Title of My Thesis

SUBMITTED BY

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Fakultät für Physik
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Faculty of Physics
Quantum Many-Body Systems/ Theoretical Nanophysics Group
Ludwig Maximilian University
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NOTATION AND SYMBOLS

λ	flow parameter; in the literature sometimes also denoted by B
$\hat{\cdot}$	denotes that $\cdot$ is an operator which does not commute with every other operator
1	indicates $1 \in \mathbb{N}$ or the identity operator $\hat{\mathbb{1}} =: \mathbb{1}$
$:\hat{A}:$	normal ordering of operator $\hat{A}$
$\hat{a}_k^\dagger$	k^{th} bosonic creation operator
$\hat{a}_k$	k^{th} bosonic annihilation operator
$[\hat{A}, \hat{B}]$	commutator of operators $\hat{A}, \hat{B}$
$\hat{A}^\dagger$	adjoint of an operator $\hat{A}$
z^*	complex conjugate of $z \in \mathbb{C}$
$\delta_{\alpha,\beta}$	Kronecker-Delta of α, β
∂_x	partial derivative $\frac{\partial}{\partial x}$ w.r.t. x
$\overset{②}{=}$	Equality up to second order, i.e. higher order terms are neglected.

ABSTRACT

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A.1 Deriving the flow equations in the case of no n-dependence

First the canonical generator $\hat{\eta}$ has to be evaluated:

$$\hat{\eta} := \hat{\eta}(\lambda) := \left[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}\right] = \left[\sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \sum_{q\neq q'} V_{q,q'} \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{p,p'}^* \hat{a}_p \hat{a}_{p'}\right)\right] \tag{A.1}$$

$$\begin{aligned}
&= \sum_k \sum_{q,q'} \omega_k V_{q,q'} \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}\right] + \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger\right] + \omega_k W_{p,p'}^* \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p \hat{a}_{p'}\right]\right) \\
&= \sum_k \sum_{q,q'} \omega_k V_{q,q'} \left(\hat{a}_k^\dagger \hat{a}_{q'} \delta_{k,q} - \hat{a}_q^\dagger \hat{a}_k \delta_{k,q'}\right) \\
&+ \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} \left(\hat{a}_k^\dagger \hat{a}_p^\dagger \delta_{k,p'} + \hat{a}_k^\dagger \hat{a}_{p'}^\dagger \delta_{k,p}\right) - \omega_k W_{p,p'}^* \left(\hat{a}_p \hat{a}_k \delta_{k,p'} + \hat{a}_{p'} \hat{a}_k \delta_{k,p}\right)\right) \\
&= \sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right)
\end{aligned} \tag{A.2}$$

Since $\hat{\eta}$ has the same form as $\hat{\mathcal{H}}_{\text{int}}$, $\left[\hat{\eta}, \hat{\mathcal{H}}_0\right]$ follows by inspection of A.2:

$$\begin{aligned}
\left[\hat{\eta}, \hat{\mathcal{H}}_0\right] &= -\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'})^2 \hat{a}_q^\dagger \hat{a}_{q'} \\
&- \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'})^2 \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{p,p'}^* (\omega_p + \omega_{p'})^2 \hat{a}_p \hat{a}_{p'}\right)
\end{aligned} \tag{A.3}$$

The commutator of the generator and $\hat{\mathcal{H}}_{\text{int}}$ needs more work:

$$\left[\hat{\eta}, \hat{\mathcal{H}}_{\text{int}}\right] = \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger\right.\right. \tag{A.4}$$

$$\left.\left. - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} + \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right]$$

$$= \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}\right] \tag{A.5}$$

$$+ \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right] \tag{A.6}$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}\right] \tag{A.7}$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right] \tag{A.8}$$

In the following, A.5-A.8 will be evaluated separately:

A.5:

$$
\begin{aligned}
&\left[\sum_{q\neq q'} V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}, \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right] \\
&=\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right] \\
&=\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\left(\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}\delta_{q',\tilde{q}}-\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'}\delta_{q,\tilde{q}'}\right) \\
&=\sum_{q\neq q'}\sum_{\tilde{q}'} V_{q',\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'} \\
&=\sum_{q,q'}\sum_{\tilde{q}'} V_{q',\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}-\sum_{q,q'}\sum_{\tilde{q}} V_{\tilde{q},q}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'} \\
&=\sum_{q,q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q,q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'} \\
&=\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'} \\
&+\sum_{k}\sum_{\tilde{q}} V_{\tilde{q},k}V_{k,\tilde{q}}(\omega_k-\omega_{\tilde{q}})\hat{a}_k^\dagger\hat{a}_k-\sum_{k}\sum_{\tilde{q}} V_{k,\tilde{q}}V_{\tilde{q},k}(\omega_{\tilde{q}}-\omega_k)\hat{a}_k^\dagger\hat{a}_k \\
&=\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'} \\
&+\sum_{k}\sum_{\tilde{q}} 2V_{\tilde{q},k}V_{k,\tilde{q}}(\omega_k-\omega_{\tilde{q}})\hat{a}_k^\dagger\hat{a}_k
\end{aligned}
\tag{A.9}
$$

A.6:

$$
\begin{aligned}
&\left[\sum_{q\neq q'} V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'}\left(W_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger+W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right)\right] \\
&=\sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'} V_{q,q'}(\omega_q-\omega_{q'})\left(W_{\tilde{p},\tilde{p}'}\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger\right]+W_{\tilde{p},\tilde{p}'}^*\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right]\right) \\
&=\sum_{q,q'}\sum_{\tilde{p},\tilde{p}'} V_{q,q'}(\omega_q-\omega_{q'})\left(W_{\tilde{p},\tilde{p}'}\left(\hat{a}_q^\dagger\hat{a}_{\tilde{p}}^\dagger\delta_{q',\tilde{p}'}+\hat{a}_q^\dagger\hat{a}_{\tilde{p}'}^\dagger\delta_{q',\tilde{p}}\right)-W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\left(\hat{a}_{\tilde{p}'}\hat{a}_{q'}\delta_{q,\tilde{p}}+\hat{a}_{\tilde{p}}\hat{a}_{q'}\delta_{\tilde{p}',q}\right)\right) \\
&=\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,p'}\hat{a}_q^\dagger\hat{a}_p^\dagger+\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{p,p'}\hat{a}_q^\dagger\hat{a}_{p'}^\dagger \\
&-\sum_{p,p'}\sum_{q'} V_{p,q'}(\omega_p-\omega_{q'})W_{p,p'}^*\hat{a}_{p'}\hat{a}_{q'}-\sum_{p,p'}\sum_{q'} V_{p',q'}(\omega_{p'}-\omega_{q'})W_{p,p'}^*\hat{a}_p\hat{a}_{q'} \\
&=\sum_{p,p'}\sum_{q} V_{p',q}(\omega_{p'}-\omega_q)W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger+\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)W_{q,p'}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\
&-\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{q,p'}^*\hat{a}_p\hat{a}_{p'}-\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,q}^*\hat{a}_p\hat{a}_{p'} \\
&=\sum_{p,p'}\sum_{q} V_{p',q}(\omega_{p'}-\omega_q)W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger+\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)W_{q,p'}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\
&-\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{q,p'}^*\hat{a}_p\hat{a}_{p'}-\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,q}^*\hat{a}_p\hat{a}_{p'} \\
&=\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)(W_{q,p'}+W_{p',q})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\
&+\sum_{p,p'}\sum_{q} V_{q,p}(\omega_p-\omega_q)(W_{q,p'}^*+W_{p',q}^*)\hat{a}_p\hat{a}_{p'}
\end{aligned}
\tag{A.10}
$$

A.7:

$$
\begin{aligned}
&\left[\sum_{p,p'}\left(W_{p,p'}(\omega_p+\omega_{p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - W_{p,p'}^*(\omega_p+\omega_{p'})\hat{a}_p\hat{a}_{p'}\right), \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right] \\
&= \sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}(\omega_p+\omega_{p'})\left(W_{p,p'}\left[\hat{a}_p^\dagger\hat{a}_{p'}^\dagger, \hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right] - W_{p,p'}^*\left[\hat{a}_p\hat{a}_{p'}, \hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right]\right) \\
&= -\sum_{p,p'}\sum_{q\neq q'} V_{q,q'}(\omega_p+\omega_{p'})W_{p,p'}\left(\hat{a}_q^\dagger\hat{a}_p^\dagger\delta_{q',p'} + \hat{a}_q^\dagger\hat{a}_{p'}^\dagger\delta_{q',p}\right) \\
&- \sum_{p,p'}\sum_{q\neq q'} V_{q,q'}(\omega_p+\omega_{p'})W_{p,p'}^*\left(\hat{a}_p\hat{a}_{q'}\delta_{q,p'} + \hat{a}_{p'}\hat{a}_{q'}\delta_{q,p}\right) \\
&= -\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_p+\omega_{p'})W_{p,p'}\hat{a}_q^\dagger\hat{a}_p^\dagger - \sum_{p,p'}\sum_{q} V_{q,p}(\omega_p+\omega_{p'})W_{p,p'}\hat{a}_q^\dagger\hat{a}_{p'}^\dagger \\
&- \sum_{p,p'}\sum_{q'} V_{p',q'}(\omega_p+\omega_{p'})W_{p,p'}^*\hat{a}_p\hat{a}_{q'} - \sum_{p,p'}\sum_{q'} V_{p,q'}(\omega_p+\omega_{p'})W_{p,p'}^*\hat{a}_{p'}\hat{a}_{q'} \\
&= -\sum_{p,p'}\sum_{q} V_{p',q}(\omega_p+\omega_{q})W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - \sum_{p,p'}\sum_{q} V_{p,q}(\omega_q+\omega_{p'})W_{q,p'}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\
&- \sum_{p,p'}\sum_{q'} V_{q',p'}(\omega_p+\omega_{q'})W_{p,q'}^*\hat{a}_p\hat{a}_{p'} - \sum_{p,p'}\sum_{q'} V_{q',p}(\omega_{q'}+\omega_{p'})W_{q',p'}^*\hat{a}_p\hat{a}_{p'} \\
&= -\sum_{p,p'}\sum_{q} V_{p,q}(\omega_q+\omega_{p'})(W_{p',q}+W_{q,p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\
&- \sum_{p,p'}\sum_{q} V_{q,p}(\omega_q+\omega_{p'})(W_{p',q}^*+W_{q,p'}^*)\hat{a}_p\hat{a}_{p'}
\end{aligned}
\tag{A.11}
$$

