On differentiating K with respect to s we get,

$dK/ds = -(3s^2 + 20s + 16)$

Put $dK/ds = 0$, $\therefore 3s^2 + 20s + 16 = 0$

$$s = \frac{-20 \pm \sqrt{20^2 - 4 \times 3 \times 16}}{2 \times 3} = -0.9 \text{ or } -5.7.$$

When $s = -0.9$, $K = -(s^3+10s^2+16s) = -((-0.9)^3 + 10(-0.9)^2 + 16(-0.9)) = 7$

When $s = -5.7$, $K = -(s^3+10s^2+16s) = -((-5.7)^3 + 10(-5.7)^2 + 16(-5.7)) = -48$

For $s = -0.9$, the value of K is positive and real and so it is actual breakaway point.

To find crossing point on imaginary axis

The characteristic equation is, $s(s+2)(s+8) + K = 0$

$\therefore s^3 + 10s^2 + 16s + K = 0$

Put $s = j\omega$.

$(j\omega)^3 + 10(j\omega)^2 + 16(j\omega) + K = 0 \quad \Rightarrow \quad -j\omega^3 - 10\omega^2 + j16\omega + K = 0$

On equating imaginary part to zero we get,

$-j\omega^3 + j16\omega = 0 \quad \Rightarrow \quad -j\omega^3 = -j16\omega \quad \Rightarrow \quad \omega^2 = 16$

$\therefore \omega = \pm\sqrt{16} = \pm 4$

Hence the root locus crosses the imaginary axis at $+j4$ and $-j4$. The complete root locus sketch is shown in fig 6.3.1.

Step-2 : Determine the dominant pole s_d.

Given that, $\%M_p = 16\%$

We know that, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$; $\therefore e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.16$

On taking natural log we get, $\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}} = ln\ 0.16 = -1.83$

On squaring we get, $\dfrac{\zeta^2\pi^2}{1-\zeta^2} = 3.3489$

On cross multiplication we get,

$\zeta^2\pi^2 = 3.3489(1-\zeta^2) \quad \Rightarrow \quad \zeta^2\pi^2 + 3.3489\zeta^2 = 3.3489 \quad \Rightarrow \quad \zeta^2(\pi^2 + 3.3489) = 3.3489$

$$\therefore \zeta = \sqrt{\frac{3.3489}{\pi^2 + 3.3489}} = 0.5$$

$\therefore \cos^{-1}\zeta = \cos^{-1}0.5 = 60°$

Draw a straight line at an angle of 60° with respect to real axis as shown in fig 6.3.1. The meeting point of this line with root locus is the dominant pole, s_d.

From fig 6.3.1. we get, $s_d = -0.75 \pm j6.35$ (Dominant poles occur as conjugate poles).

Step 3 : To find gain K, at $s = s_d$

$$K = \frac{\text{Product of vector lengths from open loop poles to } s_d}{\text{Product of vector lengths from open loop zeros to } s_d}$$

Product of vector lengths from poles $= l_1 \times l_2 \times l_3$

From root locus plot of fig 6.3.1. we get, $l_1 = 6.5$; $l_2 = 6.8$ and $l_3 = 7.35$

Note : *Vector lengths are measured to scale.*

Since there is no finite zero, the product of vector lengths from zeros is unity.

$$\therefore K = l_1 \times l_2 \times l_3 = 6.5 \times 6.8 \times 7.35 = 19.845 \approx 20$$

Step 4 : To find parameter, β

Given that, $G(s) = K/s(s+2)(s+8) = 20/s(s+2)(s+8)$

$$\left.\begin{array}{l}\text{Velocity error constant of}\\ \text{uncompensated system}\end{array}\right\} K_{vu} = \underset{s\to 0}{\text{Lt}}\ s\,G(s) = \underset{s\to 0}{\text{Lt}}\ s\,\frac{20}{s(s+2)\ (s+8)} = \frac{20}{2\times 8} = 1.25$$

It is given that steady state error, $e_{ss} \le 0.125$ for ramp input.

$$\therefore \text{The desired velocity error constant, } K_{vd} = \frac{1}{e_{ss}} = \frac{1}{0.125} = 8$$

Let A be the factor by which K_v is increased.

$$\therefore A = \frac{K_{vd}}{K_{vu}} = \frac{8}{1.25} = 6.4$$

Let, $\beta = 6.2 \times A = 6.2 \times 6.4 = 7.68$

Step-5 : Determine the transfer function of lag compensator.

Zero of the compensator, $z_c = -1/T = 0.1 \times$ second pole of G(s)

$$= 0.1\times(-2) = -0.2$$

$$\text{Now, } T = 1/0.2 = 5$$

$$\text{Pole of the compensator, } P_c = \frac{-1}{\beta T} = \frac{-1}{7.68\times 5} = \frac{-1}{38.4} = -0.026$$

$$\therefore \frac{1}{\beta T} = 0.026 \text{ and } \beta T = 38.4$$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lag compensator}\end{array}\right\} G_c(s) = \frac{s+1/T}{s+1/\beta T} = \frac{(s+0.2)}{(s+0.026)}$$

Step-6 : Transfer function of compensated system.

The lag compensator is connected in series with G(s) as shown in fig 6.3.2.

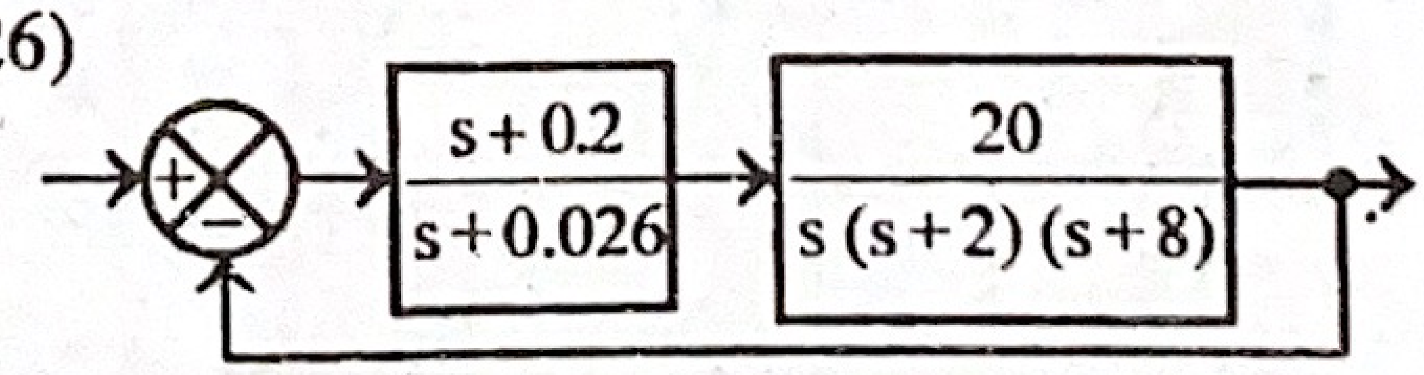

Fig 6.3.2 : Block diagram of lag compensated system.

