Duplication formula of Gamma functions -

$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$

Proof! We know that, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Put $n=m$ in $\beta(m,n)$, we get,

$$\beta(m,m) = 2\int_0^{\pi/2} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta$$

$2\theta = t$

$\theta = t/2$

$d\theta = \frac{1}{2}\,dt$

$$= \frac{\cancel{2}}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \frac{dt}{\cancel{2}}$$

when $\theta = 0$, $t=0$

$\theta = \pi/2$ $t = \pi$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} dt$$

$\sin(\pi-\theta) = \sin\theta$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x) & \text{if } f(2a-x)=f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$= \frac{1}{2^{2m-1}} \times 2\int_0^{\pi/2} (\sin t)^{2m-1} dt$$

$$\int_0^{\pi} \sin t\,dt = 2\int_0^{\pi/2} \sin t$$

$\because \sin(\pi-\theta) = \sin\theta$

$$\beta(m,m) = \frac{\cancel{2}}{2^{2m-1}} \times \frac{1}{\cancel{2}}\,\beta(m,\tfrac{1}{2})$$

$$\frac{\cancel{\Gamma(m)}\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \, \frac{\cancel{\Gamma(m)}\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$\Rightarrow \boxed{\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

① Prove that $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

Ans: consider, $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

Put $x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$

$dx = \frac{\cos\theta}{2x}\,d\theta$

$= \frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$

$x=0,\ \theta=0$

$x=1,\ \theta=\pi/2$

$$I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{\cancel{\cos^2\theta}}} \times \frac{\cancel{\cos\theta}}{2\sqrt{\sin\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\,d\theta.$$

$$= \frac{1}{2} \times \frac{1}{2}\,\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{\frac{1}{4}\,\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)}$$

$\Gamma(m+1) = m\Gamma(m)$

$$= \frac{1}{4}\,\frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4+1)}$$

$$= \frac{\cancel{1}}{4}\,\frac{\Gamma(3/4)\sqrt{\pi}}{\cancel{1/4}\,\Gamma(1/4)}$$

$$I_1 = \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}$$

Consider, $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

$x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

$dx = \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta$

$$I_2 = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \times \frac{\sec^2\theta \, d\theta}{2\sqrt{\tan\theta}}.$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}} \, d\theta$$

when $x=0$, $\theta=0$

$x=1$, $\theta = \pi/4$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta\cos\theta\left(\frac{2}{2}\right)}} \, d\theta.$$

$$\frac{\sec\theta}{\sqrt{\tan\theta}} = \frac{\frac{1}{\cos\theta}}{\sqrt{\frac{\sin\theta}{\cos\theta}}}$$

$$= \frac{1}{\cancel{\cos\theta}} \times \sqrt{\frac{\cancel{\cos\theta}}{\sin\theta}}$$

$$= \frac{1}{\sqrt{\cos\theta}\sqrt{\sin\theta}}$$

$$= \frac{1}{2} \times \int_0^{\pi/4} \frac{1}{\frac{1}{\sqrt{2}} \times \sqrt{\sin 2\theta}} \, d\theta$$

$$= \frac{1}{\cancel{2}} \times \sqrt{2} \int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}} \, d\theta.$$

$2\theta = t$

$2\,d\theta = dt$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sqrt{\sin t}} \, \frac{dt}{2}$$

when $\theta = 0$, $t=0$

$\theta = \pi/4$, $t = \pi/2$

$$I_2 = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \, dt = \frac{1}{2\sqrt{2}} \times \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{1}{4\sqrt{2}} \times \frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)}$$

$$I_1 \times I_2 = \frac{\cancel{\Gamma(3/4)}\sqrt{\pi}}{\cancel{\Gamma(1/4)}} \times \frac{1}{4\sqrt{2}} \frac{\cancel{\Gamma(1/4)}\sqrt{\pi}}{\cancel{\Gamma(3/4)}} = \frac{\sqrt{\pi}}{4\sqrt{2}}$$

2) Evaluate $\int_0^\infty x\, e^{-x^8}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx$

