机器人原理课程

05. 速度与静力a

课件来自于台湾大学 林沛群 教授 “机器人学导论”课程课件，在此表示感谢！

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速度与静力

- Differentiation of a position vector P_Q

$$ {}^{B}V_Q = \frac{d}{dt}\,{}^{B}P_Q = \lim_{\Delta t \to 0} \frac{{}^{B}P_Q(t+\Delta t) - {}^{B}P_Q(t)}{\Delta t} $$

Derivative of position vector ${}^{B}P_Q$ relative to frame $\{B\}$

$$ {}^{A}({}^{B}V_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}P_Q) $$

Expressed in frame $\{A\}$

$$ = {}^{A}_{B}R\,{}^{B}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

When both frames are the same

$$ v_C = {}^{U}V_{C\,ORG} $$

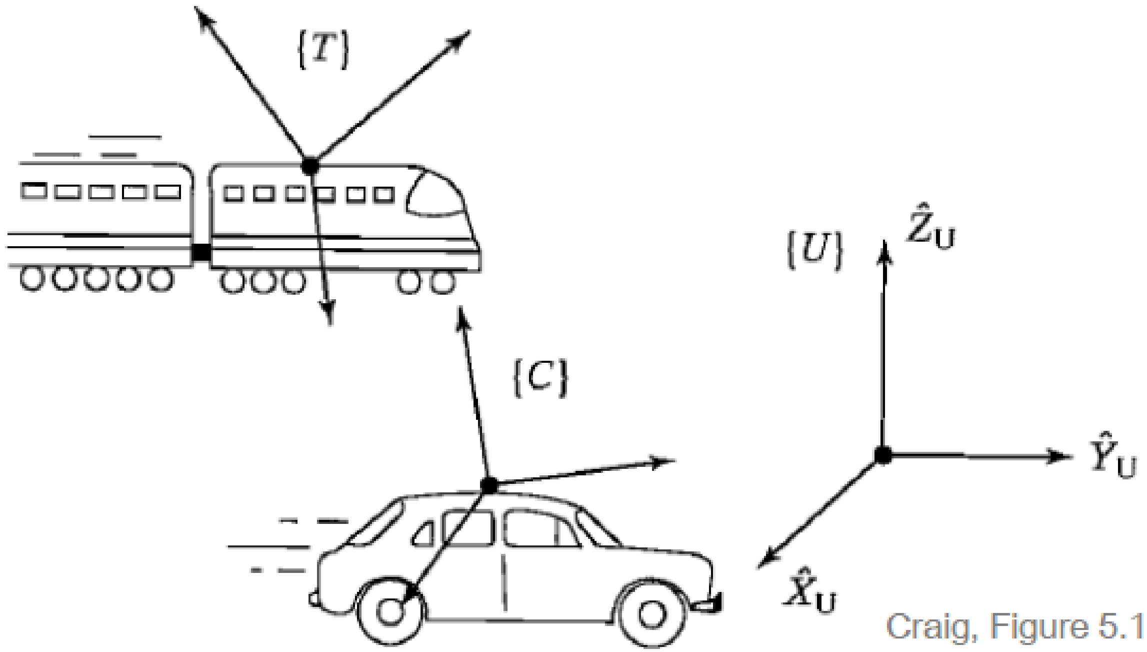
Velocity of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$

速度与静力

- Example

$$^{U}V_T = 100\hat{\imath}$$

$$^{U}V_C = 30\hat{\imath}$$



Craig, Figure 5.1

$$^{U}\left(\frac{d}{dt}\,{}^{U}P_{C\,ORG}\right) = {}^{U}V_{C\,ORG} = \upsilon_C = 30\hat{\imath}$$

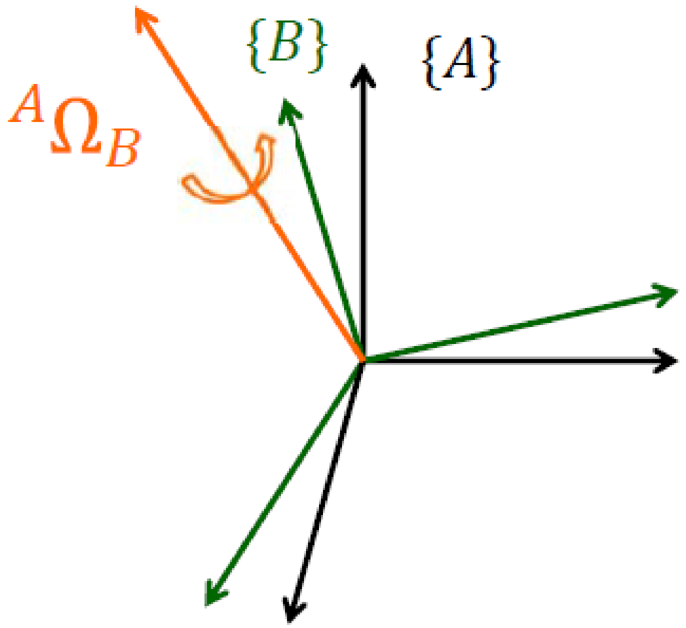
$$^{C}({}^{U}V_{T\,ORG}) = {}^{C}\upsilon_T = {}^{C}_{U}R(\upsilon_T) = {}^{C}_{U}R(100\hat{\imath}) = {}^{U}_{C}R^{-1}100\hat{\imath}$$

