2. Stability Analysis

$$AX = \lambda X$$

λ → eigen values, X → eigen vector

eigen values are given by

$$|A - \lambda I| = 0$$

$$(A - \lambda I) X = 0$$

$$\Rightarrow X = (A - \lambda I)^{-1} \cdot 0$$

$$= \frac{adj(A - \lambda I)}{|A - \lambda I|} \cdot 0$$

$$\Rightarrow |A - \lambda I| = 0$$

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ex.1:- Find the eigen values and vectors for the given matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

sol. $|\lambda I - A| = 0$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\lambda^2 + 3\lambda + 2 = 0.$$

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

$$|\lambda_1 I - A| X_1 = 0.$$

$$\left| \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| X_1 = 0.$$

$$\begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} X_1 = 0 \Rightarrow X_1 = 0$$

$$|\lambda_2 I - A| X_2 = 0.$$

$$\left| \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| X_2 = 0.$$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} X_2 = 0.$$

$$[\lambda_1 I - A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-x_1 - x_2 = 0.$$

$$2x_1 + 2x_2 = 0$$

let $x_1 = 1$

then $x_2 = -1$

eigen vector for λ_1 $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ily for λ_2.

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-2x_1 - x_2 = 0$$

$$2x_1 + x_2 = 0$$

EV for λ_2 $= \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Co-factor method.

$$[\lambda_1 I - A] = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$[\lambda_2 I - A] = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

cofactor w.r.t first row. $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

a_{11} a_{12}

a_{21} a_{22}

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$|\lambda I - A| = 0.$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} = 0.$$

$$\lambda^2 + 2\lambda + 1 = 0.$$

$$\lambda_1 = -1 \quad \lambda_2 = -1$$

$$\frac{-2 \pm \sqrt{4-8}}{2}$$

$$\frac{-2 + \sqrt{-4}}{2}$$

$\lambda_1 = -1$

EV for λ_1

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

EV for λ_2.

$$|\lambda_2 I - A| = \begin{vmatrix} \lambda_2 & -1 \\ 1 & \lambda_2+2 \end{vmatrix}$$

Cofactor matrix: $\begin{bmatrix} \lambda_2+2 \\ -1 \end{bmatrix}$

take derivate w.r.t λ_2.

$$\begin{pmatrix} \frac{d}{d\lambda_2}(\lambda_2+2) \\ \frac{d}{d\lambda_2}(-1) \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{vmatrix} \Rightarrow \lambda-4(\lambda^2-3\lambda+2)+1(-\lambda+3-2)$$

$+2(-1+\lambda)$

$\lambda^3-3\lambda^2+2\lambda-4\lambda^2+12\lambda-8-\lambda+1$

$-2+2\lambda$

$\cancel{\lambda^3-7\lambda^2-9\lambda+7}$

$$A-\lambda_1 I = \begin{pmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\lambda^3 - 7\lambda^2 + 15\lambda - 9$

$\lambda_1 = 1$ $\quad \lambda_2 = 3$.

$-3x - y + 2z = 0$

$-x + y - 2z = 0$

$\Rightarrow x + y - 2z = 0$

$$\begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & 3 & -2 \\ -1 & -1 & 0 \end{bmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

Stability.

Total response = natural response + forced response.

Stable:
A linear time invariant system is stable, if the natural response approaches to zero as time approaches to ∞.

eigen values = $-ve$

Unstable if natural response grow without bound as time $\rightarrow \infty$.

any one eigen value is $+ve$.

Marginally stable. if natural responce neither decays nor grows but remains constant or oscillates as time $\rightarrow \infty$.

any one value lie on img axis.

$$G(s) = C(sI - A)^{-1}B + D.$$

$$G(s) = C \frac{\operatorname{adj}(sI - A)}{|sI - A|} B + D.$$

$$\Rightarrow G(s) = \frac{Q(s)}{|sI - A|}$$

$\Rightarrow |sI - A| = 0$ char eq.