Ans: Consider,

$$I_1 = \int_0^\infty x e^{-x^8}\,dx$$

Put $x^8 = t$

$x = t^{1/8}$

$dx = \frac{1}{8} t^{-7/8}\,dt$

when $x=0$, $t=0$

$x=\infty$, $t=\infty$

$$= \int_0^\infty t^{1/8} e^{-t} \frac{1}{8} t^{-7/8}\,dt$$

$$= \frac{1}{8}\int_0^\infty e^{-t} t^{-6/8}\,dt$$

$-\frac{6}{8}+1$

$$I_1 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)$$

Consider $I_2 = \int_0^\infty x^2 e^{-x^4}\,dx$

$x^4 = z$

$x = z^{1/4}$

$dx = \frac{1}{4} z^{-3/4}\,dz$

$$= \int_0^\infty (z^{1/4})^2 e^{-z} \frac{1}{4} z^{-3/4}\,dz$$

$$= \frac{1}{4}\int_0^\infty e^{-z} z^{-1/4}\,dz = \frac{1}{4}\Gamma(3/4)$$

$$I_1 \times I_2 = \frac{1}{8}\Gamma(1/4) \times \frac{1}{4}\Gamma(3/4) = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma(3/4)$$

$$= \frac{1}{32}\sqrt{2}\,\pi = \frac{1}{16\times\sqrt{2}\sqrt{2}} \times \sqrt{2}\,\pi = \frac{\pi}{16\sqrt{2}}$$

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma(3/4) = \frac{\pi}{\sin \pi/4} = \sqrt{2}\,\pi$$

(3) S.T $\displaystyle\int_0^\infty \frac{x^2\,dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}$

$x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

$dx = \frac{1}{2\sqrt{\tan\theta}} \times \sec^2\theta\, d\theta.$

$x=0,\ \theta=0$

$x=\infty,\ \theta=\pi/2.$

$$\int_0^\infty \frac{x^4\,dx}{(1+x^4)^3} = \int_0^{\pi/2} \frac{\cancel{\tan}\theta}{\cancel{(1+\tan^2\theta)}^3} \times \frac{\cancel{\sec^2}\theta\, d\theta}{2\cancel{\sqrt{\tan\theta}}}.$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\tan\theta} \times \frac{1}{\sec^4\theta}\, d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{1/2}\theta}{\cos^{1/2}\theta} \times \cos^4\theta\, d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\,\cos^{7/2}\theta\, d\theta.$$

$$= \frac{1}{2} \times \frac{1}{2}\beta\left(\frac{3}{4}, \frac{9}{4}\right) = \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(9/4)}{\Gamma(3)}$$

$$= \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(9/4)}{2!}$$

$$= \frac{1}{8}\Gamma(3/4)\,\Gamma(5/4+1)$$

$$= \frac{1}{8}\Gamma(3/4) \times \frac{5}{4}\Gamma(5/4)$$

$$= \frac{1}{8} \times \frac{5}{4}\Gamma(3/4)\,\Gamma\left(\frac{1}{4}+1\right) = \frac{5}{32} \times \frac{1}{4}\Gamma(3/4)\Gamma(1/4)$$

$$\left(\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}\right)$$

$$= \frac{5}{32} \times \frac{1}{4} \times \sqrt{2}\,\pi = \frac{5\sqrt{2}\pi}{128}$$

4) Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

Ans: consider,

$$I_1 = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \sqrt{2}\pi = \frac{\pi}{\sqrt{2}}.$$

$$\left(\begin{array}{l} \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \\ \Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi \end{array} \right)$$

consider $I_2 = \int_0^{\pi/2} \sqrt{\cot\theta}. = \int_0^{\pi/2} \sqrt{\cot\left(\frac{\pi}{2}-\theta\right)}\, d\theta.$

$$= \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta. \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

$$I_1 \times I_2 = \frac{\pi}{\sqrt{2}} \times \frac{\pi}{\sqrt{2}} = \frac{\pi^2}{2}$$

5) Evaluate $\int_3^7 (x-3)^{1/4}(7-x)^{1/6}\,dx$

$$\left[\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n) \quad \text{by putting } x-a=(b-a)t\right]$$

$$I_1 = \int_0^1 (4t)^{1/4}(4-4t)^{1/6}\,4\,dt$$

$x-3=(7-3)t$

Put $x-3=4t$

$dx = 4\,dt$.