$$^{C}({}^{T}V_{C\,ORG}) = {}^{C}_{T}R({}^{T}({}^{T}V_{C\,ORG})) = {}^{C}_{T}R({}^{T}V_{C\,ORG})$$

$$= {}^{C}_{U}R\,{}^{U}_{T}R(-70\hat{\imath}) = -{}^{U}_{C}R^{-1}\,{}^{U}_{T}R70\hat{\imath}$$

速度与静力

- Angular velocity vector ${}^{A}\Omega_B$
  - The rotation of frame $\{B\}$ relative to frame $\{A\}$
  - Direction of ${}^{A}\Omega_B$: The instantaneous axis of rotation
  - Magnitude of ${}^{A}\Omega_B$: The speed of rotation



$${}^{C}({}^{A}\Omega_B)$$

Expressed in frame $\{C\}$

$$\omega_C = {}^{U}\Omega_C$$

Angular velocity of frame $\{C\}$ relative to the universe reference frame $\{U\}$

速度与静力

$$\overrightarrow{r_A} = x_A\hat{\imath} + y_A\hat{\jmath}$$

$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$

$$= (x_B\hat{\imath} + y_B\hat{\jmath}) + (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$

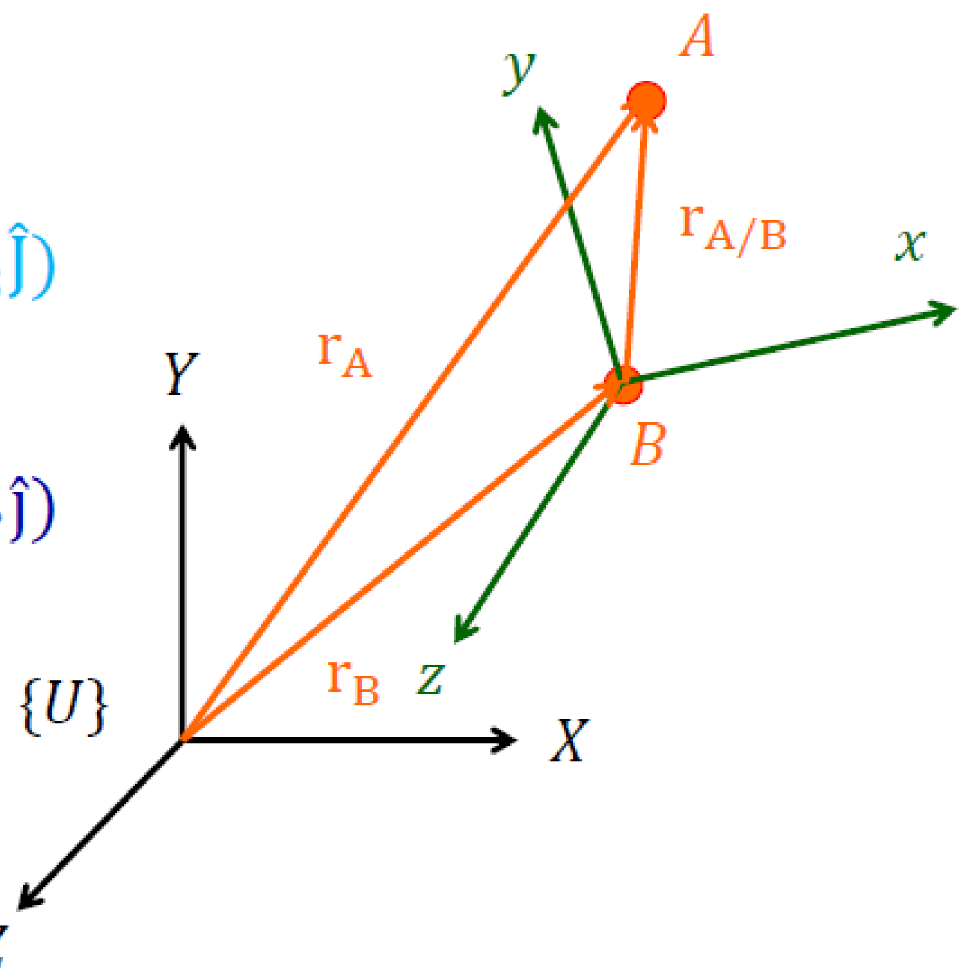
$$= (x_B\hat{\imath} + y_B\hat{\jmath}) + (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$\downarrow$ diff.

