Title of My Thesis

SUBMITTED BY

Jan-Philipp Anton Konrad Christ

Titel meiner Arbeit

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Fakultät für Physik
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Jan-Philipp Anton Konrad Christ

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Faculty of Physics
Quantum Many-Body Systems/ Theoretical Nanophysics Group
Ludwig Maximilian University
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Supervisor: Prof. Dr. Fabian Bohrdt, geb. Grusdt

NOTATION AND SYMBOLS

λ	flow parameter; in the literature sometimes also denoted by B
$\hat{\cdot}$	denotes that $\cdot$ is an operator which does not commute with every other operator
1	indicates $1 \in \mathbb{N}$ or the identity operator $\hat{\mathbb{1}} =: \mathbb{1}$
$:\hat{A}:$	normal ordering of operator $\hat{A}$
$\hat{a}_k^\dagger$	k^{th} bosonic creation operator
$\hat{a}_k$	k^{th} bosonic annihilation operator
$[\hat{A}, \hat{B}]$	commutator of operators $\hat{A}, \hat{B}$
$\hat{A}^\dagger$	adjoint of an operator $\hat{A}$
z^*	complex conjugate of $z \in \mathbb{C}$
$\delta_{\alpha,\beta}$	Kronecker-Delta of α, β
∂_x	partial derivative $\frac{\partial}{\partial x}$ w.r.t. x
$\overset{②}{=}$	Equality up to second order, i.e. higher order terms are neglected.

ABSTRACT

CONTENTS

SECTION 1

INTRODUCTION

SECTION 2
THEORETICAL BACKGROUND

2.1 The Flow Equation Approach

2.2 Normal Ordering

SECTION 3

CHAPTER 02

SECTION 4

CONCLUSION

APPENDIX A

DETAILED CALCULATIONS

A.1 Deriving the flow equations in the case of no n-dependence

First the canonical generator $\hat{\eta}$ has to be evaluated:

$$\hat{\eta} := \hat{\eta}(\lambda) := \left[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}\right] = \left[\sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \sum_{q\neq q'} V_{q,q'} \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{p,p'}^* \hat{a}_p \hat{a}_{p'}\right)\right] \tag{A.1}$$

$$= \sum_k \sum_{q,q'} \omega_k V_{q,q'} \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_q^\dagger \hat{a}_{q'}\right] + \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p^\dagger \hat{a}_{p'}^\dagger\right] + \omega_k W_{p,p'}^* \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_p \hat{a}_{p'}\right]\right)$$

$$= \sum_k \sum_{q,q'} \omega_k V_{q,q'} \left(\hat{a}_k^\dagger \hat{a}_{q'} \delta_{k,q} - \hat{a}_q^\dagger \hat{a}_k \delta_{k,q'}\right)$$

$$+ \sum_k \sum_{p,p'} \left(\omega_k W_{p,p'} \left(\hat{a}_k^\dagger \hat{a}_p^\dagger \delta_{k,p'} + \hat{a}_k^\dagger \hat{a}_{p'}^\dagger \delta_{k,p}\right) - \omega_k W_{p,p'}^* \left(\hat{a}_p \hat{a}_k \delta_{k,p'} + \hat{a}_{p'} \hat{a}_k \delta_{k,p}\right)\right)$$

$$= \sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right) \tag{A.2}$$

Since $\hat{\eta}$ has the same form as $\hat{\mathcal{H}}_{\text{int}}$, $\left[\hat{\eta}, \hat{\mathcal{H}}_0\right]$ follows by inspection of A.2:

$$\left[\hat{\eta}, \hat{\mathcal{H}}_0\right] = -\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'})^2 \hat{a}_q^\dagger \hat{a}_{q'}$$

$$- \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'})^2 \hat{a}_p^\dagger \hat{a}_{p'}^\dagger + W_{p,p'}^* (\omega_p + \omega_{p'})^2 \hat{a}_p \hat{a}_{p'}\right) \tag{A.3}$$

The commutator of the generator and $\hat{\mathcal{H}}_{\text{int}}$ needs more work:

$$\left[\hat{\eta}, \hat{\mathcal{H}}_{\text{int}}\right] = \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'} + \sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \right.\right. \tag{A.4}$$

$$\left.\left. - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{q}\neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} + \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right]$$

$$= \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{q}\neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}\right] \tag{A.5}$$

$$+ \left[\sum_{q\neq q'} V_{q,q'} (\omega_q - \omega_{q'}) \hat{a}_q^\dagger \hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right] \tag{A.6}$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{q}\neq \tilde{q}'} V_{\tilde{q},\tilde{q}'} \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}\right] \tag{A.7}$$

$$+ \left[\sum_{p,p'} \left(W_{p,p'} (\omega_p + \omega_{p'}) \hat{a}_p^\dagger \hat{a}_{p'}^\dagger - W_{p,p'}^* (\omega_p + \omega_{p'}) \hat{a}_p \hat{a}_{p'}\right), \sum_{\tilde{p},\tilde{p}'} \left(W_{\tilde{p},\tilde{p}'} \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger + W_{\tilde{p},\tilde{p}'}^* \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'}\right)\right] \tag{A.8}$$

In the following, A.5-A.8 will be evaluated separately. There will occur sums with $V_{q,q'}$ where $q = q'$. In this case, we define $V_{k,k} := 0\ \forall k$. This saves the rather tedious declaration of the constraints of several sum indices.

A.5:

$$
\begin{aligned}
&\left[\sum_{q\neq q'} V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}, \sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right]\\
&=\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{q}}^\dagger\hat{a}_{\tilde{q}'}\right]\\
&=\sum_{q\neq q'}\sum_{\tilde{q}\neq\tilde{q}'} V_{\tilde{q},\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\left(\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}\delta_{q',\tilde{q}}-\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'}\delta_{q,\tilde{q}'}\right)\\
&=\sum_{q\neq q'}\sum_{\tilde{q}'} V_{q',\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'}\\
&=\sum_{q,q'}\sum_{\tilde{q}'} V_{q',\tilde{q}'}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{\tilde{q}'}-\sum_{q,q'}\sum_{\tilde{q}} V_{\tilde{q},q}V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_{\tilde{q}}^\dagger\hat{a}_{q'}\\
&=\sum_{q,q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q,q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}\\
&=\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}\\
&+\sum_{k}\sum_{\tilde{q}} V_{\tilde{q},k}V_{k,\tilde{q}}(\omega_k-\omega_{\tilde{q}})\hat{a}_k^\dagger\hat{a}_k-\sum_{k}\sum_{\tilde{q}} V_{k,\tilde{q}}V_{\tilde{q},k}(\omega_{\tilde{q}}-\omega_k)\hat{a}_k^\dagger\hat{a}_k\\
&=\sum_{q\neq q'}\sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q-\omega_{\tilde{q}})\hat{a}_q^\dagger\hat{a}_{q'}-\sum_{q\neq q'}\sum_{\tilde{q}} V_{q,\tilde{q}}V_{\tilde{q},q'}(\omega_{\tilde{q}}-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}\\
&+\sum_{k}\sum_{\tilde{q}} 2V_{\tilde{q},k}V_{k,\tilde{q}}(\omega_k-\omega_{\tilde{q}})\hat{a}_k^\dagger\hat{a}_k
\end{aligned}
\tag{A.9}
$$

A.6:

$$
\begin{aligned}
&\left[\sum_{q\neq q'} V_{q,q'}(\omega_q-\omega_{q'})\hat{a}_q^\dagger\hat{a}_{q'}, \sum_{\tilde{p},\tilde{p}'}\left(W_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger+W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right)\right]\\
&=\sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'} V_{q,q'}(\omega_q-\omega_{q'})\left(W_{\tilde{p},\tilde{p}'}\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{p}}^\dagger\hat{a}_{\tilde{p}'}^\dagger\right]+W_{\tilde{p},\tilde{p}'}^*\left[\hat{a}_q^\dagger\hat{a}_{q'},\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right]\right)\\
&=\sum_{q,q'}\sum_{\tilde{p},\tilde{p}'} V_{q,q'}(\omega_q-\omega_{q'})\left(W_{\tilde{p},\tilde{p}'}\left(\hat{a}_q^\dagger\hat{a}_{\tilde{p}}^\dagger\delta_{q',\tilde{p}'}+\hat{a}_q^\dagger\hat{a}_{\tilde{p}'}^\dagger\delta_{q',\tilde{p}}\right)-W_{\tilde{p},\tilde{p}'}^*\hat{a}_{\tilde{p}}\left(\hat{a}_{\tilde{p}'}\hat{a}_{q'}\delta_{q,\tilde{p}}+\hat{a}_{\tilde{p}}\hat{a}_{q'}\delta_{\tilde{p}',q}\right)\right)\\
&=\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,p'}\hat{a}_q^\dagger\hat{a}_p^\dagger+\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{p,p'}\hat{a}_q^\dagger\hat{a}_{p'}^\dagger\\
&-\sum_{p,p'}\sum_{q'} V_{p,q'}(\omega_p-\omega_{q'})W_{p,p'}^*\hat{a}_{p'}\hat{a}_{q'}-\sum_{p,p'}\sum_{q'} V_{p',q'}(\omega_{p'}-\omega_{q'})W_{p,p'}^*\hat{a}_p\hat{a}_{q'}\\
&=\sum_{p,p'}\sum_{q} V_{p',q}(\omega_{p'}-\omega_q)W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger+\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)W_{q,p'}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger\\
&-\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{q,p'}^*\hat{a}_p\hat{a}_{p'}-\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,q}^*\hat{a}_p\hat{a}_{p'}\\
&=\sum_{p,p'}\sum_{q} V_{p',q}(\omega_{p'}-\omega_q)W_{p,q}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger+\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)W_{q,p'}\hat{a}_p^\dagger\hat{a}_{p'}^\dagger\\
&-\sum_{p,p'}\sum_{q} V_{q,p}(\omega_q-\omega_p)W_{q,p'}^*\hat{a}_p\hat{a}_{p'}-\sum_{p,p'}\sum_{q} V_{q,p'}(\omega_q-\omega_{p'})W_{p,q}^*\hat{a}_p\hat{a}_{p'}\\
&=\sum_{p,p'}\sum_{q} V_{p,q}(\omega_p-\omega_q)(W_{q,p'}+W_{p',q})\hat{a}_p^\dagger\hat{a}_{p'}^\dagger
\end{aligned}
$$

$$+\sum_{p,p'}\sum_{q} V_{q,p}(\omega_p-\omega_q)(W^*_{q,p'}+W^*_{p',q})\hat{a}_p\hat{a}_{p'} \tag{A.10}$$

