Digital filter	*Analog filter*
1. It operates on digital samples (or sampled version) of the signal.	1. It operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equations.	2. It is governed (or defined) by linear differential equations.
3. It consists of adders, multipliers, and delay elements implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors, and inductors.
4. In digital filters, the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters, the approximation problem is solved to satisfy the desired frequency response.

Advantages of digital filters

1. The values of resistors, capacitors and inductors used in analog filters change with temperature. Since the digital filters do not have these components, they have high thermal stability.
2. In digital filters, the precision of the filter depends on the length (or size) of the registers used to store the filter coefficients. Hence by increasing the register bit length (in hardware) the performance characteristics of the filter like accuracy, dynamic range, stability and frequency response tolerance, can be enhanced.
3. The digital filters are programmable. Hence the filter coefficients can be changed any time to implement adaptive features.
4. A single filter can be used to process multiple signals by using the techniques of multiplexing.

Disadvantages of digital filters

1. The bandwidth of the discrete signal is limited by the sampling frequency. The bandwidth of real discrete signal is half the sampling frequency.
2. The performance of the digital filter depends on the hardware (i.e., depends on the bit length of the registers in the hardware) used to implement the filter.

Important features of IIR filters

1. The physically realizable IIR filters do not have linear phase.
2. The IIR filter specifications include the desired characteristics for the magnitude response only.

Sr No	Impulse Invariance	Bilinear Transformation
1	In this method IIR filters are designed having a unit sample response h(n) that is sampled version of the impulse response of the analog filter.	This method of IIR filters design is based on the trapezoidal formula for numerical integration.
2	In this method small value of T is selected to minimize the effect of aliasing.	The bilinear transformation is a conformal mapping that transforms the j Ω axis into the unit circle in the z plane only once, thus avoiding aliasing of frequency components.
3	They are generally used for low frequencies like design of IIR LPF and a limited class of bandpass filter	For designing of LPF, HPF and almost all types of Band pass and band stop filters this method is used.
4	Frequency relationship is linear.	Frequency relationship is non-linear. Frequency warping or frequency compression is due to non-linearity.
5	All poles are mapped from the s plane to the z plane by the relationship $Z^k = e^{p_k T}$. But the zeros in two domain does not satisfy the same relationship.	All poles and zeros are mapped.

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at ∞).
2. The filter order N completely specifies the filter.
3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.

Impulse invariant methods

$$\frac{1}{s-p_i} \xrightarrow[\text{(is tranformed to)}]{} \frac{1}{1-e^{p_iT}z^{-1}}$$

Refer Class Note for derivation

1. $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left(\frac{1}{1-e^{-sT}z^{-1}}\right); \quad s=p_i$$

2. $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

3. $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

So the relationship between analog frequency Ω and digital frequency ω is $\omega = \Omega T$ or $\Omega = \dfrac{\omega}{T}$.

Refer Class Note for derivation

Impulse invariant methods

EXAMPLE 8.4 For the analog transfer function

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

determine $H(z)$ if (a) $T = 1$ s and (b) $T = 0.5$ s using impulse invariant method.

Solution: Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)\,H_a(s)\big|_{s=-1} = \frac{2}{s+3}\bigg|_{s=-1} = 1$$

$$B = (s+3)\,H_a(s)\big|_{s=-3} = \frac{2}{s+1}\bigg|_{s=-3} = -1$$

$$\therefore \quad H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{s-(-3)}$$

By impulse invariant transformation, we know that

$$\frac{A_i}{s-p_i} \xrightarrow{\text{(is transformed to)}} \frac{A_i}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = -1$ and $p_2 = -3$.

