First order and First degree DE:

A first order first degree differential equation is of the form

$$\frac{dy}{dx} = f(x,y)$$

It can also be written in the form

$$M\,dx + N\,dy = 0 \quad \text{--- (1)}$$

where M and N are functions x and y.

I Variable Separable Equations:

A differential equation of the form (1), is said to be variable separable if it can be reduced to the form,

$$f(x)\,dx + g(y)\,dy = 0 \quad \text{--- (2)}$$

where $f(x)$ is a function of x alone, $g(y)$ is a function of y alone.

Integrating (2) we get the solution as

$$\int f(x)\,dx + \int g(y)\,dy = c$$

where c is an arbitrary constant of integration.

Solve the following differential equations:

(1) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\div \tan x \tan y$

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0, \qquad x, y \neq (2n+1)\pi/2$$

$$\int \frac{\sec^2 t}{\tan t}\,dt \qquad \tan t = z, \quad \sec^2 t\, dt = dz$$

$$= \int \frac{dz}{z} = \log z = \log|\tan t|$$

Equation is variable separable.
Solution is

$$\int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = \log|c|$$

$$\log|\tan x| + \log|\tan y| = \log|c|$$

$\log m + \log n = \log mn.$

$$\log(|\tan x \tan y|) = \log|c|$$

$$\tan x \tan y = c.$$

(2) $e^x(y-1)\,dx + 2(e^x+4)\,dy = 0.$

$\div (e^x+4)(y-1)$

$$\frac{e^x}{e^x+4}\,dx + \frac{2}{y-1}\,dy = 0. \quad \text{Equation is variable separable.}$$

$e^x+4 = t$
$e^x dx = dt$

$$\int \frac{e^x}{e^x+4}\,dx + 2\int \frac{1}{y-1}\,dy = \log c$$

$$\log(|e^x+4|) + 2\log|(y-1)| = \log|c|$$

$$\log(e^x+4) + \log(y-1)^2 = \log c$$

$$\log\left((e^x+4)(y-1)^2\right) = \log c$$

$$\therefore (e^x+4)(y-1)^2 = c.$$

(3) $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

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$(\sin y + y\cos y)\, dy = x(2\log x + 1)\, dx$, Equation is variable separable.

$\int u\, v\, dx$ (I, II)

$\int \sin y\, dy + \int \underset{I}{y}\, \underset{II}{\cos y}\, dy = 2\int \underset{II}{x}\, \underset{I}{\log x}\, dx + \int x\, dx + c$

$= u\int v\, dx - \int v\, d(u)$

$-\cos y + \left[y(\sin y) - \int (\sin y)\, 1 \cdot dy\right] = 2\left[\log x \cdot \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx\right] + \frac{x^2}{2} + c$

$-\cancel{\cos y} + y\sin y \cancel{-(-\cos y)} = x^2\log x \cancel{- \frac{x^2}{2}} \cancel{+ \frac{x^2}{2}} + c$

$2\left[\log x \cdot \frac{x^2}{2} - \frac{1}{2}\int x\, dx\right]$

$y\sin y = x^2\log x + c.$

$2\left[\log x \cdot \frac{x^2}{2} - \frac{x^2}{4}\right]$

(4) $\frac{dy}{dx} = x e^{y - x^2}$, $y(0) = 0$

$\boxed{a^{m+n} = a^m a^n}$

$\frac{dy}{dx} = x\, e^y e^{-x^2}$

$\frac{dy}{e^y} = x e^{-x^2} dx$

$x^2 = t$

$2x\, dx = dt$

$x\, dx = dt/2$

$e^{-y}\, dy = x e^{-x^2} dx$

$\int e^{-y}\, dy = \int x e^{-x^2} dx - \frac{c}{2}$

$-e^{-y} = \int e^{-t} \frac{dt}{2} - \frac{c}{2}$

$\boxed{-e^{-y} = \frac{1}{2} e^{-t} + c}$ is also ok.