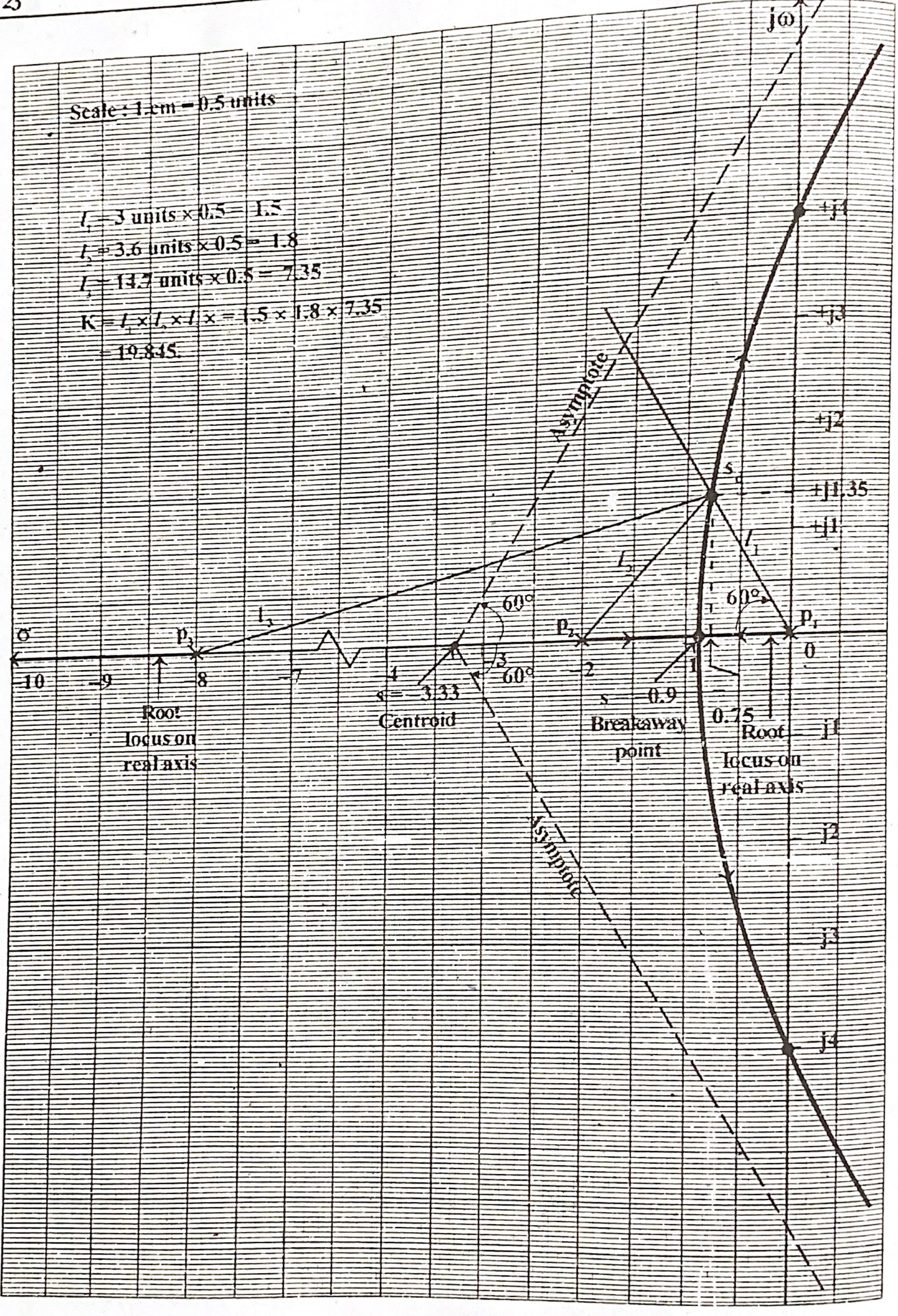

Fig 6.3.1. : Root locus sketch of $1+G(s) = 1 + K/s(s+2)(s+8)$.

Open loop transfer function of compensated system $\Big\}\ G_o(s) = \dfrac{(s+0.2)}{(s+0.026)} \times \dfrac{20}{s(s+2)(s+8)}$

$$= \frac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$$

Step-7 **:** Check steady state error of compensated system

Velocity error constant of compensated system $\Big\}\ K_{vc} = \underset{s\to 0}{Lt}\ s\,G_o(s) = \underset{s\to 0}{Lt}\ s\dfrac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$

$$= \frac{20 \times 0.2}{2 \times 8 \times 0.026} = 9.615$$

Steady state error of compensated system for unit ramp input $\Big\}\ e_{ss} = \dfrac{1}{K_{vc}} = \dfrac{1}{9.615} = 0.104$

CONCLUSION

Since the steady state error of the compensated system is less than 0.125, the design is acceptable.

RESULT

Transfer function of lag compensator, $G_c(s) = (s+0.2)/(s+0.026)$.

Open loop transfer function of lag compensated system $\Big\}\ G_o(s) = \dfrac{20(s+0.2)}{s(s+2)(s+8)(s+0.026)}$

EXAMPLE 6.4

The controlled plant of a unity feedback system is $G(s) = K/s(s+10)^2$. It is specified that velocity error constant of the system be equal to 20, while the damping ratio of the dominant roots be 0.707. Design a suitable cascade compensation scheme to meet the specifications.

SOLUTION

Step-1 **:** Sketch the root locus of uncompensated system

To find poles of open loop system

Given that, $G(s) = K/s(s+10)^2$

The poles of open loop transfer function are the roots of the equation $s(s+10)^2 = 0$.

$\therefore$ The poles are lying at $s = 0, -10, -10$

Let us denote poles by, p_1, p_2 and p_3. Here, $p_1 = 0$, $p_2 = -10$ and $p_3 = -10$.

To find root locus on real axis

The entire negative real axis will be a part of root locus. Because if we choose a test point on negative real axis then to the right of this point we have odd number of real poles and zeros.

To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is no finite zero and so all the three root locus branches will meet the zeros at infinity. Hence the number of asymptotes required is three.

$$\text{Angles of asymptotes} = \frac{\pm 180°(2q+1)}{n-m}; \quad q = 0, 1, 2, ...n-m.$$

Here n = 3 and m = 0. $\therefore q = 0, 1, 2, 3$

$$\text{When, } q = 0, \text{ Angles} = \frac{\pm 180°}{3} = \pm 60°$$

$$\text{When, } q = 1, \text{ Angles} = \frac{\pm 180°(2+1)}{3} = \pm 180°$$

$\therefore$Angles of asymptotes are, $+60°, -60°$ and $\pm 180°$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-10-10}{3} = \frac{-20}{3} = -6.6$$

To find breakaway point

$$\text{Closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(s+10)^2}}{1+\dfrac{K}{s(s+10)^2}} = \frac{K}{s(s+10)^2+K}$$

The characteristic equation is, $s(s+10)^2 + K = 0$

$$\therefore K = -s(s+10)^2 = -(s^3 + 20s^2 + 100s)$$

On differentiating K with respect to s we get, $dK/ds = -(3s^2 + 40s + 100)$.

Put $dK/ds = 0$, $\therefore 3s^2 + 40s + 100 = 0$

$$s = \frac{-40 \pm \sqrt{40^2 - 4 \times 3 \times 100}}{2 \times 3} = \frac{-40 \pm 20}{6} = -3.33, -10.$$

When $s = -3.33$, $K = -[(-3.33)^3 + 20(-3.33)^2 + 100(-3.33)] = 148.14$

When $s = -10$, $K = -[(-10)^3 + 20(-10)^2 + 100(-10)] = 0$

For $s = -3.33$, the value of K is positive and real and so it is actual breakaway point.

To find crossing point of imaginary axis

The characteristic equation is, $s(s+10)^2 + K = 0$

$$\therefore s^3 + 20s^2 + 100s + K = 0$$

Put $s = j\omega$.

$$\therefore (j\omega)^3 + 20(j\omega)^2 + 100(j\omega) + K = 0 \quad \Rightarrow \quad -j\omega^3 - 20\omega^2 + j100\omega + K = 0$$

On equating imaginary part to zero we get,

$$-j\omega^3 + j100\omega = 0 \quad \Rightarrow \quad -j\omega^3 = -j100\omega \quad \Rightarrow \quad \omega^2 = 100$$

$$\therefore \omega = \pm\sqrt{100} = \pm 10$$

Hence the root locus crosses the imaginary axis at $+j10$ and $-j10$. The complete root lo sketch is shown in fig 6.4.1.

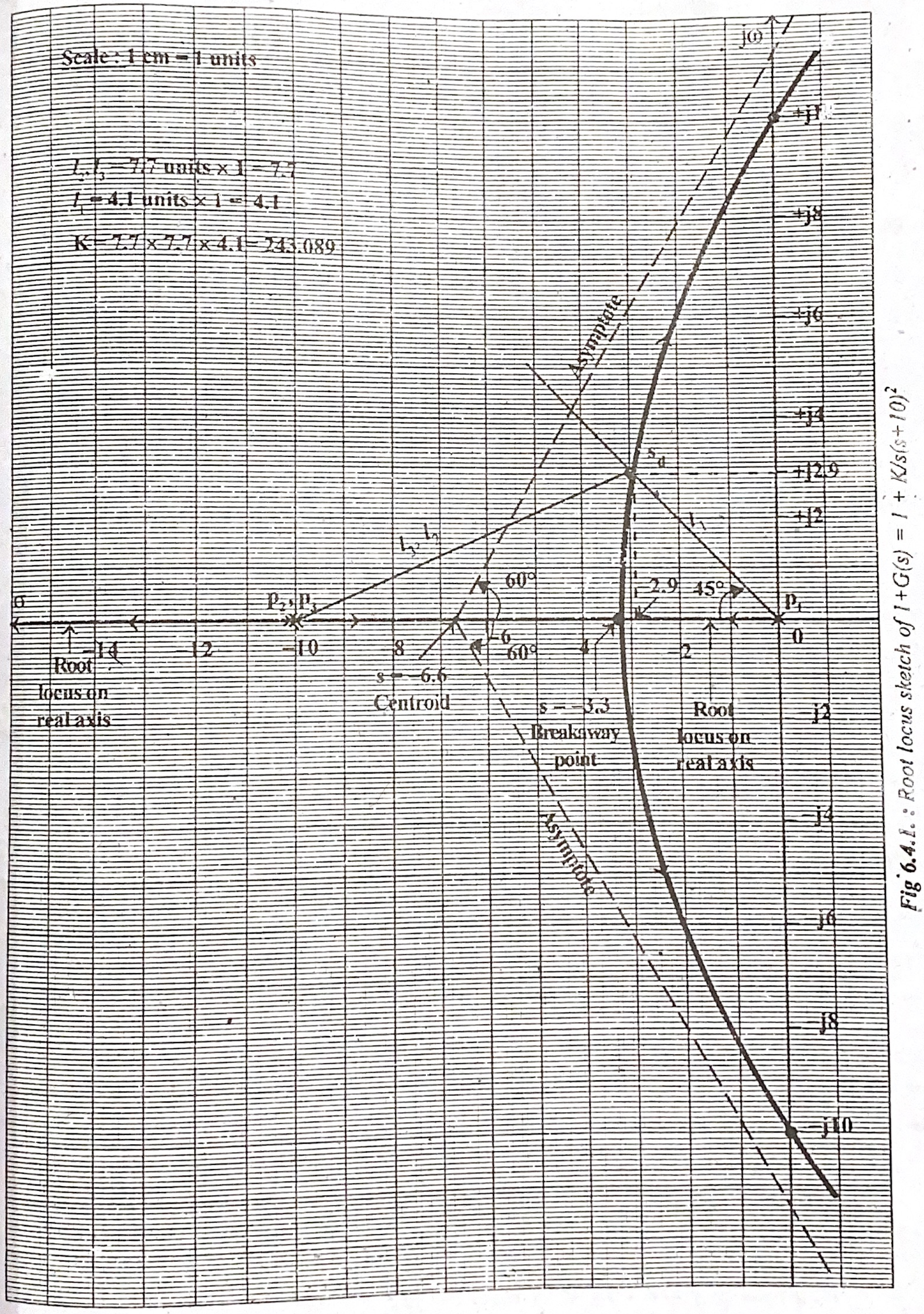

Fig 6.4.1 : Root locus sketch of $1+G(s) = 1 + K/s(s+10)^2$

Step-2 : Determine the dominant pole s_d.

