Quick Revision Convalution to FFT

Dis

Convolution sum properties

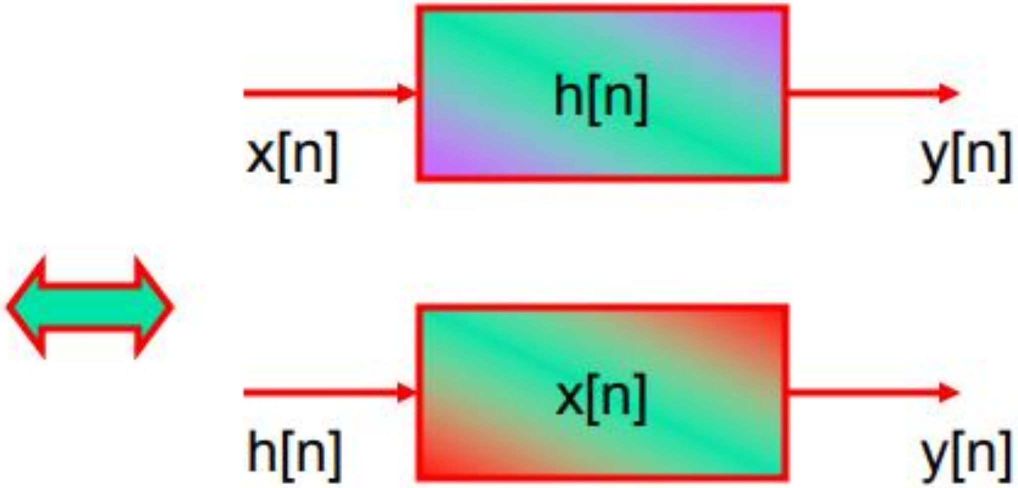
- $\delta[n] * x[n] = x[n]$

  $\delta[n-m] * x[n] = x[n-m]$

  $\delta[n] * x[n-m] = x[n-m]$
- Commutative law
- Associative law
- Distributive law

Commutative law

$$\boxed{x[n]*h[n]=h[n]*x[n]}$$



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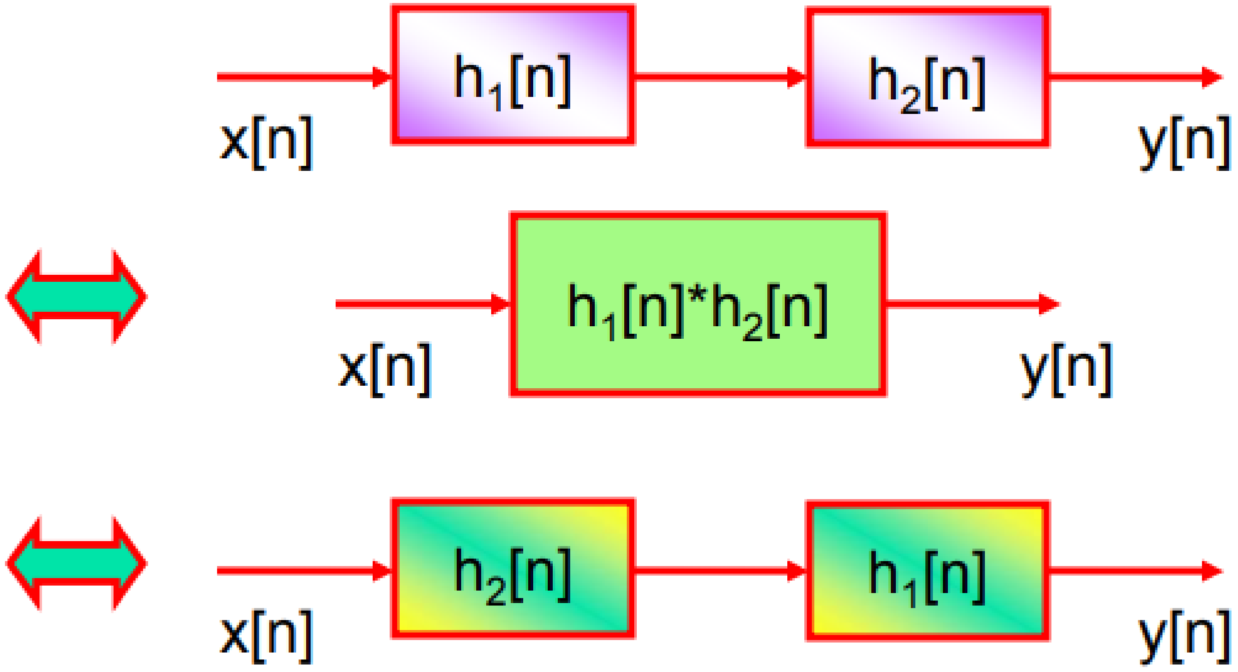
Associative law

$$\boxed{(x[n]*h_1[n])*h_2[n]=x[n]*(h_1[n]*h_2[n])}$$



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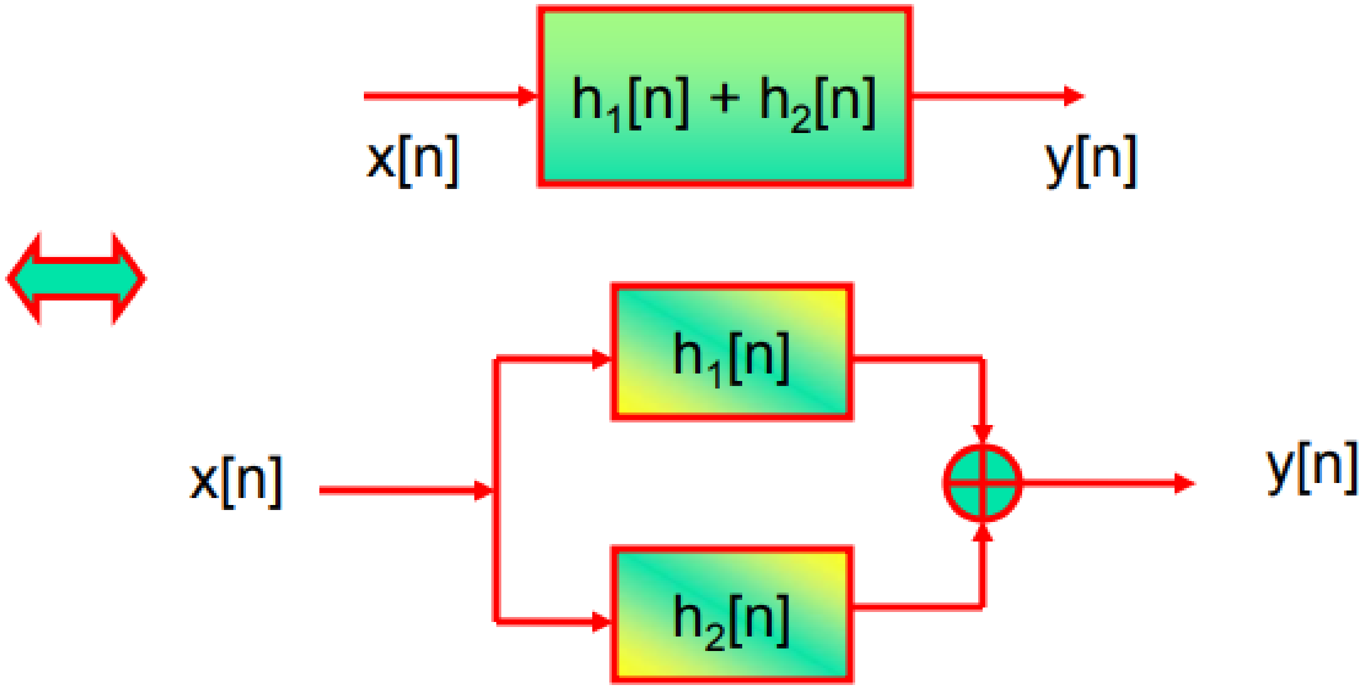
Distributive law

$$x[n]*(h_1[n]+h_2[n]) = (x[n]*h_1[n]) + (x[n]*h_2[n])$$



Convolution of finite duration sequences

In convolution of finite duration sequences it is possible to predict the length of resultant sequence.

