CS 188
Spring 2020

Introduction to
Artificial Intelligence

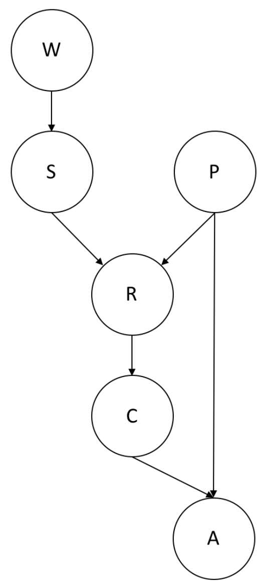
Written HW 3 Sol.

Solutions for HW 3 (Written)

Q1. [30 pts] Quadcopter: Spectator

Flying a quadcopter can be modeled using a Bayes Net with the following variables:

- W (weather) $\in$ {clear, cloudy, rainy}
- S (signal strength) $\in$ {strong, medium, weak}
- P (true position) $= (x, y, z, \theta)$ where x, y, z **each** can take on values $\in \{0, 1, 2, 3, 4\}$ and θ can take on values $\in \{0°, 90°, 180°, 270°\}$
- R (reading of the position) $= (x, y, z, \theta)$ where x, y, z **each** can take on values $\in \{0, 1, 2, 3, 4\}$ and θ can take on values $\in \{0°, 90°, 180°, 270°\}$
- C (control from the pilot) $\in$ {forward, backward, rotate left, rotate right, ascend, descend} (6 controls in total)
- A (smart alarm to warn pilot if that control could cause a collision) $\in$ {bad, good}



(a) Representation

(i) [2 pts] What is N_p, where N_p is the domain size of the variable P? Please explain your answer.
Answer: $N_p = 500$
Explanation: 5 * 5 * 5 * 4 = 500

(ii) [2 pts] Please list **all** of the Conditional Probability Tables that are needed in order to represent the Bayes Net above.
$P(W), P(S|W), P(P), P(R|S, P), P(C|R), P(A|C, P)$

(iii) [1 pt] What is the size of the Conditional Probability Table for R? You may use N_p in your answer.
$P(R|S, P)$ has size 750000 or $3 * (N_p)^2$

Now, assume that we look at this setup from the perspective of Spencer – a spectator who can observe A and W. Spencer observes A=bad and W=clear, and he now wants to infer the signal strength. In BN terminology, he wants to calculate $P(S|A = \text{bad}, W = \text{clear})$.

(b) [5 pts] Inference by Enumeration

If Spencer chooses to solve for this quantity using inference by enumeration, what is the biggest **factor** that must be calculated along the way, and what is its **size**? You may use N_p in your answer. Please show your work.

Biggest factor: $f(W = clear, S, P, R, C, A = bad)$
Size of factor: Size is $1 * 3 * N_p * N_p * 6 * 1 = 18(N_p)^2$

(c) [5 pts] Inference by Variable Elimination: Order 1

Spencer chooses to solve for this quantity by performing variable elimination in the order of $R - P - C$. Answer the following prompts to work your way through this procedure.

(1a) First, we need to eliminate R. Which factors (from the 6 CPTs above) are involved?
$P(R|S, P), P(C|R)$

(1b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
eliminating r out of $P(r|S, P) * P(C|r)$ results in $f(C|S, P)$

(2a) Second, we need to eliminate P. Which factors are involved?
$P(A = bad|C, P), P(P), f(C|S, P)$

(2b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
eliminating p out of $P(A = bad|C, p) * P(p) * f(C|S, P)$ results in $f(A = bad, C|S)$

(3a) Third, we need to eliminate C. Which factor/s are involved?
$f(A = bad, C|S)$

(3b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
eliminating c out of $f(A = bad, C|S)$ results in $f(A = bad|S)$

(4) List the 3 conditional probability factors that you calculated as a result of the 3 elimination steps above, along with their domain sizes. Which factor is the biggest? Is this bigger or smaller than the biggest factor from the "inference by enumeration" approach?
$f(C|S, P), f(A = bad, C|S), f(A = bad|S)$. Biggest is $f(C|S, P)$, whose size is $6 * 3 * N_p = 18 * N_p$. This is smaller than the biggest factor from inference by enumeration.