$\Rightarrow$ Eigen values.

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Diagonal canonical form — eigen values distinct

Jordan canonical form — repeated.

Diagonal

Case 1:- Degree of denom is greater than degree of numerator (strictly proper TF)

Consider the TF

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s^1 + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \longrightarrow (1)$$

$$\frac{Y(s)}{U(s)} = \frac{C_1}{(s+\lambda_1)} + \frac{C_2}{(s+\lambda_2)} + \frac{C_3}{(s+\lambda_3)} \longrightarrow (2)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{C_1(s+\lambda_2)(s+\lambda_3) + C_2(s+\lambda_1)(s+\lambda_3) + C_3(s+\lambda_1)(s+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} \longrightarrow (3)$$

compare numer of (3) & (1).

$$b_2 s^2 + b_1 s + b_0 = C_1(s+\lambda_2)(s+\lambda_3) + C_2(s+\lambda_1)(s+\lambda_3) + C_3(s+\lambda_1)(s+\lambda_2)$$

for C_2 sub $s = -\lambda_2$.

$$C_2(-\lambda_2+\lambda_1)(\lambda_3-\lambda_2) = b_0 - b_1\lambda_2 + b_2\lambda_2^2$$

$$C_2 = \frac{b_0 - b_1\lambda_2 + b_2\lambda_2^2}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}$$

||rly. Put $s = -\lambda_3$

$$C_3 = \frac{b_0 - b_1\lambda_3 + b_2\lambda_3^2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

Put $s=0$

$$C_1 = \frac{b_0 - C_2\lambda_1\lambda_3 + C_3\lambda_1\lambda_2}{\lambda_3\lambda_2}$$

from ②

$$Y(s) = \frac{C_1}{(S+\lambda_1)} U(s) + \frac{C_2}{S+\lambda_2} U(s) + \frac{C_3}{S+\lambda_3} U(s)$$

$$X_1(s) = \frac{U(s)}{(S+\lambda_1)} \Rightarrow s X_1(s) + \lambda_1 X_1(s) = U(s)$$

$$\Rightarrow \dot{x}_1 = -\lambda_1 x_1 + u$$

IIrly $X_2(s) = \frac{U(s)}{s+\lambda_2} \Rightarrow s X_2(s) + \lambda_2 X_2(s) = U(s)$

$$\Rightarrow \dot{x}_2 = -\lambda_2 x_2 + u$$

$$X_3(s) = \frac{U(s)}{s+\lambda_3} \Rightarrow \dot{x}_3 = -\lambda_3 x_3 + u$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = CX + DU$$

