
Title of My Thesis



SUBMITTED BY

Jan-Philipp Anton Konrad Christ

Titel meiner Arbeit

Bachelorarbeit

FAKULTÄT FÜR PHYSIK
QUANTEN VIELTEILCHENSYSTEME/ THEORETISCHE NANOPHYSIK
LUDWIG-MAXIMILIANS-UNIVERSITÄT
MÜNCHEN

VORGELEGT VON

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MÜNCHEN, 22.06.2023

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FACULTY OF PHYSICS
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NOTATION AND SYMBOLS

λ	flow parameter; in the literature sometimes also denoted by B
$\hat{\cdot}$	denotes that \cdot is an operator which does not commute with every other operator
1	indicates $1 \in \mathbb{N}$ or the identity operator $\hat{1} =: \mathbb{1}$
$:\hat{A}:$	normal ordering of operator \hat{A}
\hat{a}_k^\dagger	k^{th} bosonic creation operator
\hat{a}_k	k^{th} bosonic annihilation operator
$[\hat{A}, \hat{B}]$	commutator of operators \hat{A}, \hat{B}
\hat{A}^\dagger	adjoint of an operator \hat{A}
z^*	complex conjugate of $z \in \mathbb{C}$
$\delta_{\alpha, \beta}$	Kronecker-Delta of α, β
∂_x	partial derivative $\frac{\partial}{\partial x}$ w.r.t. x
$\stackrel{\textcircled{2}}{=}$	Equality up to second order, i.e. higher order terms are neglected.

ABSTRACT

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SECTION 1

INTRODUCTION

SECTION 2

THEORETICAL BACKGROUND

2.1 The Flow Equation Approach

2.2 Normal Ordering

SECTION 4 _____

_____ CONCLUSION

A.1 Deriving the flow equations in the case of no n-dependence

First the canonical generator $\hat{\eta}$ has to be evaluated:

$$\hat{\eta} := \hat{\eta}(\lambda) := [\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}] = \left[\sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \sum_{q \neq q'} V_{q,q'} \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} \hat{a}_p^\dagger \hat{a}_{p'} + W_{p,p'}^* \hat{a}_p \hat{a}_{p'} \right) \right] \quad (\text{A.1})$$

$$\begin{aligned} &= \sum_k \sum_{q,q'} \omega_k V_{q,q'} [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}] + \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p^\dagger \hat{a}_{p'}] + \omega_k W_{p,p'}^* [\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p \hat{a}_{p'}] \right) \\ &= \sum_k \sum_{q,q'} \omega_k V_{q,q'} (\hat{a}_k^\dagger \hat{a}_{q'} \delta_{k,q} - \hat{a}_q^\dagger \hat{a}_k \delta_{k,q'}) \\ &+ \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} (\hat{a}_k^\dagger \hat{a}_p \delta_{k,p'} + \hat{a}_k^\dagger \hat{a}_{p'} \delta_{k,p}) - \omega_k W_{p,p'}^* (\hat{a}_p \hat{a}_k \delta_{k,p'} + \hat{a}_{p'} \hat{a}_k \delta_{k,p}) \right) \\ &= \sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'} - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'} \right) \end{aligned} \quad (\text{A.2})$$

Since $\hat{\eta}$ has the same form as $\hat{\mathcal{H}}_{\text{int}}$, $[\hat{\eta}, \hat{\mathcal{H}}_0]$ follows by inspection of A.2:

$$\begin{aligned} [\hat{\eta}, \hat{\mathcal{H}}_0] &= - \sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'})^2 \hat{a}_q^\dagger \hat{a}_{q'} \\ &- \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'})^2 \hat{a}_p^\dagger \hat{a}_{p'} + W_{p,p'}^* (\omega_p + \omega_{p'})^2 \hat{a}_p \hat{a}_{p'} \right) \end{aligned} \quad (\text{A.3})$$

The commutator of the generator and $\hat{\mathcal{H}}_{\text{int}}$ needs more work:

$$[\hat{\eta}, \hat{\mathcal{H}}_{\text{int}}] = \left[\sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'} - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'} \right), \right. \quad (\text{A.4})$$

$$\left. \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} + \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} \right) \right] \\ = \left[\sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \right] \quad (\text{A.5})$$

$$+ \left[\sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} \right) \right] \quad (\text{A.6})$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'} - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'} \right), \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \right] \quad (\text{A.7})$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'} - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'} \right), \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'} + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} \right) \right] \quad (\text{A.8})$$

In the following, A.5-A.8 will be evaluated separately. There will occur sums with $V_{q,q'}$ where $q = q'$. In this case, we define $V_{k,k} := 0 \ \forall k$. This saves the rather tedious declaration of the constraints of several sum indices.

A.5:

$$\begin{aligned}
& \left[\sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \right] \\
&= \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} V_{q,q'} (\omega_q - \omega_{q'}) \left[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \right] \\
&= \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} V_{q,q'} (\omega_q - \omega_{q'}) \left(\hat{a}_q^\dagger \hat{a}_{\tilde{q}'} \delta_{q',\tilde{q}} - \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'} \delta_{q,\tilde{q}'} \right) \\
&= \sum_{q \neq q'} \sum_{\tilde{q}} V_{q',\tilde{q}'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q},q} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'} \\
&= \sum_{q,q'} \sum_{\tilde{q}'} V_{q',\tilde{q}'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{\tilde{q}'} - \sum_{q,q'} \sum_{\tilde{q}} V_{\tilde{q},q} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_{\tilde{q}}^\dagger \hat{a}_{q'} \\
&= \sum_{q,q'} \sum_{\tilde{q}} V_{\tilde{q},q'} V_{q,\tilde{q}} (\omega_q - \omega_{\tilde{q}}) \hat{a}_q^\dagger \hat{a}_{q'} - \sum_{q,q'} \sum_{\tilde{q}} V_{q,\tilde{q}} V_{\tilde{q},q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} \\
&= \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q},q'} V_{q,\tilde{q}} (\omega_q - \omega_{\tilde{q}}) \hat{a}_q^\dagger \hat{a}_{q'} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{q,\tilde{q}} V_{\tilde{q},q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} \\
&+ \sum_k \sum_{\tilde{q}} V_{\tilde{q},k} V_{k,\tilde{q}} (\omega_k - \omega_{\tilde{q}}) \hat{a}_k^\dagger \hat{a}_{\tilde{q}} - \sum_k \sum_{\tilde{q}} V_{k,\tilde{q}} V_{\tilde{q},k} (\omega_{\tilde{q}} - \omega_k) \hat{a}_k^\dagger \hat{a}_{\tilde{q}} \\
&= \sum_{q \neq q'} \sum_{\tilde{q}} V_{\tilde{q},q'} V_{q,\tilde{q}} (\omega_q - \omega_{\tilde{q}}) \hat{a}_q^\dagger \hat{a}_{q'} - \sum_{q \neq q'} \sum_{\tilde{q}} V_{q,\tilde{q}} V_{\tilde{q},q'} (\omega_{\tilde{q}} - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} \\
&+ \sum_k \sum_{\tilde{q}} 2V_{\tilde{q},k} V_{k,\tilde{q}} (\omega_k - \omega_{\tilde{q}}) \hat{a}_k^\dagger \hat{a}_{\tilde{q}} \tag{A.9}
\end{aligned}$$

A.6:

$$\begin{aligned}
& \left[\sum_{q \neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} \right) \right] \\
&= \sum_{q \neq q'} \sum_{\tilde{p},\tilde{p}'} V_{q,q'} (\omega_q - \omega_{q'}) \left(W_{\tilde{p},\tilde{p}'} \left[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger \right] + W_{\tilde{p},\tilde{p}'}^* \left[\hat{a}_q^\dagger \hat{a}_{q'}, \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} \right] \right) \\
&= \sum_{q,q'} \sum_{\tilde{p},\tilde{p}'} V_{q,q'} (\omega_q - \omega_{q'}) \left(W_{\tilde{p},\tilde{p}'} \left(\hat{a}_q^\dagger \hat{a}_{\tilde{p}}^\dagger \delta_{q',\tilde{p}'} + \hat{a}_q^\dagger \hat{a}_{\tilde{p}'}^\dagger \delta_{q,\tilde{p}} \right) - W_{\tilde{p},\tilde{p}'}^* \left(\hat{a}_{\tilde{p}'} \hat{a}_{q'} \delta_{q,\tilde{p}} + \hat{a}_{\tilde{p}} \hat{a}_{q'} \delta_{\tilde{p}',q} \right) \right) \\
&= \sum_{p,p'} \sum_q V_{q,p'} (\omega_q - \omega_{p'}) W_{p,p'} \hat{a}_q^\dagger \hat{a}_p^\dagger + \sum_{p,p'} \sum_q V_{q,p} (\omega_q - \omega_p) W_{p,p'} \hat{a}_q^\dagger \hat{a}_{p'}^\dagger \\
&- \sum_{p,p'} \sum_{q'} V_{p,q'} (\omega_p - \omega_{q'}) W_{p,p'}^* \hat{a}_{p'} \hat{a}_{q'} - \sum_{p,p'} \sum_{q'} V_{p',q'} (\omega_{p'} - \omega_{q'}) W_{p,p'}^* \hat{a}_p \hat{a}_{q'} \\
&= \sum_{p,p'} \sum_q V_{p',q} (\omega_{p'} - \omega_q) W_{p,q} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + \sum_{p,p'} \sum_q V_{p,q} (\omega_p - \omega_q) W_{q,p'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \\
&- \sum_{p,p'} \sum_q V_{q,p} (\omega_q - \omega_p) W_{q,p}^* \hat{a}_p \hat{a}_{p'} - \sum_{p,p'} \sum_q V_{q,p'} (\omega_q - \omega_{p'}) W_{p,q}^* \hat{a}_p \hat{a}_{p'} \\
&= \sum_{p,p'} \sum_q V_{p',q} (\omega_{p'} - \omega_q) W_{p,q} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + \sum_{p,p'} \sum_q V_{p,q} (\omega_p - \omega_q) W_{q,p'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \\
&- \sum_{p,p'} \sum_q V_{q,p} (\omega_q - \omega_p) W_{q,p}^* \hat{a}_p \hat{a}_{p'} - \sum_{p,p'} \sum_q V_{q,p'} (\omega_q - \omega_{p'}) W_{p,q}^* \hat{a}_p \hat{a}_{p'} \\
&= \sum_{p,p'} \sum_q V_{p,q} (\omega_p - \omega_q) (W_{q,p} + W_{p',q}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger
\end{aligned}$$

$$+ \sum_{p,p'} \sum_q V_{q,p}(\omega_p - \omega_q)(W_{q,p'}^* + W_{p',q}^*)\hat{a}_p\hat{a}_{p'} \quad (\text{A.10})$$

A.7:

$$\begin{aligned} & \left[\sum_{p,p'} \left(W_{p,p'}(\omega_p + \omega_{p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - W_{p,p'}^*(\omega_p + \omega_{p'})\hat{a}_p\hat{a}_{p'} \right), \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'}\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}^\dagger \right] \\ &= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} V_{\tilde{q},\tilde{q}'}(\omega_p + \omega_{p'}) \left(W_{p,p'} \left[\hat{a}_p^\dagger\hat{a}_{p'}^\dagger, \hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}^\dagger \right] - W_{p,p'}^* \left[\hat{a}_p\hat{a}_{p'}, \hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}^\dagger \right] \right) \\ &= - \sum_{p,p'} \sum_{q \neq q'} V_{q,q'}(\omega_p + \omega_{p'}) W_{p,p'} \left(\hat{a}_q^\dagger\hat{a}_p^\dagger\delta_{q',p'} + \hat{a}_q^\dagger\hat{a}_{p'}^\dagger\delta_{q',p} \right) \\ &\quad - \sum_{p,p'} \sum_{q \neq q'} V_{q,q'}(\omega_p + \omega_{p'}) W_{p,p'}^* \left(\hat{a}_p\hat{a}_{q'}\delta_{q,p'} + \hat{a}_{p'}\hat{a}_{q'}\delta_{q,p} \right) \\ &= - \sum_{p,p'} \sum_q V_{q,p'}(\omega_p + \omega_{p'}) W_{p,p'}\hat{a}_q^\dagger\hat{a}_p^\dagger - \sum_{p,p'} \sum_q V_{q,p}(\omega_p + \omega_{p'}) W_{p,p'}\hat{a}_q^\dagger\hat{a}_{p'}^\dagger \\ &\quad - \sum_{p,p'} \sum_{q'} V_{p',q'}(\omega_p + \omega_{p'}) W_{p,p'}^*\hat{a}_p\hat{a}_{q'} - \sum_{p,p'} \sum_{q'} V_{p,q'}(\omega_p + \omega_{p'}) W_{p,p'}^*\hat{a}_{p'}\hat{a}_{q'} \\ &= - \sum_{p,p'} \sum_q V_{p',q}(\omega_p + \omega_q) W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - \sum_{p,p'} \sum_q V_{p,q}(\omega_q + \omega_{p'}) W_{q,p}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\ &\quad - \sum_{p,p'} \sum_{q'} V_{q',p'}(\omega_p + \omega_{q'}) W_{p,q'}^*\hat{a}_p\hat{a}_{p'} - \sum_{p,p'} \sum_{q'} V_{q',p}(\omega_{q'} + \omega_{p'}) W_{q',p}^*\hat{a}_{p'}\hat{a}_{p'} \\ &= - \sum_{p,p'} \sum_q V_{p,q}(\omega_q + \omega_{p'}) (W_{p',q} + W_{q,p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger \\ &\quad - \sum_{p,p'} \sum_q V_{q,p}(\omega_q + \omega_{p'}) (W_{p',q}^* + W_{q,p'}^*)\hat{a}_p\hat{a}_{p'} \quad (\text{A.11}) \end{aligned}$$

A.8:

$$\begin{aligned} & \left[\sum_{p,p'} \left(W_{p,p'}(\omega_p + \omega_{p'})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger - W_{p,p'}^*(\omega_p + \omega_{p'})\hat{a}_p\hat{a}_{p'} \right), \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'} \right) \right] \\ &= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p + \omega_{p'}) W_{\tilde{p},\tilde{p}'}^* \left[\hat{a}_p^\dagger\hat{a}_{p'}^\dagger, \hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'} \right] - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}^* W_{\tilde{p},\tilde{p}'}(\omega_p + \omega_{p'}) \left[\hat{a}_p\hat{a}_{p'}, \hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger \right] \\ &= - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{\tilde{p},\tilde{p}'}^* \left[\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}^\dagger, \hat{a}_p^\dagger\hat{a}_{p'}^\dagger \right] \\ &= - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}}\hat{a}_p^\dagger\delta_{\tilde{p}',p'} \\ &\quad - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}}\hat{a}_{p'}^\dagger\delta_{\tilde{p}',p} \\ &\quad - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} W_{p,p'}(\omega_p + \omega_{p'} + \omega_{\tilde{p}} + \omega_{\tilde{p}'}) W_{\tilde{p},\tilde{p}'}^* \hat{a}_{p'}^\dagger\hat{a}_{\tilde{p}'}\delta_{\tilde{p},p} \\ &= - \sum_{p,p'} \sum_{\tilde{p}} W_{p,p'}(\omega_p + 2\omega_{p'} + \omega_{\tilde{p}}) W_{\tilde{p},p'}^* \hat{a}_{\tilde{p}}\hat{a}_p^\dagger - \sum_{p,p'} \sum_{\tilde{p}} W_{p,p'}(2\omega_p + \omega_{p'} + \omega_{\tilde{p}}) W_{\tilde{p},p}^* \hat{a}_{\tilde{p}}\hat{a}_{p'}^\dagger \\ &\quad - \sum_{p,p'} \sum_{\tilde{p}'} W_{p,p'}(\omega_p + 2\omega_{p'} + \omega_{\tilde{p}'}) W_{p',\tilde{p}}^* \hat{a}_p^\dagger\hat{a}_{\tilde{p}'} - \sum_{p,p'} \sum_{\tilde{p}'} W_{p,p'}(2\omega_p + \omega_{p'} + \omega_{\tilde{p}'}) W_{p,\tilde{p}'}^* \hat{a}_{p'}^\dagger\hat{a}_{\tilde{p}'} \\ &= - \sum_{p,p'} \sum_{\tilde{p}} W_{p,\tilde{p}}(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{p',\tilde{p}}^* \hat{a}_{p'}^\dagger\hat{a}_p^\dagger - \sum_{p,p'} \sum_{\tilde{p}} W_{\tilde{p},p}(2\omega_{\tilde{p}} + \omega_p + \omega_{p'}) W_{p',\tilde{p}}^* \hat{a}_{p'}^\dagger\hat{a}_p^\dagger \end{aligned}$$