A.7:

$$\begin{aligned}
&\left[\sum_{p,p'}\left(W_{p,p'}(\omega_p+\omega_{p'})\hat{a}^\dagger_p\hat{a}^\dagger_{p'}-W^*_{p,p'}(\omega_p+\omega_{p'})\hat{a}_p\hat{a}_{p'}\right),\sum_{\tilde{q}\neq\tilde{q}'}V_{\tilde{q},\tilde{q}'}\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}\right]\\
&=\sum_{p,p'}\sum_{\tilde{q}\neq\tilde{q}'}V_{\tilde{q},\tilde{q}'}(\omega_p+\omega_{p'})\left(W_{p,p'}\left[\hat{a}^\dagger_p\hat{a}^\dagger_{p'},\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}\right]-W^*_{p,p'}\left[\hat{a}_p\hat{a}_{p'},\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}\right]\right)\\
&=-\sum_{p,p'}\sum_{q\neq q'}V_{q,q'}(\omega_p+\omega_{p'})W_{p,p'}\left(\hat{a}^\dagger_q\hat{a}^\dagger_p\delta_{q',p'}+\hat{a}^\dagger_q\hat{a}^\dagger_{p'}\delta_{q',p}\right)\\
&-\sum_{p,p'}\sum_{q\neq q'}V_{q,q'}(\omega_p+\omega_{p'})W^*_{p,p'}\left(\hat{a}_p\hat{a}_{q'}\delta_{q,p'}+\hat{a}_{p'}\hat{a}_{q'}\delta_{q,p}\right)\\
&=-\sum_{p,p'}\sum_{q}V_{q,p'}(\omega_p+\omega_{p'})W_{p,p'}\hat{a}^\dagger_q\hat{a}^\dagger_p-\sum_{p,p'}\sum_{q}V_{q,p}(\omega_p+\omega_{p'})W_{p,p'}\hat{a}^\dagger_q\hat{a}^\dagger_{p'}\\
&-\sum_{p,p'}\sum_{q'}V_{p',q'}(\omega_p+\omega_{p'})W^*_{p,p'}\hat{a}_p\hat{a}_{q'}-\sum_{p,p'}\sum_{q'}V_{p,q'}(\omega_p+\omega_{p'})W^*_{p,p'}\hat{a}_{p'}\hat{a}_{q'}\\
&=-\sum_{p,p'}\sum_{q}V_{p',q}(\omega_p+\omega_q)W_{p,q}\hat{a}^\dagger_p\hat{a}^\dagger_{p'}-\sum_{p,p'}\sum_{q}V_{p,q}(\omega_q+\omega_{p'})W_{q,p'}\hat{a}^\dagger_p\hat{a}^\dagger_{p'}\\
&-\sum_{p,p'}\sum_{q'}V_{q',p'}(\omega_p+\omega_{q'})W^*_{p,q'}\hat{a}_p\hat{a}_{p'}-\sum_{p,p'}\sum_{q'}V_{q',p}(\omega_{q'}+\omega_{p'})W^*_{q',p'}\hat{a}_p\hat{a}_{p'}\\
&=-\sum_{p,p'}\sum_{q}V_{p,q}(\omega_q+\omega_{p'})(W_{p',q}+W_{q,p'})\hat{a}^\dagger_p\hat{a}^\dagger_{p'}\\
&-\sum_{p,p'}\sum_{q}V_{q,p}(\omega_q+\omega_{p'})(W^*_{p',q}+W^*_{q,p'})\hat{a}_p\hat{a}_{p'}
\end{aligned} \tag{A.11}$$

A.8:

$$\begin{aligned}
&\left[\sum_{p,p'}\left(W_{p,p'}(\omega_p+\omega_{p'})\hat{a}^\dagger_p\hat{a}^\dagger_{p'}-W^*_{p,p'}(\omega_p+\omega_{p'})\hat{a}_p\hat{a}_{p'}\right),\sum_{\tilde{p},\tilde{p}'}\left(W_{\tilde{p},\tilde{p}'}\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}+W^*_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right)\right]\\
&=\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'})W^*_{\tilde{p},\tilde{p}'}\left[\hat{a}^\dagger_p\hat{a}^\dagger_{p'},\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'}\right]-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W^*_{p,p'}W_{\tilde{p},\tilde{p}'}(\omega_p+\omega_{p'})\left[\hat{a}_p\hat{a}_{p'},\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}\right]\\
&=-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W^*_{\tilde{p},\tilde{p}'}\left[\hat{a}_{\tilde{p}}\hat{a}_{\tilde{p}'},\hat{a}^\dagger_p\hat{a}^\dagger_{p'}\right]\\
&=-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W^*_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}\hat{a}^\dagger_p\delta_{\tilde{p}',p'}\\
&-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W^*_{\tilde{p},\tilde{p}'}\hat{a}_{\tilde{p}}\hat{a}^\dagger_{p'}\delta_{\tilde{p}',p}\\
&-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W^*_{\tilde{p},\tilde{p}'}\hat{a}^\dagger_p\hat{a}_{\tilde{p}'}\delta_{\tilde{p},p'}\\
&-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}W_{p,p'}(\omega_p+\omega_{p'}+\omega_{\tilde{p}}+\omega_{\tilde{p}'})W^*_{\tilde{p},\tilde{p}'}\hat{a}^\dagger_{p'}\hat{a}_{\tilde{p}'}\delta_{\tilde{p},p}\\
&=-\sum_{p,p'}\sum_{\tilde{p}}W_{p,p'}(\omega_p+2\omega_{p'}+\omega_{\tilde{p}})W^*_{\tilde{p},p'}\hat{a}_{\tilde{p}}\hat{a}^\dagger_p-\sum_{p,p'}\sum_{\tilde{p}}W_{p,p'}(2\omega_p+\omega_{p'}+\omega_{\tilde{p}})W^*_{\tilde{p},p}\hat{a}_{\tilde{p}}\hat{a}^\dagger_{p'}\\
&-\sum_{p,p'}\sum_{\tilde{p}'}W_{p,p'}(\omega_p+2\omega_{p'}+\omega_{\tilde{p}'})W^*_{p',\tilde{p}'}\hat{a}^\dagger_p\hat{a}_{\tilde{p}'}-\sum_{p,p'}\sum_{\tilde{p}'}W_{p,p'}(2\omega_p+\omega_{p'}+\omega_{\tilde{p}'})W^*_{p,\tilde{p}'}\hat{a}^\dagger_{p'}\hat{a}_{\tilde{p}'}\\
&=-\sum_{p,p'}\sum_{\tilde{p}}W_{p,\tilde{p}}(\omega_p+2\omega_{\tilde{p}}+\omega_{p'})W^*_{p',\tilde{p}}\hat{a}_{p'}\hat{a}^\dagger_p-\sum_{p,p'}\sum_{\tilde{p}}W_{\tilde{p},p}(2\omega_{\tilde{p}}+\omega_p+\omega_{p'})W^*_{p',\tilde{p}}\hat{a}_{p'}\hat{a}^\dagger_p
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{p,p'}\sum_{\tilde{p}'} W_{p,\tilde{p}'}(\omega_p + 2\omega_{\tilde{p}'} + \omega_{p'})W^*_{\tilde{p}',p'}\hat{a}_p^\dagger\hat{a}_{p'} - \sum_{p,p'}\sum_{\tilde{p}'} W_{\tilde{p}',p}(2\omega_{\tilde{p}'} + \omega_p + \omega_{p'})W^*_{\tilde{p}',p'}\hat{a}_p^\dagger\hat{a}_{p'} \\
&= -\sum_{p,p'}\sum_{\tilde{p}}(W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'})W^*_{p',\tilde{p}}\hat{a}_{p'}\hat{a}_p^\dagger \\
&-\sum_{p,p'}\sum_{\tilde{p}'}(W_{p,\tilde{p}'} + W_{\tilde{p}',p})(\omega_p + 2\omega_{\tilde{p}'} + \omega_{p'})W^*_{\tilde{p}',p'}\hat{a}_p^\dagger\hat{a}_{p'} \\
&= -\sum_{p,p'}\sum_{\tilde{p}}(W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'})W^*_{p',\tilde{p}}(\delta_{p,p'} + \hat{a}_p^\dagger\hat{a}_{p'}) \\
&-\sum_{p,p'}\sum_{\tilde{p}}(W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'})W^*_{\tilde{p},p'}\hat{a}_p^\dagger\hat{a}_{p'} \\
&= -\sum_{p,p'}\sum_{\tilde{p}}(W_{p,\tilde{p}} + W_{\tilde{p},p})(\omega_p + 2\omega_{\tilde{p}} + \omega_{p'})(W^*_{\tilde{p},p'} + W^*_{p',\tilde{p}})\hat{a}_p^\dagger\hat{a}_{p'} \\
&-2\sum_{k}\sum_{\tilde{p}}(W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}})W^*_{k,\tilde{p}} \\
&= -\sum_{q\neq q'}\sum_{\tilde{p}}(W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + 2\omega_{\tilde{p}} + \omega_{q'})(W^*_{\tilde{p},q'} + W^*_{q',\tilde{p}})\hat{a}_q^\dagger\hat{a}_{q'} \\
&-2\sum_{k}\sum_{\tilde{p}}(W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}})(W^*_{\tilde{p},k} + W^*_{k,\tilde{p}})\hat{a}_k^\dagger\hat{a}_k \\
&-2\sum_{k}\sum_{\tilde{p}}(W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}})W^*_{k,\tilde{p}}
\end{aligned} \tag{A.12}
$$