$$\overrightarrow{v_A} = \dot{\overrightarrow{r_A}} = \dot{x}_A\hat{\imath} + \dot{y}_A\hat{\jmath}$$

$$= \dot{\overrightarrow{r_B}} + \dot{\overrightarrow{r_{A/B}}}$$

$$= (\dot{x}_B\hat{\imath} + \dot{y}_B\hat{\jmath}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath})$$



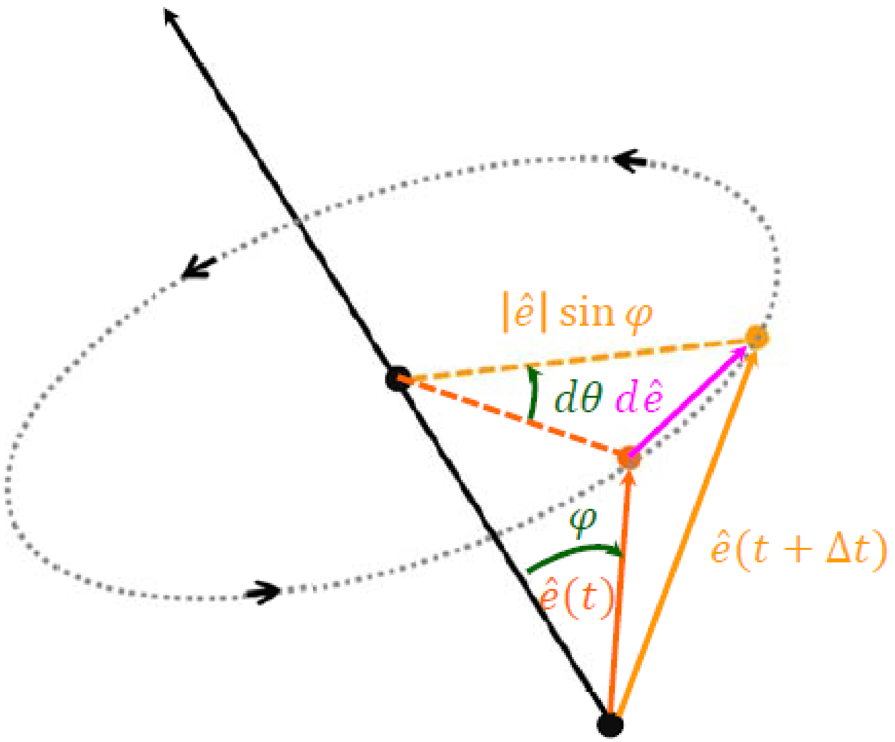
速度与静力

- $\overrightarrow{v_A} = \overrightarrow{\dot{r}_B} + \overrightarrow{\dot{r}_{A/B}}$

$$= (\dot{x}_B\hat{\imath} + \dot{y}_B\hat{\jmath}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \underline{(x_{A/B}\,\dot{\hat{\imath}} + y_{A/B}\dot{\hat{\jmath}})}$$

$$= x_{A/B}\,(\vec{\omega} \times \hat{\imath}) + y_{A/B}\,(\vec{\omega} \times \hat{\jmath})$$

$\vec{\omega}$ $\quad |\vec{\omega}| = \dot{\theta}$



Magnitude:

$$|d\hat{e}| = |\hat{e}|sin\varphi d\theta$$

$$|\dot{\hat{e}}| = |\hat{e}|sin\varphi\dot{\theta} = |\hat{e}||\vec{\omega}|sin\varphi$$

Direction:

$$d\hat{e} \perp \hat{e}$$

$$d\hat{e} \perp \vec{\omega}$$

$\Rightarrow$ $\dot{\hat{e}} = \vec{\omega} \times \hat{e}$

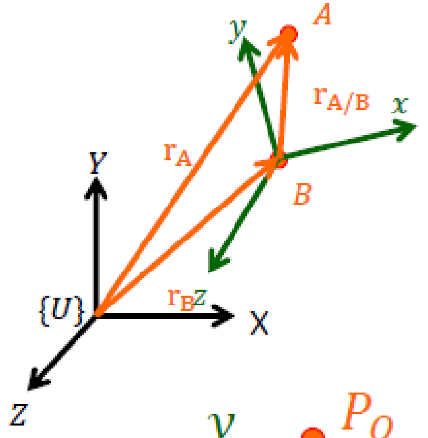
速度与静力

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

$$\overrightarrow{v_A} = \overrightarrow{v_B} + \underline{\overrightarrow{v_{rel}}} + \vec{\omega} \times \overrightarrow{r_{A/B}}$$

