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State transition Matrix using diagonalisation.

$$X = PZ$$

$$Z = P^{-1}X$$

$$\dot{Z} = P^{-1}\dot{X} = P^{-1}AX = P^{-1}APZ$$

$$\dot{Z} = \tilde{A}Z$$

$$z(t) = e^{\tilde{A}t} \cdot Z(0)$$

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

$$e^{\tilde{A}t} = I + \tilde{A}t + \frac{\tilde{A}^2 t^2}{2!} + \cdots$$

$$e^{\tilde{A}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \\ \vdots & \vdots & & \lambda_n \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & \\ 0 & \cdots & 0 \cdots & \lambda_n^2 \end{bmatrix} t^2 + \cdots$$

$$= \begin{bmatrix} 1+\lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \cdots & 0 & 0 \\ 0 & 1+\lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \cdots & 0 \\ \vdots & \vdots & \\ 0 & 0 & 1+\lambda_n t + \frac{\lambda_n^2 t^2}{2!} + \cdots \end{bmatrix}$$

$$e^{\tilde{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}$$

But we need e^{At}.

$$x(t) = PZ(t)$$

$$x(t) = P \cdot e^{\hat{A}t} \cdot z(0)$$

we have $x = PZ$, $\quad x(0) = P \cdot z(0)$

$$\Rightarrow z(0) = P^{-1} x(0)$$

$$\therefore x(t) = P \cdot e^{\tilde{A}t} \cdot P^{-1} x(0)$$

We also have,

$$x(t) = e^{At} \cdot x(0)$$

On comparin

$$e^{At} = P \cdot e^{\tilde{A}t} \cdot P^{-1}$$

Ex:- Obtain state transition matrix using diagonalization approach for a state model which has system matrix

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & -1 & -3 \\ 0 & \lambda+3 & 0 \\ 0 & -5 & \lambda+1 \end{vmatrix} = \lambda+2\left((\lambda+3)(\lambda+1)\right)$$

$$= (\lambda+2)(\lambda^2+4\lambda+3)$$

$$= \lambda^3+4\lambda^2+3\lambda+2\lambda^2+8\lambda+6$$

$$= \lambda^3+6\lambda^2+11\lambda+\cancel{6}$$

$\lambda_1 = -\cancel{3.52}$ $\quad -3$

$\lambda_2 = -1.\cancel{2+0.8}i$ $\quad -1$

$\lambda_3 = -1.2\,\cancel{-0.8}i$ $\quad -2$

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$$(\lambda_1 I - A)\,X = 0 \begin{bmatrix} -1 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$-x_1 - x_2 - 3x_3 = 0$

$-5x_2 - 2x_3 = 0.$

$\cancel{x_1 = -x_2 - 3x_3}$

$x_2 = -x_1 - 3x_3$

$A \to P$

$$-5(-x_1 - 3x_3) - 2x_3 = 0.$$

$$5x_1 + 15x_3 - 2x_3 = 0.$$

$$5x_1 + 13x_3 = 0$$

$$5x_1 = -13x_3$$

let $x_3 = 1$

then $x_1 = \frac{-13}{5}.$

$$x_2 = \frac{13}{5} - 3 = \frac{-2}{5}$$

$$\lambda_1 = -3$$

$$\lambda_1 = \begin{bmatrix} -\frac{13}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ -2 \\ 5 \end{bmatrix}$$

$$(\lambda_2 I - A) X_2 = 0$$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & 0 \\ 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\Rightarrow \lambda_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$[\lambda_3 I - A] X_3 = 0$$

$$P = \begin{bmatrix} 3 & 1 & -13 \\ 0 & 0 & -2 \\ 1 & 0 & 5 \end{bmatrix} \qquad e^{\tilde{A}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 2.5 & 1 \\ 1 & -14 & -3 \\ 0 & -0.5 & 0 \end{bmatrix}$$

