Chapter 2. One sample problem: Sign test and Wilcoxon's signed rank test

1. Review of one-sample t-test for parametric models

Example: The average score for year 2007 students in a math exam is 10. 25 year 2008 students are randomly selected to take the same exam and their mean score is 9.328 with standard deviation 1.

Is the math ability of year 2008 students same as year 2007 students or worse?

Solution:

1) Assumption: $X_1, X_2, \cdots, X_n \sim i.i.d. N(\mu, \sigma^2)$, where

$X_i --$ the score of the i-th selected students.
n=25
$\mu-$the average of all year 2008 sdudents
$N(\mu, \sigma^2) --$ the assumed distribution of the exam scores of all year 2008 sdudents

2)Estimation

μ is estimated naturely be $\bar{X} = 9.328$, the sample mean.

3)Hypothesis

$$\begin{cases} H_0 : \mu = 10 \\ H_a : \mu < 10. \end{cases}$$

4)Test statistic based on the t-method:

$$T = \frac{\sqrt{n}(\bar{X} - 10)}{\hat{\sigma}},$$

$$t_{obs} = \frac{\sqrt{25}(9.328 - 10)}{1} = -3.36,$$

where $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$
t_{obs} is the observed T statistics

5)Distribution of the T-statistic under H_0:

$$T \sim t_{n-1}$$

6) p-value:

$$P(T \le t_{obs}|H_0) \approx 0.01$$

7)Decision (reject/accept H_0) at significance level α:

$$\text{Reject} H_0 \text{when the p-value} \le \alpha$$

Paired sample t-test:

1)Assumption:

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \cdots, \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \sim \text{i.i.d.}$$

$$N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \right)$$

For example,

X_i − −blood pressure of patient before treatment
Y_i − −blood pressure of patient after treatment.

2)Hypothesis:

$$\begin{cases} H_0 : \mu_x = \mu_y & \text{ineffective treatment} \\ H_a : \mu_x > \mu_y & \text{effective treatment} \end{cases}$$

$X_1 - Y_1, X_2 - Y_2, \cdots, X_n - Y_n \sim i.i.d.$
$N(\mu_x - \mu_y, \sigma^2)$ and $\sigma^2 = \sigma_{xx} + \sigma_{yy} - \sigma_{xy}$. This is a one-sample t-test problem now .

2. Sign test

1)Assumption:

$X_1, X_2, \cdots, X_n$ is iid $\sim F$ a continuous distribution with median m.

In general, median m is the value s.t.

$$P(X \leq m) \geq 1/2, \quad P(X \geq m) \geq 1/2$$

2)Hypothesis:

$$\begin{cases} H_0 : m = m_0 \\ H_a : m > m_0 \end{cases}$$

3)Test statistic:

$$S \quad = \sum_{i=1}^{n} I(X_i > m_0)$$

where

$$I(X_i > m_0) = \begin{cases} 1 & \text{if} \quad X_i > m_0 \\ 0 & \text{if} \quad X_i \leq m_0 \end{cases}$$

As we know:

$$\text{large value of S} \Longleftrightarrow \text{evidence to reject } H_0$$

4)The distribution of S under H_0:

$$S \sim Bin(n, 1/2),$$

$$P(S = k) = \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{1}{2}\right)^n$$

5)p-value:

$$
\begin{aligned}
P(S \geq s_{obs}|H_0) &= P(Bin(n, 1/2) \geq s_{obs}) \\
&= \sum_{k=s_{obs}}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{1}{2}\right)^n
\end{aligned}
$$

Example 2.1 High Octane rating means greater ability of a fuel to resist knocking when ignited. A random sample of 15 measurements is taken from certain gasoline product:

99.0	99.4	99.8	98.8	99.7
96.2	99.1	99.6	98.3	99.9
98.0	98.9	98.1	97.4	98.2

we consider the safty standard to be strictly above 98. Is there significant evidence in data indicating the gasoline is safe?

Example 2.2 Suppose the median IQ for general public is 125. Randomly select 15 drug abusers and their IQ scores are:

99	100	90	94	135
108	107	111	119	104
127	109	117	105	125

Does this data set contain significant evidence indicating drug abusers generally have lower IQ than normal people?

The normal approximation with continuity correction

$$S \sim Bin(n, p), \quad E(S) = np,$$

$$Var(S) = np(1-p)$$

$$P(S = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k},$$

where $k = 0, 1, 2, \cdots, n$.

$$\frac{S - np}{\sqrt{np(1-p)}} \longrightarrow N(0, 1), \quad \text{as} \quad n \to \infty$$

With continuity correction:

$$P(S \leq k) \approx \Phi\left(\frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

$$
\begin{aligned}
P(S \geq s_{obs}|H_0) &= P(Bin(n, 1/2) \geq s_{obs}) \\
&\approx 1 - \Phi(\frac{s_{obs} - 1/2 - n/2}{\sqrt{n}/2})
\end{aligned}
$$

Sign test for other hypotheses

$$H_0 : m = m_0$$
$$H_a : m < m_0$$

$$P(S \leq s_{obs}|H_0) = P(Bin(n, 1/2) \leq s_{obs})$$
$$\approx \Phi(\frac{s_{obs} + 1/2 - n/2}{\sqrt{n}/2}) \quad \text{for large n}$$

$$H_0 : m = m_0$$
$$H_a : m \neq m_0$$

$$\begin{cases} 2P(S \geq s_{obs}|H_0), & \text{if} \quad s_{obs} \geq n/2 \\ 2P(S \leq s_{obs}|H_0), & \text{if} \quad s_{obs} \leq n/2 \end{cases}$$