Q5) If $X$ & $Y$ are independent Random variables then find $E(X/Y)$ and $E(Y/X)$.

Q6) suppose that $X, Y$ has pdf $f(x,y) = \begin{cases} k e^{-y}, & 0<x<y<1 \\ 0, & \text{elsewhere} \end{cases}$

find the corelation coefficient. $\boxed{K=?} \in \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1}$

$(\rho_{xy} = ?)$.

## Difference Equation:

A difference equation is a relation b/w the difference ~~of~~ of an unknown function at one or more general values of the argument.

$$\Delta y_{n+1} + y_n = 2 \longrightarrow ①$$

$$\Delta y_{n+1} + \Delta^2 y_{n-1} = 1 \longrightarrow ②$$

equation's 1 & 2 can be rewritten as

$① \Rightarrow y_{n+2} - y_{n+1} + y_n = 2 \rightarrow ③$

$② \Rightarrow y_{n+2} - y_{n+1} + y_{n+1} - y_n - y_n + y_{n-1} = 1.$

$② \Rightarrow y_{n+2} + y_{n-1} - 2y_n = 1 \rightarrow ④.$

## Formation of difference equation:

$y = ax + bx^2$

$\Delta y = a\Delta x + b\Delta(x^2)$

$= a(x+1-x) + b[(x+1)^2 - x^2].$

$= a + b(2x+1).$

$\Delta^2 y = 2b.$

$$\begin{vmatrix} y & x & x^2 \\ \Delta y & 1 & (2x+1) \\ \Delta^2 y & 0 & 2 \end{vmatrix} = 0.$$

$= y(2-0) - x(2\Delta y - \Delta^2 y(2x+1)) + x^2(-\Delta^2 y) = 0$

$2y - 2x\Delta y + (2x^2 + x - x^2)\Delta^2 y = 0$

$(x^2 + x)\Delta^2 y - 2x\Delta y + 2y = 0$

Q2) $y_n = A2^n + B(-3)^n$

$\therefore \Delta y = (y_{n+1} - y_n)$

$\Delta y_n = A2^{n+1} + B(-3)^{n+1}$
$- A2^n - B(-3)^n$

$= A2^n - 4B(-3)^n$

$\Delta^2 y_n = \Delta(\Delta y_n) = \Delta(A2^n - 4B(-3)^n)$.

$= A2^{n+1} - A2^n - \{4B(-3)^{n+1} - 4B(-3)^n\}$.

$= A2^n(2-1) - 4B(-3)^n(-3-1)$

$= A2^n + 16B(-3)^n$

$$\begin{vmatrix} y_n & 2^n & (-3)^n \\ \Delta y_n & 2^n & -4(-3)^n \\ \Delta^2 y_n & 2^n & 16(-3)^n \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y_n & 1 & 1 \\ \Delta y_n & 1 & -4 \\ \Delta^2 y_n & 1 & 16 \end{vmatrix} = 0$$

$= y_n(16+4) - 1(\Delta y_n(16) + 4(\Delta^2 y_n)) + 1(\Delta y_n - \Delta^2 y_n)$.

$= 14 y_n - 16\Delta y_n - 4\Delta^2 y_n + \Delta y_n - \Delta^2 y_n$ check

$= \cancel{15\Delta y_n - 5\Delta^2 y_n}$ $20y_n - 16\Delta y_n - 4\Delta^2 y_n + \Delta y_n - \Delta^2 y_n$

$= 20y_n + 3\Delta y_n - 4y_n = 0$.

Q3) $y = \frac{a}{x} + b$

$\cancel{\Delta y = \frac{a}{\Delta x} + b}$

$\cancel{\Delta y = \frac{a}{x+1-x} + b = \frac{a + bx_{n+1} - bx_n}{x_{n+1} - x_n}}$

$\cancel{\Delta^2 y = \Delta\left(\frac{a + bx_{n+1} - bx^{n+1}}{x_{n+1} - x_n}\right)}$

$\cancel{\Delta y = a}$

$\cancel{\Delta^2 y = 0}$

$$\begin{vmatrix} y & \frac{1}{x} & 1 \\ \Delta y & 1 & 0 \\ \Delta^2 y & 0 & 0 \end{vmatrix}$$