Given that, $\zeta = 0.707$

$$\therefore \cos^{-1}\zeta = \cos^{-1} 0.707 = 45.008 \approx 45°$$

Draw a straight line at an angle of 45° with respect to real axis as shown in fig 6.4.1. The meeting point of this line with root locus is the dominant pole, s_d.

From fig 6.4.1. we get, $s_d = -2.9 \pm j2.9$ (Dominant poles occur as conjugate poles).

Step-3 : To find gain K, at $s = s_d$

$$K = \frac{\text{Product of vector lengths from open loop poles to } s_d}{\text{Product of vector lengths from open loop zeros to } s_d}$$

Product of vector lengths from poles $= l_1 \times l_2 \times l_3$

From root locus plot of fig 6.4.1. we get,

Note : Lengths should be measured to scale

$l_1 = 4.1$; $l_2 = l_3 = 7.7$

Since there is no finite zero, the product of vector lengths from zeros is unity.

$$\therefore K = l_1 \times l_2 \times l_3 = 4.1 \times 7.7 \times 7.7\ 243.089 \approx 240$$

Step-4 : To find parameter, β

Given that, $G(s) = K/s(s + 10)^2 = 240/s(s + 10)^2$

Velocity error constant of uncompensated system $\Big\}$ $K_{vu} = \underset{s \to 0}{\text{Lt}}\ s\,G(s) = \underset{s \to 0}{\text{Lt}}\ s \dfrac{240}{s(s+10)^2} = \dfrac{240}{10^2} = 2.4$

It is given that desired velocity error constant, K_{vd} should be 20. Let A be the factor by which K_v is increased.

$$\therefore A = \frac{K_{vd}}{K_{vu}} = \frac{20}{2.4} = 8.33$$

Let, $\beta = 6.2 \times A = 6.2 \times 8.33 = 9.996 \approx 10$

Step-5 : Determine the transfer function of lag compensator

Zero of the compensator, $z_c = -1/T = 0.1 \times$ second pole of $G(s) = 0.1 \times (-10) = -1.0$

$$\therefore 1/T = 1.0 \text{ and } T = 1.0$$

Pole of the compensator, $p_c = \dfrac{-1}{\beta T} = \dfrac{-1}{10 \times 1} = -0.1$

$$\therefore \frac{1}{\beta T} = 0.1 \text{ and } \beta T = 10$$

Transfer function of lag compensator $\Big\}$ $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \dfrac{(s+1)}{(s+0.1)}$

Step-6 : Determine the open loop transfer function of compensated system.

The block diagram of lag compensated system is shown in fig 6.4.2 and in this, the lag compensator is connected in series with G(s).

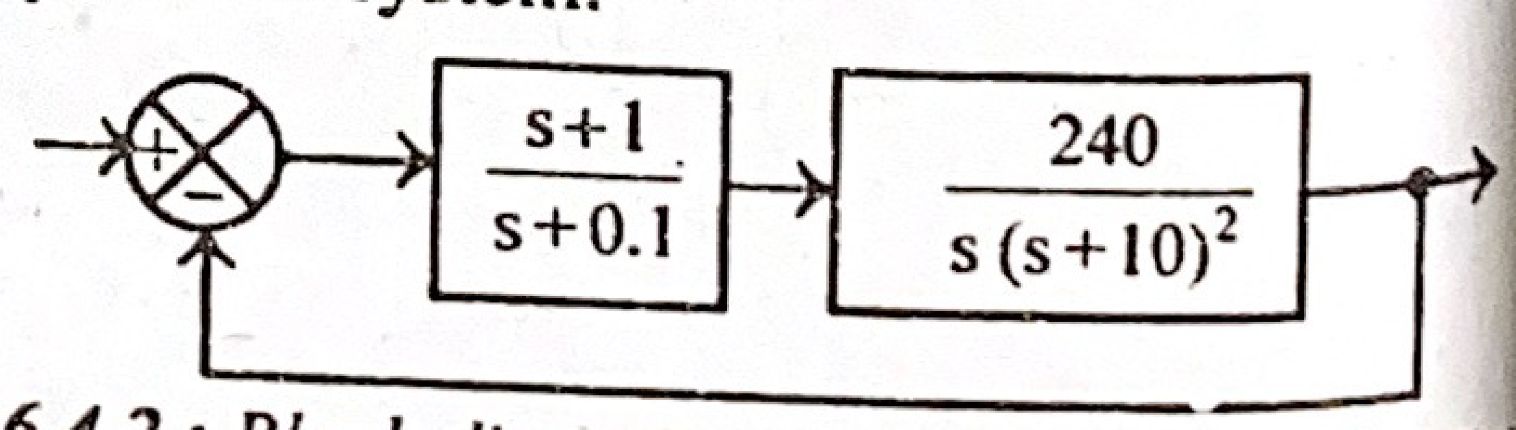

Fig 6.4.2 : *Block diagram of lag compensated system*

Open loop transfer function of compensated system $\Big\}$ $G_o(s) = \dfrac{(s+1)}{(s+0.1)} \times \dfrac{240}{s(s+10)^2} = \dfrac{240(s+1)}{s(s+10)^2(s+0.1)}$

Step-7 : Check K_v of compensated system

Velocity error constant of compensated system $\Big\}$ $K_{vc} = \underset{s \to 0}{Lt}\ s\,G_o(s)$

$$= \underset{s \to 0}{Lt}\ s\,\frac{240(s+1)}{s(s+10)^2(s+0.1)} = \frac{240}{10^2 \times 0.1} = 24.$$

CONCLUSION

Since the velocity error constant of the compensated system is greater than the desired value, the design is accepted.

RESULT

Transfer function of lag compensator $\Big\}$ $G_c(s) = \dfrac{(s+1)}{(s+0.1)}$

Open loop transfer function of lag compensated system $\Big\}$ $G_o(s) = \dfrac{240(s+1)}{s(s+10)^2(s+0.1)}$

6.3 LEAD COMPENSATOR

A compensator having the characteristics of a lead network is called a lead compensator. If a sinusoidal signal is applied to the lead network, then in steady state the output will have a phase lead with respect to the input.

The lead compensation increases the bandwidth, which improves the speed of response and also reduces the amount of overshoot. Lead compensation appreciably improves the transient response, whereas there is a small change in steady state accuracy. Generally, lead compensation is provided to make an unstable system as a stable system.

A lead compensator is basically a high pass filter and so it amplifies high frequency noise signals. If the pole introduced by the compensator is not cancelled by a zero in the system, then lead compensation increases the order of the system by one.

S-PLANE REPRESENTATION OF LEAD COMPENSATOR

The lead compensator has a zero at $s = -1/T$ and a pole at $s = -1/\alpha T$. The pole-zero plot of lead compensator is shown in fig 6.9. Here, $\alpha < 1$, so the zero is closer to the origin than the pole. The general form of lead compensator transfer function is given by equation (6.19),

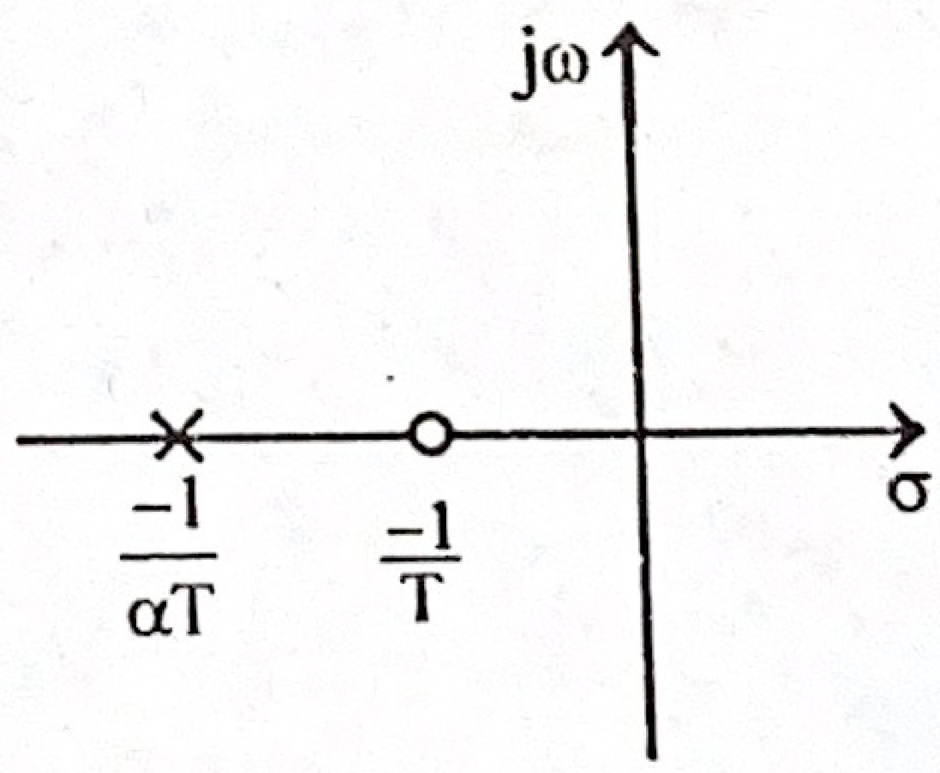

Fig 6.9 : Pole-zero plot of lead compensator.

$$G_c(s) = \frac{s+z_c}{s+p_c} = \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\alpha T}\right)} \qquad .....(6.19)$$

where, $T > 0$ and $\alpha < 1$

The zero of lead compensator, $z_c = \dfrac{1}{T}$ .....(6.20)

The pole of lead compensator, $p_c = \dfrac{1}{\alpha T}$ .....(6.21)

From equation (6.20) we get, $T = \dfrac{1}{z_c}$ .....(6.22)

From equation (6.21) we get, $\alpha = \dfrac{z_c}{p_c}$ .....(6.23)

REALISATION OF LEAD COMPENSATOR USING ELECTRICAL NETWORK

The lead compensator can be realised by the RC network shown in fig 6.10.

