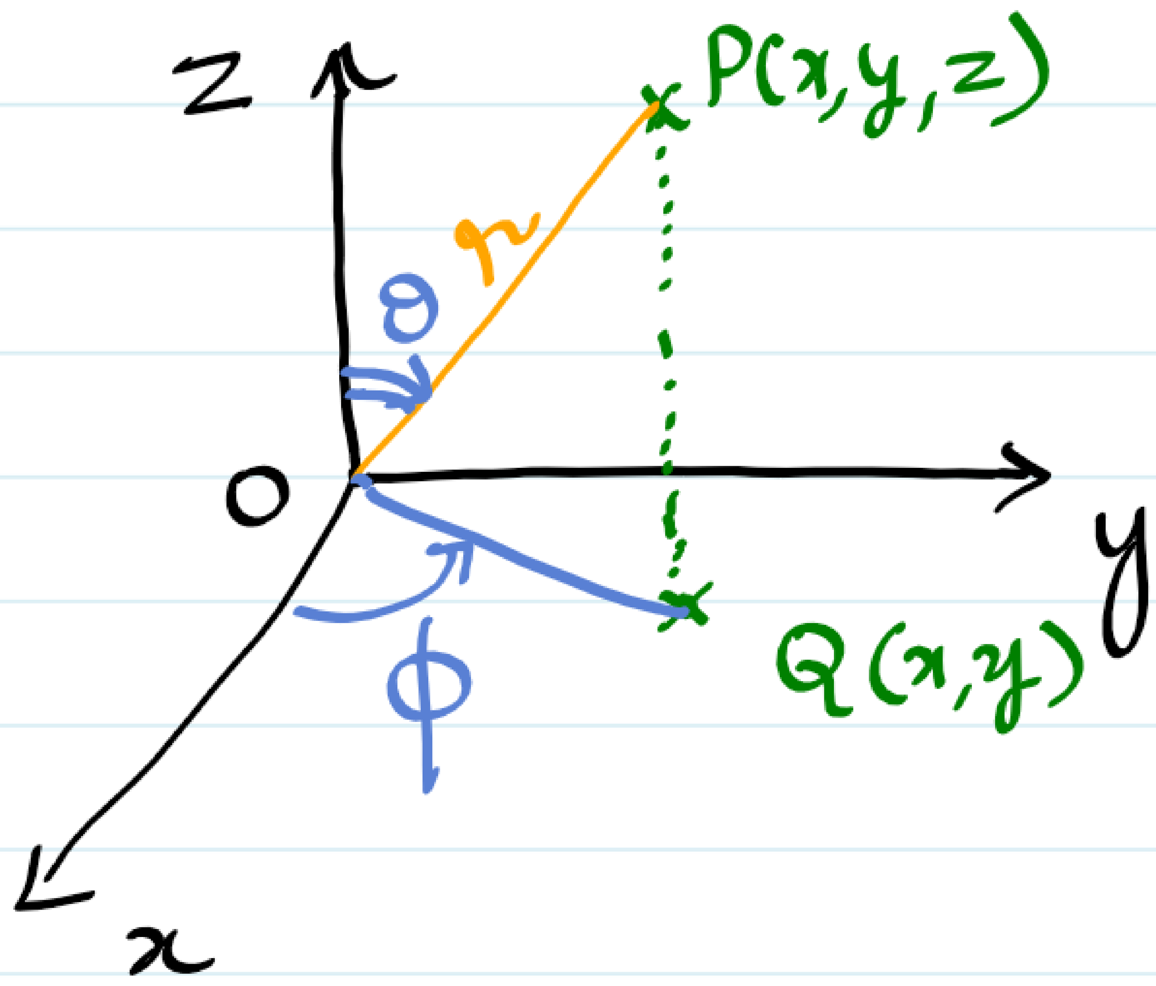
Spherical polar co-ordinates –

Let $P(x,y,z)$ be any point whose projection on the xy-plane is $Q(x,y)$. Then the spherical polar co-ordinates of P are (r,θ,ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.



The spherical polar co-ordinates are,

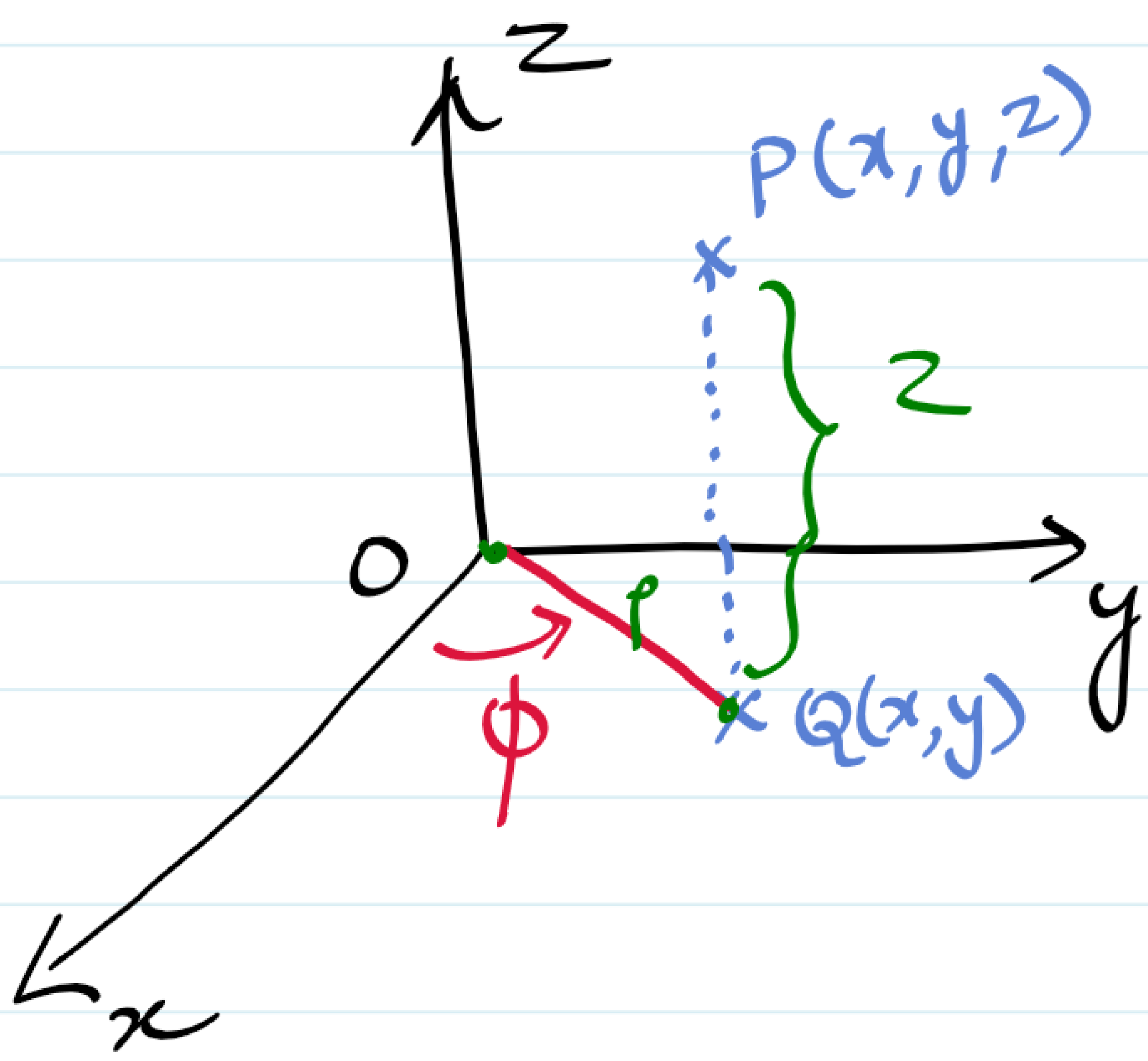
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

$$\underline{\underline{J =}}\ r^2\sin\theta \checkmark$$

Cylindrical co-ordinates –

Any point $P(x,y,z)$ whose projection on the xy-plane is $Q(x,y)$ has the cylindrical co-ordinates (ρ,ϕ,z), where

$$\rho = OQ, \quad \phi = \angle XOQ \quad \text{and} \quad z = QP.$$



The cylindrical co-ordinates are,

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\underline{\underline{J = \rho}}$$