$$e^{At} = P \cdot e^{\tilde{A}t} \cdot P^{-1} = \begin{bmatrix} 3 & 1 & -13 \\ 0 & 0 & -2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} & e^{-2t} & -13e^{-3t} \\ 0 & 0 & -2e^{-3t} \\ e^{-t} & 0 & 5e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} 3e^{-t} & e^{-2t} & -13e^{-3t} \\ 0 & 0 & -2e^{-3t} \\ e^{-t} & 0 & 5e^{-3t} \end{bmatrix} \begin{bmatrix} 0 & 2.5 & 1 \\ +1 & -14 & -3 \\ 0 & -0.5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 2.5(3e^{-t}) + 0.5(13e^{-3t}) - 14e^{-2t} & 3e^{-t} - 3e^{-2t} \\ 0 & 0.5(2e^{-3t}) & 0 \\ 0 & 2.5e^{-t} - 2.5e^{-3t} & e^{-t} \end{bmatrix}$$

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Cayley Hamilton method.

It states that for every matrix it satisfies its own characteristic eq.

$$|\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$$

consider a nxn matrix A & char eq.

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0.$$

Let f(A) be a function of matrix A and f(λ)

Q. Compute state transition matrix using Cayley Hamilton method.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Sol. $|\lambda I - A| = 0$

$$\Rightarrow \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$f(\lambda) = e^{\lambda t}$$

$$e^{\lambda_1 t} = \alpha_0 + \alpha_1 \lambda_1 + \alpha_2 \lambda_1^2$$

$$e^{-t} = \alpha_0 - \alpha_1 \rightarrow (1)$$

$$e^{\lambda_2 t} = \alpha_0 + \alpha_1 \lambda_2$$

$$e^{-2t} = \alpha_0 - 2\alpha_1 \rightarrow (2)$$

$e^{\lambda_3 t}$

solve (1) & (2)

$$e^{-t} = \alpha_0 - \alpha_1$$

$$e^{-2t} = \alpha_0 - 2\alpha_1$$

$$\boxed{e^{-t} - e^{-2t} = \alpha_1}$$

Sub α_1 in (1)

$$\alpha_0 = e^{-t} + e^{-t} - e^{-2t}$$

$$\boxed{\alpha_0 = 2e^{-t} - e^{-2t}}$$

Using C-H method.

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & 0 \\ 0 & 2e^{-t} - e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 3e^{-2t} - 3e^{-t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Q $\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ $f(\lambda) \Rightarrow e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$

Same eigen values.

$\lambda_1 = -1 \quad \lambda_2 = -1$

$f(\lambda_1) = e^{-t} = \alpha_0 - \alpha_1 \rightarrow (1)$

$\alpha_1 = te^{-t} - e^{-2t}$

$\alpha_0 = e^{-t} + te^{-t}$

$$\left.\frac{d f(\lambda)}{d\lambda}\right|_{\lambda=-1} \Rightarrow te^{\lambda t} = \alpha_1$$

$$\Rightarrow \alpha_1 = te^{-t} \rightarrow (2)$$

Sub in (1)

$$\alpha_0 = e^{-t} + te^{-t}$$

$$e^{At} = \begin{bmatrix} e^{-t}(t+1) & te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Sylvester's interpolation formula.

e^{At} can be obtained by solving

for distinct eigen values:

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} & e^{\lambda_2 t} \\ \vdots & & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} & e^{\lambda_n t} \\ I & A & A^2 & \cdots & A^{n-1} & e^{At} \end{vmatrix} = 0$$

If $\lambda_1 = \lambda_2 = \lambda_3$ & $\lambda_4 \cdots \lambda_m$ are distinct.