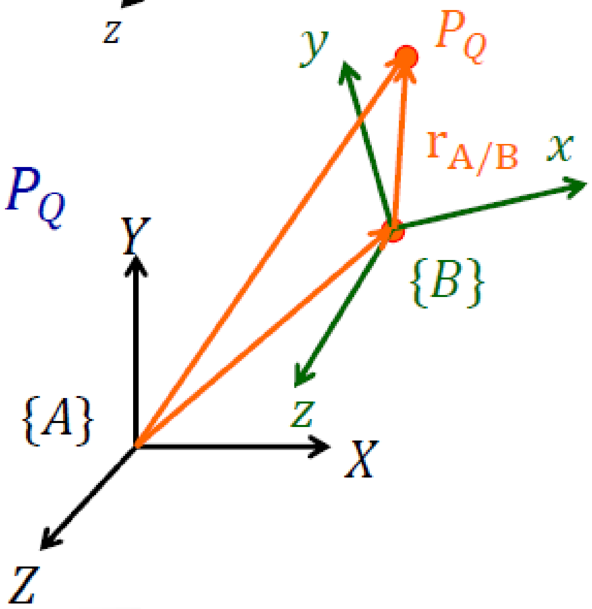
"relative" velocity



- Thus,

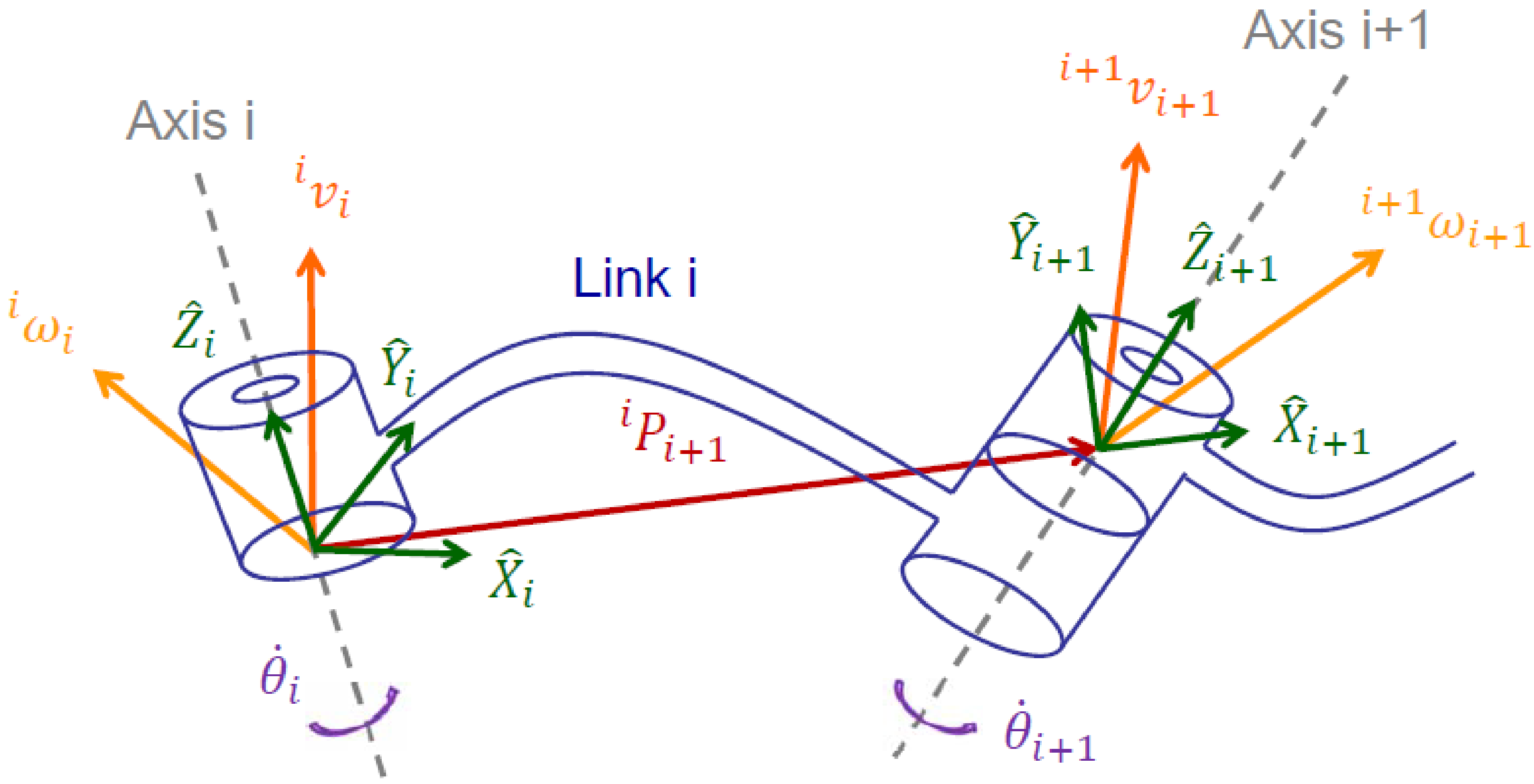
$$^{A}V_Q = {}^{A}V_{B\,ORG} + {}^{A}_{B}R\,\underline{^{B}V_Q} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