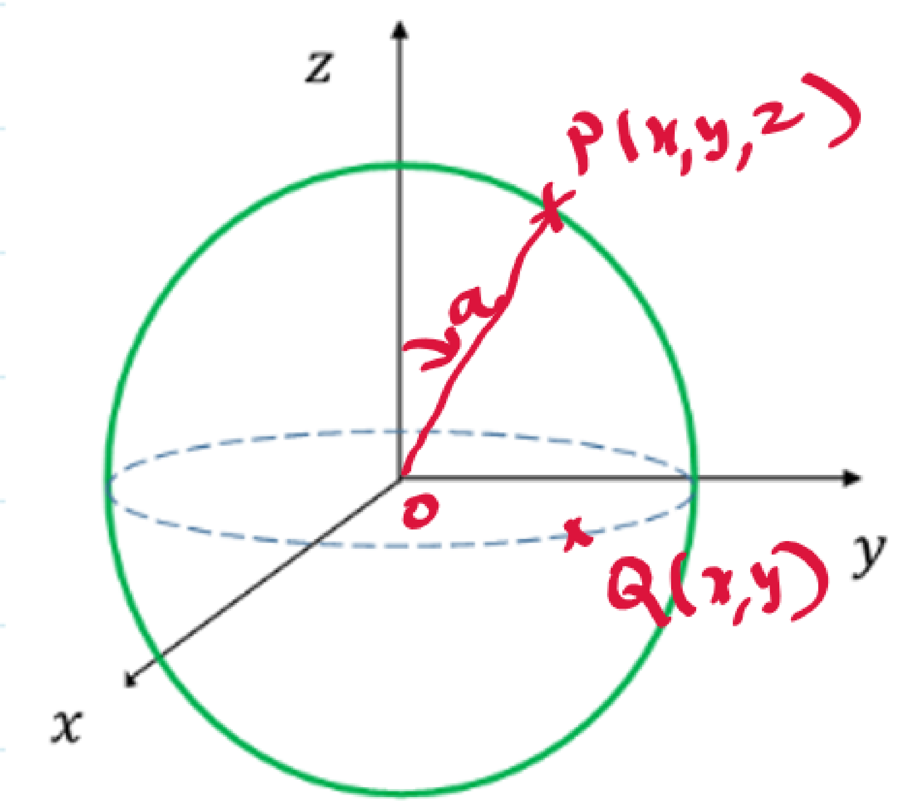
Standard limits -

① For complete sphere $x^2+y^2+z^2=a^2$

$r: 0$ to a

$\theta: 0$ to π

$\phi: 0$ to 2π

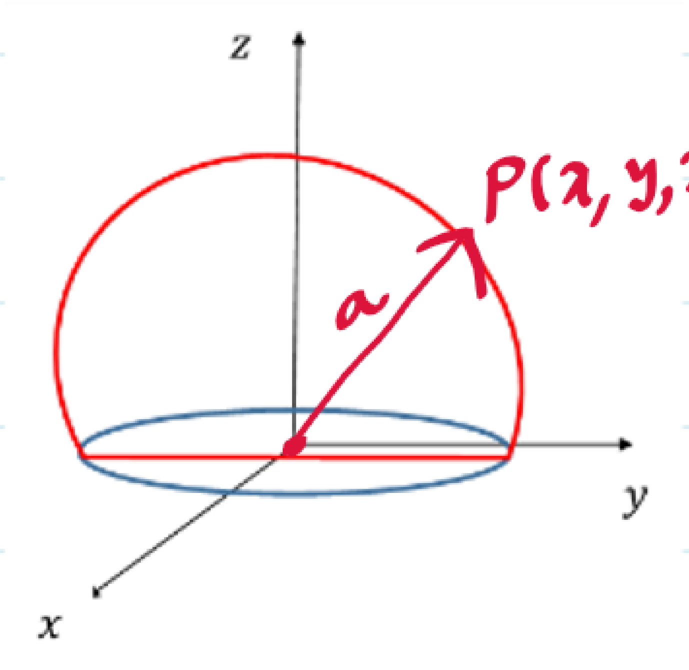


② For hemisphere $x^2+y^2+z^2=a^2$.

$r: 0$ to a

$\theta: 0$ to $\pi/2$

$\phi: 0$ to 2π

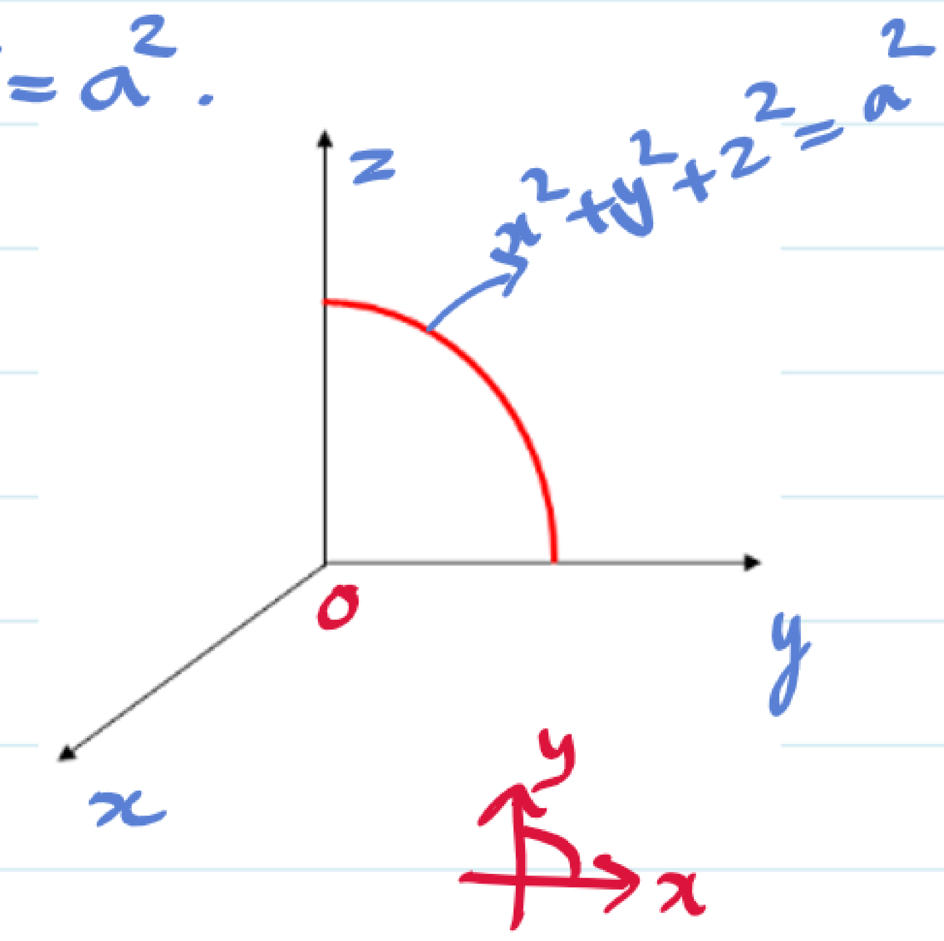


③ For positive octant of a sphere $x^2+y^2+z^2=a^2$.

$r: 0$ to a

$\theta: 0$ to $\pi/2$

$\phi: 0$ to $\pi/2$

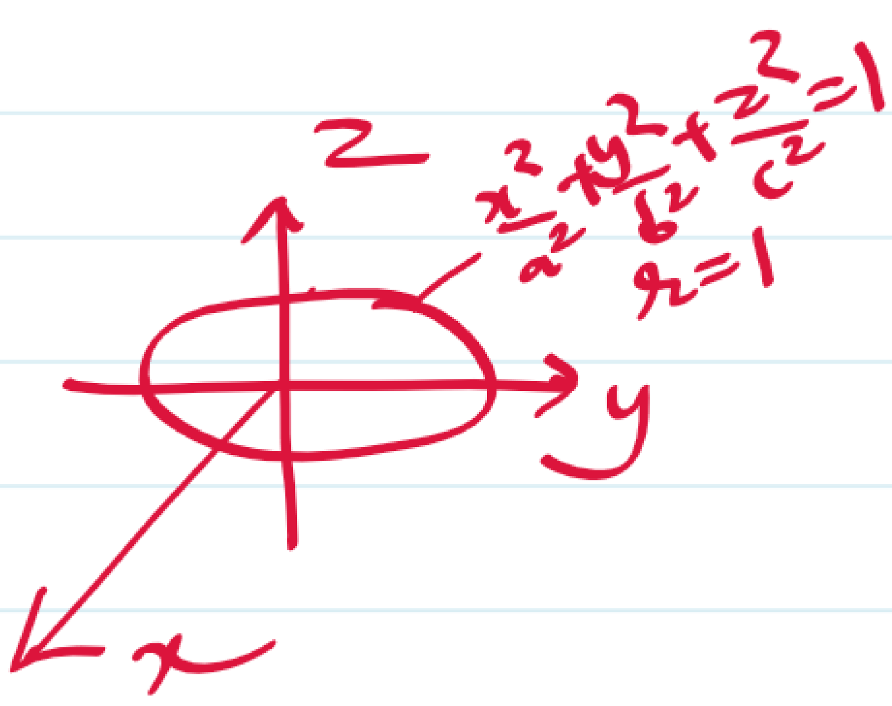


4) For ellipsoid $\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}_{r^2}=1$, we use,