A.8:

$$
\begin{aligned}
&\left[\sum_{p,p'}\left(W_{p,p'}(\omega_p+\omega_{p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - W_{p,p'}^*(\omega_p+\omega_{p'})\hat{a}_p\hat{a}_{p'}\right), \sum_{\tilde{p},\tilde{p}'}\left(W_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right)\right] \\
&= \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'})W_{\tilde{p},\tilde{p}'}^*\left[\hat{a}_p^\dagger\hat{a}_{p'}^\dagger, \hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right] - \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}^* W_{\tilde{p},\tilde{p}'}(\omega_p+\omega_{p'})\left[\hat{a}_p\hat{a}_{p'}, \hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger\right] \\
&= -\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W_{\tilde{p},\tilde{p}'}^*\left[\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger\hat{a}_{p'}^\dagger\right] \\
&= -\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_p^\dagger\delta_{\tilde{p}',p'} \\
&- \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_{p'}^\dagger\delta_{\tilde{p}',p} \\
&- \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W_{\tilde{p},\tilde{p}'}^*\hat{a}_p^\dagger\hat{a}_{\tilde{p}'}\delta_{\tilde{p},p'} \\
&- \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W_{\tilde{p},\tilde{p}'}^*\hat{a}_{p'}^\dagger\hat{a}_{\tilde{p}'}\delta_{\tilde{p},p} \\
&= -\sum_{p,p'}\sum_{\tilde{p}} W_{p,p'}(\omega_p+2\omega_{p'}+\omega_{\tilde{p}})W_{\tilde{p},p'}^*\hat{a}_{\tilde{p}}\hat{a}_p^\dagger - \sum_{p,p'}\sum_{\tilde{p}} W_{p,p'}(2\omega_p+\omega_{p'}+\omega_{\tilde{p}})W_{\tilde{p},p}^*\hat{a}_{\tilde{p}}\hat{a}_{p'}^\dagger \\
&- \sum_{p,p'}\sum_{\tilde{p}'} W_{p,p'}(\omega_p+2\omega_{p'}+\omega_{\tilde{p}'})W_{p',\tilde{p}'}^*\hat{a}_p^\dagger\hat{a}_{\tilde{p}'} - \sum_{p,p'}\sum_{\tilde{p}'} W_{p,p'}(2\omega_p+\omega_{p'}+\omega_{\tilde{p}'})W_{p,\tilde{p}'}^*\hat{a}_{p'}^\dagger\hat{a}_{\tilde{p}'} \\
&= -\sum_{p,p'}\sum_{\tilde{p}} W_{p,\tilde{p}}(\omega_p+2\omega_{\tilde{p}}+\omega_{p'})W_{p',\tilde{p}}^*\hat{a}_{p'}\hat{a}_p^\dagger - \sum_{p,p'}\sum_{\tilde{p}} W_{\tilde{p},p}(2\omega_{\tilde{p}}+\omega_p+\omega_{p'})W_{p',\tilde{p}}^*\hat{a}_{p'}\hat{a}_p^\dagger \\
&- \sum_{p,p'}\sum_{\tilde{p}'} W_{p,\tilde{p}'}(\omega_p+2\omega_{\tilde{p}'}+\omega_{p'})W_{\tilde{p}',p'}^*\hat{a}_p^\dagger\hat{a}_{p'} - \sum_{p,p'}\sum_{\tilde{p}'} W_{\tilde{p}',p}(2\omega_{\tilde{p}'}+\omega_p+\omega_{p'})W_{\tilde{p}',p'}^*\hat{a}_p^\dagger\hat{a}_{p'} \\
&= -\sum_{p,p'}\sum_{\tilde{p}}(W_{p,\tilde{p}}+W_{\tilde{p},p})(\omega_p+2\omega_{\tilde{p}}+\omega_{p'})W_{p',\tilde{p}}^*\hat{a}_{p'}\hat{a}_p^\dagger
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{p,p'} \sum_{\tilde{p}'} (W_{p,\tilde{p}'} + W_{\tilde{p}',p})(\omega_p + 2\omega_{\tilde{p}'} + \omega_{p'}) W^*_{\tilde{p}',p'} \hat{a}^\dagger_p \hat{a}_{p'} \\
&= - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W^*_{p',\tilde{p}} (\delta_{p,p'} + \hat{a}^\dagger_p \hat{a}_{p'}) \\
&- \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W^*_{\tilde{p},p'} \hat{a}^\dagger_p \hat{a}_{p'} \\
&= - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) (W^*_{\tilde{p},p'} + W^*_{p',\tilde{p}}) \hat{a}^\dagger_p \hat{a}_{p'} \\
&- 2 \sum_{k} \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) W^*_{k,\tilde{p}} \\
&= - \sum_{q \neq q'} \sum_{\tilde{p}} (W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + 2\omega_{\tilde{p}} + \omega_{q'}) (W^*_{\tilde{p},q'} + W^*_{q',\tilde{p}}) \hat{a}^\dagger_q \hat{a}_{q'} \\
&- 2 \sum_{k} \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) (W^*_{\tilde{p},k} + W^*_{k,\tilde{p}}) \hat{a}^\dagger_k \hat{a}_k \\
&- 2 \sum_{k} \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) W^*_{k,\tilde{p}}
\end{aligned}
\tag{A.12}
$$

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$
\hat{\mathcal{H}}(\lambda) = \sum_k \omega_k(\lambda) \hat{a}^\dagger_k \hat{a}_k + \sum_{q \neq q'} V_{q,q'}(\lambda) \hat{a}^\dagger_q \hat{a}_{q'} + \sum_{p,p'} \Big(W_{p,p'}(\lambda) \hat{a}^\dagger_p \hat{a}^\dagger_{p'} + W^*_{p,p'}(\lambda) \hat{a}_p \hat{a}_{p'} \Big) + \epsilon(\lambda) \tag{A.13}
$$

where $\epsilon(\lambda)$ is a constant shift in the energy scale.
Using the expressions for the commutators of the generator and $\hat{\mathcal{H}}_0$ respectively $\hat{\mathcal{H}}_{\text{int}}$ derived above, the flow $\partial_\lambda \hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ yields the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$
\partial_\lambda \omega_k = \sum_{\tilde{q}} 2 V_{\tilde{q},k} V_{k,\tilde{q}} (\omega_k - \omega_{\tilde{q}}) - 2 \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) (W^*_{\tilde{p},k} + W^*_{k,\tilde{p}}) \tag{A.14a}
$$

$$
\begin{aligned}
\partial_\lambda V_{q,q'} &= - V_{q,q'} (\omega_q - \omega_{q'})^2 - \sum_{\tilde{p}} (W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + \omega_{q'} + 2\omega_{\tilde{p}}) (W^*_{\tilde{p},q'} + W^*_{q',\tilde{p}}) \\
&+ \sum_{\tilde{q}} V_{\tilde{q},q'} V_{q,\tilde{q}} (\omega_q + \omega_{q'} - 2\omega_{\tilde{q}})
\end{aligned}
\tag{A.14b}
$$

$$
\begin{aligned}
\partial_\lambda W_{p,p'} &= - W_{p,p'} (\omega_p + \omega_{p'})^2 - \sum_q V_{p,q} (\omega_q + \omega_{p'}) (W_{p',q} + W_{q,p'}) \\
&+ \sum_q V_{p,q} (\omega_p - \omega_q) (W_{q,p'} + W_{p',q})
\end{aligned}
\tag{A.14c}
$$

$$
\begin{aligned}
\partial_\lambda W^*_{p,p'} &= - W^*_{p,p'} (\omega_p + \omega_{p'})^2 - \sum_q V_{q,p} (\omega_q + \omega_{p'}) (W^*_{p',q} + W^*_{q,p'}) \\
&+ \sum_q V_{q,p} (\omega_p - \omega_q) (W^*_{q,p'} + W^*_{p',q})
\end{aligned}
\tag{A.14d}
$$

$$
\partial_\lambda \varepsilon = -2 \sum_{p,p'} (W_{p,p'} + W_{p',p})(\omega_p + \omega_{p'}) W^*_{p,p'} \tag{A.14e}
$$

Obviously, equations A.14c and A.14d are not independent from each other, since they are related by complex conjugation. Seeing this is a good consistency check because complex conjugation was not explicitly used in the derivation of the these two equations.
Furthermore note that the flow equations A.14a-A.14e are exact in the sense that if the flow is completely traversed, the flow Hamiltonian will be exactly diagonal.

