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3 authors:

Yichuan Zhao
Georgia State University

Yueju Su

Hanfang Yang
Renmin University of China

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Jackknife empirical likelihood inference for the Pietra ratio

Yichuan Zhao [a], Yueju Su [a], Hanfang Yang [b,*]

[a] *Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States of America*
[b] *School of Statistics, Renmin University of China, Beijing, China*

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ABSTRACT

The Pietra ratio (Pietra index) is also known as the Robin Hood index or Schutz coefficient (Ricci–Schutz index). It is a measure of statistical heterogeneity in positive random variables. In this paper, we propose the jackknife empirical likelihood (JEL), the adjusted JEL, the extended JEL, and the balanced adjusted JEL method, for interval estimation of the Pietra ratio. We compare the performance of the proposed methods with the normal approximation (NA), bootstrap based methods and NA jackknife method. Simulation results indicate that under both symmetric and skewed distributions, the extended JEL method gives the best performance in terms of coverage probability. We illustrate the proposed methods by applying our methods to investigate the income data from the 2013 Current Population Survey conducted by the US Census Bureau.

1. Introduction

The Pietra ratio (Pietra index) is known as the Robin Hood index or half the relative mean deviation. It is a measure of statistical heterogeneity in the context of positive random variables (Schutz, 1951; Maio, 2007; Salverda et al., 2009; Habib, 2012). The Pietra ratio is used most frequently as a measure of income inequality, which represents the amount of resources needed to be taken from more affluent areas and given to less affluent areas in order to achieve an equal distribution (Schutz, 1951; Habib, 2012). The Pietra ratio has also been used to study the relationship between the income inequality and mortality in the United States (Kennedy et al., 1996; Shi et al., 2003; Sohler et al., 2003). As a measure of natural heterogeneity, the Pietra ratio has been shown a useful interpretation in other applications. For example, the Pietra ratio can be viewed as a benchmark measure of statistical heterogeneity in financial derivatives (Eliazar and Sokolov, 2010).

The Pietra ratio is the maximum vertical distance between the Lorenz curve and the egalitarian line (Maio, 2007; Salverda et al., 2009; Eliazar and Sokolov, 2010). It is the ratio of the area of the largest triangle, which can be inscribed in the region of concentration in a Lorenz diagram to the area under the line of equality (Kendall and Stuart, 1963). If we let X be a random variable, the Pietra ratio is defined as the ratio of the mean absolute deviation to the twice the mean of X:

$$P = \frac{1}{2\mu}\int_{-\infty}^{\infty} |x-\mu| dF(x), \tag{1.1}$$

where $\mu = E(X)$ is the mean and $F(x)$ is the distribution function for X.

Gastwirth (1974) proposed mean absolute deviation to measure the spread of the distribution function as follows: $\delta = \int_{-\infty}^{\infty} |x-\mu|\, dF(x) = E\,|X - E(X)|$. Therefore, the Pietra ratio can also be expressed as $P_0 = \delta/(2\mu)$. For large samples,

* Corresponding author.
E-mail address: hyang@ruc.edu.cn (H. Yang).

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Gastwirth (1974) established a normal approximation (NA) method to obtain confidence intervals for the Pietra ratio. However, this method is complicated to implement and performs poorly in small samples.

The empirical likelihood (EL) method (Owen, 1988, 1990) combines the reliability of nonparametric methods with the effectiveness of the likelihood approach. It yields confidence regions, which respect the range of the target parameter. One of the advantages is that the EL confidence regions often behave better than ones obtained from the NA method when the sample size is small (see Chen and Van Keilegom, 2009). When the sample size is small, the coverage probabilities of the EL confidence regions are often lower than the nominal level (Owen, 2001; Chen et al., 2009; Liu and Chen, 2010). In addition, the EL may not be properly defined because of the so-called empty set problem (Chen et al., 2008; Tsao and Wu, 2013). A number of approaches have been proposed to improve the coverage probability of the EL confidence regions to address the empty set problem. For example, Bootstrap calibration (Owen, 1988) and Bartlett correction (Chen and Cui, 2007) are two popular methods that can improve the coverage probability. Three useful approaches can tackle the problems, which are the adjusted EL (AEL), extended EL (EEL) and balanced EL (BAEL). The idea of AEL (see Chen et al., 2008; Liu and Chen, 2010; Chen and Liu, 2012; Wang et al., 2015) is to "add one pseudo-observation to the sample to ensure that the parameter of interest is always in the interior of the convex hull of the data values, and thus it eliminates the empty set problem". The EEL proposed by Tsao and Wu (2013) geometrically expands the EL domain to overcome the empty set problem. Emerson and Owen (2009) proposed the novel calibration to the mean vector in empirical likelihood setting, which is called BAEL.

The jackknife empirical likelihood (JEL) method is proposed by Jing et al. (2009). The idea of JEL is to combine both the jackknife and the empirical likelihood by turning the statistic of interest into a sample mean of jackknife pseudo-values (Quenouille, 1956) and apply the EL method for the mean of the jackknife pseudo-values. Since Jing et al. (2009) developed the JEL for the U-statistics, the JEL has been broadly applied to other statistical problems. For example, Gong et al. (2010) proposed JEL for the ROC curves, which enhanced the computational efficiency. Yang and Zhao (2013) employed the JEL method to the difference in two correlated continuous-scale ROC curves. Yang and Zhao (2017, 2018) proposed the inference procedure for the difference of two quantiles by the smoothed JEL. In Wang et al. (2013), the JEL test for the equality of two high-dimensional mean shows that it has a very robust size across the dimensions and has a good power.

Because the estimator of Pietra ratio is nonlinear, the classical EL method may be challenging to apply directly. We adopt the JEL to make inference for the Pietra ratio. In this paper, we propose JEL, the adjusted JEL (AJEL), the extended JEL (EJEL) and balanced JEL (BAJEL) for the interval estimation of Pietra ratio and compare the performance of those interval estimators to the NA method, the percentile bootstrap, the bias corrected and accelerated bootstrap method.

The rest of paper is organized as follows: In Section 2, we present inference procedures for the Pietro ratio including the NA and bootstrap methods. We also develop the JEL, the AJEL, the EJEL and the BAJEL method for the Pietra ratio. In Section 3, we carry out an extensive simulation study by comparing the proposed methods to the NA, and bootstrap methods in terms of coverage probabilities and average interval lengths from different distributions. In Section 4, we illustrate the proposed methods in a real data analysis of the U.S. household income from 2013 Current Population Survey. In Section 5, we discuss the future work related to our proposed methods. All the proofs of theorems are contained in the Appendix.

2. Inference procedure

In this section, we present inference methods for the Pietra ratio. Let $X_1, \ldots, X_n$ be i.i.d. sample of a random variable X. We adopt the same notations as Gastwirth (1974) and Zhao et al. (2015), and define $\hat{\delta}$ as an empirical estimator of the mean absolute deviation δ:

$$\hat{\delta} = \frac{1}{n}\sum_{i=1}^{n}\left|X_i - \bar{X}\right|,$$

where $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$. Define the sample Pietra ratio as

$$\hat{P} = \frac{\hat{\delta}}{2\bar{X}}.$$

2.1. NA method for the Pietra ratio

Gastwirth (1974) proved that $\hat{P}$ is asymptotically normally distributed. We construct normal approximation (NA) confidence intervals based on normal distribution with the mean $\delta/(2\mu)$ and the variance

$$\frac{1}{n}\left\{\frac{\tau^2}{\mu^2} + \frac{\delta^2\sigma^2}{4\mu^4} - \frac{\delta}{\mu^3}\left[p\sigma^2 - \int_{-\infty}^{\mu}(x-\mu)^2 dF(x)\right]\right\},$$

where $p = F(\mu)$, $\sigma^2 = Var(X)$ and τ^2 is given by

$$\tau^2 = p^2\int_{\mu}^{\infty}(x-\mu)^2 dF(x) + (1-p^2)\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) - \frac{\delta^2}{4}.$$

Then, we construct a $100(1-\alpha)\%$ NA confidence interval for the Pietra ratio:

$$\mathcal{R}_{\alpha}^{NA} = \{P : \hat{P} - \mathcal{Z}_{1-\alpha/2} * SE < P < \hat{P} + \mathcal{Z}_{1-\alpha/2} * SE\},$$

where $\mathcal{Z}_{1-\alpha/2}$ stands for $1-\alpha/2$ quantile of $N(0, 1)$ and

$$SE = \sqrt{\frac{1}{n}\left\{\frac{\hat{\tau}^2}{\bar{X}^2} + \frac{\hat{\delta}^2\hat{\sigma}^2}{4\bar{X}^4} - \frac{\hat{\delta}}{\bar{X}^3}\left[\hat{p}\hat{\sigma}^2 - \frac{1}{n}\sum_{X_i<\bar{X}}(X_i-\bar{X})^2\right]\right\}}$$

where $\hat{p} = F_n(\bar{X})$, $\hat{\sigma}^2$ is the sample variance of σ^2 and $F_n(x)$ is empirical distribution function, $\hat{\tau}^2$ is the estimator of τ^2, calculated by the following formula:

$$\hat{\tau}^2 = \frac{1}{n}\hat{p}^2\sum_{X_i\geq\bar{X}}(X_i-\bar{X})^2 + \frac{1}{n}(1-\hat{p}^2)\sum_{X_i<\bar{X}}(X_i-\bar{X})^2 - \frac{\hat{\delta}^2}{4}.$$

2.2. Bootstrap methods for the Pietra ratio

For the Pietra ratio, we construct bootstrap confidence intervals using the percentile bootstrap (Bp) and the bias corrected and accelerated bootstrap (Bca). The bootstrap is a resampling method, where we repeatedly sample with replacement from the original data set of n observations to form B bootstrap samples each of size n. In this paper, $\hat{P}^{*b}$ is based on the statistic $\hat{P}$ and computed on bootstrap sample b for $b = 1, \ldots, B$. We set $B = 999$ and obtain the bootstrap estimators $\left\{\hat{P}^{*1}, \hat{P}^{*2}, \ldots, \hat{P}^{*B}\right\}$.