Let, $E_i(s)$ = Input voltage, and $E_o(s)$ = Output voltage

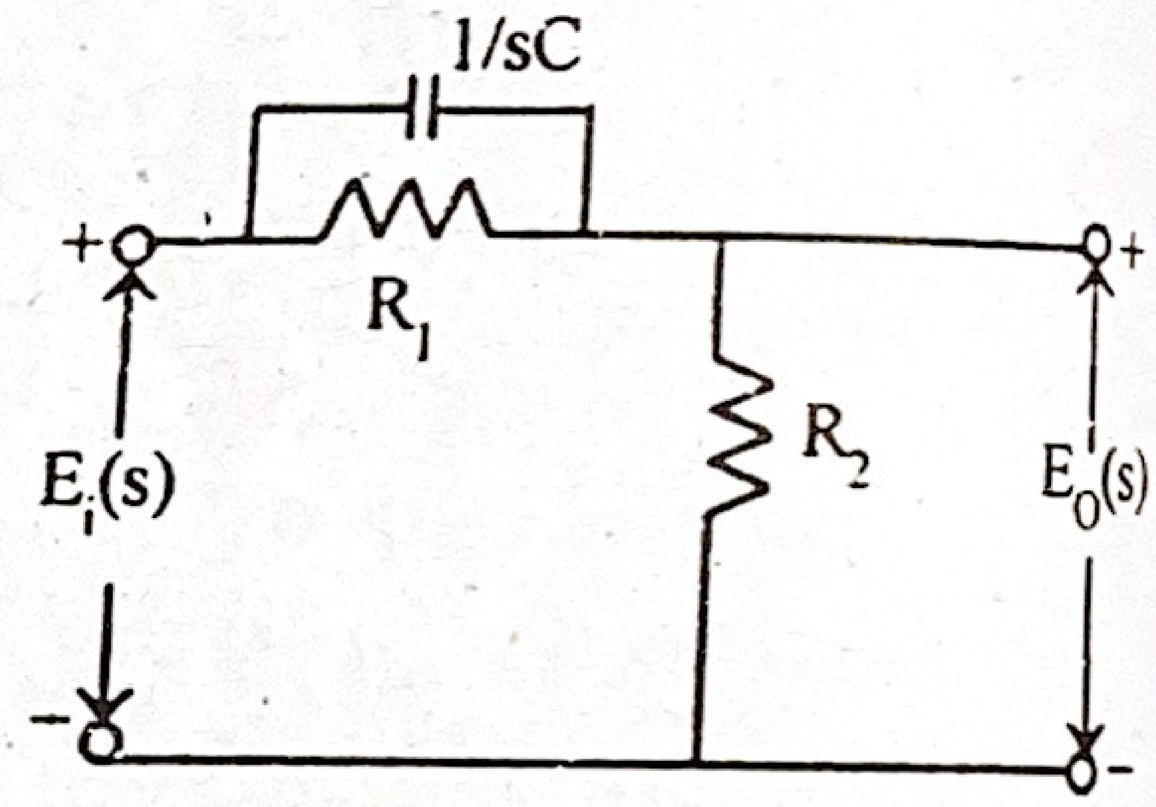

Fig 6.10 : Electrical lead compensator.

In the network shown in fig 6.10, the input voltage is applied to the series combination of $(R_1 \| C)$ and R_2. The output voltage is obtained across R_2.

By voltage division rule,

$$\text{Output voltage, } E_0(s) = E_i(s)\,\frac{R_2}{R_2 + \dfrac{\left(R_1 \times \dfrac{1}{sC}\right)}{\left(R_1 + \dfrac{1}{sC}\right)}}$$

$$E_0(s) = E_i(s)\,\frac{R_2}{R_2 + \dfrac{R_1}{(R_1Cs + 1)}} = E_i(s)\,\frac{R_2}{\dfrac{R_2(R_1Cs + 1) + R_1}{R_1Cs + 1}}$$

The transfer function of the electrical network is the ratio of output voltage to input voltage.

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{electrical network}\end{array}\right\} \frac{E_0(s)}{E_i(s)} = \frac{R_2(R_1Cs + 1)}{[R_1R_2Cs + R_2 + R_1]} = \frac{R_1CR_2\left[s + \dfrac{1}{R_1C}\right]}{R_1CR_2\left[s + \dfrac{(R_1 + R_2)}{R_1CR_2}\right]}$$

$$= \frac{\left[s + \dfrac{1}{R_1C}\right]}{\left[s + \left(\dfrac{1}{R_2/(R_1 + R_2)}\right)\dfrac{1}{R_1C}\right]} \quad .....(6.24)$$

The general form of lead compensator transfer function is,

$$G_c(s) = \frac{\left(s + \dfrac{1}{T}\right)}{\left(s + \dfrac{1}{\alpha T}\right)} \quad .....(6.25)$$

On comparing equations (6.24) and (6.25) we get,

$$\frac{E_0(s)}{E_i(s)} = \frac{s + \frac{1}{T}}{\left(s + \frac{1}{\alpha T}\right)} \qquad .....(6.26)$$

where, $T = R_1C$ and $\alpha = \frac{R_2}{R_1 + R_2}$

The transfer function of the RC network is similar to the general form of transfer function of lead compensator.

FREQUENCY RESPONSE OF LEAD COMPENSATOR

Consider the general form of lead compensator,

$$G_c(s) = \frac{s + \frac{1}{T}}{\left(s + \frac{1}{\alpha T}\right)} = \frac{(1 + sT)/T}{(s\alpha T + 1)/\alpha T} = \alpha\frac{(1 + sT)}{(1 + \alpha sT)} \qquad .....(6.27)$$

The sinusoidal transfer function of lead compensator is obtained by letting $s = j\omega$ in equation (6.27).

$$\therefore\ G_c(j\omega) = \alpha\frac{(1 + j\omega T)}{(1 + j\omega\alpha T)} \qquad .....(6.28)$$

$$\text{When } \omega = 0,\ G_c(j\omega) = \alpha \qquad .....(6.29)$$

From equation (6.29) we can say that the lead compensator provides an attenuation of α (Here $\alpha < 1$). If the attenuation of the compensator is not desirable then it can be eliminated by a suitable amplifier.

Let us assume that the attenuation α is eliminated by a suitable amplifier network. Now, $G_c(j\omega)$ is given by,

$$G_c(j\omega) = \frac{(1 + j\omega T)}{(1 + j\omega\alpha T)} = \frac{\sqrt{1 + (\omega T)^2}\angle\tan^{-1}\omega T}{\sqrt{1 + (\omega\alpha T)^2}\ \angle\tan^{-1}\omega\alpha T} \qquad .....(6.30)$$

Sinusoidal transfer function shown in equation (6.30) has two corner frequencies ω_{c1} and ω_{c2}.

Here, $\omega_{c1} = \frac{1}{T}$ and $\omega_{c2} = \frac{1}{\alpha T}$. Since, $T > \alpha T$, $\omega_{c1} < \omega_{c2}$

$$\text{Let } A = |G_c(j\omega)| \text{ in db} = 20\log\frac{\sqrt{1 + (\omega T)^2}}{\sqrt{1 + (\omega\alpha T)^2}} \qquad .....(6.31)$$

At very low frequencies i.e., upto ω_{c1}, $\omega T << 1$ and $\omega\alpha T << 1$

$$\therefore\ A \approx 20\log 1 = 0$$

In the frequency range from ω_{c1} to ω_{c2}, $\omega T >> 1$ and $\omega\alpha T << 1$

$$\therefore\ A \approx 20\log\sqrt{(\omega T)^2} = 20\log(\omega T)$$

At very high frequencies i.e., after ω_{c2}, $\omega T >> 1$ and $\omega\alpha T >> 1$

$$\therefore\ A \approx 20\log\frac{\sqrt{(\omega T)^2}}{\sqrt{(\omega\alpha T)^2}} = 20\log\frac{1}{\alpha}$$

The approximate magnitude plot of lead compensator is shown in fig 6.16. The magnitude plot of Bode plot of $G_c(j\omega)$ is a straight line through 0 db upto ω_{c1}, then it has a slope of +20 db/decade upto ω_{c2} and after ω_{c2} it is a straight line with a constant gain of 20 log $(1/\alpha)$.