A.2 Deriving the flow equations with n-dependence

A.2.1 Useful preliminaries

Consider some operator $\hat{f}$ which depends on a number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. The following relations will be used later:

$$\left[\hat{a}^\dagger, \hat{f}(\hat{n})\right] = \hat{a}^\dagger \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}+1)\right) \tag{A.15a}$$

$$\left[\hat{a}, \hat{f}(\hat{n})\right] = \hat{a} \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}-1)\right) \tag{A.15b}$$

$$\left[\hat{f}(\hat{n}), \hat{a}^\dagger\right] = \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}-1)\right) \hat{a}^\dagger \tag{A.15c}$$

$$\left[\hat{f}(\hat{n}), \hat{a}\right] = \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}+1)\right) \hat{a} \tag{A.15d}$$

These can be proved by induction for $\hat{f}(\hat{n}) = \hat{n}^k, k \in \mathbb{N}$ and from there simply extended to well-behaved $\hat{f}$ via power series. Equations A.15 are still valid for functions depending on $\{\hat{n}_k\}_k$, because all $\hat{n}_k$ pairwise commute.
We will write $\hat{f}\left(\hat{n}_1, \hat{n}_2, \ldots\right) =: \hat{f}$ and $\hat{f}\left(\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k \pm 1, \hat{n}_{k+1}, \ldots\right) =: \hat{f}(\hat{n}_k \pm 1)$. In this notation it is understood that $\hat{f}(\hat{n}_k \pm 1, \hat{n}_k \pm 1) =: \hat{f}(\hat{n}_k \pm 2)$.
Using this notation, it is evident that a simple induction for $n_1, n_2 \in \mathbb{N}_0$ yields the following relation:

$$\begin{aligned}&\left[\hat{f}(\hat{n}), \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} \cdots \hat{a}^\dagger_{k_{n_1}} \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}}\right] \\ &= \left(\hat{f} - \hat{f}\left(\hat{n}_{k_1} - 1, \hat{n}_{k_2} - 1, \ldots, \hat{n}_{k_{n_1}}, \hat{n}_{k_1} + 1, \hat{n}_{k_2} + 1 \ldots \hat{n}_{k_{n_2}} + 1\right)\right) \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} \cdots \hat{a}^\dagger_{k_{n_1}} \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}}\end{aligned} \tag{A.16}$$

Furthermore, applying the recurrence relation introduced to define the normal ordering procedure can be used to successively normal order operators. Let $\hat{O} := \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} \cdots \hat{a}^\dagger_{k_{n_1}} \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}}$. Then normal ordering w.r.t. the vacuum yields:

$$\begin{aligned}\hat{a}_q : \hat{O} : &=: \hat{O} \hat{a}_q : + \sum_k : \frac{\partial \hat{O}}{\partial \hat{a}^\dagger_k} : \\ &=: \hat{O} \hat{a}_q : + \sum_{i=1}^{n_1} \delta_{k_i, q} : \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} \cdots \hat{a}^\dagger_{k_{i-1}} \hat{a}^\dagger_{k_{i+1}} \cdots \hat{a}^\dagger_{k_{n_1}} \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}} :\end{aligned} \tag{A.17a}$$

$$\hat{a}^\dagger_q : \hat{O} : =: \hat{a}^\dagger_q \hat{O} : \tag{A.17b}$$

A.2.2 The canonical generator

The first step in the calculating the flow equations is again to calculate the canonical commutator $\hat{\eta} := \left[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}\right]$:

$$\hat{\eta} = \left[\sum_k \hat{\omega}_k : \hat{a}^\dagger_k \hat{a}_k :, \sum_{q \neq q'} \hat{V}_{q,q'} : \hat{a}^\dagger_q \hat{a}_{q'} : + \sum_{p,p'} \left(\hat{W}_{p,p'} : \hat{a}^\dagger_p \hat{a}^\dagger_{p'} : + \hat{W}^\dagger_{p,p'} : \hat{a}_p \hat{a}_{p'} :\right)\right]$$

$$= \sum_k \sum_{q \neq q'} \left[\hat{\omega}_k : \hat{a}^\dagger_k \hat{a}_k :, \hat{V}_{q,q'} : \hat{a}^\dagger_q \hat{a}_{q'} :\right] \tag{A.18a}$$

$$+ \sum_k \sum_{p,p'} \left[\hat{\omega}_k : \hat{a}^\dagger_k \hat{a}_k :, \hat{W}_{p,p'} : \hat{a}^\dagger_p \hat{a}^\dagger_{p'} :\right] \tag{A.18b}$$

$$+\sum_{k}\sum_{p,p'}\left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \tag{A.18c}$$

In the following, the terms A.18a-A.18c will be evaluated separately:

A.18a

$$\begin{aligned}
&\left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \\
&= \hat{\omega}_k \hat{V}_{q,q'} \left[: \hat{a}_k^\dagger \hat{a}_k :, : \hat{a}_q^\dagger \hat{a}_{q'} :\right] + \hat{\omega}_k \underbrace{\left[: \hat{a}_k^\dagger \hat{a}_k :, \hat{V}_{q,q'}\right]}_{=0} : \hat{a}_q^\dagger \hat{a}_{q'} : \\
&+ \hat{V}_{q,q'} \left[\hat{\omega}_k, : \hat{a}_q^\dagger \hat{a}_{q'} :\right] : \hat{a}_k^\dagger \hat{a}_k : + \underbrace{\left[\hat{\omega}_k, \hat{V}_{q,q'}\right]}_{=0} : \hat{a}_q^\dagger \hat{a}_{q'} :: \hat{a}_k^\dagger \hat{a}_k : \\
&= \hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'}\right) + \hat{V}_{q,q'} \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}_q^\dagger \hat{a}_{q'} :: \hat{a}_k^\dagger \hat{a}_k : \\
&= \hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'}\right) \\
&+ \hat{V}_{q,q'} \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) \left(: \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : + \delta_{q',k} : \hat{a}_q^\dagger \hat{a}_k :\right)
\end{aligned} \tag{A.19}$$

A.18b

$$\begin{aligned}
&\left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \\
&= \hat{W}_{p,p'} \left[\hat{\omega}_k, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] : \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left[: \hat{a}_k^\dagger \hat{a}_k :, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \\
&= \hat{W}_{p,p'} \left(\omega_k - \hat{\omega}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :: \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'}^\dagger : \delta_{k,p}\right) \\
&= \hat{W}_{p,p'} \left(\omega_k - \hat{\omega}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'}^\dagger : \delta_{k,p}\right)
\end{aligned} \tag{A.20}$$

A.18c

$$\begin{aligned}
&\left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \\
&= \hat{W}_{p,p'}^\dagger \left[\hat{\omega}_k, : \hat{a}_p \hat{a}_{p'} :\right] : \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left[: \hat{a}_k^\dagger \hat{a}_k :, : \hat{a}_p \hat{a}_{p'} :\right] \\
&= \hat{W}_{p,p'}^\dagger \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}_p \hat{a}_{p'} :: \hat{a}_k^\dagger \hat{a}_k : - \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left(: \hat{a}_p \hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'} \hat{a}_k : \delta_{k,p}\right) \\
&= \hat{W}_{p,p'}^\dagger \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) \left(: \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : + \delta_{k,p} : \hat{a}_{p'} \hat{a}_k : + \delta_{k,p'} : \hat{a}_p \hat{a}_k :\right) \\
&- \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left(: \hat{a}_p \hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'} \hat{a}_k : \delta_{k,p}\right)
\end{aligned} \tag{A.21}$$

This gives the canonical generator as:

$$\begin{aligned}
\hat{\eta} = &\sum_{k}\sum_{q\neq q'} \Bigg(\hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'}\right) \\
&+ \hat{V}_{q,q'} \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) \left(: \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : + \delta_{q',k} : \hat{a}_q^\dagger \hat{a}_k :\right)\Bigg) \\
&+ \sum_{k}\sum_{p,p'} \left(\hat{W}_{p,p'} \left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'}^\dagger : \delta_{k,p}\right)\right) \\
&+ \sum_{k}\sum_{p,p'} \Bigg(\hat{W}_{p,p'}^\dagger \left(\omega_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) \left(: \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : + \delta_{k,p} : \hat{a}_{p'} \hat{a}_k : + \delta_{k,p'} : \hat{a}_p \hat{a}_k :\right)
\end{aligned}$$

$$
\begin{aligned}
&- \hat{W}^\dagger_{p,p'}\hat{\omega}_k \left(: \hat{a}_p\hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'}\hat{a}_k : \delta_{k,p}\right)\Bigg) \\
&= \sum_{q\neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1))\hat{V}_{q,q'} : \hat{a}^\dagger_q\hat{a}_{q'} : \\
&+ \sum_{q\neq q'}\sum_k \hat{V}_{q,q'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}^\dagger_q\hat{a}^\dagger_k\hat{a}_{q'}\hat{a}_k : \\
&+ \sum_k\sum_{p,p'} \hat{W}_{p,p'}\left(\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}^\dagger_p\hat{a}^\dagger_{p'}\hat{a}^\dagger_k\hat{a}_k : \\
&+ \sum_{p,p'} \hat{W}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) : \hat{a}^\dagger_{p'}\hat{a}^\dagger_p : \\
&+ \sum_k\sum_{p,p'} \hat{W}^\dagger_{p,p'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}^\dagger_k\hat{a}_k\hat{a}_p\hat{a}_{p'} : \\
&- \sum_{p,p'} \hat{W}^\dagger_{p,p'}(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p\hat{a}_{p'} : \\
&=: \sum_{q\neq q'} \hat{\theta}_{q,q'} : \hat{a}^\dagger_q\hat{a}_{q'} : + \sum_{p,p'}\left(\hat{\phi}_{p,p'} : \hat{a}^\dagger_p\hat{a}^\dagger_{p'} : + \hat{\psi}_{p,p'} : \hat{a}_p\hat{a}_{p'} :\right) \qquad \text{(A.22)} \\
&+ \sum_{q\neq q'}\sum_k \hat{V}_{q,q'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}^\dagger_q\hat{a}^\dagger_k\hat{a}_{q'}\hat{a}_k : \\
&+ \sum_k\sum_{p,p'} \hat{W}_{p,p'}\left(\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}^\dagger_p\hat{a}^\dagger_{p'}\hat{a}^\dagger_k\hat{a}_k : \\
&+ \sum_k\sum_{p,p'} \hat{W}^\dagger_{p,p'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}^\dagger_k\hat{a}_k\hat{a}_p\hat{a}_{p'} : \\
&=: \eta^{(2)} + \eta^{(4)} \qquad \text{(A.23)}
\end{aligned}
$$