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples.

i.e., if, $\quad \text{Length of } x_1(n) = N_1$

$\qquad\qquad \text{Length of } x_2(n) = N_2$

then, $\quad \text{Length of } x_3(n) = N_1 + N_2 - 1$

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero. The final

i.e., if, $\quad x_1(n) \text{ start at } n = n_1$

$\qquad\qquad x_2(n) \text{ start at } n = n_2$

then, $\quad x_3(n) \text{ start at } n = n_1 + n_2$

Example 2.3.2

The impulse response of a linear time-invariant system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

EXAMPLE 2.1 Find the convolution of two finite duration sequences:

$$h(n) = a^n u(n) \quad \text{for all } n$$
$$x(n) = b^n u(n) \quad \text{for all } n$$

(i) When $a \neq b$

(ii) When $a = b$

Solution: The impulse response $h(n)$ and the input $x(n)$ are zero for $n < 0$, i.e. both $h(n)$ and $x(n)$ are causal.

$$\therefore \qquad y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{(n-k)} = a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k$$

$$= a^n \left[\frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)}\right] \qquad [\text{when } a \neq b]$$

When $a = b$

$$y(n) = a^n[1 + 1 + 1 + \cdots + n + 1 \text{ terms}] = a^n(n + 1)$$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

G. S. $$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}; \quad |a| < 1$$

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

EXAMPLE 3.12 Using properties of Z-transform, find the Z-transform of the sequence

(a) $x(n) = \alpha^{n-2} u(n-2)$ (b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha};\ \text{ROC};\ |z| > |\alpha|$$

EXAMPLE 3.12 Using properties of Z-transform, find the Z-transform of the sequence

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(b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

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Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m} X(z)$$

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Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m} X(z)$$

In the same way,

$$Z[\alpha^{n-2} u(n-2)] = z^{-2} Z[\alpha^n u(n)] = z^{-2} \frac{z}{z-\alpha} = \frac{1}{z(z-\alpha)}; \text{ ROC; } |z| > |\alpha|$$

Find inverse Z-Transform

$$\frac{1}{z(z-\alpha)};\ \text{ROC};\ |z|>|\alpha|$$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(d) $x(n) = 2^n u(n-2)$

step 1. Shift $\mathbf{x[n] = u(n-2)}$

$$\mathbf{X(z) = \frac{z^{-2}}{(z-1)} = \frac{1}{z^2\ (z-1))}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{1}$$

step 2. z-scaling $\mathbf{x[n] = 2^n u(n-2)}$

$$\mathbf{X(z) = \frac{1}{\frac{z^2}{2}\left(\frac{z}{2}-1\right)} = \frac{4}{z^2(z-2)}} \quad \text{ROC} \quad |\mathbf{z}| > \mathbf{2}$$

$$\mathbf{u(n) \longleftrightarrow \frac{1}{1 - z^{-1}} = \frac{z}{z-1}} \quad \text{ROC} \quad \mathbf{|z| > 1}$$

Find inverse Z-Transform

$$\mathbf{X(z)} = \frac{\mathbf{4}}{\mathbf{z^2(z-2)}} \qquad \text{ROC} \quad \mathbf{|z| > 2}$$

A.

$$h(n) = 2^n u(n)$$

$$H(z) = \frac{1}{1 - 2z^{-1}}$$

$$H(z) = \frac{z}{z - 2}$$

B.

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$H(z) = \frac{1}{1 - \left(\frac{1}{2}\right) z^{-1}}$$

$$H(z) = \frac{z}{z - \left(\frac{1}{2}\right)}$$

C.

$$h(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$$

$$H(z) = \frac{1}{1 - \left(\frac{1}{2}\right) z^{-1}}$$

$$H(z) = \frac{z}{z - \left(\frac{1}{2}\right)}$$

Stable or not?

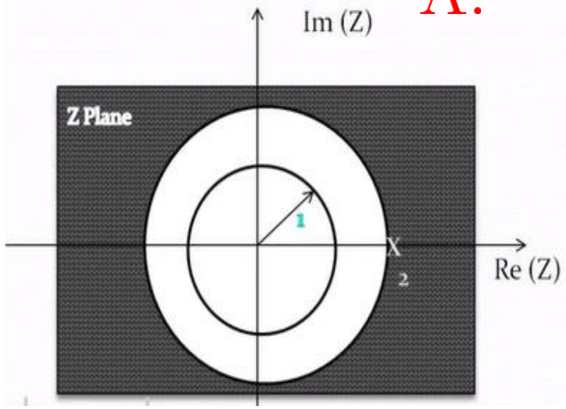
D.

$$h(n) = -2^n u(-n-1)$$

$$H(z) = \frac{1}{1 - 2z^{-1}}$$

$$H(z) = \frac{z}{z - 2}$$

A.



$$h(n) = 2^n u(n)$$

$$H(z) = \frac{1}{1 - 2z^{-1}}$$

$$H(z) = \frac{z}{z - 2}$$

$$\boxed{ROC:\ |z| > 2}$$

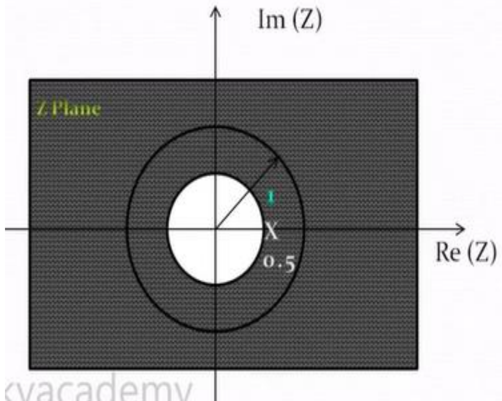
Causal and unstable

B.

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$H(z) = \frac{1}{1 - \left(\frac{1}{2}\right)z^{-1}}$$

$$H(z) = \frac{z}{z - \left(\frac{1}{2}\right)}$$



$$\boxed{ROC:\ |z| > \frac{1}{2}}$$

Causal and stable

C.

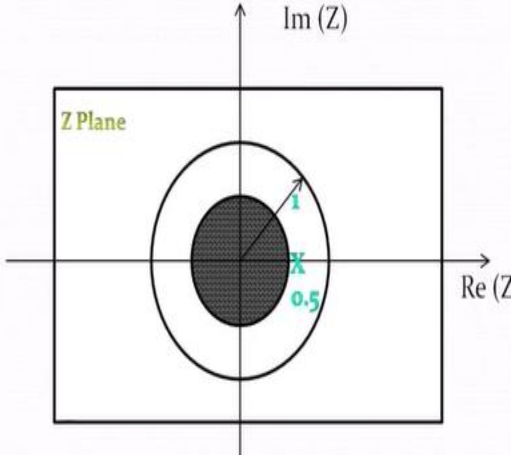
$$h(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$$

$$H(z) = \frac{1}{1 - \left(\frac{1}{2}\right)z^{-1}}$$

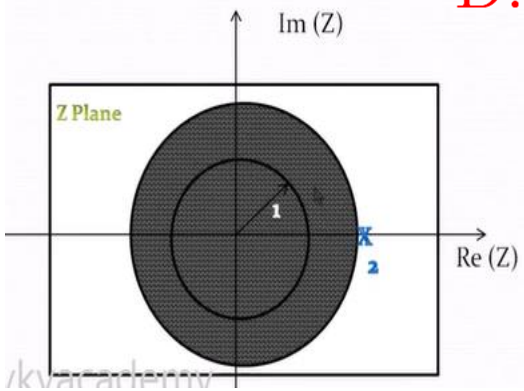
$$H(z) = \frac{z}{z - \left(\frac{1}{2}\right)}$$

$$\boxed{ROC:\ |z| < \frac{1}{2}}$$

Anticausal and unstable



D.



$$h(n) = -2^n u(-n-1)$$

$$H(z) = \frac{1}{1 - 2z^{-1}}$$

$$H(z) = \frac{z}{z - 2}$$

$$\boxed{ROC:\ |z| < 2}$$

Discrete Fourier Transform DFT

- The formulas for DFT and IDFT may be expressed as (in terms of twiddle factor)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad 0 \leq n \leq N-1$$

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\,\sin\left(\frac{2\pi}{N}\right)$$

- The relationship between $x(n)$ and $X(k)$ is denoted as

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

$\cos\left(\frac{2\pi k_0 n}{N}\right),$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right) = \frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right) = \frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right] e^{-j\frac{2\pi k n}{N}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

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$$= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{-j\frac{2\pi (k-k_0) n}{N}} + e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

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$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right] e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{-j\frac{2\pi (k-k_0)n}{N}} + e^{-j\frac{2\pi (k+k_0)n}{N}}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} e^{-j\frac{2\pi (k-k_0)n}{N}} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi (k+k_0)n}{N}}\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

G. S. $$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right),$$

$$\cos\left(\frac{2\pi k_0 n}{N}\right) = \frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right] e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{-j\frac{2\pi (k-k_0) n}{N}} + e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} e^{-j\frac{2\pi (k-k_0) n}{N}} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi (k+k_0) n}{N}}\right]$$

$$= \frac{1}{2}\left[N\delta(k-k_0) + N\delta(k+k_0)\right]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

