

## 09 级一期 A 卷参考解答

一.(每小题 6 分,共 12 分)求下列极限:

$$1. \lim_{x \rightarrow \infty} x \left( e^{\frac{2}{x}} - 1 \right);$$

$$\text{解 } \lim_{x \rightarrow \infty} x \left( e^{\frac{2}{x}} - 1 \right) = \lim_{x \rightarrow \infty} \frac{e^{2/x} - 1}{1/x} = \lim_{x \rightarrow \infty} \frac{e^{2/x} \cdot (-2/x^2)}{-1/x^2} = 2 \lim_{x \rightarrow \infty} e^{2/x} = 2e^0 = 2$$

$$2. \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}}.$$

$$\text{解 } \quad \text{令 } y = \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}}, \text{ 则}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left( \frac{\sin x}{x} \right) = \lim_{x \rightarrow 0} \frac{\ln \sin x - \ln x}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{2x^2 \sin x} = \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{4x \sin x + 2x^2 \cos x} = -\lim_{x \rightarrow 0} \frac{\sin x}{4 \sin x + 2x \cos x} \\ &= -\lim_{x \rightarrow 0} \frac{\cos x}{4 \cos x + 2 \cos x - 2x \sin x} = -\frac{1}{6}, \quad \text{于是 } \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} e^y = e^{-1/6}. \end{aligned}$$

二.(每小题 6 分,共 24 分)完成如下各题

$$1. \int \frac{2x^2 + 1}{x^2(1+x^2)} dx;$$

$$\text{解 } \quad \text{原式} = \int \left( \frac{x^2 + x^2 + 1}{x^2(x^2 + 1)} \right) dx = \int \frac{1}{x^2} dx + \int \frac{1}{x^2 + 1} dx = -\frac{1}{x} + \arctan x + C$$

$$2. \int \frac{dx}{1 + \sqrt[3]{x+2}};$$

$$\text{解 } \quad \text{令 } x+2 = t^3, dx = 3t^2 dt, \text{ 则}$$

$$\text{原式} = \int \frac{3t^2}{1+t} dt = 3 \int \frac{t^2 - 1 + 1}{1+t} dt = 3 \int (t-1) dt + 3 \int \frac{1}{1+t} dt$$

$$= \frac{3t^2}{2} - 3t + 3\ln|t+1| + C = \frac{3}{2}(x+2)^{\frac{2}{3}} - 3(x+2)^{\frac{1}{3}} + 3\ln\left|(x+2)^{\frac{1}{3}} + 1\right| + C.$$

$$3. \int_0^4 e^{\sqrt{x}} dx;$$

解 令  $t = \sqrt{x}$ , 则

$$\int_0^4 e^{\sqrt{x}} dx = 2 \int_0^2 te^t dt = 2 \left[ te^t \Big|_0^2 - \int_0^2 e^t dt \right] = 2 \left( 2e^2 - e^t \Big|_0^2 \right) = 2(e^2 + 1).$$

$$4. \text{求证: } \int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx, \text{ 并求此积分.}$$

证明 令  $t = \frac{\pi}{2} - x$ , 则

$$\begin{aligned} \text{左边} &= \int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \int_{\frac{\pi}{2}}^0 \frac{\sin^{2010} \left( \frac{\pi}{2} - t \right)}{\sin^{2010} \left( \frac{\pi}{2} - t \right) + \cos^{2010} \left( \frac{\pi}{2} - t \right)} d \left( \frac{\pi}{2} - t \right) \\ &= - \int_{\frac{\pi}{2}}^0 \frac{\cos^{2010} t}{\sin^{2010} t + \cos^{2010} t} dt = \int_0^{\frac{\pi}{2}} \frac{\cos^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \text{右边}. \end{aligned}$$

$$\text{而, 左边} + \text{右边} = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}, \quad \text{故} \quad \int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \frac{\pi}{4}.$$

三.(每小题 7 分,共 21 分)完成如下各题:

1. 设  $u(x, y) = \ln \sqrt{1+x^2+y^2}$ , 求  $du \Big|_{(1,2)}$ .

解  $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1+x^2+y^2}} \cdot \frac{2x}{2\sqrt{1+x^2+y^2}} = \frac{x}{1+x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{1+x^2+y^2},$  于是

$$\frac{\partial u}{\partial x} \Big|_{(1,2)} = \frac{x}{1+x^2+y^2} \Big|_{(1,2)} = \frac{1}{1+1+4} = \frac{1}{6}, \quad \frac{\partial u}{\partial y} \Big|_{(1,2)} = \frac{y}{1+x^2+y^2} \Big|_{(1,2)} = \frac{2}{6} = \frac{1}{3},$$

$$du \Big|_{(1,2)} = \frac{\partial u}{\partial x} \Big|_{(1,2)} dx + \frac{\partial u}{\partial y} \Big|_{(1,2)} dy = \frac{dx + 2dy}{6}.$$

2. 已知  $f(x, y, z) = 2xy - z^2$  及点  $A(2, -1, 1), B(3, 1, -1)$ , 求函数  $f(x, y, z)$  在点  $A$  处沿由  $A$  到  $B$  方向的方向导数, 并求此函数在点  $A$  处方向导数的最大值.