$y(0-0) - \frac{1}{x}(0) - 1(\Delta^2 y)$ $\Delta^2 y$

$\Delta y = \frac{a}{x+1} + b - \frac{a}{x} - b = a\left[\frac{1}{x+1} - \frac{1}{x}\right] = -\frac{a}{x(x+1)}$

$\Delta^2 y = \Delta\left(-\frac{a}{x(x+1)}\right) = \frac{-a}{(x+1)(x+2)} + \frac{a}{x(x+1)}$

$= \frac{2a}{x(x+1)(x+2)}$

$$\begin{vmatrix} y & \frac{1}{x} & 1 \\ Dy & -\frac{1}{x(x+1)} & 0 \\ D^2y & \frac{2}{x(x+1)(x+2)} & 0 \end{vmatrix} = 0$$

$$(x+2)D^2y + 2Dy = 0.$$

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## Linear Difference equation:

A linear difference eqn is that in which $y_{n+1}, y_{n+2}$ are of first degree and are not multiplied together.

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n).$$

A linear difference eqn with const coefficients is of the form.

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = 0.$$

using shift operator E.

$$(E^r + a_1 E^{r-1} + a_2 E^{r-2} + \dots + a_r) y_n = 0.$$

Auxillary equation. is,

$$E^r + a_1 E^{r-1} + a_2 E^{r-2} + \dots + a_r = 0.$$

Let, $\lambda_1, \lambda_2, \dots \lambda_r$ be the roots

Case 1: If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_r$ are real and distinct

then

$$\text{Complementary function (CF)} = c_1 \lambda_1^n + c_2 \lambda_2^n + \dots + c_r \lambda_r^n.$$

Case 2: If $\lambda_1 = \lambda_2 = \lambda_3$ and all are real.

$$CF = (c_1 + c_2 n + c_3 n^2)\lambda_1^n + c_4 \lambda_4^n + \dots + c_r \lambda_r^n.$$

Case 3: two roots are imaginary.

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta.$$

$$CF = (c_1 \cos n\theta + c_2 \sin n\theta) r^n + c_3 \lambda_3^n + \dots + c_r \lambda_r^n.$$

where, $\theta = \tan^{-1} \beta/\alpha \; ; \; r = \sqrt{\alpha^2 + \beta^2}.$

Q1) Solve $y_{n+3} - 2y_{n+2} - 5y_{n+1} + 6y_n = 0$.

Ans)

$$(E^3 - 2E^2 - 5E + 6)\, y_n = 0.$$

Auxillary equation (A.E) = $E^3 - 2E^2 - 5E + 6 = 0$.

$$E = 1, -2, 3.$$

Complementary function (CF) = $CF = c_1(-2)^n + c_2(1)^n + c_3(3)^n$.

Q2) Solve $y_{n+2} - 6y_{n+1} - 9y_n = 0$.

$$AE = (E^2 - 6E - 9)y_n = 0.$$

then, $E^2 - 6E - 9 = 0$.

~~E(E-6)-26~~

$$(E-3)^2 = 0.$$

$$\boxed{E = 3, 3}$$

$$CF = (c_1 + c_2 n)(3)^n.$$

Q3) Solve

$$\Delta^2 y_n - 2\Delta y_n + y_n = 0.$$

$\rightarrow \boxed{\Delta = E - 1}$ ~~$-2\Delta y_{n+1} + 2y_n + y_n = 0$~~

$$(E-1)^2 y_n - 2(E-1)y_n + y_n = 0.$$

$$(E^2 + 1 - 2E)y_n - (2E - 2)y + y_n = 0.$$

$$AE = E^2 + \cancel{1} - 2E - 2E + \cancel{2} + \cancel{1} = 0.$$

$$E^2 - 4E + 4 = 0$$

$\boxed{~~E(E-4)~~}$ $E(E-4) - 1(E-4)$

$$\boxed{E = 1, 4}$$

$$CF = c_1(1)^n + c_2(4)^n.$$

Q4) Solve, $y_{n+3} - 3y_{n+1} + 2y_n = 0$.

given, $y_1 = 0,\ y_2 = 8,\ y_3 = -2$.