G. S. $\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

$$\mathbf{X(k) = \sum_{n=0}^{3} x[n]W_4^{nk}}$$

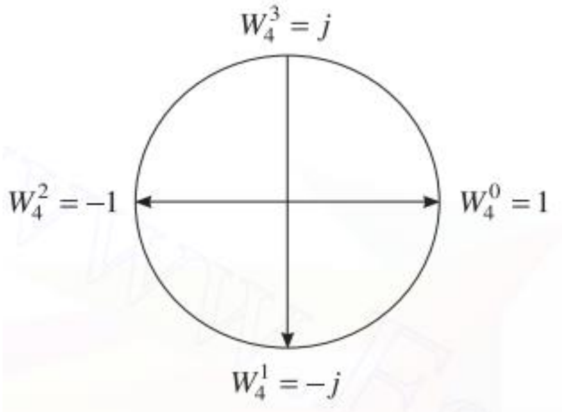
$$N_{-1}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad 0 \le k \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\mathbf{X(k) = x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}}$$

where k=0, 1, 2, 3



(a) 4-point DFT

$$\mathbf{X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0}$$

$$\mathbf{X(0) = 1 - 1 + 2 - 2 = 0}$$

$$\mathbf{X(1) = 1 - 1W_4^1 + 2W_4^2 - 2W_4^3}$$

$$\mathbf{X(1) = 1 + j - 2 - 2j = -1 - j}$$

$$\mathbf{X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6}$$

$$\mathbf{X(2) = 1 + 1 + 2 + 2 = 6}$$

$$\mathbf{X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9}$$

$$\mathbf{X(3) = 1 - j - 2 + 2j = -1 + j}$$

Property	Time domain	Frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \bmod N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \bmod N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \bmod N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Circular Time Shift

$$\mathbf{DFT}\{x(n)\} = X(k), \quad \text{then DFT } \{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}$$

Proof

$$\mathbf{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N \, e^{\frac{-j2\pi kn}{N}}$$

Let $n - m = p, \quad \therefore n = p + m$

$$= \sum_{p=-m}^{N-1-m} x(p)e^{\frac{-j2\pi(p+m)}{N}} \quad = \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi(p+m)}{N}} \quad = \sum_{p=0}^{N-1} x(p)e^{\frac{-j2\pi p}{N}} e^{\frac{-j2\pi m}{N}}$$

Periodicity

If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

Proof: By definition of DFT, the $(k + N)$th coefficient of $X(k)$ is given by

$$X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k+N)/N} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\cancelto{1}{e^{-j2\pi nN/N}}$$

But $e^{-j2\pi n} = 1$ for all n (Here n is an integer)

$$\therefore \qquad X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = X(k)$$

Multiplication

If $\quad \mathrm{DFT}\,[x_1(n)] = X_1(k)$

and $\quad \mathrm{DFT}\,[x_2(n)] = X_2(k)$

Then $\quad \mathrm{DFT}[x_1(n)x_2(n)] = \dfrac{1}{N}[X_1(k) \oplus X_2(k)]$

Proof

$$\mathbf{DFT\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Replacing $\mathbf{x_1(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}}$

$$\mathbf{= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}}$$

Rearranging the order of the summation.

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{j2\pi mn}{N}}e^{\frac{-j2\pi kn}{N}}\right]}$$

Rearranging the exponential terms

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi n(k-m)}{N}}\right]}$$

Using defination of DFT

$$\mathbf{= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)X_2((k-m))_N}$$

By defination of Circular Convalution

$$\mathrm{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$$

Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT}\ \{x_1(n)\} = X_1(k)$$

and

$$\text{DFT}\ \{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants a and b,

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

Proof:

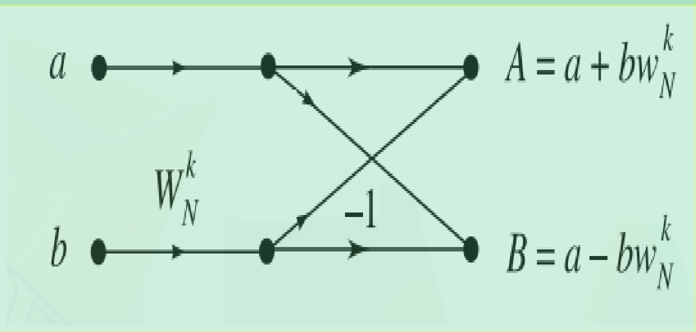
$$\text{DFT}\{ax_1(n) + bx_2(n)\} = \sum_{n=0}^{N-1}[ax_1(n) + bx_2(n)]\, e^{-j2\pi nk/N}$$

$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

$$= aX_1(k) + bX_2(k)$$

Number of Caculation in Radix-2 FFT

In radix-2 FFT, $N = 2^m$, and so there will be m stages of computations where $m = \log_2(N)$ with each stage having $N/2$ butterflies



The number of calculation in one butterflies are

1 number of complex multiplication

2 number of complex additions

There are $N/2$ butterflies in each stage

$$\frac{N}{2} \times 1 = O\left(\frac{N}{2}\right) \quad \text{number of complex multiplications}$$

$$\frac{N}{2} \times 2 = O\left(N\right) \quad \text{number of complex additions}$$

The N-point DFT involves m stages of computations.

$$\frac{N}{2} \times m = O\left(\frac{N}{2} \times \log_2(N)\right) \quad \text{number of complex multiplications}$$

$$N \times m = O\left(N \times \log_2(N)\right) \quad \text{number of complex additions}$$

Number of computation in DFT and FFT

Number of points N	Direct Computation: Complex additions $N(N-1)$	Direct Computation: Complex Multiplications N^2	Radix-2 FFT: Complex additions $N\log_2 N$	Radix-2 FFT: Complex Multiplications $(N/2)\log_2 N$
$4\,(=2^2)$	12	16	$4 \times \log_2 2^2 = 4 \times 2 = 8$	$\frac{4}{2} \times \log_2 2^2 = \frac{4}{2} \times 2 = 4$
$8\,(=2^3)$	56	64	$8 \times \log_2 2^3 = 8 \times 3 = 24$	$\frac{8}{2} \times \log_2 2^3 = \frac{8}{2} \times 3 = 12$
$16\,(=2^4)$	240	256	$16 \times \log_2 2^4 = 16 \times 4 = 64$	$\frac{16}{2} \times \log_2 2^4 = \frac{16}{2} \times 4 = 32$
$32\,(=2^5)$	992	1,024	$32 \times \log_2 2^5 = 32 \times 5 = 160$	$\frac{32}{2} \times \log_2 2^5 = \frac{32}{2} \times 5 = 80$
$64\,(=2^6)$	4,032	4,096	$64 \times \log_2 2^6 = 64 \times 6 = 384$	$\frac{64}{2} \times \log_2 2^6 = \frac{64}{2} \times 6 = 192$
$128\,(=2^7)$	16,256	16,384	$128 \times \log_2 2^7 = 128 \times 7 = 896$	$\frac{128}{2} \times \log_2 2^7 = \frac{128}{2} \times 7 = 448$

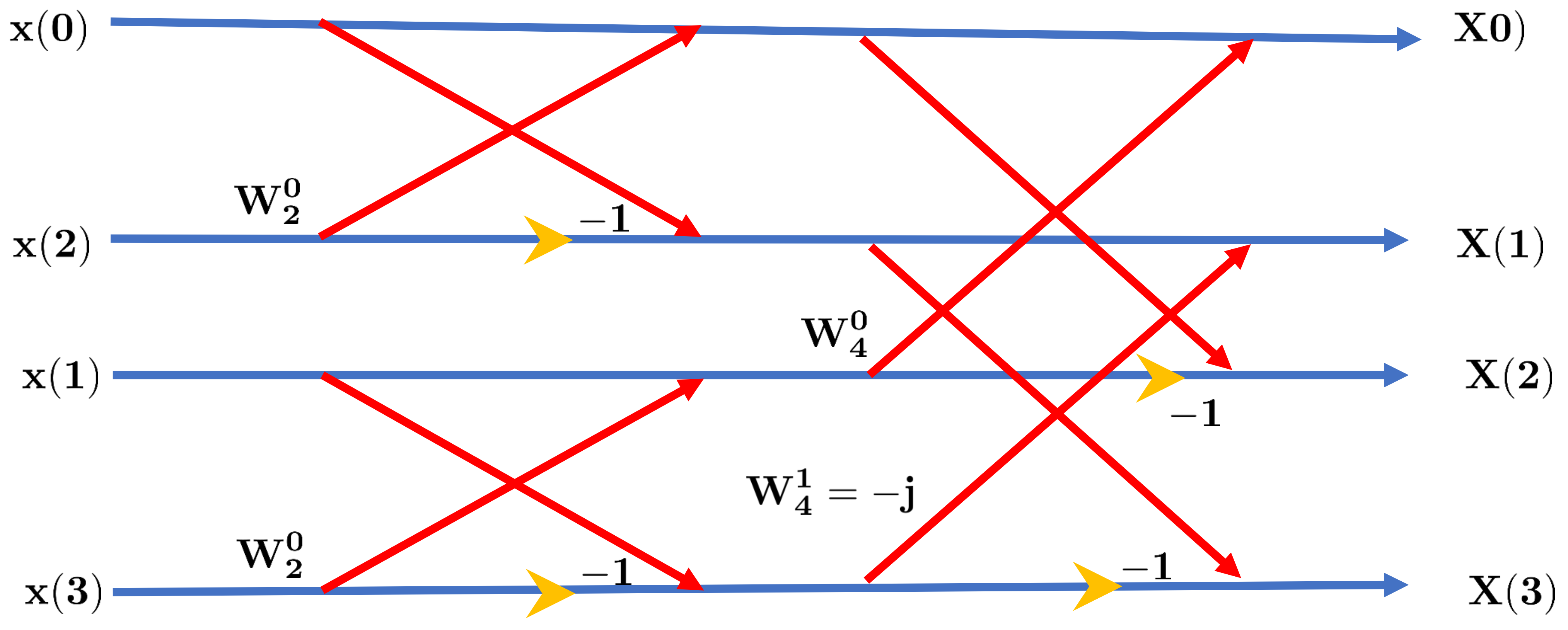
TABLE 6.1 COMPARISON OF COMPUTATIONAL COMPLEXITY FOR THE DIRECT COMPUTATION OF THE DFT VERSUS THE FFT ALGORITHM