$x = a\, r \sin\theta \cos\phi$

$y = b\, r \sin\theta \sin\phi$ $\qquad J = abc\, r^2 \sin\theta$

$z = c\, r \cos\theta$

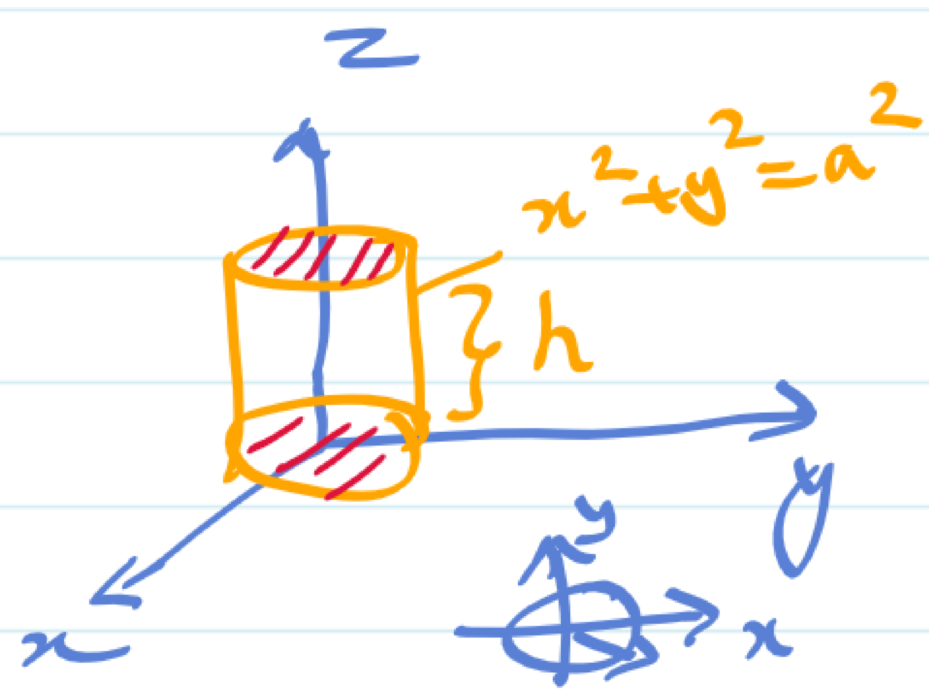


Standard limits: $r: 0$ to 1, $\theta: 0$ to π, $\phi: 0$ to 2π

⑤ For a cylinder $x^2+y^2=a^2$, $z=0$, $z=h$

$x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$

$\rho: 0$ to a, $\phi: 0$ to 2π, $z: 0$ to h



Triple integrals

① Evaluate $I = \int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

$z \quad x \quad y$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \left[(x+z)y + \frac{y^2}{2} \right]_{y=x-z}^{x+z} dx\, dz$$

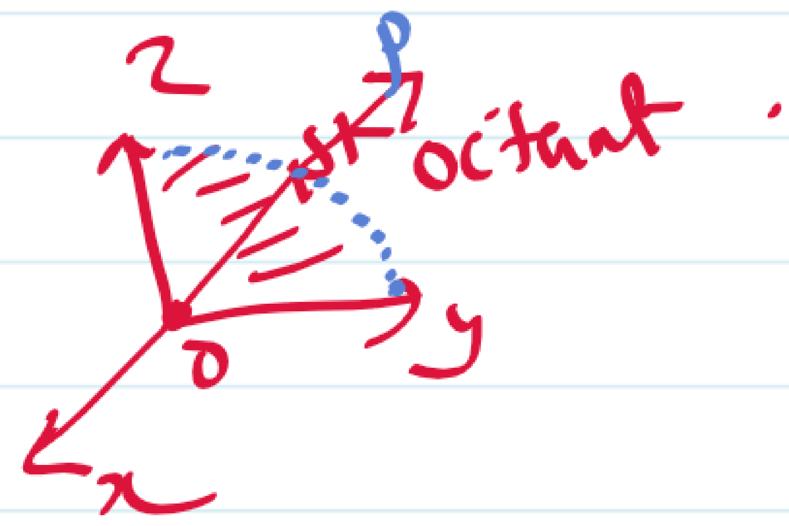
$$= \int_{-1}^{1} \int_{0}^{z} (x+z)^2 + \frac{(x+z)^2}{2} - (x^2 - z^2) - \frac{(x-z)^2}{2} \quad dx \,.\, dz$$

$$= \int_{-1}^{1} \left[\frac{(x+z)^3}{3} + \frac{1}{2}\frac{(x+z)^3}{(3)} - \frac{x^3}{3} + z^2 x - \frac{(x-z)^3}{2(3)} \right]_{x=0}^{z}$$

$$= \int_{-1}^{1} \frac{(2z)^3}{3} + \frac{1}{6}(2z)^3 - \frac{z^3}{3} + z^3 - \left(\frac{z^3}{3} + \frac{1}{6}z^3 + \frac{z^3}{6} \right) dz$$

$$= \underline{\underline{0}}$$

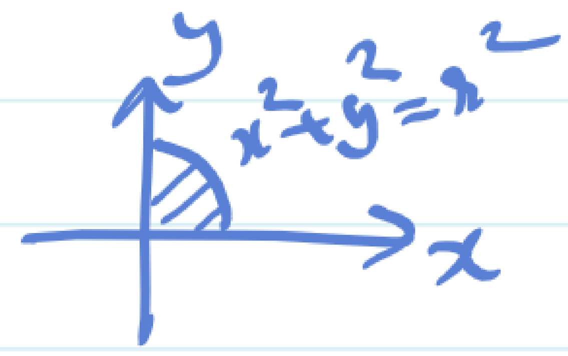
2) Evaluate $\int_0^\infty dx \int_0^\infty dy \int_0^\infty \frac{dz}{(1+x^2+y^2+z^2)^2}$ or $I = \int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$



Taking spherical polar co-ordinates,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$x^2+y^2+z^2 = r^2$, $J = r^2\sin\theta$



r: 0 to ∞, θ: 0 to $\pi/2$, ϕ: 0 to $\pi/2$

$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$= \int_{\phi=0}^{\pi/2} d\phi \times \int_{\theta=0}^{\pi/2} \sin\theta\,d\theta \times \int_{r=0}^{\infty} \frac{r^2}{(1+r^2)^2}\,dr$$

$$= \frac{\pi}{2} \times \left(-\cos\theta\right)_0^{\pi/2} \times \int_{r=0}^{\infty} \frac{r^2}{(1+r^2)^2}\,dr$$

$r = \tan t$
$dr = \sec^2 t\,dt$

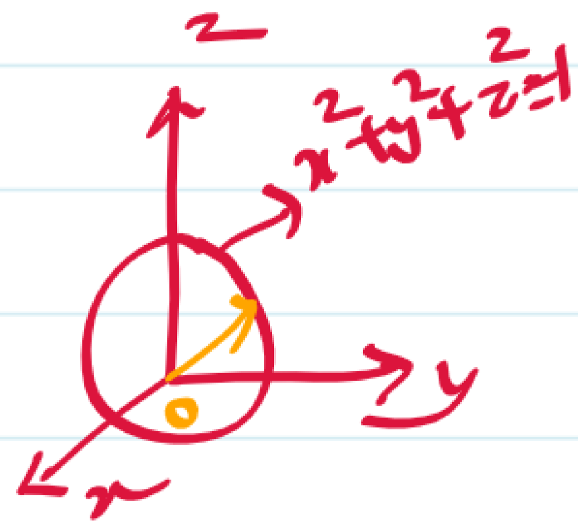
when, $r=0$, $t=0$
$r=\infty$, $t=\pi/2$

$$= \frac{\pi}{2}(0+1) \times \int_0^\infty \frac{r^2}{(1+r^2)^2}\,dr$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan^2 t}{\sec^4 t}\sec^2 t\,dt$$

$$= \frac{\pi}{2} \times \int_0^{\pi/2} \sin^2 t\,dt = \frac{\pi}{2} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi^2}{8}$$

3) Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ taken throughout the volume of the sphere $x^2+y^2+z^2=1$ in the positive octant.



Taking spherical polar co-ordinates,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$x^2+y^2+z^2 = r^2$, $J = r^2\sin\theta$.