For the Bp method, we form the order statistics of $\hat{P}^{*b}$ as $\hat{P}^{*(1)} \leq \hat{P}^{*(2)} \leq \cdots \leq \hat{P}^{*(B)}$. We obtain the $100(1-\alpha)\%$ confidence interval for P as

$$P \in \left[\hat{P}^{*((\alpha/2)B)}, \hat{P}^{*((1-\alpha/2)B)}\right],$$

where $\hat{P}^{*((\alpha/2)B)}$ and $\hat{P}^{*((1-\alpha/2)B)}$ are the $(\alpha/2)B$ and $(1-\alpha/2)B$ order statistics of $\hat{P}^{*b}$, respectively.

The Bca was introduced by Efron (1987). Based on Carpenter and Bithell (2000) and Wang and Zhao (2009), we obtain $100(1-\alpha)\%$ Bca confidence intervals for P.

2.3. NA jackknife method for the Pietra ratio

Motivated by Yang et al. (2017), we propose the normal approximation confidence intervals with jackknife variance estimator for the Pietra ratio. The jackknife pseudo-value is defined as follows:

$$\widehat{P}_i = n\widehat{P} - (n-1)\widehat{\delta}^{(-i)}/(2\bar{X}^{(-i)}),\ i = 1, \ldots, n,$$

where $\widehat{\delta}^{(-i)}$ and $\bar{X}^{(-i)}$ are the empirical estimators computed for the sample of $n-1$ values from the original data set formed by removing the ith observation.

Thus, the jackknife variance can be given as

$$\hat{\tau}^2_{jack} = \frac{1}{n-1}\sum_{i=1}^{n}(\widehat{P}_i - \frac{1}{n}\sum_{i=1}^{n}\widehat{P}_i)^2.$$

Then, we construct a $100(1-\alpha)\%$ NA jackknife confidence interval for the Pietra ratio P:

$$\mathcal{R}_{\alpha}^{Jack} = \{P : \frac{1}{n}\sum_{i=1}^{n}\widehat{P}_i - \mathcal{Z}_{1-\alpha/2} * \frac{\hat{\tau}_{jack}}{\sqrt{n}} < P < \frac{1}{n}\sum_{i=1}^{n}\widehat{P}_i + \mathcal{Z}_{1-\alpha/2} * \frac{\hat{\tau}_{jack}}{\sqrt{n}}\}.$$

2.4. Jackknife empirical likelihood for the Pietra ratio

Recall that $X_1, \ldots, X_n$ as i.i.d. random variables. Denote

$$T_n(P) = T(X_1, \ldots, X_n) = \hat{\delta} - 2P\bar{X}.$$

The jackknife pseudo-value is defined as:

$$\widehat{U}_i(P) = nT_n(P) - (n-1)T_{n-1}^{(-i)}(P),\ i = 1, \ldots, n.$$

where $T_{n-1}^{(-i)}(P) = \widehat{\delta}^{(-i)} - 2P\bar{X}^{(-i)}$. We then define the jackknife estimator as the average of the n pseudo-values

$$\widehat{T}_{n,jack}(P) := \frac{1}{n}\sum_{i=1}^{n}\widehat{U}_i(P).$$

Applying the EL to the jackknife pseudo-values, the jackknife empirical likelihood (JEL) ratio at P is defined by (see Jing et al. (2009) and Wang and Zhao (2016))

$$R(P) = \max\left\{\prod_{i=1}^{n} np_i : \sum_{i=1}^{n} p_i \widehat{U}_i(P) = 0, \sum_{i=1}^{n} p_i = 1, p_i \geq 0\right\}.$$

The jackknife empirical log-likelihood ratio is

$$logR(P) = -\sum_{i=1}^{n} log\{1 + \lambda \widehat{U}_i(P)\},$$

where λ satisfies the following equation

$$f(\lambda) \equiv \frac{1}{n}\sum_{i=1}^{n} \frac{\widehat{U}_i(P)}{1 + \lambda \widehat{U}_i(P)} = 0.$$

Condition C.1: a random variable X with finite mean μ and variance σ^2 and with a continuous density function $f(x)$ in the neighborhood of μ.

We have the following Wilk's theorem, which holds for the true value P_0.

Theorem 2.1. *Assume the condition C.1 holds, one has*

$$-2logR(P_0) \xrightarrow{\mathcal{D}} \chi_1^2.$$

By Theorem 2.1, the $100(1-\alpha)\%$ *JEL confidence interval for* P_0 *can be constructed as follows:*

$$\mathcal{R}_\alpha = \left\{P : -2logR(P) \leq \chi_1^2(\alpha)\right\}$$

where $\chi_1^2(\alpha)$ *is the upper* α*-quantile of* χ_1^2 *distribution.*

2.5. Adjusted jackknife empirical likelihood for the Pietra ratio

In order to improve the performance of the JEL method, we apply the adjusted empirical likelihood (Chen et al., 2008) to develop the adjusted JEL (AJEL) for the Pietra ratio. The adjusted jackknife empirical likelihood also eliminates the empty set issue of the classical JEL. Motivated by Zhao et al. (2015), we give the adjusted jackknife empirical likelihood ratio at P as follows

$$R^{ad}(P) = \max\left\{\prod_{i=1}^{n+1}(n+1)p_i : \sum_{i=1}^{n+1} p_i g_i^{ad}(P) = 0, \sum_{i=1}^{n+1} p_i = 1, p_i \geq 0\right\},$$

$g_i^{ad}(P) = \widehat{U}_i(P)$, $i = 1, \ldots, n$, $g_{n+1}^{ad}(P) = -a_n/n\sum_{i=1}^{n} \widehat{U}_i(P)$ and $a_n = \max(1, log(n)/2)$ (see Chen et al., 2008). We obtain the adjusted jackknife empirical log-likelihood ratio at P as

$$logR^{ad}(P) = -\sum_{i=1}^{n+1} log\left\{1 + \lambda^{ad} g_i^{ad}(P)\right\},$$

where λ^{ad} satisfies the non-linear equation

$$f^{ad}(\lambda^{ad}) \equiv \sum_{i=1}^{n+1} \frac{g_i^{ad}(P)}{1 + \lambda^{ad} g_i^{ad}(P)} = 0.$$

Following arguments from Jing et al. (2009) and Zhao et al. (2015), we derive the Wilk's theorem of AJEL for the true P_0.

Theorem 2.2. *Assume the condition C.1 holds,*

$$-2logR^{ad}(P_0) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Thus, the $100(1-\alpha)\%$ *AJEL confidence interval for* P_0 *is as follows*

$$\mathcal{R}_\alpha^{ad} = \left\{P : -2logR^{ad}(P) \leq \chi_1^2(\alpha)\right\}.$$

2.6. Extended jackknife empirical likelihood for the Pietra ratio

We use the extended empirical likelihood (see Tsao and Wu, 2013) to develop the extended JEL for the Pietra ratio. The extended empirical likelihood can overcome the empty set problem and improve the accuracy of the EL confidence regions by extending the EL domain through a similarity transformation. Let Θ_n be the interior of the convex hull of the jackknife pseudo-values and define the bijective mapping $h_n : \mathbb{R} \to \mathbb{R}$. For $P \in \Theta_n$,

$$h_n(P) = \hat{P} + \gamma(n, R(P))(P - \hat{P}),$$

where

$$\gamma\,(n, R(P)) = 1 + \frac{R(P)}{2n}.$$

Because h_n is bijective, it follows that $h_n^{-1}(P) = \hat{P} + \gamma^{-1}(n, R(P))(P - \hat{P}) \in \Theta_n, \forall P \in \mathbb{R}$. Letting $P_0 \in \Theta_n$ be the image of P_0 under h_n^{-1}, the extended jackknife empirical log-likelihood ratio for $P_0 \in \mathbb{R}$ is then given by

$$R^*(P_0) = R(h_n^{-1}\,(P_0))$$

Similar to Tsao and Wu (2013) and Zhao et al. (2015), the Wilk's theorem holds for $-2logR^*(P_0)$ for the true value P_0.

Theorem 2.3. *Assume the condition C.1 holds, we have*

$$-2logR^*(P_0) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Thus, the $100(1-\alpha)\%$ *EJEL confidence interval for* P_0 *is as follows*

$$\mathcal{R}_\alpha^* = \left\{P : -2logR^*(P) \le \chi_1^2\,(\alpha)\right\}.$$

2.7. Balanced adjusted jackknife empirical likelihood for the Pietra ratio

As Emerson and Owen (2009) did before, we try adding two observations to the jackknife pseudo-samples and employ a new method for the Pietra ratio. As Emerson and Owen (2009) mentioned, this method called balanced adjusted JEL may perform well, when the sample size is relatively small compared with the dimension.