(5) List the **1** unused conditional probability factor from the 3 that you calculated above, and also list the **2** unused conditional probability factors from the 6 original CPTs.
$f(A = bad|S), P(S|W = clear), P(W = clear)$

(6) Finally, let's solve for the original quantity of interest: $P(S|A = \text{bad}, W = \text{clear})$. After writing the equations to show how to use the factors from (5) in order to solve for $f(S|A = bad, W = clear)$, don't forget to write how to turn that into a probability $P(S|A = bad, W = clear)$.
Hint: use Bayes Rule, and use the 3 unused factors that you listed in the previous question.
By Bayes rule, $P(S|A = bad, W = clear) = \frac{P(A=bad,S,W=clear)}{P(A=bad,W=clear)}$ where $f(A = bad|S)P(S|W = clear)P(W = clear)$ gives the joint $f(A = bad, S, W = clear)$, and eliminating s out of that joint gives $f(A = bad, W = clear)$. Just normalize the whole thing at the end.

(d) [5 pts] Inference by Variable Elimination: Order 2

Spencer chooses to solve for this quantity by performing variable elimination, but this time, he wants to do elimination in the order of $P - C - R$. Answer the following prompts to work your way through this procedure.

(1a) First, we need to eliminate P. Which factors (from the 6 CPTs above) are involved?
$P(P), P(R|S, P), P(A = bad|C, P)$

(1b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
$f(R, A = bad|S, C)$

(2a) Second, we need to eliminate C. Which factors are involved? Recall that you might want to use the factor that resulted from the previous step, but you should not reuse factors that you already used.
$P(C|R), f(R, A = bad|S, C)$

(2b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
$f(R, A = bad|S)$. for staff: see this example here about why this is totally fine. students might be confused since R is on left of one and right of the other. and

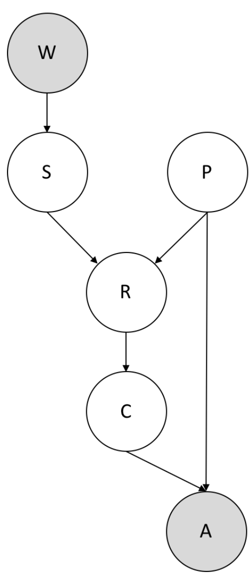
(3a) Third, we need to eliminate R. Which factor/s are involved? Recall that you might want to use the factor that resulted from the previous step, but you should not reuse factors that you already used.
$f(R, A = bad|S)$

(3b) Show the multiplication of those factors and eliminate the variable of interest. What conditional probability **factor** results from this step?
$f(A = bad|S)$

(4) List the 3 conditional probability factors that you calculated as a result of the 3 elimination steps above, along with their domain sizes. Which factor is the biggest? Is this bigger or smaller than the biggest factor from the "inference by enumeration" approach? Is this bigger or smaller than the biggest factor from R-P-C elimination order?
Biggest is $f(R, A = bad|S, C)$, and since A is observed to be bad, the size of this is $N_p * 1 * 3 * 6 = 18N_p$. This is smaller than the biggest factor from inference by enumeration, but equal to the biggest factor from the R-P-C ordering.

(e) D-Separation

(i) [5 pts] Which variable (in addition to W and A), if observed, would make $S \perp\!\!\!\perp P$? Please shade in the variable that fits the criteria and run the D-separation algorithm on the resulting graph to support your answer. If no such variable exists, please provide an explanation for why each candidate fails.



The key thing to note here is that before shading anything in, $S - A - P$ is a **special active triple through** $R - C$ as outlined on Note 6 page 12).
Shading in S and P does not make $S - A - P$ inactive, so they both fail.

Shading in R makes $S - R - P$ an active path with only 1 active triple, so it fail.

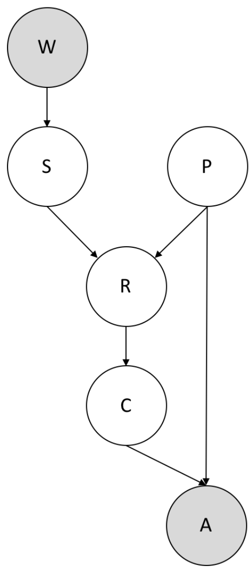
Shading in C makes $S - C - P$ an active path with only 1 active triple that is the same type as the previous $S - T - P$.