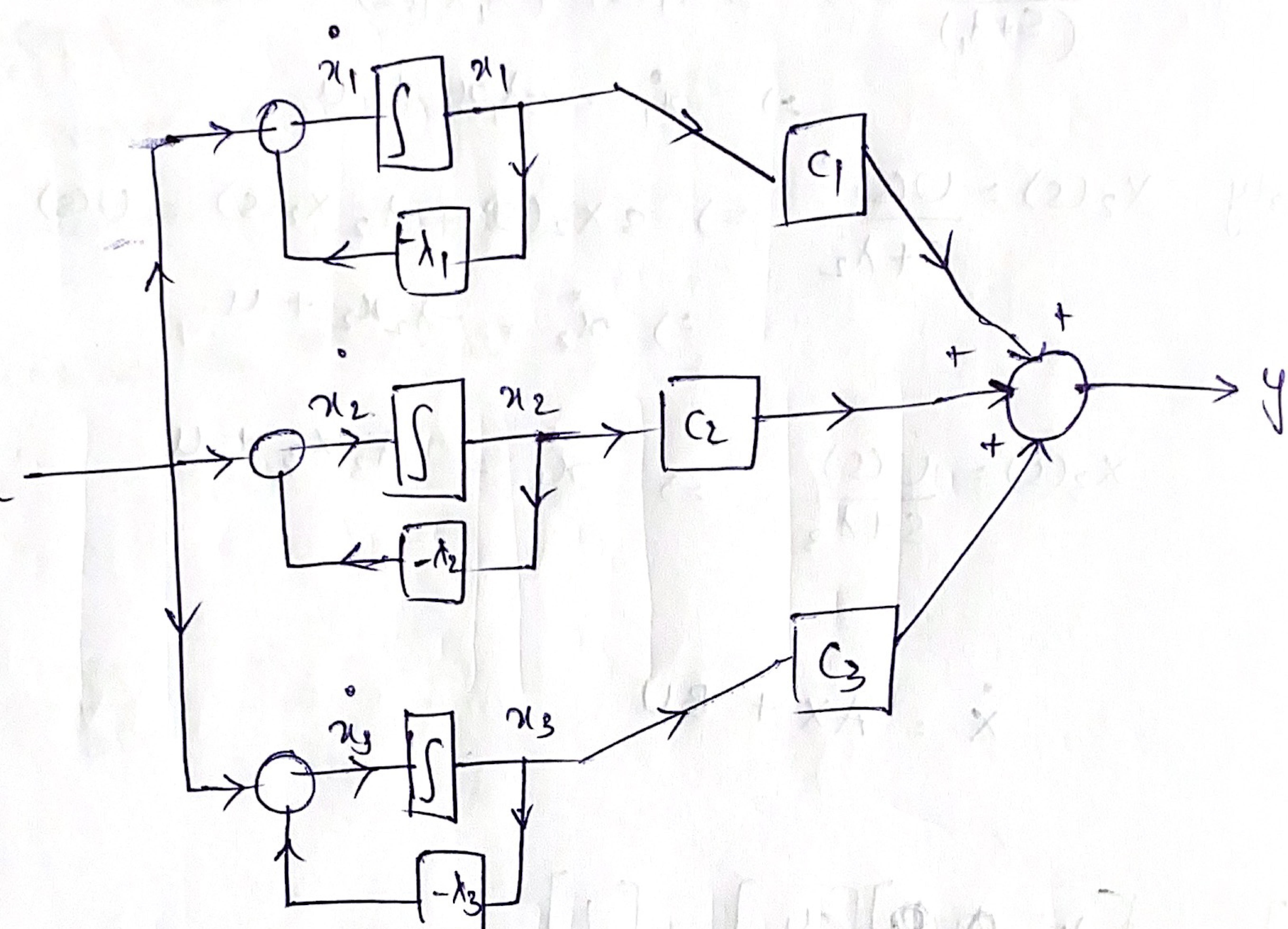
$$Y(s) = C_1 X(s) + C_2 X_2(s) + C_3 X_3(s)$$

Apply inv Laplace.

$$y = C_1 x_1 + C_2 x_2 + C_3 x_3.$$

$$y = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Block diagram.



Case 2 :- Proper TF. deg of den = deg of num.

Consider

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s^1 + b_0}{a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0}.$$

$$(b_3 s^3 + a_2 s^2 + a_1 s + a_0) - (a_2 s^2 + a_1 s + a_0) +$$

$$\Rightarrow \frac{Y(s)}{U(s)} = b_3 + \frac{(b_2 - b_3 a_2)s^2 + (b_1 - b_3 a_1)s + (b_0 - b_3 a_0)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}$$

SS model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$

Jordan canonical form. (when eigen values are repeated).

