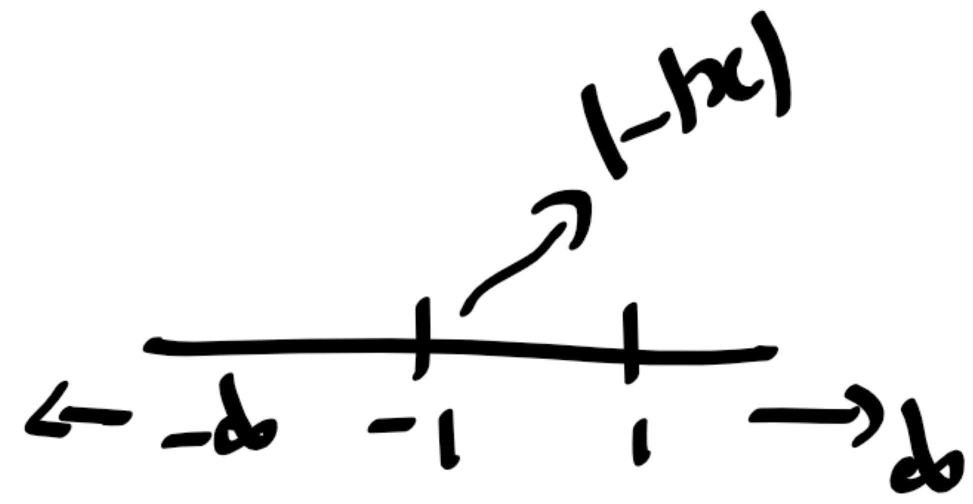
$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$



$$F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$$

To prove $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$

Using Parseval's identity, i.e. $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

$$\int_{-1}^{1} \left(1-|x|\right)^2 dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{(1-\cos s)^2}{s^4} ds$$

$$\cancel{2}\int_0^1 (1-x)^2 dx = \frac{2}{\pi} \times \cancel{2} \int_0^\infty \frac{(1-\cos s)^2}{s^4} ds$$

$$\left[-\frac{(1-x)^3}{3}\right]_0^1 = \frac{2}{\pi} \int_0^\infty \frac{(2\sin^2 s/2)^2}{s^4} ds$$

$$-\frac{1}{3}\left(0-1\right) = \frac{2\times 4}{\pi} \int_0^\infty \left(\frac{\sin s/2}{s}\right)^4 ds$$

$$\therefore \int_0^\infty \left(\frac{\sin s/2}{s}\right)^4 = \frac{\pi}{24}$$

$1-x=t$
$dx=-dt$
$-\int t^2 dt$
$= -\frac{t^3}{3}$
$= -\frac{(1-x)^3}{3}$

Put $\frac{s}{2} = t \Rightarrow s = 2t$, $ds = 2dt$

$$\therefore \int_0^\infty \left(\frac{\sin t}{2t}\right)^4 \cdot 2dt = \frac{\pi}{24} \Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{\cancel{24}\,3} \times \cancel{8}$$

$$= \pi/3$$

4. Find the F.T. of $e^{-a|x|}$, $a>0$ and deduce that (i) $\int_0^{\infty} \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a} e^{-a|x|}$

(ii) $F\left(x e^{-a|x|}\right) = i\sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[e^{-ax} \left(\frac{-a\cos sx + s\sin sx}{a^2+s^2} \right) \right]_0^{\infty} = \underline{\underline{\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}}}$$

$$\left\{ \begin{array}{l} \int e^{ax} \cos(bx+c)\,dx = e^{ax} \left[\frac{a\cos(bx+c) + b\sin(bx+c)}{a^2+b^2} \right] \\ \int e^{ax} \sin(bx+c)\,dx = e^{ax} \left[\frac{a\sin(bx+c) - b\cos(bx+c)}{a^2+b^2} \right] \end{array} \right\}$$

(i) Using inverse F.T. we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$e^{-a|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+s^2} \left(\cos sx - i\sin sx\right) ds$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{a\cos sx}{a^2+s^2}\, ds$$

$$\therefore \int_{0}^{\infty} \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a}\, e^{-a|x|}$$

(ii) Using the property, $F\left(x^n f(x)\right) = (-i)^n \frac{d^n}{ds^n} F(s)$

$$\therefore F\left(x e^{-a|x|}\right) = -i \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+s^2}\right)$$

$$= -i\sqrt{\frac{2}{\pi}}\, a \frac{d}{ds}\left(\frac{1}{a^2+s^2}\right)$$

$$= -i\sqrt{\frac{2}{\pi}}\, a \left(\frac{-2s}{(a^2+s^2)^2}\right)$$

$$= i\sqrt{\frac{2}{\pi}}\, \frac{2as}{(a^2+s^2)^2}$$

Definition If the transform of $f(x)$ is $f(s)$, then $f(x)$ is called self-reciprocal.

ie, $F\left(f(x)\right) = f(s)$ then $f(x)$ is self-reciprocal.

