Lecture 5
16-04-21

PARTIAL DIFFERENTIATION

Topics

- Partial derivatives
- Homogeneous functions
- Total derivatives
- Differentiation of Implicit functions
- Errors and approximations
- Taylor's expansion for 2 variable functions
- Maxima and minima of $f(x, y)$
- Lagrange's method of undetermined multipliers.

Defn :-

$$y = f(x)$$

$$\frac{dy}{dx}, \quad f'(x)\Big|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x}\Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$z = f(x, y)$ surface

$y = b \longrightarrow$ plane

$f(x, b)$

$x = a$

If $z = f(x, y)$ is a function of 2 independent variables then

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \left.\frac{\partial z}{\partial x}\right|_{(a,b)}$$

is called the partial derivative of z w.r.t. x

$$\text{|||ly} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

If z possesses a partial derivative w.r.t. x at every point of its domain then

we write $$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

The partial derivatives of the 1st order partial derivatives are called second order partial derivative

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

and so on.

Geometrical meaning :-

$z = f(x, y)$ represents a surface. $y = b$ denotes a plane. $\frac{\partial z}{\partial x}$ is the slope of tangent at the point (a, b) of the curve $f(x, b)$, which is the intersection of $z = f(x, y)$ and $y = b$.

order 1

1. Find the partial derivatives of,

$$u = \frac{xy}{x+y}$$

$$\frac{\partial u}{\partial x} = y \frac{\partial}{\partial x}\left(\frac{x}{x+y}\right)$$

$$= y\left[\frac{(x+y)\cdot 1 - x(1+0)}{(x+y)^2}\right]$$

$$= y \times \frac{y}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

|||^{ly}

$$\frac{\partial u}{\partial y} = \frac{x^2}{(x+y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{y^2}{(x+y)^2}\right)$$

$$= y^2\left(\frac{-2}{(x+y)^3}\right)$$

$$d\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$$

$$= -\frac{2y^2}{(x+y)^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x^2}{(x+y)^2}\right)$$

$$= \frac{(x+y)^2 \times 2x - x^2\, 2(x+y)}{(x+y)^4}$$

//

2. If $u = x^2y + y^2z + z^2x$, then compute $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$u = x^2y + y^2z + z^2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + y^2z + z^2x\right)$$

$$= 2xy + 0 + z^2 \cdot 1 = 2xy + z^2$$

$$\frac{\partial u}{\partial y} = 2yz + x^2 \quad , \quad \frac{\partial u}{\partial z} = 2zx + y^2$$

Adding we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= (x + y + z)^2$$

3. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$

then the value of

$$u_x + u_y + u_z$$

$$\frac{\partial u}{\partial x} = 4(x-y)^3 - 4(z-x)^3$$

$$\frac{\partial u}{\partial y} = 4(y-z)^3 - 4(x-y)^3$$

$$\frac{\partial u}{\partial z} = 4(z-x)^3 - 4(y-z)^3$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

4. If $u = \log_e(\tan x + \tan y + \tan z)$ then find $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 x \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z} \quad \text{— (3)}$$

$\sin 2x\,(1) + \sin 2y\,(2) + \sin 2z\,(3)$ gives

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{1}{\tan x + \tan y + \tan z}\left\{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z\right\}$$

$$= \frac{2(\tan x + \cancel{\tan y} + \tan z)}{\tan x + \cancel{\tan y} + \tan z}$$

$$\sin 2x \sec^2 x = \frac{2 \sin x \cos x}{\cos^2 x} = 2 \tan x$$

$= 2.$

4. If $u = \log(x^3+y^3+z^3-3xyz)$

S.T $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

Compute u_x, u_y & u_z

$$u_x = \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

lly $u_y = \frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$

$$u_z = \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Adding them

$$u_x+u_y+u_z = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\,\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\,\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{x+y+z}$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2} \quad //$$

5. If $u = e^x(x\cos y - y\sin y)$

$v = e^x(x\sin y + y\cos y)$

P.T a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

b) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

d) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ e) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$u = e^x(x\cos y - y\sin y)$

a) $\frac{\partial u}{\partial x} = e^x[\cos y] + (x\cos y - y\sin y)e^x$

$= e^x\{\cos y + x\cos y - y\sin y\}$

$v = e^x(-x\sin y + y\cos y)$

$\frac{\partial v}{\partial y} = e^x[x\cos y - y\sin y + \cos y]$

Note that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

b) $\frac{\partial^2 u}{\partial x \partial y}$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$= \frac{\partial}{\partial x}[e^x(-x\sin y - y\cos y - \sin y)]$