Consider 3rd order proper TF

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = b_3 + \frac{c_1}{(s+\lambda_1)^2} + \frac{c_2}{(s+\lambda_1)} + \frac{c_3}{(s+\lambda_3)}$$

$$\Rightarrow Y(s) = b_3 U(s) + c_1 \frac{U(s)}{(s+\lambda_1)^2} + c_2 \frac{U(s)}{(s+\lambda_1)} + c_3 \frac{U(s)}{(s+\lambda_3)}$$

$$X_1(s) = \frac{U(s)}{(s+\lambda_1)^2} = \frac{X_2(s)}{(s+\lambda_1)} \Rightarrow \dot{x}_1 = x_2 - \lambda_1 x_1$$

$$\Rightarrow s^2 X_1(s) + 2\lambda_1 s X_1(s) + \lambda_1^2 (X_1(s)) = U(s).$$

$$\ddot{x}_1 + 2\lambda_1 \dot{x}_1 + \lambda_1^2 x_1 = u$$

$$\Rightarrow \ddot{x}_1 = u -$$

$$X_2(s) = \frac{U(s)}{(s+\lambda_1)}$$

$$\Rightarrow \dot{x}_2 = u - \lambda_1 x_2$$

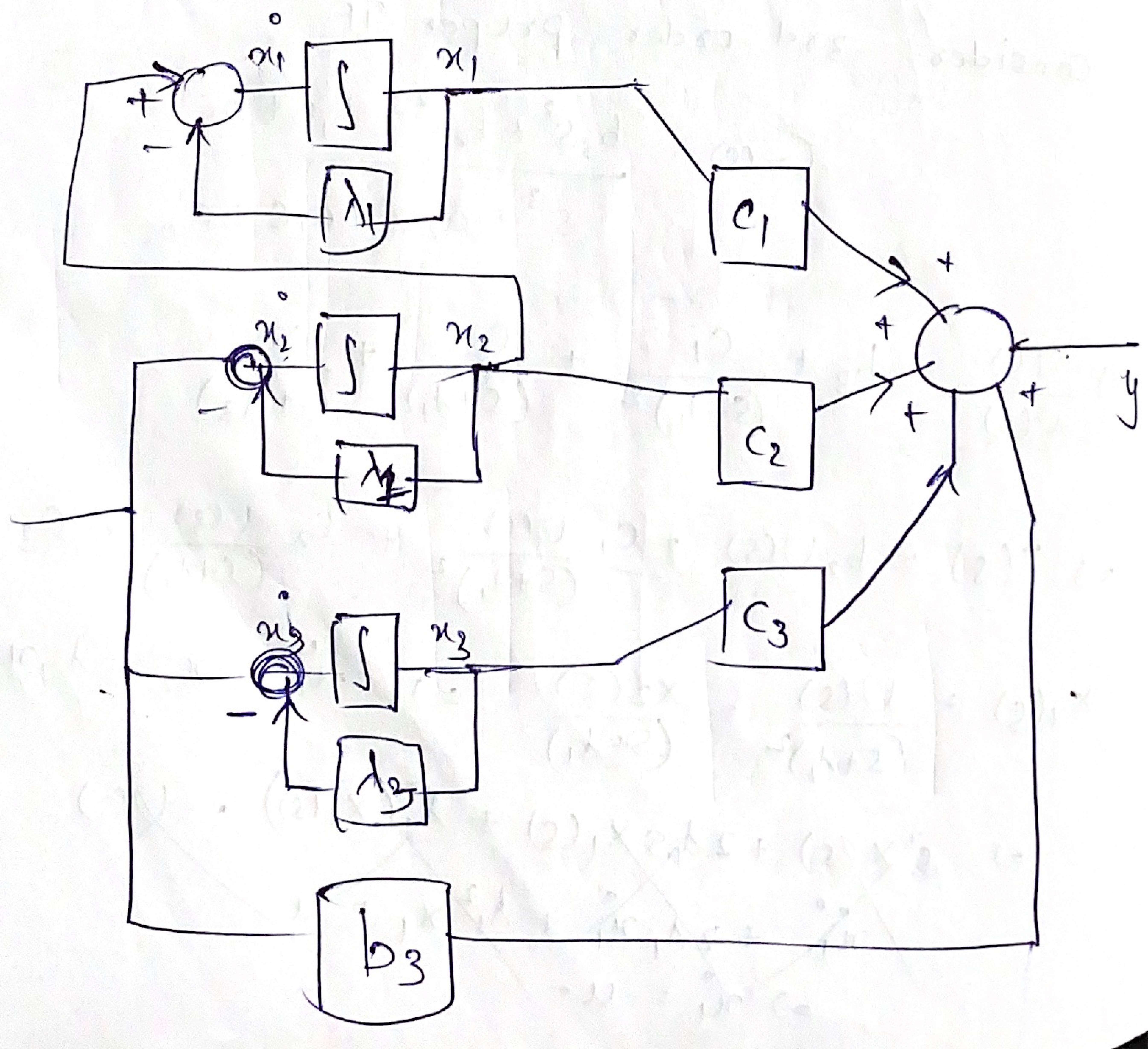
$$X_3(s) = \frac{U(s)}{(s+\lambda_3)}$$

$$\dot{x}_3 = u - \lambda_3 x_3.$$

Jordan block

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_3 u$$



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Diagonalization

$\dot{X} = AX + BU$

$Y = CX + DU$

transform State vector X

$X = PZ$ → model matrix (or) transition matrix

$Z = P^{-1}X$

$\dot{Z} = P^{-1}\dot{X}$

$= P^{-1}(AX + BU)$

$\dot{Z} = P^{-1}AX + P^{-1}BU$

$\dot{Z} = P^{-1}A(PZ) + P^{-1}BU$

$\Rightarrow \boxed{\dot{Z} = \tilde{A}Z + \tilde{B}U}$

$\tilde{A} = P^{-1}AP$

$\tilde{B} = P^{-1}B$

O/P equation.

$Y = CX + DU$

$Y = C(PZ) + DU$

$\boxed{Y = \tilde{C}Z + DU}$

$\tilde{C} = PC$

$P = [P_1 : P_2 : \dots : P_n]$

$A \longrightarrow \lambda_1, \lambda_2, \dots \lambda_n.$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are eigen values of system matrix A.

& $P_1, P_2 \dots P_3$ are eigen vector w.r.t eigen values.

Controllable canonical form or companion form or phase variable form.

if $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

then $P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$ Vander Monde Matrix.

Only possible for distinct eigen values.

