$\lambda = 1, \quad c = 1, \quad h = 1/4$

$$\frac{kc}{h^2} = \lambda \Rightarrow \quad 16k = 1 \Rightarrow k = 1/16$$

$t_1 = 1/16$

$4a - b = 25 \longrightarrow (1)$

$-2a + 4b = 37.5$

$-4a + 8b = 75$

$7b = 100 \Rightarrow b = \frac{100}{7}$

$a =$

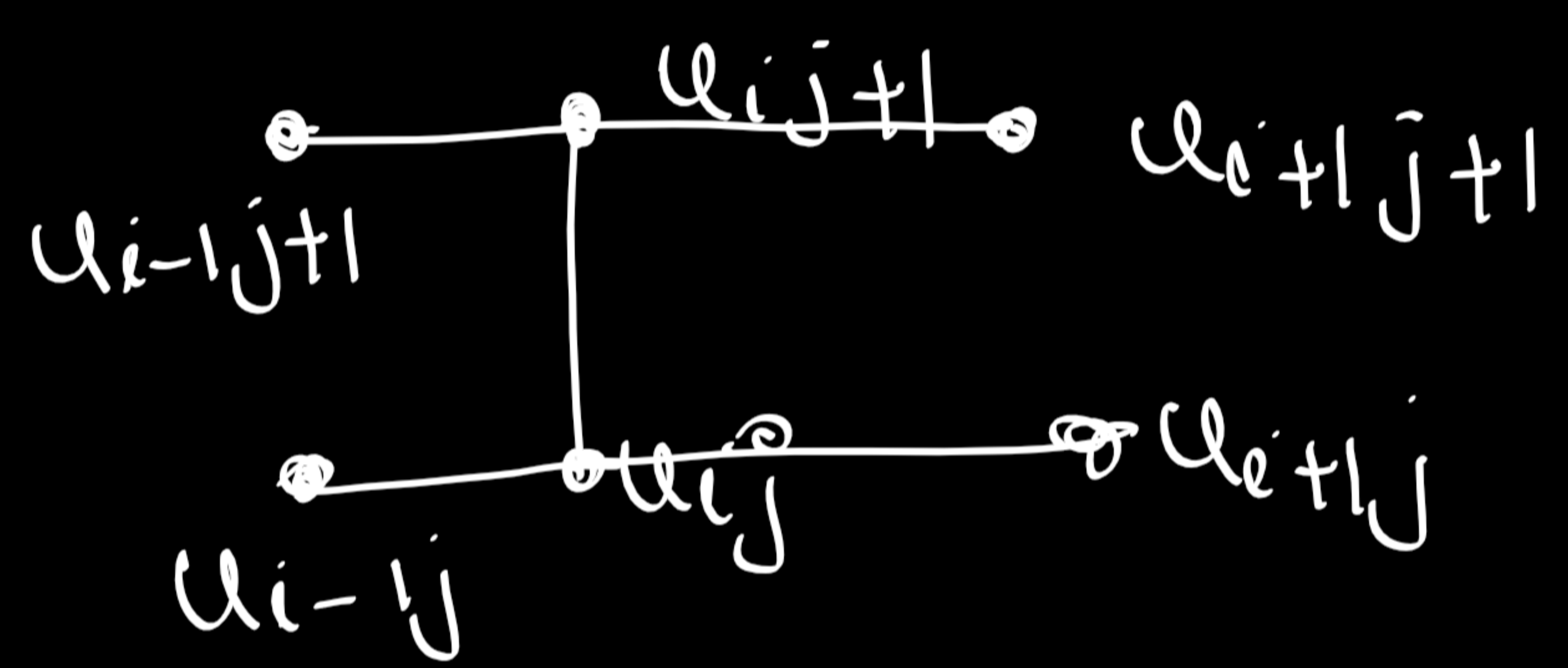
$$-\lambda u_{i-1\,j+1} + 2(1+\lambda)u_{i\,j+1} - \lambda u_{i+1\,j+1}$$

$$= \lambda u_{i-1,j} + 2(1-\lambda)u_{ij} + \lambda u_{i+1,j}$$

$\longrightarrow$ CRANK- NICOLSON's implicite finite difference formula.