Let, $\phi = \angle G_c(j\omega)$.

$$\therefore\ \phi = \tan^{-1}\omega T - \tan^{-1}\omega\alpha T$$

As $\omega \to 0$, $\phi \to 0$

As $\omega \to \infty$, $\phi \to 0$

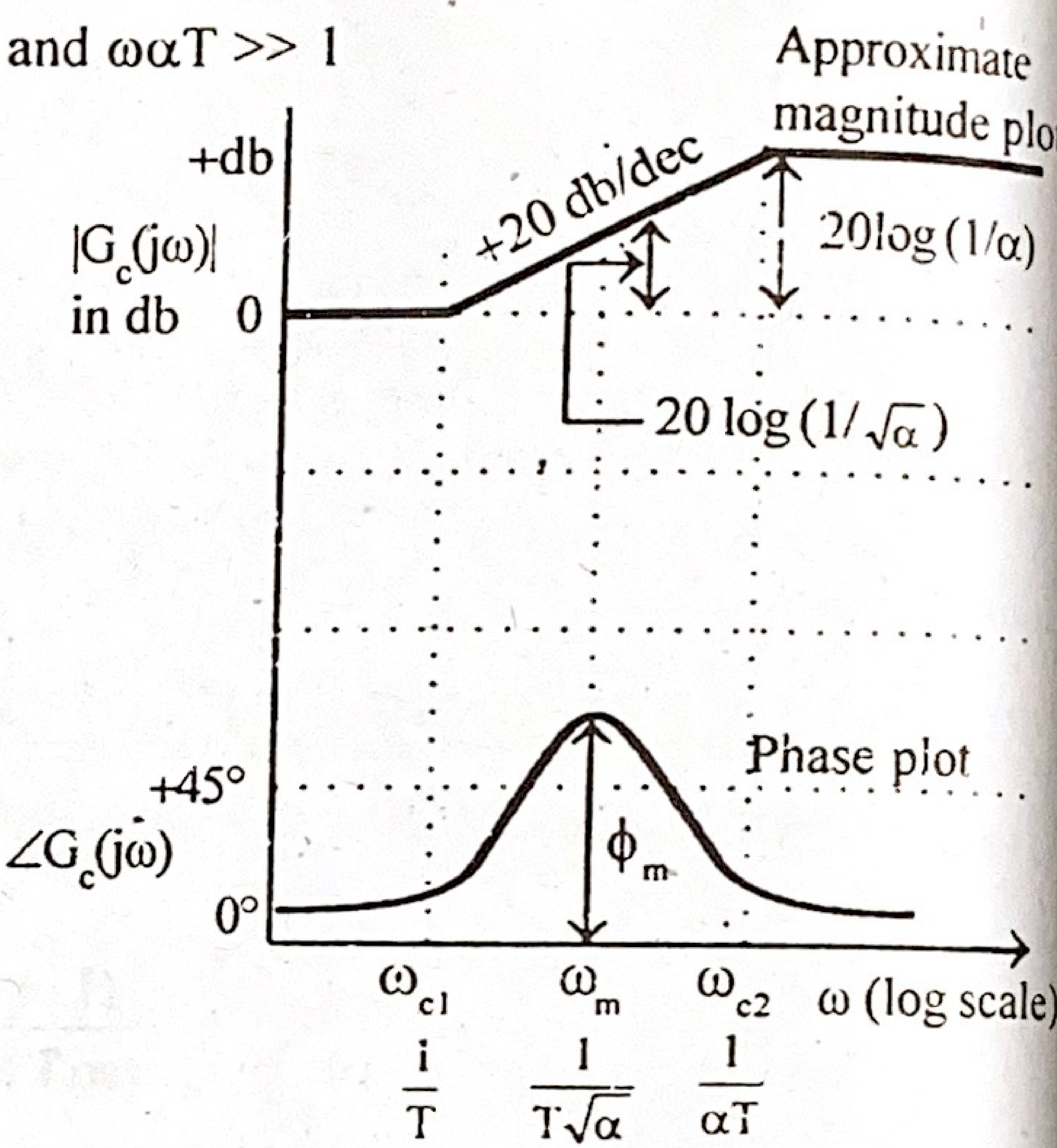

Fig 6.11 : Bode plot of lead compensator.

As ω is varied from 0 to ∞, the phase angle increases from 0 to a maximum value of ϕ_m at $\omega = \omega_m$, then decreases from this maximum value to 0.

It can be shown that the frequency at which maximum phase lead occurs is the geometric mean of the two corner frequencies.

$$\text{Frequency of maximum phase lead, } \omega_m = \sqrt{\omega_{c1}\cdot\omega_{c2}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\alpha T}} = \frac{1}{T\sqrt{\alpha}}$$

The choice of α is governed by the inherent noise in control systems. From the Bode plot of the lead network, we observe that the high frequency noise signals are amplified by a factor $1/\alpha$, while the low frequency control signals undergo unit amplification. Thus the signal/noise ratio at the output of the lead compensator is poorer than at its input. To prevent the signal/noise ratio at the output from deteriorating excessively, it is recommended that the value of α should not be less than 0.07. A typical choice of α = 0.6. Also it is advisable to provide two cascaded lead networks when ϕ_m required (i.e., phase lead required) is more than 60°.

Determination of ω_m, ϕ_m and α

The frequency ω_m can be determined by differentiating ϕ with respect to ω and equating $d\phi/d\omega$ to zero.

From equation (6.32) we get,

$$\text{Phase of } G_c(j\omega),\ \phi = \angle G_c(j\omega) = \tan^{-1}\omega T - \tan^{-1}\alpha\omega T \quad .....(6.32)$$

On differentiating the equation (6.32) with respect to ω and equating $d\phi/d\omega$ to zero, we get the frequency corresponding to maximum phase lead as, $\omega_m = 1/T\sqrt{\alpha}$.

$$\therefore \text{Frequency corresponding to maximum phase lead, } \omega_m = \frac{1}{T\sqrt{\alpha}} \quad .....(6.33)$$

Also we can express ϕ_m in terms of α and α in terms of ϕ_m as shown below.

$$\phi_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) \qquad .....(6.34)$$

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} \qquad .....(6.35)$$

> *Note : The equations (6.33), (6.34) and (6.35) can be derived by a similar analysis shown in section 6.2 for lag compensator after replacing β by α.*

PROCEDURE FOR DESIGN OF LEAD COMPENSATOR USING BODE PLOT

The following steps may be followed to design a lead compensator using bode plot and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : The open loop gain K of the given system is determined to satisfy the requirement of the error constant.

Step-2 : The bode plot is drawn for the uncompensated system using the value of K, determined from the previous step. *[Refer Chapter-4 for the procedure to sketch bode plot].*

Step-3 : The phase margin of the uncompensated system is determined from the bode plot.

Step-4 : Determine the amount of phase angle to be contributed by the lead network by using the formula given below,

$$\phi_m = \gamma_d - \gamma + \epsilon$$

where,

ϕ_m = Maximum phase lead angle of the lead compensator

γ_d = Desired phase margin

γ = Phase margin of the uncompensated system

ϵ = Additional phase lead to compensate for shift in gain crossover frequency

Choose an initial choice of ϵ as 5^0

> *Note : If ϕ_m is more than 60° then realize the compensator as cascade of two lead compensator with each compensator contributing half of the required angle).*

Step-5 : Determine the transfer function of lead compensator

Calculate α using the equation, $\alpha = \dfrac{1 - \sin\phi_m}{1 + \sin\phi_m}$

From the bode plot, determine the frequency at which the magnitude of $G(j\omega)$ is $-20 \log 1/\sqrt{\alpha}$ db. This frequency is ω_m.

Calculate T from the relation, $\omega_m = \dfrac{1}{T\sqrt{\alpha}}$ $\quad \therefore T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \frac{\alpha(1 + sT)}{(1 + \alpha sT)}$$

Step-6 : Determine the open loop transfer function of compensated system.

The lag compensator is connected in series with G(s) as shown in fig 6.12. When the lead network is inserted in series with the plant, the open loop gain of the system is attenuated by the factor α ($\because \alpha<1$), so an amplifier with the gain of $1/\alpha$ has to be introduced in series with the compensator to nullify the attenuation caused by the lead compensator.

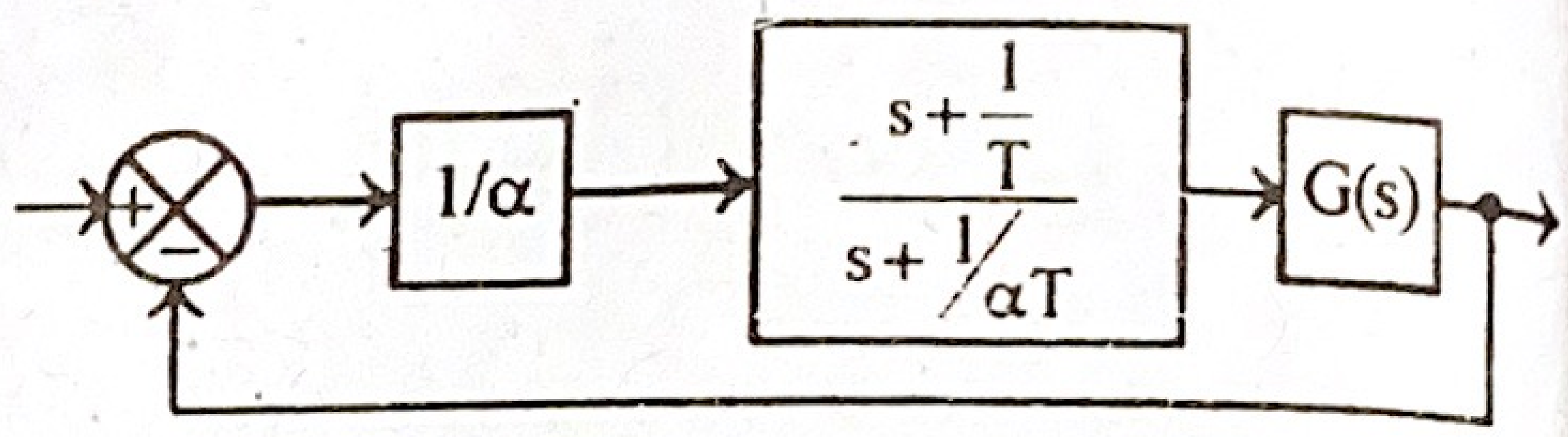

Fig 6.12 : Block diagram of lead compensated system.