"relative" velocity



速度与静力

- Strategy: Represent linear and angular velocities of link i in frame $\{i\}$, and find their relationship to those of neighboring links



速度与静力

- Rotational Joint (Link i+1)

  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\,\underline{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \Big\downarrow\ {}^{i+1}_{i}R \qquad\qquad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

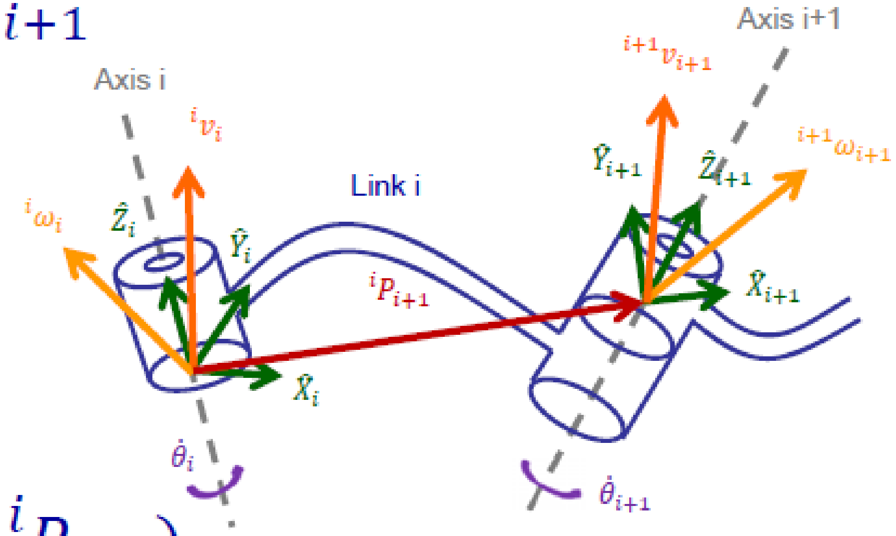
$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

  - Linear velocity propagation

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

$$ \Big\downarrow\ {}^{i+1}_{i}R $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R(\ {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$



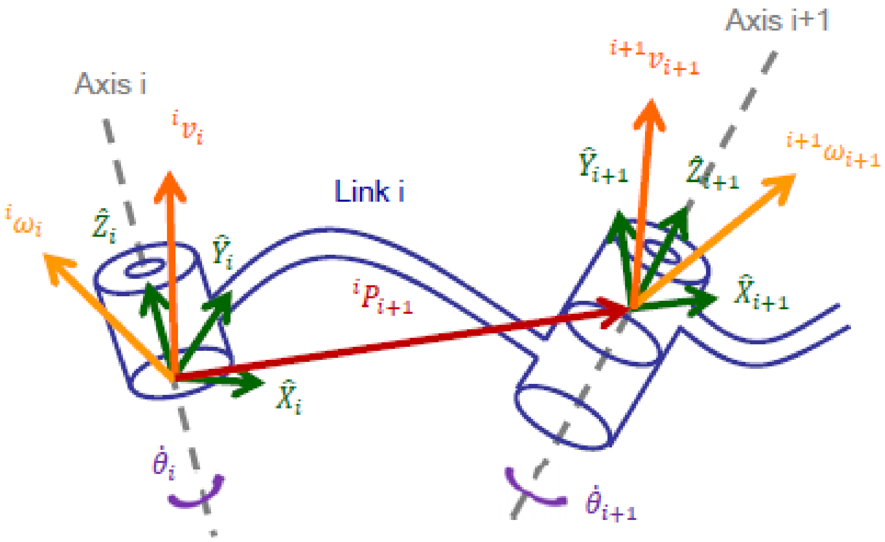
速度与静力

- Prismatic joint (Link i+1)
  - Angular velocity propagation

$$^{i}\omega_{i+1} = {}^{i}\omega_{i}$$

$$\Big\downarrow {}^{i+1}_{i}R$$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i}$$



  - Linear velocity propagation

$$^{i}v_{i+1} = \left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) + \underline{{}^{i}_{i+1}R\,\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\Big\downarrow {}^{i+1}_{i}R$$

$$\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

Jacobians -1

- A multidimensional form of the derivative

$$y_1 = f_1\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$y_2 = f_2\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\vdots$$

$$y_6 = f_6\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

⇨ $Y = F(X)$

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- Calculating the differentials of y_i as a function of differentials of x_i