$x=3,\ t=0$

$x=7,\ t=1$

$$= 4^{1+\frac{1}{4}+\frac{1}{6}} \int_0^1 t^{1/4}(1-t)^{1/6}\,dt$$

$$= 4^{17/12}\,\beta\left(\frac{5}{4},\frac{7}{6}\right)$$

$$= 4^{17/12}\,\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{5}{4}+\frac{7}{6}\right)} = 4^{17/12}\times\frac{\Gamma\left(\frac{1}{4}+1\right)\Gamma\left(\frac{1}{6}+1\right)}{\Gamma\left(\frac{17}{12}+1\right)}$$

$$= 4^{17/12}\,\frac{\frac{1}{4}\Gamma(1/4)\,\frac{1}{6}\Gamma(1/6)}{\frac{17}{12}\Gamma(17/12)}$$

$$= 4^{17/12}\times\frac{1}{24}\times\frac{12}{17}\,\frac{\Gamma(1/4)\Gamma(1/6)}{\Gamma\left(\frac{5}{12}+1\right)}$$

$$= 4^{17/12}\times\frac{1}{12\times17}\times\frac{12}{5}\,\frac{\Gamma(1/4)\Gamma(1/6)}{\Gamma(5/12)}$$

6) Evaluate $\int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx$

$e^{ix} = \cos x + i\sin x$

$e^{ibx} = \cos bx + i \sin bx$

$\sin bx = \text{Im. part of } e^{ibx}$

$$I = \int_0^\infty e^{-ax} x^{m-1} \left(\text{Im. P } e^{ibx}\right) dx$$

$$= \text{Im. P} \int_0^\infty e^{-ax} \cdot e^{ibx} x^{m-1} dx$$

$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n)$

$$= \text{Im. P} \int_0^\infty e^{-(a-ib)x} x^{m-1} dx$$

$\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma(n)}{k^n}$

$$= \text{Im. P.} \frac{\Gamma(m)}{(a-ib)^m}$$

$a = r\cos\theta$

$b = r\sin\theta$.

$$= \text{Im. P} \frac{\Gamma(m)}{r^m [\cos\theta - i\sin\theta]^m}$$

$$= \text{Im. P.} \frac{\Gamma(m)}{r^m} \times \frac{1}{e^{-im\theta}} = \text{Im. P} \frac{\Gamma(m)}{r^m} \cdot e^{im\theta}.$$

$$= \text{Im. Part.} \frac{\Gamma(m)}{r^m} \left(\cos m\theta + i \sin m\theta\right)$$

$$= \frac{\Gamma(m)}{r^m} \sin m\theta.$$

7) Evaluate $\int_0^2 x(8-x^3)^{1/3}\,dx$

$$x^3 = 8t \qquad x = 2t^{1/3}$$

$$3x^2\,dx = 8\,dt$$

$$x\,dx = \frac{8}{3x}\,dt = \frac{8}{3\times 2t^{1/3}}\,dt = \frac{8}{6t^{1/3}}\,dt$$

Ans:

$$I = \int_0^1 (8-8t)^{1/3} \times \frac{8}{6} t^{-1/3}\,dt$$

$$= \frac{8}{6} \times 8^{1/3} \int_0^1 (1-t)^{1/3} t^{-1/3}\,dt$$

$$= \frac{16}{6} \int_0^1 t^{-1/3} (1-t)^{1/3}\,dt$$

$$\Gamma(n+1) = n!$$

$$= \frac{8}{3}\,\beta\left(\tfrac{2}{3}, \tfrac{4}{3}\right)$$

$$= \frac{8}{3} \times \frac{\Gamma(2/3)\,\Gamma(4/3)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} = \frac{8}{3} \times \frac{\Gamma(2/3)\,\Gamma(1/3+1)}{\Gamma(2)}$$

$$= \frac{8}{3} \times \frac{1}{3} \times \frac{\Gamma(2/3)\,\Gamma(1/3)}{1}$$

$$= \frac{8}{9} \times \frac{2\pi}{\sqrt{3}} = \underline{\underline{\frac{16\pi}{9\sqrt{3}}}}$$

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$$

$$\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(1-\tfrac{1}{3}\right) = \frac{\pi}{\sin \pi/3} = \frac{2\pi}{\sqrt{3}}$$

Practice Questions –

Using beta/gamma functions, evaluate the following integrals:

① $\displaystyle\int_0^\infty \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx$ (Ans: $\frac{3\pi}{2^{13/4}}$)

② $\displaystyle\int_0^\infty \sqrt{x}\,e^{-x^2}\,dx \times \frac{1}{\sqrt{x}}\,e^{-x^2}\,dx$ (Ans: $\frac{\pi}{2\sqrt{2}}$)

③ $\displaystyle\int_0^\infty x^{m-1}\sin x\,dx$ Ans :- $\Gamma(m)\sin\left(\frac{m\pi}{2}\right)$