Here $\eta^{(2)}$ is the part of $\hat{\eta}$ which contains only quadratic terms and $\eta^{(4)}$ with quartic terms. The latter will be neglected for now, i.e. we assume $\hat{\eta} \approx \hat{\eta}^{(2)}$ for which we write $\hat{\eta} \overset{②}{=} \hat{\eta}^{(2)}$ because equality holds up to second order. The accuracy of this approximation will vary from model to model and must always be justified in a specific situation.

A.2.3 Evaluating the commutator of the generator with the Hamiltonian

If one notices that $\eta^{(2)}$ is structurally identical to $\hat{\mathcal{H}}_{\text{int}}$, the commutator of $\hat{\mathcal{H}}_0$ and $\eta^{(2)}$ can be written down immediately:

$$
\begin{aligned}
\left[\eta^{(2)}, \hat{\mathcal{H}}_0\right] = &- \sum_{q\neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1))\hat{\theta}_{q,q'} : \hat{a}^\dagger_q\hat{a}_{q'} : \\
&- \sum_{p,p'} \hat{\phi}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) : \hat{a}^\dagger_{p'}\hat{a}^\dagger_p : \\
&+ \sum_{p,p'} \hat{\psi}_{p,p'}(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p\hat{a}_{p'} : \\
&- \sum_k\sum_{p,p'} \hat{\psi}_{p,p'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}^\dagger_k\hat{a}_k\hat{a}_p\hat{a}_{p'} : \\
&- \sum_{q\neq q'}\sum_k \hat{\theta}_{q,q'}\left(\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}^\dagger_q\hat{a}^\dagger_k\hat{a}_{q'}\hat{a}_k : \\
&- \sum_k\sum_{p,p'} \hat{\phi}_{p,p'}\left(\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}^\dagger_p\hat{a}^\dagger_{p'}\hat{a}^\dagger_k\hat{a}_k :
\end{aligned}
$$

$$
\begin{aligned}
&\overset{②}{=} -\sum_{q\neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1))\hat{\theta}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{q'}: \\
&- \sum_{p,p'} \hat{\phi}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) :\hat{a}_{p'}^\dagger \hat{a}_p^\dagger: \\
&+ \sum_{p,p'} \hat{\psi}_{p,p'}(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) :\hat{a}_p \hat{a}_{p'}:
\end{aligned} \tag{A.24}
$$

The commutator of $\hat{\mathcal{H}}_{\text{int}}$ and $\eta^{(2)}$ requires significantly more work:

$$
\begin{aligned}
\left[\eta^{(2)}, \hat{\mathcal{H}}_0\right] = &\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, \hat{V}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{q'}:\right] && \text{(A.25a)} \\
&+ \sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, \hat{W}_{p,p'} :\hat{a}_p^\dagger \hat{a}_{p'}^\dagger:\right] && \text{(A.25b)} \\
&+ \sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, \hat{W}_{p,p'}^\dagger :\hat{a}_p \hat{a}_{p'}:\right] && \text{(A.25c)} \\
&+ \sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger:, \hat{V}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{q'}:\right] && \text{(A.25d)} \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger:, \hat{W}_{p,p'} :\hat{a}_p^\dagger \hat{a}_{p'}^\dagger:\right] && \text{(A.25e)} \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger:, \hat{W}_{p,p'}^\dagger :\hat{a}_p \hat{a}_{p'}:\right] && \text{(A.25f)} \\
&+ \sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}:, \hat{V}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{q'}:\right] && \text{(A.25g)} \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}:, \hat{W}_{p,p'} :\hat{a}_p^\dagger \hat{a}_{p'}^\dagger:\right] && \text{(A.25h)} \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} :\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}:, \hat{W}_{p,p'}^\dagger :\hat{a}_p \hat{a}_{p'}:\right] && \text{(A.25i)}
\end{aligned}
$$

For the sake of clarity, the terms A.25a-A.25i will again be evaluated one by one.

A.25a:

$$
\begin{aligned}
&\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, \hat{V}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{q'}:\right] \\
&= \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'}\hat{V}_{q,q'} \left[:\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, :\hat{a}_q^\dagger \hat{a}_{q'}:\right] \\
&+ \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{V}_{q,q'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'}, :\hat{a}_q^\dagger \hat{a}_{q'}:\right] :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}: \\
&+ \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[:\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:, \hat{V}_{q,q'}\right] :\hat{a}_q^\dagger \hat{a}_{q'}: \\
&= \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'}\hat{V}_{q,q'} \left(\delta_{\tilde{q}',q} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_q: - \delta_{\tilde{q},q'} :\hat{a}_q^\dagger \hat{a}_{\tilde{q}'}:\right) \\
&+ \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{V}_{q,q'} \left(\hat{\theta}_{\tilde{q},\tilde{q}'} - \hat{\theta}_{\tilde{q},\tilde{q}'}(\hat{n}_{q'}+1, \hat{n}_q - 1)\right) \underbrace{:\hat{a}_q^\dagger \hat{a}_{q'}::\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}:}_{\overset{②}{=}\delta_{q',\tilde{q}}:\hat{a}_q^\dagger \hat{a}_{\tilde{q}'}:} \\
&- \sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, :\hat{a}_q^\dagger \hat{a}_{q'}:\right] :\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}: \\
&\overset{②}{=} \sum_{q\neq q'}\sum_{\tilde{q}} \left(\hat{\theta}_{\tilde{q},q}\hat{V}_{q,q'} :\hat{a}_{\tilde{q}}^\dagger \hat{a}_q: - \hat{\theta}_{q',\tilde{q}}\hat{V}_{q,q'} :\hat{a}_q^\dagger \hat{a}_{\tilde{q}}:\right)
\end{aligned} \tag{A.26}
$$