$$\begin{aligned}
& - \sum_{p,p'} \sum_{\tilde{p}'} W_{p,\tilde{p}'}(\omega_p + 2\omega_{\tilde{p}'} + \omega_{p'}) W_{\tilde{p}',p}^* \hat{a}_p^\dagger \hat{a}_{p'} - \sum_{p,p'} \sum_{\tilde{p}'} W_{\tilde{p}',p}(2\omega_{\tilde{p}'} + \omega_p + \omega_{p'}) W_{\tilde{p}',p}^* \hat{a}_p^\dagger \hat{a}_{p'} \\
& = - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{p',\tilde{p}}^* \hat{a}_{p'}^\dagger \hat{a}_p^\dagger \\
& - \sum_{p,p'} \sum_{\tilde{p}'} (W_{p,\tilde{p}'} + W_{\tilde{p}',p})(\omega_p + 2\omega_{\tilde{p}'} + \omega_{p'}) W_{\tilde{p}',p}^* \hat{a}_p^\dagger \hat{a}_{p'} \\
& = - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{p',\tilde{p}}^* (\delta_{p,p'} + \hat{a}_p^\dagger \hat{a}_{p'}) \\
& - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) W_{\tilde{p},p'}^* \hat{a}_p^\dagger \hat{a}_{p'} \\
& = - \sum_{p,p'} \sum_{\tilde{p}} (W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'}) (W_{\tilde{p},p'}^* + W_{p',\tilde{p}}^*) \hat{a}_p^\dagger \hat{a}_{p'} \\
& - 2 \sum_k \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) W_{k,\tilde{p}}^* \\
& = - \sum_{q \neq q'} \sum_{\tilde{p}} (W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + 2\omega_{\tilde{p}} + \omega_{q'}) (W_{\tilde{p},q'}^* + W_{q',\tilde{p}}^*) \hat{a}_q^\dagger \hat{a}_{q'} \\
& - 2 \sum_k \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) (W_{\tilde{p},k}^* + W_{k,\tilde{p}}^*) \hat{a}_k^\dagger \hat{a}_k \\
& - 2 \sum_k \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) W_{k,\tilde{p}}^* \tag{A.12}
\end{aligned}$$

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$\hat{\mathcal{H}}(\lambda) = \sum_k \omega_k(\lambda) \hat{a}_k^\dagger \hat{a}_k + \sum_{q \neq q'} V_{q,q'}(\lambda) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} (W_{p,p'}(\lambda) \hat{a}_p^\dagger \hat{a}_{p'} + W_{p,p'}^*(\lambda) \hat{a}_p \hat{a}_{p'}) + \epsilon(\lambda) \tag{A.13}$$

where $\epsilon(\lambda)$ is a constant shift in the energy scale.

Using the expressions for the commutators of the generator and $\hat{\mathcal{H}}_0$ respectively $\hat{\mathcal{H}}_{\text{int}}$ derived above, the flow $\partial_\lambda \hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ yields the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$\partial_\lambda \omega_k = \sum_{\tilde{q}} 2V_{\tilde{q},k} V_{k,\tilde{q}} (\omega_k - \omega_{\tilde{q}}) - 2 \sum_{\tilde{p}} (W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}}) (W_{\tilde{p},k}^* + W_{k,\tilde{p}}^*) \tag{A.14a}$$

$$\begin{aligned}
\partial_\lambda V_{q,q'} &= -V_{q,q'}(\omega_q - \omega_{q'})^2 - \sum_{\tilde{p}} (W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + \omega_{q'} + 2\omega_{\tilde{p}}) (W_{\tilde{p},q'}^* + W_{q',\tilde{p}}^*) \\
&+ \sum_{\tilde{q}} V_{\tilde{q},q'} V_{q,\tilde{q}} (\omega_q + \omega_{q'} - 2\omega_{\tilde{q}}) \tag{A.14b}
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda W_{p,p'} &= -W_{p,p'}(\omega_p + \omega_{p'})^2 - \sum_q V_{p,q}(\omega_q + \omega_{p'}) (W_{p',q} + W_{q,p'}) \\
&+ \sum_q V_{p,q}(\omega_p - \omega_q) (W_{q,p'} + W_{p',q}) \tag{A.14c}
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda W_{p,p'}^* &= -W_{p,p'}^*(\omega_p + \omega_{p'})^2 - \sum_q V_{q,p}(\omega_q + \omega_{p'}) (W_{p',q}^* + W_{q,p'}^*) \\
&+ \sum_q V_{q,p}(\omega_p - \omega_q) (W_{q,p'}^* + W_{p',q}^*) \tag{A.14d}
\end{aligned}$$

$$\partial_\lambda \epsilon = -2 \sum_{p,p'} (W_{p,p'} + W_{p',p})(\omega_p + \omega_{p'}) W_{p,p'}^* \tag{A.14e}$$

Obviously, equations A.14c and A.14d are not independent from each other, since they are related by complex conjugation. Seeing this is a good consistency check because complex

conjugation was not explicitly used in the derivation of these two equations.

Furthermore note that in first order the flow equations A.14a-A.14e suggest that they are exact in the sense that if the flow is completely traversed, the flow Hamiltonian will be exactly diagonal.

A.2 Deriving the flow equations with n-dependence

A.2.1 Useful preliminaries

Consider some operator \hat{f} which depends on a number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. The following relations will be used later:

$$[\hat{a}^\dagger, \hat{f}(\hat{n})] = \hat{a}^\dagger (\hat{f}(\hat{n}) - \hat{f}(\hat{n} + 1)) \quad (\text{A.15a})$$

$$[\hat{a}, \hat{f}(\hat{n})] = \hat{a} (\hat{f}(\hat{n}) - \hat{f}(\hat{n} - 1)) \quad (\text{A.15b})$$

$$[\hat{f}(\hat{n}), \hat{a}^\dagger] = (\hat{f}(\hat{n}) - \hat{f}(\hat{n} - 1)) \hat{a}^\dagger \quad (\text{A.15c})$$

$$[\hat{f}(\hat{n}), \hat{a}] = (\hat{f}(\hat{n}) - \hat{f}(\hat{n} + 1)) \hat{a} \quad (\text{A.15d})$$

These can be proved by induction for $\hat{f}(\hat{n}) = \hat{n}^k, k \in \mathbb{N}$ and from there simply extended to well-behaved \hat{f} via power series. Equations A.15 are still valid for functions depending on $\{\hat{n}_k\}_k$, because all \hat{n}_k pairwise commute.

We will write $\hat{f}(\hat{n}_1, \hat{n}_2, \dots) =: \hat{f}$ and $\hat{f}(\hat{n}_1, \hat{n}_2, \dots, \hat{n}_k \pm 1, \hat{n}_{k+1}, \dots) =: \hat{f}(\hat{n}_k \pm 1)$. In this notation it is understood that $\hat{f}(\hat{n}_k \pm 1, \hat{n}_k \pm 1) =: \hat{f}(\hat{n}_k \pm 2)$.

Using this notation, it is evident that a simple induction for $n_1, n_2 \in \mathbb{N}_0$ yields the following relation:

$$\begin{aligned} & [\hat{f}(\hat{n}), \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \cdots \hat{a}_{k_{n_1}}^\dagger \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}}] \\ &= \left(\hat{f} - \hat{f}(\hat{n}_{k_1} - 1, \hat{n}_{k_2} - 1, \dots, \hat{n}_{k_{n_1}}, \hat{n}_{k_1} + 1, \hat{n}_{k_2} + 1 \dots \hat{n}_{k_{n_2}} + 1) \right) \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \cdots \hat{a}_{k_{n_1}}^\dagger \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}} \end{aligned} \quad (\text{A.16})$$

Furthermore, applying the recurrence relation introduced to define the normal ordering procedure can be used to successively normal order operators. Let $\hat{O} := \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \cdots \hat{a}_{k_{n_1}}^\dagger \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}}$. Then normal ordering w.r.t. the vacuum yields:

$$\begin{aligned} \hat{a}_q : \hat{O} : &= \hat{O} \hat{a}_q + \sum_k : \frac{\partial \hat{O}}{\partial \hat{a}_k^\dagger} : \\ &= \hat{O} \hat{a}_q + \sum_{i=1}^{n_1} \delta_{k_i, q} : \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \cdots \hat{a}_{k_{i-1}}^\dagger \hat{a}_{k_{i+1}}^\dagger \cdots \hat{a}_{k_{n_1}}^\dagger \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_{n_2}} : \end{aligned} \quad (\text{A.17a})$$

$$\hat{a}_q^\dagger : \hat{O} : = \hat{a}_q^\dagger \hat{O} : \quad (\text{A.17b})$$

A.2.2 The canonical generator

The first step in calculating the flow equations is again to calculate the canonical commutator $\hat{\eta} := [\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}]$:

$$\hat{\eta} = \left[\sum_k \hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \sum_{q \neq q'} \hat{V}_{q, q'} : \hat{a}_q^\dagger \hat{a}_{q'} : + \sum_{p, p'} \left(\hat{W}_{p, p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : + \hat{W}_{p, p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right) \right]$$

$$= \sum_k \sum_{q \neq q'} \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \right] \quad (\text{A.18a})$$

$$+ \sum_k \sum_{p,p'} \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'} : \right] \quad (\text{A.18b})$$

$$+ \sum_k \sum_{p,p'} \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right] \quad (\text{A.18c})$$

In the following, the terms A.18a-A.18c will be evaluated separately:

A.18a

$$\begin{aligned} & \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \right] \\ &= \hat{\omega}_k \hat{V}_{q,q'} \left[: \hat{a}_k^\dagger \hat{a}_k : , : \hat{a}_q^\dagger \hat{a}_{q'} : \right] + \underbrace{\hat{\omega}_k \left[: \hat{a}_k^\dagger \hat{a}_k : , \hat{V}_{q,q'} : \right]}_{=0} : \hat{a}_q^\dagger \hat{a}_{q'} : \\ &+ \hat{V}_{q,q'} \left[\hat{\omega}_k : \hat{a}_q^\dagger \hat{a}_{q'} : \right] : \hat{a}_k^\dagger \hat{a}_k : + \underbrace{\left[\hat{\omega}_k , \hat{V}_{q,q'} \right]}_{=0} : \hat{a}_q^\dagger \hat{a}_{q'} : : \hat{a}_k^\dagger \hat{a}_k : \\ &= \hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'} \right) + \hat{V}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_q - 1, \hat{n}_{q'} + 1)) : \hat{a}_q^\dagger \hat{a}_{q'} : : \hat{a}_k^\dagger \hat{a}_k : \\ &= \hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'} \right) \\ &+ \hat{V}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_q - 1, \hat{n}_{q'} + 1)) \left(: \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : + \delta_{q',k} : \hat{a}_q^\dagger \hat{a}_k : \right) \end{aligned} \quad (\text{A.19})$$

A.18b

$$\begin{aligned} & \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'} : \right] \\ &= \hat{W}_{p,p'} \left[\hat{\omega}_k : \hat{a}_p^\dagger \hat{a}_{p'} : \right] : \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left[: \hat{a}_k^\dagger \hat{a}_k : , : \hat{a}_p^\dagger \hat{a}_{p'} : \right] \\ &= \hat{W}_{p,p'} (\omega_k - \hat{\omega}(\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'} : : \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'} : \delta_{k,p} \right) \\ &= \hat{W}_{p,p'} (\omega_k - \hat{\omega}(\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'} : \delta_{k,p} \right) \end{aligned} \quad (\text{A.20})$$

A.18c

$$\begin{aligned} & \left[\hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : , \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right] \\ &= \hat{W}_{p,p'}^\dagger \left[\hat{\omega}_k : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left[: \hat{a}_k^\dagger \hat{a}_k : , : \hat{a}_p \hat{a}_{p'} : \right] \\ &= \hat{W}_{p,p'}^\dagger (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p \hat{a}_{p'} : : \hat{a}_k^\dagger \hat{a}_k : - \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left(: \hat{a}_p \hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'} \hat{a}_k : \delta_{k,p} \right) \\ &= \hat{W}_{p,p'}^\dagger (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_{p'} + 1, \hat{n}_p + 1)) \left(: \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : + \delta_{k,p} : \hat{a}_{p'} \hat{a}_k : + \delta_{k,p'} : \hat{a}_p \hat{a}_k : \right) \\ &- \hat{W}_{p,p'}^\dagger \hat{\omega}_k \left(: \hat{a}_p \hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'} \hat{a}_k : \delta_{k,p} \right) \end{aligned} \quad (\text{A.21})$$

This gives the canonical generator as:

$$\begin{aligned} \hat{\eta} = & \sum_k \sum_{q \neq q'} \left(\hat{\omega}_k \hat{V}_{q,q'} \left(: \hat{a}_k^\dagger \hat{a}_{q'} : \delta_{k,q} - : \hat{a}_q^\dagger \hat{a}_k : \delta_{k,q'} \right) \right. \\ & \left. + \hat{V}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_q - 1, \hat{n}_{q'} + 1)) \left(: \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : + \delta_{q',k} : \hat{a}_q^\dagger \hat{a}_k : \right) \right) \\ & + \sum_k \sum_{p,p'} \left(\hat{W}_{p,p'} (\hat{\omega}_k - \hat{\omega}_k (\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : + \hat{W}_{p,p'} \hat{\omega}_k \left(: \hat{a}_k^\dagger \hat{a}_p^\dagger : \delta_{k,p'} + : \hat{a}_k^\dagger \hat{a}_{p'} : \delta_{k,p} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_k \sum_{p,p'} \left(\hat{W}_{p,p'}^\dagger (\omega_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)) \left(: \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : + \delta_{k,p} : \hat{a}_{p'} \hat{a}_k : + \delta_{k,p'} : \hat{a}_p \hat{a}_k : \right) \right. \\
& \left. - \hat{W}_{p,p'}^\dagger \hat{\omega}_k : \hat{a}_p \hat{a}_k : \delta_{k,p'} + : \hat{a}_{p'} \hat{a}_k : \delta_{k,p} \right) \\
& = \sum_{q \neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1)) \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_{q \neq q'} \sum_k \hat{V}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)) : \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : \\
& + \sum_k \sum_{p,p'} \hat{W}_{p,p'} (\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : \\
& + \sum_{p,p'} \hat{W}_{p,p'} (\hat{\omega}_p + \hat{\omega}_{p'}) : \hat{a}_{p'}^\dagger \hat{a}_p^\dagger : \\
& + \sum_k \sum_{p,p'} \hat{W}_{p,p'}^\dagger (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : \\
& - \sum_{p,p'} \hat{W}_{p,p'}^\dagger (\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p \hat{a}_{p'} : \\
& =: \sum_{q \neq q'} \hat{\theta}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : + \sum_{p,p'} \left(\hat{\phi}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : + \hat{\psi}_{p,p'} : \hat{a}_p \hat{a}_{p'} : \right) \tag{A.22} \\
& + \sum_{q \neq q'} \sum_k \hat{V}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)) : \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k : \\
& + \sum_k \sum_{p,p'} \hat{W}_{p,p'} (\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : \\
& + \sum_k \sum_{p,p'} \hat{W}_{p,p'}^\dagger (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : \\
& =: \eta^{(2)} + \eta^{(4)} \tag{A.23}
\end{aligned}$$

Here $\eta^{(2)}$ is the part of $\hat{\eta}$ which contains only quadratic terms and $\eta^{(4)}$ with quartic terms. The latter will be neglected for now, i.e. we assume $\hat{\eta} \approx \hat{\eta}^{(2)}$ for which we write $\hat{\eta} \stackrel{(2)}{=} \hat{\eta}^{(2)}$ because equality holds up to second order. The accuracy of this approximation will vary from model to model and must always be justified in a specific situation.

A.2.3 Evaluating the commutator of the generator with the Hamiltonian

If one notices that $\eta^{(2)}$ is structurally identical to $\hat{\mathcal{H}}_{\text{int}}$, the commutator of $\hat{\mathcal{H}}_0$ and $\eta^{(2)}$ can be written down immediately:

$$\begin{aligned}
[\eta^{(2)}, \hat{\mathcal{H}}_0] & = - \sum_{q \neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1)) \hat{\theta}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& - \sum_{p,p'} \hat{\phi}_{p,p'} (\hat{\omega}_p + \hat{\omega}_{p'}) : \hat{a}_{p'}^\dagger \hat{a}_p^\dagger : \\
& + \sum_{p,p'} \hat{\psi}_{p,p'} (\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p \hat{a}_{p'} : \\
& - \sum_k \sum_{p,p'} \hat{\psi}_{p,p'} (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_k^\dagger \hat{a}_k \hat{a}_p \hat{a}_{p'} : \\
& - \sum_{q \neq q'} \sum_k \hat{\theta}_{q,q'} (\hat{\omega}_k - \hat{\omega}_k(\hat{n}_q - 1, \hat{n}_{q'} + 1)) : \hat{a}_q^\dagger \hat{a}_k^\dagger \hat{a}_{q'} \hat{a}_k :
\end{aligned}$$

$$\begin{aligned}
& - \sum_k \sum_{p,p'} \hat{\phi}_{p,p'} (\omega_k - \hat{\omega}_k(\hat{n}_{p'} - 1, \hat{n}_p - 1)) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_k^\dagger \hat{a}_k : \\
& \stackrel{\textcircled{2}}{=} - \sum_{q \neq q'} (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1)) \hat{\theta}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& - \sum_{p,p'} \hat{\phi}_{p,p'} (\hat{\omega}_p + \hat{\omega}_{p'}) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\
& + \sum_{p,p'} \hat{\psi}_{p,p'} (\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) : \hat{a}_p \hat{a}_{p'} :
\end{aligned} \tag{A.24}$$

The commutator of $\hat{\mathcal{H}}_{\text{int}}$ and $\eta^{(2)}$ requires significantly more work:

$$[\eta^{(2)}, \hat{\mathcal{H}}_0] = \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} [\hat{\theta}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :] \tag{A.25a}$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} [\hat{\theta}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :] \tag{A.25b}$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} [\hat{\theta}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :] \tag{A.25c}$$

$$+ \sum_{q \neq q'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\phi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :] \tag{A.25d}$$

$$+ \sum_{p,p'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\phi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :] \tag{A.25e}$$

$$+ \sum_{p,p'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\phi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :] \tag{A.25f}$$

$$+ \sum_{q \neq q'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\psi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :] \tag{A.25g}$$

$$+ \sum_{p,p'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\psi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :] \tag{A.25h}$$

$$+ \sum_{p,p'} \sum_{\tilde{p}, \tilde{p}'} [\hat{\psi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :] \tag{A.25i}$$

For the sake of clarity, the terms A.25a-A.25i will again be evaluated one by one.

