Problem 4.1

(a) Euler-Lagrange equation

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \quad \text{— (1)}$$

$$F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} = 0. \quad \text{— (2)}$$

$$F_u = 2(f_x u + f_y v + f_t) \cdot f_x$$

$$F_v = 2(f_x u + f_y v + f_t) \cdot f_y$$

$$F_{u_x} = \alpha \Psi'_{u_x}(|\nabla u|^2 + |\nabla v|^2) \cdot 2u_x$$

$$F_{u_y} = \alpha \Psi'_{u_y}(|\nabla u|^2 + |\nabla v|^2) \cdot 2u_y$$

$$F_{v_x} = \alpha \Psi'_{v_x}(|\nabla u|^2 + |\nabla v|^2) \cdot 2v_x$$

$$F_{v_y} = \alpha \Psi'_{v_y}(|\nabla u|^2 + |\nabla v|^2) \cdot 2v_y$$

On (1)

$\cancel{2(f_x u + f_y v + f_t) \cdot f_x}$

$$F_u - \frac{\partial}{\partial x}\left(2\alpha \Psi'_{u_x}(|\nabla u|^2 + |\nabla v|^2) \cdot u_x\right) - \frac{\partial}{\partial y}\left(2\alpha \Psi'_{u_y}(|\nabla u|^2 + |\nabla v|^2) \cdot u_y\right) = 0.$$

calculation (lec 15) - reference

$$F_u - 2\alpha \, \mathrm{div}\left(\psi'(|\nabla u|^2 + |\nabla v|^2) \cdot \nabla u\right) = 0.$$

$$f_x \cdot (f_x u + f_y v + f_t) - \alpha \, \mathrm{div}\left(\psi'(|\nabla u|^2 + |\nabla v|^2 \cdot \nabla u\right) = 0$$

Similarly on 2

$$(f_x u + f_y v + f_t) \cdot f_y - \alpha \, \mathrm{div}\left(\psi'(|\nabla u|^2 + |\nabla v|^2 \cdot \nabla v\right) = 0.$$

(b) Derivative $\psi'(s^2)$:-

$$\psi(s^2) = \lambda^2 \log\left(\frac{\lambda^2 + s^2}{\lambda^2}\right)$$

$$\psi'(s^2) = \lambda^2 \cdot \frac{1}{\frac{\lambda^2 + s^2}{\lambda^2}} \cdot \frac{\partial}{\partial(s^2)}\left(\frac{\lambda^2 + s^2}{\lambda^2}\right)$$

$$= \cancel{\lambda^2} \cdot \frac{\lambda^2}{\lambda^2 + s^2} \cdot \frac{1}{\cancel{\lambda^2}}$$

$$\psi'(s^2) = \frac{\lambda^2}{\lambda^2 + s^2}$$

Problem 3.2 Segmentation

(from lec) Solving $E(k \setminus \partial(\Omega_i, \Omega_j)) - E(k) = 0$ for λ,

$$\lambda_{i,j} = \frac{\frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|} (u_i - u_j)^2}{\ell(\partial(\Omega_i, \Omega_j))}$$

(Ω_2)

	14	6	6	2
	14	6	6	2
(Ω_3)	14	14	14	2 (Ω_1)
	14	14	14	2

$u_1 = 2 \quad u_2 = 6 \quad u_3 = 14$

$|\Omega_1| = 4 \quad |\Omega_2| = 4 \quad |\Omega_3| = 8$

Merging event:

$$\lambda_{1,2} = \frac{\frac{4 \cdot 4}{4+4} (2-6)^2}{2} = 16$$

$$\lambda_{1,3} = \frac{\frac{32}{12} (2-14)^2}{2} = 192$$

$$\lambda_{2,3} = 42.66$$

Smallest λ corresponds to $\lambda_{1,2}$!

$\boxed{\lambda = 16}$ New gray value $u_{12} = \frac{4 \cdot 2 + 4 \cdot 6}{8}$

$u_{12} = 4.$

14	4	4	4
14	4	4	4
14	14	14	4
14	14	14	4

merging event :

$$\lambda_{12,3} = \frac{\frac{8 \cdot 8}{16} (-10)^2}{6}$$

$$\lambda_{12,3} = 66.66$$

new $\lambda = 66.66$ (from $\lambda_{12,3}$).

Gray value for this merging event

$$u_{123} = \frac{8 \cdot 4 + 8 \cdot 14}{16}$$

$$u_{123} = 9.$$

Final image after merging

9	9	9	9
9	9	9	9
9	9	9	9
9	9	9	9