where $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A.

if eigen values are repeated.

for ex : if A has eigen values $\lambda_1, \lambda_1, \lambda_3$.

then $X = PZ$, $P = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$

Q. Consider the following state space model Perform diagonalization.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & -6 \end{bmatrix}$

Sol. $X = PZ$

$\cancel{\lambda_1 = 11}$ $\cancel{\lambda_2 = -6}$ $\cancel{\lambda_3 = 0}$

$P =$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$$

$\Rightarrow \lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$\tilde{A} = P^{-1}AP$

$$P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$= \dfrac{\text{adj}P}{|P|} \cdot A \cdot P$

$|P| = 1(-6) - 1(-6) + 1(-2) = -6+6-2 = -2$

$$\tilde{A} = \frac{\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}}{-2} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$P^{-1} = \begin{matrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{matrix}$$

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$\tilde{B} = P^{-1}B = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \qquad \tilde{C} = [1 \quad 1 \quad 1]$

Invariance of eigen values.

eign values should remain the same before and after transformation.

So, $|\lambda I - A|$ and $|\lambda I - P^{-1}AP|$ are identical.

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Tutorial-3

1. Find the eigen values, eigen vectors and comment on the stability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Sol. $A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$

$[\lambda I - A]$

$$\Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} = \lambda(\lambda^2 + 9 + 6\lambda) - 1(-4)$$

$$= \lambda^3 + 9\lambda + 6\lambda^2 + 4$$

$$= \lambda^3 + 3\lambda^2 + 9\lambda + 4$$

-1+3-10+7

-1+3-9

-1+3-9+4

$$= \lambda^3 + 6\lambda^2 + 9\lambda + 4$$

$\lambda_1 = -1 \quad \lambda_2 = -1 \quad \lambda_3 = -4$

All eigen values are −ve.