Number of Points, N	Complex Multiplications in Direct Computation, N^2	Complex Multiplications in FFT Algorithm, $(N/2)\log_2 N$	Speed Improvement Factor
4	16	4	4.0
8	64	12	5.3
16	256	32	8.0
32	1,024	80	12.8
64	4,096	192	21.3
128	16,384	448	36.6
256	65,536	1,024	64.0
512	262,144	2,304	113.8
1,024	1,048,576	5,120	204.8

Once a butterfly operation is performed on a pair of complex numbers (a, b) to produce (A, B), there is no need to save the input pair (a, b). Hence we can store the result (A, B) in the same locations as (a, b). Consequently, we require a fixed amount of storage, namely, $2N$ storage registers, in order to store the results (N complex numbers) of the computations at each stage. Since the same $2N$ storage locations are used throughout the computation of the N-point DFT, we say that *the computations are done in place*.

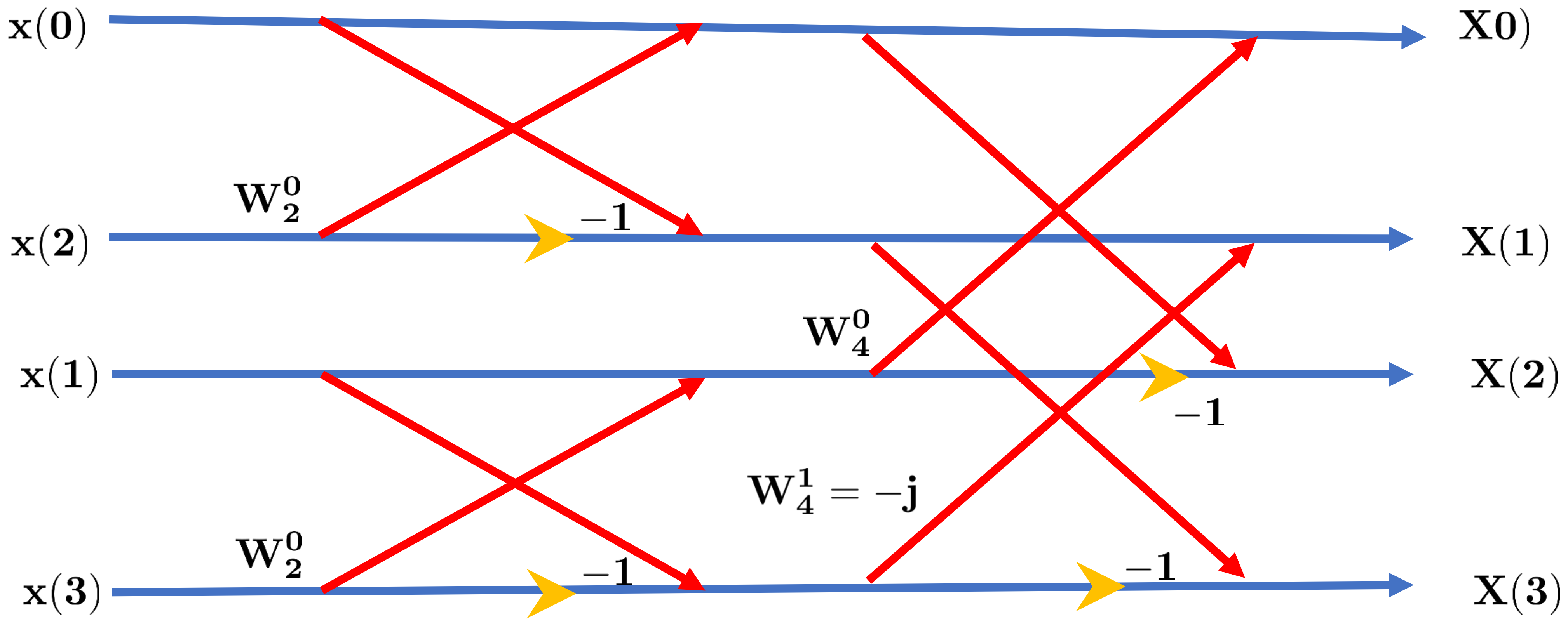
Number of computation in DFT and FFT

Find N=4 DFT with DIT FFT method $m = \log_2(N) = \log_2(4) = 2$



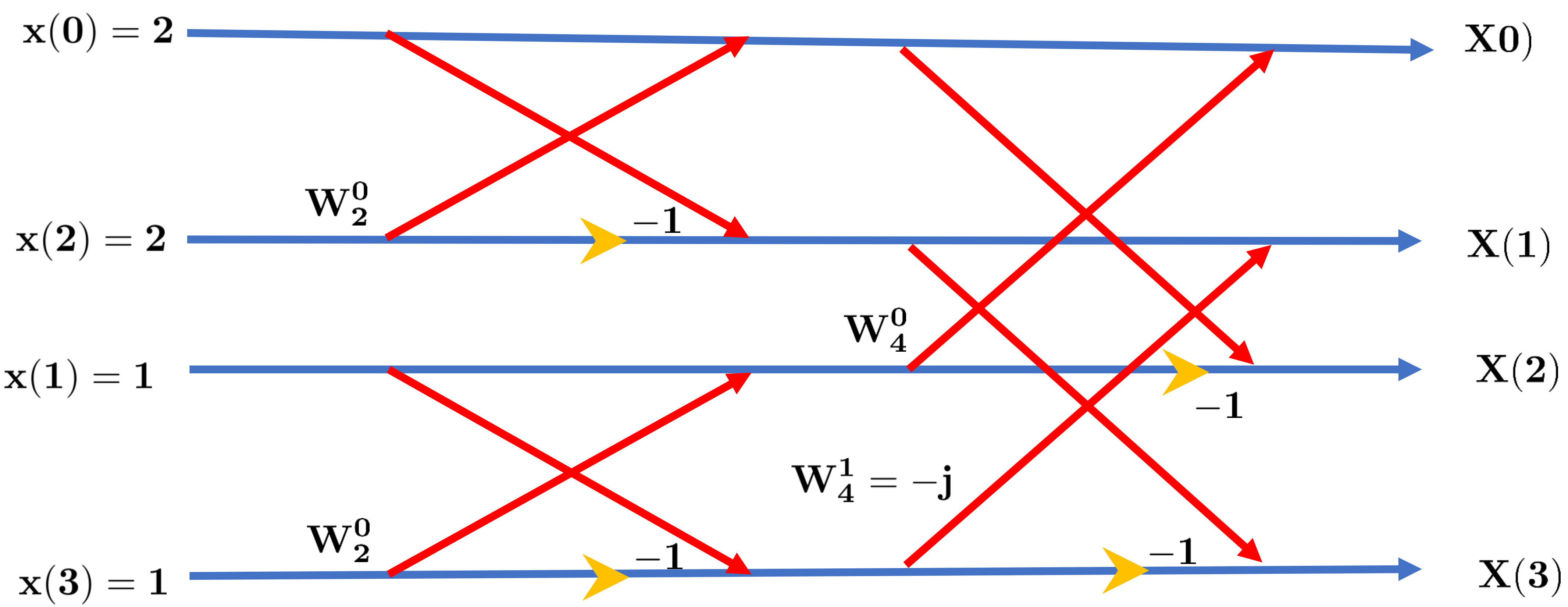
Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



