

$$\hat{H}_0 = \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \quad \omega_{k'} > \omega_k \nleftrightarrow k' > k$$

$$\hat{H}_{int}(\lambda) = \sum_{\substack{q, q' \\ q \neq q'}} V_{q, q'}(\lambda) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p, p'} (W_{pp'}(\lambda) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{pp'}^*(\lambda) \hat{a}_p \hat{a}_{p'})$$

$$= \sum_{\substack{q, q' \\ q \neq q'}} V_{q, q'} \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p, p'} (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{pp'}^* \hat{a}_p \hat{a}_{p'})$$

Canonical generator:

$$\hat{\eta}(\lambda) = [\hat{H}_0, \hat{H}_{int}] = \sum_k \sum_{\substack{q, q' \\ q \neq q'}} \omega_k V_{q, q'} [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}]$$

$$+ \sum_k \sum_{p, p'} \omega_k W_{pp'} [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] + \omega_k W_{pp'}^* [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p \hat{a}_{p'}]$$

$$[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}] = \hat{a}_k^\dagger \hat{a}_{q'} \delta_{kq} - \hat{a}_q^\dagger \hat{a}_k \delta_{kq'}$$

$$[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q \hat{a}_{q'}] = -(\hat{a}_q \hat{a}_k \delta_{kq'} + \hat{a}_{q'} \hat{a}_k \delta_{kq})$$

$$[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}^\dagger] = \hat{a}_k^\dagger \hat{a}_{q'}^\dagger \delta_{kq'} + \hat{a}_k^\dagger \hat{a}_q^\dagger \delta_{kq}$$

$$= \sum_k \sum_{\substack{q, q' \\ q \neq q'}} \omega_k V_{q, q'} (\hat{a}_k^\dagger \hat{a}_{q'} \delta_{kq} - \hat{a}_q^\dagger \hat{a}_k \delta_{kq'})$$

$$+ \sum_k \sum_{p, p'} \omega_k W_{pp'} (\hat{a}_k^\dagger \hat{a}_p^\dagger \delta_{kp'} + \hat{a}_k^\dagger \hat{a}_{p'}^\dagger \delta_{kp}) - \omega_k W_{pp'}^* (\hat{a}_p \hat{a}_k \delta_{kp'} + \hat{a}_{p'} \hat{a}_k \delta_{kp})$$

$$= \sum_{\substack{q, q' \\ q \neq q'}} V_{q, q'} (\omega_q \hat{a}_q^\dagger \hat{a}_{q'} - \omega_{q'} \hat{a}_{q'}^\dagger \hat{a}_q)$$

$$+ \sum_{p, p'} W_{pp'} (\omega_{p'} \hat{a}_{p'}^\dagger \hat{a}_p^\dagger + \omega_p \hat{a}_p^\dagger \hat{a}_{p'}^\dagger) - W_{pp'}^* (\omega_{p'} \hat{a}_p \hat{a}_{p'} + \omega_p \hat{a}_p \hat{a}_{p'})$$

$$\Rightarrow \hat{H}(\lambda) = \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p, p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'})$$

$$[\hat{H}(\lambda), \hat{H}_0] = \sum_k \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \omega_k [\hat{a}_q^\dagger \hat{a}_q, \hat{a}_k^\dagger \hat{a}_k]$$

$$+ \sum_k \sum_{p, p'} W_{pp'} (\omega_p + \omega_{p'}) [\hat{a}_p^\dagger \hat{a}_p, \hat{a}_k^\dagger \hat{a}_k] - W_{pp'}^* (\omega_p + \omega_{p'}) [\hat{a}_p \hat{a}_{p'}, \hat{a}_k^\dagger \hat{a}_k]$$

$$= \sum_k \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \omega_k (\hat{a}_q^\dagger \hat{a}_k \delta_{kq'} - \hat{a}_k^\dagger \hat{a}_{q'} \delta_{kq})$$

$$+ \sum_k \sum_{p, p'} (W_{pp'} (\omega_p + \omega_{p'}) \omega_k (\hat{a}_k^\dagger \hat{a}_p^\dagger \delta_{kp'} + \hat{a}_k^\dagger \hat{a}_{p'}^\dagger \delta_{kp}) \cdot (-1)$$