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of the overall system}\end{array}\right\} G_0(s) = \frac{1}{\alpha} \times \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \times G(s)$$

$$= \frac{1}{\alpha} \times \frac{\alpha(1 + sT)}{(1 + s\alpha T)} \times G(s) = \frac{(1 + sT)}{(1 + s\alpha T)} \times G(s)$$

Step-7 : Verify the design.

Finally the Bode plot of the compensated system is drawn and verify whether it satisfies the given specifications. If the phase margin of the compensated system is less than the required phase margin then repeat step 4 to 10 by taking $\in$ as 5° more than the previous design.

PROCEDURE FOR DESIGN OF LEAD COMPENSATOR USING ROOT LOCUS

The following steps may be followed to design a lead compensator using root locus and to be connected in series with transfer function of uncompensated system, G(s).

Step-1 : Determine the dominant pole, s_d from the given specifications.

$$\text{Dominant pole, } s_d = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

***Note ;** If ζ alone is specified and ω_n is not available, then draw the root locus and from the root locus find the dominant pole. Refer example 6.8.*

Step-2 : Mark the poles and zeros of open loop transfer function and the dominant pole on the s-plane. Let the dominant pole be point P.

Step-3 : Determine the angle to be contributed by lead network.

Let, ϕ = Angle to be contributed by lead network to make the point P as a point on root locus.

Draw vectors from all open loop poles and zeros to point P. Measure the angle contributed by the vectors. *[For procedure to find angle contribution by vectors refer root locus in Chapter-5].*

$$\text{Now, } \phi = \left(\begin{array}{c}\text{Sum of angles}\\ \text{contributed by poles}\\ \text{of uncompensated system}\end{array}\right) - \left(\begin{array}{c}\text{Sum of angles}\\ \text{contributed by zeros}\\ \text{of uncompensated system}\end{array}\right) \pm n\,180°$$

where n is an odd integer so that n180° is nearest to the difference between angles contributed by poles and zeros.

***Note :** If the angle to be contributed is more than 60° then realise the compensator as cascade of two lead compensators with each compensator contributing half of the required angle.*

Step-4 : Determine the pole and zero of the lead compensator.

Let point O be the origin of s-plane and point P be the dominant pole. Draw straight lines OP and AP such that AP is parallel to x-axis as shown in fig 6.13.

Draw a line PC so as to bisect the angle APO [∠APO] where the point C is on the real axis. With line PC as reference, draw angles BPC and CPD such that each equal to $\phi/2$. Here the points B and D are located on the real axis.

Now the point B is the location of the pole of the compensator ($-1/\alpha T$) and the point D is the location of the zero of the compensator ($-1/T$). From the values of point D and B compute T and α.

Step-5 : Determine the transfer function of lead compensator

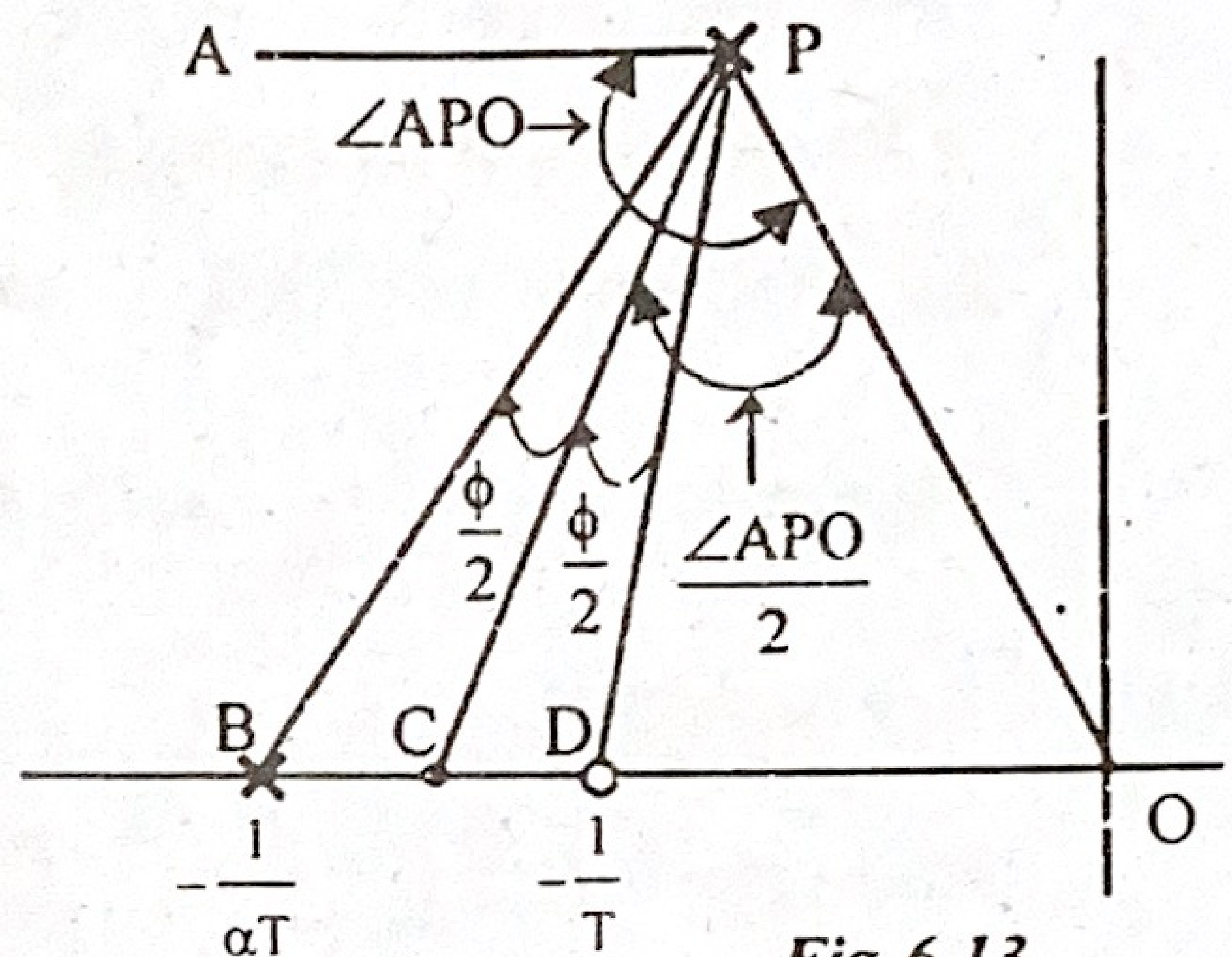

Fig 6.13

$$\left.\begin{array}{l}\text{Transfer function of}\\ \text{lead compensator}\end{array}\right\} G_c(s) = \frac{(s + \frac{1}{T})}{(s + \frac{1}{\alpha T})}$$

$$= \alpha \frac{(1 + sT)}{(1 + s\alpha T)}$$

Step-6 : Determine open loop transfer function of lead compensated system.

The lead compensator is connected in series with the plant as shown in fig 6.14.

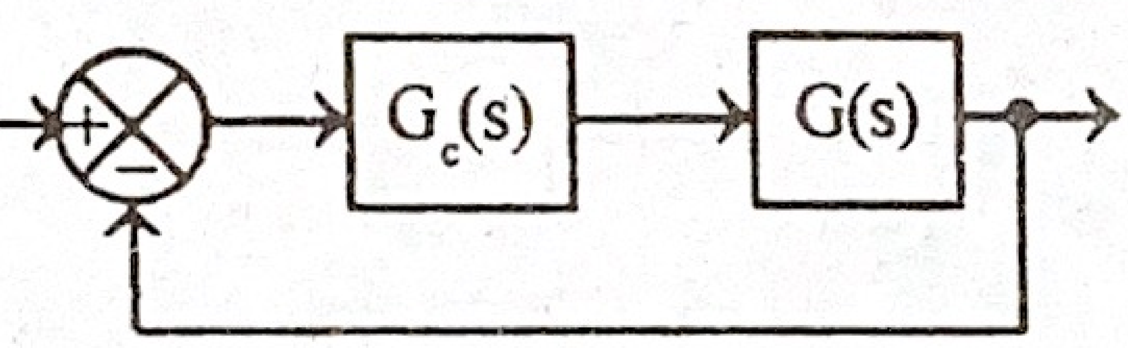

Fig 6.14 : Block diagram of lead compensated system.

Open loop transfer function of compensated system, $G_0(s) = G_c(s)\ G(s)$.