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

(a) When $T = 1$ s

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-3}z^{-1}}$$

$$= \frac{1}{1-0.3678\,z^{-1}} - \frac{1}{1-0.0497\,z^{-1}}$$

$$= \frac{(1-0.0497\,z^{-1})-(1-0.3678\,z^{-1})}{(1-0.3678\,z^{-1})(1-0.0497\,z^{-1})}$$

$$= \frac{0.3181\,z^{-1}}{1-0.4175\,z^{-1}+0.0182\,z^{-2}}$$

Impulse invariant methods

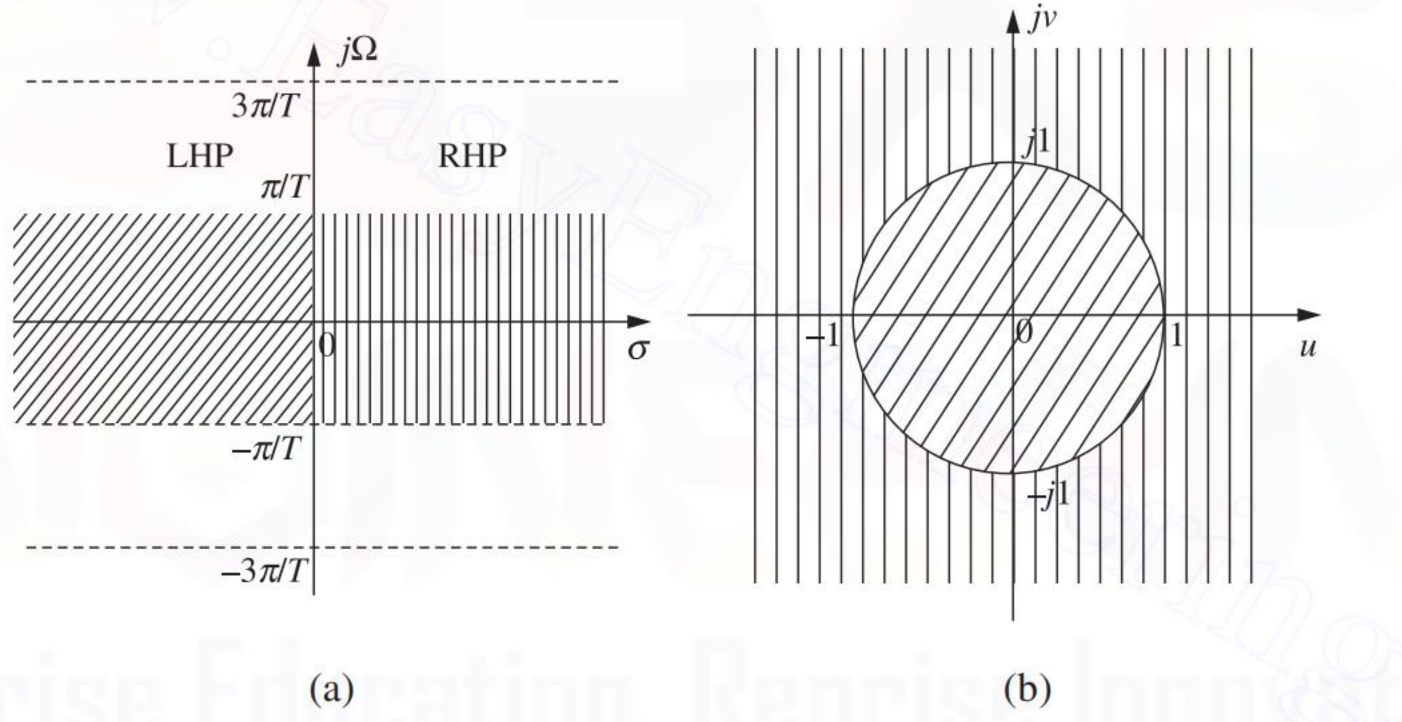


Figure 8.3 Mapping of (a) *s*-plane into (b) *z*-plane by impulse invariant transformation.

Bilinear Transformation

This is the relation between analog and digital poles in bilinear transformation. So to convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{in } H_a(s)$$

Refer Class Note for derivation

$\therefore$ The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently, we have $\omega = 2\tan^{-1}\dfrac{\Omega T}{2}$.

Refer Class Note for derivation

EXAMPLE 8.12 Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

with $T = 0.5$ s and find $H(z)$.

$$\therefore \quad H(z) = \left.\frac{4}{(s+3)\,(s+4)}\right|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \left.\frac{4}{(s+3)\,(s+4)}\right|_{s=4\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{4}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+3\right]\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+4\right]}$$

$$= \frac{4}{\left[\dfrac{4-4z^{-1}+3+3z^{-1}}{1+z^{-1}}\right]\left[\dfrac{4-4z^{-1}+4+4z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4(1+z^{-1})^2}{(7-z^{-1})\,8}$$

$$= \frac{1}{2}\,\frac{(1+z^{-1})^2}{(7-z^{-1})}$$

EXAMPLE 8.10 Convert the following analog filter with transfer function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital IIR filter by using bilinear transformation. The digital IIR filter is having a resonant frequency of $\omega_r = \pi/2$.