Put $\lambda = 1$

$$-u_{i-1\,j+1} + 4u_{i\,j+1} - u_{i+1\,j+1} = u_{i-1\,j} + u_{i+1\,j}$$

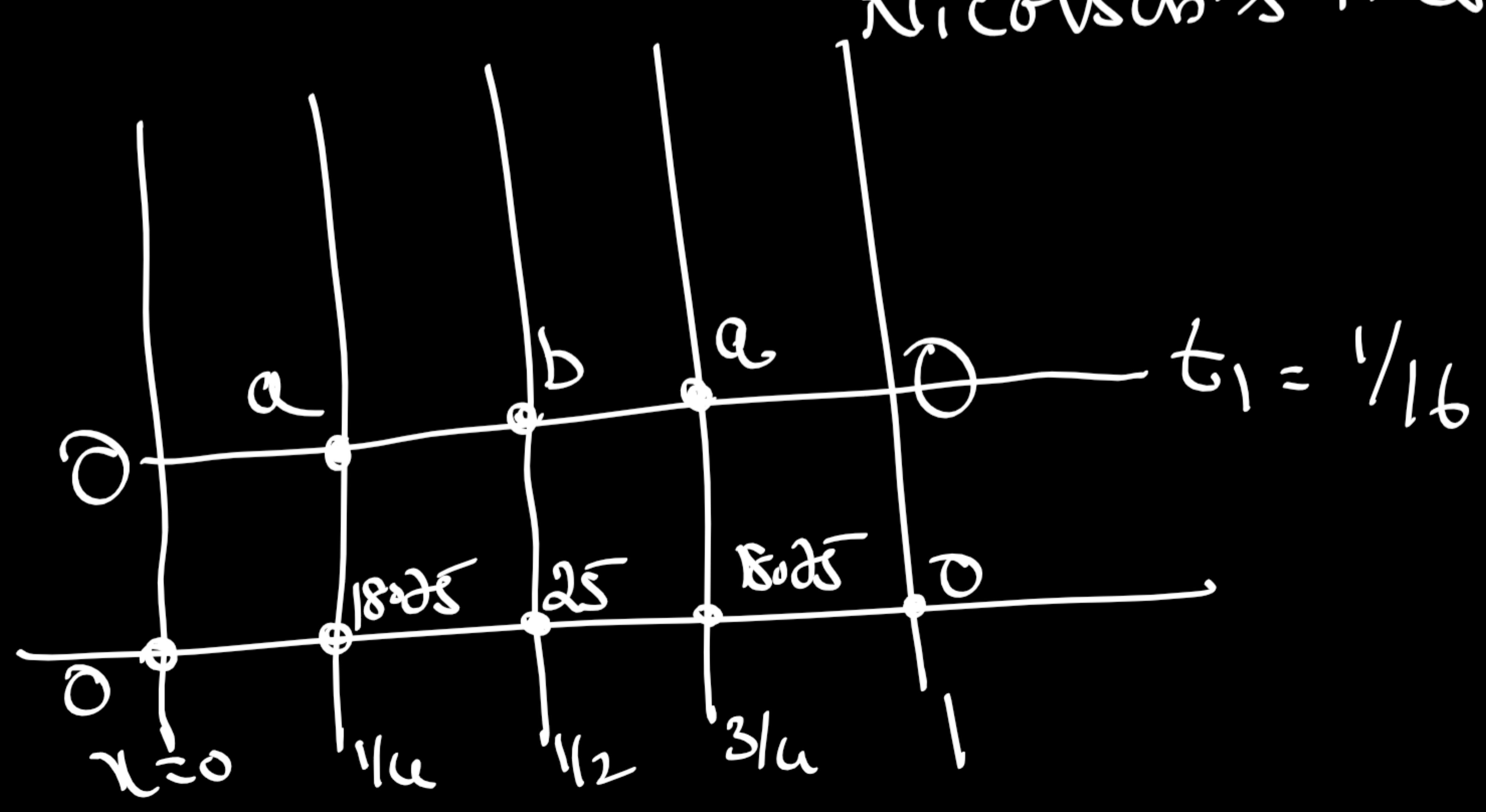


Examples

1. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$, $h = 1/4$

$u(x,0) = 100(x - x^2)$, $u(0,t) = u(1,t) = 0$

compute $u(x,t)$ for one time step using Crank-Nicolson's method.



