不确定规划课程作业 6

方言
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1 Exercise 1

1.1 What is the truth value of $(X_1 \wedge X_2) \to X_2$?

令 $f(X_1, X_2) = (X_1 \wedge X_2) \to X_2$，则有:

$$f(1,1) = 1, \quad f(1,0) = 1, \quad f(0,1) = 1, \quad f(0,0) = 1$$

由此可知:

$$\begin{cases} \sup\limits_{f(X_1,X_2)=0} \min\limits_{1 \le i \le 2} v_i(X_i) = 0 \\ \sup\limits_{f(X_1,X_2)=1} \min\limits_{1 \le i \le 2} v_i(X_i) = \max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} \end{cases}$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = \alpha_1 \wedge \alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \le i \le 2} v_i(X_i) = 1$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = \alpha_1 \wedge (1-\alpha_2) \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \le i \le 2} v_i(X_i) = 1$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = (1-\alpha_1) \wedge \alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \le i \le 2} v_i(X_i) = 1$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = (1-\alpha_1) \wedge (1-\alpha_2) \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \le i \le 2} v_i(X_i) = 1$$

综上可知，$T(Z) = 1$

1.2 What is the truth value of $(X_1 \vee X_2) \to X_2$?

令 $f(X_1, X_2) = (X_1 \vee X_2) \to X_2$，则有:

$$f(1,1) = 1, \quad f(1,0) = 0, \quad f(0,1) = 1, \quad f(0,0) = 1$$

由此可知:

$$\begin{cases} \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = \alpha_1 \wedge (1-\alpha_2) \\ \sup_{f(X_1,X_2)=1} \min_{1\le i\le 2} v_i(X_i) = \max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} \end{cases}$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = \alpha_1\wedge\alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-(1-\alpha_2) = \alpha_2$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = \alpha_2\vee(1-\alpha_1) < 0.5$，因此:

$$T(Z) = \sup_{f(X_1,X_2)=1} \min_{1\le i\le 2} v_i(X_i) = (1-\alpha_1)\vee\alpha_2$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = (1-\alpha_1)\wedge\alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-(\alpha_1\wedge(1-\alpha_2)) = (1-\alpha_1)\vee\alpha_2$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = (1-\alpha_1)\wedge(1-\alpha_2) \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-\alpha_1$$

综上可知:

$$T(Z) = \begin{cases} \alpha_2 & \alpha_1 \ge 0.5,\ \alpha_2 \ge 0.5 \\ 1-\alpha_1 & \alpha_1 < 0.5,\ \alpha_2 < 0.5 \\ (1-\alpha_1)\vee\alpha_2 & \text{otherwise} \end{cases}$$

1.3 What is the truth value of $X_1 \to (X_1 \wedge X_2)$?

令 $f(X_1,X_2) = X_1 \to (X_1\wedge X_2)$，则有:

$$f(1,1)=1,\quad f(1,0)=0,\quad f(0,1)=1,\quad f(0,0)=1$$

由此可知:

$$\begin{cases} \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = \alpha_1 \wedge (1-\alpha_2) \\ \sup_{f(X_1,X_2)=1} \min_{1\le i\le 2} v_i(X_i) = \max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} \end{cases}$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = \alpha_1\wedge\alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-(1-\alpha_2) = \alpha_2$$

当 $\alpha_1 \ge 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = \alpha_2\vee(1-\alpha_1) < 0.5$，因此:

$$T(Z) = \sup_{f(X_1,X_2)=1} \min_{1\le i\le 2} v_i(X_i) = (1-\alpha_1)\vee\alpha_2$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 \ge 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = (1-\alpha_1)\wedge\alpha_2 \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-(\alpha_1\wedge(1-\alpha_2)) = (1-\alpha_1)\vee\alpha_2$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1\wedge\alpha_2, (1-\alpha_1)\wedge\alpha_2, (1-\alpha_1)\wedge(1-\alpha_2)\} = (1-\alpha_1)\wedge(1-\alpha_2) \ge 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1\le i\le 2} v_i(X_i) = 1-\alpha_1$$

综上可知:

$$T(Z) = \begin{cases} \alpha_2 & \alpha_1 \geq 0.5,\ \alpha_2 \geq 0.5 \\ 1 - \alpha_1 & \alpha_1 < 0.5,\ \alpha_2 < 0.5 \\ (1 - \alpha_1) \vee \alpha_2 & \text{otherwise} \end{cases}$$

1.4 What is the truth value of $X_1 \to (X_1 \vee X_2)$?

令 $f(X_1, X_2) = X_1 \to (X_1 \vee X_2)$，则有:

$$f(1,1) = 1, \quad f(1,0) = 1, \quad f(0,1) = 1, \quad f(0,0) = 1$$

由此可知:

$$\begin{cases} \sup\limits_{f(X_1,X_2)=0} \min\limits_{1 \leq i \leq 2} v_i(X_i) = 0 \\ \sup\limits_{f(X_1,X_2)=1} \min\limits_{1 \leq i \leq 2} v_i(X_i) = \max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} \end{cases}$$