$$- W_{pp'}^* (\omega_p + \omega_{p'}) \omega_k (\hat{a}_p \hat{a}_k \delta_{kp'} + \hat{a}_{p'} \hat{a}_k \delta_{kp}))$$

$$= \sum_{q \neq q'} (-V_{qq'}) (\omega_q - \omega_{q'})^2 \hat{a}_q^\dagger \hat{a}_{q'}$$

$$- \sum_{p, p'} (W_{pp'} (\omega_p + \omega_{p'})^2 \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{pp'}^* (\omega_p + \omega_{p'})^2 \hat{a}_p \hat{a}_{p'})$$

$$[\hat{H}(1), \hat{H}_{int}(1)]$$

$$= \left[ \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p, p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'}) \right],$$

$$\sum_{\substack{q, q' \\ \tilde{q} \neq \tilde{q}'}} V_{q, \tilde{q}'} \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} + \sum_{\substack{p, p' \\ \tilde{p} \neq \tilde{p}'}} (W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}) \Big]$$

$$= \left[ \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{\substack{q, q' \\ \tilde{q} \neq \tilde{q}'}} V_{q, \tilde{q}'} \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} \right] \textcircled{I}$$

$$+ \left[ \sum_{q \neq q'} V_{qq'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{\substack{p, p' \\ \tilde{p} \neq \tilde{p}'}} (W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}) \right] \textcircled{II}$$

$$+ \left[ \sum_{p, p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'}) + \sum_{\substack{q, q' \\ \tilde{q} \neq \tilde{q}'}} V_{q, \tilde{q}'} \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} \right] \textcircled{III}$$

$$+ \left[ \sum_{p, p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'}) + \sum_{\substack{p, p' \\ \tilde{p} \neq \tilde{p}'}} (W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}) \right] \textcircled{IV}$$

$$\textcircled{I} \sum_{q \neq q'} \sum_{\substack{q, q' \\ \tilde{q} \neq \tilde{q}'}} V_{\tilde{q}, \tilde{q}'} V_{q, q'} (\omega_q - \omega_{q'}) \underbrace{[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}]}_{= \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} \delta_{q', \tilde{q}} - \hat{a}_{\tilde{q}}^\dagger \hat{a}_q \delta_{q, \tilde{q}'}}$$

$$= \sum_{q \neq q'} \sum_{\tilde{q}'} V_{\tilde{q}, \tilde{q}'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'}$$

$$= \sum_{q, q'} \sum_{\tilde{q}} V_{\tilde{q}, q'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}} - \sum_{q, q'} \sum_{\tilde{q}} V_{\tilde{q}, q} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'}$$

$$= \sum_{q, q'} \sum_{\tilde{q}} V_{\tilde{q}, q'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}} - \sum_{q, q'} \sum_{\tilde{q}} V_{\tilde{q}, q} V_{q, q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'}$$

$$= \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q} V_{q, q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'}$$

$$+ \sum_k \sum_{\tilde{q}} V_{\tilde{q}, k} V_{k, \tilde{q}} (\omega_k - \omega_{\tilde{q}}) \hat{a}_k^\dagger \hat{a}_{\tilde{q}} - \sum_k \sum_{\tilde{q}} V_{\tilde{q}, k} V_{k, \tilde{q}} (\omega_{\tilde{q}} - \omega_k) \hat{a}_{\tilde{q}}^\dagger \hat{a}_k$$

$$= \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q} V_{q, q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'}$$

$$+ \sum_k (2 \sum_{\tilde{q}} V_{\tilde{q}, k} V_{k, \tilde{q}} (\omega_k - \omega_{\tilde{q}})) \hat{a}_k^\dagger \hat{a}_{\tilde{q}}$$

$$= \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q}, q'} V_{q, q'} (\omega_{q'} + \omega_q - 2\omega_{\tilde{q}}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}} + \sum_k (2 \sum_{\tilde{q}} V_{\tilde{q}, k} V_{k, \tilde{q}} (\omega_k - \omega_{\tilde{q}})) \hat{a}_k^\dagger \hat{a}_{\tilde{q}}$$

$$\textcircled{II} \left[ \sum_{q \neq q'} V_{q, q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{p}, \tilde{p}'} (W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}) \right]$$

