Grouped Patterns of Heterogeneity in Panel Data

(Bonhomme and Manresa, 2015)

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2. Specification
3. Computation and an estimator
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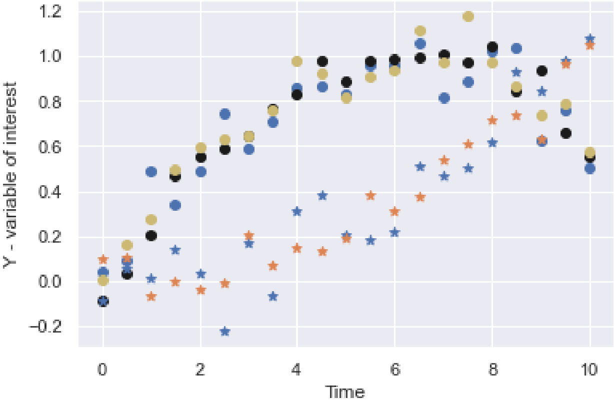
Motivation

The panel units may tend to cluster around different trends.



Common problems with usual fixed effects

- 'Incidental parameter' bias may be substantial in short panels.
- Unobserved heterogeneity is constant over time.

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Specification: Flexible way to account for unobserved heterogeneity

Group-specific, time-variant effect:

$$y_{it} = x_{it}'\theta + \alpha_{g_i t} + v_{it}, i = 1, \ldots, N, t = 1, \ldots, T$$

- y_{it} - outcome
- x_{it} - covariates
- $\alpha_{g_i t}$ - group-specific time effect, the group membership variables g_i, for all $i \in 1, \ldots, N$
- v_{it} - residual

Assumptions:

- Leaves the correlation of GFE with explanatory variables unrestricted
- No need of parallel trends assumption

Possible extensions

Unit-specific time-invariant heterogeneity:

$$y_{it} = x_{it}'\theta + \alpha_{g_i t} + \eta_i + v_{it}$$

Heterogeneous group-specific effects of covariates:

$$y_{it} = x_{it}'\theta_{g_i} + \alpha_{g_i t} + v_{it}$$

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Computation

Objective

$$(\hat{\theta}, \hat{\alpha}, \hat{\gamma}) = \underset{(\theta,\alpha,\gamma)\in\Theta\times\mathscr{A}^{GT}\times\Gamma_G}{\operatorname{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}(y_{it} - x_{it}'\theta - \alpha_{g_i t})^2$$

where $\gamma \in \Gamma_G$ denotes a particular grouping (i.e., partition) of the N units, where Γ_G is the set of all groupings of $\{1, \ldots, N\}$ to at most G groups.

Computation hurdle: Partition of N elements to G groups (explosive) - Stirlings number of the second kind.

Computation

Algorithm 1

1. Let $(\theta^{(s)}, \alpha^{(s)}) \in \Theta \times \mathscr{A}^{GT}$ be some starting value. Set $s = 0$.
2. Compute for all $i \in \{1, \ldots, N\}$:

$$g_i^{(s+1)} = \underset{g \in \{1,\ldots,G\}}{\operatorname{argmin}} \sum_{t=1}^{T} (y_{it} - x_{it}' \theta^{(s)} - \alpha_{g_i t}^{(s)})^2$$

Here, for large problems, one can apply k-means clustering algorithm on residuals after covariates

Computation:

Algorithm 1 - continued

3. Compute:

$$(\theta^{(s+1)}, \alpha^{(s+1)}) = \underset{(\theta,\alpha)\in\Theta\times\mathcal{A}^{GT}}{\operatorname{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}(y_{it} - x_{it}'\theta - \alpha_{g_i^{(s+1)}t})^2$$

   Here, one does an OLS regression that controls for the interactions of group indicators and time dummies.

4. Set $s = s + 1$ and go to Step 2 (until numerical convergence)

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Asymptotic properties: Consistency

Assumption 1

There exists a constant $M > 0$ such that:

(a) Θ and $\mathscr{A}$ are compact subsets of R^k and R, respectively.

(b) $E(\|x_{it}\|^2) \leq M$, where $\|\cdot\|$ denotes the Euclidean norm.

(c) $E(v_{it}) = 0$, and $E(v_{it}^4) \leq M$.

(d) $|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} E(v_{it}v_{is}x_{it}'x_{is})| \leq M$.

(e) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|\sum_{t=1}^{T} E(v_{it}v_{jt})| \leq M$.