A.25a:

$$\begin{aligned}
& \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} [\hat{\theta}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :] \\
& = \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q}, \tilde{q}'} \hat{V}_{q,q'} [: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, : \hat{a}_q^\dagger \hat{a}_{q'} :] \\
& + \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{V}_{q,q'} [\hat{\theta}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :] : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q}, \tilde{q}'} [: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :, \hat{V}_{q,q'}] : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& = \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q}, \tilde{q}'} \hat{V}_{q,q'} (\delta_{\tilde{q}', q} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_q : - \delta_{\tilde{q}, q'} : \hat{a}_{\tilde{q}'}^\dagger \hat{a}_{q'} :) \\
& + \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{V}_{q,q'} (\hat{\theta}_{\tilde{q}, \tilde{q}'} - \hat{\theta}_{\tilde{q}, \tilde{q}'}(\hat{n}_{q'} + 1, \hat{n}_q - 1)) : \hat{a}_q^\dagger \hat{a}_{q'} : : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger : \\
& \quad \quad \quad \stackrel{\textcircled{2}}{=} \delta_{q', \tilde{q}} : \hat{a}_q^\dagger \hat{a}_{\tilde{q}'}^\dagger : \\
& - \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{q, q'} [\hat{V}_{\tilde{q}, \tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}^\dagger :] : \hat{a}_q^\dagger \hat{a}_{q'} :
\end{aligned} \tag{A.26}$$

$$\begin{aligned}
& \stackrel{(2)}{=} \sum_{q \neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{\tilde{q},q} \hat{V}_{q,q'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_q : - \hat{\theta}_{q',\tilde{q}} \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{\tilde{q}} : \right) \\
& + \sum_{q \neq q'} \sum_{\tilde{q}} \hat{V}_{q,q'} \left(\hat{\theta}_{q',\tilde{q}} - \hat{\theta}_{q',\tilde{q}} (\hat{n}_{q'} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{\tilde{q}} : \\
& - \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, : \hat{a}_q^\dagger \hat{a}_{q'} : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \\
& = \sum_{q,q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} : \hat{a}_q^\dagger \hat{a}_{q'} : - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} : \hat{a}_q^\dagger \hat{a}_{q'} : \right) \\
& - \underbrace{\sum_k \sum_{\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} \hat{V}_{k,k} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_k : - \hat{\theta}_{k,\tilde{q}} \hat{V}_{k,k} : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : \right)}_{=0} \\
& + \sum_{q,q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'} (\hat{n}_{\tilde{q}} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& - \underbrace{\sum_k \sum_{\tilde{q}} \hat{V}_{k,k} \left(\hat{\theta}_{k,\tilde{q}} - \hat{\theta}_{k,\tilde{q}} (\hat{n}_k + 1, \hat{n}_{\tilde{q}} - 1) \right) : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} :}_{=0}
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
& - \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, : \hat{a}_q^\dagger \hat{a}_{q'} : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \\
& = \sum_{q \neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} : \hat{a}_q^\dagger \hat{a}_{q'} : - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} : \hat{a}_q^\dagger \hat{a}_{q'} : \right) \\
& + \sum_k \sum_{\tilde{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\tilde{q}} : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : - \hat{\theta}_{\tilde{q},k} \hat{V}_{k,\tilde{q}} : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : \right) \\
& + \sum_{q \neq q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'} (\hat{n}_{\tilde{q}} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_k \sum_{\tilde{q}} \hat{V}_{k,\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k} (\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : \\
& - \sum_{q \neq q'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, : \hat{a}_q^\dagger \hat{a}_{q'} : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
& = \sum_{q \neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_k \sum_{\tilde{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\tilde{q}} - \hat{\theta}_{\tilde{q},k} \hat{V}_{k,\tilde{q}} \right) : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : \\
& + \sum_{q \neq q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'} (\hat{n}_{\tilde{q}} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_k \sum_{\tilde{q}} \hat{V}_{k,\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k} (\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} : \\
& - \sum_{q \neq q'} \sum_{\tilde{q}} \hat{\theta}_{q,\tilde{q}} \left(\hat{V}_{\tilde{q},q'} - \hat{V}_{\tilde{q},q'} (\hat{n}_{\tilde{q}} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& - \sum_k \sum_{\tilde{q}} \hat{\theta}_{k,\tilde{q}} \left(\hat{V}_{\tilde{q},k} - \hat{V}_{\tilde{q},k} (\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) : \hat{a}_k^\dagger \hat{a}_{\tilde{q}} :
\end{aligned} \tag{A.29}$$

A.25b:

$$\begin{aligned}
& \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'} : \right] \\
& = \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , : \hat{a}_p^\dagger \hat{a}_{p'} : \right]
\end{aligned} \tag{A.30a}$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \quad (\text{A.30b})$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \quad (\text{A.30c})$$

We start by evaluating A.30a:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \\ &= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'} \left(\delta_{q',p'} : \hat{a}_q^\dagger \hat{a}_p^\dagger : + \delta_{q',p} : \hat{a}_q^\dagger \hat{a}_{p'}^\dagger : \right) \\ &= \sum_{p,p'} \sum_q \hat{\theta}_{q,p'} \hat{W}_{p,p'} : \hat{a}_q^\dagger \hat{a}_p^\dagger : + \sum_{p,p'} \sum_q \hat{\theta}_{q,p} \hat{W}_{p,p'} : \hat{a}_q^\dagger \hat{a}_{p'}^\dagger : \\ &= \sum_{p,p'} \sum_q \left(\hat{\theta}_{p',q} \hat{W}_{p,q} + \hat{\theta}_{p,q} \hat{W}_{q,p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \end{aligned} \quad (\text{A.31})$$

Next is A.30b:

$$\begin{aligned} & \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \left[: \hat{a}_q^\dagger \hat{a}_{q'} : , \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\ &= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \left(\hat{W}_{p,p'} (\hat{n}_{q'} + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) : \underbrace{\hat{a}_q^\dagger \hat{a}_{q'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :}_{\stackrel{\textcircled{2}}{=} \delta_{q',p} : \hat{a}_p^\dagger \hat{a}_q^\dagger : + \delta_{q',p'} : \hat{a}_p^\dagger \hat{a}_q^\dagger :} \\ &\stackrel{\textcircled{2}}{=} \sum_{p,p'} \sum_q \hat{\theta}_{q,p} \left(\hat{W}_{p,p'} (\hat{n}_p + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) : \hat{a}_p^\dagger \hat{a}_q^\dagger : \\ &+ \sum_{p,p'} \sum_q \hat{\theta}_{q,p'} \left(\hat{W}_{p,p'} (\hat{n}_{p'} + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) : \hat{a}_p^\dagger \hat{a}_q^\dagger : \\ &= \sum_{p,p'} \sum_q \hat{\theta}_{p,q} \left(\hat{W}_{q,p'} (\hat{n}_q + 1, \hat{n}_p - 1) - \hat{W}_{q,p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\ &+ \sum_{p,p'} \sum_q \hat{\theta}_{p',q} \left(\hat{W}_{p,q} (\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{W}_{p,q} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \end{aligned} \quad (\text{A.32})$$

A.30c gives no quadratic contribution:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \\ &= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left(\hat{\theta}_{\tilde{q},\tilde{q}'} - \hat{\theta}_{\tilde{q},\tilde{q}'} (\hat{n}_{p'} + 1, \hat{n}_p - 1) \right) : \underbrace{\hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :}_{= : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :} \stackrel{\textcircled{2}}{=} 0 \end{aligned} \quad (\text{A.33})$$

A.25c:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right] \\ &= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , : \hat{a}_p \hat{a}_{p'} : \right] \end{aligned} \quad (\text{A.34a})$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , \hat{W}_{p,p'}^\dagger \right] : \hat{a}_p \hat{a}_{p'} : \quad (\text{A.34b})$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'}^\dagger \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \quad (\text{A.34c})$$

