8. $32 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$

$u(x,0) = 0$, $u(0,t) = 100 \sin\left(\frac{\pi t}{6}\right)$, $u(1,t) = 0$.

Take $h = 1/4$.

(i) Compute $u(x,t)$ for 4 time steps by Schmidt's method

(ii) Compute $u(x,t)$ for one time step by Crank-Nicolson method.

9. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$

$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$

$u(0,t) = 0$, $u(1,t) = 100 \sin \pi t$.

Compute $u(x,t)$ for 4 time steps with $h = 0.25$

10. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$

$u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 100(2x - x^2)$

$u(0,t) = u(2,t) = 0$

With $h = 1/2$ compute $u(x,t)$ for 4 time steps.

Exercise

solve:

1. $y'' + y = x, \quad y(0) = 0, \quad y(1) = 2, \quad h = 0.25$

2. $y'' + xy' + y = 3x^2 + 2, \quad y(0) = 0, \quad y(1) = 1, \quad h = 0.25$

3. $xy'' + y = 0, \quad y'(1) = 0, \quad y(2) = 1, \quad h = 1/2$

4. $y'' - 2x^2 y' + 2y = 0, \quad y(0) + y'(0) = 5, \quad y(1) = 0, \quad h = 1/2$

5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -1, \quad |x| < 1, \quad |y| < 1, \quad h = \frac{1}{2},$

   $u(\pm 1, y) = u(x, \pm 1) = 0$

6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 4, \quad 0 < y < 4, \quad h = 1.$

   $u(x, 0) = x^2 + 2x, \quad u(0, y) = -2y - y^2$

   $u(x, 4) = x^2 + 2x - 24, \quad u(4, y) = 24 - y^2 - 2y$

7. $\dfrac{\partial u}{\partial t} = \dfrac{1}{16} \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad h = 1/4$

   $u(x, 0) = 100 \sin \pi x, \quad u(0, t) = u(1, t) = 0$

   Compute (i) $u(x, t)$ for four time steps by Schmidt's method

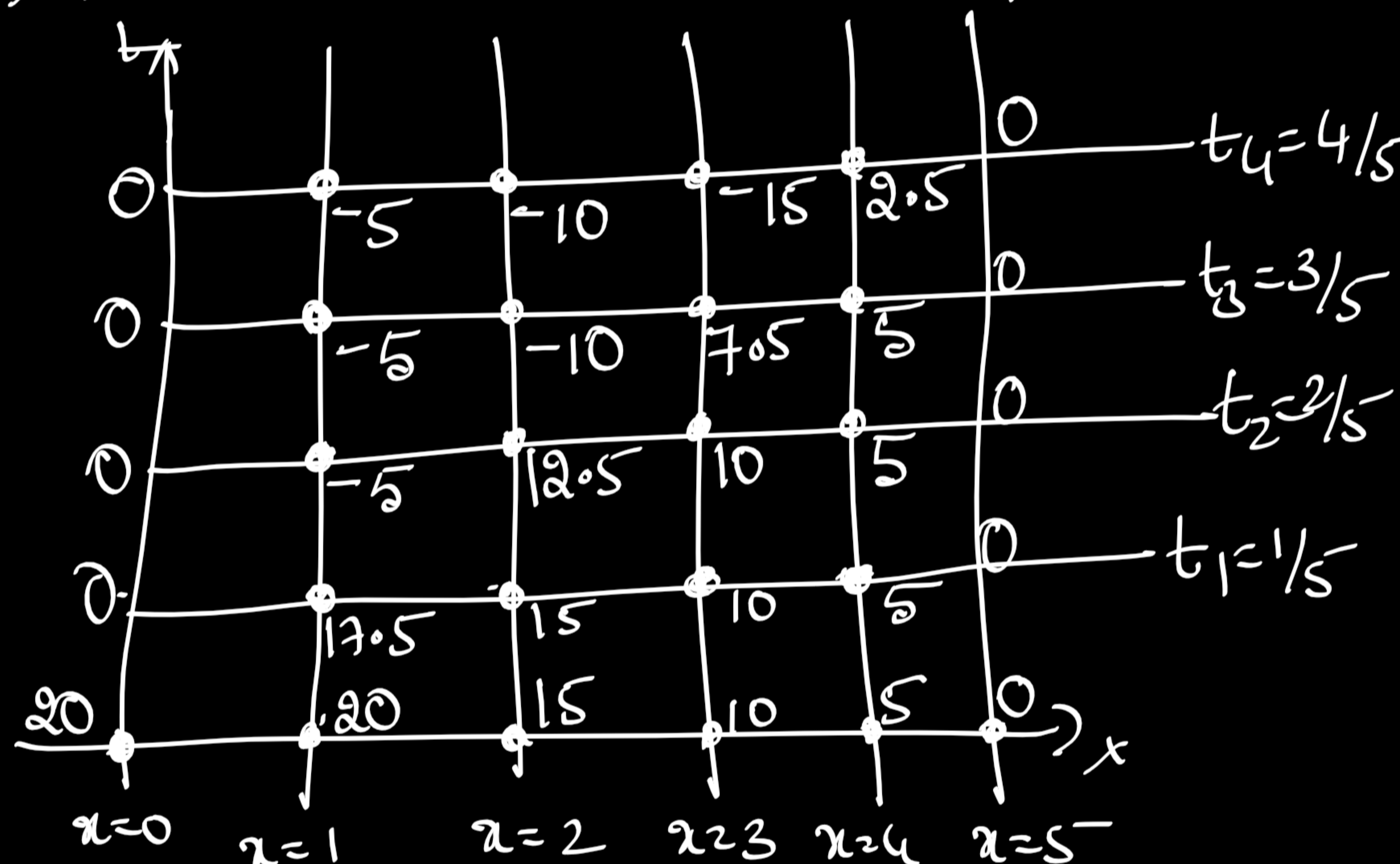
   (ii) $u(x, t)$ for one time step by Crank–Nicolson method.