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

Jacobian, “linear transformation”

$$\Rightarrow \quad \delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

Function of X, if f_i is nonlinear

$$\Rightarrow \quad \dot{Y} = J(X)\dot{X}$$

速度与静力

- In robotics
  - Relating joint velocities to Cartesian velocities of the tip of the arm

$$^{0}\mathbf{v} = \begin{bmatrix} ^{0}v \\ ^{0}\omega \end{bmatrix} = {}^{0}J(\Theta)\dot{\Theta}$$

3x1 : plane motion

6x1 : spatial motion

- Changing a Jacobian's frame of reference (spatial motion)

$$^{B}\mathbf{v} = \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\dot{\Theta}$$

$$^{A}\mathbf{v} = \begin{bmatrix} ^{A}v \\ ^{A}\omega \end{bmatrix} = {}^{A}J(\Theta)\dot{\Theta} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix}$$

$$\Rightarrow \quad {}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)$$

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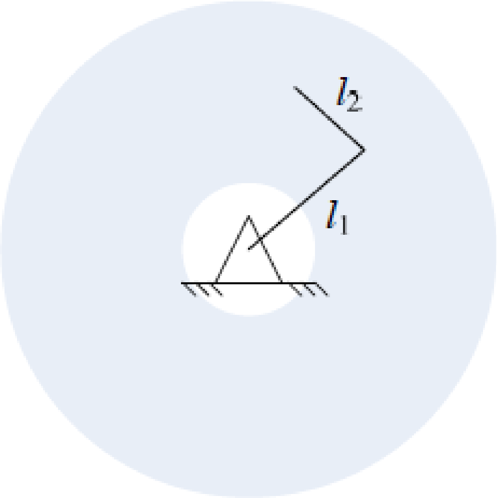
- Invertibility

$$\dot{\Theta} = J^{-1}(\Theta) v$$

- Singular: When the Jacobian J is NOT invertible
  - Workspace-boundary singularities

    Ex: When the manipulator is fully stretch out or folded back on itself
  - Workspace-interior singularities
- When a manipulator is in a singular configuration
  - Lost one or more DOF