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$
\hat{\mathcal{H}}(\lambda) = \sum_k \omega_k(\lambda)\hat{a}_k^\dagger\hat{a}_k + \sum_{q\neq q'} V_{q,q'}(\lambda)\hat{a}_q^\dagger\hat{a}_{q'} + \sum_{p,p'}\Big(W_{p,p'}(\lambda)\hat{a}_p^\dagger\hat{a}_{p'}^\dagger + W^*_{p,p'}(\lambda)\hat{a}_p\hat{a}_{p'}\Big) + \epsilon(\lambda) \tag{A.13}
$$

where $\epsilon(\lambda)$ is a constant shift in the energy scale.
Using the expressions for the commutators of the generator and $\hat{\mathcal{H}}_0$ respectively $\hat{\mathcal{H}}_{\text{int}}$ derived above, the flow $\partial_\lambda\hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ yields the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$
\partial_\lambda\omega_k = \sum_{\tilde{q}} 2V_{\tilde{q},k}V_{k,\tilde{q}}(\omega_k - \omega_{\tilde{q}}) - 2\sum_{\tilde{p}}(W_{k,\tilde{p}} + W_{\tilde{p},k})(\omega_k + \omega_{\tilde{p}})(W^*_{\tilde{p},k} + W^*_{k,\tilde{p}}) \tag{A.14a}
$$

$$
\begin{aligned}
\partial_\lambda V_{q,q'} &= -V_{q,q'}(\omega_q - \omega_{q'})^2 - \sum_{\tilde{p}}(W_{q,\tilde{p}} + W_{\tilde{p},q})(\omega_q + \omega_{q'} + 2\omega_{\tilde{p}})(W^*_{\tilde{p},q'} + W^*_{q',\tilde{p}}) \\
&+ \sum_{\tilde{q}} V_{\tilde{q},q'}V_{q,\tilde{q}}(\omega_q + \omega_{q'} - 2\omega_{\tilde{q}})
\end{aligned} \tag{A.14b}
$$

$$
\begin{aligned}
\partial_\lambda W_{p,p'} &= -W_{p,p'}(\omega_p + \omega_{p'})^2 - \sum_q V_{p,q}(\omega_q + \omega_{p'})(W_{p',q} + W_{q,p'}) \\
&+ \sum_q V_{p,q}(\omega_p - \omega_q)(W_{q,p'} + W_{p',q})
\end{aligned} \tag{A.14c}
$$

$$
\begin{aligned}
\partial_\lambda W^*_{p,p'} &= -W^*_{p,p'}(\omega_p + \omega_{p'})^2 - \sum_q V_{q,p}(\omega_q + \omega_{p'})(W^*_{p',q} + W^*_{q,p'}) \\
&+ \sum_q V_{q,p}(\omega_p - \omega_q)(W^*_{q,p'} + W^*_{p',q})
\end{aligned} \tag{A.14d}
$$

$$
\partial_\lambda\varepsilon = -2\sum_{p,p'}(W_{p,p'} + W_{p',p})(\omega_p + \omega_{p'})W^*_{p,p'} \tag{A.14e}
$$

Obviously, equations A.14c and A.14d are not independent from each other, since they are related by complex conjugation. Seeing this is a good consistency check because complex

conjugation was not explicitly used in the derivation of the these two equations.
Furthermore note that in first order the flow equations A.14a-A.14e suggest that they are exact in the sense that if the flow is completely traversed, the flow Hamiltonian will be exactly diagonal.

A.2 Deriving the flow equations with n-dependence

A.2.1 Useful preliminaries

Consider some operator $\hat{f}$ which depends on a number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. The following relations will be used later:

$$\left[\hat{a}^\dagger, \hat{f}(\hat{n})\right] = \hat{a}^\dagger \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}+1)\right) \tag{A.15a}$$
$$\left[\hat{a}, \hat{f}(\hat{n})\right] = \hat{a} \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}-1)\right) \tag{A.15b}$$
$$\left[\hat{f}(\hat{n}), \hat{a}^\dagger\right] = \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}-1)\right) \hat{a}^\dagger \tag{A.15c}$$
$$\left[\hat{f}(\hat{n}), \hat{a}\right] = \left(\hat{f}(\hat{n}) - \hat{f}(\hat{n}+1)\right) \hat{a} \tag{A.15d}$$

These can be proved by induction for $\hat{f}(\hat{n}) = \hat{n}^k, k \in \mathbb{N}$ and from there simply extended to well-behaved $\hat{f}$ via power series. Equations A.15 are still valid for functions depending on $\{\hat{n}_k\}_k$, because all $\hat{n}_k$ pairwise commute.
We will write $\hat{f}\left(\hat{n}_1, \hat{n}_2, \ldots\right) =: \hat{f}$ and $\hat{f}\left(\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k \pm 1, \hat{n}_{k+1}, \ldots\right) =: \hat{f}(\hat{n}_k \pm 1)$. In this notation it is understood that $\hat{f}(\hat{n}_k \pm 1, \hat{n}_k \pm 1) =: \hat{f}(\hat{n}_k \pm 2)$.
Using this notation, it is evident that a simple induction for $n_1, n_2 \in \mathbb{N}_0$ yields the following relation:

$$\begin{aligned}
&\left[\hat{f}(\hat{n}), \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_{n_1}}\hat{a}_{k_1}\hat{a}_{k_2}\cdots\hat{a}_{k_{n_2}}\right] \\
&= \left(\hat{f} - \hat{f}\left(\hat{n}_{k_1}-1, \hat{n}_{k_2}-1, \ldots, \hat{n}_{k_{n_1}}, \hat{n}_{k_1}+1, \hat{n}_{k_2}+1 \ldots \hat{n}_{k_{n_2}}+1\right)\right) \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_{n_1}}\hat{a}_{k_1}\hat{a}_{k_2}\cdots\hat{a}_{k_{n_2}}
\end{aligned} \tag{A.16}$$

Furthermore, applying the recurrence relation introduced to define the normal ordering procedure can be used to successively normal order operators. Let $\hat{O} := \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_{n_1}}\hat{a}_{k_1}\hat{a}_{k_2}\cdots\hat{a}_{k_{n_2}}$. Then normal ordering w.r.t. the vacuum yields:

$$\begin{aligned}
\hat{a}_q : \hat{O} : &=: \hat{O}\hat{a}_q : + \sum_k : \frac{\partial \hat{O}}{\partial \hat{a}^\dagger_k} : \\
&=: \hat{O}\hat{a}_q : + \sum_{i=1}^{n_1} \delta_{k_i,q} : \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_{i-1}}\hat{a}^\dagger_{k_{i+1}}\cdots\hat{a}^\dagger_{k_{n_1}}\hat{a}_{k_1}\hat{a}_{k_2}\cdots\hat{a}_{k_{n_2}} :
\end{aligned} \tag{A.17a}$$
$$\hat{a}^\dagger_q : \hat{O} : =: \hat{a}^\dagger_q \hat{O} : \tag{A.17b}$$

A.2.2 The canonical generator

Our Hamiltonian $\hat{\mathcal{H}}$ is of the form:

$$\hat{\mathcal{H}} = \sum_k \hat{\omega}_k : \hat{a}^\dagger_k \hat{a}_k : + \sum_{p,p'} \left(\hat{W}_{p,p'} : \hat{a}^\dagger_p \hat{a}^\dagger_{p'} : + \hat{W}^\dagger_{p,p'} : \hat{a}_p \hat{a}_{p'} :\right) + \hat{\epsilon} \tag{A.18}$$

Upon realizing that $\sum_k \hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k :$ is also just a function of the number operators, we can consider $\hat{\mathcal{H}}_0 := \hat{H} := \sum_k \hat{\omega}_k : \hat{a}_k^\dagger \hat{a}_k : +\hat{\epsilon}$ as the diagonal part of $\hat{\mathcal{H}}$.

