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Modern Control Theory
(ICE 3153)

Solution of State Equaion

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Why you need solution of state equation ?

What type of equation is a state equation?

What are the components in solution of a differential equation ?

(i) Homogeneous solution, that describes the response to an arbitrary set of initial conditions $x(0)$
(ii) Particular solution, that describes the response for the given input $u(t)$.

The two components are then combined to form the total response.

Two cases of system response:

Case (i) – Response of system without excitation

Case (ii) – Response of system with excitation

Case (i) – Response of system without excitation (Homogeneous State Equations.)

- Before we solve vector-matrix differential equations, let us review the solution of the scalar differential equation

$$\dot{x} = ax \qquad (1)$$

In solving this equation, we may assume a solution *x(t)* of the form

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + .. \qquad (2)$$

By substituting this assumed solution into Equation (1), we obtain

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots k b_k t^{k-1} + .. = a(b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + ..) \qquad (3)$$

- If the assumed solution is to be the true solution, Equation (3) must hold for any ***t.***
- Hence, equating the coefficients of the equal powers of *t,* we obtain

$$b_1 = ab_0$$

$$b_2 = \frac{1}{2} ab_1 = \frac{1}{2} a^2 b_0$$

$$b_3 = \frac{1}{3} ab_2 = \frac{1}{3 \times 2} a^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

- The value of *bo* is determined by substituting *t = 0* into Equation (2), or

$$x(0) = b_0$$

- Hence, the solution *x(t)* can be written as

$$\begin{aligned} x(t) &= \left(1 + at + \frac{1}{2!} a^2 t^2 + \cdots + \frac{1}{k!} a^k t^k + \cdots \right) x(0) \\ &= e^{at} x(0) \end{aligned}$$

- We shall now solve the vector-matrix differential equation
  $\dot{X} = AX \ where \ X = n - vector \ and \ A = nxn \ constant \ matrix$
- By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t, or

$$X(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + .. \quad (4)$$

- By substituting this assumed solution into Equation **(4),** we obtain

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots k b_k t^{k-1} + .. = A(b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + ..) \quad (5)$$

- If the assumed solution is to be the true solution, Equation **(5)** must hold for all **t.** Thus, by equating the coefficients of like powers oft on both sides of Equation **(5),** we obtain

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{A}\mathbf{b}_0 \\
\mathbf{b}_2 &= \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{b}_0 \\
\mathbf{b}_3 &= \frac{1}{3}\mathbf{A}\mathbf{b}_2 = \frac{1}{3 \times 2}\mathbf{A}^3\mathbf{b}_0 \\
&\vdots \\
\mathbf{b}_k &= \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0
\end{aligned}$$

- by substituting $t = 0$ into Equation (4), or

$$x(0) = b_0$$

- Hence, the solution $x(t)$ can be written as

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots\right)\mathbf{x}(0)$$

- The expression in the parentheses on the right-hand side of this last equation is an n X n matrix.
- Because of its similarity to the infinite power series for a scalar exponential, we call it the matrix exponential and write

$$\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots = e^{\mathbf{A}t}$$

- In terms of the matrix exponential, the solution of State Equation can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

Laplace Transform Approach

- Consider the homogeneous state equation,
$$\dot{x} = Ax(t)$$
- Taking LT on both the sides of the above equation,
$$SX(s) - x(0) = AX(s)$$
- Rearranging the terms to get,
$$(SI - A)X(s) = x(0)$$
$$X(s) = (SI - A)^{-1}x(0)$$
- Take Inverse LT of the above equation to get the solution,
$$x(t) = \mathcal{L}^{-1}[(SI - A)^{-1}]x(0)$$
- By comparing the above solution with solution in matrix exponential, we get
$$e^{At} = \mathcal{L}^{-1}[(SI - A)^{-1}]$$
- The inverse Laplace transform of a matrix is the matrix consisting of the inverse Laplace transforms of all elements

State-Transition Matrix

- We can write the solution of the homogeneous state equation

$$\dot{x} = Ax(t) \qquad (6) \text{ as}$$

$$x(t) = \Phi(t)x(0) \qquad (7)$$

$$\Phi(t) = e^{At} = \mathcal{L}^{-1}[(SI - A)^{-1}]$$

- From Equation (7), we see that the solution of Equation (6) is simply a
- transformation of the initial condition.
- Hence, the unique matrix $\Phi(t)$ is called the state transition matrix.
- The state-transition matrix contains all the information about the free motions of the system defined by Equation (6).