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\ \vdots & \vdots & \vdots & & & & \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & A^3 & \cdots & A^{m-1} & e^{At} \end{vmatrix}$$

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Compute state transition matrix

$$[A] = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$ using Sylvester's Interpolation Formula.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda+2 \end{vmatrix} \Rightarrow \lambda^2 + 2\lambda = 0$$

$\cancel{\lambda^2 = 2\lambda}$

$\lambda(\lambda+2) = 0$

$\Rightarrow \lambda = 0, -2$

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ I & A & e^{At} \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ I & A & e^{At} \end{vmatrix} = 0.$$

$$1(-2e^{At} - Ae^{-2t}) + 1(A + 2I) = 0.$$

$$-2e^{At} - Ae^{-2t} + A + 2I = 0.$$

$$A(1 - e^{-2t}) + 2I = 2e^{At}$$

$$A\left(\frac{1-e^{-2t}}{2}\right) + I = e^{At}.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\left(\frac{1-e^{-2t}}{2}\right) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{At}.$$

$$\begin{bmatrix} 0 & \frac{1-e^{-2t}}{2} \\ 0 & e^{-2t} - 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{At}$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1 & \frac{1-e^{-2t}}{2} \\ 0 & e^{-2t} \end{bmatrix}$$

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Q. using diagonalization.

★ Repeated eigen values of q multiplicity
general structure of vandermonde matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \cdots & \lambda_{q+1} \\ \lambda_1^2 & 2\lambda_1 & 1 & \cdots & \lambda_{q+1}^2 \\ \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{(n-1)} & \frac{d(\lambda^{n-1})}{d\lambda_1} & \frac{1}{2!}\frac{d^2(\lambda_1^{n-1})}{d\lambda_1^2} & \cdots & \lambda_{q+1}^{n-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

Sol. $[\lambda I - A] = 0$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 3 & \lambda - 3 \end{vmatrix} = 0.$$

$$\lambda(\lambda^2 - 3\lambda + 3) + 1(-1) = 0.$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$$

$$\lambda = 1, 1, 1$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_1^2 & 2\lambda_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

★ General form of $e^{\tilde{A}t}$ for repeated eigen values.

$$\begin{bmatrix} e^{\lambda_1 t} & \frac{d}{d\lambda} e^{\lambda_1 t} = te^{\lambda_2 t} & \frac{1}{2!}t^2 e^{\lambda_3 t} \\ 0 & e^{\lambda_2 t} & te^{\lambda_3 t} \\ 0 & 0 & e^{\lambda_3 t} \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{\tilde{A}t} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

$$e^{At} = S\, e^{\tilde{A}t}\, S^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

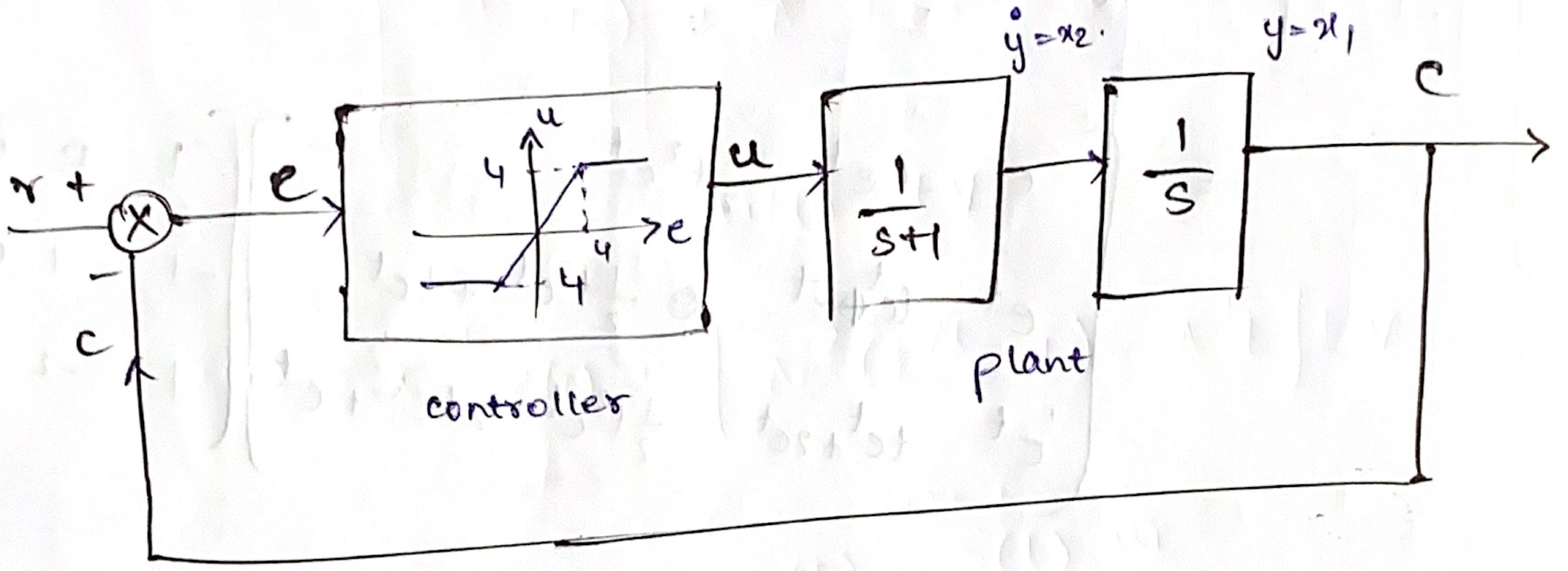
$$= \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ e^t & te^t + e^t & \frac{1}{2}t^2e^t + te^t \\ e^t & te^t + 2e^t & \frac{1}{2}t^2e^t + 2te^t + e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{matrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \cancel{e^t} - \cancel{te^t} - \cancel{e^t} + \frac{1}{2}t^2e^t + \cancel{te^t} & te^t + e^t - t^2e^t - 2te^t & \frac{1}{2}t^2e^t + te^t \\ \cancel{e^t} - te^t - 2\cancel{e^t} + \frac{1}{2}t^2e^t + 2te^t + \cancel{e^t} & \cancel{te^t} + \cancel{2e^t} - t^2e^t - \cancel{4te^t} - \cancel{2e^t} & \frac{1}{2}t^2e^t + 2te^t + e^t \end{matrix}$$