$$
\begin{aligned}
&+ \sum_{q\neq q'} \sum_{\tilde q} \hat V_{q,q'} \left(\hat\theta_{q',\tilde q} - \hat\theta_{q',\tilde q}(\hat n_{q'}+1, \hat n_q - 1)\right) : \hat a^\dagger_q \hat a_{\tilde q} : \\
&- \sum_{q\neq q'} \sum_{\tilde q\neq \tilde q'} \hat\theta_{q,q'} \left[\hat V_{\tilde q,\tilde q'}, : \hat a^\dagger_q \hat a_{q'} :\right] : \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} : \\
&= \sum_{q,q'} \sum_{\tilde q} \left(\hat\theta_{q,q'} \hat V_{q',\tilde q} : \hat a^\dagger_q \hat a_{q'} : - \hat\theta_{\tilde q,q'} \hat V_{q,\tilde q} : \hat a^\dagger_q \hat a_{q'} :\right) && \text{(A.27)} \\
&\underbrace{- \sum_{k} \sum_{\tilde q} \left(\hat\theta_{\tilde q,k} \hat V_{k,k} : \hat a^\dagger_{\tilde q} \hat a_k : - \hat\theta_{k,\tilde q} \hat V_{k,k} : \hat a^\dagger_k \hat a_{\tilde q} :\right)}_{=0} \\
&+ \sum_{q,q'} \sum_{\tilde q} \hat V_{q,\tilde q} \left(\hat\theta_{\tilde q,q'} - \hat\theta_{\tilde q,q'}(\hat n_{\tilde q}+1, \hat n_q - 1)\right) : \hat a^\dagger_q \hat a_{q'} : \\
&\underbrace{- \sum_{k} \sum_{\tilde q} \hat V_{k,k} \left(\hat\theta_{k,\tilde q} - \hat\theta_{k,\tilde q}(\hat n_k+1, \hat n_k - 1)\right) : \hat a^\dagger_k \hat a_{\tilde q} :}_{=0} \\
&- \sum_{q\neq q'} \sum_{\tilde q\neq \tilde q'} \hat\theta_{q,q'} \left[\hat V_{\tilde q,\tilde q'}, : \hat a^\dagger_q \hat a_{q'} :\right] : \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} : \\
&= \sum_{q\neq q'} \sum_{\tilde q} \left(\hat\theta_{q,q'} \hat V_{q',\tilde q} : \hat a^\dagger_q \hat a_{q'} : - \hat\theta_{\tilde q,q'} \hat V_{q,\tilde q} : \hat a^\dagger_q \hat a_{q'} :\right) && \text{(A.28)} \\
&+ \sum_{k} \sum_{\tilde q} \left(\hat\theta_{k,k} \hat V_{k,\tilde q} : \hat a^\dagger_k \hat a_k : - \hat\theta_{\tilde q,k} \hat V_{k,\tilde q} : \hat a^\dagger_k \hat a_k :\right) \\
&+ \sum_{q\neq q'} \sum_{\tilde q} \hat V_{q,\tilde q} \left(\hat\theta_{\tilde q,q'} - \hat\theta_{\tilde q,q'}(\hat n_{\tilde q}+1, \hat n_q - 1)\right) : \hat a^\dagger_q \hat a_{q'} : \\
&+ \sum_{k} \sum_{\tilde q} \hat V_{k,\tilde q} \left(\hat\theta_{\tilde q,k} - \hat\theta_{\tilde q,k}(\hat n_{\tilde q}+1, \hat n_k - 1)\right) : \hat a^\dagger_k \hat a_k : \\
&- \sum_{q\neq q'} \sum_{\tilde q\neq \tilde q'} \hat\theta_{q,q'} \left[\hat V_{\tilde q,\tilde q'}, : \hat a^\dagger_q \hat a_{q'} :\right] : \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} : \\
&= \sum_{q\neq q'} \sum_{\tilde q} \left(\hat\theta_{q,q'} \hat V_{q',\tilde q} - \hat\theta_{\tilde q,q'} \hat V_{q,\tilde q}\right) : \hat a^\dagger_q \hat a_{q'} : && \text{(A.29)} \\
&+ \sum_{k} \sum_{\tilde q} \left(\hat\theta_{k,k} \hat V_{k,\tilde q} - \hat\theta_{\tilde q,k} \hat V_{k,\tilde q}\right) : \hat a^\dagger_k \hat a_k : \\
&+ \sum_{q\neq q'} \sum_{\tilde q} \hat V_{q,\tilde q} \left(\hat\theta_{\tilde q,q'} - \hat\theta_{\tilde q,q'}(\hat n_{\tilde q}+1, \hat n_q - 1)\right) : \hat a^\dagger_q \hat a_{q'} : \\
&+ \sum_{k} \sum_{\tilde q} \hat V_{k,\tilde q} \left(\hat\theta_{\tilde q,k} - \hat\theta_{\tilde q,k}(\hat n_{\tilde q}+1, \hat n_k - 1)\right) : \hat a^\dagger_k \hat a_k : \\
&- \sum_{q\neq q'} \sum_{\tilde q} \hat\theta_{q,\tilde q} \left(\hat V_{\tilde q,q'} - \hat V_{\tilde q,q'}(\hat n_{\tilde q}+1, \hat n_q - 1)\right) : \hat a^\dagger_q \hat a_{q'} : \\
&- \sum_{k} \sum_{\tilde q} \hat\theta_{k,\tilde q} \left(\hat V_{\tilde q,k} - \hat V_{\tilde q,k}(\hat n_{\tilde q}+1, \hat n_k - 1)\right) : \hat a^\dagger_k \hat a_k :
\end{aligned}
$$

A.25b:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{\tilde q\neq \tilde q'} \left[\hat\theta_{\tilde q,\tilde q'} : \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} :, \hat W_{p,p'} : \hat a^\dagger_p \hat a^\dagger_{p'} :\right] \\
&= \sum_{p,p'} \sum_{\tilde q\neq \tilde q'} \hat\theta_{\tilde q,\tilde q'} \hat W_{p,p'} \left[: \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} :, : \hat a^\dagger_p \hat a^\dagger_{p'} :\right] && \text{(A.30a)} \\
&+ \sum_{p,p'} \sum_{\tilde q\neq \tilde q'} \hat\theta_{\tilde q,\tilde q'} \left[: \hat a^\dagger_{\tilde q} \hat a_{\tilde q'} :, \hat W_{p,p'}\right] : \hat a^\dagger_p \hat a^\dagger_{p'} : && \text{(A.30b)}
\end{aligned}
$$

$$+\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat W_{p,p'}\left[\hat\theta_{\tilde q,\tilde q'},:\hat a_p^\dagger\hat a_{p'}^\dagger:\right]:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}: \tag{A.30c}$$

We start by evaluating A.30a:

$$\begin{aligned}
&\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat\theta_{\tilde q,\tilde q'}\hat W_{p,p'}\left[:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:,:\hat a_p^\dagger\hat a_{p'}^\dagger:\right]\\
&=\sum_{p,p'}\sum_{q\neq q'}\hat\theta_{q,q'}\hat W_{p,p'}\left(\delta_{q',p'}:\hat a_q^\dagger\hat a_p^\dagger:+\delta_{q',p}:\hat a_q^\dagger\hat a_{p'}^\dagger:\right)\\
&=\sum_{p,p'}\sum_{q}\hat\theta_{q,p'}\hat W_{p,p'}:\hat a_q^\dagger\hat a_p^\dagger:+\sum_{p,p'}\sum_{q}\hat\theta_{q,p}\hat W_{p,p'}:\hat a_q^\dagger\hat a_{p'}^\dagger:\\
&=\sum_{p,p'}\sum_{q}\left(\hat\theta_{p',q}\hat W_{p,q}+\hat\theta_{p,q}\hat W_{q,p'}\right):\hat a_p^\dagger\hat a_{p'}^\dagger:
\end{aligned} \tag{A.31}$$

Next is A.30b:

$$\begin{aligned}
&\sum_{p,p'}\sum_{q\neq q'}\hat\theta_{q,q'}\left[:\hat a_q^\dagger\hat a_{q'}:,\hat W_{p,p'}\right]:\hat a_p^\dagger\hat a_{p'}^\dagger:\\
&=\sum_{p,p'}\sum_{q\neq q'}\hat\theta_{q,q'}\left(\hat W_{p,p'}(\hat n_{q'}+1,\hat n_q-1)-\hat W_{p,p'}\right)\quad\underbrace{:\hat a_q^\dagger\hat a_{q'}::\hat a_p^\dagger\hat a_{p'}^\dagger:}_{\overset{②}{=}\delta_{q',p}:\hat a_{p'}^\dagger\hat a_q^\dagger:+\delta_{q',p'}:\hat a_p^\dagger\hat a_q^\dagger:}\\
&\overset{②}{=}\sum_{p,p'}\sum_{q}\hat\theta_{q,p}\left(\hat W_{p,p'}(\hat n_p+1,\hat n_q-1)-\hat W_{p,p'}\right):\hat a_{p'}^\dagger\hat a_q^\dagger:\\
&+\sum_{p,p'}\sum_{q}\hat\theta_{q,p'}\left(\hat W_{p,p'}(\hat n_{p'}+1,\hat n_q-1)-\hat W_{p,p'}\right):\hat a_p^\dagger\hat a_q^\dagger:\\
&=\sum_{p,p'}\sum_{q}\hat\theta_{p,q}\left(\hat W_{q,p'}(\hat n_q+1,\hat n_p-1)-\hat W_{q,p'}\right):\hat a_p^\dagger\hat a_{p'}^\dagger:\\
&+\sum_{p,p'}\sum_{q}\hat\theta_{p',q}\left(\hat W_{p,q}(\hat n_q+1,\hat n_{p'}-1)-\hat W_{p,q}\right):\hat a_p^\dagger\hat a_{p'}^\dagger:
\end{aligned} \tag{A.32}$$

A.30c gives no quadratic contribution:

$$\begin{aligned}
&\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat W_{p,p'}\left[\hat\theta_{\tilde q,\tilde q'},:\hat a_p^\dagger\hat a_{p'}^\dagger:\right]:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}\\
&=\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat W_{p,p'}\left(\hat\theta_{\tilde q,\tilde q'}-\hat\theta_{\tilde q,\tilde q'}(\hat n_{p'},\hat n_p-1)\right)\underbrace{:\hat a_p^\dagger\hat a_{p'}^\dagger::\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:}_{=:\hat a_p^\dagger\hat a_{p'}^\dagger\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:}\overset{②}{=}0
\end{aligned} \tag{A.33}$$

A.25c:

$$\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\left[\hat\theta_{\tilde q,\tilde q'}:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:,\hat W_{p,p'}^\dagger:\hat a_p\hat a_{p'}:\right]$$

$$=\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat\theta_{\tilde q,\tilde q'}\hat W_{p,p'}^\dagger\left[:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:,:\hat a_p\hat a_{p'}:\right] \tag{A.34a}$$

$$+\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat\theta_{\tilde q,\tilde q'}\left[:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}:,\hat W_{p,p'}^\dagger\right]:\hat a_p\hat a_{p'}: \tag{A.34b}$$

$$+\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\hat W_{p,p'}^\dagger\left[\hat\theta_{\tilde q,\tilde q'},:\hat a_p\hat a_{p'}:\right]:\hat a_{\tilde q}^\dagger\hat a_{\tilde q'}: \tag{A.34c}$$