(f) $|\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(v_{it}v_{jt}, v_{is}v_{js})| \leq M$.

Asymptotic properties: Consistency

Assumption 1 - continued

(g) Let $\bar{x}_{g \wedge \tilde{g},t}$ denote the mean of x_{it} in the intersection of groups $g_i^0 = g$, and $g_i = \tilde{g}$. For all groupings $\gamma = \{g_1, \ldots, g_N\} \in \Gamma_G$, define $\hat{\rho}(\gamma)$ as the minimum eigenvalue of the following matrix [a]:

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it} - \bar{x}_{g_i^0 \wedge g_i,t})(x_{it} - \bar{x}_{g_i^0 \wedge g_i,t})'$$

Then $plim_{N,T\to\infty} min_{\gamma\in\Gamma_G}\hat{\rho}(\gamma) = \rho > 0$.

[a] $\tilde{x}$ refers to be the infeasible version of the GFE estimator where group membership g_i, instead of being estimated, is fixed to its population counterpart g_i^0, and then estimated with OLS.

Asymptotic properties: Consistency

Theorem 1

Let Assumption 1 hold. Then, as N and T tend to infinity:

$$\hat{\theta} \xrightarrow{p} \theta^0 \quad \text{and}$$

$$\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{\alpha}_{\hat{g}_i t} - \alpha^0_{g_i^0 t})^2 \xrightarrow{p} 0.$$

where the ^{0} superscripts refer to true parameter values.

Asymptotic properties: Distribution

Assumption 2

(a) For all $g \in \{1, \dots, G\}$: $plim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} 1\{g_i^0 = g\} = \pi_g > 0$.

(b) For all $(g, \tilde{g}) \in \{1, \dots, G\}^2$ such that $g \neq \tilde{g}$: $plim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt}^0 - \alpha_{\tilde{g}t}^0)^2 = c_{g,\tilde{g}} > 0$.

(c) There exist constants $a > 0$ and $d_1 > 0$ and a sequence $\alpha[t] \leq e^{-at^{d_1}}$ such that, for all $i \in \{1, \dots, N\}$ and $(g, \tilde{g}) \in \{1, \dots, G\}^2$ such that $g \neq \tilde{g}, \{v_{it}\}_t, \{\alpha_{gt}^0 - \alpha_{\tilde{g}t}^0\}_t$, and $(\alpha_{gt}^0 - \alpha_{\tilde{g}t}^0)v_{it\,t}$ are strongly mixing processes with mixing coefficients $\alpha[t]$. Moreover, $E((\alpha_{gt}^0 - \alpha_{\tilde{g}t}^0)v_{it}) = 0$.

Asymptotic properties: Distribution

Assumption 2 - continued

(d) There exist constants $b > 0$ and $d_2 > 0$ such that $Pr(|v_{it}| > m) \leq e^{1-(\frac{m}{b})^{d_2}}$ for all i, t, and $m > 0$.

(e) There exists a constant $M^* > 0$ such that, as N, T tend to infinity:

$$\underset{i \in \{1,...,N\}}{sup} Pr\Big(\frac{1}{T}\sum_{t=1}^{T} \|x_{it}\| \geq M^*\Big) = o(T^{-\Sigma})$$

for all $\delta > 0$.

Asymptotic properties: Distribution

Theorem 2

Asymptotic Equivalence: Let Assumptions 1 and 2 hold. Then, for all $\Sigma > 0$ and as N and T tend to infinity:

(16) $Pr(\underset{i \in 1,\ldots,N}{|\hat{g}_i - g_i^0|} > 0) = o(1) + o(NT^{-\Sigma})$

(17) $\hat{\theta} = \tilde{\theta} + o_p(T^{-\Sigma})$

(18) $\hat{\alpha}_{gt} = \tilde{\alpha}_{gt} + o_p(T^{-\Sigma})$ for all g, t.

Asymptotic properties: Distribution

Assumption 3

(a) For all i, j, and t: $E(x_{jt} v_{it}) = 0$

(b) There exist positive definite matrices Σ_θ and Ω_θ such that

$$\Sigma_\theta = \underset{N,T\to\infty}{plim} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{g_i^0 t})(x_{it} - \bar{x}_{g_i^0 t})',$$

$$\Omega_\theta = \lim_{N,T\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[v_{it} v_{js} (x_{it} - \bar{x}_{g_i^0 t})(x_{js} - \bar{x}_{g_j^0 s})']$$

Asymptotic properties: Distribution

Assumption 3 - continued

(c) As N and T tend to infinity:
$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{g_i^0 t}) v_{it} \xrightarrow{d} N(0, \Omega_\theta)$.