2. $$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 5, \ t > 0$$

$$u(x,0) = \begin{cases} 20x, & 0 \le x < 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad u(0,t) = u(5,t) = 0, \quad h = 1$$

Compute $u(x,t)$ for four time steps.



$$c^2 = 25 \Rightarrow c = 5$$

$$k = h/c = 1/5$$

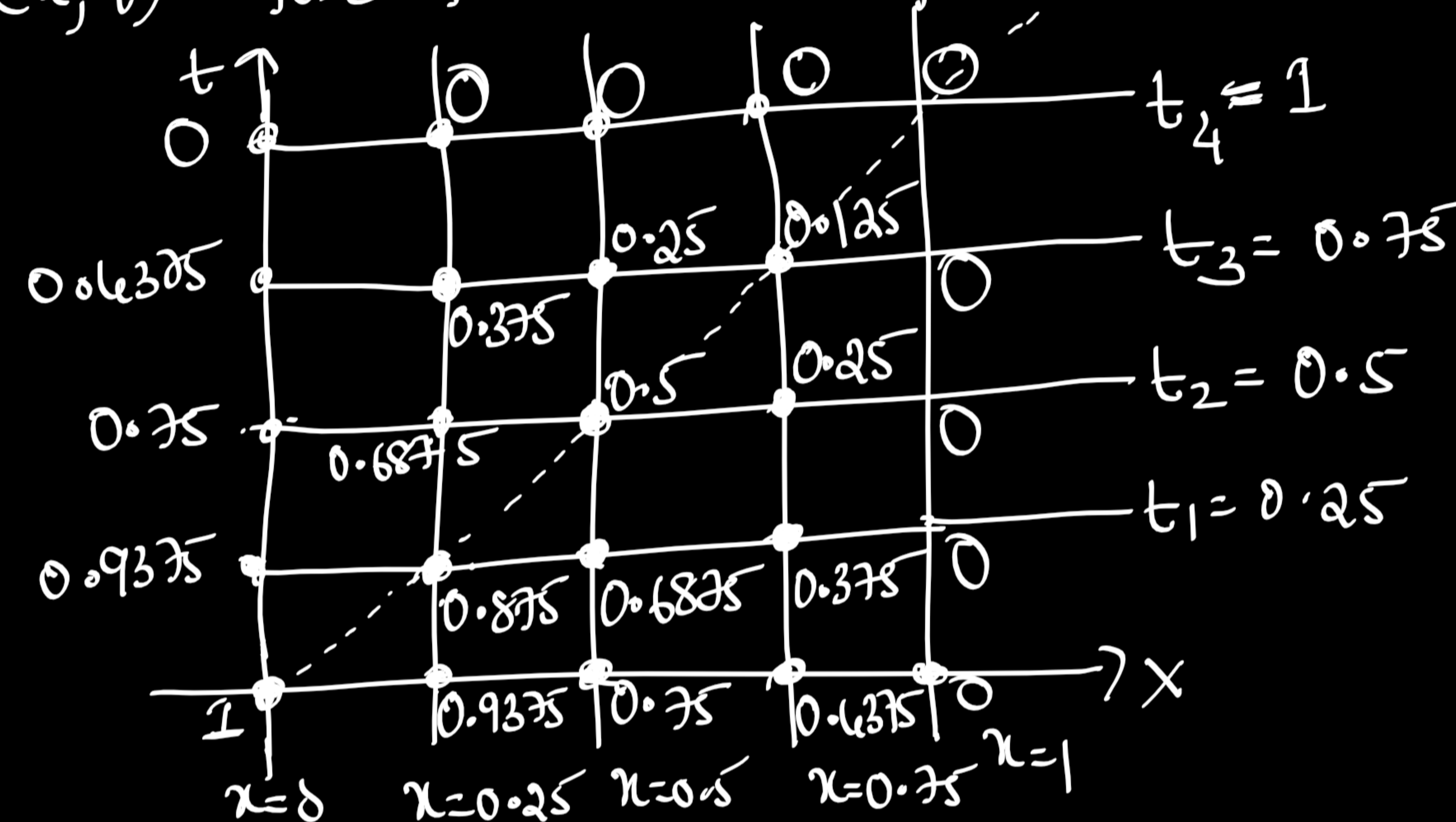
Examples

1. Given $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$