We will again start by evaluating A.34a:

$$\sum_{p,p'}\sum_{q\neq q'}\hat\theta_{q,q'}\hat W_{p,p'}^\dagger\left[:\hat a_q^\dagger\hat a_{q'}:,:\hat a_p\hat a_{p'}:\right]$$

$$
\begin{aligned}
&= \sum_{p,p'}\sum_{q\neq q'} \hat{\theta}_{q,q'}\hat{W}^{\dagger}_{p,p'}\left(\delta_{q,p'} : \hat{a}_p\hat{a}_{q'} : +\delta_{q,p} : \hat{a}_{p'}\hat{a}_{q'} :\right) \\
&= \sum_{p,p'}\sum_{q'} \hat{\theta}_{p',q'}\hat{W}^{\dagger}_{p,p'} : \hat{a}_p\hat{a}_{q'} : + \sum_{p,p'}\sum_{q'} \hat{\theta}_{p,q'}\hat{W}^{\dagger}_{p,p'} : \hat{a}_{p'}\hat{a}_{q'} : \\
&= \sum_{p,p'}\sum_{q} \left(\hat{\theta}_{q,p'}\hat{W}^{\dagger}_{p,q} + \hat{\theta}_{q,p}\hat{W}^{\dagger}_{q,p'}\right) : \hat{a}_p\hat{a}_{p'} : \qquad \text{(A.35)}
\end{aligned}
$$

A.34b gives no quadratic contribution:

$$
\begin{aligned}
&\sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'}\left[: \hat{a}^{\dagger}_{\tilde{q}}\hat{a}_{\tilde{q}'} :, \hat{W}^{\dagger}_{p,p'}\right] : \hat{a}_p\hat{a}_{p'} : \\
&= \sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'}\left(W^{\dagger}_{p,p'}(\hat{n}_{q'}+1,\hat{n}_q-1) - W^{\dagger}_{p,p'}\right)\underbrace{: \hat{a}^{\dagger}_{\tilde{q}}\hat{a}_{\tilde{q}'} :: \hat{a}_p\hat{a}_{p'} :}_{=:\hat{a}^{\dagger}_{\tilde{q}}\hat{a}_{\tilde{q}'}\hat{a}_p\hat{a}_{p'}:} \overset{②}{=} 0 \qquad \text{(A.36)}
\end{aligned}
$$

A.34c:

$$
\begin{aligned}
&\sum_{p,p'}\sum_{q\neq q'} \hat{W}^{\dagger}_{p,p'}\left[\hat{\theta}_{q,q'}, : \hat{a}_p\hat{a}_{p'} :\right] : \hat{a}^{\dagger}_q\hat{a}_{q'} : \\
&= \sum_{p,p'}\sum_{q\neq q'} \hat{W}^{\dagger}_{p,p'}\left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'}(\hat{n}_{p'}+1,\hat{n}_p+1)\right) : \hat{a}_p\hat{a}_{p'} :: \hat{a}^{\dagger}_q\hat{a}_{q'} : \qquad \text{(A.37)} \\
&\overset{②}{=} \sum_{p,p'}\sum_{q\neq q'} \hat{W}^{\dagger}_{p,p'}\left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'}(\hat{n}_{p'}+1,\hat{n}_p+1)\right)\left(\delta_{p,q} : \hat{a}_{p'}\hat{a}_{q'} : +\delta_{p',q} : \hat{a}_p\hat{a}_{q'} :\right) \\
&= \sum_{p,p'}\sum_{q'} \hat{W}^{\dagger}_{p,p'}\left(\hat{\theta}_{p,q'} - \hat{\theta}_{p,q'}(\hat{n}_{p'}+1,\hat{n}_p+1)\right) : \hat{a}_{p'}\hat{a}_{q'} : \qquad \text{(A.38)} \\
&+ \sum_{p,p'}\sum_{q'} \hat{W}^{\dagger}_{p,p'}\left(\hat{\theta}_{p',q'} - \hat{\theta}_{p',q'}(\hat{n}_{p'}+1,\hat{n}_p+1)\right) : \hat{a}_p\hat{a}_{q'} : \\
&= \sum_{p,p'}\sum_{q} \hat{W}^{\dagger}_{q,p'}\left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'}+1,\hat{n}_q+1)\right) : \hat{a}_p\hat{a}_{p'} : \qquad \text{(A.39)} \\
&+ \sum_{p,p'}\sum_{q} \hat{W}^{\dagger}_{p,q}\left(\hat{\theta}_{q,p'} - \hat{\theta}_{q,p'}(\hat{n}_q+1,\hat{n}_p+1)\right) : \hat{a}_p\hat{a}_{p'} : \\
&= \sum_{p,p'}\sum_{q} \left(\hat{W}^{\dagger}_{q,p'} + \hat{W}^{\dagger}_{p',q}\right)\left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'}+1,\hat{n}_q+1)\right) : \hat{a}_p\hat{a}_{p'} : \qquad \text{(A.40)}
\end{aligned}
$$

A.25d: Follows immediately from the calculations already done for A.25b:

$$
\begin{aligned}
&\sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}^{\dagger}_{\tilde{p}}\hat{a}^{\dagger}_{\tilde{p}'} :, \hat{V}_{q,q'} : \hat{a}^{\dagger}_q\hat{a}_{q'} :\right] \qquad \text{(A.41a)} \\
&= -\sum_{q\neq q'}\sum_{p,p'} \left[\hat{V}_{q,q'} : \hat{a}^{\dagger}_q\hat{a}_{q'} :, \hat{\phi}_{p,p'} : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} :\right] \qquad \text{(A.41b)} \\
&= -\sum_{p,p'}\sum_{q} \left(\hat{V}_{p',q}\hat{\phi}_{p,q} + \hat{V}_{p,q}\hat{\phi}_{q,p'}\right) : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} : \qquad \text{(A.41c)} \\
&- \sum_{p,p'}\sum_{q} \hat{V}_{p,q}\left(\hat{\phi}_{q,p'}(\hat{n}_q+1,\hat{n}_p-1) - \hat{\phi}_{q,p'}\right) : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} : \\
&- \sum_{p,p'}\sum_{q} \hat{V}_{p',q}\left(\hat{\phi}_{p,q}(\hat{n}_q+1,\hat{n}_{p'}-1) - \hat{\phi}_{p,q}\right) : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} :
\end{aligned}
$$

A.25e:

$$
\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}^{\dagger}_{\tilde{p}}\hat{a}^{\dagger}_{\tilde{p}'} :, \hat{W}_{p,p'} : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} :\right]
$$

$$= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}\right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \tag{A.42a}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'}, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : \tag{A.42b}$$

A.42a will be analyzed first:

$$\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}\right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :$$

$$= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}_{p,p'}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{\tilde{p}'} + 1) - \hat{W}_{p,p'}\right) : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :: \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \overset{②}{=} 0 \tag{A.43}$$

Similiarly, A.42b also gives no quadratic contribution.

A.25f

$$\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right]$$

$$= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, : \hat{a}_p \hat{a}_{p'} :\right] \tag{A.44a}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}^\dagger\right] : \hat{a}_p \hat{a}_{p'} : \tag{A.44b}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left[\hat{\phi}_{\tilde{p},\tilde{p}'}, : \hat{a}_p \hat{a}_{p'} :\right] : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : \tag{A.44c}$$

A.44a:

$$\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\delta_{p',\tilde{p}'} \hat{a}_p \hat{a}_{\tilde{p}}^\dagger + \delta_{p',\tilde{p}} \hat{a}_p \hat{a}_{\tilde{p}'}^\dagger + \delta_{p,\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} + \delta_{p,\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'}\right)$$

$$= -\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p',\tilde{p}'} \left(: \hat{a}_{\tilde{p}}^\dagger \hat{a}_p : + \delta_{p,\tilde{p}}\right) \tag{A.45}$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p',\tilde{p}} \left(: \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p : + \delta_{\tilde{p}',p}\right)$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p,\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p,\tilde{p}} : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} :$$

$$= -\sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}}^\dagger \hat{a}_p : \tag{A.46}$$

$$- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p',\tilde{p}'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p :$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p,\tilde{p}'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}_{p,p'}^\dagger + \hat{\phi}_{p',p} \hat{W}_{p,p'}^\dagger\right)$$

$$= -\sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{p,\tilde{p}} \hat{W}_{p',\tilde{p}}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} : \tag{A.47}$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}_{p',\tilde{p}}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} :$$

$$
\begin{aligned}
&-\sum_{p,p'}\sum_{\tilde{p}}\hat{\phi}_{p,\tilde{p}}\hat{W}^\dagger_{\tilde{p},p'} : \hat{a}^\dagger_p\hat{a}_{p'} : \\
&-\sum_{p,p'}\sum_{\tilde{p}}\hat{\phi}_{\tilde{p},p}\hat{W}^\dagger_{\tilde{p},p'} : \hat{a}^\dagger_p\hat{a}_{p'} : \\
&-\sum_{p,p'}\left(\hat{\phi}_{p,p'}\hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p}\hat{W}^\dagger_{p,p'}\right) \\
&= -\sum_{p,p'}\sum_{\tilde{p}}\left(\hat{\phi}_{p,\tilde{p}}\hat{W}^\dagger_{p',\tilde{p}} + \hat{\phi}_{\tilde{p},p}\hat{W}^\dagger_{p',\tilde{p}} + \hat{\phi}_{p,\tilde{p}}\hat{W}^\dagger_{\tilde{p},p'} + \hat{\phi}_{\tilde{p},p}\hat{W}^\dagger_{\tilde{p},p'}\right) : \hat{a}^\dagger_p\hat{a}_{p'} : \qquad (A.48) \\
&-\sum_{p,p'}\left(\hat{\phi}_{p,p'}\hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p}\hat{W}^\dagger_{p,p'}\right)
\end{aligned}
$$