$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\theta\,dr\,d\theta\,d\phi$

$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \frac{1}{\sqrt{1-r^2}}\, r^2\sin\theta\; dr\,d\theta\,d\phi$$

$$= \int_{0}^{\pi/2} d\phi \times \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \times \int_{r=0}^{1} \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$r = \sin t$

$dr = \cos t\, dt$

$r=0$, $t=0$

$r=1$, $t=\pi/2$

$$= \frac{\pi}{2} \times \left(-\cos\theta\right)_0^{\pi/2} \times \int_0^{\pi/2} \frac{\sin^2 t}{\cancel{\cos t}}\, \cancel{\cos t}\, dt$$

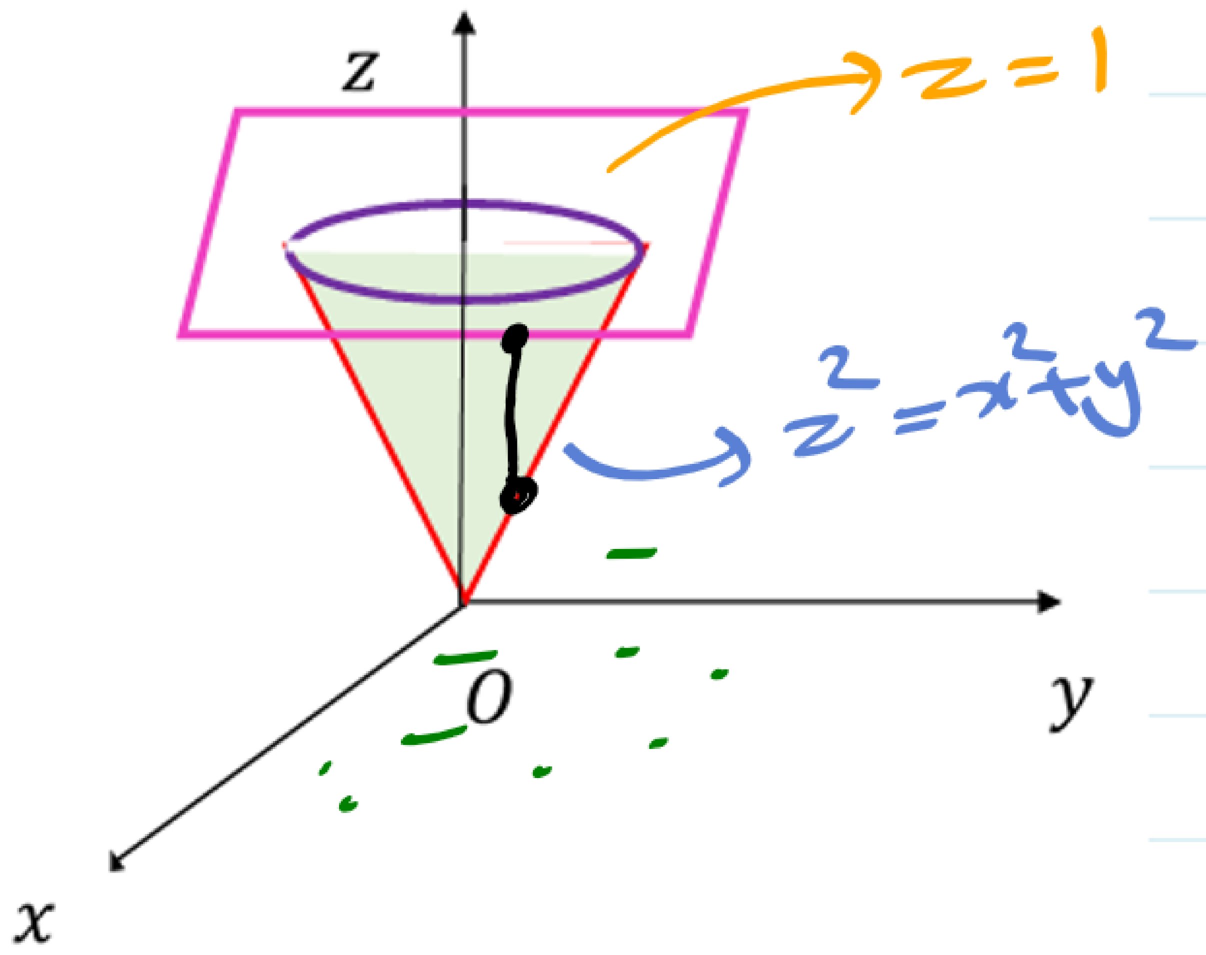
$$= \frac{\pi}{2} \times (1) \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

4) Evaluate $\displaystyle\iiint_V \sqrt{x^2+y^2}\; dx\,dy\,dz$,

where V is $x^2+y^2=z^2$, $z \geq 0$ and $z=0$, $z=1$

cone

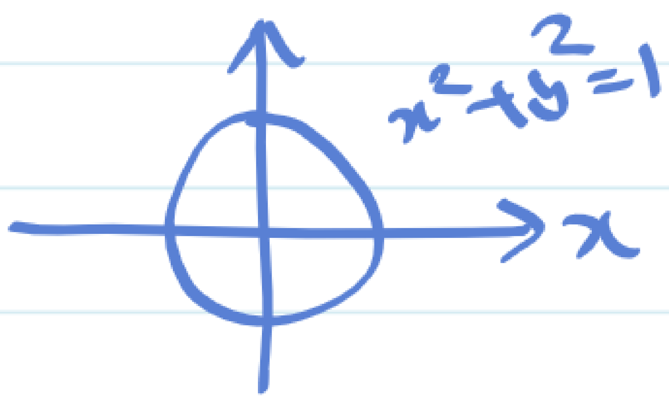


$$I = \iint_R \int_{z=\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\; dz\,dy\,dz$$

$$= \iint \left(\sqrt{x^2+y^2}\right) z \Big]_{\sqrt{x^2+y^2}}^{1} dx\,dy$$

$z^2=x^2+y^2$, $z=1$

$x^2+y^2=1$



$$= \iint \sqrt{x^2+y^2} - (x^2+y^2)\; dx\,dy \,.$$

Using polaral co-ordinates

$x = r\cos\theta$ $y = \sin\theta$, $J = r$

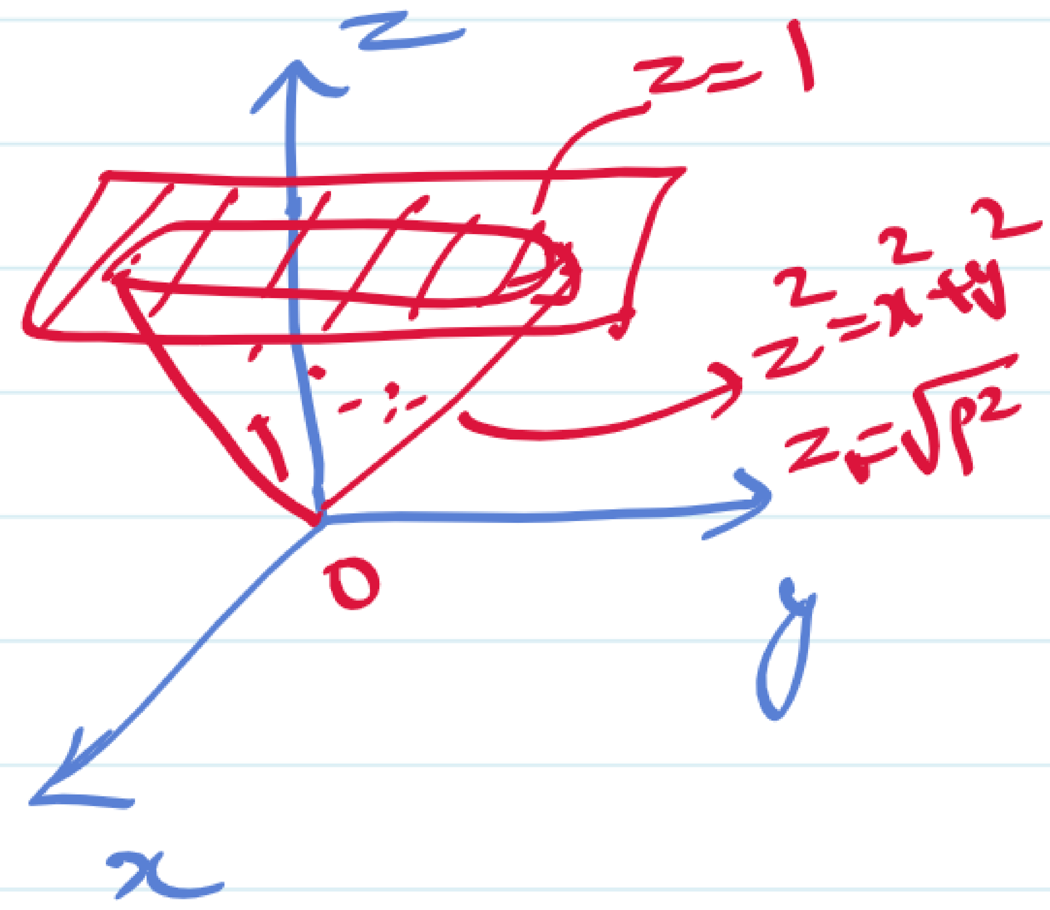
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r - r^2)\; r\; dr\,d\theta = \pi/6$$

4) Evaluate $\iiint_V \sqrt{x^2+y^2}\, dx\,dy\,dz$,

where V is $x^2+y^2=z^2$, $z \geq 0$ and $z=0$, $z=1$

cone

Using cylindrical co-ordinates.