We will again start by evaluating A.34a:

$$\begin{aligned}
& \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_q^\dagger \hat{a}_{q'} : , : \hat{a}_p \hat{a}_{p'} : \right] \\
&= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'}^\dagger (\delta_{q,p'} : \hat{a}_p \hat{a}_{q'} : + \delta_{q,p} : \hat{a}_{p'} \hat{a}_{q'} :) \\
&= \sum_{p,p'} \sum_{q'} \hat{\theta}_{p',q'} \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{q'} : + \sum_{p,p'} \sum_{q'} \hat{\theta}_{p,q'} \hat{W}_{p,p'}^\dagger : \hat{a}_{p'} \hat{a}_{q'} : \\
&= \sum_{p,p'} \sum_q \left(\hat{\theta}_{q,p'} \hat{W}_{p,q}^\dagger + \hat{\theta}_{q,p} \hat{W}_{q,p'}^\dagger \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.35}
\end{aligned}$$

A.34b gives no quadratic contribution:

$$\begin{aligned}
& \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : , : \hat{W}_{p,p'}^\dagger : \right] : \hat{a}_p \hat{a}_{p'} : \\
&= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left(W_{p,p'}^\dagger (\hat{n}_{q'} + 1, \hat{n}_q - 1) - W_{p,p'}^\dagger \right) : \underbrace{\hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : : \hat{a}_p \hat{a}_{p'} :}_{=: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \hat{a}_p \hat{a}_{p'} :} \stackrel{\textcircled{2}}{=} 0 \tag{A.36}
\end{aligned}$$

A.34c:

$$\begin{aligned}
& \sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left[\hat{\theta}_{q,q'} , : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_q^\dagger \hat{a}_{q'} : \\
&= \sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'} (\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{p'} : : \hat{a}_q^\dagger \hat{a}_{q'} : \tag{A.37}
\end{aligned}$$

$$\begin{aligned}
& \stackrel{\textcircled{2}}{=} \sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'} (\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) (\delta_{p,q} : \hat{a}_{p'} \hat{a}_{q'} : + \delta_{p',q} : \hat{a}_p \hat{a}_{q'} :) \\
&= \sum_{p,p'} \sum_{q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{p,q'} - \hat{\theta}_{p,q'} (\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_{p'} \hat{a}_{q'} : \tag{A.38}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p,p'} \sum_{q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{p',q'} - \hat{\theta}_{p',q'} (\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{q'} : \\
&= \sum_{p,p'} \sum_q \hat{W}_{q,p'}^\dagger \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p} (\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.39}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p,p'} \sum_q \hat{W}_{p,q}^\dagger \left(\hat{\theta}_{q,p'} - \hat{\theta}_{q,p'} (\hat{n}_q + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{p'} : \\
&= \sum_{p,p'} \sum_q \left(\hat{W}_{q,p'}^\dagger + \hat{W}_{p',q}^\dagger \right) \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p} (\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.40}
\end{aligned}$$

A.25d: Follows immediately from the calculations already done for A.25b:

$$\sum_{q \neq q'} \sum_{\tilde{p}, \tilde{p}'} \left[\hat{\phi}_{\tilde{p}, \tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \right] \tag{A.41a}$$

$$= - \sum_{q \neq q'} \sum_{p,p'} \left[\hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : , \hat{\phi}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \tag{A.41b}$$

$$= - \sum_{p,p'} \sum_q \left(\hat{V}_{p',q} \hat{\phi}_{p,q} + \hat{V}_{p,q} \hat{\phi}_{q,p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \tag{A.41c}$$

$$- \sum_{p,p'} \sum_q \hat{V}_{p,q} \left(\hat{\phi}_{q,p'} (\hat{n}_q + 1, \hat{n}_p - 1) - \hat{\phi}_{q,p} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :$$

$$- \sum_{p,p'} \sum_q \hat{V}_{p',q} \left(\hat{\phi}_{p,q} (\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{\phi}_{p,q} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :$$

A.25e:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \\ &= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \end{aligned} \quad (\text{A.42a})$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \quad (\text{A.42b})$$

A.42a will be analyzed first:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\ &= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}_{p,p'} (\hat{n}_{\tilde{p}} + 1, \hat{n}_{\tilde{p}'} + 1) - \hat{W}_{p,p'} \right) : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \stackrel{\textcircled{2}}{=} 0 \end{aligned} \quad (\text{A.43})$$

Similiarly, A.42b also gives no quadratic contribution.

A.25f

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right] \\ &= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , : \hat{a}_p \hat{a}_{p'} : \right] \end{aligned} \quad (\text{A.44a})$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'}^\dagger \right] : \hat{a}_p \hat{a}_{p'} : \quad (\text{A.44b})$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} : \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \quad (\text{A.44c})$$

A.44a:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\delta_{p',\tilde{p}'} \hat{a}_p \hat{a}_{\tilde{p}}^\dagger + \delta_{p',\tilde{p}} \hat{a}_p \hat{a}_{\tilde{p}'}^\dagger + \delta_{p,\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} + \delta_{p,\tilde{p}} \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} \right) \\ &= - \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p',\tilde{p}'} \left(: \hat{a}_{\tilde{p}}^\dagger \hat{a}_p : + \delta_{p,\tilde{p}} \right) \end{aligned} \quad (\text{A.45})$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p',\tilde{p}} \left(: \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p : + \delta_{\tilde{p}',p} \right)$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p,\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \delta_{p,\tilde{p}} : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} :$$

$$= - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}}^\dagger \hat{a}_p : \quad (\text{A.46})$$

$$- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p',\tilde{p}'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p :$$

$$- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p,\tilde{p}'} \hat{W}_{p,p'}^\dagger : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} :$$

$$- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}_{p,p'}^\dagger + \hat{\phi}_{p',p} \hat{W}_{p,p'}^\dagger \right)$$

$$= - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{p,\tilde{p}} \hat{W}_{p',\tilde{p}}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} : \quad (\text{A.47})$$

$$\begin{aligned} & - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}_{p',\tilde{p}}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} : \\ & - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{p,\tilde{p}} \hat{W}_{\tilde{p},p'}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} : \\ & - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}_{\tilde{p},p'}^\dagger : \hat{a}_p^\dagger \hat{a}_{p'} : \\ & - \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}_{p,p'}^\dagger + \hat{\phi}_{p',p} \hat{W}_{p,p'}^\dagger \right) \\ & = - \sum_{p,p'} \sum_{\tilde{p}} \left(\hat{\phi}_{p,\tilde{p}} \hat{W}_{p',\tilde{p}}^\dagger + \hat{\phi}_{\tilde{p},p} \hat{W}_{p',\tilde{p}}^\dagger + \hat{\phi}_{p,\tilde{p}} \hat{W}_{\tilde{p},p'}^\dagger + \hat{\phi}_{\tilde{p},p} \hat{W}_{\tilde{p},p'}^\dagger \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \quad (\text{A.48}) \\ & - \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}_{p,p'}^\dagger + \hat{\phi}_{p',p} \hat{W}_{p,p'}^\dagger \right) \end{aligned}$$

A.44b gives no quadratic contribution:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : , \hat{W}_{p,p'}^\dagger \right] : \hat{a}_p \hat{a}_{p'} : \\ & = \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}_{p,p'}^\dagger (\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}_{p,p'}^\dagger \right) : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : : \hat{a}_p \hat{a}_{p'} : \\ & = \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}_{p,p'}^\dagger (\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}_{p,p'}^\dagger \right) : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p \hat{a}_{p'} : \stackrel{\textcircled{2}}{=} 0 \quad (\text{A.49}) \end{aligned}$$

A.44c:

$$\begin{aligned} & \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left[\hat{\phi}_{\tilde{p},\tilde{p}'} , : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : \\ & = \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_p \hat{a}_{p'} : : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger : \quad (\text{A.50}) \end{aligned}$$

$$\stackrel{\textcircled{2}}{=} \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p,\tilde{p}} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} : \quad (\text{A.51})$$

$$\begin{aligned} & + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p,\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} : \\ & + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p',\tilde{p}} : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p : \\ & + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p',\tilde{p}'} : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p : \\ & + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p',\tilde{p}} \delta_{p,\tilde{p}'} \\ & + \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p',\tilde{p}'} \delta_{p,\tilde{p}} \\ & = \sum_{p,p'} \sum_{\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p,\tilde{p}'} - \hat{\phi}_{p,\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_{p'} : \quad (\text{A.52}) \\ & + \sum_{p,p'} \sum_{\tilde{p}} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\tilde{p},p} - \hat{\phi}_{\tilde{p},p} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{p'} : \\ & + \sum_{p,p'} \sum_{\tilde{p}'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p',\tilde{p}'} - \hat{\phi}_{p',\tilde{p}'} (\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_{\tilde{p}'}^\dagger \hat{a}_p : \end{aligned}$$

$$\begin{aligned}
& + \sum_{p,p'} \sum_{\bar{p}} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{\bar{p},p'} - \hat{\phi}_{\bar{p},p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_{\bar{p}}^\dagger \hat{a}_p : \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& = \sum_{p,p'} \sum_{\bar{p}} \hat{W}_{\bar{p},p'}^\dagger \left(\hat{\phi}_{\bar{p},p} - \hat{\phi}_{\bar{p},p}(\hat{n}_{\bar{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_{\bar{p}}^\dagger \hat{a}_{p'} : \\
& + \sum_{p,p'} \sum_{\bar{p}} \hat{W}_{\bar{p},p'}^\dagger \left(\hat{\phi}_{p,\bar{p}} - \hat{\phi}_{p,\bar{p}}(\hat{n}_{\bar{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& + \sum_{p,p'} \sum_{\bar{p}} \hat{W}_{p',\bar{p}}^\dagger \left(\hat{\phi}_{\bar{p},p} - \hat{\phi}_{\bar{p},p}(\hat{n}_{p'} + 1, \hat{n}_{\bar{p}} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& + \sum_{p,p'} \sum_{\bar{p}} \hat{W}_{p',\bar{p}}^\dagger \left(\hat{\phi}_{p,\bar{p}} - \hat{\phi}_{p,\bar{p}}(\hat{n}_{p'} + 1, \hat{n}_{\bar{p}} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)
\end{aligned} \tag{A.53}$$