Crank – Nicolson's Method

To solve: $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$

By Schmidt's formula

$$\frac{u_{i,j+1} - u_{ij}}{k} = \frac{c}{h^2}\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right] \longrightarrow (1)$$

LHS is the forward difference approximation to $\frac{\partial u}{\partial t}$. If backward difference is used we get,

$$\frac{u_{ij} - u_{i,j-1}}{k} = \frac{c}{h^2}\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right]$$

Replace j by j+1, we get

$$\frac{u_{i,j+1} - u_{ij}}{k} = \frac{c}{h^2}\left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] \longrightarrow (2)$$

$\frac{(1)+(2)}{2} \Rightarrow$

$$\frac{u_{i,j+1} - u_{ij}}{k} = \frac{c}{2h^2}\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right]$$

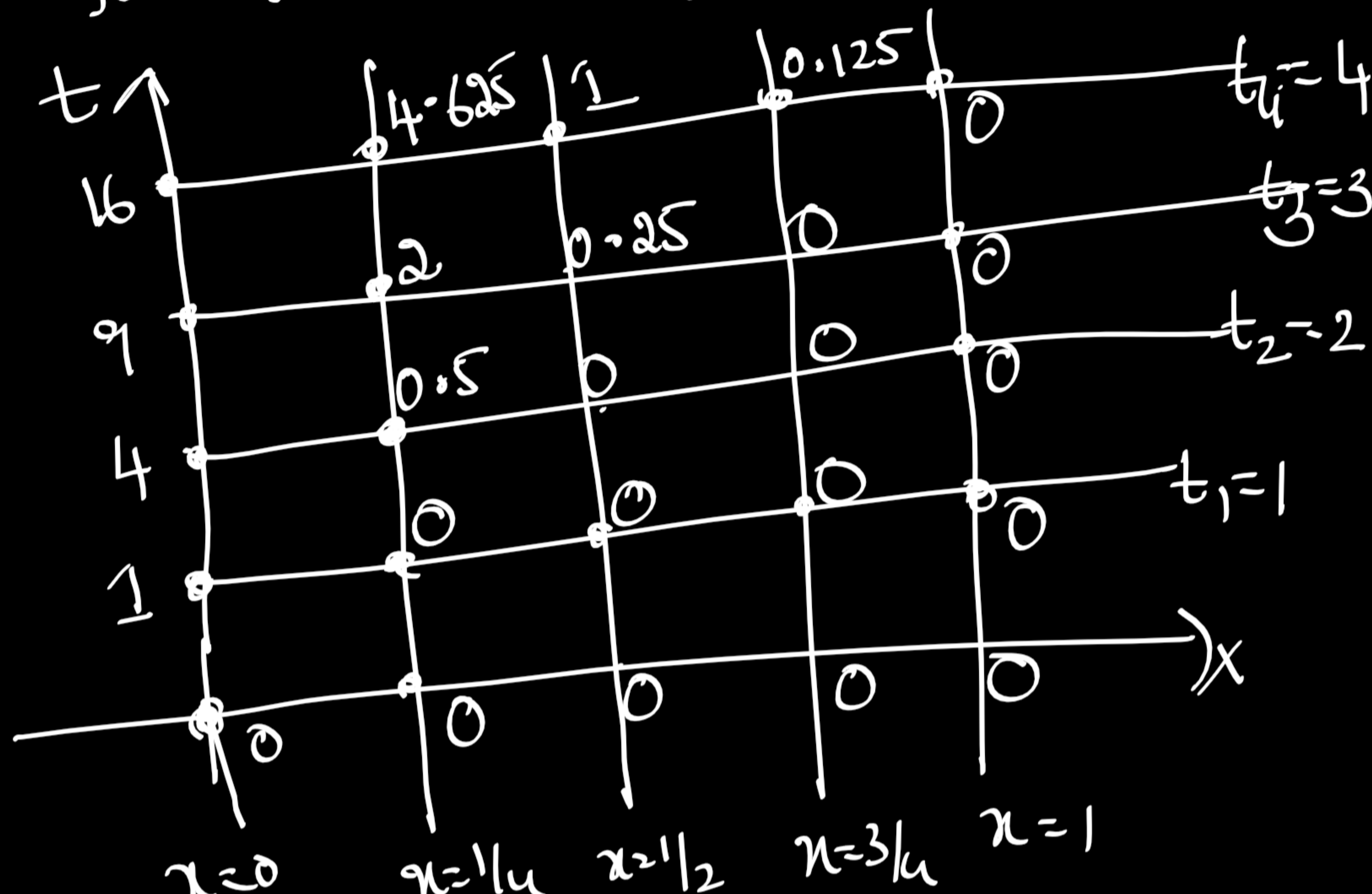
Let $\frac{kc}{h^2} = \lambda$

$$2u_{i,j+1} - 2u_{ij} = \lambda\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right]$$

2. $\dfrac{\partial u}{\partial t} = \dfrac{1}{32}\dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad h = 1/4.$

$u(x,0) = 0, \quad u(0,t) = t^2, \quad u(1,t) = 0$

Compute $u(x,t)$ for 4 time steps.



$t_j = jk, \quad j = 1, 2, 3, 4$

$\dfrac{kc}{h^2} = \lambda$

$\dfrac{k \times \frac{1}{32}}{1/16} = 1/2 \Rightarrow k = 1$

Examples

1. Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$, $h = 1/4$

$u(x,0) = 100(x - x^2)$, $u(0,t) = u(1,t) = 0$

Compute $u(x,t)$ for 4 time steps.

$t_j = jk$, $j = 1, 2, 3, 4$

$\frac{kc}{h^2} = \lambda$

Let $\lambda = 1/2$

$c = 1$, $h = 1/4$

$\therefore k = 1/32$

$\therefore t_1 = \frac{1}{32}$, $t_2 = 2 \times \frac{1}{32} = \frac{1}{16}$, $t_3 = 3/32$, $t_4 = 1/8$

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$ becomes

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = \frac{c}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