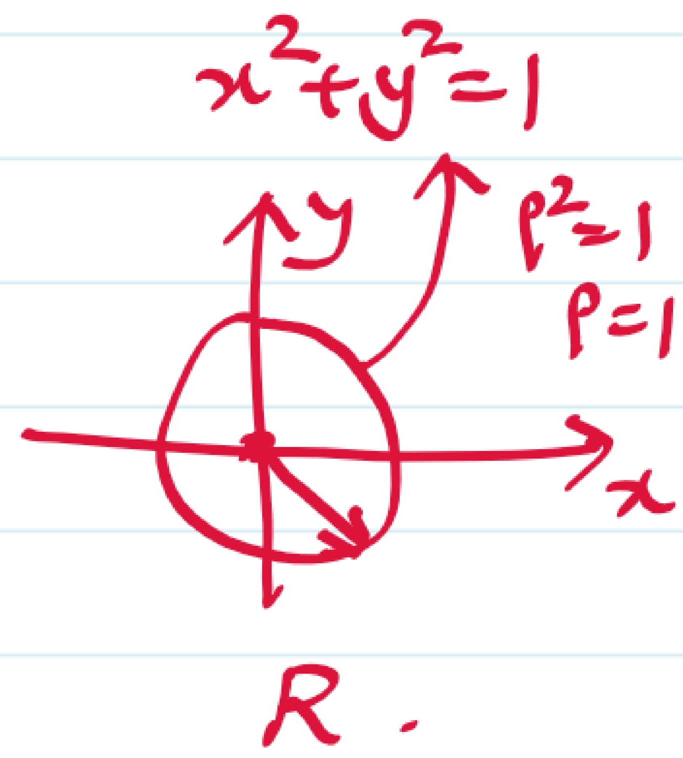
Ans: Cylindrical co-ordinates

$x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$

$J=\rho$

$x^2+y^2=\rho^2$

$z^2=x^2+y^2$, $z=1$



$$I=\int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\int_{z=\rho}^{1}\sqrt{\rho^2}\;\rho\,d\rho\,d\phi\,dz$$

$$=\int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\rho^2\left(z\right)_{\rho}^{1}\,d\phi\,d\rho$$

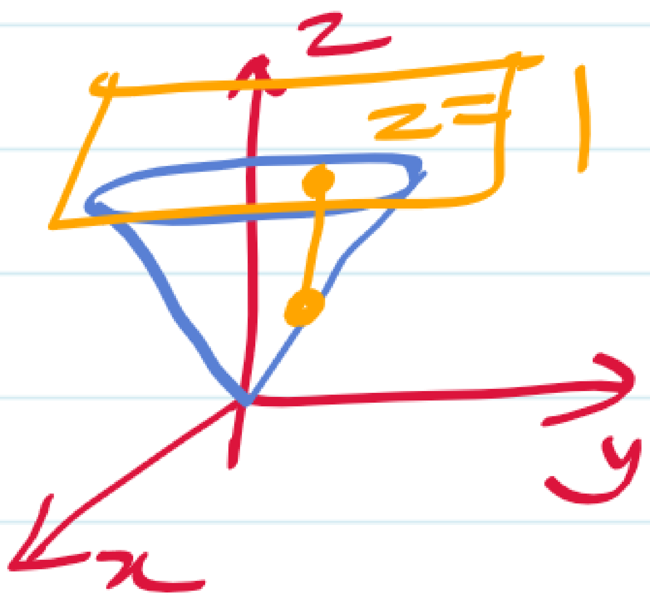
$$=\int_{\rho=0}^{1}\int_{0}^{2\pi}\left(\rho^2-\rho^3\right)d\rho\,d\phi$$

$$=\int_{\rho=0}^{1}\rho^2-\rho^3\,d\rho \times \int_0^{2\pi}d\phi$$

$$=\left(\frac{\rho^3}{3}-\frac{\rho^4}{4}\right)_0^1 \times 2\pi = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$$

5) Changing to spherical polar co-ordinates, evaluate.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$$



$z^2 = x^2+y^2$

Here $z: \sqrt{x^2+y^2}$ to 1

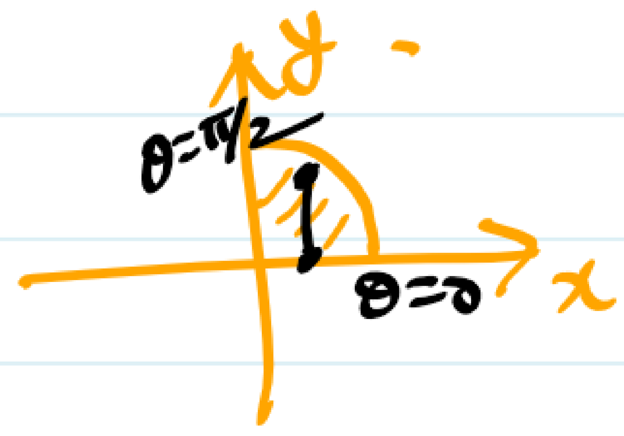
$y: 0 \to \sqrt{1-x^2}$

$x = 0$ to 1

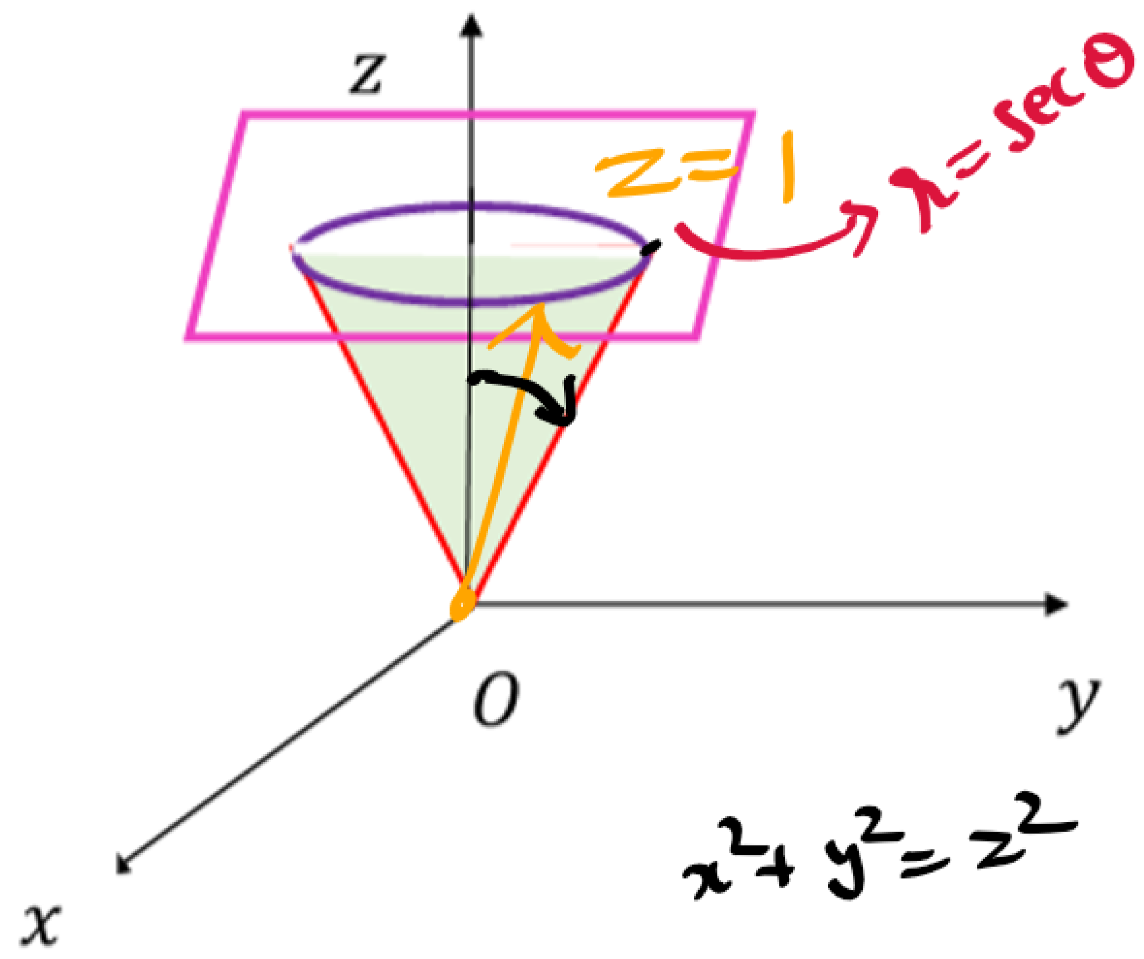
Using spherical polar co-ordinates,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$J = r^2\sin\theta$. , $x^2 + {}^2 + {}^2 = r^2$



$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sec\theta} \frac{r^2\sin\theta}{\sqrt{r^2}}\, dr\, d\theta\, d\phi$$



$z = 1$

$\downarrow$

$r\cos\theta = 1$

$r = \sec\theta$.

$$= \int_{\phi=0}^{\pi/2} \int_{0}^{\pi/4} \sin\theta \cdot \left(\frac{r^2}{2}\right)_0^{\sec\theta} d\theta\, d\phi$$

$r =$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/4} \frac{1}{2} \sin\theta\sec^2\theta\, d\theta \cdot d\phi$$

$$= \int_{\phi=0}^{\pi/2} d\phi \times \frac{1}{2} \int_{\theta=0}^{\pi/4} \tan\theta\sec\theta\, d\theta$$

