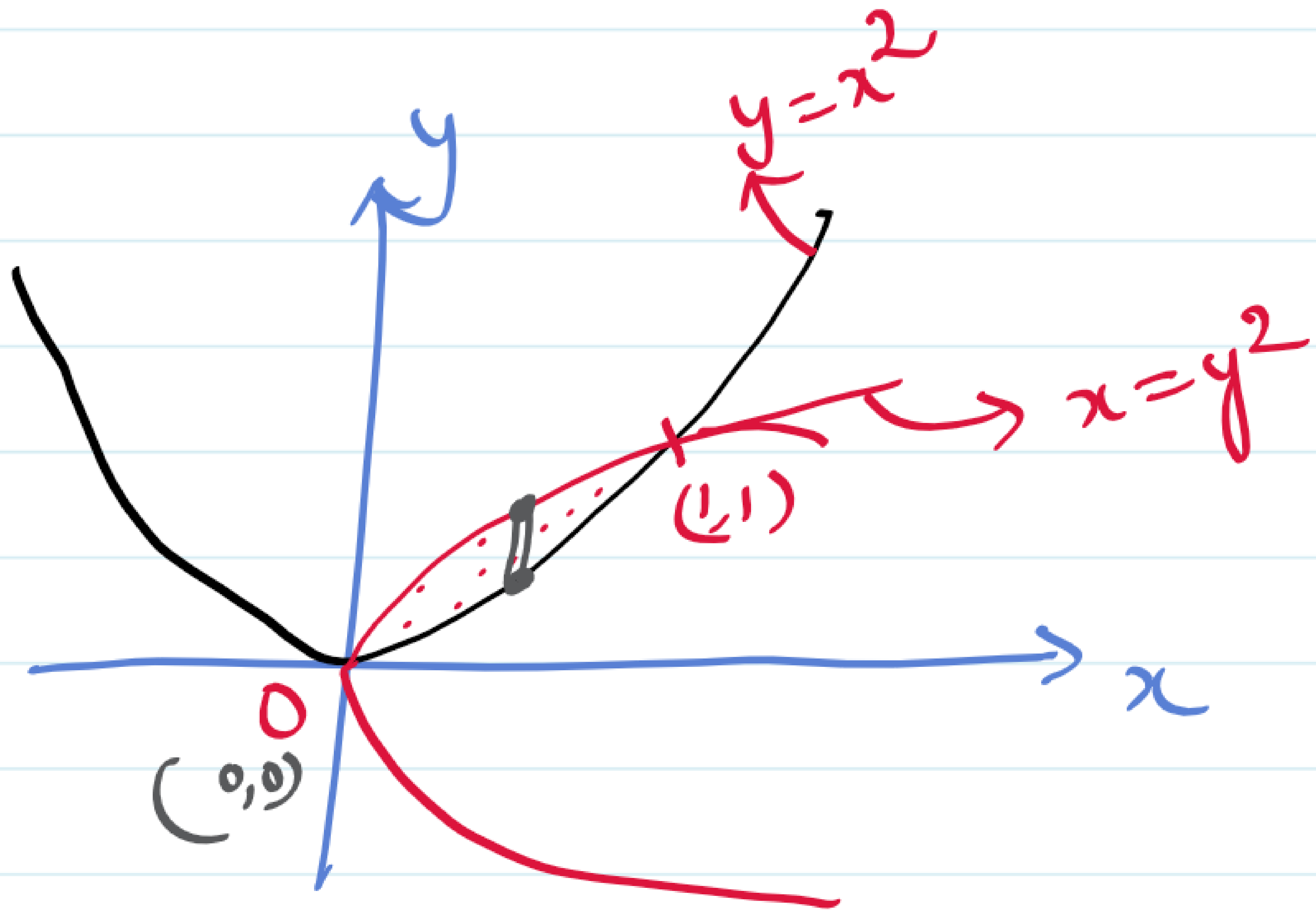
$$I = \iint_{R_1} y \, dx\, dy + \iint_{R_2} y \, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=-\sqrt{y}}^{\sqrt{y}} y \, dx\, dy + \int_{y=1}^{4} \int_{x=-\sqrt{y}}^{2-y} y \, dx\, dy = 36/5$$

4) Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by $y^2 = x$ and $x^2 = y$.

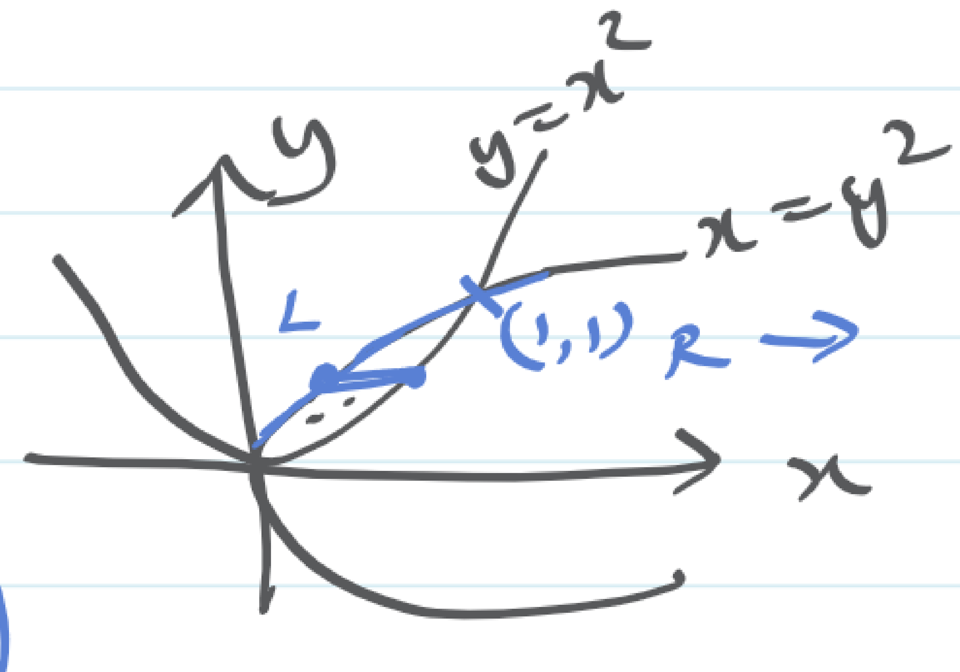
Taking vertical strip

$$I = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx$$




By taking horizontal strip,

$$I = \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} xy(x+y)\,dx\,dy = \frac{3}{28}$$




Practice Questions:

① Evaluate $\int_0^1 \int_{\sqrt{y}}^{2-y} y^2\,dx\,dy$ (Ans: $11/84$)

② Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{(1+x^2)(1+y^2)}$ (Ans: $\frac{\pi^2}{16}$)

③ Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$ (Ans: $\frac{a^4}{3}$)