$$e^{At} = \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t & e^t - te^t - t^2e^t & te^t + \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t & -3te^t - t^2e^t & e^t + 2te^t + \frac{1}{2}t^2e^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3!}t^3e^{\lambda_4 t} \\ \frac{1}{2!}t^2e^{\lambda_4 t} \\ te^{\lambda_4 t} \\ e^{\lambda_4 t} \end{bmatrix}$$

Q. Consider the position servo system shown below. Find the response to a step input $r(t) = 10$. Assume that the output position and velocity are both zero initially.



$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s} \qquad \frac{X_2(s)}{U(s)} = \frac{1}{s+1} \Rightarrow \dot{x}_2 + x_2 = u$$

$$\Rightarrow \dot{x}_1 = x_2 \qquad \Rightarrow \dot{x}_2 = -x_2 + u.$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here $x_1(0) = 0$, $x_2(0) = 0$

So, $x(0) = 0$

Using Cayley Hamilton.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda+1 \end{vmatrix}$$

$$\lambda(\lambda+1) = 0$$

$$\lambda = 0, -1$$

$$f(\lambda) \Rightarrow \alpha_0 + \lambda_1 \alpha_1 = e^{\lambda_1 t} \Rightarrow \alpha_0 = e^{0t} = 1$$

$$\alpha_0 + \lambda_2 \alpha_1 = e^{\lambda_2 t} \Rightarrow \alpha_0 - \alpha_1 = e^{-t}.$$

$$1 - \alpha_1 = e^{-t}$$

$$\alpha_1 = 1 - e^{-t}$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 - e^{-t} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1-e^{-t} \\ 0 & e^{-t}-1 \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$x_1(t) = 4(t - 1 + e^{-t})$$

$$x_2(t) = 4(1 - e^{-t})$$

Response

$$x(t) = e^{At}\cdot x(0) + \int_0^t e^{A(t-\tau)}\cdot B\cdot u(\tau)\cdot d\tau$$

$$= \int_0^t \begin{bmatrix} 1 & 1-e^{-t+\tau} \\ 0 & e^{-t+\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 4$$