So, system is stable.

$$[\lambda_1 I - A] = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

cofactor = $\begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$

EV_1

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & 0 & -1 \\ 2 & \lambda_2+3 & 0 \\ 0 & -2 & \lambda_2+3 \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_2+3)^2 \\ 2(\lambda_2+3) \\ (-4) \end{bmatrix} = \begin{bmatrix} \frac{d}{d\lambda_2}(\lambda^2+6\lambda+9) \\ \frac{d}{d\lambda}(2\lambda_2+6) \\ \frac{d}{d\lambda}(-4) \end{bmatrix} = \begin{bmatrix} 2\lambda_2+6 \\ 2 \\ 0 \end{bmatrix}$$

$\lambda_2 = -1$

$$EV_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$[\lambda_3 I - A] = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$$EV_3 = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

Q. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Q. CLTF, $\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$. Construct state model in diagonal canonical form and draw block diagram

sol. $$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$$

$$\frac{Y(s)}{U(s)} = \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+3)}$$

Sub $s = 0$, $C_1 = 40/3$

$s = -1$, $C_2 = -15$

$s = -3$, $C_3 = 5/3$

$$Y(s) = \frac{40}{3} \cdot \frac{U(s)}{s} + \frac{-15\,(U(s))}{(s+1)} + \frac{5\,U(s)}{3(s+3)}$$

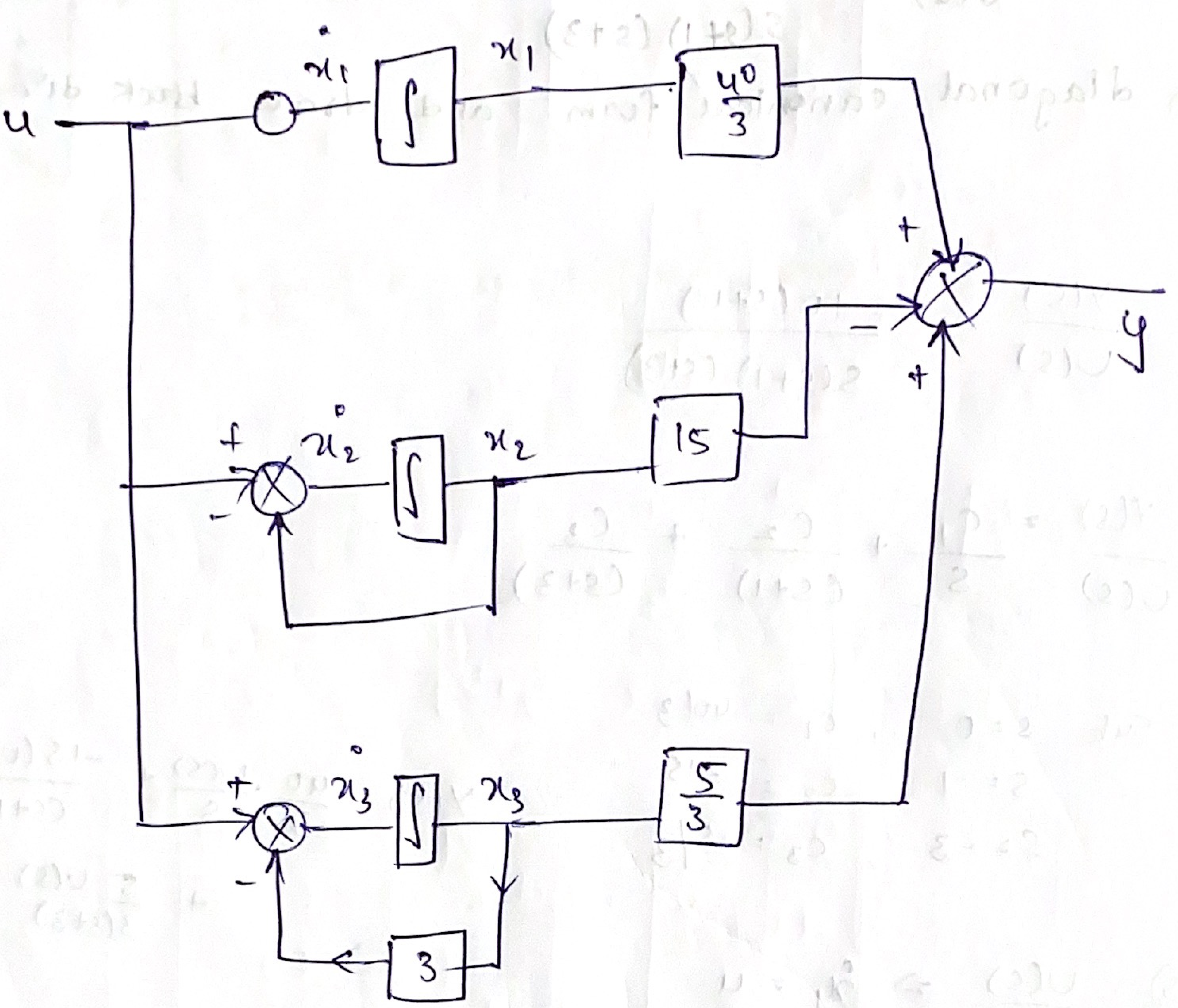
$x_1(s) = \frac{U(s)}{s} \Rightarrow \dot{x}_1 = u$