The open loop gain K is given by the value of gain of $s = s_d$. The value of gain, K is determined from pole-zero plot of lead compensated system and by using the magnitude condition given below.

$$K = \frac{\text{Product of vector lengths from all poles to } s = s_d}{\text{Product of vector lengths from all zeros to } s = s_d}$$

***Note :** The length of vectors should be measured to scale. For details of magnitude condition, refer root locus in chapter-5.*

Step-7 : Check whether the compensated system satisfies the error requirement. If the error requirement is satisfied then the design is accepted. Otherwise repeat the design by altering the location of poles and zeros by trial and error, without changing the value of ϕ.

Note : *If the open loop gain K is specified in the problem, then take the gain at $s = s_d$ as K_6. Find a parameter, A where $A = K_l/K$. Now introduce an amplifier with gain, A in cascade with compensator to account for reduction in gain due to attenuation by parameter, α. Now, $G_0(s) = A\ G_c(s)\ G(s)$.*

EXAMPLE 6.5

Design a phase lead compensator for the system shown in fig 6.5.1 to satisfy the following specifications. (i) The phase margin of the system $\geq 45°$. (ii) Steady state error for a unit ramp input $\leq 1/15$. (iii) The gain crossover frequency of the system must be less than 7.5 rad/sec.

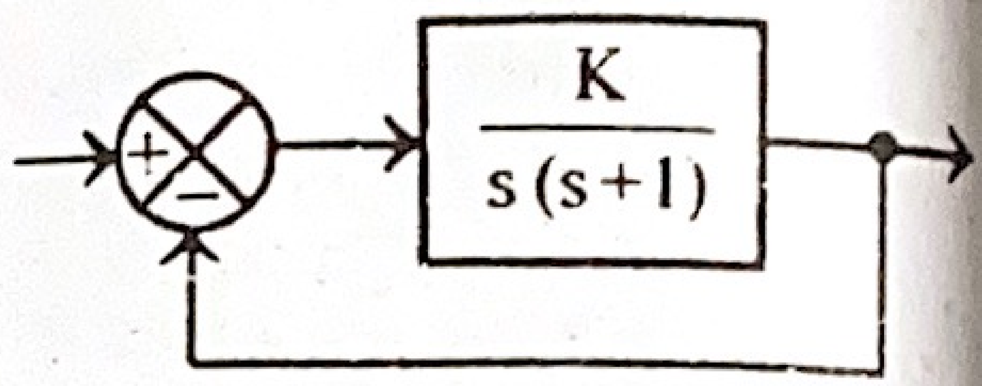

Fig 6.5.1

SOLUTION

Step-1 : Determine K .

Given that, steady state error, $e_{ss} \leq 1/15$ for unit ramp input

When the input is unit ramp, $e_{ss} = 1/K_v = 1/15$. $\therefore K_v = 15$

By definition of velocity error constant we get, $K_V = \underset{S\to 0}{Lt}\ s\,G(s)\,H(s)$

Here, $G(s) = \dfrac{K}{s(s+1)}$ and $H(s) = 1$. $\therefore K_v = \underset{s\to 0}{Lt}\ s\dfrac{K}{s(s+1)} = K$

Step-2 : Draw bode plot .

Given that, $G(s) = \dfrac{K}{s(s+1)} = \dfrac{15}{s(s+1)}$

Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{15}{j\omega(1+j\omega)}$

MAGNITUDE PLOT

The corner frequency is, $\omega_{c1} = 1$ rad/sec.

The various terms of G(jω) are listed in table-6. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{15}{j\omega}$	–	−20	–
$\dfrac{1}{1+j\omega}$	$\omega_{c1} = 1$	−20	$-20+(-20) = -40$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c0}$.
Let, $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec.

Let, $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c1} and ω_h.

At $\omega = \omega_l = 0.1$ rad / sec, $A = 20\log\left|\dfrac{15}{j\omega}\right| = 20\log\dfrac{15}{0.1} = 43.5\text{ db} \approx 44\text{ db}$

At $\omega = \omega_{c1} = 1$ rad / sec, $A = 20\log\left|\dfrac{15}{j\omega}\right| = 20\log\dfrac{15}{1} = 23.5\text{ db} \approx 24\text{ db}$

At $\omega = \omega_h = 10$ rad / sec, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_h \times \log\dfrac{\omega_h}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$

$$= -40 \times \log\frac{10}{1} + 24 = -16\text{ db}$$

Let the points a, b and c be the points corresponding to frequencies ω_l, ω_{c1}, and ω_h respectively on the magnitude plot. In a semilog graph sheet choose appropriate scales and fix the points a, b and c. Join the points by straight lines and mark the slope on respective region. Magnitude plot is shown in fig 6.5.2.

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\phi = \angle G(j\omega) = -90° - \tan^{-1}\omega$$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	0.1	0.5	1	2	5	10
ϕ deg	–96	–117	–135	–153	–169	–174

On the same semilog sheet take another y-axis, choose appropriate scale and draw phase plot as shown in fig 6.5.2.

Step-3 : Determine the phase margin of uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain crossover frequency (ω_{gc}).

γ = Phase margin of uncompensated system.

From the bode plot of uncompensated system we get, $\phi_{gc} = -167°$.

Now, $\gamma = 180° + \phi_{gc} = 180° - 167° = 13°$

The system requires a phase margin of 45°, but the available phase margin is 13° and so lead compensation should be employed to improve the phase margin.

Step-4 : Find ϕ_m

The desired phase margin, $\gamma_d \geq 45°$

Let additional phase lead required, $\epsilon = 5°$

Maximum lead angle, $\phi_m = \gamma_d - \gamma + \epsilon = 45° - 13° + 5° = 37°$

Step-5 : Determine the transfer function of lead compensator.

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin 37°}{1 + \sin 37°} = 0.2486 \approx 0.25$$

The db magnitude corresponding to ω_m $= -20 \log \frac{1}{\sqrt{\alpha}} = -20 \log \frac{1}{\sqrt{0.25}} = -6$ db.

From the bode plot of uncompensated system the frequency ω_m corresponding to a db gain of -6 db is found to be 5.6 rad/sec.

$\therefore \omega_m = 5.6$ rad/sec.

$$\text{Now, } T = \frac{1}{\omega_m \sqrt{\alpha}} = \frac{1}{5.6\sqrt{0.25}} = 0.357 \approx 0.36$$

Transfer function of the lead compensator $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \alpha \dfrac{(1 + sT)}{(1 + s\alpha T)} = 0.25 \dfrac{(1 + 0.36s)}{(1 + 0.09s)}$

Step-6 : Open loop transfer function of compensated system.

The block diagram of the lead compensated system is shown in fig 6.5.3.

The compensator will provide an attenuation of α. To compensate for that, an amplifier of gain $1/\alpha$ is introduced in series with compensator.

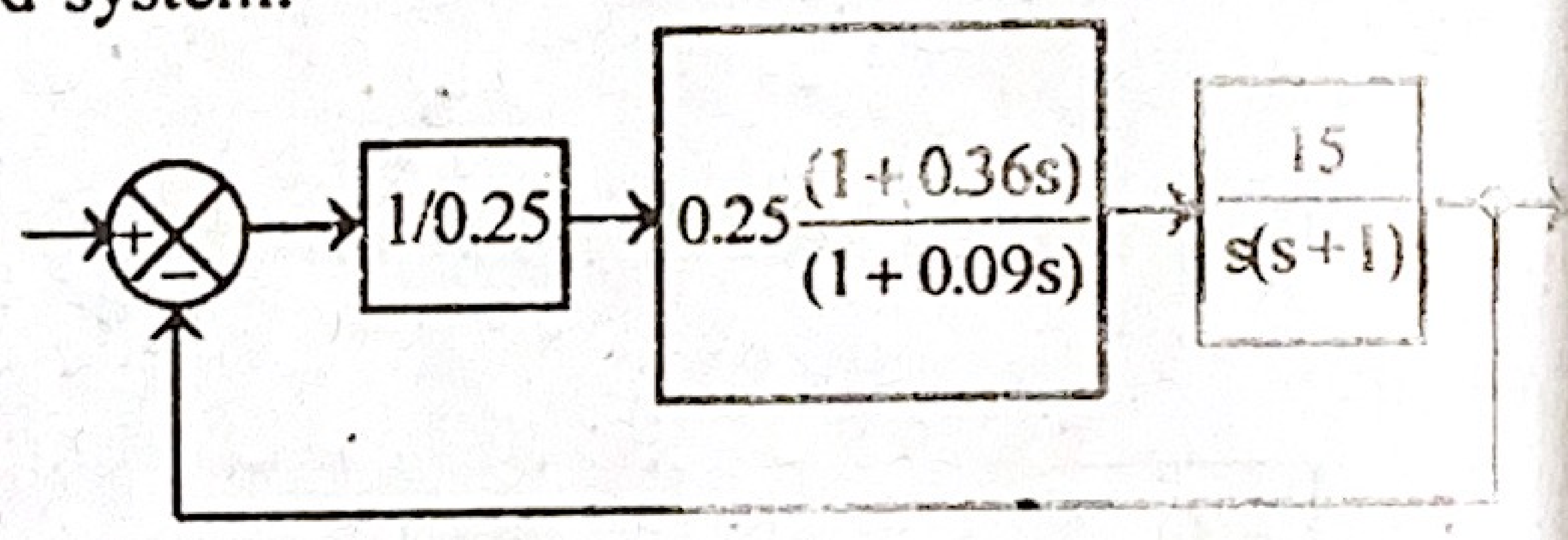

Fig 1.5.3 : Block diagram of lead compensated system.