Ans)

A.E. $E^3 - 3E + 2 = 0$.

one root $E = 1$

$$
\begin{array}{r|l}
 & E^2 + E - 2 \\
E-1 & E^3 - 0E^2 - 3E + 2 \\
 & E^3 - E^2 \\
\hline
 & E^2 - 3E \\
 & E^2 - E \\
\hline
 & -2E + 2 \\
 & -2E + 2 \\
\hline
 & 0
\end{array}
$$

$$E^3 - 3E + 2 = (E-1)(E^2 + E - 2).$$

$$= (E-1)(E+2)(E-1).$$

$y_1 = 0 \Rightarrow c_1 + c_2 - 2c_3 = 0.$ $\quad CF = (c_1 + c_2 n) + c_3(-2)^n$

$y_2 = 8 \Rightarrow c_1 + 2c_2 + 4c_3 = 8$

$y_3 = -2 \Rightarrow c_1 + 3c_2 - 8c_3 = 2.$

$c_1 = 0 \quad c_2 = 2 \quad c_3 = 1.$

$= 2n + (-2)^n$

HW

① $y_{n+2} - 2y_{n+1} + y_n = 0.$

② $(D^2 - 3D + 2)\, y_n = 0.$

③ $(E^3 - 5E^2 + 8E - 4)\, y_n = 0$, Given, $y_0 = 3, y_1 = 2, y_3 = 4.$

Rules for finding particular integrals:

consider $y_{n+r} + a_1 y_{n+r-1} + \cdots + a_r y_n = f(n).$

$(E^r + a_1 E^{r-1} + \ldots a_r)\, y_n = f(n)$

$\phi(E)\, y_n = f(n)$

(Particular Integral): $\therefore PI = \frac{1}{\phi E} f(n)$

$\boxed{E^k y_n = y_{n+k}}$

Case 1: $f(n) = a^n$

$PI = \frac{1}{\phi(E)} a^n = \frac{1}{\phi(a)} \cdot a^n, \; \phi(a) \neq 0.$

If $\phi(a) = 0$?

$(E-a)\, y_n = a^n$, $\quad PI = \frac{1}{(E-a)} a^n = n a^{n-1}$

$(E-a)^2 y_n = a^n$, $\quad PI = \frac{1}{(E-a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$

$(E-a)^3 y_n = a^n$, $\quad PI = \frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$

Case 2: $f(n) = \sin kn.$

$PI = \frac{1}{\phi(E)} \sin kn.$

$= \frac{1}{\phi(E)} \frac{e^{ikn} - e^{-ikn}}{2i}.$

$= \frac{1}{2i}\left[\frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n\right].$ where, $a = e^{ik}$, $b = e^{-ik}$

$f(n) = \cos kn$

$$PI = \frac{1}{2}\left[\frac{1}{\phi(E)}a^n + \frac{1}{\phi(E)}b^n\right]$$ where, $a = e^{ik}$, $b = e^{-ik}$.

Case-3:

$f(n) = n^p$ (Polynomial function).

$$PI = \frac{1}{\phi(E)} n^p$$

$$= \frac{1}{\phi(\Delta+1)} [n]^p.$$ where, $[n]^p = (n-1)(n-2) \ldots (n-p+1)$.

$PI = [\phi(\Delta+1)]^{-1} [n]^p \rightarrow$ (factorial of $n$) ($[\,] \rightarrow$ factorial).

Expand $[\phi(1+\Delta)]^{-1}$ in assending powers of delta by using Binomial theorem as far as the term $\Delta^p$.

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Case-4: $f(n) = a^n F(n)$

where, $F(n)$ is a polynomial of finite degree

$$PI = \frac{1}{\phi(E)} a^n F(n)$$

$$= a^n \frac{1}{\phi(aE)} F(n)$$, proceed as in case III.

Solve:

(i) $(E^2 - 4E + 3) y_n = 5^n$

A.E, $E^2 - 4E + 3 = 0$,

$E = 1, 3$.

$CF = c_1(1)^n + c_2(3)^n = c_1 + c_2 3^n$

$$PI = \frac{1}{E^2 - 4E + 3} 5^n = [\because \text{replace by 5 as in case 1}]$$