Properties of State-Transition Matrices

1. $\boldsymbol{\Phi}(0) = e^{\mathbf{A}0} = \mathbf{I}$
2. $\boldsymbol{\Phi}(t) = e^{\mathbf{A}t} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left[\boldsymbol{\Phi}(-t)\right]^{-1}$ or $\boldsymbol{\Phi}^{-1}(t) = \boldsymbol{\Phi}(-t)$
3. $\boldsymbol{\Phi}(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = \boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(t_2) = \boldsymbol{\Phi}(t_2)\boldsymbol{\Phi}(t_1)$
4. $\left[\boldsymbol{\Phi}(t)\right]^n = \boldsymbol{\Phi}(nt)$
5. $\boldsymbol{\Phi}(t_2 - t_1)\boldsymbol{\Phi}(t_1 - t_0) = \boldsymbol{\Phi}(t_2 - t_0) = \boldsymbol{\Phi}(t_1 - t_0)\boldsymbol{\Phi}(t_2 - t_1)$

Computation of State-Transition Matrices

Method 1: Using matrix exponential

Method 2: Using Laplace transform

Method 3: By canonical form

Method 4: Using Cayley – Hamilton theorem

Question: 1

Ex 11-5, MCE 4th Edition K. Ogata

Obtain the state-transition matrix e^{At} of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] \quad s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Case (ii) – Response of system with excitation (Nonhomogeneous State Equations.)

- Using Matrix Exponential
- Let us now consider the nonhomogeneous state equation described by

$$\dot{X} = AX + BU$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- Laplace Transform Approach

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- **Solution in Terms of** x(t0).

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Question: 2

Ex 11-6, MCE 4th Edition K. Ogata

Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is the unit-step function occurring at $t = 0$, or

$$u(t) = 1(t)$$

Initial state of the system is X(0) = 0.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

State Transition Matrix using Diagonalization

- Consider the state equation without input, $\dot{X} = AX$
- The solution of the above state equation is $x(t) = e^{At}x(0)$
- Assume that the system matrix A is non-diagonal and has distinct eigenvalues.
- Let us transform the original state X to Z by diagonalization.
- New state is defined as,

$$X = PZ$$
$$Z = P^{-1}X$$
$$\dot{Z} = P^{-1}\dot{X} = P^{-1}AX = P^{-1}APZ$$
$$\dot{Z} = \tilde{A}Z$$
$$z(t) = e^{\widetilde{At}}Z(0)$$

Where, $\tilde{A} = \begin{bmatrix} \lambda_1 & 0 \cdots & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$

$$e^{\widetilde{At}} = I + \tilde{A}t + \frac{\tilde{A}^2t^2}{2!} + \frac{\tilde{A}^3t^3}{3!} + \cdots \ldots$$

- $$e^{\widetilde{A}t} = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0\cdots & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} t + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & 0\cdots & 0 \\ 0 & \lambda_2^2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix} t^2 + \frac{1}{3!}\begin{bmatrix} \lambda_1^3 & 0\cdots & 0 \\ 0 & \lambda_3^2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^3 \end{bmatrix} t^3 + \ldots\ldots\ldots$$

- $$e^{\widetilde{A}t} = \begin{bmatrix} 1+\lambda_1 t+\frac{1}{2!}\lambda_1^2 t^2+\frac{1}{3!}\lambda_1^3 t^3+\cdots & 0\cdots & 0 \\ 0 & 1+\lambda_2 t+\frac{1}{2!}\lambda_2^2 t^2+\frac{1}{3!}\lambda_2^3 t^3+\cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1+\lambda_n t+\frac{1}{2!}\lambda_n^2 t^2+\frac{1}{3!}\lambda_n^3 t^3+\cdots \end{bmatrix}$$

- $$e^{\widetilde{A}t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- But we need to find e^{At}
- From new state definition,

$$X(t) = PZ(t)$$
$$X(t) = Pe^{\widetilde{A}t}Z(0)$$

- We have $X = PZ$, if we consider $X(0) = PZ(0)$

$$Z(0) = P^{-1}X(0)$$

- Therefore,

$$X(t) = Pe^{\widetilde{A}t}P^{-1}X(0)$$

- We also have

$$X(t) = e^{At}X(0)$$

- From the above 2 equations we can write,

$$e^{At} = Pe^{\widetilde{A}t}P^{-1}$$

Question:3

Obtain the state transition matrix using diagonalization approach for a state model which has the system matrix as given below

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

First step is to find the model Matrix P

Eigenvalues are,

$$[\lambda I - A] = \begin{bmatrix} \lambda + 2 & -1 & -3 \\ 0 & \lambda + 3 & 0 \\ 0 & -5 & \lambda + 1 \end{bmatrix}$$

$$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$$

$$P = \begin{bmatrix} 3 & 1 & -13 \\ 0 & 0 & 2 \\ 1 & 0 & 5 \end{bmatrix} \qquad e^{\widetilde{A}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

- $e^{At} = Pe^{\widetilde{A}t}P^{-1} = \begin{bmatrix} e^{-2t} & 7.5e^{-t} + 14e^{-2t} - 6.5e^{-3t} & 3e^{-t} - 3e^{-2t} \\ 0 & e^{-3t} & 0 \\ 0 & 2.5e^{-t} + 2.5e^{-3t} & e^{-t} \end{bmatrix}$

Cayley-Hamilton Theorem.