The first step in the calculating the flow equations is again to calculate the canonical commutator $\hat{\eta} := \left[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_{\text{int}}\right]$:

$$\begin{aligned}
\hat{\eta} &= \left[\hat{H}, \sum_{q\neq q'} \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : + \sum_{p,p'} \left(\hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : + \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right)\right] \\
&= \sum_{q\neq q'} \left[\hat{H}, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] && \text{(A.19a)} \\
&+ \sum_{p,p'} \left[\hat{H}, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] && \text{(A.19b)} \\
&+ \sum_{p,p'} \left[\hat{H}, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] && \text{(A.19c)}
\end{aligned}$$

In the following, the terms A.19a-A.19c will be evaluated separately:

A.19a

$$\begin{aligned}
&\sum_{q\neq q'} \left[\hat{H}, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \\
&= \sum_{q\neq q'} V_{q,q'} \left[\hat{H}, : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \\
&= \sum_{q\neq q'} \hat{V}_{q,q'} \left(\hat{H} - \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}_q^\dagger \hat{a}_{q'} : && \text{(A.20)} \\
& && \text{(A.21)}
\end{aligned}$$

A.19b

$$\begin{aligned}
&\sum_{p,p'} \left[\hat{H}, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \\
&= \sum_{p,p'} \hat{W}_{p,p'} \left[\hat{H}, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \\
&= \sum_{p,p'} \hat{W}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : && \text{(A.22)}
\end{aligned}$$

A.19c

$$\begin{aligned}
&\sum_{p,p'} \left[\hat{H}, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \\
&= \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}_p \hat{a}_{p'} : && \text{(A.23)}
\end{aligned}$$

This gives the canonical generator as:

$$\begin{aligned}
\hat{\eta} &= \sum_{q\neq q'} \hat{V}_{q,q'} \left(\hat{H} - \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}_q^\dagger \hat{a}_{q'} : && \text{(A.24)} \\
&+ \sum_{p,p'} \hat{W}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\
&+ \sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}_p \hat{a}_{p'} : \\
&=: \sum_{q\neq q'} \hat{\theta}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} : + \sum_{p,p'} \left(\hat{\phi}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : + \hat{\psi}_{p,p'} : \hat{a}_p \hat{a}_{p'} :\right) && \text{(A.25)}
\end{aligned}$$

A.2.3 Evaluating the commutator of the generator with the Hamiltonian

If one notices that η is structurally identical to $\hat{\mathcal{H}}_{\text{int}}$, the commutator of $\hat{\mathcal{H}}_0$ and η can be written down immediately:

$$\begin{aligned}\left[\eta^{(2)}, \hat{\mathcal{H}}_0\right] = &-\sum_{q\neq q'} \hat{\theta}_{q,q'} \left(\hat{H} - \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\ &-\sum_{p,p'} \hat{\phi}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\ &-\sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right) : \hat{a}_p \hat{a}_{p'} :\end{aligned} \tag{A.26}$$

$$\begin{aligned}= &-\sum_{q\neq q'} \hat{V}_{q,q'} \left(\hat{H} - \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1)\right)^2 : \hat{a}_q^\dagger \hat{a}_{q'} : \\ &-\sum_{p,p'} \hat{W}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} - 1, \hat{n}_p - 1)\right)^2 : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\ &-\sum_{p,p'} \hat{W}_{p,p'}^\dagger \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1)\right)^2 : \hat{a}_p \hat{a}_{p'} :\end{aligned} \tag{A.27}$$

The commutator of $\hat{\mathcal{H}}_{\text{int}}$ and η requires significantly more work:

$$\left[\eta^{(2)}, \hat{\mathcal{H}}_{\text{int}}\right] = \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \tag{A.28a}$$

$$+ \sum_{p,p'} \sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \tag{A.28b}$$

$$+ \sum_{p,p'} \sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \tag{A.28c}$$

$$+ \sum_{q\neq q'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \tag{A.28d}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \tag{A.28e}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}}^\dagger \hat{a}_{\tilde{p}'}^\dagger :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \tag{A.28f}$$

$$+ \sum_{q\neq q'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \tag{A.28g}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :\right] \tag{A.28h}$$

$$+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} :\right] \tag{A.28i}$$

For the sake of clarity, the terms A.28a-A.28i will again be evaluated one by one.

A.28a:

$$\begin{aligned}&\sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{V}_{q,q'} : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \\ &= \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{V}_{q,q'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, : \hat{a}_q^\dagger \hat{a}_{q'} :\right] \\ &+ \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{V}_{q,q'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'}, : \hat{a}_q^\dagger \hat{a}_{q'} :\right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[:\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}:, \hat{V}_{q,q'} \right] :\hat{a}^\dagger_{q}\hat{a}_{q'}: \\
&= \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{V}_{q,q'} \left(\delta_{\tilde{q}',q} :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{q}: - \delta_{\tilde{q},q'} :\hat{a}^\dagger_{q}\hat{a}_{\tilde{q}'}: \right) \\
&+ \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{V}_{q,q'} \left(\hat{\theta}_{\tilde{q},\tilde{q}'} - \hat{\theta}_{\tilde{q},\tilde{q}'}(\hat{n}_{q'}+1, \hat{n}_{q}-1) \right) \underbrace{:\hat{a}^\dagger_{q}\hat{a}_{q'}::\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}:}_{\overset{\textcircled{2}}{=}\delta_{q',\tilde{q}}:\hat{a}^\dagger_{q}\hat{a}_{\tilde{q}'}:} \\
&- \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right] :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}:
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
&\overset{\textcircled{2}}{=} \sum_{q\neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{\tilde{q},q} \hat{V}_{q,q'} :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{q}: - \hat{\theta}_{q',\tilde{q}} \hat{V}_{q,q'} :\hat{a}^\dagger_{q}\hat{a}_{\tilde{q}}: \right) \\
&+ \sum_{q\neq q'} \sum_{\tilde{q}} \hat{V}_{q,q'} \left(\hat{\theta}_{q',\tilde{q}} - \hat{\theta}_{q',\tilde{q}}(\hat{n}_{q'}+1, \hat{n}_{q}-1) \right) :\hat{a}^\dagger_{q}\hat{a}_{\tilde{q}}: \\
&- \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right] :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}: \\
&= \sum_{q,q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} :\hat{a}^\dagger_{q}\hat{a}_{q'}: - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right) \\
&\underbrace{- \sum_{k} \sum_{\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} \hat{V}_{k,k} :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{k}: - \hat{\theta}_{k,\tilde{q}} \hat{V}_{k,k} :\hat{a}^\dagger_{k}\hat{a}_{\tilde{q}}: \right)}_{=0} \\
&+ \sum_{q,q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'}(\hat{n}_{\tilde{q}}+1, \hat{n}_{q}-1) \right) :\hat{a}^\dagger_{q}\hat{a}_{q'}: \\
&\underbrace{- \sum_{k} \sum_{\tilde{q}} \hat{V}_{k,k} \left(\hat{\theta}_{k,\tilde{q}} - \hat{\theta}_{k,\tilde{q}}(\hat{n}_{k}+1, \hat{n}_{k}-1) \right) :\hat{a}^\dagger_{k}\hat{a}_{\tilde{q}}:}_{=0} \\
&- \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right] :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}:
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
&= \sum_{q\neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} :\hat{a}^\dagger_{q}\hat{a}_{q'}: - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right) \\
&+ \sum_{k} \sum_{\tilde{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\tilde{q}} :\hat{a}^\dagger_{k}\hat{a}_{k}: - \hat{\theta}_{\tilde{q},k} \hat{V}_{k,\tilde{q}} :\hat{a}^\dagger_{k}\hat{a}_{k}: \right) \\
&+ \sum_{q\neq q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'}(\hat{n}_{\tilde{q}}+1, \hat{n}_{q}-1) \right) :\hat{a}^\dagger_{q}\hat{a}_{q'}: \\
&+ \sum_{k} \sum_{\tilde{q}} \hat{V}_{k,\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k}(\hat{n}_{\tilde{q}}+1, \hat{n}_{k}-1) \right) :\hat{a}^\dagger_{k}\hat{a}_{k}: \\
&- \sum_{q\neq q'} \sum_{\tilde{q}\neq\tilde{q}'} \hat{\theta}_{q,q'} \left[\hat{V}_{\tilde{q},\tilde{q}'}, :\hat{a}^\dagger_{q}\hat{a}_{q'}: \right] :\hat{a}^\dagger_{\tilde{q}}\hat{a}_{\tilde{q}'}:
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
&= \sum_{q\neq q'} \sum_{\tilde{q}} \left(\hat{\theta}_{q,q'} \hat{V}_{q',\tilde{q}} - \hat{\theta}_{\tilde{q},q'} \hat{V}_{q,\tilde{q}} \right) :\hat{a}^\dagger_{q}\hat{a}_{q'}: \\
&+ \sum_{k} \sum_{\tilde{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\tilde{q}} - \hat{\theta}_{\tilde{q},k} \hat{V}_{k,\tilde{q}} \right) :\hat{a}^\dagger_{k}\hat{a}_{k}: \\
&+ \sum_{q\neq q'} \sum_{\tilde{q}} \hat{V}_{q,\tilde{q}} \left(\hat{\theta}_{\tilde{q},q'} - \hat{\theta}_{\tilde{q},q'}(\hat{n}_{\tilde{q}}+1, \hat{n}_{q}-1) \right) :\hat{a}^\dagger_{q}\hat{a}_{q'}: \\
&+ \sum_{k} \sum_{\tilde{q}} \hat{V}_{k,\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k}(\hat{n}_{\tilde{q}}+1, \hat{n}_{k}-1) \right) :\hat{a}^\dagger_{k}\hat{a}_{k}:
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
&- \sum_{q \neq q'} \sum_{\tilde{q}} \hat{\theta}_{q,\tilde{q}} \left(\hat{V}_{\tilde{q},q'} - \hat{V}_{\tilde{q},q'}(\hat{n}_{\tilde{q}} + 1, \hat{n}_q - 1) \right) : \hat{a}_q^\dagger \hat{a}_{q'} : \\
&- \sum_{k} \sum_{\tilde{q}} \hat{\theta}_{k,\tilde{q}} \left(\hat{V}_{\tilde{q},k} - \hat{V}_{\tilde{q},k}(\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) : \hat{a}_k^\dagger \hat{a}_k :
\end{aligned}
$$

Here we introduced the symbol $\overset{②}{=}$ which is used for equalities which are exact up to second order.