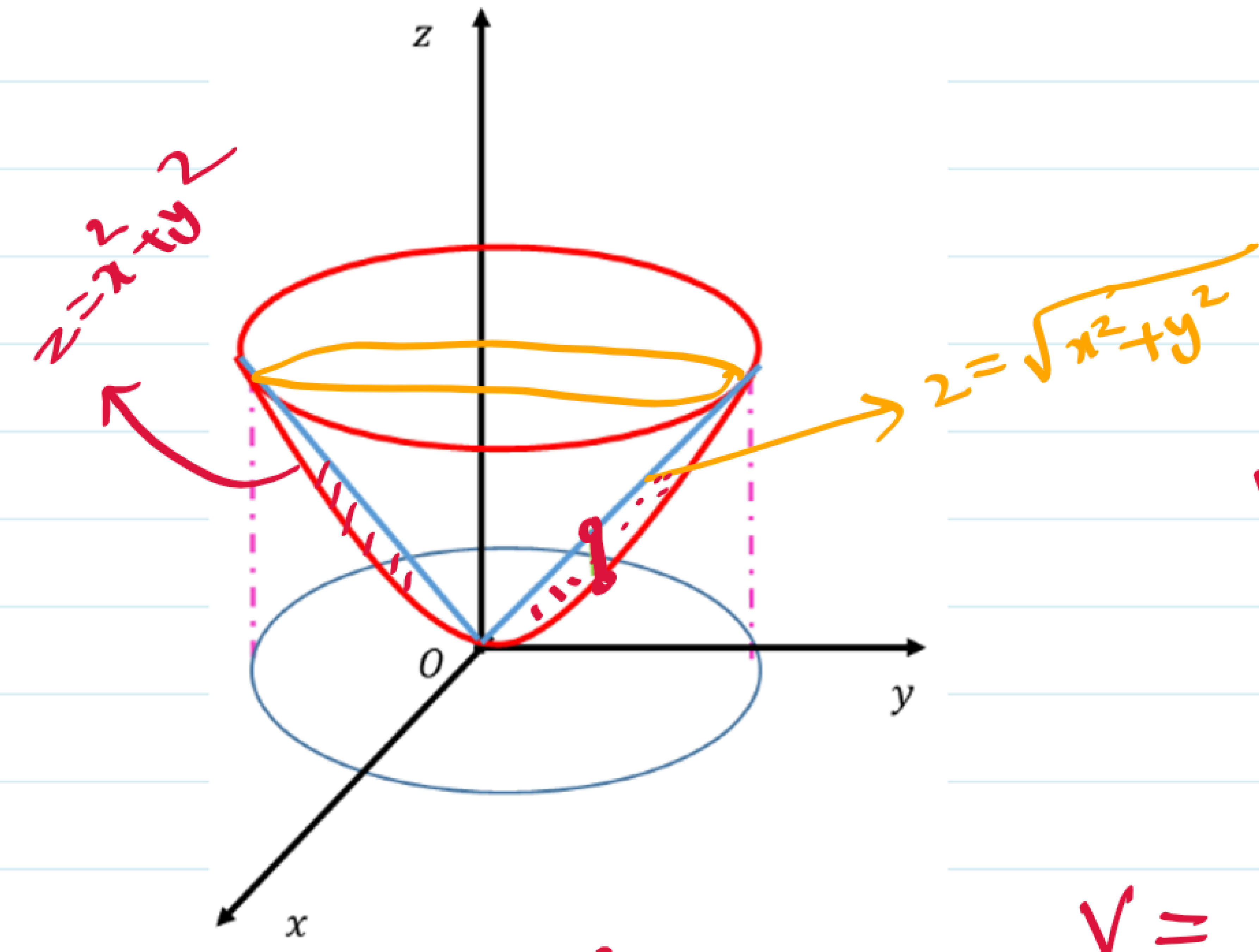
$$= \pi/2 \times \frac{1}{2}(\sec\theta)_0^{\pi/4} = \frac{\pi}{4}(\sqrt{2}-1)$$

Volumes of solids –

To express the volume of a solid as a triple integral, we note that the volume of elementary cuboid is $dx\,dy\,dz$, and so the volume of the solid is given by,

$$V = \iiint_V dx\,dy\,dz$$

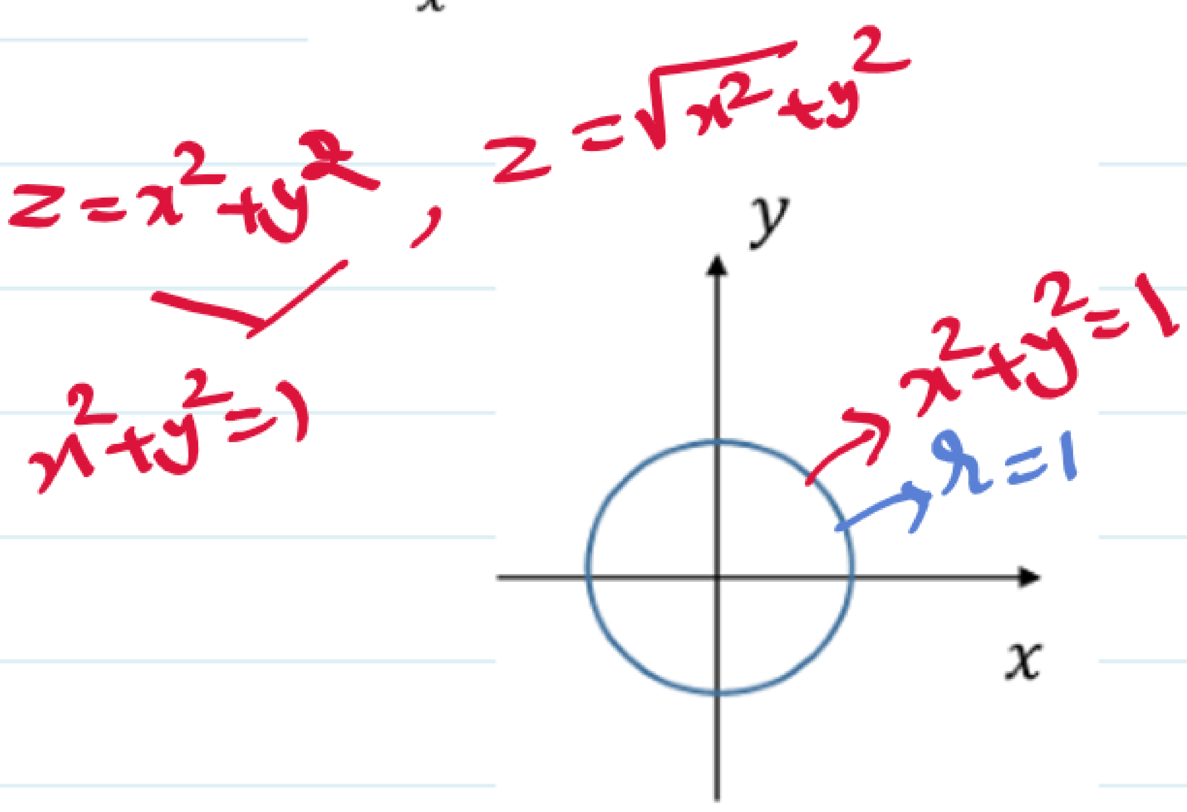
① Find the volume of the region enclosed by the cone $z = \sqrt{x^2+y^2}$ and paraboloid $z = x^2+y^2$



$$V = \iiint dx\,dy\,dz\,.$$

$$V = \int\int\int_{z=x^2+y^2}^{\sqrt{x^2+y^2}} dz\;dy\;dz$$

$$V = \iint_R (z)_{x^2+y^2}^{\sqrt{x^2+y^2}}\;dy\,dx$$



$$V = \iint \sqrt{x^2+y^2} - (x^2+y^2)\;dy\,dx$$

Using Polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$ $\quad J = r$

$x^2+y^2 = \sqrt{x^2+y^2}$

$\Rightarrow x^2+y^2 = 1$

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r - r^2)\,r\;dr\,d\theta\,.$$