解  $\boldsymbol{l} = \overrightarrow{AB} = (1, 2, -2), (\cos \alpha, \cos \beta, \cos \gamma) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right),$

$$\frac{\partial f}{\partial x} = 2y, \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = -2z. \quad \left. \frac{\partial f}{\partial x} \right|_{(2, -1, 1)} = -2, \left. \frac{\partial f}{\partial y} \right|_{(2, -1, 1)} = 4, \left. \frac{\partial f}{\partial z} \right|_{(2, -1, 1)} = -2.$$

因此,  $\frac{\partial f}{\partial \boldsymbol{l}} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta + \frac{\partial f}{\partial z} \cos \gamma = (-2) \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} + (-2) \cdot \left(-\frac{2}{3}\right) = \frac{10}{3}.$

而在点  $A$  处方向导数的最大值为  $|\boldsymbol{g}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = 2\sqrt{6}.$

3. 设函数  $z = z(x, y)$  由方程  $z^3 - 3xyz = 1$  给出, 求  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$  及  $\frac{\partial^2 z}{\partial x^2}.$

解 令  $F(x, y) = z^3 - 3xyz - 1, \quad \frac{\partial F}{\partial x} = -3yz, \frac{\partial F}{\partial y} = -3xz, \frac{\partial F}{\partial z} = 3z^2 - 3xy,$

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial z} = -\frac{-3yz}{3z^2 - 3xy} = \frac{yz}{z^2 - xy},$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} / \frac{\partial F}{\partial z} = -\frac{-3xz}{3z^2 - 3xy} = \frac{xz}{z^2 - xy},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{yz}{z^2 - xy} \right) = \frac{y \frac{\partial z}{\partial x} \cdot (z^2 - xy) - yz \left( 2z \frac{\partial z}{\partial x} - y \right)}{(z^2 - xy)^2},$$

$$= \frac{y^2 z - y(xy + z^2) \frac{\partial z}{\partial x}}{(z^2 - xy)^2} = \frac{y^2 z - y(xy + z^2) \cdot \frac{yz}{z^2 - xy}}{(z^2 - xy)^2}$$

$$= -\frac{2xy^3 z}{(z^2 - xy)^3}.$$

四.(第一小题 4 分,第二小题 6 分,共 10 分)

1. 已知点  $A(2, 2, 2), B(4, 4, 2), C(4, 2, 4)$ , 求向量  $\overrightarrow{AB}, \overrightarrow{AC}$  的夹角.

解  $\overrightarrow{AB} = (2, 2, 0), \overrightarrow{AC} = (2, 0, 2)$ , 设所求夹角为  $\alpha$ , 则

$$\cos \alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{2 \times 2 + 2 \times 0 + 0 \times 2}{\sqrt{2^2 + 2^2 + 0^2} \cdot \sqrt{2^2 + 0^2 + 2^2}} = \frac{1}{2}, \quad \alpha = \frac{\pi}{3}.$$

2. 求经过直线  $L_1: \begin{cases} x+y=0, \\ x-y-z-2=0, \end{cases}$  且平行于直线  $L_2: x=y=z$  的平面方程.

解  $L_1$  的参数方程为  $x=t, y=-t, z=2(t-1)$ , 化为标准方程为

$$\frac{x}{1} = \frac{y}{-1} = \frac{z+2}{2},$$

其方向向量为  $\boldsymbol{l}_1 = (1, -1, 2)$ , 而直线  $L_2$  的方向向量为  $\boldsymbol{l}_2 = (1, 1, 1)$ , 故所求平面法向量为

$$\boldsymbol{n} = \boldsymbol{l}_1 \times \boldsymbol{l}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} = (-3, 1, 2).$$

所求平面过点  $(0, 0, -2)$ , 故所求平面方程为  $-3x + y + 2(z+2) = 0$ , 即  $3x - y - 2z = 4$ .

五.(7 分) 求函数  $f(x) = \int_0^x (t-1)(t-2)^2 dt$  的极值.

解  $f'(x) = (x-1)(x-2)^2$ , 从而驻点为  $x_1 = 1, x_2 = 2$ . 列表如下

$x$	$(-\infty, 1)$	1	$(1, 2)$	2	$(2, +\infty)$
$f'(x)$	—	0	+	0	+
$f(x)$	$\searrow$	极小值	$\nearrow$	非极值	$\nearrow$

所求函数最小值为

$$\begin{aligned} f(1) &= \int_0^1 (t-1)(t-2)^2 dt = \int_0^1 (t^3 - 5t^2 + 8t - 4) dt \\ &= \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}. \end{aligned}$$

六.(12 分) 设函数  $f(x) = \frac{x^3}{2(1+x)^2}$ , 求(1)函数的单调区间与极值点;(2)函数的凹凸区间与拐点;(3)函数的渐近线.