$$= \frac{1}{8} 5^n$$

$y_n = CF + PI$

$$y_n = c_1 + c_2 3^n + \frac{1}{8} 5^n$$

(2) $y_{n+2}-4y_{n+1}+4y_n=2^n$

$(E^2-4E+4)y_n=2^n$

AE. $(E-2)^2=0 \Rightarrow E=2,2$

$CF=(c_1+c_2n)2^n$

$PI=\frac{1}{(E-2)^2}2^n$

$=\frac{n(n-1)}{2!}2^{n-2}=(n-1)n\,2^{n-3}$

(3) $y_{n+2}-2\cos\alpha\, y_{n+1}+y_n=\cos\alpha n$

$(E^2-2\cos\alpha E+1)y_n=\cos\alpha n.$

AE $= E^2-2\cos\alpha E+1=0.$

$E=\frac{+2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$

$E=\cos\alpha\pm i\sin\alpha$

$r=\sqrt{\cos^2\alpha+\sin^2\alpha}=1$

$\theta=\tan^{-1}\frac{\sin\alpha}{\cos\alpha}=\alpha$

$CF=c_1\cos n\alpha+c_2\sin n\alpha$

$PI=\frac{1}{E^2-2\cos\alpha E+1}\cos\alpha n.$

$\boxed{\begin{array}{l} e^{i\theta}=\cos\theta+i\sin\theta \\ e^{-i\theta}=\cos\theta-i\sin\theta \end{array}}$

$PI=\frac{1}{E^2-2E\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)+1}\left(\frac{e^{i\alpha n}+e^{-i\alpha n}}{2}\right)$

$PI=\frac{1}{2}\cdot\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}\left[e^{i\alpha n}+e^{-i\alpha n}\right].$

$=\frac{1}{2}\left\{\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{i\alpha n}+\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{-i\alpha n}\right\}$

$=\frac{1}{2}\left\{\frac{1}{(E-e^{i\alpha})(e^{i\alpha}-e^{-i\alpha})}e^{i\alpha n}+\frac{1}{(e^{i\alpha}-e^{-i\alpha})(E-e^{-i\alpha})}e^{-i\alpha n}\right\}$

$=\frac{1}{2(e^{i\alpha}-e^{-i\alpha})}\left\{n\,e^{i\alpha(n-1)}-n e^{-i\alpha(n-1)}\right\}$

$=\frac{n}{4i\left(\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\right)}2i\left(\frac{e^{i\alpha(n-1)}-e^{-i\alpha(n-1)}}{2i}\right)$

$=\frac{n}{2\sin\alpha}\cdot\sin(n-1)\alpha$

(4) $y_{n+2} - 4y_n = n^2 + n - 1$   A.E $= E^2 - 4 = 0$   $E = \pm 2$.

CF $= c_1 2^n + c_2(-2)^n$.

$\left[\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots\right]$

PI $= \frac{1}{E^2-4}\, n^2+n-1$   (Replace $E$ by $1+\Delta$)

$= \frac{1}{\Delta^2+2\Delta-3}\{[n]^2 - \cancel{[n]} + 2[n]\}$

$= \frac{1}{-3}\cdot\frac{1}{1-\left(\frac{\Delta^2+2\Delta}{3}\right)}\{[n]^2+2[n]-1\}$

$\therefore = -\frac{1}{3}\left(1-\frac{\Delta^2+2\Delta}{3}\right)^{-1}\{[n]^2+2[n]-1\}$

$\left[\Delta[n]^r = r[n]^{r-1}\right]$ $= -\frac{1}{3}\left[1+\left(\frac{\Delta^2+2\Delta}{3}\right)+\left(\frac{\Delta^2+2\Delta}{3}\right)^2+\dots\right]\{[n]^2+2[n]-1\}$

$= -\frac{1}{3}\left\{[n]^2+2[n]-1+\frac{2}{3}+\frac{2}{3}\cdot 2[n]+\frac{2}{3}\cdot 2+\frac{4}{9}\times 2\right\}$

$= -\frac{1}{3}\left\{n^2+n-1+\frac{2}{3}+\frac{4}{3}n+\frac{4}{3}+\frac{8}{9}\right\}$

$= -\frac{1}{3}\left\{\frac{9n^2+9n-9+6+12n+12+8}{9}\right\}$

$= -\frac{1}{3}\left[\frac{9n^2+21n+17}{9}\right] = -\frac{1}{3}\left[n^2+\frac{7n}{3}+\frac{17}{9}\right]$.

$= \cancel{-\frac{1}{3}\frac{n^2+21n}{9}} - \frac{1}{3}\cancel{\left[n^2+\frac{13n}{3}+\frac{8}{9}\right]}$.

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Q) Solve $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n$.

A.E $\rightarrow E^3 - 5E^2 + 3E + 9 = 0$.

$E = -1$, $E = 3, 3$.