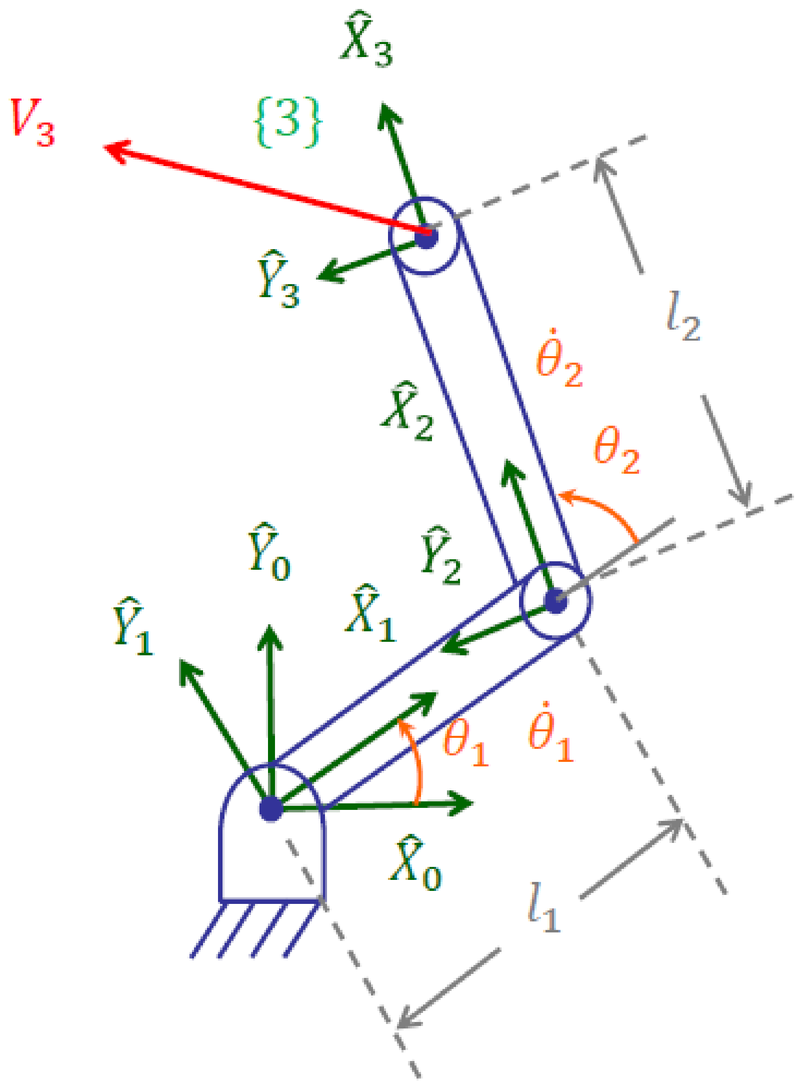
速度与静力

- Method 1: Velocity "propagation" from link to link

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



速度与静力

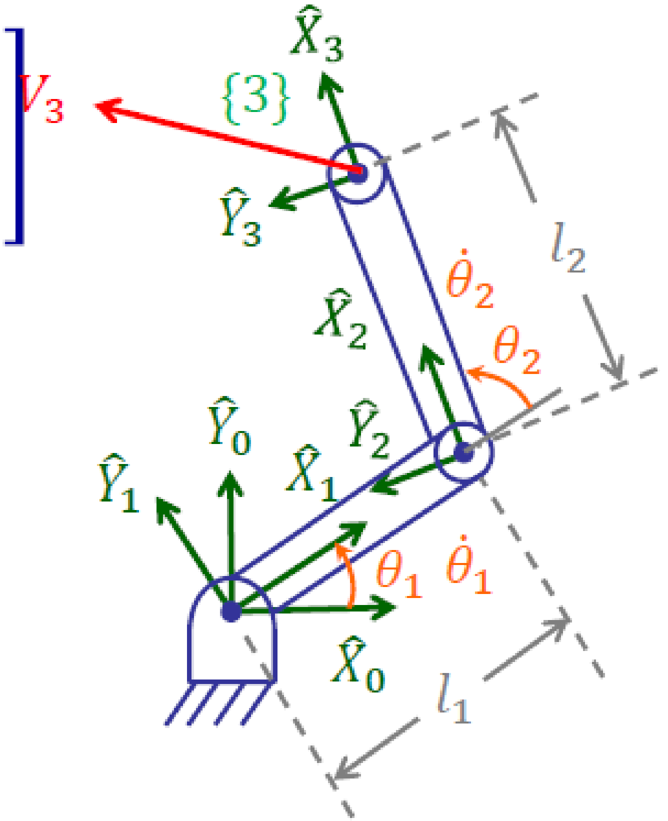
- Link "propagation"

$${}^{1}\omega_1 = {}^{1}_{0}R\, {}^{0}\cancel{\omega_0} + \dot\theta_1\, {}^{1}\hat Z_1 = \dot\theta_1\, {}^{1}\hat Z_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}$$

$${}^{1}v_1 = {}^{1}_{0}R(\, {}^{0}\cancel{v_0} + {}^{0}\cancel{\omega_0} \times {}^{0}\cancel{P_1}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^{2}\omega_2 = {}^{2}_{1}R\, {}^{1}\omega_1 + \dot\theta_2\, {}^{2}\hat Z_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}$$

$${}^{2}v_2 = {}^{2}_{1}R(\, {}^{1}\cancel{v_1} + {}^{1}\omega_1 \times {}^{1}P_2) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot\theta_1 \\ l_1 c_2 \dot\theta_1 \\ 0 \end{bmatrix}$$



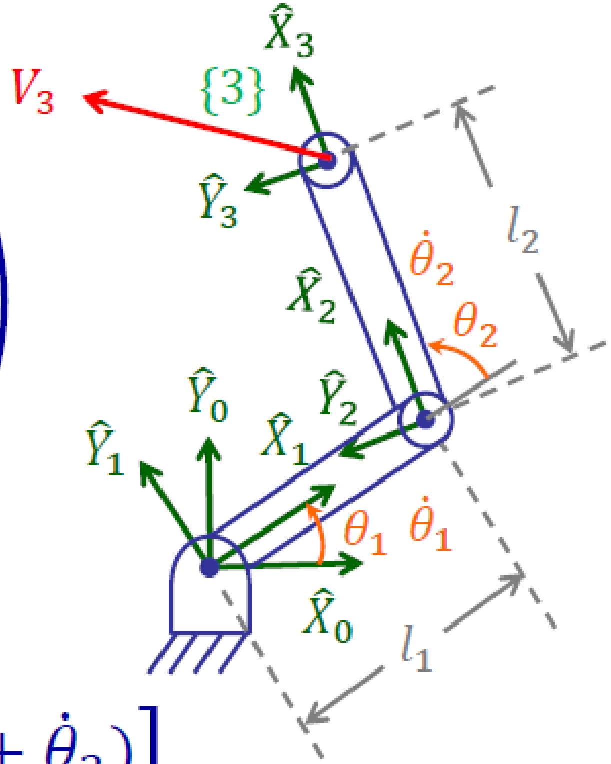
速度与静力

$$^{3}\omega_3 = {}^{2}\omega_2$$

$$^{3}v_3 = {}^{3}_{2}R\left({}^{2}v_2 + {}^{2}\omega_2 \times {}^{2}P_3\right)$$

$$= I\left(\begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$



$$^{0}v_3 = \underline{{}^{0}_{3}R}\ {}^{3}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$= {}^{0}_{1}R\,{}^{1}_{2}R\,{}^{2}_{3}R$$