$u(x,0) = 1 - x^2$, $\dfrac{\partial u}{\partial t}(x,0) = 0$,

$u(0,t) = 1 - t^2$, $u(1,t) = 0$, $h = 0.25$

Compute $u(x,t)$ for four time steps.

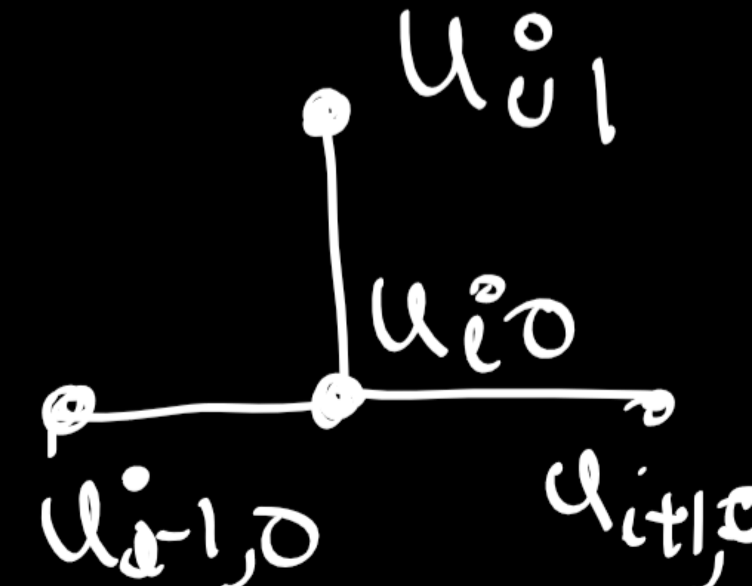


$t_j = jk$, $j = 1, 2, 3, 4$

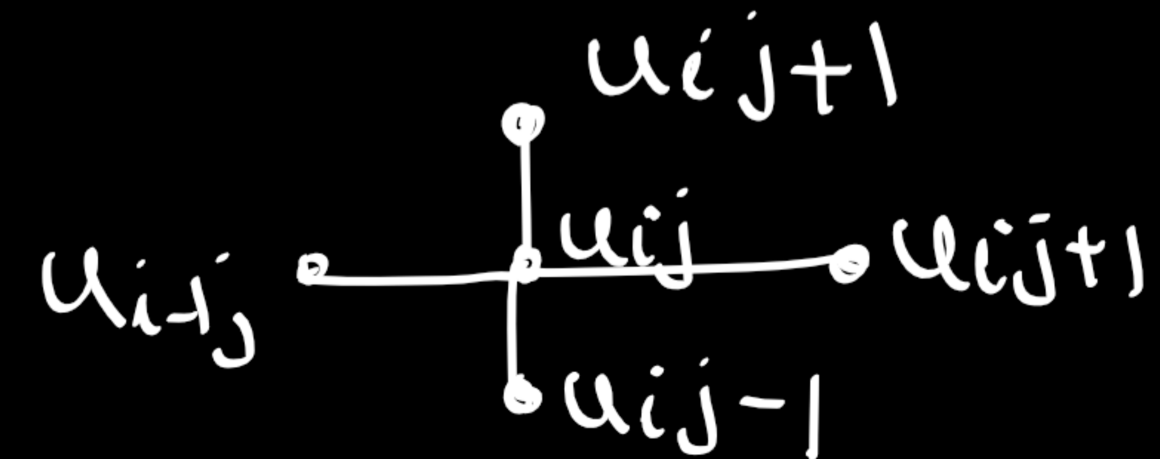
$k = h/c = \dfrac{0.25}{1} = 0.25$

$g_i = \dfrac{\partial u}{\partial t}(x_i, 0) = 0$

$\therefore u_{i1} = \dfrac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right]$