The balanced adjusted jackknife empirical likelihood (BAJEL) ratio is defined by

$$R^{ba}(P) = \max\left\{\prod_{i=1}^{n+2}(n+2)p_i : \sum_{i=1}^{n+2} p_i b_i^{ba}(P) = 0, \sum_{i=1}^{n+2} p_i = 1, p_i \ge 0\right\},$$

for $i = 1, \ldots, n$, $b_i^{ba}(P) = \widehat{U}_i(P)$ and $b_{n+1}^{ba}(P) = -s^{ba}c_{u^\star}u^\star$ and $b_{n+2}^{ba}(P) = 2\widehat{T}_{n,jack}(P) + s^{ba}c_{u^\star}u^\star$, where $u^\star = \widehat{T}_{n,jack}(P)/|\widehat{T}_{n,jack}(P)|$, $s_{jack}^2 = (n-1)^{-1}\sum_{i=1}^n\{\widehat{U}_i(P) - \widehat{T}_{n,jack}(P)\}^2$, $c_{u^\star} = u^\star\sqrt{s_{jack}^2}$, $\widehat{T}_{n,jack}(P) = 1/n\sum_{i=1}^n \widehat{U}_i(P)$ and s^{ba} is chosen to be 1.6 to 2.5 for different dimensions of an estimation equation by Section 6 in Emerson and Owen (2009). We set s^{ba} as 1.9 like Emerson and Owen (2009). We obtain the balanced adjusted jackknife empirical log-likelihood ratio at P as

$$logR^{ba}(P) = -\sum_{i=1}^{n+2} log\left\{1 + \lambda^{ba} b_i^{ba}(P)\right\} \tag{2.1}$$

where λ^{ba} satisfies the equation

$$f(\lambda^{ba}) \equiv \sum_{i=1}^{n+2} \frac{b_i^{ba}(P)}{1 + \lambda^{ba} b_i^{ba}(P)} = 0. \tag{2.2}$$

Following arguments from Jing et al. (2009) and Emerson and Owen (2009), the Wilk's theorem holds for the true P_0.

Theorem 2.4. *Assume the C.1 hold, one has*

$$-2logR^{ba}(P_0) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Thus, the $100(1-\alpha)\%$ *BAJEL confidence interval for* P_0 *is as follows*

$$\mathcal{R}_\alpha^{ba} = \left\{P : -2logR^{ba}\,(P) \le \chi_1^2\,(\alpha)\right\}.$$

Table 1
Coverage probability of 100(1 − α)% CI under the normal distribution.

n	$1-\alpha$	NA	Bp	Bca	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.769	0.761	0.784	0.843	1.000	0.939	0.886	0.893
	0.95	0.698	0.713	0.754	0.767	1.000	0.860	0.821	0.796
	0.90	0.647	0.667	0.717	0.712	0.947	0.799	0.781	0.716
10	0.99	0.914	0.912	0.932	0.941	0.981	0.977	0.939	0.908
	0.95	0.843	0.848	0.893	0.874	0.916	0.921	0.879	0.824
	0.90	0.801	0.802	0.833	0.816	0.862	0.860	0.847	0.746
20	0.99	0.956	0.964	0.978	0.959	0.970	0.977	0.963	0.958
	0.95	0.908	0.904	0.933	0.919	0.934	0.941	0.920	0.885
	0.90	0.848	0.859	0.876	0.860	0.881	0.884	0.865	0.820
30	0.99	0.967	0.971	0.982	0.974	0.978	0.981	0.970	0.973
	0.95	0.916	0.924	0.937	0.922	0.932	0.940	0.925	0.911
	0.90	0.868	0.875	0.887	0.878	0.890	0.895	0.877	0.848
100	0.99	0.984	0.983	0.988	0.988	0.989	0.990	0.984	0.986
	0.95	0.937	0.939	0.944	0.945	0.947	0.949	0.939	0.938
	0.90	0.889	0.888	0.897	0.892	0.895	0.896	0.892	0.886
300	0.99	0.991	0.988	0.988	0.989	0.989	0.990	0.991	0.992
	0.95	0.946	0.950	0.949	0.947	0.948	0.948	0.952	0.953
	0.90	0.892	0.901	0.904	0.897	0.898	0.898	0.892	0.903

Table 2
Average length of 100(1 − α)% CI under the normal distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.145	0.138	0.125	0.143	2.105	0.238	0.187	0.236
	0.95	0.111	0.113	0.108	0.116	2.105	0.160	0.140	0.195
	0.90	0.093	0.099	0.093	0.100	0.095	0.127	0.119	0.101
10	0.99	0.102	0.122	0.121	0.124	0.175	0.166	0.131	0.135
	0.95	0.078	0.094	0.101	0.096	0.112	0.114	0.100	0.102
	0.90	0.065	0.080	0.085	0.078	0.092	0.092	0.084	0.071
20	0.99	0.082	0.091	0.095	0.088	0.095	0.103	0.092	0.089
	0.95	0.062	0.069	0.074	0.068	0.072	0.076	0.070	0.070
	0.90	0.052	0.058	0.061	0.057	0.061	0.062	0.059	0.062
30	0.99	0.073	0.075	0.078	0.072	0.076	0.081	0.075	0.071
	0.95	0.055	0.056	0.059	0.055	0.058	0.060	0.057	0.054
	0.90	0.047	0.047	0.049	0.047	0.049	0.049	0.048	0.043
100	0.99	0.040	0.041	0.042	0.042	0.042	0.043	0.041	0.043
	0.95	0.031	0.031	0.032	0.031	0.031	0.032	0.031	0.037
	0.90	0.026	0.026	0.026	0.026	0.026	0.027	0.026	0.033
300	0.99	0.024	0.024	0.024	0.024	0.024	0.024	0.024	0.025
	0.95	0.018	0.018	0.018	0.018	0.018	0.018	0.018	0.019
	0.90	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.017

3. Numerical studies

We conducted extensive simulation studies to evaluate the performance of the JEL, the AJEL, the EJEL and BAJEL method. For the Pietra ratio P, we considered data from normal distribution $N(4, 1)$, exponential distribution $Exp(1)$, gamma distribution $gamma(2, 2)$, $t(10)+4$ and chi-square distribution χ_3^2. We compared performance of the JEL methods with other nonparametric methods, including the percentile bootstrap (Bp), the bias-corrected and accelerated bootstrap (Bca), the NA and the normal approximation jackknife (NAJ). We computed the coverage probability and the average length of the confidence interval (CI) of the Pietra ratio for confidence levels of 0.99, 0.95 and 0.90 and sample sizes of $n = 5, 10, 20, 30, 100,$ and 300. We calculated coverage probability and average length of confidence intervals using 5000 repetitions.

3.1. Simulation study under the normal distribution

We simulated data from $N(4, 1)$. Results are displayed in Tables 1 and 2. Among the four JEL methods, the EJEL had coverage probabilities which were the closest to the confidence levels for all sample sizes. For $n \geq 20$, the AJEL had better coverage than JEL. For $n = 5$, the AJEL showed over-coverage. For 0.99 confidence level, its coverage probability for $n = 20$ was lower than that for $n = 10$, which indicates that the AJEL method is not suitable for very small sample sizes ($n \leq 10$). The BAJEL method has poor coverage for all confidence levels when $n \leq 20$ and for 0.90 level when $n = 30$.

Table 3
Coverage probability of $100(1-\alpha)\%$ CI under the exponential distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
	0.99	0.359	0.833	0.852	0.881	1.000	0.960	0.918	0.942
5	0.95	0.285	0.790	0.822	0.831	1.000	0.900	0.864	0.865
	0.90	0.243	0.736	0.777	0.770	0.973	0.848	0.824	0.788
	0.99	0.764	0.946	0.965	0.951	0.988	0.987	0.961	0.908
10	0.95	0.699	0.889	0.922	0.888	0.927	0.925	0.912	0.840
	0.90	0.636	0.841	0.864	0.832	0.876	0.873	0.868	0.774
	0.99	0.937	0.982	0.987	0.976	0.981	0.989	0.978	0.951
20	0.95	0.860	0.931	0.939	0.918	0.934	0.944	0.929	0.888
	0.90	0.808	0.876	0.881	0.857	0.881	0.891	0.893	0.821
	0.99	0.962	0.982	0.987	0.978	0.981	0.987	0.984	0.972
30	0.95	0.909	0.931	0.940	0.927	0.938	0.946	0.943	0.919
	0.90	0.837	0.876	0.882	0.870	0.882	0.892	0.889	0.857
	0.99	0.986	0.987	0.988	0.983	0.986	0.987	0.989	0.986
100	0.95	0.940	0.946	0.946	0.942	0.945	0.948	0.950	0.943
	0.90	0.882	0.895	0.898	0.890	0.893	0.902	0.896	0.890
	0.99	0.988	0.989	0.988	0.988	0.988	0.988	0.990	0.990
300	0.95	0.944	0.951	0.951	0.946	0.947	0.951	0.951	0.944
	0.90	0.903	0.902	0.901	0.891	0.892	0.896	0.899	0.896