A.44b gives no quadratic contribution:

$$
\begin{aligned}
&\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left[: \hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'} :, \hat{W}^\dagger_{p,p'}\right] : \hat{a}_p\hat{a}_{p'} : \\
&= \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left(\hat{W}^\dagger_{p,p'}(\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}^\dagger_{p,p'}\right) : \hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'} :: \hat{a}_p\hat{a}_{p'} : \\
&= \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left(\hat{W}^\dagger_{p,p'}(\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}^\dagger_{p,p'}\right) : \hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}\hat{a}_p\hat{a}_{p'} : \overset{2}{=} 0 \qquad (A.49)
\end{aligned}
$$

A.44c:

$$
\begin{aligned}
&\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'}, : \hat{a}_p\hat{a}_{p'} :\right] : \hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'} : \\
&= \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right) : \hat{a}_p\hat{a}_{p'} :: \hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'} : \qquad (A.50) \\
&\overset{2}{=} \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p,\tilde{p}} : \hat{a}^\dagger_{\tilde{p}'}\hat{a}_{p'} : \qquad (A.51) \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p,\tilde{p}'} : \hat{a}^\dagger_{\tilde{p}}\hat{a}_{p'} : \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p',\tilde{p}} : \hat{a}^\dagger_{\tilde{p}'}\hat{a}_p : \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p',\tilde{p}'} : \hat{a}^\dagger_{\tilde{p}}\hat{a}_p : \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p',\tilde{p}}\delta_{p,\tilde{p}'} \\
&+ \sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)\delta_{p',\tilde{p}'}\delta_{p,\tilde{p}} \\
&= \sum_{p,p'}\sum_{\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{p,\tilde{p}'} - \hat{\phi}_{p,\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right) : \hat{a}^\dagger_{\tilde{p}'}\hat{a}_{p'} : \qquad (A.52) \\
&+ \sum_{p,p'}\sum_{\tilde{p}}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},p} - \hat{\phi}_{\tilde{p},p}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right) : \hat{a}^\dagger_{\tilde{p}}\hat{a}_{p'} : \\
&+ \sum_{p,p'}\sum_{\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{p',\tilde{p}'} - \hat{\phi}_{p',\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right) : \hat{a}^\dagger_{\tilde{p}'}\hat{a}_p : \\
&+ \sum_{p,p'}\sum_{\tilde{p}}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{\tilde{p},p'} - \hat{\phi}_{\tilde{p},p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right) : \hat{a}^\dagger_{\tilde{p}}\hat{a}_p : \\
&+ \sum_{p,p'}\hat{W}^\dagger_{p,p'}\left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1)\right)
\end{aligned}
$$

$$+ \sum_{p,p'} \hat{W}^{\dagger}_{p,p'} \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)$$

$$= \sum_{p,p'} \sum_{\tilde{p}} \hat{W}^{\dagger}_{\tilde{p},p'} \left(\hat{\phi}_{\tilde{p},p} - \hat{\phi}_{\tilde{p},p}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} : \tag{A.53}$$

$$+ \sum_{p,p'} \sum_{\tilde{p}} \hat{W}^{\dagger}_{\tilde{p},p'} \left(\hat{\phi}_{p,\tilde{p}} - \hat{\phi}_{p,\tilde{p}}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$+ \sum_{p,p'} \sum_{\tilde{p}} \hat{W}^{\dagger}_{p',\tilde{p}} \left(\hat{\phi}_{\tilde{p},p} - \hat{\phi}_{\tilde{p},p}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$+ \sum_{p,p'} \sum_{\tilde{p}} \hat{W}^{\dagger}_{p',\tilde{p}} \left(\hat{\phi}_{p,\tilde{p}} - \hat{\phi}_{p,\tilde{p}}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$+ \sum_{p,p'} \hat{W}^{\dagger}_{p,p'} \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)$$

$$+ \sum_{p,p'} \hat{W}^{\dagger}_{p,p'} \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)$$

A.25g: Follows immediately from A.25c:

$$\sum_{q \neq q'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{V}_{q,q'} : \hat{a}^{\dagger}_q \hat{a}_{q'} : \right]$$

$$= - \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \left[\hat{V}_{\tilde{q},\tilde{q}'} : \hat{a}^{\dagger}_{\tilde{q}} \hat{a}_{\tilde{q}'} :, \hat{\psi}_{p,p'} : \hat{a}_p \hat{a}_{p'} : \right]$$

$$\overset{②}{=} - \sum_{p,p'} \sum_{q} \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.54a}$$

$$- \sum_{p,p'} \sum_{q} \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} :$$

A.25h Follows immediately from A.25f:

$$\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}_{p,p'} : \hat{a}^{\dagger}_p \hat{a}^{\dagger}_{p'} : \right]$$

$$= - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{W}_{\tilde{p},\tilde{p}'} : \hat{a}^{\dagger}_{\tilde{p}} \hat{a}^{\dagger}_{\tilde{p}'} :, \hat{\psi}_{p,p'} : \hat{a}_p \hat{a}_{p'} : \right] \tag{A.55}$$

$$= \sum_{p,p'} \sum_{\tilde{p}} \left(\hat{W}_{p,\tilde{p}} \hat{\psi}_{p',\tilde{p}} + \hat{W}_{\tilde{p},p} \hat{\psi}_{p',\tilde{p}} + \hat{W}_{p,\tilde{p}} \hat{\psi}_{\tilde{p},p'} + \hat{W}_{\tilde{p},p} \hat{\psi}_{\tilde{p},p'} \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} : \tag{A.56}$$

$$+ \sum_{p,p'} \left(\hat{W}_{p,p'} \hat{\psi}_{p,p'} + \hat{W}_{p',p} \hat{\psi}_{p,p'} \right)$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{\tilde{p},p'} \left(\hat{W}_{\tilde{p},p} - \hat{W}_{\tilde{p},p}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{\tilde{p},p'} \left(\hat{W}_{p,\tilde{p}} - \hat{W}_{p,\tilde{p}}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{p',\tilde{p}} \left(\hat{W}_{\tilde{p},p} - \hat{W}_{\tilde{p},p}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{p',\tilde{p}} \left(\hat{W}_{p,\tilde{p}} - \hat{W}_{p,\tilde{p}}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^{\dagger}_p \hat{a}_{p'} :$$

$$- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)$$

$$- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)$$

A.25i: Similiar to A.25e:

$$\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'} :, \hat{W}^{\dagger}_{p,p'} : \hat{a}_{p}\hat{a}_{p'} :\right] \overset{②}{=} 0 \tag{A.57}$$

A.3 The flow equations

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$\begin{aligned}\hat{\mathcal{H}}(\lambda) \overset{②}{=} &\sum_{k}\hat{\omega}_k(\lambda) : \hat{a}^{\dagger}_k\hat{a}_k : + \sum_{q\neq q'}\hat{V}_{q,q'}(\lambda) : \hat{a}^{\dagger}_q\hat{a}_{q'} : \\ &+\sum_{p,p'}\left(\hat{W}_{p,p'}(\lambda) : \hat{a}^{\dagger}_p\hat{a}^{\dagger}_{p'} : + \hat{W}^{\dagger}_{p,p'}(\lambda) : \hat{a}_p\hat{a}_{p'} :\right) + \epsilon(\lambda)\end{aligned} \tag{A.58}$$

where $\epsilon(\lambda)$ is a constant which indicates a shift in the energy scale.
Collecting all terms in $\partial_\lambda\hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ gives the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$\begin{aligned}\partial_\lambda\hat{\omega}_k \overset{②}{=} &\sum_{\tilde{q}}\left(\hat{\theta}_{k,k}\hat{V}_{k,\tilde{q}} - \hat{\theta}_{\tilde{q},k}\hat{V}_{k,\tilde{q}}\right) \\ &+\sum_{\tilde{q}}\hat{V}_{k,\tilde{q}}\left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k}(\hat{n}_{\tilde{q}}+1, \hat{n}_k-1)\right) \\ &-\sum_{\tilde{q}}\hat{\theta}_{k,\tilde{q}}\left(\hat{V}_{\tilde{q},k} - \hat{V}_{\tilde{q},k}(\hat{n}_{\tilde{q}}+1, \hat{n}_k-1)\right) \\ &-\sum_{\tilde{p}}\left(\hat{\phi}_{k,\tilde{p}}\hat{W}^{\dagger}_{k,\tilde{p}} + \hat{\phi}_{\tilde{p},k}\hat{W}^{\dagger}_{k,\tilde{p}} + \hat{\phi}_{k,\tilde{p}}\hat{W}^{\dagger}_{\tilde{p},k} + \hat{\phi}_{\tilde{p},k}\hat{W}^{\dagger}_{\tilde{p},k}\right) \\ &+\sum_{\tilde{p}}\hat{W}^{\dagger}_{\tilde{p},k}\left(\hat{\phi}_{\tilde{p},k} - \hat{\phi}_{\tilde{p},k}(\hat{n}_{\tilde{p}}+1, \hat{n}_k+1)\right) \\ &+\sum_{\tilde{p}}\hat{W}^{\dagger}_{\tilde{p},k}\left(\hat{\phi}_{k,\tilde{p}} - \hat{\phi}_{k,\tilde{p}}(\hat{n}_{\tilde{p}}+1, \hat{n}_k+1)\right) \\ &+\sum_{\tilde{p}}\hat{W}^{\dagger}_{k,\tilde{p}}\left(\hat{\phi}_{\tilde{p},k} - \hat{\phi}_{\tilde{p},k}(\hat{n}_k+1, \hat{n}_{\tilde{p}}+1)\right) \\ &+\sum_{\tilde{p}}\hat{W}^{\dagger}_{k,\tilde{p}}\left(\hat{\phi}_{k,\tilde{p}} - \hat{\phi}_{k,\tilde{p}}(\hat{n}_k+1, \hat{n}_{\tilde{p}}+1)\right) \\ &+\sum_{\tilde{p}}\left(\hat{W}_{k,\tilde{p}}\hat{\psi}_{k,\tilde{p}} + \hat{W}_{\tilde{p},k}\hat{\psi}_{k,\tilde{p}} + \hat{W}_{k,\tilde{p}}\hat{\psi}_{\tilde{p},k} + \hat{W}_{\tilde{p},k}\hat{\psi}_{\tilde{p},k}\right) \\ &-\sum_{\tilde{p}}\hat{\psi}_{\tilde{p},k}\left(\hat{W}_{\tilde{p},k} - \hat{W}_{\tilde{p},k}(\hat{n}_{\tilde{p}}+1, \hat{n}_k+1)\right) \\ &-\sum_{\tilde{p}}\hat{\psi}_{\tilde{p},k}\left(\hat{W}_{k,\tilde{p}} - \hat{W}_{k,\tilde{p}}(\hat{n}_{\tilde{p}}+1, \hat{n}_k+1)\right) \\ &-\sum_{\tilde{p}}\hat{\psi}_{k,\tilde{p}}\left(\hat{W}_{\tilde{p},k} - \hat{W}_{\tilde{p},k}(\hat{n}_k+1, \hat{n}_{\tilde{p}}+1)\right) \\ &-\sum_{\tilde{p}}\hat{\psi}_{k,\tilde{p}}\left(\hat{W}_{k,\tilde{p}} - \hat{W}_{k,\tilde{p}}(\hat{n}_k+1, \hat{n}_{\tilde{p}}+1)\right)\end{aligned} \tag{A.59a}$$