A.25g: Follows immediately from A.25c:

$$\begin{aligned}
& \sum_{q \neq q'} \sum_{\bar{p}, \bar{p}'} \left[\hat{\psi}_{\bar{p}, \bar{p}'} : \hat{a}_{\bar{p}} \hat{a}_{\bar{p}'} : , \hat{V}_{q, q'} : \hat{a}_q^\dagger \hat{a}_{q'} : \right] \\
& = - \sum_{p, p'} \sum_{\bar{q} \neq \bar{q}'} \left[\hat{V}_{\bar{q}, \bar{q}'} : \hat{a}_{\bar{q}}^\dagger \hat{a}_{\bar{q}'} : , \hat{\psi}_{p, p'} : \hat{a}_p \hat{a}_{p'} : \right] \\
& \stackrel{\textcircled{2}}{=} - \sum_{p, p'} \sum_q \left(\hat{V}_{q, p'} \hat{\psi}_{p, q} + \hat{\theta}_{q, p} \hat{\psi}_{q, p'} \right) : \hat{a}_p \hat{a}_{p'} : \\
& - \sum_{p, p'} \sum_q \left(\hat{\psi}_{q, p'} + \hat{\psi}_{p', q} \right) \left(\hat{V}_{q, p} - \hat{V}_{q, p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} :
\end{aligned} \tag{A.54a}$$

A.25h Follows immediately from A.25f:

$$\begin{aligned}
& \sum_{p, p'} \sum_{\bar{p}, \bar{p}'} \left[\hat{\psi}_{\bar{p}, \bar{p}'} : \hat{a}_{\bar{p}} \hat{a}_{\bar{p}'} : , \hat{W}_{p, p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \\
& = - \sum_{p, p'} \sum_{\bar{p}, \bar{p}'} \left[\hat{W}_{\bar{p}, \bar{p}'} : \hat{a}_{\bar{p}}^\dagger \hat{a}_{\bar{p}'}^\dagger : , \hat{\psi}_{p, p'} : \hat{a}_p \hat{a}_{p'} : \right]
\end{aligned} \tag{A.55}$$

$$\begin{aligned}
& = \sum_{p, p'} \sum_{\bar{p}} \left(\hat{W}_{p, \bar{p}} \hat{\psi}_{p', \bar{p}} + \hat{W}_{\bar{p}, p} \hat{\psi}_{p', \bar{p}} + \hat{W}_{p, \bar{p}} \hat{\psi}_{\bar{p}, p'} + \hat{W}_{\bar{p}, p} \hat{\psi}_{\bar{p}, p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& + \sum_{p, p'} \left(\hat{W}_{p, p'} \hat{\psi}_{p, p'} + \hat{W}_{p', p} \hat{\psi}_{p, p'} \right) \\
& - \sum_{p, p'} \sum_{\bar{p}} \hat{\psi}_{\bar{p}, p'} \left(\hat{W}_{\bar{p}, p} - \hat{W}_{\bar{p}, p}(\hat{n}_{\bar{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& - \sum_{p, p'} \sum_{\bar{p}} \hat{\psi}_{\bar{p}, p'} \left(\hat{W}_{p, \bar{p}} - \hat{W}_{p, \bar{p}}(\hat{n}_{\bar{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& - \sum_{p, p'} \sum_{\bar{p}} \hat{\psi}_{p', \bar{p}} \left(\hat{W}_{\bar{p}, p} - \hat{W}_{\bar{p}, p}(\hat{n}_{p'} + 1, \hat{n}_{\bar{p}} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} : \\
& - \sum_{p, p'} \sum_{\bar{p}} \hat{\psi}_{p', \bar{p}} \left(\hat{W}_{p, \bar{p}} - \hat{W}_{p, \bar{p}}(\hat{n}_{p'} + 1, \hat{n}_{\bar{p}} + 1) \right) : \hat{a}_p^\dagger \hat{a}_{p'} :
\end{aligned} \tag{A.56}$$

$$\begin{aligned}
& - \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& - \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)
\end{aligned}$$

A.25i: Similiar to A.25e:

$$\sum_{p,p'} \sum_{\bar{p},\bar{p}'} \left[\hat{\psi}_{\bar{p},\bar{p}'} : \hat{a}_{\bar{p}} \hat{a}_{\bar{p}'}^\dagger : , \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'}^\dagger : \right] \stackrel{(2)}{=} 0 \quad (\text{A.57})$$

A.3 The flow equations

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$\begin{aligned}
\hat{\mathcal{H}}(\lambda) & \stackrel{(2)}{=} \sum_k \hat{\omega}_k(\lambda) : \hat{a}_k^\dagger \hat{a}_k : + \sum_{q \neq q'} \hat{V}_{q,q'}(\lambda) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
& + \sum_{p,p'} \left(\hat{W}_{p,p'}(\lambda) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : + \hat{W}_{p,p'}^\dagger(\lambda) : \hat{a}_p \hat{a}_{p'} : \right) + \epsilon(\lambda)
\end{aligned} \quad (\text{A.58})$$

where $\epsilon(\lambda)$ is a constant which indicates a shift in the energy scale.

Collecting all terms in $\partial_\lambda \hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ gives the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$\begin{aligned}
\partial_\lambda \hat{\omega}_k & \stackrel{(2)}{=} \sum_{\bar{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\bar{q}} - \hat{\theta}_{\bar{q},k} \hat{V}_{k,\bar{q}} \right) \\
& + \sum_{\bar{q}} \hat{V}_{k,\bar{q}} \left(\hat{\theta}_{\bar{q},k} - \hat{\theta}_{\bar{q},k}(\hat{n}_{\bar{q}} + 1, \hat{n}_k - 1) \right) \\
& - \sum_{\bar{q}} \hat{\theta}_{k,\bar{q}} \left(\hat{V}_{\bar{q},k} - \hat{V}_{\bar{q},k}(\hat{n}_{\bar{q}} + 1, \hat{n}_k - 1) \right) \\
& - \sum_{\bar{p}} \left(\hat{\phi}_{k,\bar{p}} \hat{W}_{k,\bar{p}}^\dagger + \hat{\phi}_{\bar{p},k} \hat{W}_{k,\bar{p}}^\dagger + \hat{\phi}_{k,\bar{p}} \hat{W}_{\bar{p},k}^\dagger + \hat{\phi}_{\bar{p},k} \hat{W}_{\bar{p},k}^\dagger \right) \\
& + \sum_{\bar{p}} \hat{W}_{\bar{p},k}^\dagger \left(\hat{\phi}_{\bar{p},k} - \hat{\phi}_{\bar{p},k}(\hat{n}_{\bar{p}} + 1, \hat{n}_k + 1) \right) \\
& + \sum_{\bar{p}} \hat{W}_{\bar{p},k}^\dagger \left(\hat{\phi}_{k,\bar{p}} - \hat{\phi}_{k,\bar{p}}(\hat{n}_{\bar{p}} + 1, \hat{n}_k + 1) \right) \\
& + \sum_{\bar{p}} \hat{W}_{k,\bar{p}}^\dagger \left(\hat{\phi}_{\bar{p},k} - \hat{\phi}_{\bar{p},k}(\hat{n}_k + 1, \hat{n}_{\bar{p}} + 1) \right) \\
& + \sum_{\bar{p}} \hat{W}_{k,\bar{p}}^\dagger \left(\hat{\phi}_{k,\bar{p}} - \hat{\phi}_{k,\bar{p}}(\hat{n}_k + 1, \hat{n}_{\bar{p}} + 1) \right) \\
& + \sum_{\bar{p}} \left(\hat{W}_{k,\bar{p}} \hat{\psi}_{k,\bar{p}} + \hat{W}_{\bar{p},k} \hat{\psi}_{k,\bar{p}} + \hat{W}_{k,\bar{p}} \hat{\psi}_{\bar{p},k} + \hat{W}_{\bar{p},k} \hat{\psi}_{\bar{p},k} \right) \\
& - \sum_{\bar{p}} \hat{\psi}_{\bar{p},k} \left(\hat{W}_{\bar{p},k} - \hat{W}_{\bar{p},k}(\hat{n}_{\bar{p}} + 1, \hat{n}_k + 1) \right) \\
& - \sum_{\bar{p}} \hat{\psi}_{\bar{p},k} \left(\hat{W}_{k,\bar{p}} - \hat{W}_{k,\bar{p}}(\hat{n}_{\bar{p}} + 1, \hat{n}_k + 1) \right) \\
& - \sum_{\bar{p}} \hat{\psi}_{k,\bar{p}} \left(\hat{W}_{\bar{p},k} - \hat{W}_{\bar{p},k}(\hat{n}_k + 1, \hat{n}_{\bar{p}} + 1) \right)
\end{aligned} \quad (\text{A.59a})$$