A.28b:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'} : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \\
&= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] && \text{(A.33a)} \\
&+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : && \text{(A.33b)} \\
&+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'}, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : && \text{(A.33c)}
\end{aligned}
$$

We start by evaluating A.33a:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] \\
&= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'} \left(\delta_{q',p'} : \hat{a}_q^\dagger \hat{a}_p^\dagger : + \delta_{q',p} : \hat{a}_q^\dagger \hat{a}_{p'}^\dagger : \right) \\
&= \sum_{p,p'} \sum_{q} \hat{\theta}_{q,p'} \hat{W}_{p,p'} : \hat{a}_q^\dagger \hat{a}_p^\dagger : + \sum_{p,p'} \sum_{q} \hat{\theta}_{q,p} \hat{W}_{p,p'} : \hat{a}_q^\dagger \hat{a}_{p'}^\dagger : \\
&= \sum_{p,p'} \sum_{q} \left(\hat{\theta}_{p',q} \hat{W}_{p,q} + \hat{\theta}_{p,q} \hat{W}_{q,p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : && \text{(A.34)}
\end{aligned}
$$

Next is A.33b:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \left[: \hat{a}_q^\dagger \hat{a}_{q'} :, \hat{W}_{p,p'} \right] : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\
&= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \left(\hat{W}_{p,p'}(\hat{n}_{q'} + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) \underbrace{: \hat{a}_q^\dagger \hat{a}_{q'} :: \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :}_{\overset{②}{=} \delta_{q',p} : \hat{a}_{p'}^\dagger \hat{a}_q^\dagger : + \delta_{q',p'} : \hat{a}_p^\dagger \hat{a}_q^\dagger :} \\
&\overset{②}{=} \sum_{p,p'} \sum_{q} \hat{\theta}_{q,p} \left(\hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) : \hat{a}_{p'}^\dagger \hat{a}_q^\dagger : \\
&+ \sum_{p,p'} \sum_{q} \hat{\theta}_{q,p'} \left(\hat{W}_{p,p'}(\hat{n}_{p'} + 1, \hat{n}_q - 1) - \hat{W}_{p,p'} \right) : \hat{a}_p^\dagger \hat{a}_q^\dagger : \\
&= \sum_{p,p'} \sum_{q} \hat{\theta}_{p,q} \left(\hat{W}_{q,p'}(\hat{n}_q + 1, \hat{n}_p - 1) - \hat{W}_{q,p'} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \\
&+ \sum_{p,p'} \sum_{q} \hat{\theta}_{p',q} \left(\hat{W}_{p,q}(\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{W}_{p,q} \right) : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : && \text{(A.35)}
\end{aligned}
$$

A.33c gives no quadratic contribution:

$$
\sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'}, : \hat{a}_p^\dagger \hat{a}_{p'}^\dagger : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'}
$$

$$= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'} \left(\hat{\theta}_{\tilde{q},\tilde{q}'} - \hat{\theta}_{\tilde{q},\tilde{q}'}(\hat{n}_{p'}, \hat{n}_p - 1) \right) \underbrace{: \hat{a}_p^\dagger \hat{a}_{p'}^\dagger :: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :}_{=: \hat{a}_p^\dagger \hat{a}_{p'}^\dagger \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :} \overset{\textcircled{2}}{=} 0 \tag{A.36}$$

A.28c:

$$\sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \left[\hat{\theta}_{\tilde{q},\tilde{q}'} : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{p'} : \right]$$

$$= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, : \hat{a}_p \hat{a}_{p'} : \right] \tag{A.37a}$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'}^\dagger \right] : \hat{a}_p \hat{a}_{p'} : \tag{A.37b}$$

$$+ \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{W}_{p,p'}^\dagger \left[\hat{\theta}_{\tilde{q},\tilde{q}'}, : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} : \tag{A.37c}$$

We will again start by evaluating A.37a:

$$\sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'}^\dagger \left[: \hat{a}_q^\dagger \hat{a}_{q'} :, : \hat{a}_p \hat{a}_{p'} : \right]$$

$$= \sum_{p,p'} \sum_{q \neq q'} \hat{\theta}_{q,q'} \hat{W}_{p,p'}^\dagger \left(\delta_{q,p'} : \hat{a}_p \hat{a}_{q'} : + \delta_{q,p} : \hat{a}_{p'} \hat{a}_{q'} : \right)$$

$$= \sum_{p,p'} \sum_{q'} \hat{\theta}_{p',q'} \hat{W}_{p,p'}^\dagger : \hat{a}_p \hat{a}_{q'} : + \sum_{p,p'} \sum_{q'} \hat{\theta}_{p,q'} \hat{W}_{p,p'}^\dagger : \hat{a}_{p'} \hat{a}_{q'} :$$

$$= \sum_{p,p'} \sum_{q} \left(\hat{\theta}_{q,p'} \hat{W}_{p,q}^\dagger + \hat{\theta}_{q,p} \hat{W}_{q,p'}^\dagger \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.38}$$

A.37b gives no quadratic contribution:

$$\sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left[: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :, \hat{W}_{p,p'}^\dagger \right] : \hat{a}_p \hat{a}_{p'} :$$

$$= \sum_{p,p'} \sum_{\tilde{q} \neq \tilde{q}'} \hat{\theta}_{\tilde{q},\tilde{q}'} \left(W_{p,p'}^\dagger(\hat{n}_{q'} + 1, \hat{n}_q - 1) - W_{p,p'}^\dagger \right) \underbrace{: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} :: \hat{a}_p \hat{a}_{p'} :}_{=: \hat{a}_{\tilde{q}}^\dagger \hat{a}_{\tilde{q}'} \hat{a}_p \hat{a}_{p'} :} \overset{\textcircled{2}}{=} 0 \tag{A.39}$$

A.37c:

$$\sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left[\hat{\theta}_{q,q'}, : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}_q^\dagger \hat{a}_{q'} :$$

$$= \sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{p'} :: \hat{a}_q^\dagger \hat{a}_{q'} : \tag{A.40}$$

$$\overset{\textcircled{2}}{=} \sum_{p,p'} \sum_{q \neq q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{q,q'} - \hat{\theta}_{q,q'}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) \left(\delta_{p,q} : \hat{a}_{p'} \hat{a}_{q'} : + \delta_{p',q} : \hat{a}_p \hat{a}_{q'} : \right)$$

$$= \sum_{p,p'} \sum_{q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{p,q'} - \hat{\theta}_{p,q'}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_{p'} \hat{a}_{q'} : \tag{A.41}$$

$$+ \sum_{p,p'} \sum_{q'} \hat{W}_{p,p'}^\dagger \left(\hat{\theta}_{p',q'} - \hat{\theta}_{p',q'}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{q'} :$$

$$= \sum_{p,p'} \sum_{q} \hat{W}_{q,p'}^\dagger \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.42}$$

$$+ \sum_{p,p'} \sum_{q} \hat{W}_{p,q}^\dagger \left(\hat{\theta}_{q,p'} - \hat{\theta}_{q,p'}(\hat{n}_q + 1, \hat{n}_p + 1) \right) : \hat{a}_p \hat{a}_{p'} :$$

$$= \sum_{p,p'} \sum_{q} \left(\hat{W}_{q,p'}^\dagger + \hat{W}_{p',q}^\dagger \right) \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) : \hat{a}_p \hat{a}_{p'} : \tag{A.43}$$

A.28d: Follows immediately from the calculations already done for A.28b:

$$\sum_{q\neq q'}\sum_{\tilde{p},\tilde{p}'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'}:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{V}_{q,q'}:\hat{a}^\dagger_q\hat{a}_{q'}:\right] \tag{A.44a}$$

$$=-\sum_{q\neq q'}\sum_{p,p'}\left[\hat{V}_{q,q'}:\hat{a}^\dagger_q\hat{a}_{q'}:,\hat{\phi}_{p,p'}:\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:\right] \tag{A.44b}$$

$$=-\sum_{p,p'}\sum_{q}\left(\hat{V}_{p',q}\hat{\phi}_{p,q}+\hat{V}_{p,q}\hat{\phi}_{q,p'}\right):\hat{a}^\dagger_p\hat{a}^\dagger_{p'}: \tag{A.44c}$$

$$-\sum_{p,p'}\sum_{q}\hat{V}_{p,q}\left(\hat{\phi}_{q,p'}(\hat{n}_q+1,\hat{n}_p-1)-\hat{\phi}_{q,p'}\right):\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:$$

$$-\sum_{p,p'}\sum_{q}\hat{V}_{p',q}\left(\hat{\phi}_{p,q}(\hat{n}_q+1,\hat{n}_{p'}-1)-\hat{\phi}_{p,q}\right):\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:$$

A.28e:

$$\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'}:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{W}_{p,p'}:\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:\right]$$

$$=\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left[:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{W}_{p,p'}\right]:\hat{a}^\dagger_p\hat{a}^\dagger_{p'}: \tag{A.45a}$$

$$+\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}_{p,p'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'},:\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:\right]:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}: \tag{A.45b}$$

A.45a will be analyzed first:

$$\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left[:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{W}_{p,p'}\right]:\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:$$

$$=\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left(\hat{W}_{p,p'}(\hat{n}_{\tilde{p}}+1,\hat{n}_{\tilde{p}'}+1)-\hat{W}_{p,p'}\right):\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}::\hat{a}^\dagger_p\hat{a}^\dagger_{p'}:\overset{2}{=}0 \tag{A.46}$$

Similiarly, A.45b also gives no quadratic contribution.