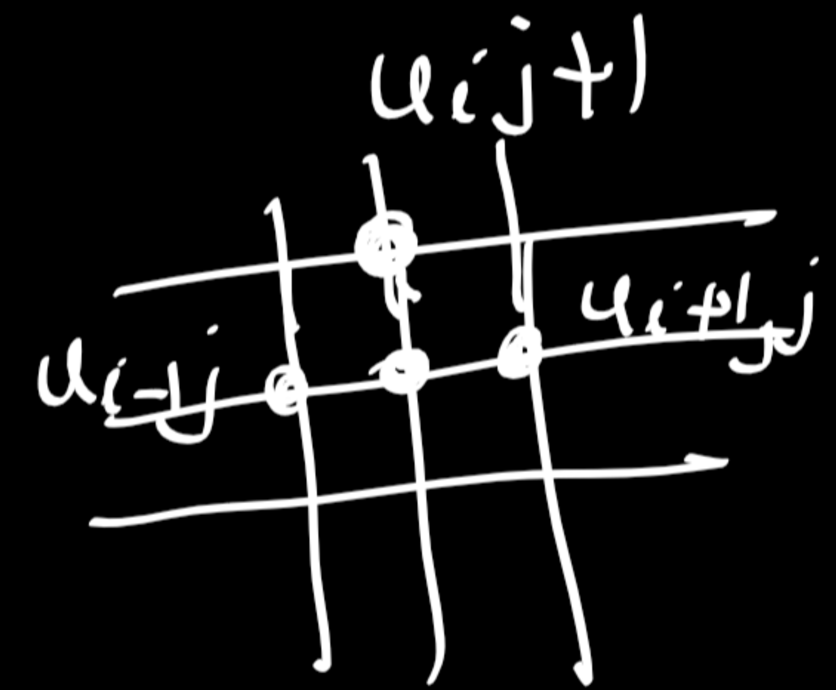
Let $\frac{kc}{h^2} = \lambda$

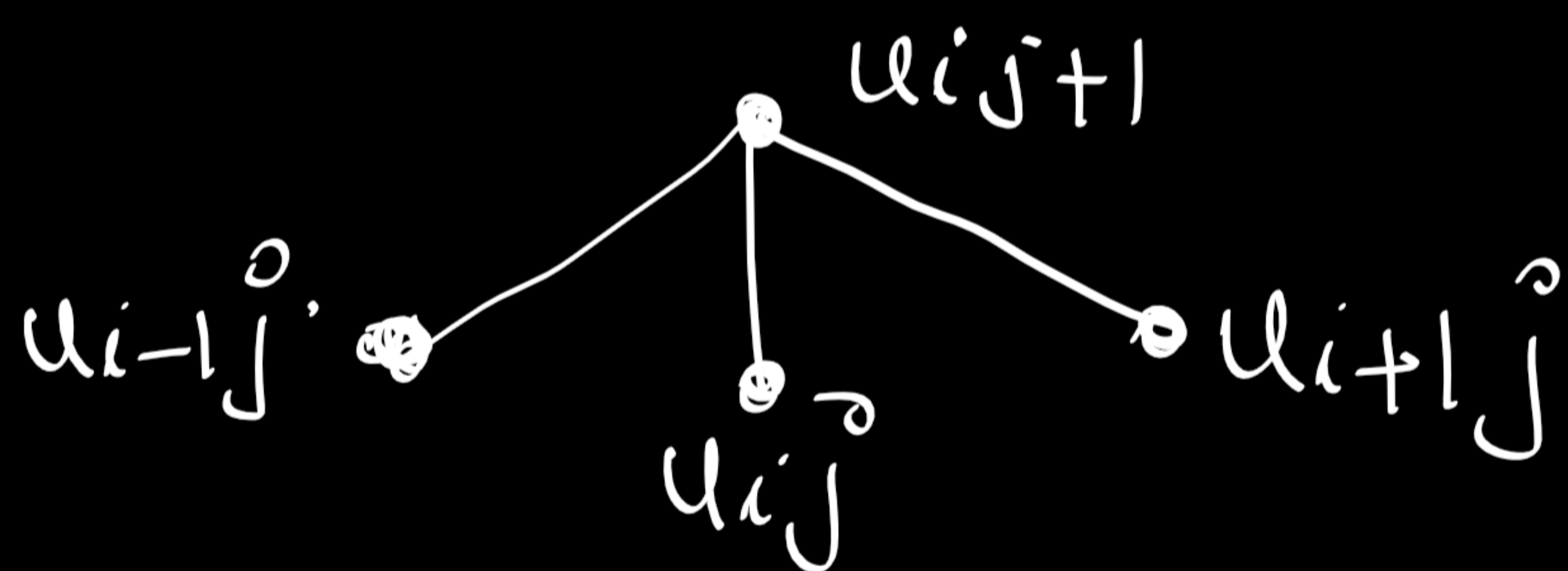
$$\therefore u_{i,j+1} = \lambda\left[u_{i+1,j} + u_{i-1,j}\right] + (1-2\lambda)u_{i,j}$$