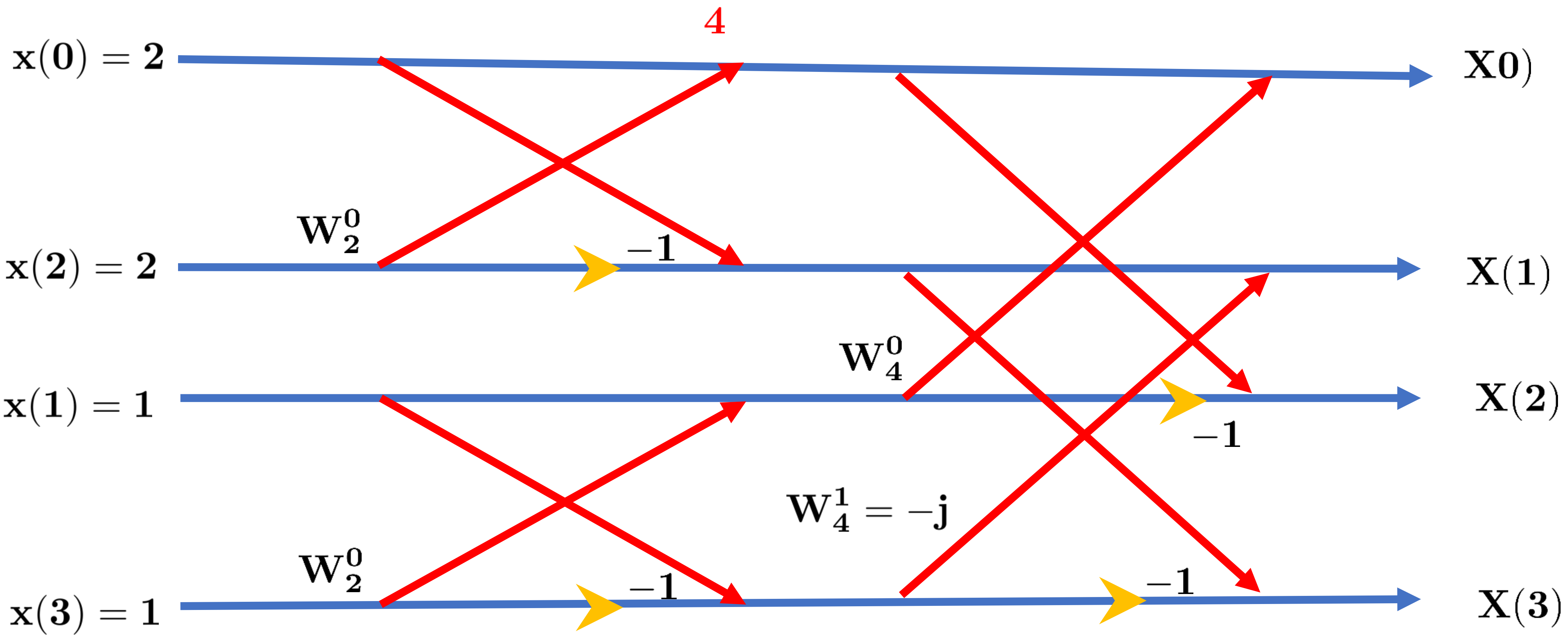
Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



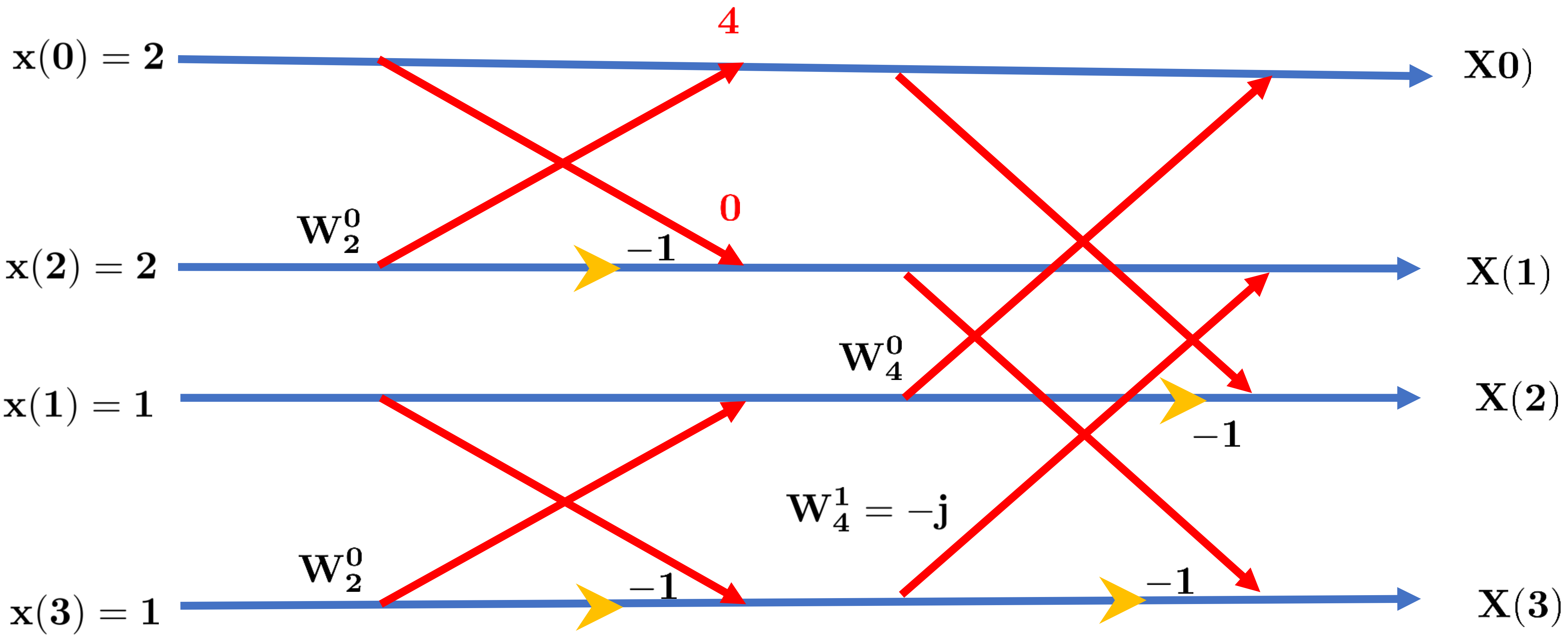
Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



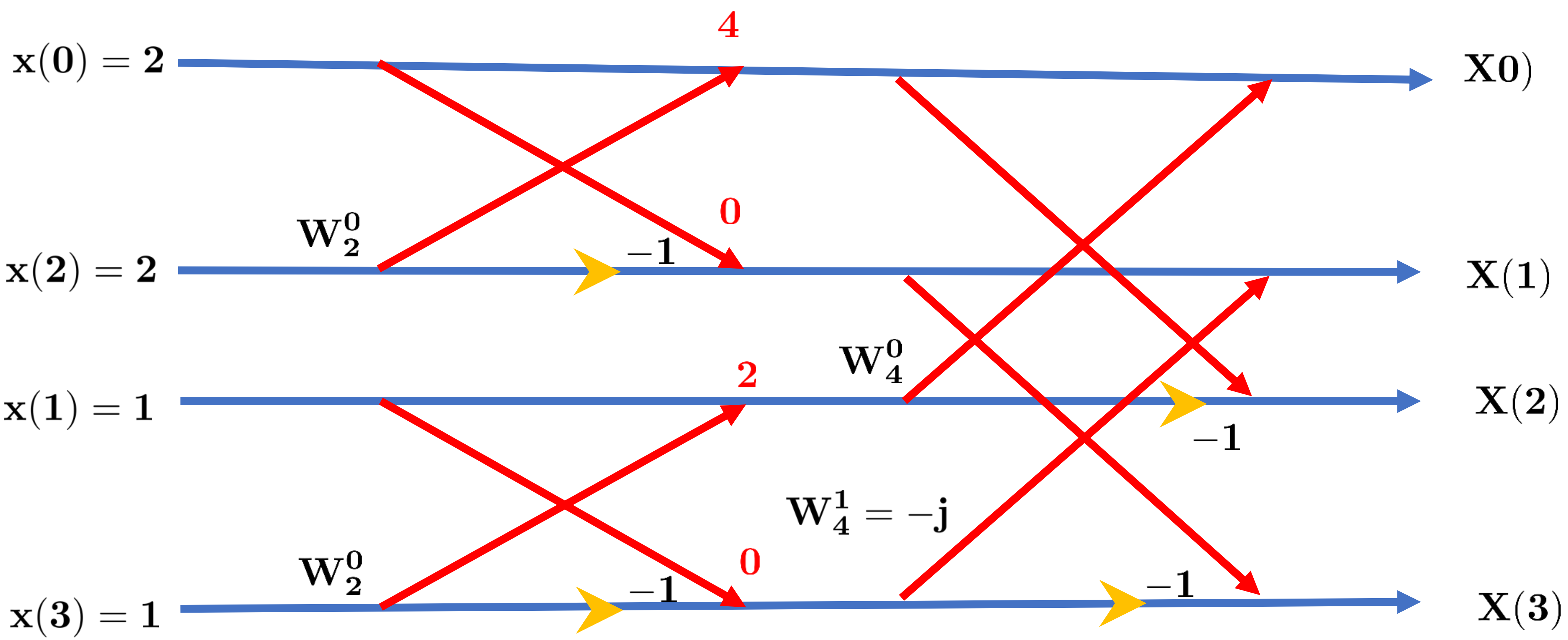
Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