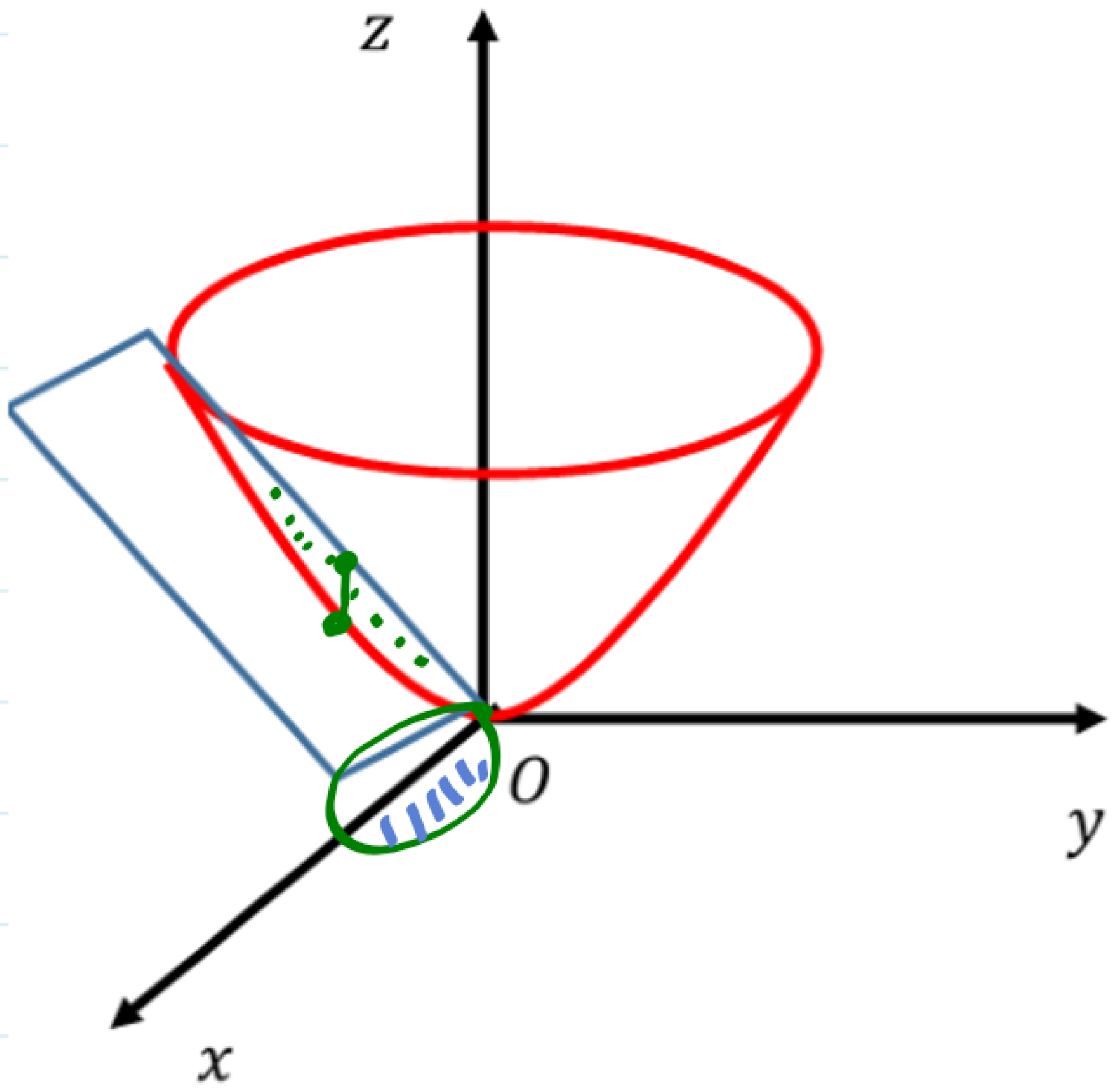
$$= 2\pi \times \left(\frac{r^3}{3} - \frac{r^4}{4}\right)_0^1 = \underline{\underline{\pi/6}}$$

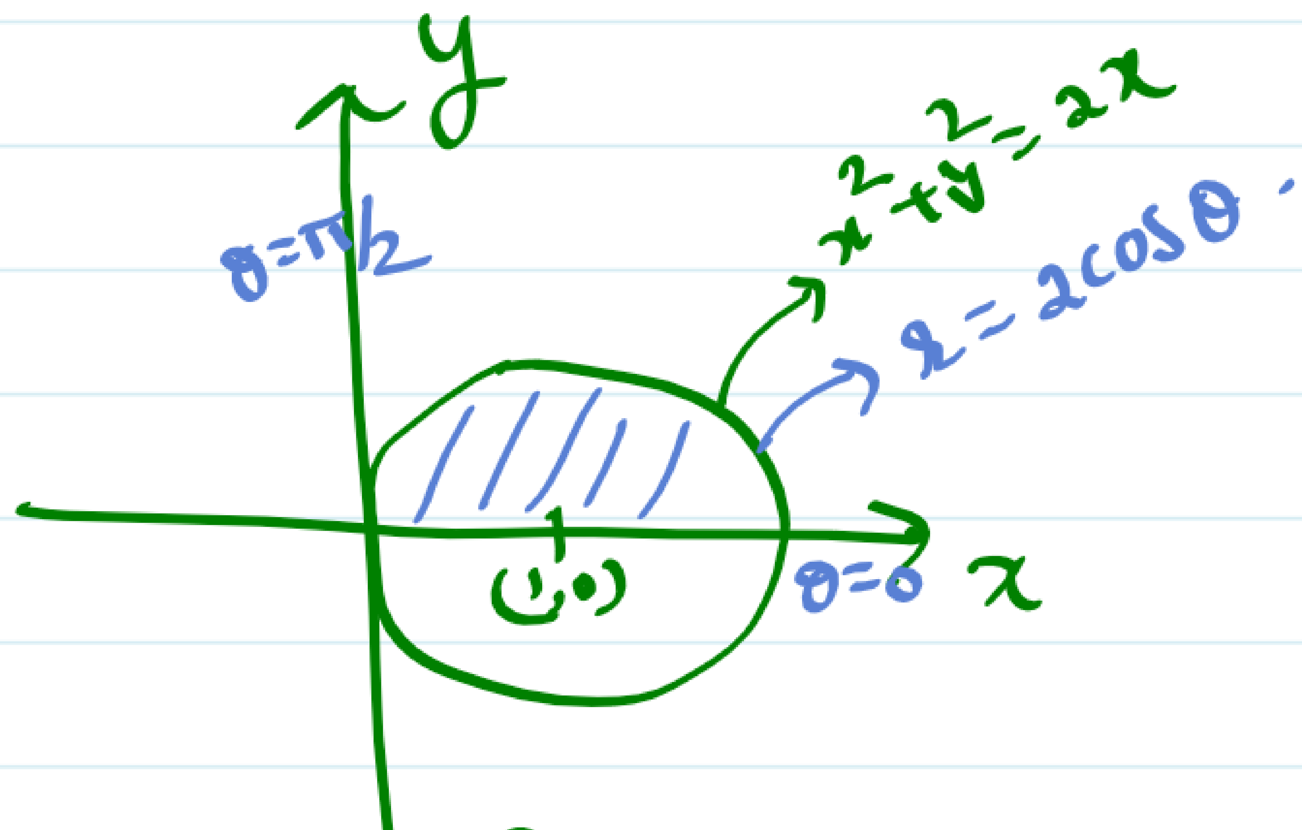
Using Cylindrical co-ordinates,

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$V = \iiint dx\,dy\,dz = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\int_{z=\rho^2}^{\rho} \rho\, dz\, d\phi\, d\rho = \frac{\pi}{6}$$

2) Find the volume of region bounded by x^2+y^2 and $z=2x$.





R: $x^2+y^2=2x$.

$z=x^2+y^2$, $z=2x$

$x^2+y^2=2x$.

↳ circle.