当 $\alpha_1 \geq 0.5$ 且 $\alpha_2 \geq 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = \alpha_1 \wedge \alpha_2 \geq 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \leq i \leq 2} v_i(X_i) = 1$$

当 $\alpha_1 \geq 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = \alpha_1 \wedge (1-\alpha_2) \geq 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \leq i \leq 2} v_i(X_i) = 1$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 \geq 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = (1-\alpha_1) \wedge \alpha_2 \geq 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \leq i \leq 2} v_i(X_i) = 1$$

当 $\alpha_1 < 0.5$ 且 $\alpha_2 < 0.5$ 时，$\max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2), (1-\alpha_1) \wedge (1-\alpha_2)\} = (1-\alpha_1) \wedge (1-\alpha_2) \geq 0.5$，因此:

$$T(Z) = 1 - \sup_{f(X_1,X_2)=0} \min_{1 \leq i \leq 2} v_i(X_i) = 1$$

综上可知，$T(Z) = 1$

2 Exercise 2

2.1 What is the truth value of $A \to B$?

令 A, B 的 truth value 为 α_1, α_2，令:

$$Y_1 = A \vee B, \quad Y_2 = A \wedge B, \quad Z = A \to B$$

可知:

$$\begin{cases} T(Y_1) = \alpha_1 \vee \alpha_2 = a \\ T(Y_2) = \alpha_1 \wedge \alpha_2 = b \\ T(Z) = (1 - \alpha_1) \vee \alpha_2 \end{cases}$$

由此可得:

$$\begin{cases} \min|(1-\alpha_1)\vee\alpha_2-0.5| \\ \text{subject to}: \\ \quad 0\le\alpha_1\le 1 \\ \quad 0\le\alpha_2\le 1 \\ \quad \alpha_1\vee\alpha_2=a \\ \quad \alpha_1\wedge\alpha_2=b \end{cases}$$

当 $a=b$ 时，只有一个解 (a,a)，此时 $T(Z)=(1-a)\vee a$
当 $a>b$，有解 $\{(a,b),(b,a)\}$，此时在最优解情况下:

$$T(Z)=(1-a)\vee b=\begin{cases} 1-a, & a\ge b,\ a+b<1 \\ a \text{ or } b, & a\ge b,\ a+b=1 \\ b, & a\ge b,\ a+b>1 \end{cases}$$

当 $a<b$ 时，无解
综上:

$$T(Z)=\begin{cases} 1-a, & a\ge b,\ a+b<1 \\ a \text{ or } b, & a\ge b,\ a+b=1 \\ b, & a\ge b,\ a+b>1 \\ \text{illness}, & a<b \end{cases}$$

2.2 What is the truth value of C?

令 A,B,C 的 truth value 为 $\alpha_1,\alpha_2,\alpha_3$，令:

$$Y_1=A\to C,\quad Y_2=B\to C,\quad Y_3=A\vee B,\quad Z=(A\vee B)\to C$$

可知:

$$\begin{cases} T(Y_1)=(1-\alpha_1)\vee\alpha_3=a \\ T(Y_2)=(1-\alpha_2)\vee\alpha_3=b \\ T(Y_3)=\alpha_1\vee\alpha_2=c \end{cases}$$

由此可得:

$$\begin{aligned} T(Z)&=T(\neg(A\vee B)\vee C) \\ &=T((\neg A\vee C)\wedge(\neg B\vee C)) \\ &=T((A\to C)\wedge(B\to C)) \\ &=a\wedge b \end{aligned}$$

由不确定假言推理可知:

$$T(C)=\begin{cases} a\wedge b, & c+a\wedge b>1 \\ 0.5\wedge(a\wedge b), & c+a\wedge b=1 \\ \text{illness}, & c+a\wedge b<1 \end{cases}$$

2.3 What is the truth value of $C \vee D$?

令 A, B, C, D 的 truth value 为 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$，令:

$$Y_1 = A \to C,\quad Y_2 = B \to D,\quad Y_3 = A \vee B,\quad Z = C \vee D$$

可知:

$$\begin{cases} T(Y_1) = (1-\alpha_1) \vee \alpha_3 = a \\ T(Y_2) = (1-\alpha_2) \vee \alpha_4 = b \\ T(Y_3) = \alpha_1 \vee \alpha_2 = c \\ T(Z) = \alpha_3 \vee \alpha_4 \end{cases}$$

由此可得:

$$\begin{cases} \min |\alpha_3 \vee \alpha_4 - 0.5| \\ \text{subject to}: \\ \quad 0 \le \alpha_1 \le 1 \\ \quad 0 \le \alpha_2 \le 1 \\ \quad 0 \le \alpha_3 \le 1 \\ \quad 0 \le \alpha_4 \le 1 \\ \quad (1-\alpha_1) \vee \alpha_3 = a \\ \quad (1-\alpha_2) \vee \alpha_4 = b \\ \quad \alpha_1 \vee \alpha_2 = c \end{cases}$$