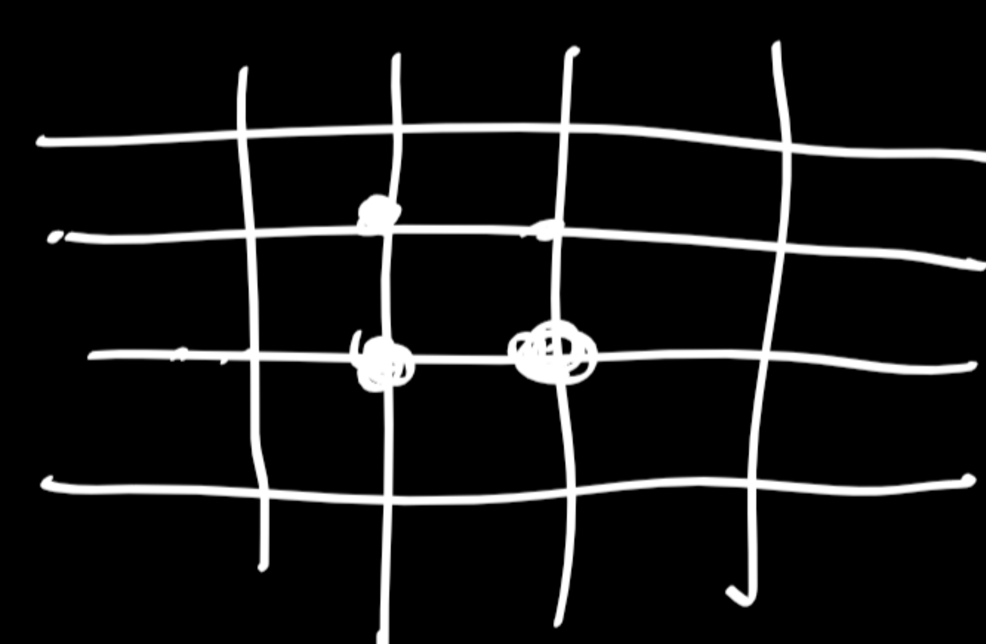
→ Schmidt's explicit finite difference formula.

If we take $\lambda = \frac{1}{2}$

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$$







Heat Equation (Parabolic eqn)

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}, \quad a<x<b,\ t>0$$

$u(x,0) = f(x)$
(Initial condition)

$u(a,t) = \varphi(t), \quad u(b,t) = \psi(t), \quad t>0$

Divide $[a,b]$ into n equal parts each of length h.

Let $x_i = a + ih, \quad i = 0, 1, 2, \dots n$

Let k be the time interval size and $t_j = jk$, $j = 1, 2, 3, \dots$

Put $u(x_i, t_j) = u_{ij}$

By Taylor's series

$$u_{i,j+1} = u(x_i, t_j + k)$$
$$= u_{ij} + k\frac{\partial u_{ij}}{\partial t} + \dots$$

$$\frac{\partial u_{ij}}{\partial t} = \frac{1}{k}\left[u_{i,j+1} - u_{ij}\right]$$

Also $$\frac{\partial^2 u_{ij}}{\partial x^2} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right]$$

$$c = \frac{1}{4}\left[2b + 200 - \frac{1}{9}\left(-81 \times \frac{2}{3} \times \frac{2}{3}\right)\right]$$