A.28f

$$\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'}:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{W}^\dagger_{p,p'}:\hat{a}_p\hat{a}_{p'}:\right]$$

$$=\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left[:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,:\hat{a}_p\hat{a}_{p'}:\right] \tag{A.47a}$$

$$+\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\left[:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}:,\hat{W}^\dagger_{p,p'}\right]:\hat{a}_p\hat{a}_{p'}: \tag{A.47b}$$

$$+\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left[\hat{\phi}_{\tilde{p},\tilde{p}'},:\hat{a}_p\hat{a}_{p'}:\right]:\hat{a}^\dagger_{\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}: \tag{A.47c}$$

A.47a:

$$\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\left(\delta_{p',\tilde{p}'}\hat{a}_p\hat{a}^\dagger_{\tilde{p}}+\delta_{p',\tilde{p}}\hat{a}_p\hat{a}^\dagger_{\tilde{p}'}+\delta_{p,\tilde{p}'}\hat{a}^\dagger_{\tilde{p}}\hat{a}_{p'}+\delta_{p,\tilde{p}}\hat{a}^\dagger_{\tilde{p}'}\hat{a}_{p'}\right)$$

$$=-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\delta_{p',\tilde{p}'}\left(:\hat{a}^\dagger_{\tilde{p}}\hat{a}_p:+\delta_{p,\tilde{p}}\right) \tag{A.48}$$

$$-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\delta_{p',\tilde{p}}\left(:\hat{a}^\dagger_{\tilde{p}'}\hat{a}_p:+\delta_{\tilde{p}',p}\right)$$

$$-\sum_{p,p'}\sum_{\tilde{p},\tilde{p}'}\hat{\phi}_{\tilde{p},\tilde{p}'}\hat{W}^\dagger_{p,p'}\delta_{p,\tilde{p}'}:\hat{a}^\dagger_{\tilde{p}}\hat{a}_{p'}:$$

$$
\begin{aligned}
&- \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \delta_{p,\tilde{p}} : \hat{a}^\dagger_{\tilde{p}'} \hat{a}_{p'} : \\
&= - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p'} \hat{W}^\dagger_{p,p'} : \hat{a}^\dagger_{\tilde{p}} \hat{a}_p : \qquad \text{(A.49)} \\
&- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p',\tilde{p}'} \hat{W}^\dagger_{p,p'} : \hat{a}^\dagger_{\tilde{p}'} \hat{a}_p : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}^\dagger_{p,p'} : \hat{a}^\dagger_{\tilde{p}} \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}'} \hat{\phi}_{p,\tilde{p}'} \hat{W}^\dagger_{p,p'} : \hat{a}^\dagger_{\tilde{p}'} \hat{a}_{p'} : \\
&- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p} \hat{W}^\dagger_{p,p'} \right) \\
&= - \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{p,\tilde{p}} \hat{W}^\dagger_{p',\tilde{p}} : \hat{a}^\dagger_{p} \hat{a}_{p'} : \qquad \text{(A.50)} \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}^\dagger_{p',\tilde{p}} : \hat{a}^\dagger_{p} \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{p,\tilde{p}} \hat{W}^\dagger_{\tilde{p},p'} : \hat{a}^\dagger_{p} \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\phi}_{\tilde{p},p} \hat{W}^\dagger_{\tilde{p},p'} : \hat{a}^\dagger_{p} \hat{a}_{p'} : \\
&- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p} \hat{W}^\dagger_{p,p'} \right) \\
&= - \sum_{p,p'} \sum_{\tilde{p}} \left(\hat{\phi}_{p,\tilde{p}} \hat{W}^\dagger_{p',\tilde{p}} + \hat{\phi}_{\tilde{p},p} \hat{W}^\dagger_{p',\tilde{p}} + \hat{\phi}_{p,\tilde{p}} \hat{W}^\dagger_{\tilde{p},p'} + \hat{\phi}_{\tilde{p},p} \hat{W}^\dagger_{\tilde{p},p'} \right) : \hat{a}^\dagger_{p} \hat{a}_{p'} : \qquad \text{(A.51)} \\
&- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p} \hat{W}^\dagger_{p,p'} \right)
\end{aligned}
$$

A.47b gives no quadratic contribution:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left[: \hat{a}^\dagger_{\tilde{p}} \hat{a}^\dagger_{\tilde{p}'} :, \hat{W}^\dagger_{p,p'} \right] : \hat{a}_p \hat{a}_{p'} : \\
&= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}^\dagger_{p,p'}(\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}^\dagger_{p,p'} \right) : \hat{a}^\dagger_{\tilde{p}} \hat{a}^\dagger_{\tilde{p}'} :: \hat{a}_p \hat{a}_{p'} : \\
&= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{\phi}_{\tilde{p},\tilde{p}'} \left(\hat{W}^\dagger_{p,p'}(\hat{n}_p - 1, \hat{n}_{p'} - 1) - \hat{W}^\dagger_{p,p'} \right) : \hat{a}^\dagger_{\tilde{p}} \hat{a}^\dagger_{\tilde{p}'} \hat{a}_p \hat{a}_{p'} : \overset{②}{=} 0 \qquad \text{(A.52)}
\end{aligned}
$$

A.47c:

$$
\begin{aligned}
&\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \left[\hat{\phi}_{\tilde{p},\tilde{p}'}, : \hat{a}_p \hat{a}_{p'} : \right] : \hat{a}^\dagger_{\tilde{p}} \hat{a}^\dagger_{\tilde{p}'} : \\
&= \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) : \hat{a}_p \hat{a}_{p'} :: \hat{a}^\dagger_{\tilde{p}} \hat{a}^\dagger_{\tilde{p}'} : \qquad \text{(A.53)} \\
&\overset{②}{=} \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p,\tilde{p}} : \hat{a}^\dagger_{\tilde{p}'} \hat{a}_{p'} : \qquad \text{(A.54)} \\
&+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p,\tilde{p}'} : \hat{a}^\dagger_{\tilde{p}} \hat{a}_{p'} : \\
&+ \sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{\tilde{p},\tilde{p}'} - \hat{\phi}_{\tilde{p},\tilde{p}'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \delta_{p',\tilde{p}} : \hat{a}^\dagger_{\tilde{p}'} \hat{a}_p :
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{p,p'}\sum_{\tilde p,\tilde p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{\tilde p,\tilde p'}-\hat\phi_{\tilde p,\tilde p'}(\hat n_p+1,\hat n_{p'}+1)\right)\delta_{p',\tilde p'}:\hat a^\dagger_{\tilde p}\hat a_p:\\
&+\sum_{p,p'}\sum_{\tilde p,\tilde p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{\tilde p,\tilde p'}-\hat\phi_{\tilde p,\tilde p'}(\hat n_p+1,\hat n_{p'}+1)\right)\delta_{p',\tilde p}\delta_{p,\tilde p'}\\
&+\sum_{p,p'}\sum_{\tilde p,\tilde p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{\tilde p,\tilde p'}-\hat\phi_{\tilde p,\tilde p'}(\hat n_p+1,\hat n_{p'}+1)\right)\delta_{p',\tilde p'}\delta_{p,\tilde p}\\
&=\sum_{p,p'}\sum_{\tilde p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p,\tilde p'}-\hat\phi_{p,\tilde p'}(\hat n_p+1,\hat n_{p'}+1)\right):\hat a^\dagger_{\tilde p'}\hat a_{p'}: &&\text{(A.55)}\\
&+\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{p,p'}\left(\hat\phi_{\tilde p,p}-\hat\phi_{\tilde p,p}(\hat n_p+1,\hat n_{p'}+1)\right):\hat a^\dagger_{\tilde p}\hat a_{p'}:\\
&+\sum_{p,p'}\sum_{\tilde p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p',\tilde p'}-\hat\phi_{p',\tilde p'}(\hat n_p+1,\hat n_{p'}+1)\right):\hat a^\dagger_{\tilde p'}\hat a_p:\\
&+\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{p,p'}\left(\hat\phi_{\tilde p,p'}-\hat\phi_{\tilde p,p'}(\hat n_p+1,\hat n_{p'}+1)\right):\hat a^\dagger_{\tilde p}\hat a_p:\\
&+\sum_{p,p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p',p}-\hat\phi_{p',p}(\hat n_p+1,\hat n_{p'}+1)\right)\\
&+\sum_{p,p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p,p'}-\hat\phi_{p,p'}(\hat n_p+1,\hat n_{p'}+1)\right)\\
&=\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{\tilde p,p'}\left(\hat\phi_{\tilde p,p}-\hat\phi_{\tilde p,p}(\hat n_{\tilde p}+1,\hat n_{p'}+1)\right):\hat a^\dagger_p\hat a_{p'}: &&\text{(A.56)}\\
&+\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{\tilde p,p'}\left(\hat\phi_{p,\tilde p}-\hat\phi_{p,\tilde p}(\hat n_{\tilde p}+1,\hat n_{p'}+1)\right):\hat a^\dagger_p\hat a_{p'}:\\
&+\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{p',\tilde p}\left(\hat\phi_{\tilde p,p}-\hat\phi_{\tilde p,p}(\hat n_{p'}+1,\hat n_{\tilde p}+1)\right):\hat a^\dagger_p\hat a_{p'}:\\
&+\sum_{p,p'}\sum_{\tilde p}\hat W^\dagger_{p',\tilde p}\left(\hat\phi_{p,\tilde p}-\hat\phi_{p,\tilde p}(\hat n_{p'}+1,\hat n_{\tilde p}+1)\right):\hat a^\dagger_p\hat a_{p'}:\\
&+\sum_{p,p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p',p}-\hat\phi_{p',p}(\hat n_p+1,\hat n_{p'}+1)\right)\\
&+\sum_{p,p'}\hat W^\dagger_{p,p'}\left(\hat\phi_{p,p'}-\hat\phi_{p,p'}(\hat n_p+1,\hat n_{p'}+1)\right)
\end{aligned}
$$