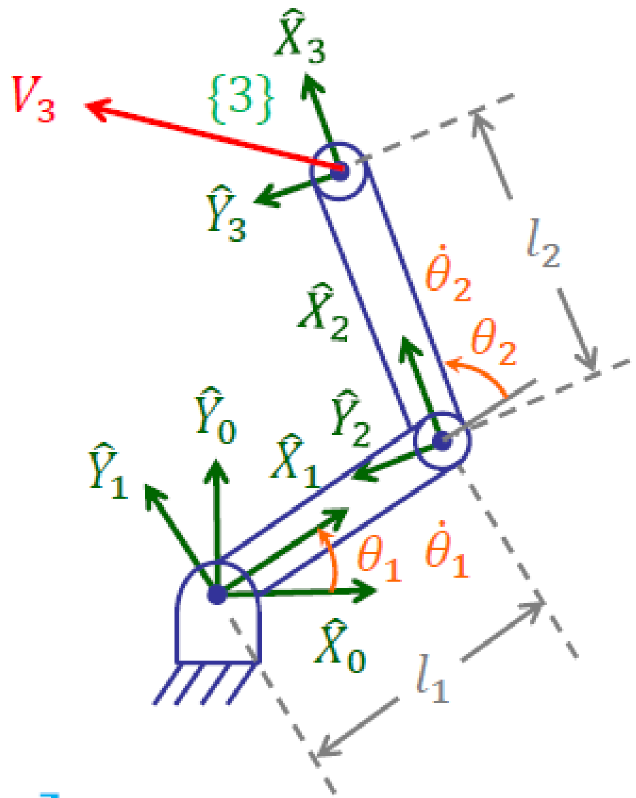
$$= \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

速度与静力

- Therefore

$$^{3}\boldsymbol{v}=\begin{bmatrix}v_x\\v_y\\\omega\end{bmatrix}=\begin{bmatrix}l_1s_2 & 0\\l_1c_2+l_2 & l_2\\1 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\end{bmatrix}$$

$$=\ {}^{3}J(\Theta)\dot{\Theta}$$



$$det\begin{vmatrix}l_1s_2 & 0\\l_1c_2+l_2 & l_2\end{vmatrix}=l_1l_2s_2=0$$

$\Rightarrow$ $\theta_2=0\ or\ 180$

$$^{0}\boldsymbol{v}=\begin{bmatrix}v_x\\v_y\\\omega\end{bmatrix}=\begin{bmatrix}-l_1s_1-l_2s_{12} & -l_2s_{12}\\l_1c_1+l_2c_{12} & l_2c_{12}\\1 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\end{bmatrix}$$

$$=\ {}^{0}J(\Theta)\dot{\Theta}$$

速度与静力

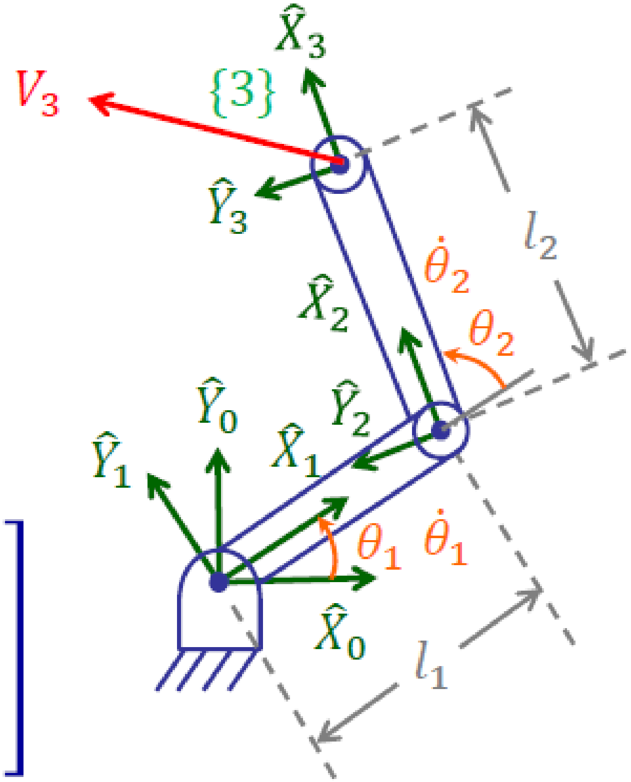
- Method 2: Direct differentiation

$$
{}^{0}\begin{bmatrix} p_x \\ p_y \\ \theta \end{bmatrix} = {}^{0}\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix}
$$

$\downarrow$ diff.

$$
{}^{0}\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = {}^{0}\begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
$$

$$
= {}^{0}\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$



$\dot{X} = {}^{0}J(\Theta)\dot{\Theta}$ Note: NO 3x1 orientation vector whose derivative is ω

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