Table 4
Average length of $100(1-\alpha)\%$ CI under the exponential distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
	0.99	0.377	0.484	0.444	0.440	2.347	0.752	0.607	0.533
5	0.95	0.287	0.388	0.368	0.364	2.340	0.479	0.464	0.498
	0.90	0.241	0.335	0.314	0.318	0.459	0.371	0.388	0.490
	0.99	0.285	0.380	0.370	0.376	0.355	0.492	0.420	0.473
10	0.95	0.217	0.290	0.291	0.295	0.343	0.351	0.322	0.398
	0.90	0.182	0.244	0.243	0.250	0.343	0.284	0.267	0.314
	0.99	0.235	0.276	0.278	0.279	0.296	0.333	0.290	0.301
20	0.95	0.178	0.210	0.211	0.212	0.224	0.237	0.221	0.188
	0.90	0.150	0.176	0.177	0.178	0.189	0.193	0.186	0.143
	0.99	0.205	0.228	0.231	0.227	0.236	0.254	0.234	0.163
30	0.95	0.155	0.173	0.175	0.173	0.180	0.184	0.179	0.207
	0.90	0.129	0.145	0.147	0.145	0.150	0.152	0.149	0.132
	0.99	0.122	0.126	0.127	0.125	0.126	0.129	0.126	0.134
100	0.95	0.093	0.095	0.096	0.095	0.096	0.097	0.096	0.113
	0.90	0.078	0.080	0.080	0.080	0.080	0.081	0.080	0.098
	0.99	0.072	0.073	0.074	0.073	0.073	0.073	0.073	0.072
300	0.95	0.055	0.055	0.055	0.055	0.055	0.056	0.055	0.058
	0.90	0.046	0.046	0.046	0.046	0.046	0.046	0.046	0.050

The NA method had the lowest coverage probability for $n \le 30$, and the EJEL had the highest coverage probability for $n \le 10$. For $n = 5$, the NA method had very poor coverage, but the EJEL still had coverage probability close to the nominal level. For example, for 0.99 nominal level, the NA coverage probability was about 77% while the EJEL's coverage probability was about 94%. For $n > 100$, the eight methods had similar coverage probability. For the bootstrap method, the Bca performed well for $n \ge 100$, but had lower coverage for $n \le 100$. The EJEL which generally had the best coverage also had the longest average CI length among the eight methods. The NA had the poorest coverage and also had the lowest average CI length. For all methods, average length decreased with the increasing sample size.

3.2. Simulation study under the exponential distribution

In the simulation study, we generated data from an exponential distribution with $\lambda = 1$. Results are displayed in Tables 3 and 4. In terms of coverage probability, the EJEL outperformed the other three JEL methods. For very small sample size ($n = 5$), the JEL, AJEL and BAJEL also showed the good performance among the eight methods. Comparing coverage probability between exponential and normal distributions, for $n \le 10$, the NA had lower coverage probability with exponential data. Conversely, the other methods showed better performance with exponential data. The NAJ's coverage probability was close to the confidence level, especially for small sample sizes. The bootstrap methods had better coverage probability compared with the NA method.

Table 5
Coverage probability of 100(1 − α)% CI under the gamma distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.486	0.799	0.822	0.840	1.000	0.946	0.902	0.897
	0.95	0.424	0.751	0.801	0.760	1.000	0.869	0.840	0.778
	0.90	0.383	0.701	0.749	0.699	0.963	0.802	0.804	0.682
10	0.99	0.862	0.939	0.963	0.953	0.986	0.984	0.949	0.934
	0.95	0.781	0.891	0.917	0.886	0.927	0.933	0.905	0.841
	0.90	0.716	0.838	0.865	0.826	0.872	0.871	0.858	0.749
20	0.99	0.950	0.967	0.981	0.974	0.983	0.990	0.974	0.972
	0.95	0.887	0.912	0.936	0.927	0.942	0.949	0.925	0.896
	0.90	0.824	0.865	0.878	0.874	0.892	0.894	0.873	0.828
30	0.99	0.968	0.977	0.986	0.981	0.985	0.989	0.979	0.980
	0.95	0.914	0.929	0.942	0.938	0.945	0.952	0.931	0.917
	0.90	0.863	0.879	0.886	0.868	0.881	0.884	0.892	0.852
100	0.99	0.985	0.986	0.989	0.985	0.987	0.988	0.987	0.988
	0.95	0.936	0.937	0.941	0.945	0.947	0.949	0.941	0.941
	0.90	0.889	0.891	0.890	0.897	0.901	0.901	0.897	0.883
300	0.99	0.988	0.992	0.992	0.990	0.990	0.991	0.989	0.992
	0.95	0.954	0.953	0.953	0.949	0.950	0.951	0.955	0.952
	0.90	0.899	0.904	0.904	0.899	0.899	0.899	0.902	0.902

Table 6
Average length of 100(1 − α)% CI under the gamma distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.311	0.359	0.328	0.335	2.268	0.575	0.466	0.546
	0.95	0.236	0.292	0.277	0.279	2.263	0.351	0.353	0.469
	0.90	0.198	0.252	0.239	0.240	0.246	0.280	0.298	0.257
10	0.99	0.244	0.291	0.285	0.298	0.413	0.394	0.326	0.263
	0.95	0.185	0.222	0.228	0.231	0.270	0.277	0.246	0.236
	0.90	0.156	0.187	0.191	0.195	0.223	0.223	0.209	0.207
20	0.99	0.195	0.217	0.221	0.215	0.231	0.252	0.225	0.230
	0.95	0.149	0.165	0.169	0.164	0.175	0.180	0.173	0.178
	0.90	0.124	0.138	0.141	0.138	0.146	0.148	0.144	0.148
30	0.99	0.167	0.178	0.182	0.174	0.182	0.194	0.183	0.187
	0.95	0.127	0.135	0.137	0.134	0.140	0.143	0.139	0.134
	0.90	0.107	0.113	0.115	0.113	0.117	0.118	0.117	0.113
100	0.99	0.096	0.099	0.100	0.097	0.098	0.100	0.099	0.082
	0.95	0.074	0.075	0.075	0.074	0.075	0.076	0.075	0.077
	0.90	0.062	0.063	0.063	0.062	0.063	0.064	0.063	0.061
300	0.99	0.056	0.057	0.057	0.057	0.057	0.057	0.057	0.058
	0.95	0.043	0.043	0.043	0.043	0.043	0.044	0.043	0.044
	0.90	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.033

3.3. Simulation study under the other distributions

We also simulated data from $gamma(2, 2)$, $t(10)+4$ distribution and χ^2_3 distributions. Results are displayed in Tables 5–10. The results are similar to those of exponential and normal distributions. EJEL outperformed the other three JEL methods in terms of coverage probability when $n \leq 30$. AJEL methods showed the problem of over-coverage when $n = 5$. Bp, Bca and NA methods had lower coverage for the small sample size ($n = 5$) compared with JEL methods.

3.4. Simulation study based on a real data

We illustrate the simulation performance of the proposed methods under finite empirical population of the income data from the 2013 Current Population Survey (CPS). We construct a sample as the finite population which consists of 41 individuals with the range 25–30 years old. We take bootstrap procedure and repeatedly draw 41 observations with replacement as a sample from this finite population, using six methods evaluated in this simulation: the NA, the NAJ, the JEL, BAJEL, the AJEL, and the EJEL. In terms of coverage probability, the simulation results showed that JEL methods have better performance than the NA method in Table 11. Jackknife method has good performances in terms of coverage probabilities and average lengths. AJEL and EJEL methods keep the advantage over other methods in terms of coverage probabilities and average lengths.

Table 7
Coverage probability of $100(1-\alpha)\%$ CI under the t distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.742	0.743	0.761	0.838	1.000	0.931	0.869	0.897
	0.95	0.674	0.691	0.731	0.765	1.000	0.860	0.812	0.803
	0.90	0.642	0.646	0.688	0.705	0.945	0.795	0.777	0.729
10	0.99	0.905	0.901	0.929	0.926	0.969	0.968	0.926	0.909
	0.95	0.837	0.840	0.884	0.869	0.916	0.919	0.871	0.827
	0.90	0.790	0.791	0.829	0.804	0.858	0.856	0.835	0.754
20	0.99	0.946	0.959	0.974	0.954	0.969	0.973	0.956	0.951
	0.95	0.899	0.907	0.930	0.903	0.924	0.935	0.910	0.875
	0.90	0.845	0.858	0.881	0.847	0.870	0.876	0.862	0.814
30	0.99	0.959	0.968	0.980	0.964	0.970	0.976	0.961	0.970
	0.95	0.915	0.917	0.933	0.916	0.927	0.933	0.925	0.908
	0.90	0.858	0.865	0.879	0.863	0.875	0.880	0.867	0.845
100	0.99	0.984	0.983	0.986	0.985	0.986	0.987	0.984	0.985
	0.95	0.943	0.933	0.939	0.937	0.940	0.941	0.944	0.941
	0.90	0.879	0.878	0.882	0.884	0.887	0.888	0.882	0.885
300	0.99	0.987	0.986	0.987	0.989	0.989	0.989	0.987	0.989
	0.95	0.952	0.944	0.945	0.949	0.950	0.952	0.952	0.945
	0.90	0.900	0.888	0.894	0.900	0.901	0.901	0.901	0.896

