(i) $x_0 = 0.1$, $x = 0.12$, $h = 0.05$, $p = \frac{0.12 - 0.1}{0.05} = 0.4$

$\tan 0.12 = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots$

$= 0.1003 + (0.4)(0.0508) + \frac{(0.4)(-0.6)}{2}(0.0008) + \frac{(0.4)(-0.6)(-1.6)}{6}(0.0002)$
$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(0.0002)$

$= 0.12052848$ Actual value: $\tan 12 = 0.120579$

(ii) $x_n = 0.3$, $x = 0.26$, $p = \frac{x - x_n}{h} = \frac{0.26 - 0.3}{0.05} = -0.8$

$\tan 26 = 0.3093 + (-0.8)(0.0540) + \frac{(-0.8)(0.2)}{2}(0.0014) + \frac{(-0.8)(0.2)(1.2)}{6}(0.0004)$
$+ \frac{(-0.8)(0.2)(1.2)(2.2)}{24}(0.0002)$

$= 0.265971$

Actual Value $\tan 0.26 = 0.266021$

(iii) $x_n = 0.3$, $x = 0.4$, $p = \frac{0.4 - 0.3}{0.05} = 2$

$\tan 0.4 = 0.3093 + 2(0.0540) + \frac{2\times 3}{2}(0.0014) + \frac{2\times 3\times 4}{6}\times(0.0004) + \frac{2\times 3\times 4\times 5}{24}(0.0002)$

$= 0.4241$ Actual value $\tan 0.4 = 0.42279$

(iv) $x_n = 0.3$, $x = 0.5$ $p = \frac{0.5 - 0.3}{0.05} = 4$

$\tan 0.5 = 0.3093 + 4(0.0540) + \frac{4\times 5}{2}(0.0014) + \frac{4\times 5\times 6}{6}(0.0004) + \frac{4\times 5\times 6\times 7}{24}(0.0002)$

$= 0.5427$ Actual value = $\tan 0.5 = 0.5463$

(5) From the following table find the number of students who obtained less than 45 marks.

Marks :	<40	40-50	50-60	60-70	70-80
Number of students :	31	42	51	35	31

Marks $<x$	Number of students	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

At $x = 45$, $p = \frac{45 - 40}{10} = 0.5$

$y = 31 + (0.5)(42)$
$+ \frac{(0.5)(-0.5)}{2}9 + \ldots + \ldots$

$= 47.86 \approx 48$

$\therefore$ Number of students with marks <45 is 48.

Note: No. of students scored between 40 & 45 $= 48 - 31 = 17$.

③ value of x (in degrees) and $\sin x$ are given in the following table:

x :	15	20	25	30	35	40
$\sin x$:	0.2588190	0.3420201	0.4226183	0.5	0.5735764	0.6427876

find $\sin 38^\circ$

Solution:

x	$\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	0.2588190					
		0.0832011				
20	0.3420201		−0.0026029			
		0.0805982		−0.0006136		
25	0.4226183		−0.0032165		0.0000248	
		0.0773817		−0.0005888		0.0000041
30	0.5		−0.0038053		0.0000289	
		0.0735764		−0.0005599		
35	0.5735764		−0.0043652			
		0.0692112				
40	0.6427876					

$x_n = 40$, $h = 5$ $\quad x = 38^\circ$ $\quad \therefore p = \dfrac{x - x_n}{h} = \dfrac{38-40}{5} = -0.4$

$$y = 0.6427876 + (-0.4)(0.0692112) + \frac{(-0.4)(0.6)}{2!}(-0.0043652) + \frac{(-0.4)(0.6)(1.6)}{3!}(-0.0005599)$$
$$+ \frac{(-0.4)(0.6)(1.6)(2.6)}{4!}(0.0000289) + \frac{(-0.4)(0.6)(1.6)(2.6)(3.6)}{5!}(0.0000041)$$
$$= 0.6427876 - 0.02768448 + 0.000523824 + 0.0000358336 - 0.0000120224$$
$$- 0.000000128232$$