$$= \sum_{q \neq q'} \sum_{\tilde{p}, \tilde{p}'} V_{q, q'} (\omega_q - \omega_{q'}) \left( W_{\tilde{p}\tilde{p}'} \underbrace{[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger]}_{= \hat{a}_q^\dagger \hat{a}_{\tilde{p}'} \delta_{q', \tilde{p}} + \hat{a}_q^\dagger \hat{a}_{\tilde{p}}^\dagger \delta_{q, \tilde{p}'}} + W_{\tilde{p}\tilde{p}'}^* \underbrace{[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}]}_{= \hat{a}_{\tilde{p}} \hat{a}_{q'} \delta_{q, \tilde{p}} + \hat{a}_{\tilde{p}'} \hat{a}_{q'} \delta_{q, \tilde{p}'}} \right)$$

$$= \sum_{\tilde{p}, \tilde{p}'} \sum_q V_{q, \tilde{p}'} (\omega_q - \omega_{\tilde{p}'}) W_{\tilde{p}\tilde{p}'} \hat{a}_q^\dagger \hat{a}_{\tilde{p}'}^\dagger \quad (q \in \tilde{p})$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_q V_{q, \tilde{p}} (\omega_q - \omega_{\tilde{p}}) W_{\tilde{p}\tilde{p}'} \hat{a}_q^\dagger \hat{a}_{\tilde{p}'}^\dagger \quad (q \in \tilde{p})$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_{q'} V_{\tilde{p}', q'} (\omega_{\tilde{p}'} - \omega_{q'}) W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{q'} \quad (q' \in \tilde{p}')$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_{q'} V_{\tilde{p}, q'} (\omega_{\tilde{p}} - \omega_{q'}) W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}'} \hat{a}_{q'} \quad (q' \in \tilde{p})$$

$$= \sum_{\tilde{p}, \tilde{p}'} \sum_q V_{\tilde{p}\tilde{p}'} (\omega_{\tilde{p}} - \omega_{\tilde{p}'}) W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_q V_{\tilde{p}\tilde{p}'} (\omega_{\tilde{p}} - \omega_q) W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_{q'} V_{\tilde{p}\tilde{p}'} (\omega_{\tilde{p}'} - \omega_{q'}) W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{q'}$$

$$+ \sum_{\tilde{p}, \tilde{p}'} \sum_{q'} V_{\tilde{p}\tilde{p}'} (\omega_{\tilde{p}} - \omega_{q'}) W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}'} \hat{a}_{q'}$$



$$\left[ \sum_{p,p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'}) , \sum_{\substack{q,q' \\ q \neq q'}} V_{q,q'} \hat{a}_q^\dagger \hat{a}_{q'} \right]$$

$$= \sum_{p,p'} \sum_{q \neq q'} V_{qq'} (\omega_p + \omega_{p'}) \left( W_{pp'} [\hat{a}_p^\dagger \hat{a}_{p'}^\dagger, \hat{a}_q^\dagger \hat{a}_{q'}] - W_{pp'}^* [\hat{a}_p \hat{a}_{p'}, \hat{a}_q^\dagger \hat{a}_{q'}] \right)$$

$$= -(\hat{a}_q^\dagger \hat{a}_p^\dagger \delta_{q',p'} + \hat{a}_q^\dagger \hat{a}_{p'}^\dagger \delta_{q',p}) = -(\hat{a}_p \hat{a}_q \delta_{q,p'} + \hat{a}_{p'} \hat{a}_q \delta_{q,p})$$

$$= \sum_{p,p'} \sum_{q'} W_{pp'}^* V_{p'q'} (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{q'} \quad (q' \in p')$$

$$+ \sum_{p,p'} \sum_{q'} W_{pp'}^* V_{pq'} (\omega_p + \omega_{p'}) \hat{a}_{q'} \hat{a}_{p'} \quad (q' \in p)$$

$$- \sum_{p,p'} \sum_q W_{pp'} V_{q,p'} (\omega_p + \omega_{p'}) \hat{a}_q^\dagger \hat{a}_p^\dagger \quad (q \in p')$$

$$- \sum_{p,p'} \sum_q W_{pp'} V_{qp} (\omega_p + \omega_{p'}) \hat{a}_q^\dagger \hat{a}_{p'}^\dagger \quad (q \in p)$$

$$= \sum_{p,p'} \sum_{q'} W_{p'q'}^* V_{p'p'} (\omega_p + \omega_{q'}) \hat{a}_p \hat{a}_{p'}$$