$$= \int_0^t \begin{bmatrix} 4(1-e^{-t+\tau}) \\ 4(e^{-t+\tau}) \end{bmatrix}$$

$$= 4(t - e^{-t+\tau}$$

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$$\dot{x} = Ax + Bu$$

Step response $\quad u(t) = k$

$$x(t) = e^{At} \cdot x(0) + e^{At}\left[-(A^{-1})(e^{-At} - I)\right] Bk$$

$$= e^{At} x(0) + A^{-1}(e^{At} - I) Bk.$$

Ramp response: $\quad u(t) = tV.$

$$x(t) = e^{At} \cdot x(0) + \left[A^{-2}(e^{At} - I) - A^{-1}t\right] BV.$$

Impulse response: $\quad u(t) = \delta(t)\, w$

$$x(t) = e^{At} \cdot x(0) + e^{At} \cdot Bw$$

Tutorial-5

Q. Obtain $y(t)$. where $u(t)$ = unit step input
$u(t) = 1(t)$.

Inverse Laplace transform

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$|\lambda I - A| = 0.$

Sol. $e^{At} = L^{-1}\left[(SI - A)^{-1}\right]$ $\qquad$ adj $= \begin{bmatrix} S & +1 \\ -0.5 & S+1 \end{bmatrix}^T$

$$SI - A = \begin{bmatrix} S+1 & 0.5 \\ -1 & S \end{bmatrix} \qquad \text{adj} = \begin{bmatrix} S & -0.5 \\ 1 & S+1 \end{bmatrix}$$

$$SI - A = \frac{\text{adj}(SI - A)}{|SI - A|} = \frac{1}{S^2 + S + 0.5} \begin{bmatrix} S & -0.5 \\ 1 & S+1 \end{bmatrix}$$

$$\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$$

$$\frac{b}{(s-a)^2+b^2} = e^{at}\sin bt.$$

$$= \begin{bmatrix} \frac{s}{s^2+s+0.5} & \frac{-0.5}{s^2+s+0.5} \\ \frac{1}{s^2+s+0.5} & \frac{s+1}{s^2+s+0.5} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{s+0.5-0.5}{(s+0.5)^2+0.5^2} & (-1)\frac{(0.5)}{(s+0.5)^2+0.5^2} \\ (2)\frac{(0.5)}{(s+0.5)^2+(0.5)^2} & \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-0.5t}\cdot\cos 0.5t - e^{0.5t}\cdot\sin 0.5t & -e^{-0.5t}\cdot\sin 0.5t \\ 2\cdot e^{-0.5t}\cdot\sin 0.5t & e^{-0.5t}\cos 0.5t + e^{-0.5t}\sin 0.5t \end{bmatrix}$$

$$x(t) = e^{At}\cdot x(0) + A^{-1}(e^{At}-I)Bk$$

$$= 0 + A^{-1}\cdot(e^{At}-I)\cdot B(1)$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$A^{-1}(e^{At}-I) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t)-1 & -e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t)-1 \end{bmatrix}$$

$$= \cancel{\begin{matrix} 2\cdot e^{-0.5t}\cdot\sin 0.5t & e^{-0.5t}(\cos 0.5t+\sin 0.5t)-1 \\ -2e^{-0.5t}(\cos 0.5t-\sin 0.5t)+2 & \end{matrix}}$$

$$(e^{At}-I)\cdot B = \begin{bmatrix} 0.5\cdot e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t}\sin 0.5t \end{bmatrix} = C$$

$$\| \qquad \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$A^{-1}\cdot C = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0.5\, e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t}\cdot \sin 0.5t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t}\cdot \sin 0.5t \\ -e^{-0.5t}(\cos 0.5t - \sin 0.5t) + 1 \end{bmatrix}$$

y_r

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Controllability

Kalman's test

$\dot{X} = Ax + Bu$ is completely controllable

if and only if vectors $B, AB \dots A^{n-1}B$ are linearly independent or $n \times n$ matrix

$$C = [B \vdots AB \vdots \dots \vdots A^{n-1}B]$$ is of rank n

or we can say $|C| \neq 0$.

Matrix C is called controllability matrix.

Uncontrollable states cannot be identified.

ex:. Check the controllability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

sol.

$$C = [B \vdots AB]$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$|C| = 0$.