A.28g: Follows immediately from A.28c:

$$
\begin{aligned}
&\sum_{q\neq q'}\sum_{\tilde p,\tilde p'}\left[\hat\psi_{\tilde p,\tilde p'}:\hat a_{\tilde p}\hat a_{\tilde p'}:,\hat V_{q,q'}:\hat a^\dagger_q\hat a_{q'}:\right]\\
&=-\sum_{p,p'}\sum_{\tilde q\neq\tilde q'}\left[\hat V_{\tilde q,\tilde q'}:\hat a^\dagger_{\tilde q}\hat a_{\tilde q'}:,\hat\psi_{p,p'}:\hat a_p\hat a_{p'}:\right]\\
&\overset{②}{=}-\sum_{p,p'}\sum_{q}\left(\hat V_{q,p'}\hat\psi_{p,q}+\hat\theta_{q,p}\hat\psi_{q,p'}\right):\hat a_p\hat a_{p'}: &&\text{(A.57a)}\\
&-\sum_{p,p'}\sum_{q}\left(\hat\psi_{q,p'}+\hat\psi_{p',q}\right)\left(\hat V_{q,p}-\hat V_{q,p}(\hat n_{p'}+1,\hat n_q+1)\right):\hat a_p\hat a_{p'}:
\end{aligned}
$$

A.28h Follows immediately from A.28f:

$$
\begin{aligned}
&\sum_{p,p'}\sum_{\tilde p,\tilde p'}\left[\hat\psi_{\tilde p,\tilde p'}:\hat a_{\tilde p}\hat a_{\tilde p'}:,\hat W_{p,p'}:\hat a^\dagger_p\hat a^\dagger_{p'}:\right]\\
&=-\sum_{p,p'}\sum_{\tilde p,\tilde p'}\left[\hat W_{\tilde p,\tilde p'}:\hat a^\dagger_{\tilde p}\hat a^\dagger_{\tilde p'}:,\hat\psi_{p,p'}:\hat a_p\hat a_{p'}:\right] &&\text{(A.58)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{p,p'} \sum_{\tilde{p}} \left(\hat{W}_{p,\tilde{p}} \hat{\psi}_{p',\tilde{p}} + \hat{W}_{\tilde{p},p} \hat{\psi}_{p',\tilde{p}} + \hat{W}_{p,\tilde{p}} \hat{\psi}_{\tilde{p},p'} + \hat{W}_{\tilde{p},p} \hat{\psi}_{\tilde{p},p'} \right) : \hat{a}^\dagger_p \hat{a}_{p'} : \\
&+ \sum_{p,p'} \left(\hat{W}_{p,p'} \hat{\psi}_{p,p'} + \hat{W}_{p',p} \hat{\psi}_{p,p'} \right) \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{\tilde{p},p'} \left(\hat{W}_{\tilde{p},p} - \hat{W}_{\tilde{p},p}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^\dagger_p \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{\tilde{p},p'} \left(\hat{W}_{p,\tilde{p}} - \hat{W}_{p,\tilde{p}}(\hat{n}_{\tilde{p}} + 1, \hat{n}_{p'} + 1) \right) : \hat{a}^\dagger_p \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{p',\tilde{p}} \left(\hat{W}_{\tilde{p},p} - \hat{W}_{\tilde{p},p}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^\dagger_p \hat{a}_{p'} : \\
&- \sum_{p,p'} \sum_{\tilde{p}} \hat{\psi}_{p',\tilde{p}} \left(\hat{W}_{p,\tilde{p}} - \hat{W}_{p,\tilde{p}}(\hat{n}_{p'} + 1, \hat{n}_{\tilde{p}} + 1) \right) : \hat{a}^\dagger_p \hat{a}_{p'} : \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right)
\end{aligned}
\tag{A.59}
$$

A.28i: Similiar to A.28e:

$$
\sum_{p,p'} \sum_{\tilde{p},\tilde{p}'} \left[\hat{\psi}_{\tilde{p},\tilde{p}'} : \hat{a}_{\tilde{p}} \hat{a}_{\tilde{p}'} :, \hat{W}^\dagger_{p,p'} : \hat{a}_p \hat{a}_{p'} : \right] \overset{②}{=} 0 \tag{A.60}
$$

A.3 The flow equations

We conclude that $\hat{\mathcal{H}}(\lambda)$ is of the form

$$
\begin{aligned}
\hat{\mathcal{H}}(\lambda) \overset{②}{=} &\sum_k \hat{\omega}_k(\lambda) : \hat{a}^\dagger_k \hat{a}_k : + \sum_{q \neq q'} \hat{V}_{q,q'}(\lambda) : \hat{a}^\dagger_q \hat{a}_{q'} : \\
&+ \sum_{p,p'} \left(\hat{W}_{p,p'}(\lambda) : \hat{a}^\dagger_p \hat{a}^\dagger_{p'} : + \hat{W}^\dagger_{p,p'}(\lambda) : \hat{a}_p \hat{a}_{p'} : \right) + \epsilon(\lambda)
\end{aligned}
\tag{A.61}
$$

where $\epsilon(\lambda)$ is a constant which indicates a shift in the energy scale.
Collecting all terms in $\partial_\lambda \hat{\mathcal{H}}(\lambda) = [\hat{\eta}(\lambda), \hat{\mathcal{H}}(\lambda)]$ gives the following flow equations $\forall k, p, p', q, q'$ where $q \neq q'$:

$$
\begin{aligned}
\partial_\lambda \hat{\omega}_k \overset{②}{=} &\sum_{\tilde{q}} \left(\hat{\theta}_{k,k} \hat{V}_{k,\tilde{q}} - \hat{\theta}_{\tilde{q},k} \hat{V}_{k,\tilde{q}} \right) \\
&+ \sum_{\tilde{q}} \hat{V}_{k,\tilde{q}} \left(\hat{\theta}_{\tilde{q},k} - \hat{\theta}_{\tilde{q},k}(\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) \\
&- \sum_{\tilde{q}} \hat{\theta}_{k,\tilde{q}} \left(\hat{V}_{\tilde{q},k} - \hat{V}_{\tilde{q},k}(\hat{n}_{\tilde{q}} + 1, \hat{n}_k - 1) \right) \\
&- \sum_{\tilde{p}} \left(\hat{\phi}_{k,\tilde{p}} \hat{W}^\dagger_{k,\tilde{p}} + \hat{\phi}_{\tilde{p},k} \hat{W}^\dagger_{k,\tilde{p}} + \hat{\phi}_{k,\tilde{p}} \hat{W}^\dagger_{\tilde{p},k} + \hat{\phi}_{\tilde{p},k} \hat{W}^\dagger_{\tilde{p},k} \right) \\
&+ \sum_{\tilde{p}} \hat{W}^\dagger_{\tilde{p},k} \left(\hat{\phi}_{\tilde{p},k} - \hat{\phi}_{\tilde{p},k}(\hat{n}_{\tilde{p}} + 1, \hat{n}_k + 1) \right) \\
&+ \sum_{\tilde{p}} \hat{W}^\dagger_{\tilde{p},k} \left(\hat{\phi}_{k,\tilde{p}} - \hat{\phi}_{k,\tilde{p}}(\hat{n}_{\tilde{p}} + 1, \hat{n}_k + 1) \right)
\end{aligned}
\tag{A.62a}
$$