Solution: From the transfer function, we observe that $\Omega_c = 3$. The sampling period T can be determined using the equation:

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$\therefore \qquad T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\frac{\pi/2}{2} = 0.6666 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H_a(s)\Big|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$\therefore \qquad H(z) = \frac{s+0.1}{(s+0.1)^2+9}\Bigg|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{3\dfrac{1-z^{-1}}{1+z^{-1}}+0.1}{\left[3\dfrac{1-z^{-1}}{1+z^{-1}}+0.1\right]^2+9}$$

$$= \frac{\left[3(1-z^{-1})+0.1(1+z^{-1})\right][1+z^{-1}]}{\left[3(1-z^{-1})+0.1(1+z^{-1})\right]^2+9(1+z^{-1})^2}$$

$$= \frac{3.1+0.2z^{-1}-2.9z^{-2}}{18.61+0.02z^{-1}+17.41z^{-2}}$$

Frequency Warping

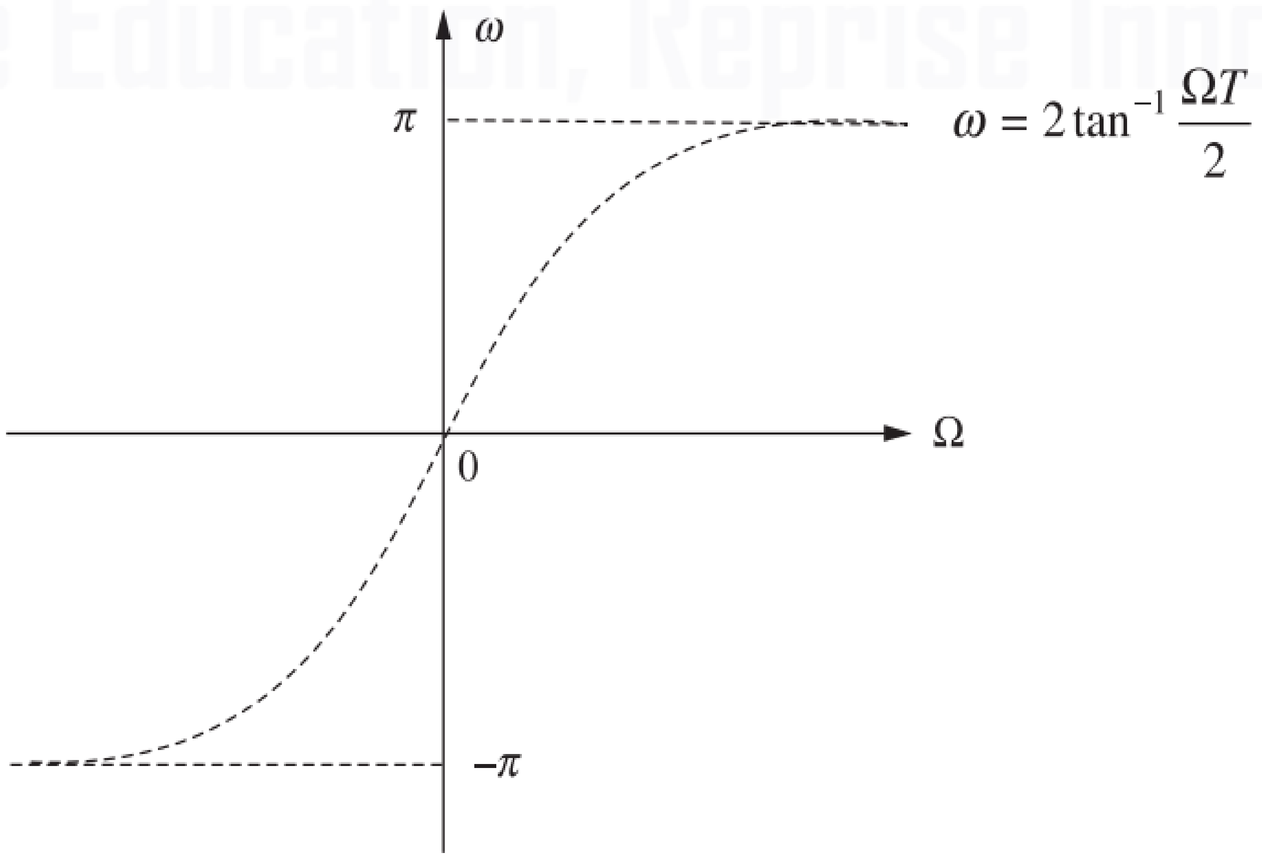


Figure 8.4 Mapping between Ω and ω in bilinear transformation.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 8.5(b). The phase response of corresponding digital filter will be nonlinear.

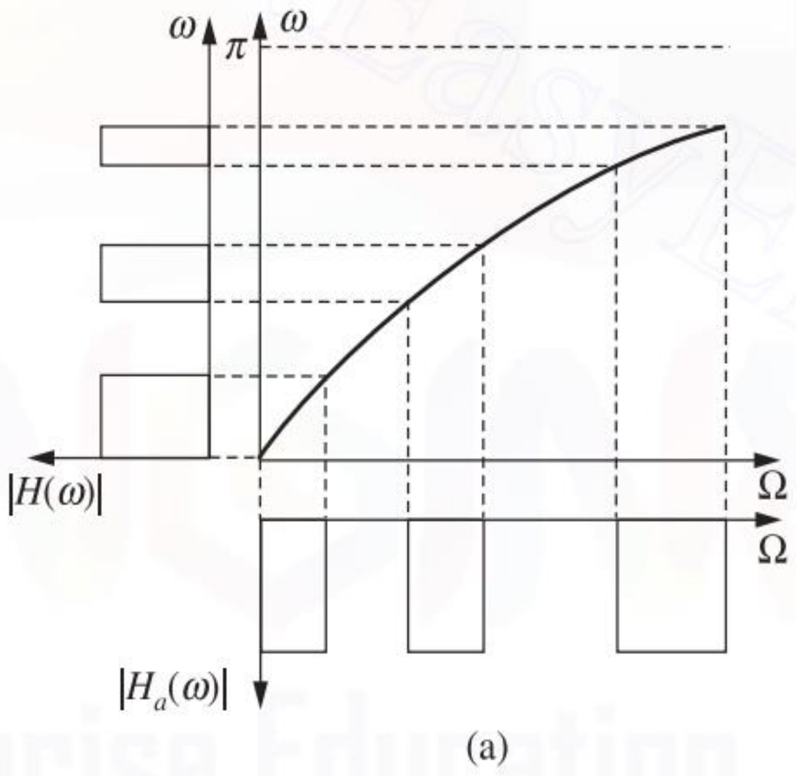
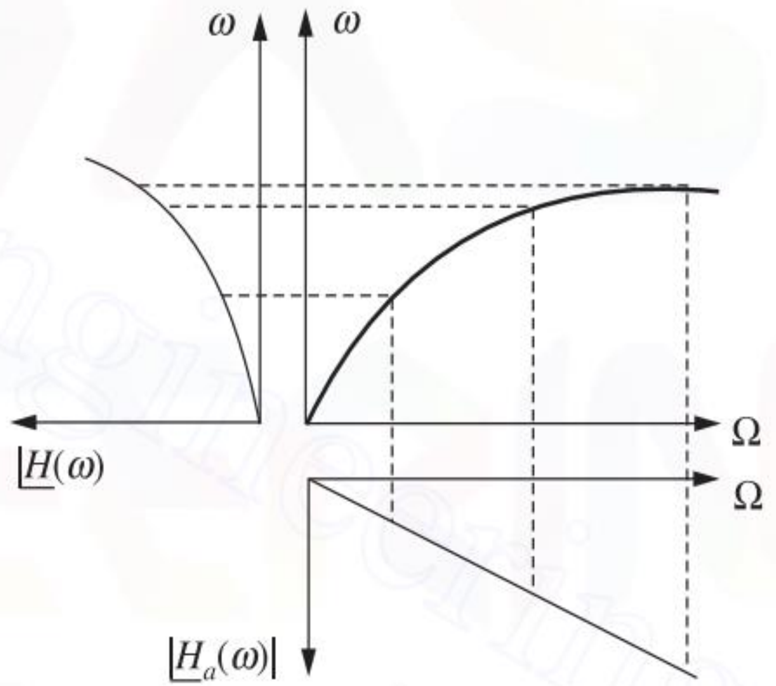


Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 8.5(b). The phase response of corresponding digital filter will be nonlinear.



(b)

Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 8.5(b). The phase response of corresponding digital filter will be nonlinear.

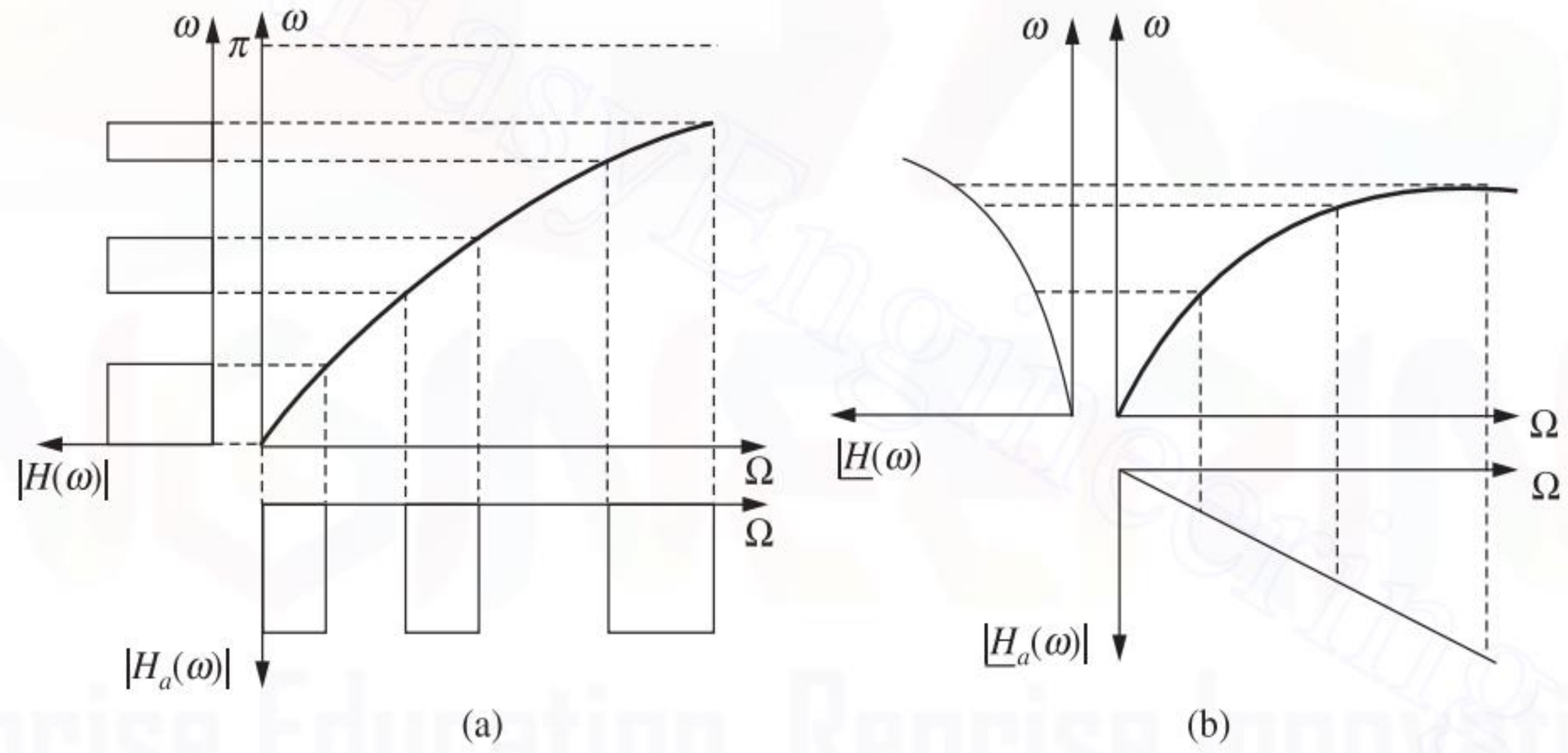


Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

Low Pass Filter Design

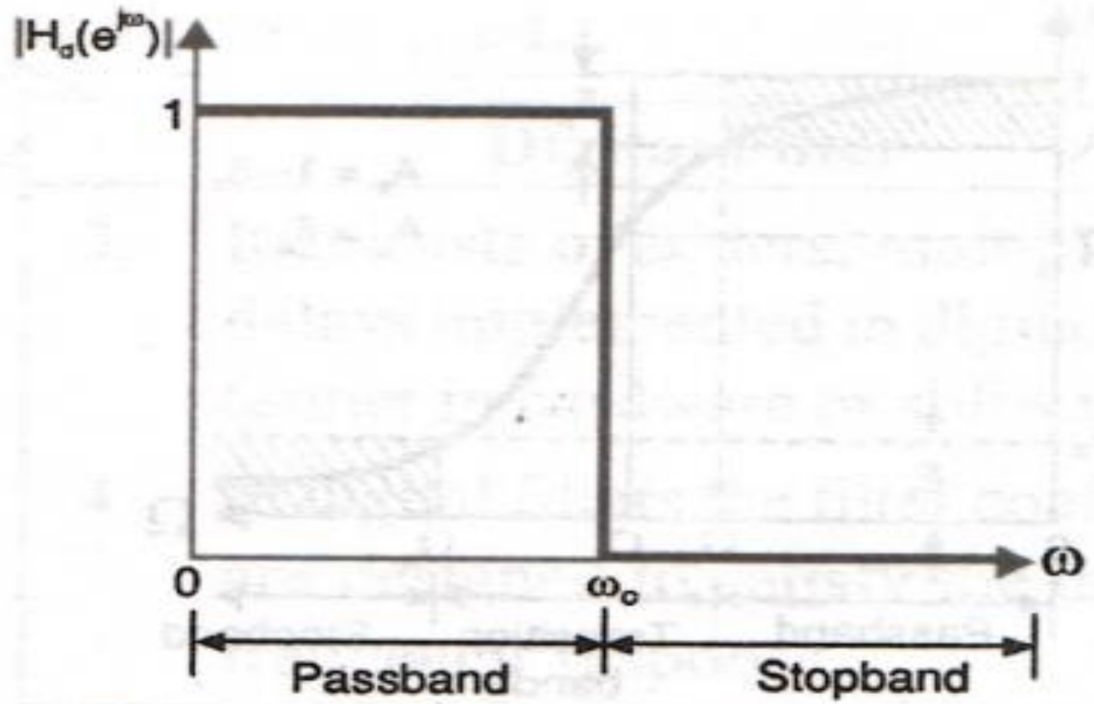


Fig a : Normalized magnitude response of ideal digital IIR lowpass filter.

Low Pass Filter Design

- Specification:

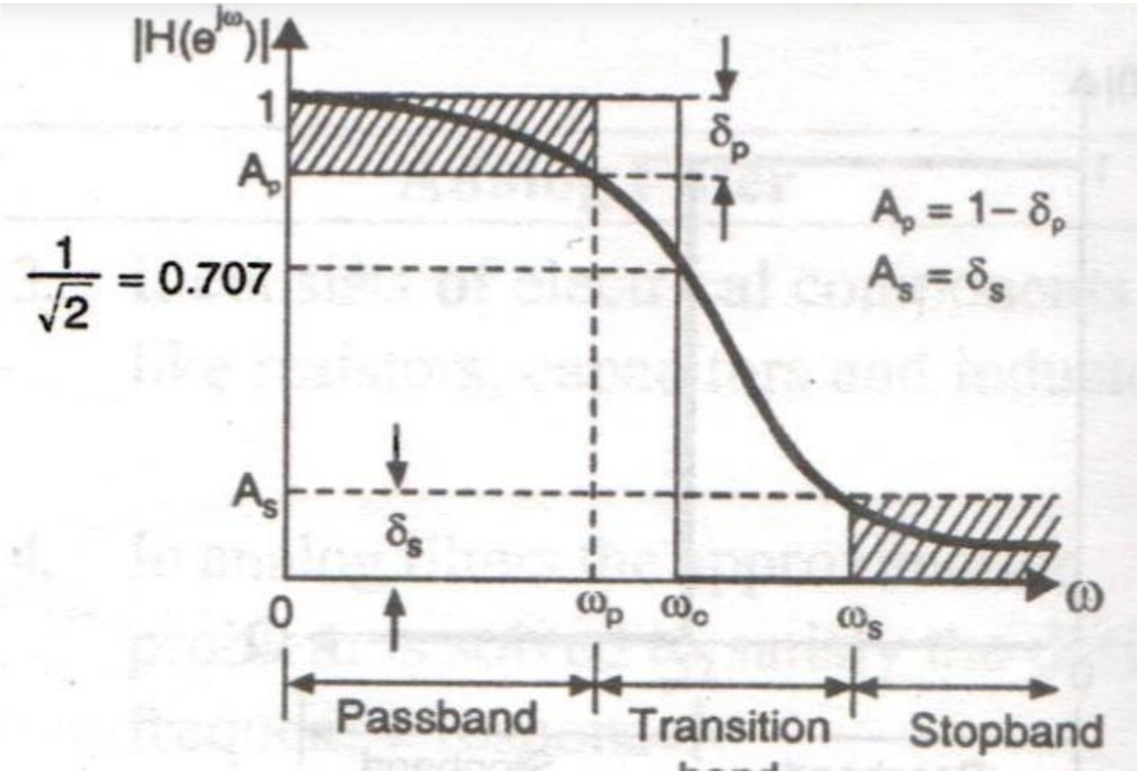
Gain at passband edge frequency $(\mathbf{A_p})$

Gain at stopband edge frequency $(\mathbf{A_s})$

Passband edge digital frequency $(\omega_\mathbf{p})$

Stopband edge digital frequency $(\omega_\mathbf{s})$

Sampling time $(\mathbf{T})$



EXAMPLE 8.17 Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \le |H(\omega)| \le 1; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ s

$\therefore$ The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

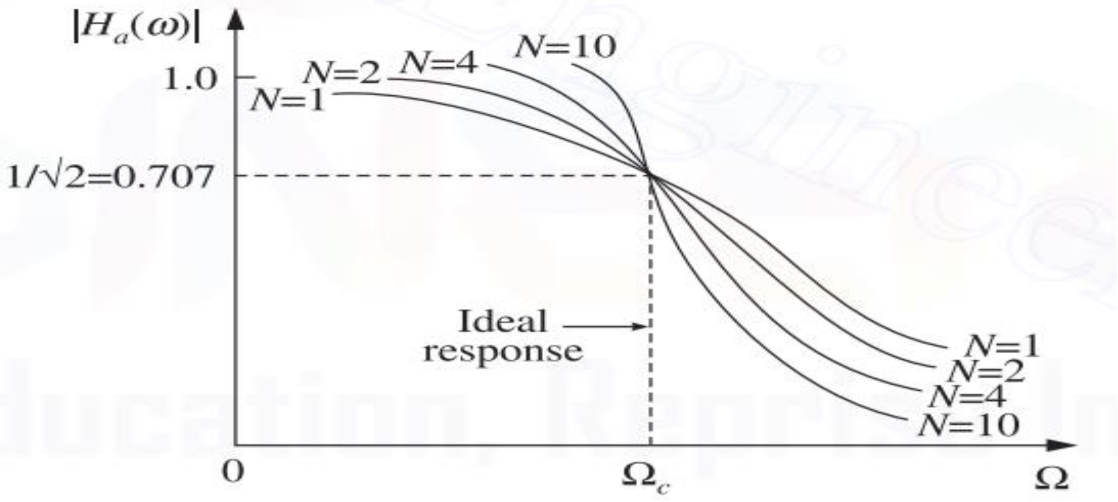
Analog Butterworth Filter

The magnitude response of low-pass filter obtained by this approximation is given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the 3 dB cutoff frequency and N is the order of the filter.