(d) For all (g, t):
$lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} E(1\{g_i^0 = g\} 1\{g_j^0 = g\} v_{it} v_{jt}) = \omega_{gt} > 0$

(e) For all (g, t), and as N and T tend to infinity:
$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} 1\{g_i^0 = g\} v_{it} \xrightarrow{d} N(0, \omega_{gt})$

Asymptotic properties: Distribution

Corollary 1 (Theorem 2): Asymptotic Distribution

Let Assumptions 1, 2, and 3 hold, and let N and T tend to infinity such that, for some $v > 0, N/T^v \longrightarrow 0$. Then, we have

(19) $\sqrt{NT}(\hat{\theta} - \theta^0) \xrightarrow{d} N(0, \Sigma_\theta^{-1}\Omega_\theta\Sigma_\theta^{-1})$, and, for all (g, t),

(20) $\sqrt{N}(\hat{\alpha}_{gt} - \alpha_{gt}^0) \xrightarrow{d} N(0, \frac{\omega_{gt}}{\pi_g^2})$,

where π_g is defined in Assumption 2, and where $\Sigma_\theta, \Omega_\theta$, and ω_{gt} are defined in Assumption 3.

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Does income affect democracy?

Acemoglu et al. (2008) found that the positive association between income and democracy disappears when controlling for additive country and time-effects.

$$democracy_{it} = \theta_1 democracy_{it-1} + \theta_2 logGDPpc_{it-1} + \alpha_i + v_{it}$$

The authors interpreted the country fixed-effects as reflecting long-run, historical factors that have shaped the political and economic development of countries.

Does income affect democracy?

Estimate an alternative grouped fixed effects for countries:

$$democracy_{it} = \theta_1 democracy_{it-1} + \theta_2 logGDPpc_{it-1} + \alpha_{g_i t} + v_{it}$$

The cumulative income effect on democracy and the number of groups

After controlling for group-specific, time-variant effects, the impact of the cumulative income becomes trivial.

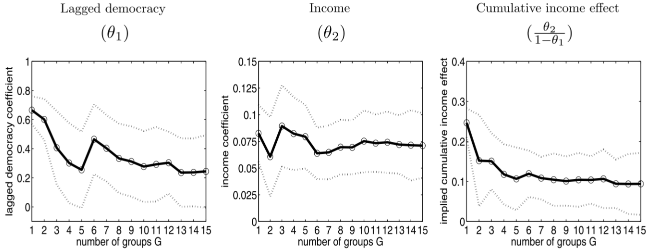

FIGURE 1.—Coefficients of income and lagged democracy. *Note*: Balanced panel from Acemoglu et al. (2008). The x-axis shows the number of groups G used in estimation, the y-axis reports parameter values. 95%-confidence intervals clustered at the country level are shown in dashed lines. Confidence intervals are based on bootstrapped standard errors (100 replications).

Group-specific time-effects explain the majority of differences in democratic changes

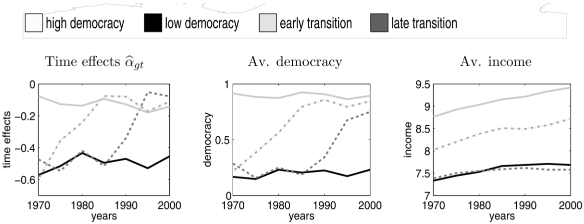

FIGURE 2.—Patterns of heterogeneity, $G = 4$. *Note*: See the notes to Figure 1. On the bottom panel, the left graph shows the group-specific time effects $\widehat{\alpha}_{gt}$. The other two graphs show the group-specific averages of democracy and lagged log-GDP per capita, respectively. Calendar years (1970–2000) are shown on the x-axis. Light solid lines correspond to Group 1 ("high-democracy"), dark solid lines to Group 2 ("low-democracy"), light dashed lines to Group 3 ("early transition"), and dark dashed lines to Group 4 ("late transition"). The top panel shows group membership. The list of countries by group is given in the Supplemental Material.

Simulation

Conclusion and discussion

- Offers a flexible yet parsimonious approach to model unobserved heterogeneity.
- Useful in applications where time-varying grouped effects may be present in the data.
- GFE should be well-suited in difference-in-difference designs, as a way to relax parallel trends assumptions.
- The extension to nonlinear models is a natural next step, however, statistical and computational challenges increase.

Thank you!

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