$u_{ij+1} = u_{i+1,j} + u_{i-1j} - u_{ij-1}$



$\therefore$ To solve wave eqn we need

$$u_{i,0} = f_i \longrightarrow (6)$$

$$u_{i,1} = \frac{1}{2}\left(f_{i+1} + f_{i-1}\right) + k g_i \longrightarrow (7)$$

where $f_{i+1} = u_{i+1,0}$, $f_{i-1} = u_{i-1,0}$,
$g_i = \frac{\partial u}{\partial t}(x_i, 0)$

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \longrightarrow (5)$$

Eqn: (6) is used to obtain $u(x,t)$ at the zeroth level (along x-axis)

Eqn: (7) is used to obtain $u(x,t)$ at first time step.

Eqn: (5) is used to obtain $u(x,t)$ at second and higher time steps.

Then 4 simplifies to

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \longrightarrow 5$$

Initial condition $u(x_i, 0) = f(x_i)$ i.e. $u_{i0} = f_i \longrightarrow 6$

To find u_{i1}

from eqn 5

$$u_{i1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$
$$= f_{i+1} + f_{i-1} - u_{i,-1}$$

To find $u_{i,-1}$

consider $u_{i,j-1} = u_{ij} - k\frac{\partial u_{ij}}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_{ij}}{\partial t^2} + \dots$

Put $j=0$

$$u_{i,-1} = u_{i0} - k\frac{\partial u_{i0}}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_{i0}}{\partial t^2}$$
$$= f_i - k g_i + \frac{h^2}{2\cancel{c^2}}\cancel{c^2}\frac{\partial^2 u_{i0}}{\partial x^2}$$
$$= f_i - k g_i + \frac{\cancel{h^2}}{2}\left[\frac{u_{i+1,0} - 2u_{i,0} + u_{i-1,0}}{\cancel{h^2}}\right]$$
$$= f_i - k g_i + \frac{1}{2}\left[f_{i+1} - 2f_i + f_{i-1}\right]$$
$$= \frac{1}{2}\left(f_{i+1} + f_{i-1}\right) - k g_i$$

$$\therefore u_{i1} = f_{i+1} + f_{i-1} - \left\{\frac{1}{2}\left(f_{i+1} + f_{i+1}\right) - k g_i\right\}$$
$$= \frac{1}{2}\left(f_{i+1} + f_{i-1}\right) + k g_i \longrightarrow 7$$

One dimensional wave equation

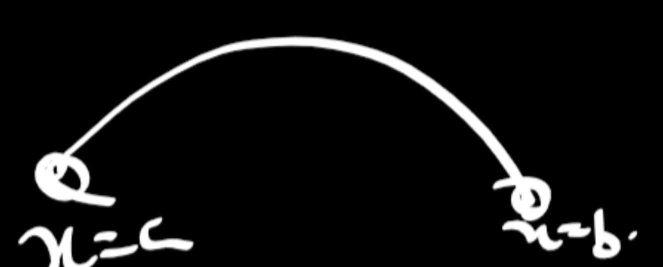
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1), \quad a < x < b, \quad t > 0$$

(Hyperbolic)

$A = c^2$
$B = 0$
$C = -1$
$B^2 - 4AC$
$0 + 4c^2 > 0$

$$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

(initial velocity)



$$u(a,t) = \phi(x), \qquad u(b,t) = \psi(x)$$

Divide $[a,b]$ into n equal parts each of length h.
Let k be the time interval size. Let $t_j = jk$

Let $u(x_i, t_j) = u_{ij}$

We have
$$\frac{\partial^2 u_{ij}}{\partial x^2} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} \rightarrow (2)$$

and
$$\frac{\partial^2 u_{ij}}{\partial t^2} = \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{k^2} \rightarrow (3)$$

Use (2) and (3) in (1)

$$\frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{k^2} = \frac{c^2}{h^2}\left[u_{i+1j} - 2u_{ij} + u_{i-1j}\right] \rightarrow (4)$$

Let $k = h/c$

Crank- Nicolson's method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 5, \quad t > 0$$

$$u(x,0) = 20, \quad u(0,t) = 0, \quad u(5,t) = 100$$

Compute $u(x,t)$ for one time step with $h = 1$.