Table 8
Average length of $100(1-\alpha)\%$ CI under the t distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.167	0.189	0.536	0.164	2.122	0.276	0.213	0.266
	0.95	0.127	0.136	0.144	0.134	2.122	0.186	0.162	0.234
	0.90	0.107	0.117	0.117	0.115	0.119	0.143	0.136	0.195
10	0.99	0.137	0.144	0.146	0.145	0.188	0.194	0.152	0.167
	0.95	0.107	0.110	0.124	0.111	0.131	0.133	0.115	0.118
	0.90	0.089	0.094	0.104	0.093	0.108	0.107	0.097	0.089
20	0.99	0.104	0.108	0.115	0.097	0.106	0.113	0.107	0.108
	0.95	0.079	0.082	0.091	0.077	0.082	0.085	0.082	0.085
	0.90	0.065	0.068	0.074	0.065	0.069	0.069	0.069	0.071
30	0.99	0.086	0.087	0.093	0.081	0.085	0.090	0.088	0.086
	0.95	0.065	0.066	0.071	0.063	0.065	0.067	0.067	0.079
	0.90	0.055	0.055	0.058	0.054	0.056	0.057	0.056	0.070
100	0.99	0.048	0.049	0.051	0.047	0.048	0.049	0.048	0.049
	0.95	0.037	0.037	0.038	0.036	0.037	0.037	0.037	0.037
	0.90	0.031	0.031	0.031	0.030	0.030	0.031	0.031	0.031
300	0.99	0.028	0.028	0.029	0.028	0.028	0.029	0.028	0.034
	0.95	0.021	0.021	0.022	0.021	0.021	0.022	0.021	0.022
	0.90	0.018	0.018	0.018	0.017	0.017	0.018	0.018	0.021

3.5. Numerical results

From those simulation results, JEL-based methods had good performances. Among the eight methods, the EJEL demonstrated the best result. For larger sample sizes ($n \geq 100$), all methods had similar performance. The difference of performances was even larger with small sample cases. Comparing results among data generated from different distributions, the NA method had lower coverage for skewed data whereas the JEL-based methods had better performance for the skewed data. Considering the bootstrap, the Bca had large coverage probability than the Bp. The Bp performed better than the NA method for the skewed data. NAJ methods outperformed bootstrap and NA methods. For small sample sizes ($n = 5$), NAJ may also outperform BAJEL and JEL methods. Computational speed for the JEL methods was faster than for both bootstrap methods. Among JEL methods, the JEL, AJEL, and BAJEL had faster computation performance than the EJEL method.

4. Real data analysis

We applied proposed JEL-based methods to analysis a relatively large sample data (1390 observations) from 2013 Current Population Survey, and compute the point estimate for the Pietra ratio P and confidence intervals (CI) at three confidence levels: 0.99, 0.95 and 0.90. We displayed the lower bound (LB), upper bound (UB), and interval length of confidence intervals for each method in Table 12.

Table 9
Coverage probability of $100(1-\alpha)\%$ CI under the chi-square distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.742	0.804	0.836	0.946	0.995	0.956	0.902	0.916
	0.95	0.635	0.771	0.812	0.881	0.993	0.899	0.853	0.805
	0.90	0.606	0.706	0.777	0.827	0.959	0.846	0.819	0.698
10	0.99	0.896	0.936	0.962	0.965	0.998	0.968	0.952	0.940
	0.95	0.795	0.882	0.913	0.912	0.951	0.918	0.903	0.846
	0.90	0.734	0.828	0.846	0.850	0.906	0.861	0.851	0.758
20	0.99	0.950	0.968	0.977	0.984	0.993	0.985	0.972	0.973
	0.95	0.880	0.925	0.932	0.931	0.955	0.936	0.927	0.905
	0.90	0.820	0.866	0.880	0.878	0.905	0.882	0.887	0.828
30	0.99	0.968	0.969	0.983	0.986	0.991	0.987	0.975	0.976
	0.95	0.900	0.935	0.942	0.933	0.950	0.936	0.939	0.917
	0.90	0.846	0.874	0.892	0.878	0.902	0.882	0.886	0.847
100	0.99	0.985	0.984	0.986	0.991	0.992	0.991	0.990	0.987
	0.95	0.942	0.944	0.949	0.950	0.956	0.951	0.944	0.939
	0.90	0.887	0.891	0.891	0.897	0.905	0.898	0.901	0.885
300	0.99	0.988	0.987	0.988	0.989	0.989	0.989	0.991	0.990
	0.95	0.947	0.946	0.948	0.950	0.952	0.951	0.952	0.950
	0.90	0.889	0.900	0.897	0.904	0.907	0.904	0.892	0.897

Table 10
Average length of $100(1-\alpha)\%$ CI under the chi-square distribution.

n	$1-\alpha$	NA	Bp	BCa	JEL	AJEL	EJEL	NAJ	BAJEL
5	0.99	0.346	0.407	0.382	0.374	2.286	0.467	0.521	0.458
	0.95	0.260	0.333	0.312	0.349	2.276	0.351	0.400	0.378
	0.90	0.221	0.284	0.267	0.311	0.397	0.341	0.335	0.363
10	0.99	0.261	0.325	0.317	0.322	0.486	0.328	0.365	0.281
	0.95	0.196	0.250	0.249	0.280	0.368	0.351	0.277	0.263
	0.90	0.164	0.211	0.209	0.225	0.316	0.252	0.233	0.238
20	0.99	0.213	0.237	0.234	0.219	0.271	0.276	0.252	0.261
	0.95	0.162	0.182	0.182	0.148	0.171	0.181	0.192	0.207
	0.90	0.136	0.153	0.154	0.133	0.125	0.128	0.161	0.170
30	0.99	0.184	0.195	0.194	0.143	0.167	0.187	0.204	0.211
	0.95	0.139	0.149	0.151	0.110	0.118	0.127	0.155	0.161
	0.90	0.117	0.125	0.127	0.101	0.102	0.117	0.130	0.118
100	0.99	0.107	0.108	0.108	0.090	0.092	0.096	0.110	0.087
	0.95	0.081	0.083	0.083	0.082	0.086	0.090	0.084	0.073
	0.90	0.068	0.070	0.070	0.078	0.077	0.077	0.070	0.071
300	0.99	0.063	0.063	0.063	0.058	0.077	0.080	0.063	0.060
	0.95	0.048	0.048	0.048	0.049	0.056	0.057	0.048	0.052
	0.90	0.040	0.040	0.040	0.037	0.049	0.047	0.040	0.040

Table 11
Average length of CI and coverage probability based on the real data.

Length	99%	95%	90%	Coverage probability	99%	95%	90%
NA	0.131	0.099	0.084	NA	0.972	0.915	0.870
NAJ	0.142	0.109	0.091	NAJ	0.980	0.925	0.884
JEL	0.121	0.076	0.056	JEL	0.975	0.926	0.875
AJEL	0.128	0.086	0.061	AJEL	0.981	0.936	0.891
EJEL	0.131	0.090	0.071	EJEL	0.985	0.938	0.883
BAJEL	0.123	0.084	0.072	BAJEL	0.956	0.886	0.816

The point estimate of P was 0.236. Results in Table 12 showed that the methods produced CIs that were similar in length. These results are consistent with results from the simulation studies, where we observed the longest average length for the EJEL and similar lengths for the other methods. The CIs based on the NA method were symmetric about the point estimate of P, whereas the CIs based on the other nonparametric methods were asymmetric, particularly for the EJEL. This perhaps indicates that the symmetry imposed by the NA confidence intervals may be unrealistic for the skewed distribution of the income data.

5. Discussion

In this paper, we introduce nonparametric interval estimators for the Pietra ratio and its asymptotic properties, including JEL based methods, which are the JEL, the AJEL, the EJEL and the BAJEL method. Simulation studies showed

Table 12
Lower bound (LB), upper bound (UB), and length of confidence intervals (CI) of the Pietra ratio for the income distribution among white individuals, aged 25–30 years, in the U.S.A.

$1-\alpha$		99%		95%		90%	
NA	LB	Length	0.224	Length	0.227	Length	0.228
	UB	0.025	0.249	0.019	0.246	0.016	0.244
Bp	LB	Length	0.225	Length	0.227	Length	0.228
	UB	0.023	0.248	0.019	0.246	0.016	0.244
Bca	LB	Length	0.225	Length	0.227	Length	0.229
	UB	0.026	0.251	0.019	0.246	0.015	0.244
JEL	LB	Length	0.225	Length	0.228	Length	0.229
	UB	0.023	0.248	0.017	0.245	0.014	0.243
EJEL	LB	Length	0.226	Length	0.223	Length	0.227
	UB	0.030	0.246	0.022	0.246	0.018	0.246
AJEL	LB	Length	0.225	Length	0.228	Length	0.229
	UB	0.023	0.248	0.017	0.245	0.014	0.243
BAJEL	LB	Length	0.225	Length	0.228	Length	0.229
	UB	0.023	0.248	0.017	0.245	0.014	0.243
NAJ	LB	Length	0.224	Length	0.227	Length	0.228
	UB	0.025	0.249	0.019	0.246	0.016	0.244

that the JEL methods demonstrate two advantages over the other methods. First, the JEL-based confidence intervals have better coverage probability than the NA confidence intervals particularly for the small sample size. For very small sample size ($n \le 10$), the EJEL shows the best performance. Second, although the bootstrap methods perform well for most sample sizes ($n \ge 100$), the computation speed for the JEL methods is much faster than that for the bootstrap methods.