当 $a = b$ 时，
当 $a + c > 1$: 最优解为 $(\alpha_3, \alpha_4) = (a, b)$，此时 $T(Z) = a \vee b = a$
当 $a + c = 1$: 最优解为 $(\alpha_3, \alpha_4) = (a \wedge 0.5, a \wedge 0.5)$，此时 $T(Z) = a \wedge 0.5$
当 $a + c < 1$: 无解
当 $a \neq b$ 时，由对称性，不妨假设 $a > b$
当 $a + c < 1$ 或 $b + c < 1$: 无解
当 $b + c = 1, a + c > 1$ 时，
当 $b = c = 0.5$: 最优解对应 $T(Z) = 0.5$
当 $a > b > 0.5 > c$: 最优解对应 $T(Z) = b$
当 $b + c > 1, a + c > 1$ 时，
当 $a > b = c > 0.5$: 最优解对应 $T(Z) = b$
当 $a > b > c \ge 0.5$: 最优解对应 $T(Z) = b$
当 $a > b > 0.5 \ge c$: 最优解对应 $T(Z) = b$
当 $a > c > b \ge 0.5$: 最优解对应 $T(Z) = b$
当 $a > c > 0.5 \ge b$: 最优解对应 $T(Z) = 0.5$
当 $c \ge a > b \ge 0.5$: 最优解对应 $T(Z) = b$
当 $c \ge a > 0.5 \ge b$: 最优解对应 $T(Z) = 0.5$
当 $c > 0.5 \ge a > b$: 最优解对应 $T(Z) = a$
同理，由对称性可知，当 $a < b$ 时，只需要交换上述 a, b 即可。
整理可得:

$$T(Z) = \begin{cases} 0.5, & (a > 0.5 > b \text{ or } b > 0.5 > a) \text{ and } a + c \geq 1 \text{ and } b + c \geq 1 \\ a, & (b \geq a \geq 0.5 \text{ or } 0.5 \geq a \geq b) \text{ and } a + c \geq 1 \text{ and } b + c \geq 1 \\ b, & (a \geq b \geq 0.5 \text{ or } 0.5 \geq b \geq a) \text{ and } a + c \geq 1 \text{ and } b + c \geq 1 \\ \text{illness}, & a + c < 1 \text{ or } b + c < 1 \end{cases}$$

2.4 hat is the truth value of $B \vee C$?

令 A, B, C 的 truth value 为 $\alpha_1, \alpha_2, \alpha_3$，令:

$$Y_1 = A \vee B, \quad Y_2 = \neg A \vee C, \quad Z = B \vee C$$

可知:

$$\begin{cases} T(Y_1) = \alpha_1 \vee \alpha_2 = a \\ T(Y_2) = (1 - \alpha_1) \vee \alpha_3 = b \\ T(Z) = \alpha_2 \vee \alpha_3 \end{cases}$$

由此可得:

$$\begin{cases} \min |\alpha_2 \vee \alpha_3 - 0.5| \\ \text{subject to :} \\ \quad 0 \leq \alpha_1 \leq 1 \\ \quad 0 \leq \alpha_2 \leq 1 \\ \quad 0 \leq \alpha_3 \leq 1 \\ \quad \alpha_1 \vee \alpha_2 = a \\ \quad (1 - \alpha_1) \vee \alpha_3 = b \end{cases}$$

当 $a + b < 1$ 时，无解
当 $a + b = 1$ 时，可行解为 $\{a\} \times [0, a] \times [0, b]$
当 $a > 0.5 > b$ 时，最优解为 $\{a\} \times \{0.5\} \times [0, b]$，此时 $T(Z) = 0.5$
当 $b > 0.5 > a$ 时，最优解为 $\{a\} \times [0, a] \times \{0.5\}$，此时 $T(Z) = 0.5$
当 $a = b = 0.5$ 时，最优解对应 $T(Z) = 0.5$
当 $a + b > 1$ 时，不妨假设 $a \geq b$
当 $a = b > 0.5$ 时，最优解对应 $T(Z) = a$
当 $a > b \geq 0.5$ 时，最优解对应 $T(Z) = b$
当 $a > 0.5 \geq b$ 时，最优解对应 $T(Z) = 0.5$
同理，由对称性可知，当 $a < b$ 时，只需要交换上述 a, b 即可。
整理可得:

$$T(C) = \begin{cases} 0.5, & (a > 0.5 > b \text{ or } b > 0.5 > a), a + b \geq 1 \\ a, & b \geq a \geq 0.5 \\ b, & a \geq b \geq 0.5 \\ \text{illness}, & a + b < 1 \end{cases}$$