$-e^{-y} = -\frac{1}{2} e^{-t} - \frac{c}{2}$

$2e^{-y} = e^{-t} + c.$

$2e^{-y} = e^{-x^2} + c$

$y(0) = 0 \Rightarrow 2e^{-0} = e^{-0} + c \Rightarrow 2 = 1 + c$ $\therefore c = 1$

$\therefore$ Required solution is $\underline{\underline{2e^{-y} = e^{-x^2} + 1}}$

(5) $xy\frac{dy}{dx} = 1 + x + y + xy$

$xy\frac{dy}{dx} = (1+x) + y(1+x)$

$xy\frac{dy}{dx} = (1+x)(1+y)$

$\frac{y}{1+y}\, dy = \frac{1+x}{x}\, dx$

$\int \frac{y}{y+1}\, dy = \int \frac{1+x}{x}\, dx + c$

$\int \frac{y+1-1}{y+1}\, dy = \int \left(\frac{1}{x} + 1\right) dx + c$

$\int \left(1 - \frac{1}{y+1}\right) dy = \log|x| + x + c$

$y - \log|y+1| = \log|x| + x + c.$ is the required solution.

Equations Reducible to Variable separable equations:

Solve the following DEs:

(1) $\frac{dy}{dx} = (9x+y+1)^2$

$9x+y+1 = z$

Diff. w.r.t x

$9 + \frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 9$

$\frac{dz}{dx} - 9 = z^2$

$\frac{dz}{dx} = z^2 + 9$

$\frac{dz}{z^2+9} = dx$

$\int \frac{1}{z^2+9} dz = \int dx + c$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\frac{1}{3}\tan^{-1}\left(\frac{z}{3}\right) = x + c$

$\frac{1}{3}\tan^{-1}\left(\frac{9x+y+1}{3}\right) = x + c.$

(2) $\frac{dy}{dx} - x\tan(y-x) = 1$

$y - x = z$

$\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} + 1$

$\frac{dz}{dx} + 1 - x\tan z = 1$

$\frac{dz}{dx} = x\tan z$

$\frac{dz}{\tan z} = x\,dx$

$\cot z\, dz = x\, dx$

$\therefore \int \cot z\, dz = \int x\, dx + c$

$\log|\sin z| = \frac{x^2}{2} + c$

$\log|\sin(y-x)| = \frac{x^2}{2} + c.$

Problems for Practice:

Solve the following:

(1) $(xy+x)\,dx + (x^2y^2 + x^2 + y^2 + 1)\,dy = 0$

(2) $\frac{y}{x}\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$

(3) $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

(4) $\tan y \frac{dy}{dx} = \cos(x+y) + \cos(x-y)$

(5) $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

(6) $(x+y+1)^2 \frac{dy}{dx} = 1$

II Homogeneous Differential Equations:

Definition: Let $z = f(x,y)$ be a function of two variables x and y. z is said to be a homogeneous function of degree n, if it can be written in the form

$$z = x^n \varphi\left(\frac{y}{x}\right) \quad \text{OR} \quad z = y^n \psi\left(\frac{x}{y}\right).$$

$\varphi\left(\frac{y}{x}\right)$ is a function of y/x.

$\psi\left(\frac{x}{y}\right)$ is a function of x/y.

Examples:

① $z = x^2 - xy + y^2$

$$= x^2\left[1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2\right]$$

$$= x^2 \varphi\left(\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree 2.

② $z = x - y\sin(y/x) + x\tan(x/y)$

$$= x\left[1 - \frac{y}{x}\sin(y/x) + \tan(x/y)\right]$$

$$= x\,\varphi(y/x).$$

$\therefore$ z is homogeneous function of degree 1.

Definition: A differential equation of the form

$$M\,dx + N\,dy = 0 \quad \text{——— ①}$$

is said to be a homogeneous differential equation if both M and N are homogeneous functions of same degree.