The current study indicates that our proposed interval estimators, the AJEL and EJEL, for the Pietra ratio outperform other interval estimators. However, our research was limited to the one-dimensional data. It may be beneficial to adapt our estimators to a multi-dimensional framework as a future work. Also our work assumed that data were completely observed, and it would be useful to extend the proposed JEL methods for the Pietra ratio to the more realistic setting with data that are missing at random or right censored or stratified sample. In addition, it would be advantageous to apply JEL inference for the difference of two Pietra ratios. Finally a robust Pietra ratio based on the median rather than the mean of the sampling distribution may be interesting, and nonparametric inference methods for a robust estimation of Pietra ratio would be appealing.

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Appendix. Proofs of theorems

Lemma A.1. *Define* $\upsilon^2 = \int_{-\infty}^{+\infty}\{(p-P_0)x - p\mu\}^2 dF(x) - (P_0\mu)^2 + \int_{-\infty}^{\mu}(x-\mu)^2 dF(x) - (\delta/2)^2 - 2\int_{-\infty}^{\mu}\{(p-P_0)x - p\mu + P_0\mu\}\{I_{\{x<\mu\}}(x-\mu) + \delta/2\}dF(x)$ *and* $p = F(\mu)$*. Assume that C.1 holds, one has*

$$\sqrt{n}\hat{T}_{n,jack} \xrightarrow{\mathfrak{D}} N\left(0, 4\upsilon^2\right).$$

Proof of Lemma A.1. Using the same methods as those in Gastwirth (1974) and Zhao et al. (2015), denote N as the number of the observations which are less than $\bar{X}$, $X_i < \bar{X}$. From the delete-one sample

$$\mathbf{X}_i = \{X_1, X_2, \ldots X_{i-1}, X_{i+1}, \ldots X_n\},$$

we have $\bar{X}_{(-i)} = \frac{1}{n-1}\sum_{j\ne i}^{n} X_j$, $\bar{X} - \bar{X}_{(-i)} = (X_i - \bar{X})/(n-1)$ and $\sum_{i=1}^{n}\{\bar{X} - \bar{X}_{(-i)}\} = 0$, where

$$\bar{X} - \bar{X}_{(-i)} = O_p(n^{-1}).$$

Following Zhao et al. (2015), we denote $N(\mathbf{X})$ as the number of elements in the set $\{X_j < \bar{X}, j = 1, \ldots, n\}$ and $N(\mathbf{X}_i)$ as the number of elements in the set $\{X_j < \bar{X}_{(-i)}, j = 1, \ldots, n, i \ne j\}$. We define $N(\triangle\mathbf{X}_i)$ as the number of elements between

$\bar{X}$ and $\bar{X}_{(-i)}$, i.e., the set $\{X_j \in (\bar{X}, \bar{X}_{(-i)}) \cup (\bar{X}_{(-i)}, \bar{X}), j = 1, \ldots, n\}$. Then,

$$\triangle N(\mathbf{X}, \mathbf{X}_i) = N(\mathbf{X}) - N(\mathbf{X}_i) = \begin{cases} 1 - N(\triangle \mathbf{X}_i), & X_i \le \bar{X}, \\ N(\triangle \mathbf{X}_i), & X_i > \bar{X}. \end{cases}$$

From $\bar{X} - \bar{X}_{(-i)} = O_p(n^{-1})$ and arguments in Gastwirth (1974) and Zhao et al. (2015), $N(\triangle \mathbf{X}_i)$ is approximately $nf(\mu)|\bar{X} - \bar{X}_{(-i)}|$, i.e.,

$$N(\triangle \mathbf{X}_i) = O_p(f(\mu)).$$

And, we have

$$\sum_{X_j < \bar{X}} X_j - \sum_{X_j < \bar{X}_{(-i)}} X_j = \begin{cases} X_i - \sum_{\bar{X} < X_j < \bar{X}_{(-i)}} X_j, & X_i \le \bar{X}, \\ \sum_{\bar{X}_{(-i)} < X_j < \bar{X}} X_j, & X_i > \bar{X}. \end{cases}$$

Based on the above equations, we know that (see Zhao et al. (2015))

$$\begin{cases} \sum_{\bar{X} < X_j < \bar{X}_{(-i)}} X_j = N(\triangle \mathbf{X}_i)\bar{X} + O_p(n^{-1}), & X_i \le \bar{X}, \\ \sum_{\bar{X}_{(-i)} < X_j < \bar{X}} X_j = N(\triangle \mathbf{X}_i)\bar{X} + O_p(n^{-1}), & X_i > \bar{X}. \end{cases}$$

Hence, combining the above results, we have

$$\begin{aligned}
&\hat{T}_{n,jack}(P_0) \\
=&\frac{1}{n}\sum_{i=1}^{n} \widehat{U}_i(P_0) \\
=&\frac{1}{n}\sum_{i=1}^{n} \{n\hat{T}_n(P_0) - (n-1)\,\hat{T}_{n-1}^{(-i)}(P_0)\} \\
=&2\left(N(\mathbf{X})\bar{X} - \sum_{X_j < \bar{X}} X_j\right) - \frac{2}{n}\sum_{i=1}^{n}\left(N(\mathbf{X}_i)\bar{X}_{(-i)} - \sum_{X_j < \bar{X}_{(-i)}, i \ne j} X_j\right) \\
&-\frac{2}{n}\sum_{i=1}^{n}\left\{n\bar{X}P_0 - (n-1)\bar{X}_{(-i)}P_0\right\} \\
=&\frac{2}{n}\sum_{i=1}^{n}\left(N(\mathbf{X})\bar{X} - N(\mathbf{X}_i)\bar{X}_{(-i)}\right) - \frac{2}{n}\sum_{i=1}^{n}\left(\sum_{X_j < \bar{X}} X_j - \sum_{X_j < \bar{X}_{(-i)}, i \ne j} X_j\right) \\
&+\frac{2(n-1)}{n}\sum_{i=1}^{n}\left(\bar{X}P_0 - \bar{X}_{(-i)}P_0\right) - 2\bar{X}P_0 \\
=&\frac{2}{n}\sum_{i=1}^{n}\left(N(\mathbf{X})\bar{X} - N(\mathbf{X}_i)\bar{X}_{(-i)}\right) - \frac{2}{n}\sum_{i=1}^{n}\left(\sum_{X_j < \bar{X}} X_j - \sum_{X_j < \bar{X}_{(-i)}, i \ne j} X_j\right) - 2\bar{X}P_0 \\
=&\frac{2}{n}\sum_{i=1}^{n}\left(N(\mathbf{X})\bar{X} - \{N(\mathbf{X}) - \triangle N(\mathbf{X}, \mathbf{X}_i)\}\bar{X}_{(-i)}\right) - \frac{2}{n}\sum_{i=1}^{n}\left(\sum_{X_j < \bar{X}} X_j - \sum_{X_j < \bar{X}_{(-i)}, i \ne j} X_j\right) - 2\bar{X}P_0 \\
=&\frac{2}{n}\sum_{i=1}^{n}\left(\triangle N(\mathbf{X}, \mathbf{X}_i)\bar{X}_{(-i)}\right) - \frac{2}{n}\sum_{X_i < \bar{X}}^{n} X_i + \frac{2}{n}\sum_{X_i < \bar{X}}^{n}\left(\sum_{\bar{X} < X_j < \bar{X}_{(-i)}} X_j\right) \\
&-\frac{2}{n}\sum_{X_i > \bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)} < X_j < \bar{X}} X_j\right) - 2\bar{X}P_0 \\
=&\frac{2}{n}\sum_{i=1}^{n}\left(\triangle N(\mathbf{X}, \mathbf{X}_i)\bar{X}\right) - \frac{2}{n}\sum_{X_i < \bar{X}}^{n} X_i + \frac{2}{n}\sum_{X_i < \bar{X}}^{n}\left(\sum_{\bar{X} < X_j < \bar{X}_{(-i)}} X_j\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}} X_j\right) + o_p(n^{-1/2}) - 2\bar{X}P_0 \\
=&\frac{2N(\mathbf{X})\bar{X}}{n} - \frac{2}{n}\sum_{X_i<\bar{X}}^{n} X_i + \frac{2\bar{X}}{n}\sum_{X_i>\bar{X}} N(\triangle \mathbf{X}_i) - \frac{2}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}} X_j\right) \\
&- \frac{2\bar{X}}{n}\sum_{X_i<\bar{X}} N(\triangle \mathbf{X}_i) + \frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left(\sum_{\bar{X}<X_j<\bar{X}_{(-i)}} X_j\right) - 2\bar{X}P_0 + o_p(n^{-1/2}) \\
=&\frac{2N(\mathbf{X})\bar{X}}{n} - \frac{2}{n}\sum_{X_i<\bar{X}}^{n} X_i - 2\bar{X}P_0 + o_p(n^{-1/2}).
\end{aligned}
$$