$$-2b + 4c = 204 \longrightarrow (3)$$

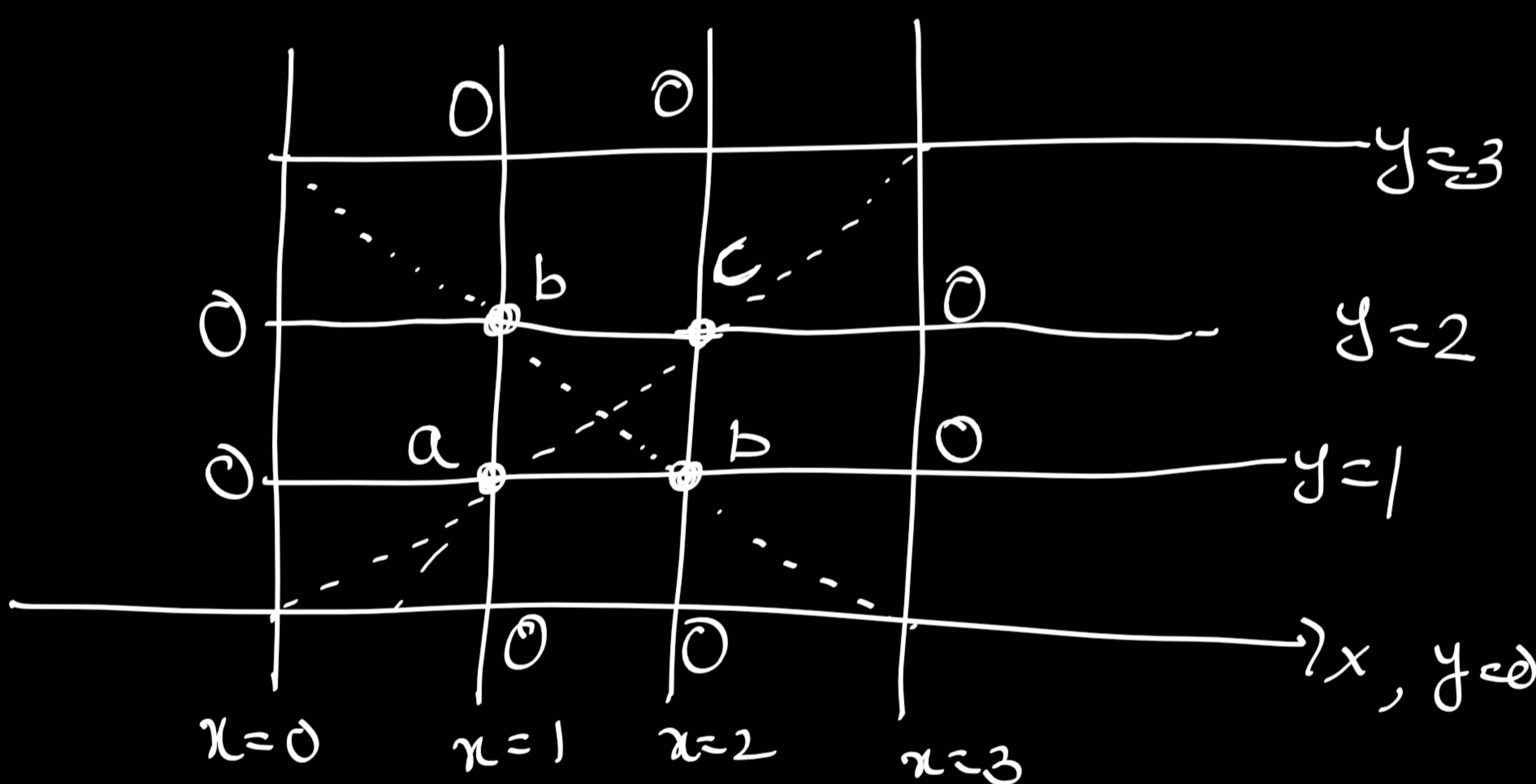
On solving (1), (2) and (3), we get

$$a = 25.8, \quad \underline{\underline{b = 51.08}}, \quad c = 76.54$$

2) solve: $u_{xx} + u_{yy} = -10(x^2 + y^2 + 10)$, $0 < x < 3$, $0 < y < 3$

$h = 1$

$u(x, y) = 0$ on the boundary.



$$a = \frac{1}{4}\left[2b + 120\right] \Rightarrow \quad 4a - 2b = 120 \longrightarrow (1)$$

$$b = \frac{1}{4}\left[a + c + 150\right] \Rightarrow \quad -a + 4b - c = 150 \longrightarrow (2)$$