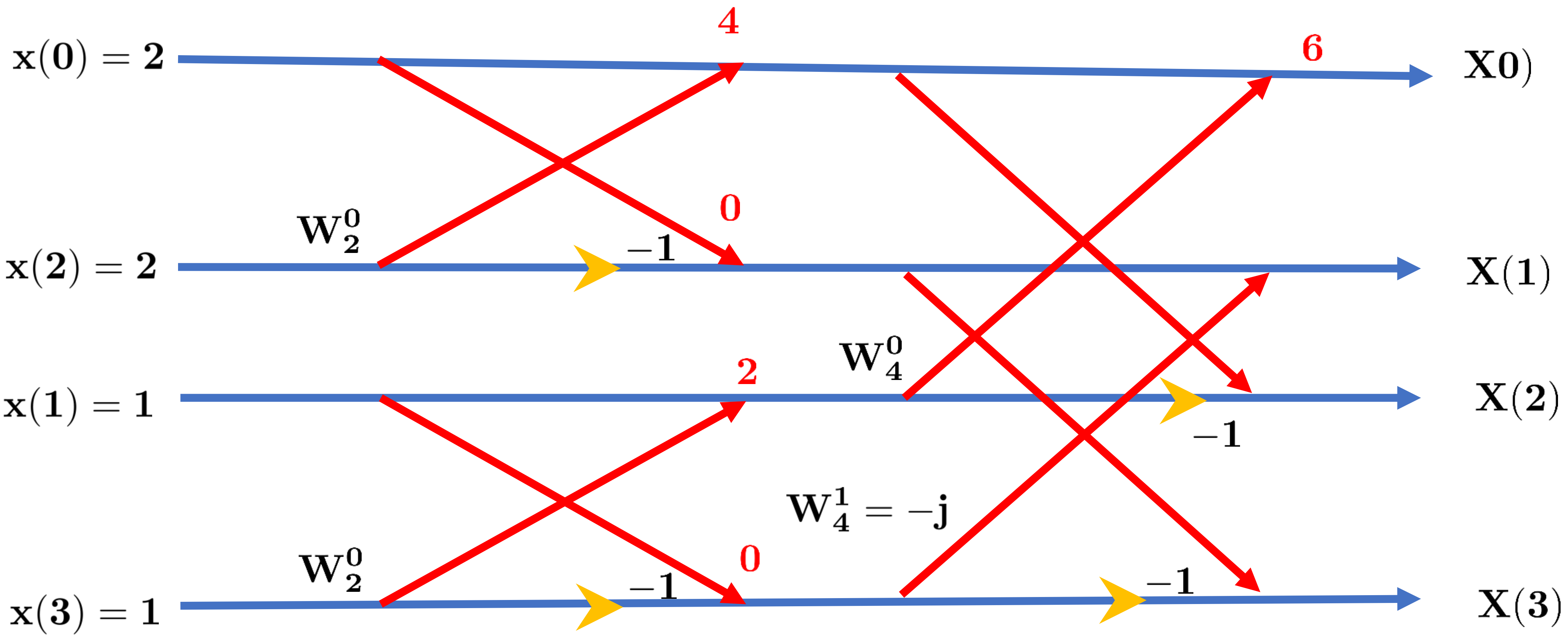
Number of computation in DFT and FFT

$$x[n] = \{2, 1, 2, 1\}$$



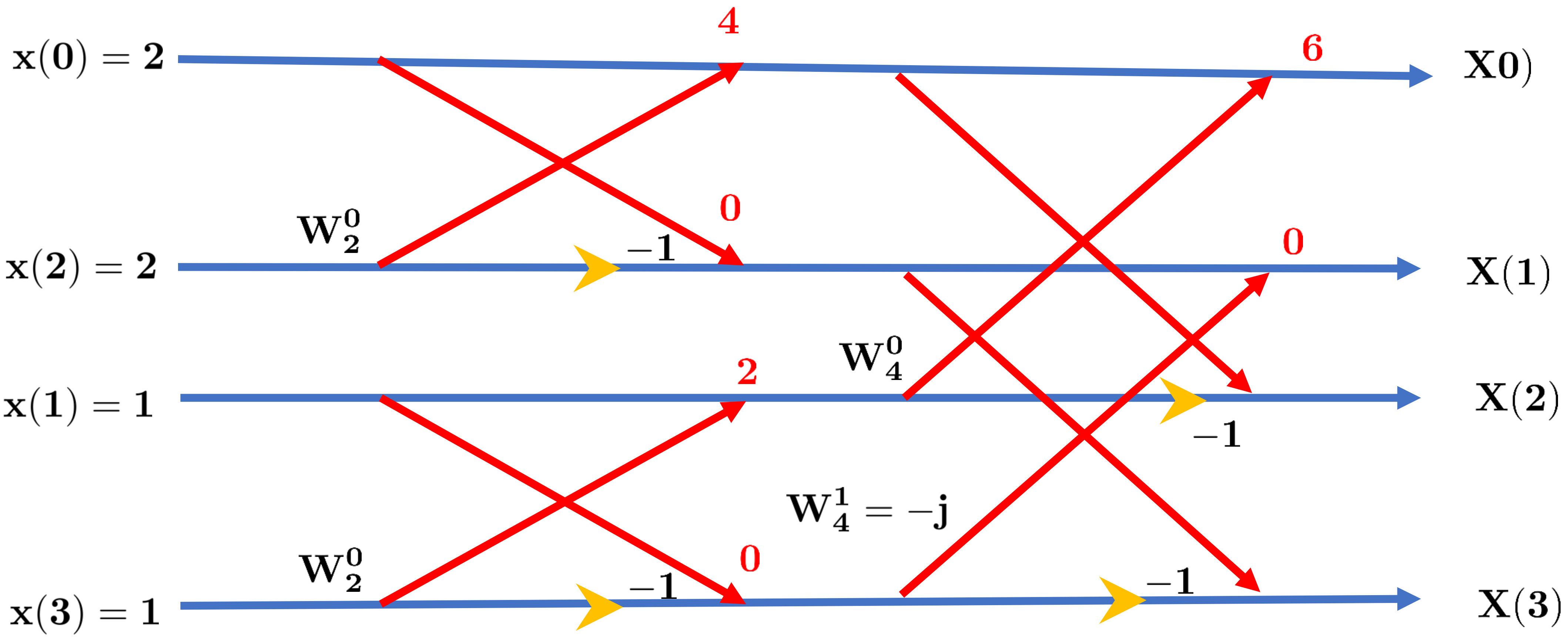
Number of computation in DFT and FFT

$$x[n] = \{2, 1, 2, 1\}$$



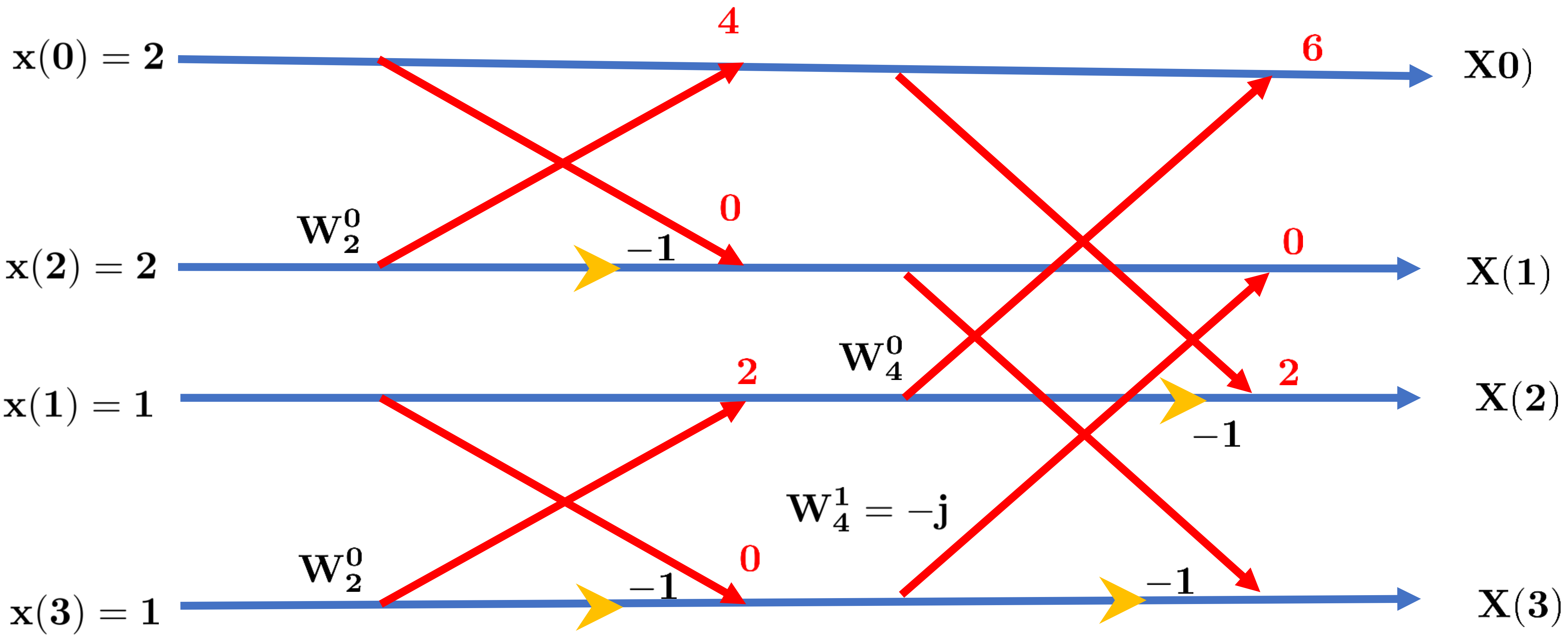
Number of computation in DFT and FFT

$$x[n] = \{2, 1, 2, 1\}$$



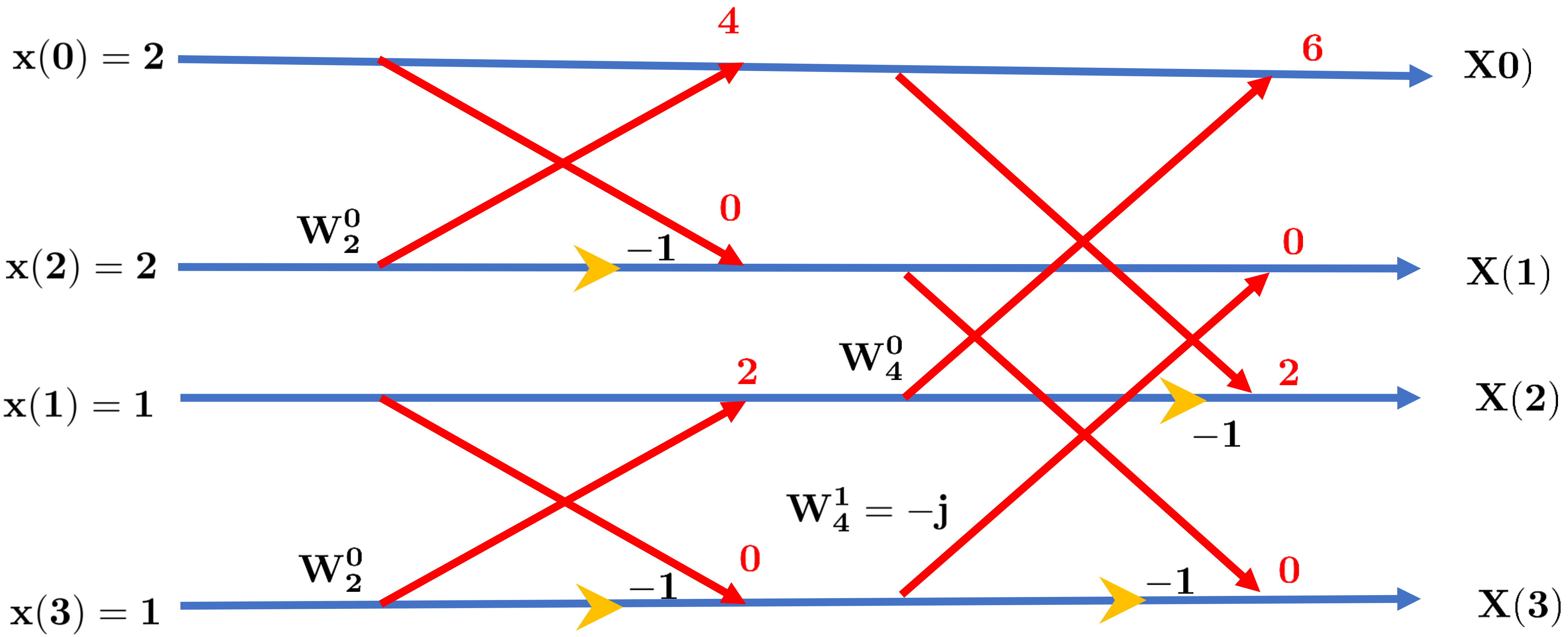
Number of computation in DFT and FFT

$$x[n] = \{2, 1, 2, 1\}$$



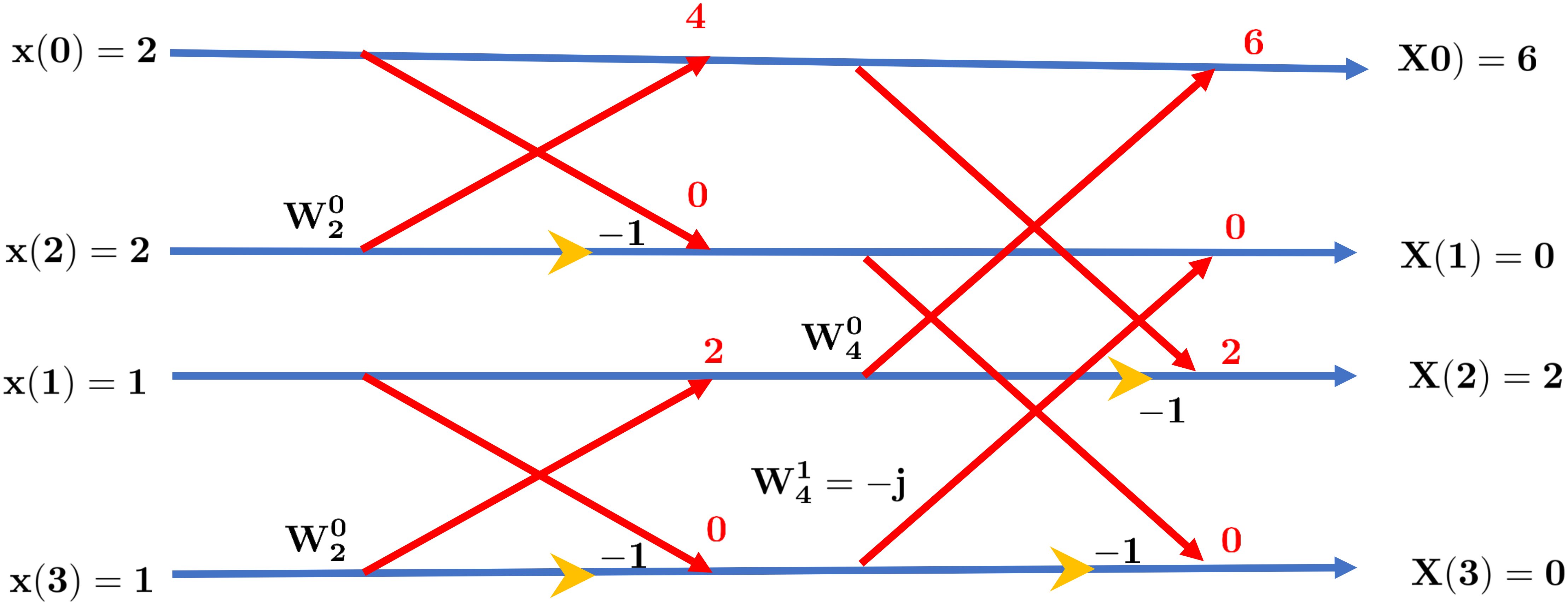
Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



Number of computation in DFT and FFT

$$\mathbf{x[n] = \{2, 1, 2, 1\}}$$



$$\mathbf{X[k] = \{6, 0, 2, 0\}}$$

Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$

$x(0) \longrightarrow X(0)$

$x(4) \longrightarrow X(1)$

$x(2) \longrightarrow X(2)$

$x(6) \longrightarrow X(3)$

$x(1) \longrightarrow X(4)$

$x(5) \longrightarrow X(5)$

$x(3) \longrightarrow X(6)$

$x(7) \longrightarrow X(7)$

Original	Binary Form	Reversed Form	Final
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$

Original	Binary Form	Reversed Form	Final
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