Recall that $p = F(\mu)$. From Gastwirth (1974), we know that

$$
\begin{aligned}
&T_n(P_0) \\
=&\frac{2}{n}\left(N(\mathbf{X})\bar{X} - \sum_{X_j<\bar{X}} X_j\right) - 2\bar{X}P_0 \\
=&\frac{2p}{n}\sum_{i=1}^{n}(X_i-\mu) - \frac{2}{n}\sum_{i=1}^{n} I_{\{X_i<\mu\}}(X_i-\mu) - 2\bar{X}P_0 + o_p(n^{-1/2}) \\
=&\frac{2}{n}\sum_{i=1}^{n}\{(p-P_0)X_i - p\mu\} - \frac{2}{n}\sum_{i=1}^{n} I_{\{X_i<\mu\}}(X_i-\mu) + o_p(n^{-1/2}).
\end{aligned}
$$

Denote $Y_i = (p-P_0)X_i - p\mu$ and $Z_i = I_{\{X_i<\mu\}}(X_i-\mu)$. We note that $E(Y_i) = -P_0\mu$ and $E(Z_i) = -\delta/2$, $\mathrm{Var}(Y_i) = \int_{-\infty}^{+\infty}\{(p-P_0)x - p\mu\}^2 \mathrm{d}F(x) - (P_0\mu)^2$, $\mathrm{Var}(Z_i) = \int_{-\infty}^{\mu}(x-\mu)^2\mathrm{d}F(x) - (\delta/2)^2$ and $\mathrm{Cov}(Y_i, Z_i) = \int_{-\infty}^{\mu}\{(p-P_0)x - p\mu + P_0\mu\}\{I_{\{x<\mu\}}(x-\mu) + \delta/2\}\mathrm{d}F(x)$. One has that

$$
\begin{aligned}
\mathrm{Var}(Y_i - Z_i) =&\mathrm{Var}(Y_i) + \mathrm{Var}(Z_i) - 2\mathrm{Cov}(Y_i, Z_i) \\
=&\int_{-\infty}^{+\infty}\{(p-P_0)x - p\mu\}^2\mathrm{d}F(x) - (P_0\mu)^2 \\
&+ \int_{-\infty}^{\mu}(x-\mu)^2\mathrm{d}F(x) - (\delta/2)^2 \\
&+ \int_{-\infty}^{\mu}\{(p-P_0)x - p\mu + P_0\mu\}\{I_{\{x<\mu\}}(x-\mu) + \delta/2\}\mathrm{d}F(x) \\
=&v^2.
\end{aligned}
$$

Note that $\delta - 2P_0\mu = 0$. Hence, we have

$$
\sqrt{n}\hat{T}_{n,jack} = \sqrt{n}\{\hat{T}_{n,jack} - (\delta - 2P_0\mu)\} \xrightarrow{\mathfrak{D}} N\left(0, 4v^2\right). \quad \square
$$

Define the jackknife estimated variance as

$$
\hat{\sigma}^2_{n,jack} = \frac{1}{n}\sum_{i=1}^{n}\hat{U}_i^2(P_0). \tag{A.1}
$$

Lemma A.2. *Assume that C.1 holds, one has*

$$
\frac{\hat{\sigma}^2_{n,jack}}{4} \xrightarrow{\mathcal{P}} v^2.
$$

Proof of Lemma A.2. We know that

$$
\frac{1}{4n}\sum_{i=1}^{n}\hat{U}_i^2(P_0)
$$

$$
\begin{aligned}
&=\frac{1}{n}\sum_{i=1}^{n}\left\{N(\mathbf{X})\bar{X}-\sum_{X_j<\bar{X}}X_j-N(\mathbf{X}_i)\bar{X}_{(-i)}+\sum_{X_j<\bar{X}_{(-i)},i\neq j}X_j-\left(n\bar{X}P_0-(n-1)\bar{X}_{(-i)}P_0\right)\right\}^2\\
&=\frac{1}{n}\sum_{i=1}^{n}\left\{N(\mathbf{X})\bar{X}-\sum_{X_j<\bar{X}}X_j-N(\mathbf{X}_i)\bar{X}_{(-i)}+\sum_{X_j<\bar{X}_{(-i)},i\neq j}X_j-X_iP_0\right\}^2\\
&=\underbrace{\frac{1}{n}\sum_{i=1}^{n}\left\{N(\mathbf{X})\bar{X}-N(\mathbf{X}_i)\bar{X}_{(-i)}-X_iP_0\right\}^2}_{\mathrm{I}}+\underbrace{\frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{X_j<\bar{X}}X_j-\sum_{X_j<\bar{X}_{(-i)},i\neq j}X_j\right\}^2}_{\mathrm{II}}\\
&\quad-\underbrace{\frac{2}{n}\sum_{i=1}^{n}\left\{N(\mathbf{X})\bar{X}-N(\mathbf{X}_i)\bar{X}_{(-i)}-X_iP_0\right\}\left\{\sum_{X_j<\bar{X}}X_j-\sum_{X_j<\bar{X}_{(-i)},i\neq j}X_j\right\}}_{\mathrm{III}}.
\end{aligned}
$$

Then, we investigate I, II and III, respectively.

$$
\begin{aligned}
\mathrm{I}=&\frac{1}{n}\sum_{i=1}^{n}\left\{N(\mathbf{X})\bar{X}-N(\mathbf{X}_i)\bar{X}-X_iP_0+N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}\right\}^2\\
=&\frac{1}{n}\sum_{i=1}^{n}\left\{\bar{X}\triangle N(\mathbf{X},\mathbf{X}_i)-X_iP_0+N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}\right\}^2\\
=&\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{N(\mathbf{X}_i)}{n-1}(X_i-\bar{X})-X_iP_0\right\}^2+\frac{1}{n}\sum_{X_i>\bar{X}}^{n}\bar{X}^2\left\{N(\triangle\mathbf{X}_i)\right\}^2+\underbrace{\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\bar{X}^2\left\{1-N(\triangle\mathbf{X}_i)\right\}^2}_{\mathrm{IV}}\\
&+\underbrace{\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}}_{\mathrm{V}}\\
&+\frac{2}{n}\sum_{X_i>\bar{X}}^{n}\bar{X}N(\triangle\mathbf{X}_i)\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}
\end{aligned}
$$

and

$$
\mathrm{II}=\frac{1}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}}X_j\right)^2+\underbrace{\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)^2}_{\mathrm{VI}}
$$

and

$$
\begin{aligned}
\mathrm{III}=&\frac{2}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}}X_j\right)\left\{\bar{X}N(\triangle\mathbf{X}_i)-X_iP_0+N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}\right\}\\
&+\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\left\{\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}-X_iP_0+N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}\right\}\\
=&\frac{2}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}}X_j\right)\left\{\bar{X}N(\triangle\mathbf{X}_i)\right\}+\frac{2}{n}\sum_{X_i>\bar{X}}^{n}\left(\sum_{\bar{X}_{(-i)}<X_j<\bar{X}}X_j-X_iP_0\right)\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}\right\}
\end{aligned}
$$

$$
+ \underbrace{\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\left\{\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}\right\}}_{\mathrm{VII}}
$$

$$
+ \underbrace{\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}}_{\mathrm{VIII}}.
$$

Combining IV, VI and VII, we have

$$
\begin{aligned}
&\mathrm{IV}+\mathrm{VI}-\mathrm{VII}\\
=&\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}\left[\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}-\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\right]\\
&-\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\left[\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}-\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\right]\\
=&\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i-\left\{\bar{X}N(\triangle\mathbf{X}_i)-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right\}\right]^2\\
=&\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]^2+o_p(1).
\end{aligned}
$$

Then, from V and VIII, one has

$$
\begin{aligned}
&\mathrm{V}+\mathrm{VIII}\\
=&\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}\left\{1-N(\triangle\mathbf{X}_i)\right\}-\left(X_i-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right)\right]\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}\\
=&\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i-\left\{\bar{X}N(\triangle\mathbf{X}_i)-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right\}\right]\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}\\
=&-\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]^2\frac{N(\mathbf{X}_i)}{n-1}+\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]X_iP_0\\
&+\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left\{\bar{X}N(\triangle\mathbf{X}_i)-\sum_{\bar{X}<X_j<\bar{X}_{(-i)}}X_j\right\}\left\{N(\mathbf{X}_i)\frac{X_i-\bar{X}}{n-1}-X_iP_0\right\}\\
=&-\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]^2\frac{N(\mathbf{X}_i)}{n-1}+\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]X_iP_0+o_p(1).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathrm{I}+\mathrm{II}-\mathrm{III}\\
=&\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{N(\mathbf{X}_i)}{n-1}(X_i-\bar{X})-X_iP_0\right\}^2-\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]^2\frac{N(\mathbf{X}_i)}{n-1}\\
&+\frac{2}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]X_iP_0+\frac{1}{n}\sum_{X_i<\bar{X}}^{n}\left[\bar{X}-X_i\right]^2+o_p(1)
\end{aligned}
$$

$$= \int_{-\infty}^{+\infty} \{(p-P_0)x - p\mu\}^2 \mathrm{d}F(x) + \int_{-\infty}^{\mu} (x-\mu)^2 \mathrm{d}F(x) + \int_{-\infty}^{\mu} ((p-P_0)x - p\mu)(x-\mu)\mathrm{d}F(x) + o_p(1),$$

which shows

$$\begin{aligned}
&\frac{\hat{\sigma}^2_{n,jack}}{4} \\
&= \int_{-\infty}^{+\infty} \{(p-P_0)x - p\mu\}^2 \mathrm{d}F(x) + \int_{-\infty}^{\mu} (x-\mu)^2 \mathrm{d}F(x) \\
&\quad + \int_{-\infty}^{\infty} \{(p-P_0)x - p\mu\}\{I_{x<\mu}(x-\mu)\}\mathrm{d}F(x) - (\delta/2 - P_0\mu)^2 + o_p(1) \\
&= \int_{-\infty}^{+\infty} \{(p-P_0)x - p\mu\}^2 \mathrm{d}F(x) + \int_{-\infty}^{\mu} (x-\mu)^2 \mathrm{d}F(x) \\
&\quad + \int_{-\infty}^{\infty} \{(p-P_0)x - p\mu\}\{I_{x<\mu}(x-\mu)\}\mathrm{d}F(x) + o_p(1) \\
&\xrightarrow{\mathcal{P}} v^2. \quad \square
\end{aligned}$$