解 函数的定义域为  $(-\infty, -1) \cup (-1, +\infty)$ , 且

$$f'(x) = \frac{3x^2(1+x)^2 - x^3 \cdot 2(1+x)}{2(1+x)^4} = \frac{x^2(x+3)}{2(1+x)^3}.$$

$$f''(x) = \frac{(3x^2 + 6x)(1+x)^3 - (x^3 + 3x^2) \cdot 3(1+x)^2}{2(1+x)^6} = \frac{3x}{(1+x)^4}.$$

从而函数的驻点为 0, -3. 又二阶导数为零的点为 0, 列表如下:

$x$	$(-\infty, -3)$	$-3$	$(-3, -1)$	$(-1, 0)$	$0$	$(0, +\infty)$
$f'(x)$	+	0	-	+	0	+
$f''(x)$	-	-	-	-	0	+
$f(x)$	凸 ↗	极大	凸 ↘	凸 ↗	拐点	凹 ↗

函数的单调增加区间为  $(-\infty, -3)$  和  $(0, +\infty)$ , 单调减少区间为  $(-3, -1)$ . 极小值点为  $-3$ . 凹区间为  $(0, +\infty)$ , 凸区间为  $(-\infty, -1)$  和  $(-1, 0)$ , 拐点为  $(0, 0)$ . 下面再求渐近线. 显然, 直线  $x = -1$  是垂直渐近线. 而

$$\lim_{x \rightarrow \infty} \frac{x^3}{2(1+x)^2} = \infty$$

因而曲线无水平渐近线, 但

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^3}{2x(1+x)^2} = \frac{1}{2},$$

$$\lim_{x \rightarrow \infty} \left[ f(x) - \frac{x}{2} \right] = \lim_{x \rightarrow \infty} \left[ \frac{x^3}{2(1+x)^2} - \frac{x}{2} \right] = -1,$$

因而曲线有斜渐近线  $y = \frac{1}{2}x - 1$ .

七. (每小题 7 分, 共 14 分)

1. 求证:  $1 + x \ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2}, x \in \mathbb{R}$ .

证 令  $f(x) = 1 + x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$ , 则当  $x > 0$  时,

$$f'(x) = \ln\left(x + \sqrt{1+x^2}\right) + \frac{x\left(1 + \frac{x}{\sqrt{1+x^2}}\right)}{x + \sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{1+x^2}\right) > 0,$$

故此函数单调增加.而容易验证  $f(0)=0$ , 故当  $x>0$  时,  $f(x) \geq 0$ , 此即

$$1 + x \ln\left(x + \sqrt{1+x^2}\right) \geq \sqrt{1+x^2}, x > 0.$$

$$\text{又, } f(-x) = 1 - x \ln\left(-x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} = 1 - x \ln\left(\frac{1}{x + \sqrt{1+x^2}}\right) - \sqrt{1+x^2} = f(x),$$

$$\text{从而 } 1 + x \ln\left(x + \sqrt{1+x^2}\right) \geq \sqrt{1+x^2}, x \in \mathbb{R}.$$

2. 设函数  $f(x)$  在闭区间  $[0,1]$  上连续, 在开区间  $(0,1)$  内可导, 且  $f(0)=0, f(1)=1$ , 求证:

(1) 存在  $\alpha \in (0,1)$ , 使得  $f(\alpha) = 1 - \alpha$ ;

(2) 存在两个不同的点  $\xi \in (0,1), \eta \in (0,1)$ , 满足  $f'(\xi)f'(\eta) = 1$ .

证 (1) 令  $g(x) = f(x) + x - 1$ , 则  $g(0) = f(0) - 1 < 0, g(1) = f(1) + 1 - 1 = 1 > 0$ .

于是由介值定理, 存在  $\alpha \in (0,1)$ , 使得  $g(\alpha) = 0$ , 即  $f(\alpha) = 1 - \alpha$ ;

(2) 由 Lagrange 定理, 在区间  $(0, \alpha)$  内存在  $\xi$ , 使得

$$f'(\xi) = \frac{f(\alpha) - f(0)}{\alpha - 0} = \frac{1 - \alpha}{\alpha}.$$

在区间  $(\alpha, 1)$  内, 存在  $\eta$ , 使得

$$f'(\eta) = \frac{f(1) - f(\alpha)}{1 - \alpha} = \frac{\alpha}{1 - \alpha}.$$

于是存在两个不同的点  $\xi \in (0,1), \eta \in (0,1)$ , 满足  $f'(\xi)f'(\eta) = 1$ .