$$V=\iint_{R}\int_{z=x^2+y^2}^{2x} dz\,dx\,dy$$

$$=\iint_{R} 2x-(x^2+y^2)\,dx\,dy$$

Using polar co-ordinates

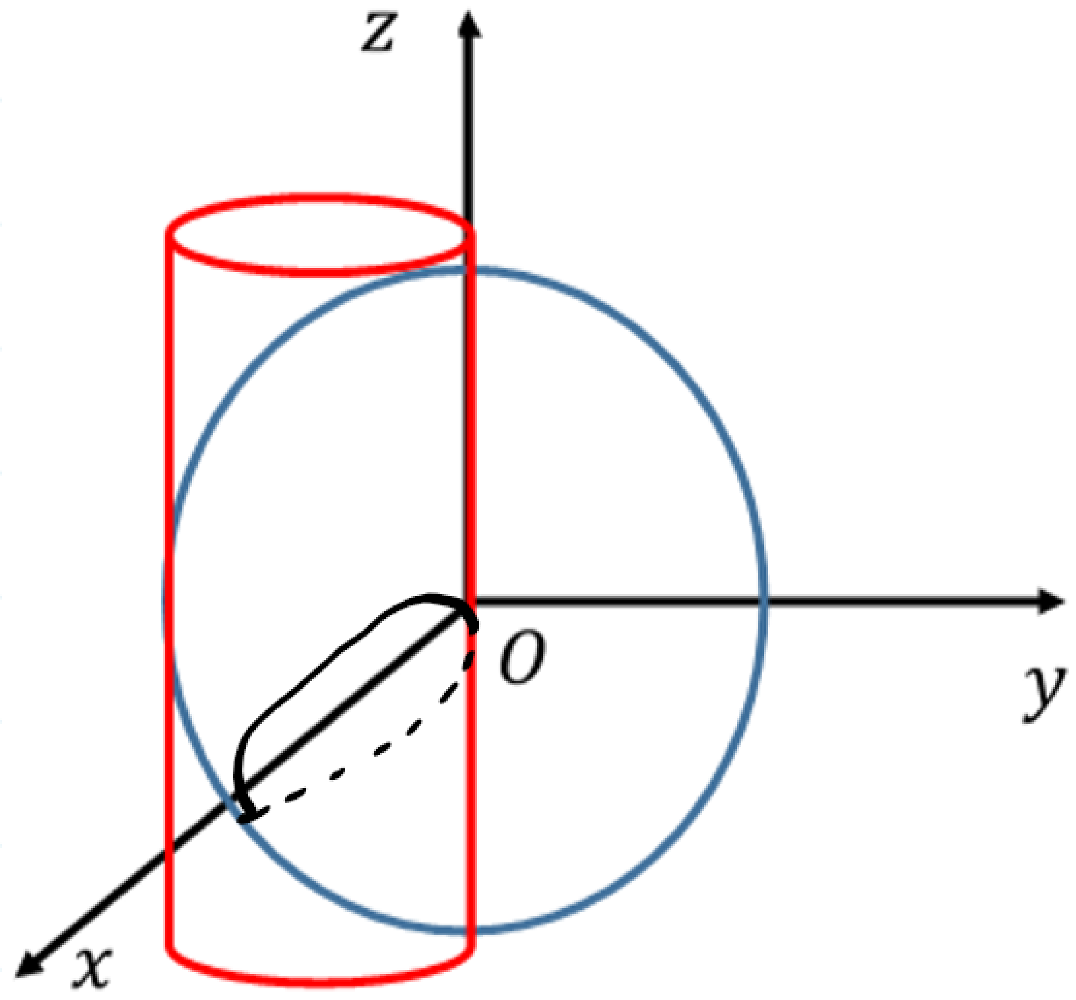
$$=2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta}\left(2r\cos\theta-r^2\right)r\,dr\,d\theta$$

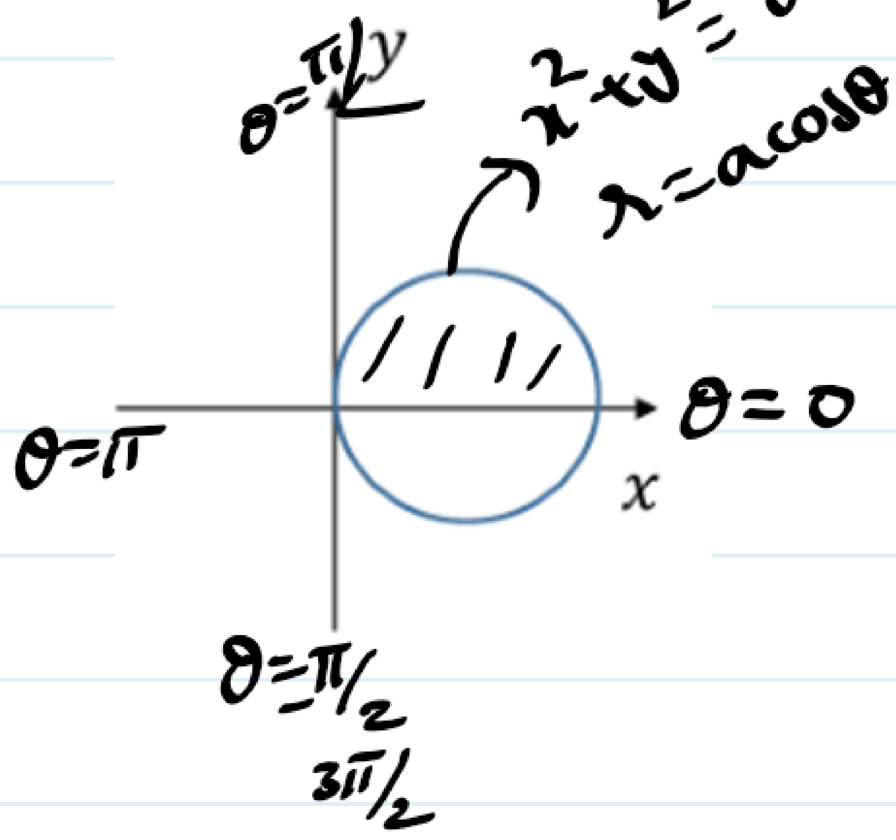
$$=2\int_0^{\pi/2}\left[2\cos\theta\left(\frac{r^3}{3}\right)-\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta$$

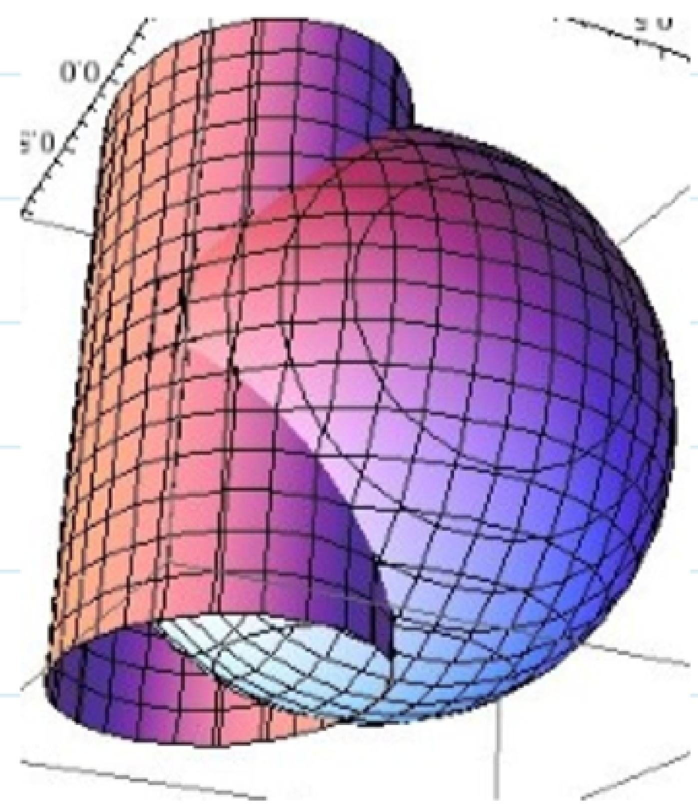
$$=2\int_0^{\pi/2}\frac{16}{3}\cos^4\theta-4\cos^4\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta=\frac{8}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\underline{\underline{\pi/2}}$$

3) Find the volume of the solid cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$.







$$V = 2\iint \int_{z=0}^{\sqrt{a^2-(x^2+y^2)}} dz\,dx\,dy$$

$$= 2\iint \sqrt{a^2-(x^2+y^2)}\;dx\,dy$$

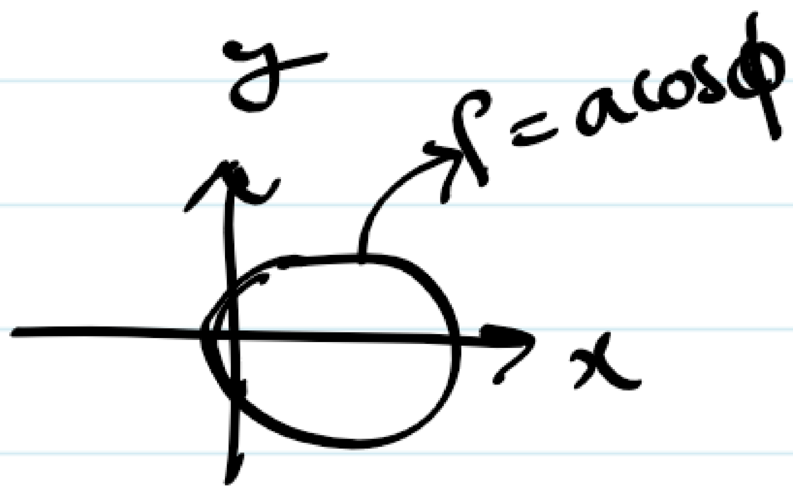
Using polar co-ordinates, $x = r\cos\theta$, $y = r\sin\theta$
$J = r$

$$= 2\times 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$= \frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

Using cylindrical co-ordinates, $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$

$$V = 2\times 2\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a\cos\phi}\int_{z=0}^{\sqrt{a^2-\rho^2}} \rho\,d\rho\,d\phi\,dz$$



Practice questions -

① Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$

(Ans: $1/48$)

② Evaluate $\iiint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}} \, dx\, dy\, dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Ans: $\frac{abc\,\pi^2}{4}$

③ Calculate the volume bounded by the planes $x=0$, $y=0$, $x+y+z=a$ and $z=0$.

Ans: $a^3/6$

④ Find the volume of $x^2+y^2+z^2=a^2$ using spherical polar co-ordinates.

⑤ Find the volume of the cylinder $x^2+y^2=2ax$ intercepted by the paraboloid $x^2+y^2=2az$ and the xy-plane.

Ans: $\frac{3\pi a^3}{4}$