Proof of Theorem 2.1. Motivated by Zhao et al. (2015), we give the proof by Lemmas A.1 and A.2. We have $1/n \sum_{i=1}^{n} \widehat{U}_i^2(P_0) = \hat{\sigma}^2_{n,jack} + o_p(1)$ a.s. Therefore, we have

$$\begin{aligned}
\lambda &= \frac{1}{n}\sum_{i=1}^{n} \widehat{U}_i(P_0) \Big/ \left\{ \frac{1}{n}\sum_{i=1}^{n} \widehat{U}_i^2(P_0) \right\} + o_p(n^{-1/2}) \\
&= \{\hat{T}_{n,jack}(P_0)\}^2 \hat{\sigma}^{-2}_{n,jack} + o_p(n^{-1/2}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
-2\log R(P_0) &= 2\sum_{i=1}^{n} \{\lambda \widehat{U}_i(P_0) - \frac{1}{2}\lambda^2 \widehat{U}_i^2(P_0)\} + o_p(1) \\
&= 2n\lambda \hat{T}_{n,jack}(P_0) - n\lambda^2 \hat{\sigma}^2_{n,jack} + o_p(1) \\
&= n\hat{\sigma}^{-2}_{n,jack}\{\hat{T}_{n,jack}(P_0)\}^2 + o_p(1) \\
&\xrightarrow{\mathcal{D}} \chi^2_1. \quad \square
\end{aligned}$$

Proof of Theorem 2.2. We follow the argument of Chen et al. (2008) and Zhao et al. (2015). We have $1/(n+1)\sum_{i=1}^{n+1}\{g_i^{ad}(P_0)\}^2 = \hat{\sigma}^2_{n,jack} + o_p(1)$. The adjusted JEL ratio can be shown as follows as in Zhao et al. (2015).

$$-2\log R^{ad}(P_0) \quad \xrightarrow{\mathcal{D}} \quad \chi^2_1. \quad \square$$

Proof of Theorem 2.3. Following Tsao and Wu (2013) and Zhao et al. (2015), we define

$$h_n(P) = \hat{P} + \gamma(n, R(P))(P - \hat{P}),$$

where the expansion factor $\gamma(n, l(P)) = 1 + R(P)/(2n)$ and $\kappa_n = 1 - \gamma^{-1}(n, R(P))$. Details are omitted. $\square$

Proof of Theorem 2.4. From Eq. (2.2), we follow Emerson and Owen (2009) to obtain that

$$\begin{aligned}
0 &= \left| \frac{1}{n+2}\sum_{i=1}^{n+2} b_i^{ba}(P_0) - \frac{\lambda^{ba}}{n+2}\sum_{i=1}^{n+2} \frac{\{b_i^{ba}(P_0)\}^2}{1+\lambda^{ba} b_i^{ba}(P_0)} \right| \\
&\ge \frac{|\lambda^{ba}|}{n+2}\sum_{i=1}^{n+2} \frac{\{b_i^{ba}(P_0)\}^2}{1+\lambda^{ba} b_i^{ba}(P_0)} - \left| \frac{1}{n+2}\sum_{i=1}^{n+2} b_i^{ba}(P_0) \right| \\
&\ge \frac{|\lambda^{ba}|}{1+|\lambda^{ba}|Z^{ba}_{n+2}} \frac{\sum_{i=1}^{n+2}\{b_i^{ba}(P_0)\}^2}{n+2} - \left|\bar{b}_n(P_0)\right|,
\end{aligned}$$

where $\bar{b}_n(P_0) = (n+2)^{-1}\sum_{i=1}^{n+2} b_i^{ba}(P_0)$ and λ^{ba} is the Lagrange multiplier in Eq. (2.2). Note that $b^{ba}_{n+1}(P_0) = -s^{ba}c_{u^\star}u^\star$ and $b^{ba}_{n+2}(P_0) = 2\widehat{T}_{n,jack}(P_0) + s^{ba}c_{u^\star}u^\star$. One has that $\bar{b}_n(P_0) = \widehat{T}_{n,jack}(P_0) = O_p(n^{-1/2})$ and

$$Z^{ba}_{n+2} = \max_{1\le i\le n+2} |b_i^{ba}(P_0)|$$

$$
\begin{aligned}
&= \max\{\max_{1\le i\le n} |\widehat{U}_i(P_0)|, -s^{ba}c_{u^\star}u^\star, 2\widehat{T}_{n,jack}(P_0)+s^{ba}c_{u^\star}u^\star\} \\
&= o_p(n^{1/2}).
\end{aligned}
$$

From Eq. (A.1), we have $1/(n+2)\sum_{i=1}^{n+2}\{b_i^{ba}(P_0)\}^2 = \hat{\sigma}^2_{n,jack} + o_p(1)$. Hence, $\lambda^{ba} = O_p(n^{-1/2})$ and

$$
\begin{aligned}
\left|\lambda^{ba} Z_{n+2}^{ba}\right| &= \max_{1\le i\le n+2} \left|\lambda^{ba} b_i^{ba}(P_0)\right| \\
&= O_p(n^{-1/2})o(n^{1/2}) \\
&= o_p(1).
\end{aligned}
$$

From Eq. (2.2), we have,

$$
\begin{aligned}
0 &= \frac{1}{n+2}\sum_{i=1}^{n+2}\frac{b_i^{ba}(P_0)}{1+\lambda^{ba}b_i^{ba}(P_0)} \\
&= \frac{1}{n+2}\sum_{i=1}^{n+2} b_i^{ba}(P_0) - \lambda^{ba}\frac{1}{n+2}\sum_{i=1}^{n+2}(b_i^{ba}(P_0))^2 + \frac{1}{n+2}\sum_{i=1}^{n+2} b_i^{ba}(P_0)\frac{(\lambda^{ba})^2(b_i^{ba}(P_0))^2}{1+\lambda^{ba}b_i^{ba}(P_0)},
\end{aligned}
$$

where the last term is bounded by $o_p(n^{-1/2})$. Therefore, we can write that

$$
\begin{aligned}
\lambda^{ba} &= \frac{1}{n+2}\sum_{i=1}^{n+2} b_i^{ba}(P_0) \Big/ \left\{\frac{1}{n+2}\sum_{i=1}^{n+2}(b_i^{ba}(P_0))^2\right\} + o_p(n^{-1/2}) \\
&= \bar{b}_n(P_0)\hat{\sigma}^{-2}_{n,jack} + o_p(n^{-1/2}) \\
&= \widehat{T}_{n,jack}(P_0)\hat{\sigma}^{-2}_{n,jack} + o_p(n^{-1/2}).
\end{aligned}
$$

From Lemmas A.1 and A.2, the balanced adjusted JEL ratio can be shown as follows:

$$
\begin{aligned}
-2\log R^{ba}(P_0) &= 2\sum_{i=1}^{n+2}\left\{\lambda^{ba}b_i^{ba}(P_0) - \frac{(\lambda^{ba})^2(b_i^{ba}(P_0))^2}{2}\right\} + o_p(1) \\
&= 2(n+2)\lambda^{ba}\bar{b}_n(P_0) - (n+2)(\lambda^{ba})^2\frac{1}{n+2}\sum_{i=1}^{n+2}\{b_i^{ba}(P_0)\}^2 + o_p(1) \\
&= 2(n+2)\bar{b}_n(P_0)\widehat{T}_{n,jack}(P_0)\hat{\sigma}^{-2}_{n,jack} - (n+2)(\lambda^{ba})^2\frac{1}{n+2}\sum_{i=1}^{n+2}\{b_i^{ba}(P_0)\}^2 + o_p(1) \\
&= \frac{n+2}{n} n\{\widehat{T}_{n,jack}(P_0)\}^2\hat{\sigma}^{-2}_{n,jack} + o_p(1) \\
&\xrightarrow{\mathcal{D}} \chi_1^2. \quad \square
\end{aligned}
$$

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