Therefore all candidates fail

(ii) [5 pts] Ivan claims that there exist two variables (which are NOT directly connected) such that if you know the value of all **other** variables, then those two variables are guaranteed to be independent. Is Ivan right?

● Yes, and I will shade in all but two variables in the graph below, and run D-separation algorithm to prove that those two variables are guaranteed to be independent conditioned on all other variables in the graph.

○ No, there is no such two variables, and I will provide a proof below.



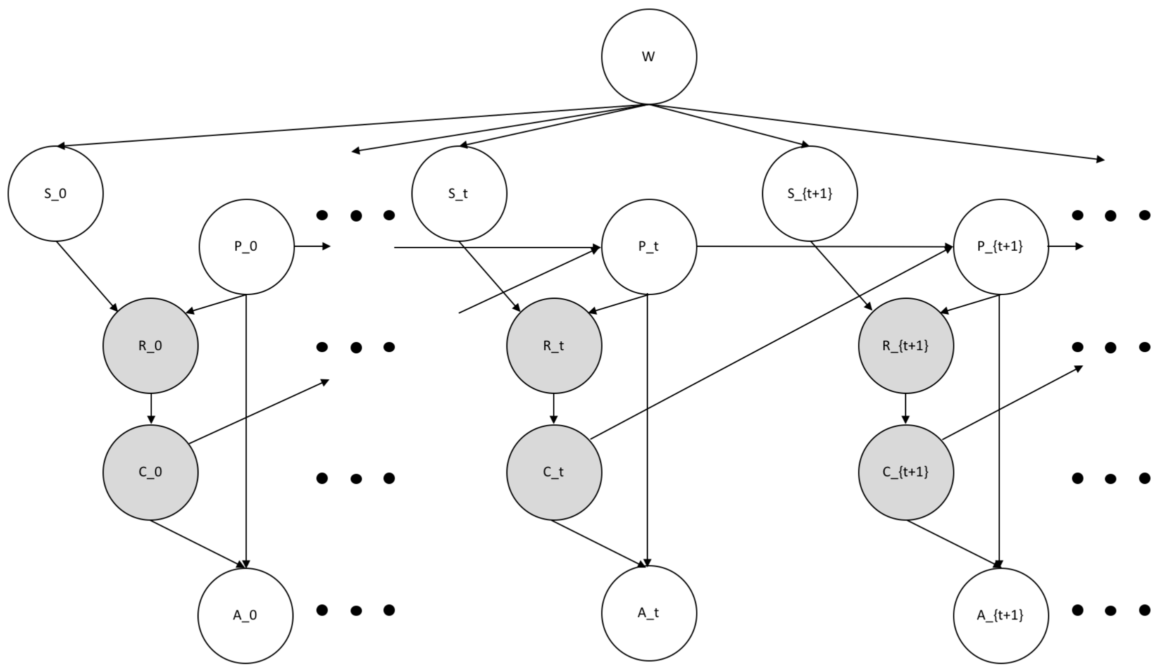
S is independent of C given all other variables.

Q2. [40 pts] Quadcopter: Data Analyst

In this question, we look at the setup from the previous problem, but we consider the quadcopter flight over time. Here, flight can be considered in discrete time-steps: $t \in 0, 1, 2, ..., N-1$ with, for example, P_t representing the true position P at discrete time-step t. Suppose the weather (W) does not change throughout the quadcopter flight.

One key thing to note here is that there are edges going between time t and time $t+1$: The true position at time $t+1$ depends on the true position at time t as well as the control input from time t.

Let's look at this setup from the perspective of Diana, a data analyst who can only **observe** the output from a data-logger, which stores **R (reading of position) and C (control from the pilot)**.



(a) Hidden Markov Model

(i) [4 pts] List all the hidden variables and observed variables in this setup. In a few sentences, how is this setup different from the vanilla Hidden Markov Model you saw in lecture? You should identify at least 2 major differences.

Hidden variables: W, $S_i \forall i$, $P_i \forall i$, $A_i \forall i$

Observed variables: $R_i \forall i$, $C_i \forall i$

Differences: Include but not limited to

1. There is one overarching hidden variable that doesn't change with respect to time.
2. There are multiple observation variables and multiple hidden variables at any time step.
3. There is a hidden variable (A) at the tail of observed variables (R and C).

(ii) [3 pts] As a data analyst, Diana's responsibility is to infer the true positions of the quadcopter throughout its flight. In other words, she wants to find a list of true positions $p_0, p_1, p_2, ..., p_N$ that are the most likely to have happened, given the recorded readings $r_0, r_1, r_2, ..., r_N$ and controls $c_0, c_1, c_2, ..., c_N$.

Write down the probability that Diana tries to maximize in terms of a **joint probability**, and interpret the meaning of that probability. Note that the objective that you write below is such that Diana is solving the following problem: $\max_{p_0, p_1, ..., p_N}$ (maximization objective).

Maximization objective: $P(P_0 = p_0, P_1 = p_1, \ldots, P_N = p_N, R_0 = r_0, R_N = r_N, C_0 = c_0, C_N = c_N)$

Explanation: Probability of the positions being there and the observations of reading/controls being there also.

(iii) [3 pts] Morris, a colleague of Diana's, points out that maximizing the joint probability is the same as maximizing a **conditional probability** where all evidence ($r_0, r_1, r_2, \dots$ and $c_0, c_1, c_2, \dots$) are moved to the right of the conditional bar. Is Morris right?

- ● Yes, and I will provide a proof/explanation below.
- ○ No, and I will provide a counter example below.

Yes. $max_p P(p, r, c)$ is the same as $max_p P(p|r, c)$ because when you factor the joint, the $P(r, c)$ is independent of the optimization variables p and can thus be considered a constant.

(b) The Markov Property

(i) [5 pts] In this setup, conditioned on all observed evidence, does $P_0, P_2, ..., P_N$ follow the Markov property? Please justify your answer. No. Because of W, there is an active path $(P_{t-2}, R_{t-2}, S_{t-2})$-$(W)$-$(P_t, R_t, S_t)$

(ii) [5 pts] In this setup, conditioned on all observed evidence, does $S_0, S_2, ..., S_N$ follow the Markov property? Please justify your answer. No because S_{t+1} is not independent of S_{t-1}. Knowing S_t doesn't block the active triple $S_{t-1} - W - S_{t+1}$.

(c) Forward Algorithm Proxy

Conner, a colleague of Diana's, would like to use this model (with the R_t and C_t observations) to perform something analogous to the forward algorithm for HMMs to infer the true positions. Let's analyze below the effects that certain decisions can have on the outcome of running the forward algorithm.

Note that when we say to **not include** some variable in the algorithm, we mean that we marginalize/sum out that variable. For example, if we do not want to include W in the algorithm, then we replace $P(S_t|W)$ everywhere with $P(S_t)$, where $P(S_t) = \sum_W P(S_t|W)P(W)$.

(i) [4 pts] He argues that since W (weather) does not depend on time, and is not something he is directly interested in, he does not need to include it in the forward algorithm. What effect does not including W in the forward algorithm have on (a) the accuracy of the resulting belief state calculations, and on (b) the efficiency of calculations? Please justify your answer.

Accuracy: This makes everything less informed than if you were to include the W, so accuracy is worse.

Efficiency: Efficiency is better since you only need to include $P(S_t)$ everywhere, instead of $P(S_t|W)$

(ii) [3 pts] He also argues that he does not need to include hidden state A (smart alarm warning) in the forward algorithm. What effect does not including A in the forward algorithm have on (a) the accuracy of the resulting belief state calculations, and on (b) the efficiency of calculations? Please justify your answer.

Accuracy: Since A is downstream of the observations and also doesn't effect the next timesteps in any way, not including A in the forward algorithm has no effect on accuracy.

Efficiency: No effect of efficiency, since it shouldn't be included anyway. Note: if student says that it reduces efficiency because of the extra marginalization step mentioned above (since you wouldn't have done that otherwise), consider that correct, even though you don't need to do it anyway.

(iii) [3 pts] Last but not least, Conner recalls that for the forward algorithm, one should calculate the belief at time-step t by conditioning on evidence up to $t-1$, instead of conditioning on evidence from the entire trajectory (up to N). Let's assume that some other algorithm allows us to use evidence from the full trajectory ($t = 0$ to $t = N$) in order to infer each belief state. What is an example of a situation (in this setup, with the quadcopter variables) that illustrates that incorporating evidence from the full trajectory can result in better belief states than incorporating evidence only from the prior steps?

If the signal strength is bad at the beginning of the traj but gets better later, the information from later in the trajectory can greatly improve the belief state estimates earlier in the trajectory (whereas if only evidence up to $t-1$ is allowed to be used at step t, then those early belief states would not have been good).

(d) Policy Reconstruction

(i) [2 pts] Denise states that the probabilistic model for the pilot's **policy** is entirely captured in one Conditional Probability Table from the Bayes Net Representation. Which table do you think this is, and explain why this table captures the pilot's policy.

Table: $P(C|R)$

Explanation: The probability of control given a reading of position is exactly what a policy looks like in this representation

(ii) [4 pts] Denise argues that if we were given a lot of data from the data logger, we could reconstruct the probabilistic model for the pilot's policy. Is she right?

● Yes, and I will provide an overview of how to reconstruct the pilot's policy from the data.

○ No, and I will provide a list of reasons for why we cannot reconstruct the policy.

Counting is your friend.

(iii) [4 pts] Supposed Diana (from earlier) now wants to use particle filtering as a way to estimate the hidden states. In order to do this, assume that she has the R and C observations (as above), along with the necessary transition models, sensor models, etc.

Let's assume that Diana has multiple logs of data, where each log contains many flights from a specific human operator. Diana reasons that the transition model $P(P_{t+1}|P_t, C_t)$ is determined by the dynamics of the quadcopter itself (so it doesn't change across human operators), and other models such as $P(R_t|S_t, P_t)$ also are pre-defined for this problem setting (and don't change across human operators). However, the factor discovered by Denise (above) is actually one that is slightly different per human operator.

(a) If Diana uses Denise's approach of reconstructing a policy from the observed data (from a particular human operator's log), would this be a more or less accurate model for representing that operator's control choices, as compared to some generic policy model that summarizes overall/average human behavior? Why?

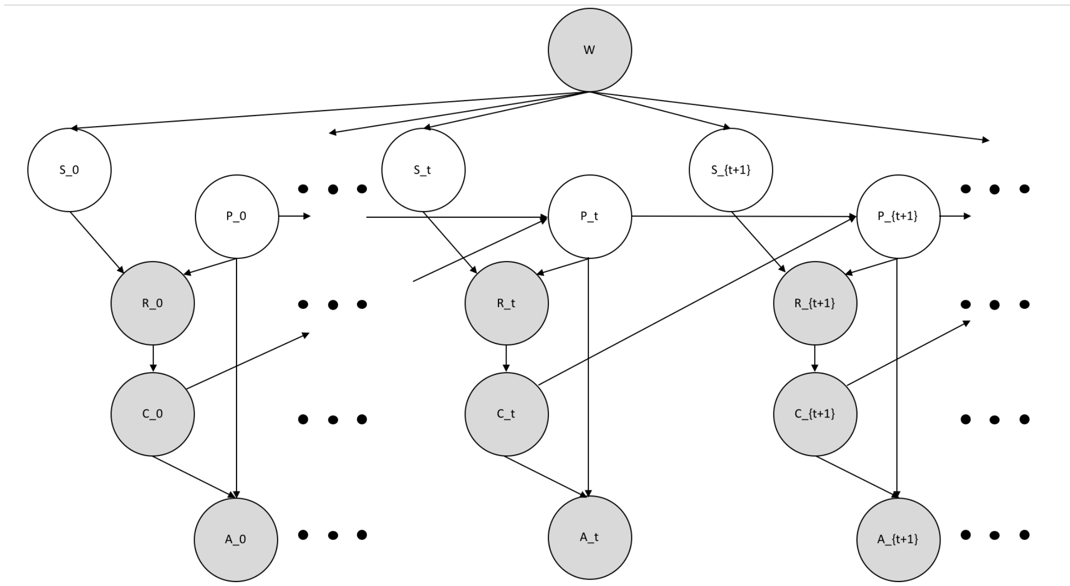
It's more accurate to reconstruct the policy from the observed data. An average human's $P(C|R)$ policy model would not describe the data from this operator's log as well as a policy which was explicitly fit to this operator's data.

(b) How exactly would this new model be used in the particle filtering process? Would it lead to improvements in the results of particle filtering? Why?

When reweighting particles based on $P(evidence|particles)$, that looks like $P(R, C|S, P)$ which can be broken into $P(C|R) * P(R|S, P)$. So if this new $P(C|R)$ is more accurate than some generic model of human behavior, then the resulting particle filtering results would be more accurate

Q3. [30 pts] Quadcopter: Pilot

In this question, we look at same setup from Question 2, but we now look at it from the perspective of Paul, a quadcopter pilot who can **observe W (weather), R (reading of position), C (control from the pilot), and A (smart alarm warning)**. As before, suppose weather (W) does not change throughout the quadcopter's flight.



(a) Forward Algorithm: The real deal

(i) [2 pts] Now that the only hidden states are S_t and P_t, is this graph a well-behaving HMM (where $E_{t+1} \perp\!\!\!\perp E_t \mid X_{t+1}$ and $X_{t+1} \perp\!\!\!\perp E_t \mid X_t$, recall that X is the hidden variable and E is the evidence variable)? Please explain your reasoning. A subset of your responses from Q2 might apply here.

No, it is not a well-behaving HMM, because the observed state C_t can affect the hidden state P_{t+1} (in the next time-step). Also, there is one over-arching observed state (the weather) that affects every time step.

(ii) [4 pts] What is the **time-elapsed update** from time-step t to time-step $t+1$? Be sure to include all hidden states and observed states, and show how to assemble the update from the conditional probability tables corresponding to the graph.

The solution follows the notation from lecture note 8 (https://inst.eecs.berkeley.edu/ cs188/sp20/assets/notes/n8.pdf)

$B(S_t, P_t) = P(S_t, P_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$B'(S_{t+1}, P_{t+1})$

$= P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$= \sum_{s_t} \sum_{p_t} P(S_{t+1}, P_{t+1}, s_t, p_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$= \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t)) * P(s_t, p_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$= \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t)) * B(S_t, P_t)$

(iii) [4 pts] What is the **observation update** at time-step $t+1$? Be sure to include all hidden states and observed states, and show how to assemble the update from the conditional probability tables corresponding to the graph.

$B'(S_{t+1}, P_{t+1}) = P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$B(S_{t+1}, P_{t+1})$

$= P(S_{t+1}, P_{t+1} \mid W, R_{0:t+1}, C_{0:t+1}, A_{0:t+1})$

$\propto P(S_{t+1}, P_{t+1}, R_{t+1}, C_{t+1}, A_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$= P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}) * P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$

$= P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}) * B'(S_{t+1}, P_{t+1})$

(b) Consider a simpler scenario where we only track the 2D position (x, y) of the quadcopter, where x, y **each** can take on values $\in \{0, 1, 2\}$. Consequently, we now only have four controls: forward, backward, left, and right. Paul, the pilot, wants to infer the quadcopter's true position P as accurately as possible. In this problem, consider an additional variable E_R, which represents Paul's estimate of the current position, and this variable depends on the reading R. Consider a penalty that is equal to the L1 norm of the difference between the estimate of current position E_R and the actual position P. Thus, the utility is defined to be $U(P, E_R) = U_{max} - ||P - E_R||_1$, in dollars.

Suppose for the rest of this question that your **reading R is** $(1, 0)$ and that the weather is always cloudy. Given this cloudy weather, the signal strength can take on 3 values with equal probability: weak, medium, and strong. These signal strengths correspond to the following errors in readings:

- Weak: The reading R returns a random number (for each position element) sampled uniformly from the domain of possible positions.
- Medium: The reading R has an error of at most 1, and that error occurs in only one of the position elements.
- Strong: The reading R is identical to the true position.

Additionally, suppose that when the signal strength is unknown, the reading R is assumed to have a 50% chance of being correct and 50% chance of being incorrect (where, if the reading is incorrect, it takes on a uniformly random value from the list of possible incorrect positions).

(i) [2 pts] Which variable should intuitively have the greatest VPI? Explain your answer. You should not do any calculations for this part.

P would be the best possible variable to observe, since our utility captures how well we can guess P.

(ii) [8 pts] Suppose Paul's coworker offers to tell him the signal strength (S) in exchange for some cash. What is the most Paul should pay to know the value of S?
Hint: Recall the current reading R.
$VPI(S) = MEU(S) - MEU(\varnothing)$.

First, we will calculate $MEU(\varnothing)$. We will make our guess (E) of the position to be equal to our reading R. When we don't know the signal strength, with $\frac{1}{2}$ probability, we will incur no penalty because $R = P$. With $\frac{1}{2} * \frac{1}{8}$ probability, the actual position is a different position. If we are in square $(0, 0), (2, 0), (1, 1)$, we will receive utility $U_{max} - 1$. If we are in square $(0, 1), (1, 2), (2, 1)$, we achieve utility $U_{max} - 2$. If we are in $(0, 2), (2, 2)$, we achieve utility $U_{max} - 3$. Our expected utility is then $MEU(\varnothing) = \frac{1}{2} * U_{max} + \frac{1}{2}\frac{1}{8}(U_{max} - (1 + 1 + 1 + 2 + 2 + 2 + 3 + 3)) = U_{max} - \frac{15}{16}$.

Now, we will calculate $MEU(S)$ using the fact that each signal strength is possible with equal probability, so $MEU(S) = \sum_s P(S = s)MEU(S = s)$.

If the reading is weak, then the reading is random, and we gain no information about P. Then, we can achieve $MEU(S = weak)$ by guessing our position is in the middle square at $(1, 1)$, which will give us utility $MEU(S = weak) = U_{max} - \frac{1}{9}(0 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 2) = U_{max} - \frac{4}{3}$. (If we chose a corner square, we would get expected utility $U_{max} - 2$. If we had chosen any other square, we would get expected utility $U_{max} - \frac{5}{3}$.)

If the reading is medium, then we know that it's off by at most 1 in one of the directions x or y. Then, based off our reading of $(1, 0)$, we know that our true position must be one of the following: $\{(0, 0), (1, 0), (2, 0), (1, 1)\}$. We then find that we can achieve $MEU(S = medium) = U_{max} - \frac{1}{4}(1 + 0 + 1 + 1) = U_{max} - \frac{3}{4}$ if we guess our true position to be the reading $(1, 0)$. (All the other options give expected utility $U_{max} - \frac{5}{4}$).

Finally, if the reading is strong, then it's the same as the true position. In this case, we would be able to achieve $MEU(S = strong) = U_{max}$ by guessing $(1, 0)$.

$MEU(S) = \frac{1}{3}\sum_s MEU(S = s) = \frac{1}{3}(U_{max} - \frac{4}{3} + U_{max} - \frac{3}{4} + U_{max}) = U_{max} - \frac{25}{36}$.

$VPI(S) = MEU(S) - MEU(\varnothing) = (U_{max} - \frac{25}{36}) - (U_{max} - \frac{15}{16}) = \frac{35}{144} = 0.243.$
You should pay your coworker up to \$0.24.

(c) **(i)** [5 pts] Suppose your coworker only tells you the signal strength with probability p, and with probability $1 - p$, they don't tell you the signal strength even after payment. How much would you be willing to pay in this scenario?

Let us denote S' as the response we get from our coworker.
With probability p, we can achieve the previously calculated $MEU(S)$ of $U_{max} - \frac{7}{9}$.
With probability $1-p$ our coworker doesn't tell us anything, and we can only achieve the value $MEU(\varnothing) = U_{max} - \frac{15}{16}$.

$MEU(S') = p * (U_{max} - \frac{25}{36}) + (1-p)(U_{max} - \frac{15}{16}) = U_{max} - \frac{15}{16} + \frac{35p}{144}.$
$VPI(S') = MEU(S') - MEU(\varnothing) = U_{max} - \frac{15}{16} + \frac{35p}{144} - U_{max} + \frac{15}{16} = \frac{35p}{144}.$
We would pay the highest cent amount less than $\frac{35p}{144}$.

(ii) [5 pts] How much would you pay to know the true position (P)?

$VPI(P) = MEU(P) - MEU(\varnothing).$
$MEU(\varnothing) = U_{max} - \frac{15}{16}.$
We also note that $MEU(P) = U_{max}$ because we can just report the known position.
Then, $VPI(P) = U_{max} - U_{max} + \frac{15}{16} = \frac{15}{16} = 0.9375.$
Thus, you should pay at most \$0.93.