$x(0) \rightarrow X(0)$

$x(4) \rightarrow X(1)$

$x(2) \rightarrow X(2)$

$x(6) \rightarrow X(3)$

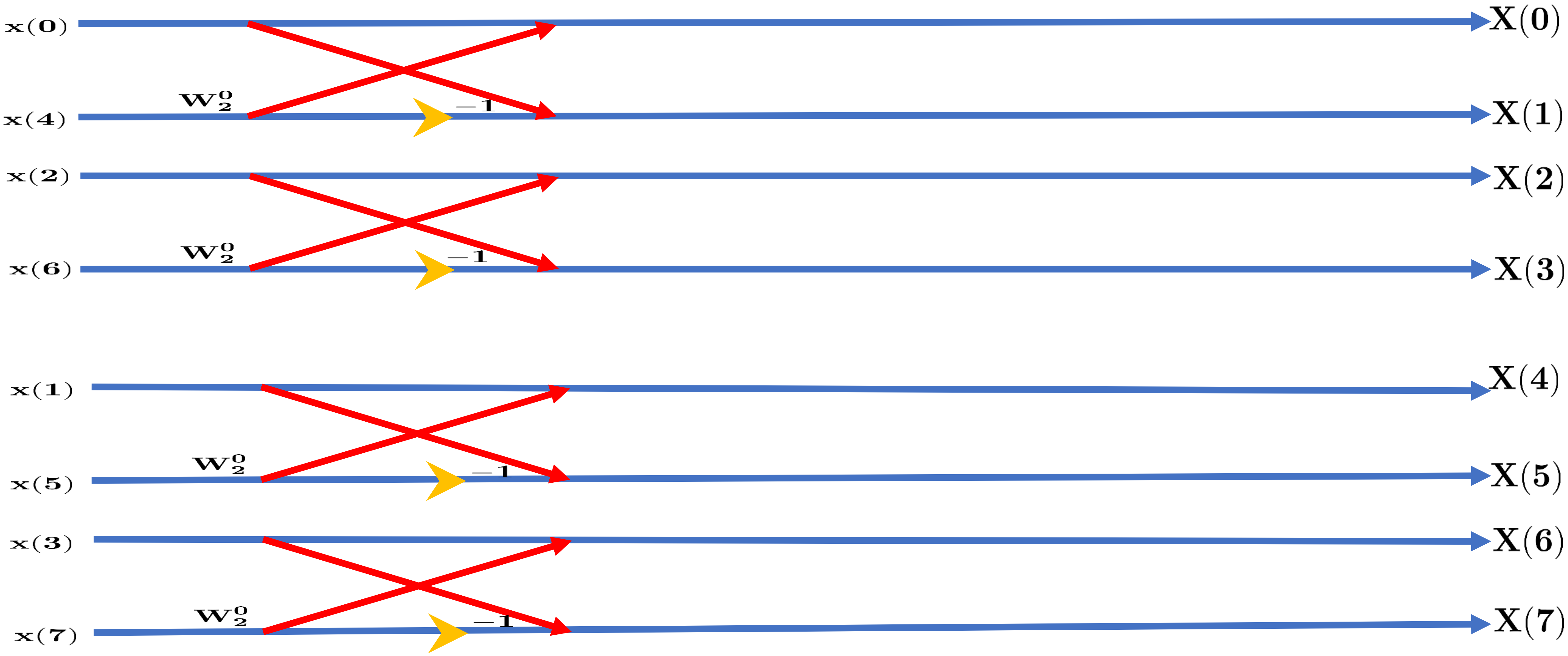
$x(1) \rightarrow X(4)$

$x(5) \rightarrow X(5)$

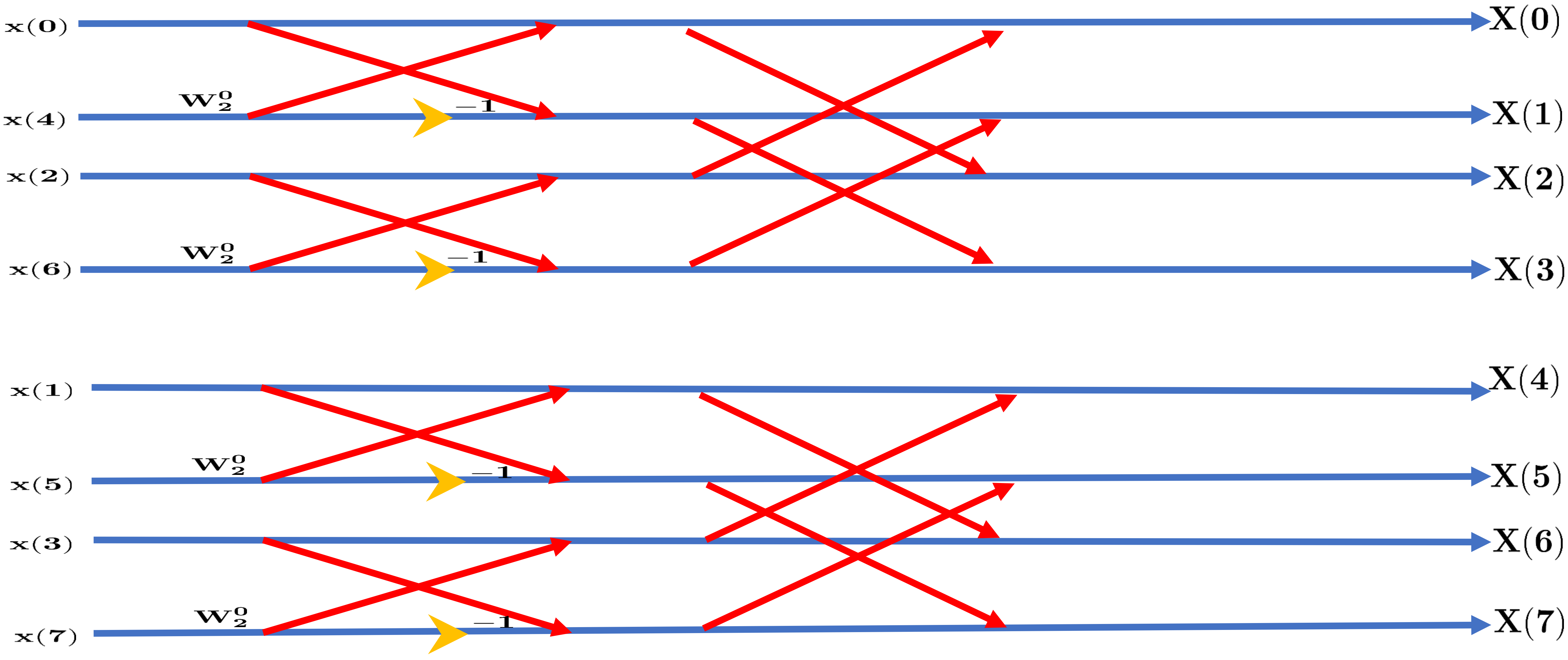
$x(3) \rightarrow X(6)$

$x(7) \rightarrow X(7)$

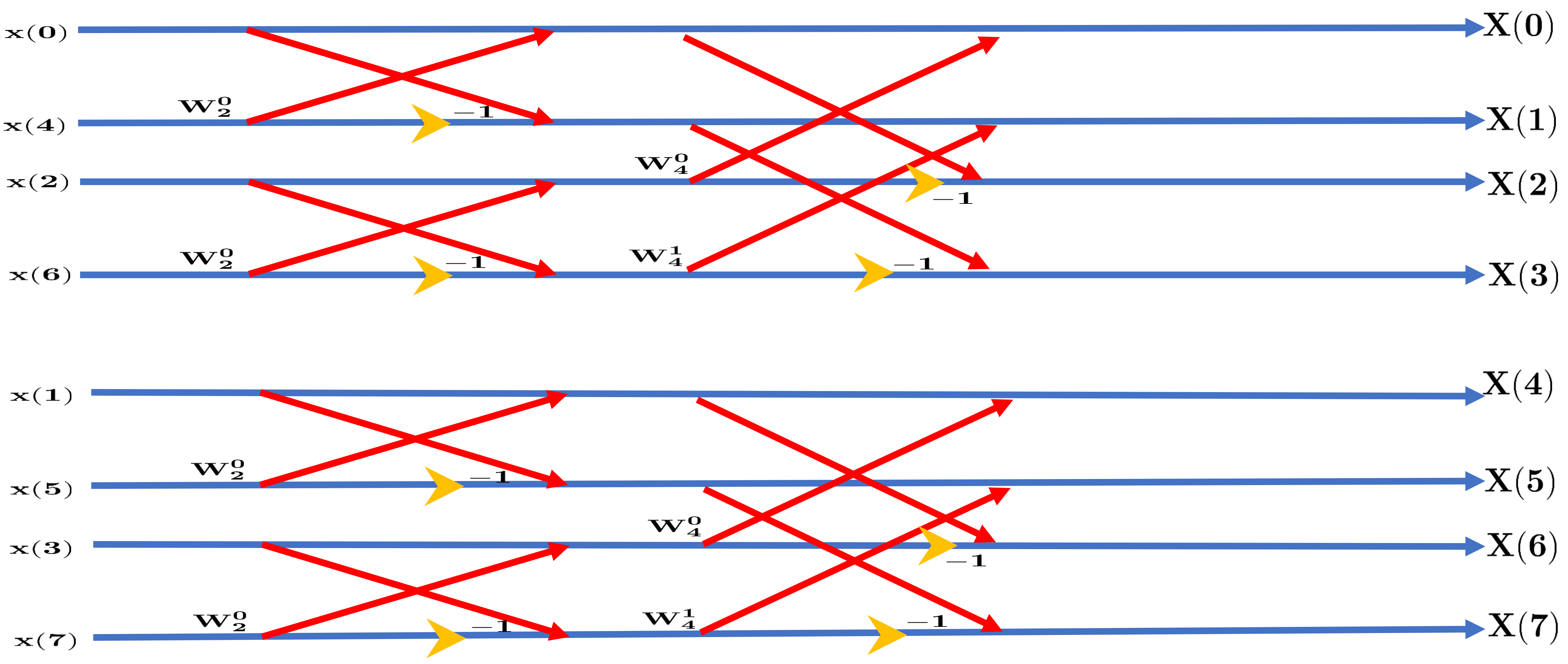
Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$



Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$



Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$



Find N=8 DFT with DIT FFT method $m = \log_2(N) = \log_2(8) = 3$