Open loop transfer function of compensated system $G_0(s) = \dfrac{1}{0.25} \times \dfrac{0.25\,(1 + 0.36s)}{(1 + 0.09s)} \times \dfrac{15}{s\,(s+1)}$

$$= \frac{15\,(1 + 0.36s)}{s\,(1 + 0.09s)\,(1 + s)}$$

Step-7 : Draw the bode plot of compensated system to verify the design.

Put $s = j\omega$ in $G_0(s)$, $\therefore G_0(j\omega) = \dfrac{15\,(1 + j0.36\omega)}{j\omega\,(1 + j0.09\omega)\,(1 + j\omega)}$

MAGNITUDE PLOT

The corner frequencies are ω_{c1}, ω_{c2} and ω_{c3}.

$$\omega_{c1} = \frac{1}{1} = 1 \text{ rad / sec}; \quad \omega_{c2} = \frac{1}{0.36} = 2.8 \text{ rad / sec}; \quad \omega_{c3} = \frac{1}{0.09} = 11.1 \text{ rad / sec}$$

The various terms of $G_0(j\omega)$ are listed in table-3. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c3}$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec

Let $A_0 = |G_0(j\omega)|$ in db

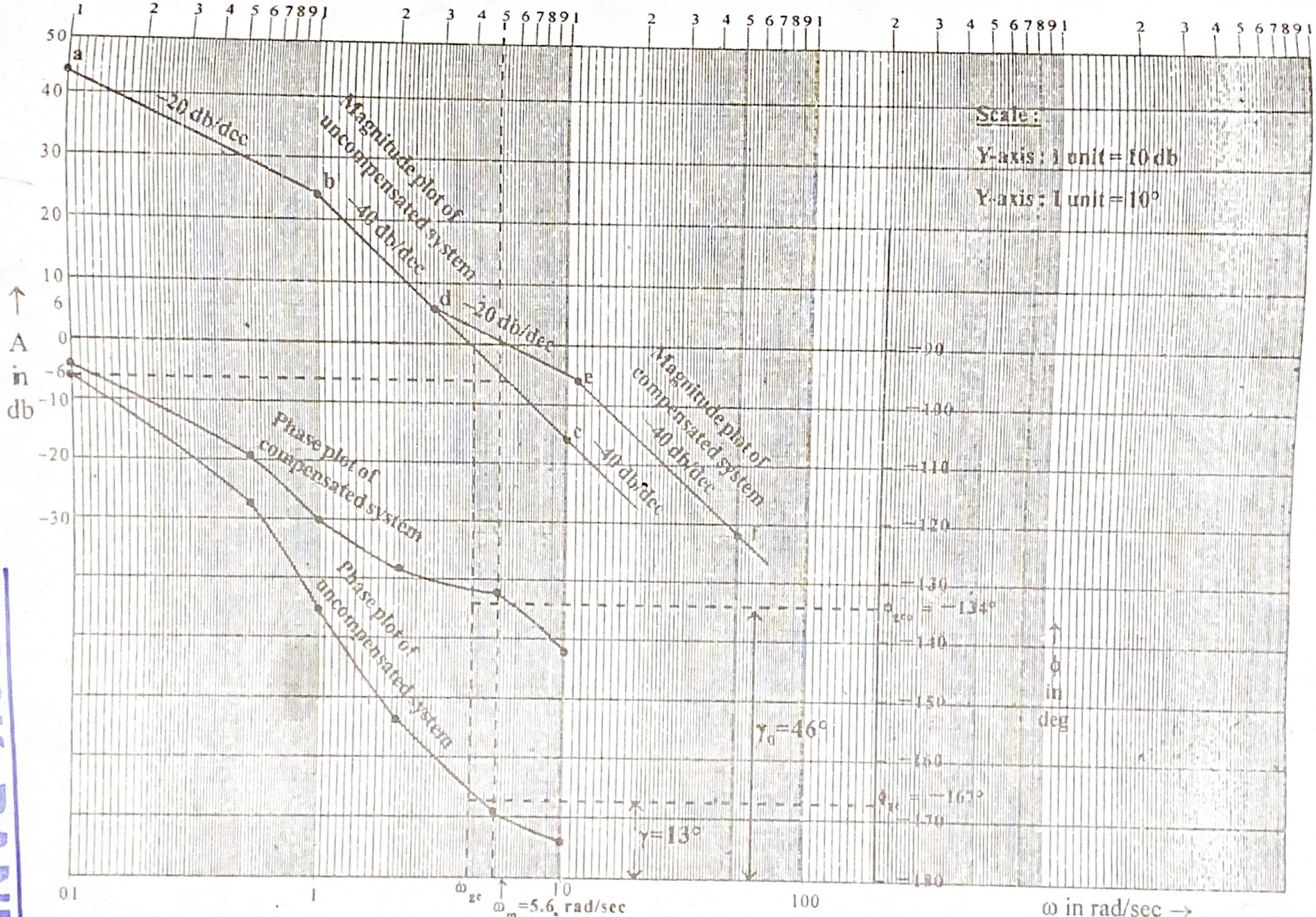

Fig 6.5.2 : Bode plot of G(jω) = 15/jω(1 + jω).

$$\text{At } \omega = \omega_l = 0.1 \text{ rad/sec}, \quad A_0 = 20\log\left|\frac{15}{j\omega}\right| = 20\log\frac{15}{0.1} = 43.5 \text{ db} \approx 44 \text{ db}$$

$$\text{At } \omega = \omega_{c1} = 1 \text{ rad/sec}, \quad A_0 = 20\log\left|\frac{15}{j\omega}\right| = 20\log\frac{15}{1} = 23.5 \text{ db} \approx 24 \text{ db}$$

$$\text{At } \omega = \omega_{c2} = 2.8 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c1}\end{matrix}\right)$$

$$= -40 \times \log\frac{2.8}{1} + 24 = 6 \text{ db}$$

$$\text{At } \omega = \omega_{c3} = 11.1 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c2}\end{matrix}\right)$$

$$= -20 \times \log\frac{11.1}{2.8} + 6 = -6 \text{ db}$$

$$\text{At } \omega = \omega_h = 50 \text{ rad/sec}, \quad A_0 = \left[\text{slope from } \omega_{c3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c3}}\right] + \left(\begin{matrix}\text{gain at}\\ \omega = \omega_{c3}\end{matrix}\right)$$

$$= -40 \times \log\frac{50}{11.1} + (-6) = -32 \text{ db}$$

TABLE-3

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{15}{j\omega}$	–	–20	–
$\frac{1}{1+j\omega}$	$\omega_{c1} = 1$	–20	–20 –20 = –40
$1+j0.36\omega$	$\omega_{c2} = \frac{1}{0.36} = 2.8$	+20	–40+20 = –20
$\frac{1}{1+j0.09\omega}$	$\omega_{c3} = \frac{1}{0.09} = 11.1$	–20	–20–20=–40

Let the points a, b, d, e and f be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot of compensated system. The magnitude plot of compensated system is drawn on the same semilog graph sheet by using the same scales as shown in fig 6.5.2.

PHASE PLOT

The phase angle of $G_0(j\omega)$ as a function of ω is given by,

$$\phi_0 = \angle G_0(j\omega) = \tan^{-1}0.36\omega - 90^\circ - \tan^{-1}0.09\omega - \tan^{-1}\omega.$$

The phase angle of $G_0(j\omega)$ are calculated for various values of ω and listed in table- 4.

TABLE-4

ω rad/sec	0.1	0.5	1	2	5	10
ϕ_0 deg	−94	−109	−120	−128	−132	−142

In the same semilog sheet and by using the same scales, the phase plot of compensated system is sketched as shown in fig 6.5.2.

Let, ϕ_{gc0} = Phase of $G_0(j\omega)$ at new gain crossover frequency.

γ_0 = Phase margin of compensated system.

From the bode plot of compensated system we get, $\phi_{gc0} = -134°$.

Now, $\gamma_0 = 180° + \phi_{gc0} = 180° - 134° = 46°$

CONCLUSION

The phase margin of the compensated system is satisfactory. Hence the design is acceptable.

RESULT

The transfer function of lead compensator, $G_c(s) = \dfrac{0.25\,(1+0.36s)}{(1+0.09s)} = \dfrac{(s+2.78)}{(s+11.11)}$

$$\left.\begin{array}{l}\text{Open loop transfer function}\\ \text{of lead compensated system}\end{array}\right\} G_0(s) = \frac{15(1+0.36s)}{s\,(1+0.09s)\,(1+s)}$$

EXAMPLE 6.6

Design a lead compensator for a unity feedback system with open loop transfer function, $G(s) = K/s\,(s+1)(s+5)$ to satisfy the following specifications (i) Velocity error constant, $K_v \geq 50$ (ii) Phase margin is $\geq 20°$

SOLUTION

Step-1 : Determine K

Given that, $K_v \geq 50$, Let $K_v = 50$

By definition of velocity error constant, K_v we get,

$$K_v = \underset{s\to 0}{\text{Lt}}\; s\,G(s) = \underset{s\to 0}{\text{Lt}}\; s\frac{K}{s(s+1)(s+5)} = \frac{K}{5}$$

$$\therefore\; K = 5 \times K_v = 5 \times 50 = 250.$$

Step-2 : Draw bode plot

$$\text{Given that, } G(s) = \frac{K}{s\,(s+1)\,(s+5)} = \frac{250}{s\,(s+1)\,(s+5)}$$

$$= \frac{250}{s\,(1+s) \times 5 \times (1+s/5)} = \frac{50}{s(1+s)\,(1+0.2s)}$$