$= e^x[-2\sin y - x\sin y - y\cos y]$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(e^{x}(x\cos y - y\sin y + \cos y)\right)$$

$$= e^{x}\left[-x\sin y - y\cos y - \sin y - \sin y\right]$$

$$= e^{x}\left[-x\sin y - y\cos y - 2\sin y\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad //$$

e) To prove $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(e^{x}(x\cos y - y\sin y + \cos y)\right)$$

$$= e^{x}\left[\cos y + x\cos y - y\sin y\right] \quad (1)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[e^{x}(-x\sin y - y\cos y - \sin y)\right]$$

$$= -e^{x}\left[x\cos y + \cos y - y\sin y + \cos y\right]$$

$$= -e^{x}\left[x\cos y + 2\cos y - y\sin y\right]$$

(2)

(1) + (2) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\longrightarrow$ Laplace eqn.

$u(x, y)$ satisfy Laplace eqn.

$u(x, y)$ is called harmonic function.

6. If $x^x y^y z^z = c$, prove that $\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e ex)^{-1}$ when $x = y = z$.

Taking log on both sides

$x \log x + y \log y + z \log z = \log c$

Here z is dependent variable.

x & y are independent variables

Differentiating partially w.r.t. x we get

$$\cancel{x} \times \frac{1}{\cancel{x}} + \log x \cdot 1 + 0 + \cancel{z} \frac{1}{\cancel{z}} \cdot \frac{\partial z}{\partial x}$$

$$+ \log z \cdot 1 \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(1+\log z\right) = -\left(1+\log x\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\left(1+\log x\right)}{1+\log z} \quad \text{—— } (1)$$

$$\text{||ly} \quad \frac{\partial z}{\partial y} = -\frac{\left(1+\log y\right)}{1+\log z} \quad \text{—— } (2)$$

differentiating (2) w.r.t. x

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\left(1+\log y\right)}{1+\log z}\right)$$

$$= +\frac{\left(1+\log y\right)}{\left(1+\log z\right)^2}\left\{\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right\}$$

$$= -\frac{\left(1+\log y\right)}{z\left(1+\log z\right)^3}\left(1+\log x\right)$$

$$\therefore \left.\frac{\partial^2 z}{\partial x\,\partial y}\right|_{x=y=z} = -\frac{\cancel{\left(1+\log x\right)}}{x\left(1+\log x\right)^{\overset{2}{\cancel{3}}}}$$

$$= -\frac{1}{x\left(\log_e e+\log_e x\right)^2} = -\left(x\log ex\right)^{-1}$$

7. If $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$

P.T $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$

$$u \longrightarrow r \begin{array}{l} \nearrow x \\ \rightarrow y \\ \searrow z \end{array}$$

$2r \dfrac{\partial r}{\partial x} = 2x$

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$u = f(r)$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = f'(r) \times \frac{x}{r}$$
$$= \frac{x}{r} f'(r)$$

|||ly $\dfrac{\partial u}{\partial y} = \dfrac{y}{r} f'(r)$, $\dfrac{\partial u}{\partial z} = \dfrac{z}{r} f'(r)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r} f'(r)\right)$$
$$= \frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x} + f'(r) \frac{r \cdot 1 - x \frac{\partial r}{\partial x}}{r^2}$$
$$= \frac{x^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r)$$

$\longrightarrow$ (1)

|||ly we can write

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) \quad \text{--- } (2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{z^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{z^2}{r^3} f'(r) \quad \text{--- } (3)$$

Adding (1) + (2) + (3) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3}{r} f'(r) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{f''(r)}{\cancel{r^2}} \times \cancel{r^2} + \frac{3}{r} f'(r) - \frac{f'(r)}{r^{\cancel{3}}} \times \cancel{r^2}$$

$$= f''(r) + \frac{2}{r} f'(r)$$

HW 1) If $u = (y-z)(z-x)(x-y)$

find $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$

2) If $u = \frac{1}{r}$ and $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

3) $v = (x^2+y^2+z^2)^{-1/2}$

find $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$