$$
\begin{aligned}
&+ \sum_{\tilde p} \hat W^\dagger_{k,\tilde p} \Big(\hat\phi_{\tilde p,k} - \hat\phi_{\tilde p,k}(\hat n_k + 1, \hat n_{\tilde p} + 1) \Big) \\
&+ \sum_{\tilde p} \hat W^\dagger_{k,\tilde p} \Big(\hat\phi_{k,\tilde p} - \hat\phi_{k,\tilde p}(\hat n_k + 1, \hat n_{\tilde p} + 1) \Big) \\
&+ \sum_{\tilde p} \Big(\hat W_{k,\tilde p} \hat\psi_{k,\tilde p} + \hat W_{\tilde p,k} \hat\psi_{k,\tilde p} + \hat W_{k,\tilde p} \hat\psi_{\tilde p,k} + \hat W_{\tilde p,k} \hat\psi_{\tilde p,k} \Big) \\
&- \sum_{\tilde p} \hat\psi_{\tilde p,k} \Big(\hat W_{\tilde p,k} - \hat W_{\tilde p,k}(\hat n_{\tilde p} + 1, \hat n_k + 1) \Big) \\
&- \sum_{\tilde p} \hat\psi_{\tilde p,k} \Big(\hat W_{k,\tilde p} - \hat W_{k,\tilde p}(\hat n_{\tilde p} + 1, \hat n_k + 1) \Big) \\
&- \sum_{\tilde p} \hat\psi_{k,\tilde p} \Big(\hat W_{\tilde p,k} - \hat W_{\tilde p,k}(\hat n_k + 1, \hat n_{\tilde p} + 1) \Big) \\
&- \sum_{\tilde p} \hat\psi_{k,\tilde p} \Big(\hat W_{k,\tilde p} - \hat W_{k,\tilde p}(\hat n_k + 1, \hat n_{\tilde p} + 1) \Big)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\lambda \hat V_{q,q'} \overset{②}{=} &- \hat V_{q,q'} \Big(\hat H - \hat H(\hat n_q - 1, \hat n_{q'} + 1) \Big)^2 \\
&+ \sum_{\tilde q} \Big(\hat\theta_{q,q'} \hat V_{q',\tilde q} - \hat\theta_{\tilde q,q'} \hat V_{q,\tilde q} \Big) \\
&+ \sum_{\tilde q} \hat V_{q,\tilde q} \Big(\hat\theta_{\tilde q,q'} - \hat\theta_{\tilde q,q'}(\hat n_{\tilde q} + 1, \hat n_q - 1) \Big) \\
&- \sum_{\tilde q} \hat\theta_{q,\tilde q} \Big(\hat V_{\tilde q,q'} - \hat V_{\tilde q,q'}(\hat n_{\tilde q} + 1, \hat n_q - 1) \Big) \\
&- \sum_{\tilde q} \Big(\hat\phi_{q,\tilde q} \hat W^\dagger_{q',\tilde q} + \hat\phi_{\tilde q,q} \hat W^\dagger_{q',\tilde q} + \hat\phi_{q,\tilde q} \hat W^\dagger_{\tilde q,q'} + \hat\phi_{\tilde q,q} \hat W^\dagger_{\tilde q,q'} \Big) \\
&+ \sum_{\tilde q} \hat W^\dagger_{\tilde q,q'} \Big(\hat\phi_{\tilde q,q} - \hat\phi_{\tilde q,q}(\hat n_{\tilde q} + 1, \hat n_{q'} + 1) \Big) \\
&+ \sum_{\tilde q} \hat W^\dagger_{\tilde q,q'} \Big(\hat\phi_{q,\tilde q} - \hat\phi_{q,\tilde q}(\hat n_{\tilde q} + 1, \hat n_{q'} + 1) \Big) \\
&+ \sum_{\tilde q} \hat W^\dagger_{q',\tilde q} \Big(\hat\phi_{\tilde q,q} - \hat\phi_{\tilde q,q}(\hat n_{q'} + 1, \hat n_{\tilde q} + 1) \Big) \\
&+ \sum_{\tilde q} \hat W^\dagger_{q',\tilde q} \Big(\hat\phi_{q,\tilde q} - \hat\phi_{q,\tilde q}(\hat n_{q'} + 1, \hat n_{\tilde q} + 1) \Big) \\
&+ \sum_{\tilde q} \Big(\hat W_{q,\tilde q} \hat\psi_{q',\tilde q} + \hat W_{\tilde q,q} \hat\psi_{q',\tilde q} + \hat W_{q,\tilde q} \hat\psi_{\tilde q,q'} + \hat W_{\tilde q,q} \hat\psi_{\tilde q,q'} \Big) \\
&- \sum_{\tilde q} \hat\psi_{\tilde q,q'} \Big(\hat W_{\tilde q,q} - \hat W_{\tilde q,q}(\hat n_{\tilde q} + 1, \hat n_{q'} + 1) \Big) \\
&- \sum_{\tilde q} \hat\psi_{\tilde q,q'} \Big(\hat W_{q,\tilde q} - \hat W_{q,\tilde q}(\hat n_{\tilde q} + 1, \hat n_{q'} + 1) \Big) \\
&- \sum_{\tilde q} \hat\psi_{q',\tilde q} \Big(\hat W_{\tilde q,q} - \hat W_{\tilde q,q}(\hat n_{q'} + 1, \hat n_{\tilde q} + 1) \Big) \\
&- \sum_{\tilde q} \hat\psi_{q',\tilde q} \Big(\hat W_{q,\tilde q} - \hat W_{q,\tilde q}(\hat n_{q'} + 1, \hat n_{\tilde q} + 1) \Big)
\end{aligned} \tag{A.62b}
$$

$$
\begin{aligned}
\partial_\lambda \hat W_{p,p'} \overset{②}{=} &- \hat W_{p,p'} \Big(\hat H - \hat H(\hat n_{p'} - 1, \hat n_p - 1) \Big)^2 \\
&+ \sum_{q} \Big(\hat\theta_{p',q} \hat W_{p,q} + \hat\theta_{p,q} \hat W_{q,p'} \Big)
\end{aligned} \tag{A.62c}
$$

$$
\begin{aligned}
&+ \sum_q \hat{\theta}_{p,q} \left(\hat{W}_{q,p'}(\hat{n}_q + 1, \hat{n}_p - 1) - \hat{W}_{q,p'} \right) \\
&+ \sum_q \hat{\theta}_{p',q} \left(\hat{W}_{p,q}(\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{W}_{p,q} \right) \\
&- \sum_q \left(\hat{V}_{p',q} \hat{\phi}_{p,q} + \hat{V}_{p,q} \hat{\phi}_{q,p'} \right) \\
&- \sum_q \hat{V}_{p,q} \left(\hat{\phi}_{q,p'}(\hat{n}_q + 1, \hat{n}_p - 1) - \hat{\phi}_{q,p'} \right) \\
&- \sum_q \hat{V}_{p',q} \left(\hat{\phi}_{p,q}(\hat{n}_q + 1, \hat{n}_{p'} - 1) - \hat{\phi}_{p,q} \right)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\lambda \hat{W}^\dagger_{p,p'} \overset{②}{=} &-W^\dagger_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right)^2 \\
&+ \sum_q \left(\hat{\theta}_{q,p'} \hat{W}^\dagger_{p,q} + \hat{\theta}_{q,p} \hat{W}^\dagger_{q,p'} \right) \\
&+ \sum_q \left(\hat{W}^\dagger_{q,p'} + \hat{W}^\dagger_{p',q} \right) \left(\hat{\theta}_{q,p} - \hat{\theta}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
&- \sum_q \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
&- \sum_q \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right)
\end{aligned}
\tag{A.62d}
$$

$$
\begin{aligned}
\partial_\lambda \epsilon \overset{②}{=} &- \sum_{p,p'} \left(\hat{\phi}_{p,p'} \hat{W}^\dagger_{p,p'} + \hat{\phi}_{p',p} \hat{W}^\dagger_{p,p'} \right) \\
&+ \sum_{p,p'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{p',p} - \hat{\phi}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&+ \sum_{p,p'} \hat{W}^\dagger_{p,p'} \left(\hat{\phi}_{p,p'} - \hat{\phi}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&- \sum_q \left(\hat{\psi}_{q,p'} + \hat{\psi}_{p',q} \right) \left(\hat{V}_{q,p} - \hat{V}_{q,p}(\hat{n}_{p'} + 1, \hat{n}_q + 1) \right) \\
&- \sum_q \left(\hat{V}_{q,p'} \hat{\psi}_{p,q} + \hat{\theta}_{q,p} \hat{\psi}_{q,p'} \right) \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p',p} - \hat{W}_{p',p}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&- \sum_{p,p'} \hat{\psi}_{p,p'} \left(\hat{W}_{p,p'} - \hat{W}_{p,p'}(\hat{n}_p + 1, \hat{n}_{p'} + 1) \right) \\
&+ \sum_{p,p'} \left(\hat{W}_{p,p'} \hat{\psi}_{p,p'} + \hat{W}_{p',p} \hat{\psi}_{p,p'} \right)
\end{aligned}
\tag{A.62e}
$$

The three operators $\hat{\psi}, \hat{\theta}, \hat{\phi}$ are based on their definition in equation A.25:

$$\hat{\theta}_{q,q'} = \hat{V}_{q,q'} \left(\hat{H} - \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1) \right) \tag{A.63}$$

$$\hat{\phi}_{p,p'} = \hat{W}_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} - 1, \hat{n}_p - 1) \right) \tag{A.64}$$

$$\hat{\psi}_{p,p'} = W^\dagger_{p,p'} \left(\hat{H} - \hat{H}(\hat{n}_{p'} + 1, \hat{n}_p + 1) \right) \tag{A.65}$$

In first order, we can expect the off-diagonal elements to vanish if $\hat{H} \neq \hat{H}(\hat{n}_q - 1, \hat{n}_{q'} + 1)\ \forall q, q'$ and $\hat{H} \neq \hat{H}(\hat{n}_p \pm 1, \hat{n}_{p'} \pm 1)\ \forall p, p'$

APPENDIX B

THE SECOND APPENDIX

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DECLARATION OF AUTHORSHIP

I hereby declare that I have written this thesis independently and by myself and that I have not used any sources or auxiliary materials other than those indicated in the thesis.

Munich, 22.06.2023 . . . .