If ① is a homogeneous differential equation, we put

$$y = vx$$

then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

With this substitution, ① reduces to a variable separable equation and hence can be solved.

Solve the following differential equations:

① $(x^2 - y^2)\,dx = xy\,dy$

$$xy\frac{dy}{dx} = x^2 - y^2$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy} \qquad \text{Equation is homogeneous.}$$

$x = uy$

$\frac{dx}{dy} = u + y\frac{du}{dy}$

Put $y = vx$, $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 - x^2v^2}{x \cdot vx} = \frac{x^2(1-v^2)}{x^2 v} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-v^2-v^2}{v} = \frac{1-2v^2}{v} = -\frac{(2v^2-1)}{v}$$

$$\frac{v}{2v^2-1}\,dv = -\frac{dx}{x}$$

$$\frac{1}{4}\int \frac{4v}{2v^2-1}\,dv = -\int \frac{dx}{x} + \frac{\log|C|}{4}$$

$$\frac{1}{4}\log|2v^2-1| = -\log|x| + \frac{\log|C|}{4}$$

$$\log|2v^2-1| + 4\log|x| = \log|C|$$

$$\log|2v^2-1| + \log|x|^4 = \log|C|$$

$$\log|(2v^2-1)x^4| = \log|C|$$

$$x^4(2v^2-1) = C$$

$$x^4\left(2\frac{y^2}{x^2}-1\right) = C$$

$$\underline{\underline{x^2(2y^2-x^2) = C}}$$

$2v^2-1 = t$

$4v\,dv = dt.$

$\int \frac{v}{2v^2-1}\,dv$

$= \frac{1}{4}\int \frac{4v\,dv}{2v^2-1}$

$= \frac{1}{4}\int \frac{dt}{t}$

② $x^2y\,dx - (x^3+y^3)\,dy =$

$$x^2y\,dx = (x^3+y^3)\,dy$$

$$\frac{dx}{dy} = \frac{x^3+y^3}{x^2y}$$

put $x = uy$

$\frac{dx}{dy} = u + y\frac{du}{dy}$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{y^2}{x^2}$$

$$u + y\frac{du}{dy} = u + \frac{1}{u^2}$$

$$y\frac{du}{dy} = \frac{1}{u^2}$$

$$u^2\,du = \frac{dy}{y}$$

$$\frac{u^3}{3} = \log(y) + c$$

$$\frac{x^3}{3y^3} = \log y + c$$

$$x^3 = 3y^3(\log y + c).$$

③ $$\frac{dy}{dx} = \frac{y}{x-\sqrt{xy}}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x-\sqrt{vx^2}}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x(1-\sqrt{v})} = \frac{v}{1-\sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v}{1-\sqrt{v}} - v = \frac{v - v + v\sqrt{v}}{1-\sqrt{v}} = \frac{v\sqrt{v}}{1-\sqrt{v}}$$

$$\frac{(1-\sqrt{v})\,dv}{v\sqrt{v}} = \frac{dx}{x}$$

$$\left(v^{-3/2} - \frac{1}{v}\right)dv = \frac{dx}{x}$$

$$\frac{v^{-3/2+1}}{-3/2+1} - \log v = \log x + c$$

$$\frac{v^{-1/2}}{-1/2} - \log v = \log x + \log C$$

$$\frac{-2}{\sqrt{v}} = \log(Cxv)$$

$$\frac{-2\sqrt{x}}{\sqrt{y}} = \log(cy).$$

$$cy = e^{\frac{-2\sqrt{x}}{\sqrt{y}}}$$

Problems for Practice:

Solve the following:

① $(y^3 - 3x^2y)\,dx - (x^3 - 3xy^2)\,dy = 0$

② $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

③ $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

④ $y\,dy + \sin^2\left(\frac{x}{y}\right)[x\,dy - y\,dx] = 0.$