- The Cayley-Hamilton theorem is very useful in proving theorems involving matrix equations or solving problems involving matrix equations.
- ***The theorem states that for every matrix it satisfies its own characteristic equation.***
- Consider an *n* X *n* matrix A and its characteristic equation:

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

- ***According to theorem,***

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n \mathbf{I} = \mathbf{0}$$

State transition matrix using Cayley-Hamilton Theorem.

- Consider an *n* X *n* matrix A and its characteristic equation:

$$q(\lambda) = |\lambda I - A| = \lambda^n + a_1 \lambda^{n-1} + \ldots\ldots + a_{n-1} \lambda + a_n = 0$$

- Let $f(A)$ be a function of matrix A and $f(A)$ can be expressed as a matrix polynomial.
- Let $f(\lambda)$ be a scalar polynomial obtained from $f(A)$ after substituting A by λ.
- On dividing $f(\lambda)$ by $\mathrm{q}(\lambda)$ we get

$$\frac{f(\lambda)}{q(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{q(\lambda)} \quad (1)$$

- Where $Q(\lambda)$ is the Quotient polynomial and $\mathrm{R}(\lambda)$ is reminder polynomial.
- Equation (1) can be rewritten as,

$$\frac{f(\lambda)}{q(\lambda)} = \frac{Q(\lambda)q(\lambda) + R(\lambda)}{q(\lambda)}$$

- From the above equation we can write,

$$f(\lambda) = Q(\lambda)q(\lambda) + R(\lambda)$$

- $q(\lambda)$ is the characteristic equation so $q(\lambda)$=0 and the above equation becomes,

$$f(\lambda) = R(\lambda) \text{ (2)}$$

We can also write the above equation as,

$$f(A) = R(A) \text{ (3)}$$

If we evaluate the eqn (2) using the eigenvalues λ_1, λ_2,…. λ_n then,

$$f(\lambda_i) = R(\lambda_i) \text{ i=0,1,2,…n}$$

The reminder polynomial $R(\lambda)$ will be in the form of,

$$R(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots \ldots . + \alpha_{n-1}\lambda^{n-1}$$

- Where α_0, α_1…. α_{n-1} are constants.
- From eqn (3) we can write,

$$R(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots \ldots . + \alpha_{n-1} A^{n-1} \text{ (4)}$$

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots \ldots . + \alpha_{n-1} A^{n-1}$$

Where α_0, α_1-----α_{n-1} are constants.

- On substituting number of eigenvalues in the above equation we get n number of equations. These equations can be solved to find the constants α_i.
- Now assume that $f(A) = e^{At}$ or $f(\lambda) = e^{\lambda t}$ we can compute state transition matrix using eqn (4)
- Suppose if we have 2 eigenvalues $\lambda_1 \text{ and } \lambda_2$.
- For λ_1, $e^{\lambda_1 t} = \alpha_0 + \alpha_1\lambda_1$
- For λ_2, $e^{\lambda_2 t} = \alpha_0 + \alpha_1\lambda_2$
- By solving the above equations we get the constants α_i
- Also we get,

$$e^{At} = e^{\lambda t}$$

Question:4

Ex 5.1, Advanced Control Theory , Nagoor Kani

- Compute state transition matrix using Cayley-Hamilton theorem for the state model has system matrix as given below.

$$[A]=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$[\lambda I - A] = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}$$

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2$$

$$f(\lambda_1) = e^{\lambda_1 t} = R(\lambda_1) = \alpha_0 + \alpha_1 \lambda_1$$

For $\lambda_1 = -1$

$$e^{-t} = \alpha_0 - \alpha_1 \quad (1)$$

For $\lambda_2 = -2$

$$e^{-2t} = \alpha_0 - 2\alpha_1 \quad (2)$$

Solve eqn (1) and (2) for unknown constants

- (1)-(2) gives,
- $\alpha_1 = e^{-t} - e^{-2t}$

- And $\alpha_0 = 2e^{-t} - e^{-2t}$
- By C-H theorem,

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Question:5

Ex 5.2, Advanced Control Theory , Nagoor Kani

- Compute state transition matrix using Cayley-Hamilton theorem for the state model has system matrix as given below.