$x_2(s) = \frac{U(s)}{(s+1)} \Rightarrow \dot{x}_2 = u - x_2$

$X_3(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_3 = u - 3x_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{40}{3} & -15 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$



Q. Perform diagonalization & obtain canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & -2 & -2\\ 0 & -1 & 1\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}2\\ 0\\ 1\end{bmatrix}u \qquad y = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}^T \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + 0u$$

sol. $P = [P_1 : P_2 : P_3]$

$$|\lambda I - A| = \begin{vmatrix}\lambda+1 & 2 & 2\\ 0 & \lambda+1 & -1\\ -1 & 0 & \lambda+1\end{vmatrix} = \lambda+1\left((\lambda+1)^2\right) - 1\left(-2-2\lambda-2\right)$$

$$= (\lambda+1)^3 + 2\lambda + 4$$

$$= \lambda^3 + 1 + 3\lambda^2 + 3\lambda + 2\lambda + 4$$

$$= \lambda^3 + 3\lambda^2 + 5\lambda + 5$$

$\tilde{A} = P^{-1}AP$

$\tilde{B} = P^{-1}B$

$C = PC$

$D = D$

$\lambda_1 = -1.77$

$\lambda_2 = -0.61$

$\lambda_3 = -0.61$

$$P = \begin{bmatrix}1 & 1 & 1\\ \lambda_1 & \lambda_2 & 1\\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 1\\ -1.77 & -0.61 & 1\\ 3.13 & 0.37 & -1.22\end{bmatrix}$$

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Solution of State equation

Consider $\dot{X} = AX \rightarrow (1)$.

Assume, solution $X(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + \cdots$

Sub in (1)

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + k b_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots) \rightarrow (2)$$

Assume the solution to be true solution, then the above eq should hold for all t.

So, equate equal powers of t.

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2} Ab_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} Ab_2 = \frac{1}{3\times 2} A^3 b_0$$

$\vdots$

$$b_k = \frac{1}{k!} \cdot A^k b_0$$

Put $t=0$ in (2).

we get

$$X(0) = b_0$$

So, $x(t) = \left(I + At + \frac{1}{2!} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots\right) x(0)$.

It is of the form matrix exponential.