$$\begin{aligned}\partial_\lambda\hat{V}_{q,q'} \overset{②}{=} &-(\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q-1, \hat{n}_{q'}+1))\hat{\theta}_{q,q'} \\ &+\sum_{\tilde{q}}\left(\hat{\theta}_{q,q'}\hat{V}_{q',\tilde{q}} - \hat{\theta}_{\tilde{q},q'}\hat{V}_{q,\tilde{q}}\right)\end{aligned} \tag{A.59b}$$

$$
\begin{aligned}
&+\sum_{\tilde q}\hat V_{q,\tilde q}\left(\hat\theta_{\tilde q,q'}-\hat\theta_{\tilde q,q'}(\hat n_{\tilde q}+1,\hat n_q-1)\right)\\
&-\sum_{\tilde q}\hat\theta_{q,\tilde q}\left(\hat V_{\tilde q,q'}-\hat V_{\tilde q,q'}(\hat n_{\tilde q}+1,\hat n_q-1)\right)\\
&-\sum_{\tilde q}\left(\hat\phi_{q,\tilde q}\hat W^\dagger_{q',\tilde q}+\hat\phi_{\tilde q,q}\hat W^\dagger_{q',\tilde q}+\hat\phi_{q,\tilde q}\hat W^\dagger_{\tilde q,q'}+\hat\phi_{\tilde q,q}\hat W^\dagger_{\tilde q,q'}\right)\\
&+\sum_{\tilde q}\hat W^\dagger_{\tilde q,q'}\left(\hat\phi_{\tilde q,q}-\hat\phi_{\tilde q,q}(\hat n_{\tilde q}+1,\hat n_{q'}+1)\right)\\
&+\sum_{\tilde q}\hat W^\dagger_{\tilde q,q'}\left(\hat\phi_{q,\tilde q}-\hat\phi_{q,\tilde q}(\hat n_{\tilde q}+1,\hat n_{q'}+1)\right)\\
&+\sum_{\tilde q}\hat W^\dagger_{q',\tilde q}\left(\hat\phi_{\tilde q,q}-\hat\phi_{\tilde q,q}(\hat n_{q'}+1,\hat n_{\tilde q}+1)\right)\\
&+\sum_{\tilde q}\hat W^\dagger_{q',\tilde q}\left(\hat\phi_{q,\tilde q}-\hat\phi_{q,\tilde q}(\hat n_{q'}+1,\hat n_{\tilde q}+1)\right)\\
&+\sum_{\tilde q}\left(\hat W_{q,\tilde q}\hat\psi_{q',\tilde q}+\hat W_{\tilde q,q}\hat\psi_{q',\tilde q}+\hat W_{q,\tilde q}\hat\psi_{\tilde q,q'}+\hat W_{\tilde q,q}\hat\psi_{\tilde q,q'}\right)\\
&-\sum_{\tilde q}\hat\psi_{\tilde q,q'}\left(\hat W_{\tilde q,q}-\hat W_{\tilde q,q}(\hat n_{\tilde q}+1,\hat n_{q'}+1)\right)\\
&-\sum_{\tilde q}\hat\psi_{\tilde q,q'}\left(\hat W_{q,\tilde q}-\hat W_{q,\tilde q}(\hat n_{\tilde q}+1,\hat n_{q'}+1)\right)\\
&-\sum_{\tilde q}\hat\psi_{q',\tilde q}\left(\hat W_{\tilde q,q}-\hat W_{\tilde q,q}(\hat n_{q'}+1,\hat n_{\tilde q}+1)\right)\\
&-\sum_{\tilde q}\hat\psi_{q',\tilde q}\left(\hat W_{q,\tilde q}-\hat W_{q,\tilde q}(\hat n_{q'}+1,\hat n_{\tilde q}+1)\right)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\lambda\hat W_{p,p'}\overset{②}{=}&-\hat\phi_{p,p'}(\hat\omega_p+\hat\omega_{p'})\\
&+\sum_q\left(\hat\theta_{p',q}\hat W_{p,q}+\hat\theta_{p,q}\hat W_{q,p'}\right)\\
&+\sum_q\hat\theta_{p,q}\left(\hat W_{q,p'}(\hat n_q+1,\hat n_p-1)-\hat W_{q,p'}\right)\\
&+\sum_q\hat\theta_{p',q}\left(\hat W_{p,q}(\hat n_q+1,\hat n_{p'}-1)-\hat W_{p,q}\right)\\
&-\sum_q\left(\hat V_{p',q}\hat\phi_{p,q}+\hat V_{p,q}\hat\phi_{q,p'}\right)\\
&-\sum_q\hat V_{p,q}\left(\hat\phi_{q,p'}(\hat n_q+1,\hat n_p-1)-\hat\phi_{q,p'}\right)\\
&-\sum_q\hat V_{p',q}\left(\hat\phi_{p,q}(\hat n_q+1,\hat n_{p'}-1)-\hat\phi_{p,q}\right)
\end{aligned}
\tag{A.59c}
$$

$$
\begin{aligned}
\partial_\lambda\hat W^\dagger_{p,p'}\overset{②}{=}&\,\hat\psi_{p,p'}(\hat\omega_p(\hat n_{p'}+1,\hat n_p+1)+\hat\omega_{p'}(\hat n_{p'}+1,\hat n_p+1))\\
&+\sum_q\left(\hat\theta_{q,p'}\hat W^\dagger_{p,q}+\hat\theta_{q,p}\hat W^\dagger_{q,p'}\right)\\
&+\sum_q\left(\hat W^\dagger_{q,p'}+\hat W^\dagger_{p',q}\right)\left(\hat\theta_{q,p}-\hat\theta_{q,p}(\hat n_{p'}+1,\hat n_q+1)\right)\\
&-\sum_q\left(\hat\psi_{q,p'}+\hat\psi_{p',q}\right)\left(\hat V_{q,p}-\hat V_{q,p}(\hat n_{p'}+1,\hat n_q+1)\right)\\
&-\sum_q\left(\hat V_{q,p'}\hat\psi_{p,q}+\hat\theta_{q,p}\hat\psi_{q,p'}\right)
\end{aligned}
\tag{A.59d}
$$

$$\begin{aligned}
\partial_\lambda \epsilon \overset{②}{=} &- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p} \hat{W}^\dagger_{p,p'} \right) \\
&+ \sum_{p,p'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&+ \sum_{p,p'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&- \sum_{q} \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
&- \sum_{q} \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right) \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&+ \sum_{p,p'} \left(\hat{W}_{p,p'} \hat{\psi}_{p,p'} + \hat{W}_{p',p} \hat{\psi}_{p,p'} \right)
\end{aligned} \tag{A.59e}$$

The three operators $\hat{\psi}, \hat{\theta}, \hat{\phi}$ are based on their definition in equation A.22:

$$\hat{\theta}_{q,q'} = (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1))\hat{V}_{q,q'} \tag{A.60}$$

$$\hat{\phi}_{p,p'} = \hat{W}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) \tag{A.61}$$

$$\hat{\psi}_{p,p'} = -\hat{W}^\dagger_{p,p'}(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) \tag{A.62}$$

APPENDIX B

THE SECOND APPENDIX

Here comes the second appendix.

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DECLARATION OF AUTHORSHIP

I hereby declare that I have written this thesis independently and by myself and that I have not used any sources or auxiliary materials other than those indicated in the thesis.

Munich, 22.06.2023 . . . .