Complexity Solutions (1)

Date:______________

a) $T(n) = 2T(n/2) + n^4$

Ans)

• Akra Bazzi:

$$a = 2 \;,\; b = \frac{1}{2} \;,\; g(n) = n^4$$

$$2 \cdot \left(\frac{1}{2}\right)^p = 1 \;,\; \therefore p = 1$$

[Here, $p <$ power of $g(n)$
$\therefore$ We already know, $T(n) = O(n^4)$ ↓ $g(n)$]

Using Formula: $x^p + x^p \int_1^x \frac{g(n)}{n^{p+1}}\, dn$

$$\cancel{= x + x\int \frac{n^4}{n^3}\, dn = x + x\Big[\log n\Big]_1}$$

$$\cancel{= x + x\log x \rightarrow O(n + n\log n) \rightarrow O(n\log n)}$$

$$\cancel{\therefore T(n) = O(n\log n)}$$

$$= x + x\int_1^x \frac{n^4}{n^2}\, dn = x + x\left[\frac{n^3}{3}\right]_1^x = x + x\left(\frac{x^3 - 1}{3}\right)$$

$$= x + \frac{x^4}{3} - \frac{x}{3} \rightarrow O\left(n + \frac{n^4}{3} - \frac{n}{3}\right) \rightarrow O(n^4)$$

• Master's Theorem:

$$a = 2 \;,\; b = 2 \;,\; k = 4 \;,\; \log_b a = \log_2 2 = 1$$

$\therefore \log_2 2 < 4$

$\therefore \log_b a < k$ ∴ Case 3

In $(n^4 \log^0 n)$, $p = 0$, First condition

$$\therefore T(n) = O(n^k \log^p n) = O(n^4 \log^0 n)$$

$$= O(n^4)$$

Date

b) $T(n) = T(7n/10) + n$

Ans

• Akra Bazzi:

$a = 1$, $b = 7/10$, $g(n) = n$

$1 \cdot \left(\frac{7}{10}\right)^p = 1$, For $p = 0$, condition is satisfied.

again, $p = 0 <$ power of 'n' in $g(n) = 1$

$\therefore$ We know, $T(n) = O(g(n)) = O(n)$

• Master's Theorem:

$a = 1$, $b = 10/7 \approx 1.4$, $k = 1$, $f(n) = n^1 \log^0 n$

$\therefore \log_b a = \log_{1.4} 1 = 0$

Again, $\log_b a < k$ and $p = 0$ for $f(n)$

Thus, by Case 3, $T(n) = O(n^k \log^p n) = O(n)$

c) $T(n) = 16T(n/4) + n^2$

Ans

• Akra Bazzi:

$a = 16$, $b = 1/4$, $g(n) = n^2$

$16 \cdot \left(\frac{1}{4}\right)^p = 1$, $\therefore p = 2$

By formula, $x^p + x^p \int_1^x \frac{g(n)}{n^{p+1}} dn = x^2 + x^2 \int_1^x \frac{n^2}{n^3} dn$

$= x^2 + x^2 [\log n]_1^x = x^2 + x^2 \log x = O(n^2 + n^2 \log n)$

Date:

$= O(n^2 \log n)$ //

- Master's Theorem:

$a = 16,\ b = 4,\ k = 2,\ p = 0$

$\therefore \log_b a = 2 = k$ ... Case 2

$p = 0 > -1$ ... First Condition

$\therefore T(n) = O(n^k \log^{p+1} n) = O(n^2 \log n)$ //

d) $T(n) = 7T(n/3) + n^2$

Ans)

- Akra Bazzi:

$a = 7,\ b = 1/3,\ g(n) = n^2$

$7 \cdot \left(\frac{1}{3}\right)^p = 1$, For $p = 0$, we get $7 > 1$

For $p = 1$, we get $7/3 \approx 2.33 > 1$

For $p = 2$, we get $7/9 \approx 0.77 < 1$

$\therefore$ Actual 'p' value lies between 1 & 2.

$\therefore p < 2 =$ power of 'n' in $g(n)$

$\therefore T(n) = O(g(n)) = O(n^2)$ //

- Master's Theorem:

$a = 7,\ b = 3,\ k = 2,\ p = 0$

$\therefore \log_b a = \log_3 7 \approx 1.77 < k = 2$ ... Case 3

As, $p = 0$, first condition applies.

$\therefore T(n) = O(n^k \log^p n) = O(n^2)$ //

Date:

e) $T(n) = 7T(n/2) + n^2$

Ans)

- Akra Bazzi:

$a = 7, \quad b = 1/2, \quad g(n) = n^2$

$7 \cdot \left(\frac{1}{2}\right)^p = 1$, For $p = 0$, we get $7 > 1$
For $p = 1$, we get $3.5 > 1$
For $p = 2$ we get $1.75 > 1$ $\Big\}$ $p = \log_2 7$
For $p = 3$ we get $0.8 < 1$

$\therefore$ 'p' falls between 2 & 3 and $p > 2$ = pow. of 'n' in $g(n)$

Using formula: $x^p + x^p \int_1^x \frac{n^2}{n^{p+1}} dn = x^p + x^p \int_1^x n^{1-p} dn$

$\cancel{= x^p + x^p \left[\frac{n^{2-p}}{2-p}\right]_1^x = x^p + x^p\left(x^{2-p} - 1\right) = x^p + x^2}$ ~~ignore const~~

$= x^p + x^p \left[\frac{n^{2-p}}{(2-p)}\right]_1^x$ As $p > 2$, assume -1 $\quad = x^p + x^p \left[\frac{x^{2-p} - 1}{-1}\right]$

$= x^p + x^p\left(1 - x^{2-p}\right) = x^p + x^p - x^2 = 2x^p - x^2$

$\because$ We know, $p > 2$, 'x^2' is less dominant

$\therefore T(n) = O(2x^p) = O(x^p) \approx O(x^{\log_2 7})$

$= O(n^{\log_2 7})$ // $\boxed{\text{Thus, if } p > \text{pow. of 'n' in } g(n),\ T(n) = O(n^p)}$

- Master's Theorem:

$a = 7, \quad b = 1/2^{-1} \to 2, \quad g(n) = n^2, \quad k = 2, \quad p = 0$ $\quad$ Imp. Rule

$\therefore \log_b a = \log_2 7 \approx 2.8 > 2 = k \quad \therefore$ Case 1

$\therefore T(n) = O(n^{\log_b a}) = O(n^{\log_2 7})$ //