$\sin 38^\circ = 0.6156614526$

From the calculator, $\sin 38^\circ = 0.61566147532$

④ Table gives the values of $\tan x$ for $0.1 \le x \le 0.3$

x :	0.10	0.15	0.20	0.25	0.30
$\tan x$:	0.1003	0.1511	0.2027	0.2553	0.3093

Find (i) $\tan 0.12$ (ii) $\tan 0.26$ (iii) $\tan 0.4$ (iv) $\tan 0.5$

Solution:

x	$\tan x$	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

② If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$, $f(1.30) = 1.1401$, find $f(1.18)$ and $f(1.28)$

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 1.15$	1.0723			
		0.0231		
1.20	1.0954		-0.0005	
		0.0226		0
1.25	1.1180		-0.0005	
		0.0221		
$x_n = 1.30$	1.1401			

To find $f(1.18)$:

$$x_0 = 1.15, \quad h = 0.05, \quad p = \frac{x - x_0}{h} = \frac{1.18 - 1.15}{0.05} = 0.6$$

$$y_n = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$= 1.0723 + (0.6)(0.0231) + \frac{(0.6)(-0.4)}{2!}(-0.0005)$$

$$= 1.0723 + 0.01386 + 0.00006$$

$$= 1.08622$$

To find $f(1.28)$:

$$x_n = 1.30, \quad x = 1.28, \quad h = 0.05 \qquad p = \frac{x - x_n}{h} = \frac{1.28 - 1.30}{0.05} = -0.4$$

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$$

$$= 1.1401 + (-0.4)(0.0221) + \frac{(-0.4)(0.6)}{2!}(-0.0005)$$

$$= 1.13132$$

Note: We can use Forward difference formula to find $f(1.28)$

$$x_0 = 1.15, \quad x = 1.28, \quad h = 0.05 \qquad p = \frac{x - x_0}{h} = \frac{1.28 - 1.15}{0.05} = 2.6$$

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$= 1.0723 + (2.6)(0.0231) + \frac{(2.6)(1.6)}{2!}(-0.0005)$$

$$= 1.0723 + 0.06006 - 0.00104$$

$$= 1.13132$$

(Newton–Gregory)

Note: Given $(n+1)$ points, we can find a polynomial of degree atmost n.

Newton's Forward Difference Formula:-

Given a set of points $(x_0,y_0), (x_1,y_1), \ldots, (x_n,y_n)$ satisfying $y=f(x)$ where the explicit nature of $f(x)$ is not known, the n^{th} degree polynomial $y_n(x)$ such that $y_n(x)$ and $f(x)$ agree at the tabulated points is given by
(at other points $y_n(x)$ is a approximation for $f(x)$)

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n y_0$$

where $x = x_0 + ph$.

Here, the values of x are equally spaced. i.e. $x_i = x_0 + ih$, $i=1,2,..,n$, $h>0$.

Proof:
$$y_n(x) = f(x_0+ph) = E^p f(x_0)$$
$$= (1+\Delta)^p y_0$$
$$= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right) y_0$$
$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0$$

Newton's Backward Difference Formula:

Given a set of points $(x_0,y_0), (x_1,y_1), \ldots, (x_n,y_n)$ satisfying $y=f(x)$, where $f(x)$ is not known explicitly, values of x are equally spaced, the n^{th} degree polynomial $y_n(x)$ such that $y_n(x)$ and $f(x)$ agree at the tabulated points, is given by

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n y_n$$

where $x = x_n + ph$.

Example 1: Find the cubic polynomial which takes the following values:
$y(1)=24$, $y(3)=120$, $y(5)=336$, $y(7)=720$. Hence or otherwise find $y(8)$.

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0=1$	$y_0=24$			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

$h = 3-1 = 5-3 = 7-5 = 2$

$$p = \frac{x-x_0}{h} = \frac{x-1}{2}$$

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$y = 24 + \left(\frac{x-1}{2}\right)96 + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2!}120 + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{3!}(48)$$

$$= 24 + 48(x-1) + 15(x-1)(x-3) + (x-1)(x-3)(x-5)$$

$$= 24 + 48x - 48 + 15x^2 - 60x + 45 + x^3 - 9x^2 + 23x - 15$$

$$y = x^3 + 6x^2 + 11x + 6$$

$$y(8) = 8^3 + 6\times 8^2 + 11\times 8 + 6 = 990$$

$y(1) = 1+6+11+6 = 24$ ✓
$y(3) = 27+54+33+6 = 120$ ✓
$y(5) = 125+150+55+6 = 336$, $y(7) = 720$ ✓