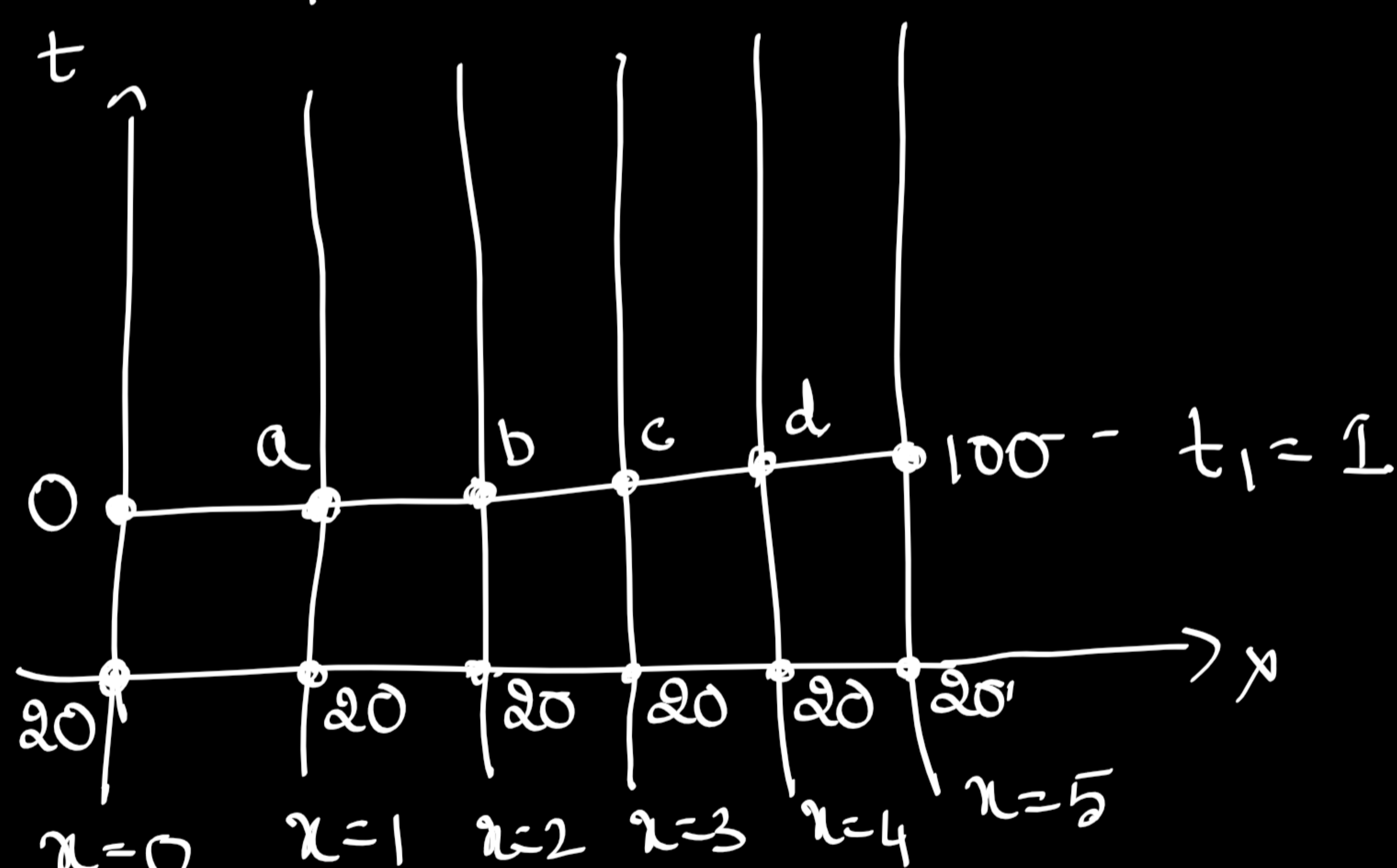
$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

$$t_j = jk, \quad j = 1$$

$$kc/h^2 = \lambda \qquad c = 1, \; h = 1, \; \lambda = 1$$

$$\therefore k = 1$$

$$\therefore t_1 = 1$$



$0 + 4a - b = 20 + 20 \Rightarrow 4a - b = 40 \longrightarrow (1)$

$-a + 4b - c = 40,$

$-a + 4b - c = 40 \longrightarrow (2)$

$-b + 4c - d = 40 \longrightarrow (3)$

$-c + 4d - 100 = 40$

$-c + 4d = 140 \longrightarrow (4)$

On solving (1), (2), (3), (4)

$a = 15.02$

$b = 20.1$

$c = 25.36$

$d = 41.3$