$$+ \sum_{p,p'} \sum_{q'} W_{q'p'}^* V_{pp} (\omega_{q'} + \omega_{p'}) \hat{a}_p \hat{a}_{p'}$$

$$- \sum_{p,p'} \sum_q W_{pq} V_{p'q} (\omega_p + \omega_q) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger$$

$$- \sum_{p,p'} \sum_q W_{q,p'} V_{pq} (\omega_q + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger$$

$$\textcircled{IV} \left[ \sum_{p,p'} (\omega_p + \omega_{p'}) (W_{pp'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{pp'}^* \hat{a}_p \hat{a}_{p'}) , \sum_{\tilde{p},\tilde{p}'} (W_{\tilde{p}\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} ) \right]$$

$$= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'}) W_{pp'} W_{\tilde{p}\tilde{p}'} [\hat{a}_p^\dagger \hat{a}_{p'}^\dagger, \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger] \\ + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* [\hat{a}_p^\dagger \hat{a}_{p'}^\dagger, \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}] \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'}) W_{pp'}^* W_{\tilde{p}\tilde{p}'} [\hat{a}_p \hat{a}_{p'}, \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger] \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'}) W_{pp'}^* W_{\tilde{p}\tilde{p}'} [\hat{a}_p \hat{a}_{p'}, \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}] \\ = - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* [\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* [\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger]$$

$$\begin{aligned} [\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] &= \hat{a}_{\tilde{p}} [\hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] + [\hat{a}_{\tilde{p}}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] \hat{a}_{\tilde{p}'} \\ &= \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger [\hat{a}_{\tilde{p}'}, \hat{a}_{p'}^\dagger] + \hat{a}_{\tilde{p}} [\hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger] \hat{a}_{p'}^\dagger \\ &\quad + \hat{a}_{\tilde{p}}^\dagger [\hat{a}_{\tilde{p}'}, \hat{a}_{p'}^\dagger] \hat{a}_p^\dagger + [\hat{a}_{\tilde{p}}, \hat{a}_{p'}^\dagger] \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p^\dagger \\ &= \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger \delta_{\tilde{p}',p'} + \underbrace{\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger}_{\delta_{\tilde{p},p} + \hat{a}_p^\dagger \hat{a}_{\tilde{p}}} \delta_{\tilde{p}',p} + \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p'} + \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p} \\ &\quad (\delta_{\tilde{p},p} + \hat{a}_p^\dagger \hat{a}_{\tilde{p}} \delta_{\tilde{p}',p} + \delta_{\tilde{p},p} \delta_{\tilde{p}',p} + \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}} \delta_{\tilde{p}',p} + \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p'} + \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p}) \end{aligned}$$

$$= - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* [\hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger] \\ = - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger \delta_{\tilde{p}',p'} \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger \delta_{\tilde{p}',p} \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p'} \\ - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}\tilde{p}'}^* \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} \delta_{\tilde{p},p}$$

$$= - \sum_{p,p'} \sum_{\tilde{p}} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{p'}) W_{pp'} W_{\tilde{p}p'}^* \hat{a}_{\tilde{p}} \hat{a}_p^\dagger \quad (\tilde{p} \leftrightarrow p') \\ - \sum_{p,p'} \sum_{\tilde{p}} (\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_p) W_{pp'} W_{\tilde{p}p}^* \hat{a}_{\tilde{p}} \hat{a}_{p'}^\dagger \quad (\tilde{p} \leftrightarrow p) \\ - \sum_{p,p'} \sum_{\tilde{p}'} (\omega_p + \omega_{p'} + \omega_{p'} + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}'p'}^* \hat{a}_p^\dagger \hat{a}_{\tilde{p}'} \quad (p' \in \tilde{p}') \\ - \sum_{p,p'} \sum_{\tilde{p}'} (\omega_p + \omega_{p'} + \omega_p + \omega_{\tilde{p}'}) W_{pp'} W_{\tilde{p}'p}^* \hat{a}_p^\dagger \hat{a}_{\tilde{p}'} \quad (\tilde{p}' \in p)$$