$\therefore$ System is uncontrollable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u].$$

$$C = [B \vdots AB] \qquad AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|C| = -1 \neq 0$$

$\therefore$ It is controllable.

Gilbert test for Controllability

$$\dot{x} = Ax + Bu$$

After diagonalization.

we get $\dot{z} = P^{-1}APz + P^{-1}Bu$

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix} u$$

The system is controllable if the elements $\tilde{b}_1, \tilde{b}_2, \ldots \tilde{b}_n$ are non zero. (for distinct eigen values)

If u is an ~~e~~ r vector

if and only if no row of $P^{-1}B$ are all zero

if repeated eigen values.

transforming A into Jordan Canonical form

$$\dot{z} = S^{-1}ASz + S^{-1}Bu$$

$\lambda_1 = \lambda_2 = \lambda_3$

$$= Jz + S^{-1}Bu$$

$\lambda_4 = \lambda_5$

$$\begin{bmatrix} \lambda_1 & 1 & 0 & & & \\ 0 & \lambda_1 & 1 & & & \\ 0 & 0 & \lambda_1 & & & \\ & & & \lambda_4 & 1 & \\ & & & 0 & \lambda_4 & \\ & & & & & \lambda_6 \end{bmatrix}$$

(2) the elements of any row of $S^{-1}B$ that correspond to the last row of each jordan block are not all zero,

(3) the elements of each row of $S^{-1}B$ that correspond to distinct eigenvalues are not all zero.

ex!. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$

Controllability
in Transfer function.

no cancellation should occur in the TF. If the cancellation occurs, the system cannot be controlled in the direction of the cancelled mode.

Output controllability.

if and only if the $m \times (n+1) r$ matrix C_{out} is of rank m.

Output controllability matrix

$$C_{out} = [CB \vdots CAB \vdots CA^2B \vdots \cdots \vdots CA^{n-1}B \vdots D]$$

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Stabilizability

Uncontrolled states are stable

Unstable modes are controllable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

x_1 is controllable with $\lambda_1 = 1$ (unstable)

x_2 is uncontrollable with $\lambda_2 = -1$ (stable)

So, the overall system is stabilizable.

Observability

$\dot{x} = Ax$; $y = Cx$

The system is observable if every state $x(t_0)$ can be determined from the observation of $y(t)$ over a finite interval $t_0 \le t \le t_1$

if every transition of the state eventually affects every element of the output vector.

Test for Observability.

Kalman's test

matrix 'O' $= [C^* \vdots A^* C^* \vdots \cdots \vdots A^{*(n-1)} C^*]$

$|O| \neq 0$

ex:- Check o/p controllability & observability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol. $C_{out} = [CB \quad CAB]$

$$CB = [1 \quad 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$CAB = \underset{1\times 2}{[1 \quad 0]} \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

m = no. of o/p's.

$$= [1 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$C_{out} = [0 \quad 1]$ rank = 1 = m.

∴ It is output controllable.

$$O' = \left[C^{*} \quad A^{*} C^{*} \right]$$

$$O = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{rank} = 2 \neq 0$$

$\therefore$ The system is Completely observable.

$$A^{*} C^{*} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^{*} C^{*} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Gilbert's test:

i) distinct eigen values.

After diagon $y = CPz$

The system is completely observable if none of the columns of $m \times n$ matrix CP consists of all zero elements.

ii) repeated eigen values.

we get $y = CSz$

A) no two Jordan blocks in J are associated with same eigen values

B) no columns of CS that correspond to the first row of each Jordan block consist of zero elements.

c) no columns of CS that correspond to distinct eigen values consists of zero elements.

Condition for Observability in s-plane.

Relationship b/w controlability and observability.

Principle of duality.

System. S1 :

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

dual system S2 :

$$\dot{z} = A^* z + C^* u$$

$$n = B^* z$$

The system S1 is completely controllable if and only if system S2 is completely observable vice versa.

Detectability.

If unobservable are stable

and observable are unstable

then system is detectable.