$$\begin{aligned}
& - \sum_{\bar{p}} \hat{\psi}_{k,\bar{p}} \left(\hat{W}_{k,\bar{p}} - \hat{W}_{k,\bar{p}}(\hat{n}_k + 1, \hat{n}_{\bar{p}} + 1) \right) \\
\partial_\lambda \hat{V}_{q,q'} & \stackrel{\textcircled{2}}{=} -(\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1))\hat{\theta}_{q,q'} \tag{A.59b} \\
& + \sum_{\bar{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\bar{q}} - \hat{\theta}_{\bar{q},q'} \hat{V}_{q,\bar{q}} \right) \\
& + \sum_{\bar{q}} \hat{V}_{q,\bar{q}} \left(\hat{\theta}_{\bar{q},q'} - \hat{\theta}_{\bar{q},q'}(\hat{n}_{\bar{q}} + 1, \hat{n}_q - 1) \right) \\
& - \sum_{\bar{q}} \hat{\theta}_{q,\bar{q}} \left(\hat{V}_{\bar{q},q'} - \hat{V}_{\bar{q},q'}(\hat{n}_{\bar{q}} + 1, \hat{n}_q - 1) \right) \\
& - \sum_{\bar{q}} \left(\hat{\phi}_{q,\bar{q}} \hat{W}_{q',\bar{q}}^\dagger + \hat{\phi}_{\bar{q},q} \hat{W}_{q',\bar{q}}^\dagger + \hat{\phi}_{q,\bar{q}} \hat{W}_{\bar{q},q'}^\dagger + \hat{\phi}_{\bar{q},q} \hat{W}_{\bar{q},q'}^\dagger \right) \\
& + \sum_{\bar{q}} \hat{W}_{\bar{q},q'}^\dagger \left(\hat{\phi}_{\bar{q},q} - \hat{\phi}_{\bar{q},q}(\hat{n}_{\bar{q}} + 1, \hat{n}_{q'} + 1) \right) \\
& + \sum_{\bar{q}} \hat{W}_{\bar{q},q'}^\dagger \left(\hat{\phi}_{q,\bar{q}} - \hat{\phi}_{q,\bar{q}}(\hat{n}_{\bar{q}} + 1, \hat{n}_{q'} + 1) \right) \\
& + \sum_{\bar{q}} \hat{W}_{q',\bar{q}}^\dagger \left(\hat{\phi}_{\bar{q},q} - \hat{\phi}_{\bar{q},q}(\hat{n}_{q'} + 1, \hat{n}_{\bar{q}} + 1) \right) \\
& + \sum_{\bar{q}} \hat{W}_{q',\bar{q}}^\dagger \left(\hat{\phi}_{q,\bar{q}} - \hat{\phi}_{q,\bar{q}}(\hat{n}_{q'} + 1, \hat{n}_{\bar{q}} + 1) \right) \\
& + \sum_{\bar{q}} \left(\hat{W}_{q,\bar{q}} \hat{\psi}_{q',\bar{q}} + \hat{W}_{\bar{q},q} \hat{\psi}_{q',\bar{q}} + \hat{W}_{q,\bar{q}} \hat{\psi}_{\bar{q},q'} + \hat{W}_{\bar{q},q} \hat{\psi}_{\bar{q},q'} \right) \\
& - \sum_{\bar{q}} \hat{\psi}_{\bar{q},q'} \left(\hat{W}_{\bar{q},q} - \hat{W}_{\bar{q},q}(\hat{n}_{\bar{q}} + 1, \hat{n}_{q'} + 1) \right) \\
& - \sum_{\bar{q}} \hat{\psi}_{\bar{q},q'} \left(\hat{W}_{q,\bar{q}} - \hat{W}_{q,\bar{q}}(\hat{n}_{\bar{q}} + 1, \hat{n}_{q'} + 1) \right) \\
& - \sum_{\bar{q}} \hat{\psi}_{q',\bar{q}} \left(\hat{W}_{\bar{q},q} - \hat{W}_{\bar{q},q}(\hat{n}_{q'} + 1, \hat{n}_{\bar{q}} + 1) \right) \\
& - \sum_{\bar{q}} \hat{\psi}_{q',\bar{q}} \left(\hat{W}_{q,\bar{q}} - \hat{W}_{q,\bar{q}}(\hat{n}_{q'} + 1, \hat{n}_{\bar{q}} + 1) \right)
\end{aligned}$$

$$\partial_\lambda \hat{W}_{p,p'} \stackrel{\textcircled{2}}{=} -\hat{\phi}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) \tag{A.59c}$$

$$\begin{aligned}
& + \sum_q \left(\hat{\theta}_{p',q} \hat{W}_{p,q} + \hat{\theta}_{p,q} \hat{W}_{q,p'} \right) \\
& + \sum_q \hat{\theta}_{p,q} \left(\hat{W}_{q,p'}(\hat{n}_q + 1, \hat{n}_p - 1) - \hat{W}_{q,p'} \right) \\
& + \sum_q \hat{\theta}_{p',q} \left(\hat{W}_{p,q}(\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{W}_{p,q} \right) \\
& - \sum_q \left(\hat{V}_{p',q} \hat{\phi}_{p,q} + \hat{V}_{p,q} \hat{\phi}_{q,p'} \right) \\
& - \sum_q \hat{V}_{p,q} \left(\hat{\phi}_{q,p'}(\hat{n}_q + 1, \hat{n}_p - 1) - \hat{\phi}_{q,p'} \right) \\
& - \sum_q \hat{V}_{p',q} \left(\hat{\phi}_{p,q}(\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{\phi}_{p,q} \right)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \hat{W}_{p,p'}^\dagger & \stackrel{\textcircled{2}}{=} \hat{\psi}_{p,p'}(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) \\
& + \sum_q \left(\hat{\theta}_{q,p'} \hat{W}_{p,q}^\dagger + \hat{\theta}_{q,p} \hat{W}_{q,p'}^\dagger \right) \tag{A.59d}
\end{aligned}$$

$$\begin{aligned}
& + \sum_q \left(\hat{W}_{q,p'}^\dagger + \hat{W}_{p',q}^\dagger \right) \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
& - \sum_q \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
& - \sum_q \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right) \\
\partial_\lambda \epsilon & \stackrel{\textcircled{2}}{=} - \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}_{p,p'}^\dagger + \hat{\phi}_{p',p} \hat{W}_{p,p'}^\dagger \right) \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& + \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& - \sum_q \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
& - \sum_q \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right) \\
& - \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& - \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
& + \sum_{p,p'} \left(\hat{W}_{p,p'} \hat{\psi}_{p,p'} + \hat{W}_{p',p} \hat{\psi}_{p,p'} \right)
\end{aligned} \tag{A.59e}$$

The three operators $\hat{\psi}, \hat{\theta}, \hat{\phi}$ are based on their definition in equation A.22:

$$\hat{\theta}_{q,q'} = (\hat{\omega}_q - \hat{\omega}_{q'}(\hat{n}_q - 1, \hat{n}_{q'} + 1)) \hat{V}_{q,q'} \tag{A.60}$$

$$\hat{\phi}_{p,p'} = \hat{W}_{p,p'}(\hat{\omega}_p + \hat{\omega}_{p'}) \tag{A.61}$$

$$\hat{\psi}_{p,p'} = -\hat{W}_{p,p'}^\dagger(\hat{\omega}_p(\hat{n}_{p'} + 1, \hat{n}_p + 1) + \hat{\omega}_{p'}(\hat{n}_{p'} + 1, \hat{n}_p + 1)) \tag{A.62}$$

APPENDIX B _____

_____ THE SECOND APPENDIX

BIBLIOGRAPHY

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DECLARATION OF AUTHORSHIP

I hereby declare that I have written this thesis independently and by myself and that I have not used any sources or auxiliary materials other than those indicated in the thesis.

Munich, 22.06.2023

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