5. Find the F.T. of $e^{-a^2x^2}$, $a>0$ and show that $e^{-x^2/2}$ is self-reciprocal.

$$F\left(e^{-a^2x^2}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx-\frac{s^2}{4a^2}+\frac{s^2}{4a^2}\right)}\,dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}\,dx$$

Put $ax-\frac{is}{2a} = t$, $dx = \frac{dt}{a}$

$$\therefore F\left(e^{-a^2x^2}\right) = \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{e^{-s^2/4a^2}}{a}\int_{0}^{\infty} e^{-t^2}\,dt\,.$$

To evaluate $\int_0^\infty e^{-t^2}\,dt$

Put $t^2 = z$

$$2t\,dt = dz$$

$$dt = \frac{dz}{2t} = \frac{dz}{2\sqrt{z}}$$

$$\int_0^\infty e^{-t^2}\,dt = \int_0^\infty e^{-z}\,\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty e^{-z}\,z^{-\frac{1}{2}}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z}\,z^{\frac{1}{2}-1}\,dz$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

$\Gamma(n) = \int_0^\infty e^{-t}\,t^{n-1}\,dt$

↓

Gamma function.

$$\therefore F\left(e^{-a^2x^2}\right) = \frac{e^{-s^2/4a^2}}{a}\sqrt{\frac{2}{\pi}} \times \frac{\sqrt{\pi}}{2}$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2}\,a}$$

Put $a = 1/\sqrt{2}$

$$F\left(e^{-x^2/2}\right) = \frac{e^{-s^2/4\times\frac{1}{2}}}{\sqrt{2} \times \frac{1}{\sqrt{2}}} = e^{-s^2/2}$$

$\Rightarrow e^{-x^2/2}$ is <u>self-reciprocal</u> under the F.T

Fourier cosine transform

Fourier cosine transform of a function $f(x)$ is defined as

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = F_c(s)$$

and inverse Fourier cosine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(f(x)) \cos sx \, ds$$

Fourier Sine transform of $f(x)$ is

$$F_s(f(x)) = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

and inverse sine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds.$$

Note: (1) Fourier cosine transform of the cosine transform of a given function is itself.

$$F_c(F_c(f(x))) = f(s)$$

(2) Fourier sine transform of the sine transform of a given function is itself.

$$F_s(F_s(f(x))) = f(s).$$

Properties

1. $F_c\left[af(x)+bg(x)\right] = a\,F_c(f(x)) + b\,F_c(g(x))$

2. $F_s\left[af(x)+bg(x)\right] = a\,F_s(f(x)) + b\,F_s(g(x))$

3. $F_c\left[f(x)\cos ax\right] = \frac{1}{2}\left\{F_c(s+a) + F_c(s-a)\right\}$

Pf:- $F_c\left[f(x)\cos ax\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^\infty f(x)\left[\cos(s+a)+\cos(s-a)\right]dx$$

$$= \frac{1}{2}\left\{\underline{\underline{F_c(s+a) + F_c(s-a)}}\right\}$$

4. $F_c\left[f(x)\sin ax\right] = \frac{1}{2}\left\{F_s(a+s) + F_s(a-s)\right\}$

5. $F_s\left[f(x)\cos ax\right] = \frac{1}{2}\left\{F_s(s+a) + F_s(s-a)\right\}$

6. $F_s\left[f(x)\sin ax\right] = \frac{1}{2}\left\{F_c(s-a) - F_c(s+a)\right\}$

7. $F_c(f(ax)) = \frac{1}{a}F_c(s/a)$

8. $F_s(f(ax)) = \frac{1}{a}F_s(s/a)$

9. $F_c(f'(x)) = -\sqrt{\frac{2}{\pi}}f(0) + sF_s(s)$

10. $F_s(f'(x)) = -sF_c(s)$

Parseval's identity

(1) $$\int_0^\infty |F_c(s)|^2 ds = \int_0^\infty |f(x)|^2 dx$$

(2) $$\int_0^\infty |F_s(s)|^2 ds = \int_0^\infty |f(x)|^2 dx.$$

Result

$$F_c(x f(x)) = \frac{d}{ds} F_s(f(x))$$

and $$F_s(x f(x)) = -\frac{d}{ds} F_c(f(x))$$

Ans

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

$$\frac{d}{ds} F_s(f(x)) = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s} f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$= \underline{F_c(x f(x))}$$

Exercises

1. Find the F.T. of $f(x) = \begin{cases} |x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

2. Find the F.T. of $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

3. Find the F.T. of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

   Hence deduce that $\int_0^{\pi/2} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

4. Find the F.T. of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

   Hence deduce that (i) $\int_0^{\infty} \frac{\sin t - t\cos t}{t^3} dt = \pi/4$

   (ii) $\int_0^{\infty} \frac{(\sin t - t\cos t)^2}{t^6} dt = \pi/15$.