$$[A]=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\lambda_1 = -1 \text{ and } \lambda_2 = -1$$

$$f(\lambda\ \ ) = e^{\lambda\ \ t} = R(\lambda\ \ ) = \alpha_0 + \alpha_1\lambda$$

For $\lambda_1 = -1$

$$f(\lambda_1) = e^{\lambda_1 t} = R(\lambda_1) = \alpha_0 - \alpha_1$$

$$e^{-t} = \alpha_0 - \alpha_1 \ \ (1)$$

Evaluate $\frac{d}{d\lambda}f(\lambda)$ at $\lambda = -1$

And we get, $\alpha_1 = te^{-t}$

- $\alpha_1 = te^{-t}$
- And $\alpha_0 = e^{-t} + te^{-t} = e^{-t}(t+1)$
- By C-H theorem,

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{At} = \begin{bmatrix} e^{-t}(t+1) & te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Sylvester's Interpolation method.

- In order to solve $f(A) = e^{At}$, when the eigenvalues are distinct in the eqn

$$f(\lambda) = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots \ldots . + \alpha_{n-1}\lambda_i^{n-1}$$

e^{At} can be obtained by solving the below equation.

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \ldots\ldots\ldots & \lambda_1^{n-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 \ldots\ldots\ldots & \lambda_2^{n-1} & e^{\lambda_2 t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 \ldots\ldots\ldots & \lambda_n^{n-1} & e^{\lambda_n t} \\ I & A & A^2 \ldots\ldots\ldots & A^{n-1} & e^{At} \end{vmatrix} = 0$$

A Involves Multiple Roots.

- As an example, consider the case where the minimal polynomial of **A** involves three equal roots $\lambda_1 = \lambda_2 = \lambda_3$ and has other roots $(\lambda_4, \lambda_5 \ldots . \lambda_m)$ that are all distinct.
- By applying Sylvester's interpolation formula, it can be shown that e^{At} can be obtained from the following determinant equation:

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \dfrac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \dfrac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Tutorial -4

Question:1

Ex 11-8, Modern Control Engineering , 4th Edition Ogata

- Consider the system matrix given below, compute state transition matrix using Sylvester's interpolation formula.

$$[A]=\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Using Sylvester's interpolation formula the determinant equation is,

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Substitute the eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = -2$ the above equation become,

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

- Expanding the determinant, we obtain

$$-2e^{At} + A + 2I - Ae^{-2t} = 0$$

Or, $e^{At} = \frac{1}{2}(A + 2I - Ae^{-2t})$

$$e^{At} = \begin{bmatrix} 1 & 1/2(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Question:2

Example page no-774, Modern Control Engineering , 4th Edition Ogata

- Obtain the state transition matrix using diagonalization approach for a state model which has the system matrix as given below

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

Eigenvalues are,

$$[\lambda I - A] = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

- Since A has repeated eigenvalues, the transformation matrix that will transform A matrix into Jordan Canonical form,

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$ Slide 34

$$e^{Jt} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

- In general, let λ_1 has multiplicity of q and other eigenvalues are, $\lambda_{q=1}, \lambda_{q+2} \dots \lambda_n$ then the transformation matrix S can be given as,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 1 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \cdots\cdots & \lambda_{q+1} & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 1 & \cdots\cdots & \lambda_{q+1}^2 & \cdots & \lambda_n^2 \\ \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 & \cdots\cdots & \lambda_{q+1}^3 & \cdots & \lambda_n^3 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \lambda_1^{(n-1)} & \dfrac{d(\lambda_1^{n-1})}{d\lambda_1} & \dfrac{1}{2!}\dfrac{d^2(\lambda_1^{n-1})}{d\lambda_1^2} & \cdots\cdots & \lambda_{q+1}^{n-1} & \cdots & \lambda_n{}^{n-1} \end{bmatrix}$$

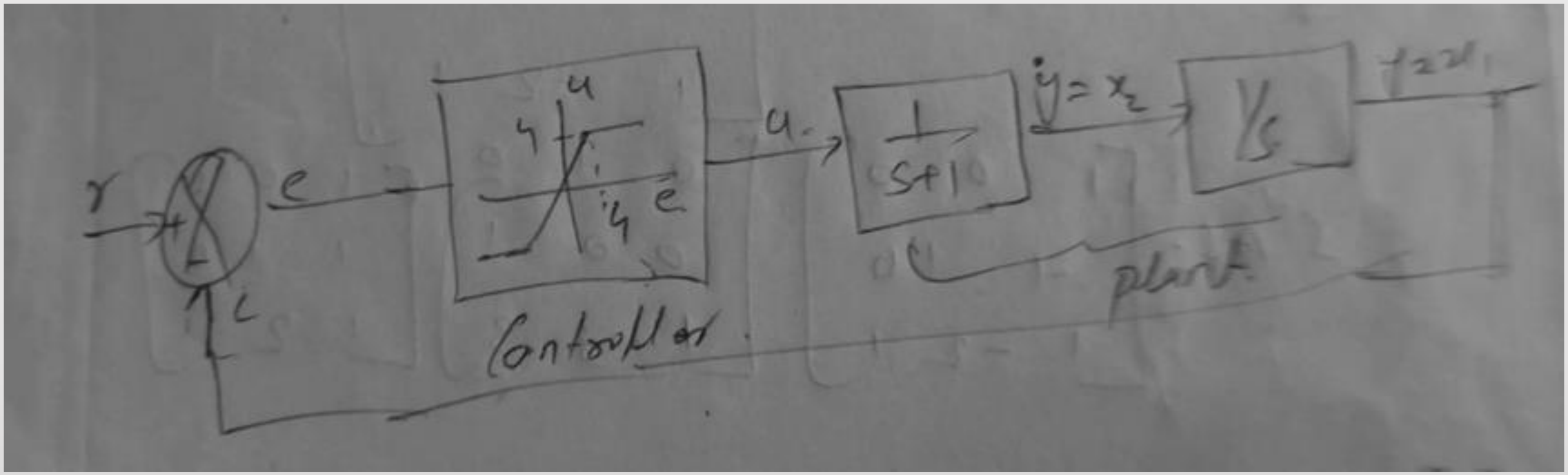
- $e^{At} = Se^{Jt}S^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t & e^t - te^t - t^2e^t & te^t + \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t & -3te^t - t^2e^t & e^t + 2te^t + \frac{1}{2}t^2e^t \end{bmatrix}$$

Question:3

- Consider the position servo system shown below. Find the response to a step input r(t)=10. Assume that the output position and velocity are both zero initially.



- By considering o/p of each integrator as state variable, we have

$$y = x_1 \text{ and } \dot{y} = x_2$$

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s} \text{ and we get } \dot{x_1} = x_2$$

$$\frac{X_2(s)}{U(s)} = \frac{1}{s+1} \text{ and we get } \dot{x_2} = -x_2 + u$$

- The state space model can be written as,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Eigenvalues are, $\lambda_1 = 0, \lambda_2 = -1$

$$e^{At} = \begin{bmatrix} 1 & 1 - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$$

- The zero state response is give by,

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

- At time $t = 0, e(0) = r(0) - c(0) = 10$
- Therefore the controller will operate in its positive saturation zone and the plant will have an input $u = 4$.

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \int_0^t e^{At} * e^{-A\tau} Bu(\tau) d\tau$$

$$x(t) = \begin{bmatrix} 4(t - 1 + e^{-t}) \\ 4(1 - e^{-t}) \end{bmatrix}$$

Tutorial -5

For the system defined by,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Step Response:

Let us write the step input u(t) as $\mathbf{u}(t) = \mathbf{k}$

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}\end{aligned}$$

Ramp Response:

Let us write the ramp input ***u(t)*** as $\mathbf{u}(t) = t\mathbf{v}$

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{B}\mathbf{v}\end{aligned}$$

Impulse Response:

Let us write the impulse input u(t) as $\mathbf{u}(t) = \delta(t)\mathbf{w}$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

Question:1

Example A-11-7, Modern Control Engineering , 4th Edition Ogata

- Obtain the response ***y(t)*** of the following system: where ***u(t)*** is the unit-step input occurring at ***t*** = 0, or ***u(t)*** = ***1(t). (Use inverse Laplace transform method)***

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$= \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t} \sin 0.5t \\ 2e^{-0.5t} \sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k$$

$$= \begin{bmatrix} e^{-0.5t} \sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t} \sin 0.5t$$

Summary of the topic:

Case I: Solution of Homogeneous State Equations

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \qquad OR \qquad x(t) = L^{-1}[\emptyset(s)]\, X(0)$$

Case II: Solution of Non - Homogeneous State Equations

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau} b u(\tau)\, d\tau$$

$$OR$$

$$x(t) = L^{-1}[\emptyset(s)]\, X(0) + L^{-1}[\emptyset(s) B\, U(s)]$$

Computation of state transition matrix using Laplace Transform

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$