Frequency Response of the Butterworth Filter



Designing Low Pass Butterworth Filters

Poles of Low-pass Butterworth filter

Refer Class Note for derivation

when N odd $\quad \mathbf{s_n = e^{\frac{jk\pi}{N}}}$

where $\mathbf{k = 1, \ldots, 2N}$ and $\mathbf{s_n = \frac{s}{\Omega_c}}$

when N even $\quad \mathbf{s_n = e^{\frac{j(2k-1)\pi}{2N}}}$

Determine the pole of lowpass Butterworth filter for $N = 1$. Sketch the location of poles on s-plane.

Refer Class Note for Solution

Design procedure for low-pass digital Butterworth IIR filter

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1
A_2 = Gain at a stopband frequency ω_2
Ω_1 = Analog frequency corresponding to ω_1
Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.

For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2}, \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore \ \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T}, \quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore \ \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \geq \frac{1}{2} \frac{\log\left\{\left[\frac{1}{A_2^2}-1\right] \Big/ \left[\frac{1}{A_1^2}-1\right]\right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Step 4 Calculate the analog cutoff frequency $\Omega_c = \dfrac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}}$

Step 5 Determine the transfer function of the analog filter.

Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For normalized case, $\Omega_c = 1$ rad/s

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

EXAMPLE 8.17 Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \leq |H(\omega)| \leq 1; \quad 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(\omega)| \leq 0.2; \quad \frac{3\pi}{4} \leq \omega \leq \pi$$

with $T = 1$ s

EXAMPLE 8.17 Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

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$$|H(\omega)| \leq 0.2; \quad \frac{3\pi}{4} \leq \omega \leq \pi$$

with $T = 1$ s

Solution: The Butterworth digital filter is designed as per the following steps.

From the given specification, we have

$$A_1 = 0.9 \text{ and } \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \text{ and } \omega_2 = \frac{3\pi}{4} \qquad \text{and } T = 1 \text{ s}$$

Step 1 Choice of the type of transformation

Here the bilinear transformation is already specified.

Step 2 Determination of the ratio of the analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

Step 3 Determination of the order of the filter N

$$N \geq \frac{1}{2}\,\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}} \geq \frac{1}{2}\,\frac{\log\{24/0.2345\}}{\log 2.414} \geq 2.626$$

Since $N \geq 2.626$, choose $N = 3$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., –3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{2}{\left[\dfrac{1}{0.9^2} - 1\right]^{1/2\times 3}} = 2.5467$$

Step 5 Determination of the transfer function of the analog Butterworth filter $H_a(s)$

For odd N, we have $H_a(s) = \dfrac{\Omega_c}{s + \Omega_c} \displaystyle\prod_{k=1}^{\frac{N-1}{2}} \dfrac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$

where $b_k = 2 \sin\left[\dfrac{(2k-1)\pi}{2N}\right]$

For $N = 3$, we have

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

where $b_1 = 2 \sin\left[\dfrac{(2\times 1-1)\pi}{2\times 3}\right] = 2\sin\dfrac{\pi}{6} = 1$

Therefore, $H_a(s) = \left(\dfrac{2.5467}{s + 2.5467}\right)\left(\dfrac{(2.5467)^2}{s^2 + 1(2.5467)\,s + (2.5467)^2}\right)$

EXAMPLE 8.19 Design a low-pass Butterworth digital filter to give response of 3 dB or less for frequencies upto 2 kHz and an attenuation of 20 dB or more beyond 4 kHz. Use the bilinear transformation technique and obtain $H(z)$ of the desired filter.

the sampling frequency be 10000 Hz.

Find $\mathbf{A_1}$ and $\mathbf{A_2}$

A_1 in dB is given by

$$A_1 \text{ dB} = -20 \log A_1$$

i.e.

$$\log A_1 = -\frac{A_1 \text{ dB}}{20}$$

or

$$A_1 = 10^{-\frac{A_1 \text{ dB}}{20}}$$

Find $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$

$$\omega_1 = \mathbf{2 * pi * \frac{f_1}{f_s}}$$

$$\omega_2 = \mathbf{2 * pi * \frac{f_1}{f_s}}$$