$$= - \sum_{p,p'} \sum_{\tilde{p}} (\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{p\tilde{p}} W_{p'\tilde{p}}^* \hat{a}_{p'}^\dagger \hat{a}_p^\dagger \\ - \sum_{p,p'} \sum_{\tilde{p}} (\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{\tilde{p}p} W_{p'\tilde{p}}^* \hat{a}_{p'}^\dagger \hat{a}_p^\dagger \\ - \sum_{p,p'} \sum_{\tilde{p}'} (\omega_{p'} + 2\omega_{\tilde{p}'} + \omega_p) W_{p\tilde{p}'} W_{p'\tilde{p}'}^* \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \\ - \sum_{p,p'} \sum_{\tilde{p}'} (\omega_{p'} + 2\omega_{\tilde{p}'} + \omega_p) W_{\tilde{p}'p} W_{p'\tilde{p}'}^* \hat{a}_p^\dagger \hat{a}_{p'}^\dagger$$

$$= - 2 \sum_{q \neq q'} \sum_{\tilde{q}} (\omega_q + 2\omega_{\tilde{q}} + \omega_{q'}) W_{q\tilde{q}} W_{q'\tilde{q}}^* \hat{a}_q^\dagger \hat{a}_{q'}^\dagger$$

$$- 2 \cdot \sum_u \sum_q 2 \cdot (\omega_u + \omega_q) |W_{uq}|^2 (\hat{a}_u^\dagger \hat{a}_u + 1)$$

$$- \sum_{q \neq q'} \sum_{\tilde{q}} (\omega_{q'} + \omega_q + 2\omega_{\tilde{q}}) (W_{q\tilde{q}} W_{q'\tilde{q}}^* + W_{\tilde{q}q} W_{\tilde{q}q'}^*) \hat{a}_q^\dagger \hat{a}_{q'}$$

$$- \sum_u \sum_q 2 \cdot (\omega_u + \omega_q) (W_{uq} W_{uu}^* + W_{qu} W_{qu}^*) \hat{a}_u^\dagger \hat{a}_u$$

$$\partial_\lambda \omega_k = 2 \sum_{\tilde{q}} V_{\tilde{q}k} V_{k\tilde{q}} (\omega_k - \omega_{\tilde{q}}) - \sum_q 2(\omega_k + \omega_q) (W_{kq} W_{kk}^* + W_{qk} W_{qk}^*) \\ - 4 \sum_q (\omega_k + \omega_q) |W_{kq}|^2 \quad \forall k$$

$$\partial_\lambda V_{qq'} = -V_{qq'} (\omega_q - \omega_{q'})^2 + \sum_{\tilde{q}} V_{\tilde{q}q'} V_{q\tilde{q}} (\omega_{q'} + \omega_q - 2\omega_{\tilde{q}}) \\ - 2 \sum_{\tilde{q}} (\omega_q + 2\omega_{\tilde{q}} + \omega_{q'}) W_{q\tilde{q}} W_{q'\tilde{q}}^* - \sum_{\tilde{q}} (\omega_{q'} + \omega_q + 2\omega_{\tilde{q}}) (W_{q\tilde{q}} W_{q'\tilde{q}}^* + W_{\tilde{q}q} W_{\tilde{q}q'}^*) \\ \quad \forall q \neq q'$$

$$\partial_\lambda W_{pp'} = -W_{pp'} (\omega_p + \omega_{p'})^2 + \sum_q V_{pp'} (\omega_p - \omega_{p'}) W_{qp} + \sum_q V_{pq} (\omega_p - \omega_q) W_{q p'} \\ - \sum_q W_{pq} V_{p'q} (\omega_p + \omega_q) - \sum_q W_{q p'} V_{pq} (\omega_q + \omega_{p'}) \quad \forall p, p'$$

$$\partial_\lambda W_{pp'}^* = -W_{pp'}^* (\omega_p + \omega_{p'})^2 + \sum_q V_{q p'} (\omega_q - \omega_{p'}) W_{pq}^* + \sum_q V_{pp'} (\omega_p - \omega_{p'}) W_{pq'}^* \\ + \sum_{q'} W_{pq'}^* V_{p'p'} (\omega_p + \omega_{q'}) + \sum_{q'} W_{q' p'}^* V_{pp} (\omega_q + \omega_{p'}) \quad \forall p, p'$$

and also:

$$-4 \sum_q (\omega_k + \omega_q) |W_{kq}|^2 \stackrel{!}{=} 0 \quad \forall k$$