CF $\rightarrow c_1(-1)^n + (c_2 + c_3 n)(3)^n$

PI $\rightarrow \frac{1}{(E+1)(E-3)^2}(2^n + 3n)$.   $[\because E = \Delta + 1]$.

$= \frac{1}{3}2^n + 3\frac{1}{(2+\Delta)(\Delta-2)^2}[n]$.   $(1+\Delta+1)(1+\Delta-3)^2$

$(2+\Delta)(-2+\Delta)^2$

$= \frac{1}{3}2^n - 3\frac{1}{(4+\Delta^2)(\Delta-2)}[n]$

$= \frac{1}{3}2^n - \frac{3}{4}\frac{1}{(1-\frac{D^2}{4})(D-2)}[n]$

$= \frac{1}{3}2^n - \frac{3}{4}\frac{1}{(D-2)}(1+D^2/4+\cdots)[n]$.

$= \frac{1}{3}2^n - \frac{3}{4(D-2)}[n]$

$= \frac{1}{3}2^n + \frac{3}{8}(1-\frac{D}{2})^{-1}[n]$

$= \frac{1}{3}2^n + \frac{3}{8}(1+\frac{D}{2}+(\frac{D}{2})^2+\cdots)[n]$

$= \frac{1}{3}2^n + \frac{3}{8}(n+\frac{1}{2})$.

Q6) solve $(E^2-5E+6)y_n = 4^n(n^2-n+5)$

$E = 2, 3$ (case 4)

$CF = c_1 2^n + c_2 3^n$.

$PI = \frac{1}{E^2-5E+6} \cdot 4^n(n^2-n+5)$

$= 4^n \frac{1}{(4E)^2-5(4E)+6}(n^2-n+5)$. $n^2-n = [n]$

$= 4^n \frac{1}{16E^2-20E+6}\{[n]^2+5\}$

$= 4^n \frac{1}{16(1+D)^2-20(1+D)+6}\{[n]^2+5\}$.

$= 4^n \frac{1}{16D^2+32D+16-20-20D+6}\{\quad\}$ $\boxed{D[n]^k = k[D]^{k-1}}$

$= 4^n \frac{1}{2+12D+16D^2}\{[n]^2+5\}$.

$= 2^{2n-1}\frac{1}{1+(6D+8D^2)}\{[n]^2+5\}$

$= 2^{2n-1}[1+(6D+8D^2)]^{-1}\{[n]^2+5\}$.

$= 2^{2n-1}\{1-(6D+8D^2)+(6D+8D^2)^2\cdots\}\{[n]^2+5\}$

$= 2^{2n-1}\{[n]^2+5-12[n]-16+72\}$

$= 2^{2n-1}\{n^2-n+5-12n+66\}$.

$= 2^{2n-1}\{n^2-13n+61\}$

(HW)

① $\Delta^2 y_n - 3\Delta y_n + 4y_n = n + 2^n$

② $y_{n+1} - 3y_n = 3^n(n+2)$

③ $y_{n+2} - (2\cos\frac{1}{2})y_{n+1} + y_n = \sin(\frac{n}{2})$

④ $y_{n+2} - 6y_{n+1} + 9y_n = n2^n + 3^n + 7$

⑤ $(E^2 - 3E + 2)y_n = n^2 + 2n - 1$.

## Z-Transforms

If funtion $u_n$ is defined for discrete values $n = 0, 1, 2, 3 \ldots$ and $u_n = 0$, $n < 0$ then its z-transform is defined $Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$.

$$z^{-1}[U(z)] = u_n.$$

Standard z-transforms:

(1) $z(a^n) = \frac{z}{z-a}$

(2) $z(n^p) = -z\frac{d}{dz}z(n^{p-1})$ [(p is +ve integer)]

[Consider $z(n^{p-1}) = \sum n^{p-1} z^{-n}$].

$$= \sum_{n=0}^{\infty} n^p z^{-n} \rightarrow ①$$

$$\frac{d}{dz}(z(n^{p-1})) = \sum n^{p-1}(-n)z^{-(n+1)} \rightarrow ②$$

$$z(n^p) = -z \cdot \frac{d}{dz}(z(n^{p-1}))$$

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If $a = 1$

$z(1) = \frac{z}{z-1}$ (from Ⓐ)

from Ⓑ

$$z(n) = -z \cdot \frac{d}{dz} z(1)$$

$$= -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}$$