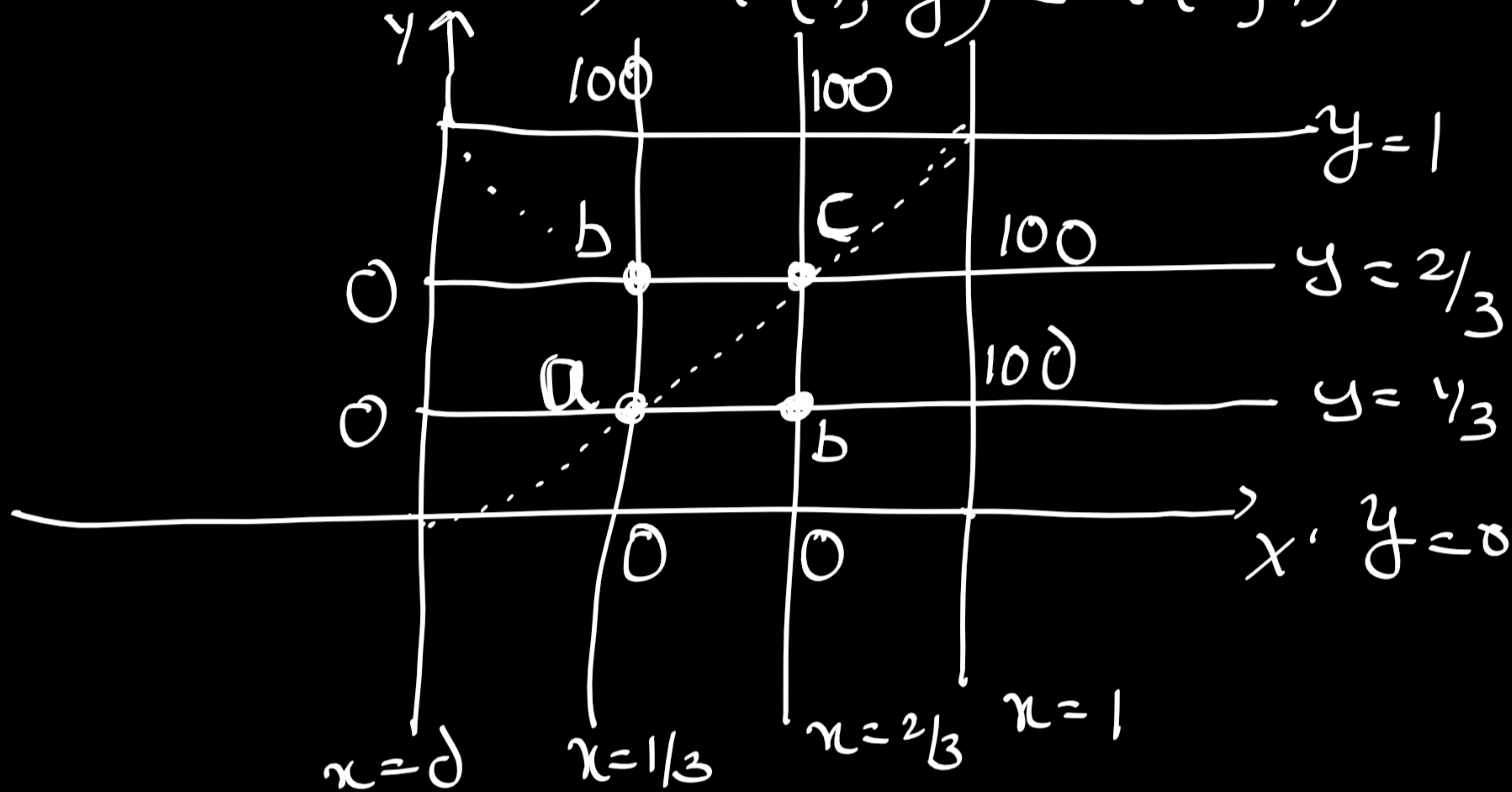
$$-2b + 4c = 180 \longrightarrow (3)$$

$$c = \frac{1}{4}\left[2b + 180\right] \Rightarrow \quad a = 67.5, \ b = 75, \ \underline{\underline{c = 82.5}}$$

1) solve: $u_{xx} + u_{yy} = -81xy, \; 0 < x < 1, \; 0 < y < 1,$

$h = 1/3$

$u(0, y) = u(x, 0) = 0, \quad u(1, y) = u(x, 1) = 100$



$a = u(1/3, 1/3), \quad b = u(2/3, 1/3) = u(1/3, 2/3)$

$c = u(2/3, 2/3)$

$$u_{ij} = \frac{1}{4}\left[u_{i+1\,j} + u_{i-1\,j} + u_{i\,j+1} + u_{i\,j-1} - h^2 f_{ij}\right]$$

$$a = \frac{1}{4}\left[2b - \frac{1}{9}\left(-81 \times \frac{1}{3} \times \frac{1}{3}\right)\right]$$

$$\therefore 4a - 2b = 1 \longrightarrow (1)$$

$$b = \frac{1}{4}\left[100 + a + c - \frac{1}{9}\left(-81 \times \frac{2}{3} \times \frac{1}{3}\right)\right.$$

$$-a + 4b - c = 2 \longrightarrow (2)$$