So, $X(t) = e^{At} \cdot x(0)$

Properties of State transition matrix

i) $\phi(0) = e^{A0} = I$

ii) $\phi(t) = e^{(At)} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$ or $\phi^{-1}(t) = \phi(-t)$

iii) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

iv) $[\phi(t)]^n = \phi(nt)$

v) $\phi(t_2-t_1)\,\phi(t_1-t_0) = \phi(t_2-t_0) = \phi(t_1-t_0)\cdot\phi(t_2-t_1)$

Q1. Obtain state-transition matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol. laplace Inverse method.

$$e^{At} = L^{-1}(SI-A)^{-1}$$

$$\begin{bmatrix} S+2 & -2 \\ -1 & S+3 \end{bmatrix}$$

$S+3 \quad 2$

$S+2$

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$S^2+5S+6-2$

S^2+5S+4

$$SI-A = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{(S^2+3S-2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S+3}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

$\mathcal{L}^{-1}(SI-A)^{-1}$:

$$\frac{s+3}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$As + 4A + Bs + B$

$\boxed{A+B = 1}$

$\boxed{4A+B} = 3$.

$3A = 2$

$\cancel{A=1}$

$B=-1$

$A = \frac{2}{3}$

$B = \frac{1}{3}$

$\frac{8}{3} + \frac{1}{3}$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{s+1} - \frac{1}{s+2}$$

$A = 2$ $B = -1$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$A+B = 0$

$4A+B = \cancel{1}$

$3A = 2$

$A = \frac{\cancel{1}}{3}$

$2A + B = 1$ $A = 1$ $B = -1$

$A+B = 0$

$A = 1$

$A+B = 1$

$4A+B = 2$

$4A+1-A = 2$

$3A = 1$

$A = \frac{1}{3}$

$B = \frac{2}{3}$

$$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}.$$

$A+B = 1$ $2A+B = 0$.

$A = -1$ $B = 2$.

$$\frac{-2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-2}{s+1} + \frac{2}{s+1}$$

$A+B = 0$ $2A+B = -2$

$A = -2$ $B = 2$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

laplace inverse.

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{2t} \end{bmatrix}$$

~~9/10~~ 9/9/22.

Case (ii) Response of system with excitation.

(Non-Homogeneous state equations)

$$\dot{X} = AX + BU$$

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau) \cdot d\tau$$

Q. Obtain the time response of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is the unit step function occuring at $t = 0$ or $u(t) = 1(t)$.

Initial state ~~stat~~ is $X(0) = 0$

Sol. $\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-t} & -e^{-t} + 2e^{2t} \end{bmatrix}$

Here natural response $= 0$ because initial state is zero.

$$x(t) = \phi(t) \cdot x(0) + \int_0^t \phi(t-\tau) \cdot B \cdot u(\tau) \cdot d\tau$$

coz. $u(\tau) = 1 \Rightarrow$ step input

$$\phi(t-\tau)*B = \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}$$

$$\int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)} \cdot d\tau$$

$$= \left[\frac{-e^{-(t-\tau)}}{-1} + e^{-2(t-\tau)} \right]_0^t$$

$$= -e^{0} + e^{0} - \left(-e^{-t} + e^{-2t}\right)$$

$$= e^{-t} - e^{-2t}$$

$$\Rightarrow \int_0^t e^{-(t-\tau)} \cdot d\tau - \int_0^t e^{-2(t-\tau)} \cdot d\tau$$

$$\Rightarrow \left[\frac{e^{-(t-\tau)}}{1}\right]_0^t - \left[\frac{e^{-2(t-\tau)}}{2}\right]_0^t$$

$$\Rightarrow 1 - e^{-t} - \frac{1}{2} + \frac{e^{-2t}}{2} = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$$

$$\int_0^t \left(-e^{-(t-\tau)} + 2e^{-2(t-\tau)}\right) \cdot d\tau$$

$$= e^{-t} - e^{-2t}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$e^{-(t-\tau)}$$

$$\frac{e^{-(t-\tau)}}{-1(-1)}$$

$$e^{-(t-\tau)}$$

$$e^{-2(t-\tau)}$$

$$\frac{e^{-2(t-\tau)}}{2}$$

ex:. Obtain response $y(t)$ with zero initial condition when the system is excited with a unit step input at $t=0$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol: