09 级一期 B 卷参考解答

一.(每小题 6 分,共 12 分)求极限:

(1) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{2x^3}$;

解 原式 $=\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{2x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{4x^3}$

$=\lim\limits_{x\to 0}\dfrac{\sec^2 x-\cos x}{12x^2}=\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{12x^2\cos^2 x}=\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{12x^2}=\lim\limits_{x\to 0}\dfrac{3\cos^2 x\sin x}{24x}=\dfrac{1}{8}.$

(2) $\lim\limits_{x\to 0}\left(\cos x\right)^{\frac{1}{\sin^2 x}}$.

解 $\lim\limits_{x\to 0}\left(\cos x\right)^{\frac{1}{\sin^2 x}}=\lim\limits_{x\to 0}e^{\frac{1}{\sin^2 x}\ln\cos x}.$ 而

$\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{-\sin x/\cos x}{2\sin x\cos x}=-\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{1}{\cos^2 x}=-\dfrac{1}{2}.$

故 $\lim\limits_{x\to 0}\left(\cos x\right)^{\frac{1}{\sin^2 x}}=e^{-\frac{1}{2}}=\dfrac{1}{\sqrt{e}}.$

二. (每小题 6 分,共 24 分)求下列积分:

(1) $\displaystyle\int\dfrac{dx}{x\left(2+x^{10}\right)}$;

解 $\displaystyle\int\dfrac{dx}{x\left(2+x^{10}\right)}=\int\dfrac{x^9 dx}{x^{10}\left(2+x^{10}\right)}=\dfrac{1}{10}\int\dfrac{dx^{10}}{x^{10}\left(2+x^{10}\right)};$

$=\dfrac{1}{20}\left(\displaystyle\int\dfrac{dx^{10}}{x^{10}}-\int\dfrac{dx^{10}}{2+x^{10}}\right)=\dfrac{1}{20}\ln\left|\dfrac{x^{10}}{2+x^{10}}\right|+C.$

(2) $\displaystyle\int\cos(\ln x)dx$;

解 $\displaystyle\int\cos(\ln x)dx\overset{u=\ln x}{=}\int e^u\cos u\,du=\int e^u d\sin u=e^u\sin u-\int\sin u\,de^u$

$=e^u\sin u+\displaystyle\int e^u d\cos u=e^u\sin u+e^u\cos u-\int e^u\cos u\,du$

所以 $\displaystyle\int\cos(\ln x)dx=\dfrac{e^u}{2}\left[\sin u+\cos u\right]+C=\dfrac{x}{2}\left[\sin\ln x+\cos\ln x\right]+C.$

(3) $\int_1^e \dfrac{dx}{x\left(2+\ln^2 x\right)}$;

解　$\int_1^e \dfrac{dx}{x\left(2+\ln^2 x\right)} \overset{u=\ln x}{=\!=} \int_0^1 \dfrac{du}{2+u^2} = \dfrac{1}{\sqrt{2}}\int_0^1 \dfrac{d(u/\sqrt{2})}{1+(u/\sqrt{2})^2}$

$$= \dfrac{1}{\sqrt{2}} \arctan \dfrac{u}{\sqrt{2}} \Bigg|_0^1 = \dfrac{1}{\sqrt{2}} \arctan \dfrac{1}{\sqrt{2}}.$$

(4) $\int_0^{\frac{\pi}{2}} \dfrac{\cos t}{\sin t + \cos t} dt$

解　$\int_0^{\frac{\pi}{2}} \dfrac{\cos t}{\sin t + \cos t} dt \overset{s=\frac{\pi}{2}-t}{=\!=} \int_{\frac{\pi}{2}}^0 \dfrac{\cos\left(\dfrac{\pi}{2}-s\right)}{\sin\left(\dfrac{\pi}{2}-t\right)+\cos\left(\dfrac{\pi}{2}-t\right)} d\left(\dfrac{\pi}{2}-t\right)$

$$= \int_0^{\frac{\pi}{2}} \dfrac{\sin s}{\sin s + \cos s} ds = \int_0^{\frac{\pi}{2}} \dfrac{\sin t}{\sin t + \cos t} dt$$

而　$\int_0^{\frac{\pi}{2}} \dfrac{\cos t}{\sin t + \cos t} dt + \int_0^{\frac{\pi}{2}} \dfrac{\sin t}{\sin t + \cos t} dt = \int_0^{\frac{\pi}{2}} dt = \dfrac{\pi}{2}$,

于是　$\int_0^{\frac{\pi}{2}} \dfrac{\cos t}{\sin t + \cos t} dt = \dfrac{\pi}{4}$.

三. (每小题 7 分, 共 21 分)

(1)设 $z(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, 求 $dz\big|_{(0,1)}$;

解　$\dfrac{\partial z}{\partial x} = \dfrac{\sqrt{x^2+y^2} - x\dfrac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{y^2}{(x^2+y^2)^{3/2}}$,　$\dfrac{\partial z}{\partial y} = \dfrac{-x\dfrac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{-xy}{(x^2+y^2)^{3/2}}$,

于是　$\dfrac{\partial z}{\partial x}\bigg|_{(0,1)} = 1$,　$\dfrac{\partial z}{\partial y}\bigg|_{(0,1)} = 0$,　故　$dz\big|_{(0,1)} = \dfrac{\partial z}{\partial x}\bigg|_{(0,1)} dx + \dfrac{\partial z}{\partial y}\bigg|_{(0,1)} dy = dx$.

(2)已知 $f(x,y,z) = \ln\left(x+\sqrt{y^2+z^2}\right)$ 及点 $A(1,0,1), B(3,-2,-2)$, 求函数 $f(x,y,z)$ 在点 A 处

沿由 A 到 B 的方向导数, 并求此函数在点 A 处方向导数的最大值.

解　$\overrightarrow{AB}=\boldsymbol{l}=(2,-2,-3)$, 故 $(\cos\alpha,\cos\beta,\cos\gamma)=\left(\dfrac{2}{\sqrt{17}},\dfrac{-2}{\sqrt{17}},\dfrac{-3}{\sqrt{17}}\right).$,　$\dfrac{\partial f}{\partial x}=\dfrac{1}{x+\sqrt{y^2+z^2}}$,

$$\dfrac{\partial f}{\partial y}=\dfrac{\dfrac{y}{\sqrt{y^2+z^2}}}{x+\sqrt{y^2+z^2}}=\dfrac{y}{x\sqrt{y^2+z^2}+y^2+z^2},\qquad \dfrac{\partial f}{\partial y}=\dfrac{\dfrac{z}{\sqrt{y^2+z^2}}}{x+\sqrt{y^2+z^2}}=\dfrac{z}{x\sqrt{y^2+z^2}+y^2+z^2}$$

于是　$\left.\dfrac{\partial f}{\partial x}\right|_{(1,0,1)}=\dfrac{1}{2},\left.\dfrac{\partial f}{\partial y}\right|_{(1,0,1)}=0,\left.\dfrac{\partial f}{\partial z}\right|_{(1,0,1)}=\dfrac{1}{2}$,

$$\left.\dfrac{\partial f}{\partial \boldsymbol{l}}\right|_{(1,0,1)}=\dfrac{1}{2}\times\dfrac{2}{\sqrt{17}}+0\times\dfrac{(-2)}{\sqrt{17}}+\dfrac{1}{2}\dfrac{(-3)}{\sqrt{17}}=-\dfrac{1}{2\sqrt{17}}.$$

在 A 点的方向导数最大值为 $\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2+\left(\dfrac{\partial f}{\partial z}\right)^2}=\sqrt{\left(\dfrac{1}{2}\right)^2+0+\left(\dfrac{1}{2}\right)^2}=\dfrac{\sqrt{2}}{2}.$

(3)设函数 $z=z(x,y)$ 由方程 $x+y+z=e^z$ 给出,求 $\dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y}$ 及 $\dfrac{\partial^2 z}{\partial x^2}$.

解　令 $F(x,y,z)=e^z-x-y-z$,　则　$\dfrac{\partial F}{\partial x}=-1,\dfrac{\partial F}{\partial y}=-1,\dfrac{\partial F}{\partial z}=e^z-1.$ 于是

$$\dfrac{\partial z}{\partial x}=-\dfrac{\partial F}{\partial x}\Big/\dfrac{\partial F}{\partial z}=\dfrac{1}{e^z-1},\dfrac{\partial z}{\partial y}=-\dfrac{\partial F}{\partial y}\Big/\dfrac{\partial F}{\partial z}=\dfrac{1}{e^z-1},$$

$$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{1}{e^z-1}\right)=\dfrac{-e^z}{(e^z-1)^2}\dfrac{\partial z}{\partial x}=\dfrac{-e^z}{(e^z-1)^3}.$$

四. (第一小题 4 分, 第二小题 6 分, 共 10 分)

(1)给定空间三点: $A(1,2,0),B(-1,3,1),C(2,-1,2)$,求 $\triangle ABC$ 的面积 S.

解　根据向量积的定义，可知三角形 ABC 的面积

$$2S_{\triangle ABC}=|AB|\,|AC|\sin\angle A=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

由于 $AB=(-2，1，1)$, $AC=(1，-3，1)$, 故

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-2&1&1\\1&-3&2\end{vmatrix}=5\boldsymbol{i}+5\boldsymbol{j}+5\boldsymbol{k},\qquad \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{5^2+5^2+5^2}=5\sqrt{3}$$

$$S_{\triangle ABC}=\dfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\dfrac{5\sqrt{3}}{2}.$$

(2)求经过直线 $L_1: \dfrac{x-1}{2}=\dfrac{y+2}{3}=\dfrac{z+3}{4}$ 且平行于直线 $L_2: x=y=\dfrac{z}{2}$ 的平面方程.

解　平面的法向量为　$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix}=2\boldsymbol{i}+0\boldsymbol{j}-\boldsymbol{k}$，又显然所求平面过点 $(1,-2,-3)$

故所求平面方程为　$2(x-1)+0(y+2)-(z+3)=0$, 即　$2x-z-5=0$.

五.（7 分）求函数 $f(x)=x^{\frac{1}{x}},x>0$ 的极值.

解　$f'(x)=\dfrac{d}{dx}(e^{\frac{\ln x}{x}})=e^{\frac{\ln x}{x}}\cdot\dfrac{1-\ln x}{x^2}=x^{\frac{1}{x}}\cdot\dfrac{1-\ln x}{x^2}$，

令　$f'(x)=0$, 得驻点 $x=e$. 又　当 $0<x<e$ 时，$f'(x)>0$，当 $x>e$ 时，$f'(x)<0$, 故此点

为极大值点, 极大值为 $f(e)=e^{\frac{1}{e}}$.

六.（12 分）设函数 $f(x)=\dfrac{(x-1)^3}{(x+1)^2}$，求(1)此函数的单调区间与极值点；(2)此函数的凹凸区间与拐点；(3)此函数的渐近线.

解　$f'(x)=\dfrac{3(x-1)^2(x+1)^2-2(x-1)^3(x+1)}{(x+1)^4}$

$\qquad=\dfrac{(x-1)^2[3(x+1)-2(x-1)]}{(x+1)^3}=\dfrac{(x-1)^2(x+5)}{(x+1)^3}$，

$f''(x)=\dfrac{[2(x-1)(x+5)+(x-1)^2](x+1)^3-3(x-1)^2(x+5)(x+1)^2}{(x+1)^6}$

$\qquad=\dfrac{(x-1)[(3x+9)(x+1)-3(x-1)(x+5)]}{(x+1)^4}=\dfrac{24(x-1)}{(x+1)^4}$.

x	$(-\infty,-5)$	-5	$(-5,-1)$	$(-1,1)$	1	$(1,+\infty)$
$f'(x)$	+	0	−	+	0	+
$f''(x)$	−	−	−	−	0	+
$f(x)$	凸, ↗	极大值-27/2	凸, ↘	凸, ↗	拐点(1,0)	凹, ↗

单调增加区间为 $(-\infty,-5)$，$(-1,1)$ 和 $(1,+\infty)$，单调减少区间为 $(-5,-1)$ 函数在点

x=-5 处取到极大值, 极大值为 f (-5)= -27/2.   <mark>因为极值点要在稳定点中寻求，故没有极小值点</mark>

曲线的凸区间为 $(-\infty,-1)$ 和 $(-1,1)$, 凹区间为 $(1, +\infty)$, 曲线拐点为 $(1,0)$.

显然曲线有垂直渐近线 x=-1, 曲线无水平渐近线. 由于

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{(x-1)^3}{x(x+1)^2} = 1,$$

$$b = \lim_{x \to \infty}[f(x)-kx] = \lim_{x \to \infty}\left[\frac{(x-1)^3}{(x+1)^2} - x\right] = \lim_{x \to \infty}\frac{-5x^2+2x-1}{(x+1)^2} = -5,$$

因此曲线有斜渐近线 $y = x$-5.

(七. (每小题 7 分, 共 14 分)

1.求证不等式 $\sin x + \tan x > 2x, \quad 0 < x < \frac{\pi}{2}$;

证   令 $f(x) = \sin x + \tan x - 2x, \quad 0 < x < \frac{\pi}{2}$, 则 $f'(x) = \cos x + \sec^2 x - 2$,

$$f''(x) = -\sin x + 2\sec x \cdot \sec x \tan x = \frac{\sin x(2 - \cos^3 x)}{\cos^3 x} \geq 0,$$

故 $f'(x) = \cos x + \sec^2 x - 2$, 在区间 $0 < x < \frac{\pi}{2}$ 内单调增加, 而 $f'(0) = 0$, 于是

在区间 $0 < x < \frac{\pi}{2}$ 内 $f'(x) > 0$, 从而 $f(x) = \sin x + \tan x - 2x$ 在区间 $0 < x < \frac{\pi}{2}$ 内单调增加,

于是 $f(x) = \sin x + \tan x - 2x > f(0) = 0$. 证毕.

2.设函数 $f(x)$ 在闭区间 $[a,b]$ 上二阶可导, 且 $f(a) = f(b) = 0, f''(x) \neq 0, x \in (a,b)$. 求证: $f(x) \neq 0, x \in (a,b)$.

证   用反证法. 设存在 $c \in (a,b)$, 使得 $f(c)$=0. 于是由罗尔定理,

$$\exists \xi \in (a,c), \exists \eta \in (c,b), 使得 f'(\xi) = f'(\eta) = 0.$$

同样由罗尔定理, $\exists \zeta \in (\xi,\eta) \subset (a,b)$, 使得 $f''(\zeta) = 0$. 矛盾.