Answers to the Problem Set 2

Macroeconomis (Ph.D.)

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Question 1: Pareto Optimality in OLG economies

Since there is neither money nor government in the economy to facilitate the transfer of resources between periods, the inter-generational trade is impossible. Therefore, as the utility function is strictly increasing over consumption, the agents will consume all of its endowment in a given period himself/herself and the uniques equilibrium is an autarky.

The representative generation will have the following objective and constrait if there was a way to shift consumption between periods.

$$\max_{c_t^t, c_{t+1}^t} u(c_t^t, c_{t+1}^t) = log(c_t^t) + log(c_{t+1}^t)$$

$$\text{s.t.}$$

$$c_t^t + c_{t+1}^t = w_1 + w_2 \text{ - equality since log-utility}$$

$$0 < c_t^t \leq w_1$$

$$0 < c_{t+1}^t \leq w_2$$

If the λ is the Lagrange multiplier, then at the optimum

$$\lambda = \frac{\partial u(c_t^t, c_{t+1}^t)}{\partial c_t^t} = \frac{1}{c_t^t},$$

$$\lambda = \frac{\partial u(c_t^t, c_{t+1}^t)}{\partial c_{t+1}^t} = \frac{1}{c_{t+1}^t}$$

or

$$\frac{\partial u(c_t^t, c_{t+1}^t)}{\partial c_t^t} = \frac{\partial u(c_t^t, c_{t+1}^t)}{\partial c_{t+1}^t} \Rightarrow c_t^t = c_{t+1}^t$$

and

$$0 < c_t^t \leq w_1, \ 0 < c_{t+1}^t \leq w_2$$

must be satisfied. However, as mentioned before, there is no way to shift consumption between periods, thus each generation should consume its own endowment, in other words, an autarky,

$$c_t^t = w_1,\ c_{t+1}^t = w_2$$

If $w_1 \neq w_1$, then the marginal utility of the consumption of the two periods will not be equal. Thus, the agent will be willing to give up some consumption in the period when it has higher endowment to consume more in the other period, thus smooting the consumption over periods. This willingness to give up some consumption to get other later can be expressed as the interest rate she/he is willing to pay. Denote r as the interest rate. Then the optimal interest rate she/he will be indifferent is:

$$1 + r = \frac{u_1(w_1, 0)}{u_2(0, w_2)}$$

1. If $w_1 > w_2$, then $u_1(w_1) < u_1(w_1)$ and $r < 0$. Hence, the young is willing to exchange one unit of its young endowment with less than one unit of good in the next period. If there is a means of saving for the young to consume when old and the amount of such saving is up to the point where the consumption of young and old periods are equal, then this will leave all the generation, except the initial old, equally satisfied and the initial old better off. Therefore, it is clear that the unique equilibrium of autarky is not Pareta optimal.

2. If $w_1 < w_2$, then $u_1(w_1) > u_1(w_1)$ and $r > 0$. The young wants to consume more than its endowment when young by borrowing from the old generation and pay off its debt from the endowment at old when she/he is old with higher amount. But, if the inter-generational trade is implemented, it means that the initial old needs to lend some of its endowment to the first period young which will make him/her strictly worse off. If the initial old doesn't participate, then the first young will neither participate when he/she is old. Therefore, the unique equilibrium of autarky is Pareta optimal in this case.

Question 2: Money and Social Security in OLG models

1. The generations except the initial will have the following life-time utility maximization problem (sequential):

$$\max_{c_t^t, c_{t+1}^t, s_t^t} u(c_t^t, c_{t+1}^t) = log(c_t^t) + log(c_{t+1}^t)$$

$$\text{s.t}$$

$$c_t^t + s_t^t \leq w_1$$

$$c_{t+1}^t \leq (1 + r_{t+1}) s_t^t$$

or (given log utility)

$$c_t^t + \frac{c_{t+1}^t}{1 + r_{t+1}} = w_1$$

For the initial, the problem is:

$$\max_{c_1^0} u(c_1^0)$$

$$\text{s.t.}$$

$$c_1^0 \leq (1 + r_1) m$$

The FOCs for the representative generation is (λ_1 and λ_2 for the constraint, respectively):

$$\frac{1}{c_t^t} = \lambda_1,\ \frac{1}{c_{t+1}^t} = \lambda_2,\ \lambda_1 = \lambda_2 (1 + r_{t+1})$$

or

$$\frac{1}{c_t^t} = \frac{1}{c_{t+1}^t}(1 + r_{t+1})$$

Taking into account the budget constraints, it will yield:

$$c_t^t = \frac{1}{2} w_1,\ c_{t+1}^t = \frac{1}{2}(1 + r_{t+1}) w_1 \tag{1}$$

On top of this, the goods market clearing condition and the budget constraints for the two generations will provide the following identities (inuequalities turn to equalities given the log utility):

$$c_t^{t-1} + (1+n)c_t^t = (1+n)w_1 \tag{2}$$

$$c_t^{t-1} + (1+n)(c_t^t + s_t^t) = (1+n)w_1 + (1+r_t)s_t^{t-1}$$

Then by Walras' law, the asset market will clear at:

$$s_t^t = \frac{(1+r_t)s_t^{t-1}}{1+n}$$

Using goods market clearing condition in the equation 2 and equilibrium consumptions in equation 1, I get:

$$c_t^{t-1} + (1+n)c_t^t = \frac{1}{2}(1+r_t)w_1 + \frac{1}{2}(1+n)w_1 = (1+n)w_1 \tag{3}$$

$$r_t = n \tag{4}$$

Thus,

$$c_t^t = \frac{1}{2}w_1,\ c_{t+1}^t = \frac{1}{2}(1+n)w_1 \tag{5}$$

While the initial old had an endowment of m units of money for which he purchased $\frac{m}{p_1}$ units of goods from the initial young, the initial young will be able to purchase $\frac{m}{p_2}$ unit of goods when he/she is old. Since there is no alternative means of savings available in the economy, the interest raid paid on the money will be the change in the value of the money in real terms or the real effect of the changes in prices, i.e.,

$$\frac{\frac{m}{p_2}}{\frac{m}{p_1}} = \frac{p_1}{p_2} = 1 + r_2$$

This will repeat infinitely. Hence, $1 + r_{t+1} = \frac{p_t}{p_{t+1}}$. The first young generation's savings in real terms $m^R = \frac{m}{p_1}$. Plugging it into the asset market clearing condition, I get:

$$s_1^1 = \frac{(1+r_1)m^R}{(1+n)}$$

$$s_t^t = \prod_{\tau=1}^{t} \frac{(1+r_\tau)}{1+n} m^R$$

Taking into account the equation 4, $s_t^t = m^R$, in other words, the price is decreasing at the rate n percent (deflation) and keeps the savings per capita constant in real terms in the equilibrium.

From the consumption of the initial old, the price level will be drived easily as:

$$\begin{aligned} c_1^0 = \frac{1}{2}(1+n)w_1 &= \frac{m}{p_1} \\ p_1 &= \frac{2m}{(1+n)w_1} \end{aligned}$$

Since the price is declining at rate n

$$p_t = \frac{1}{(1+n)^t}\frac{2m}{w_1} \tag{6}$$

Saving less than the equilibrium will hurt the initial old and the consumption of the future generations in old. Even worse, if there is no trade at all, each generation will starve to death in the old period and the marginal utility will be infinite. Thus autarky cannot be an equilibrium. On the other hand, saving more will lead the each generation to consume less when young. Neither of which will be less desirable than the optimum. Moreover, the initial price is picked just such that it ensures the initial old consumes the optimal consumption of old. Thus the equilibrium is unique.

2. The problem setting is the same to the previous one only with the social security system embedded.

$$\max_{c_t^t, c_{t+1}^t, s_t^t} u(c_t^t, c_{t+1}^t) = log(c_t^t) + log(c_{t+1}^t)$$

s.t

$$c_t^t + s_t^t \leq w_1 - \tau$$
$$c_{t+1}^t \leq (1+r_{t+1})s_t^t + (1+n)\tau$$

or (given log utility)

$$c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = w_1 - \tau + \frac{1+n}{1+r_{t+1}}\tau$$

For the initial, the problem is:

$$\max_{c_1^0} u(c_1^0) \text{ s.t.}$$

$$c_1^0 \leq (1+r_1)m + (1+n)\tau$$

The Euler equation is the same as previous, namely,

$$\frac{1}{c_t^t} = \frac{1}{c_{t+1}^t}(1+r_{t+1})$$

The consumptions are

$$\begin{aligned} c_t^t &= \frac{1}{2}(w_1 - \tau + \frac{1+n}{1+r_{t+1}}\tau) \\ c_{t+1}^t &= \frac{(1+r_{t+1})}{2}(w_1 - \tau + \frac{1+n}{1+r_{t+1}}\tau) \end{aligned}$$

$$c_t^{t-1} + (1+n)c_t^t = (1+n)w_1 \tag{7}$$

$$c_t^{t-1} + (1+n)(c_t^t + s_t^t) = (1+n)(w_1 - \tau) + (1+r_t)s_t^{t-1} + (1+n)\tau$$

Then by Walras' law, the asset market will clear at, the same as previous,:

$$s_t^t = \frac{(1+r_t)s_t^{t-1}}{1+n}$$

The feasibility condition and the optimal consumptions will yield:

$$\begin{aligned} c_t^{t-1} + (1+n)c_t^t = \frac{1}{2}(1+r_t)(w_1 - \tau + \frac{1+n}{1+r_t}\tau) + \\ \frac{1}{2}(1+n)(w_1 - \tau + \frac{1+n}{1+r_{t+1}}\tau) = (1+n)w_1 \end{aligned}$$

Thus, the stationary equilibrium can be attained at:

$$r_t = r_{t+1} = n \tag{8}$$

In which, $c_t^t = \frac{1}{2}w_1$ and $c_{t+1}^t = \frac{1}{2}(1+n)w_1$.
That means, the young is saving:

$$s_t^t = w_1 - \tau - \frac{1}{2}w_1 = \frac{1}{2}w_1 - \tau$$

The consumption and income when old look:

$$\begin{aligned}\text{consumption} = \text{savings+social benefit} &= (1+n)(\frac{1}{2}w_1 - \tau) + (1+n)\tau \\ &= \frac{1}{2}(1+n)w_1\end{aligned}$$

as driven above.

From the savings equation of the young, the young will want to borrow, instead of saving, if the tax is so high that $\tau > \frac{1}{2}w_1$. In this case, the initial old (and future olds) will not be willing to give up any consumption and the equilibrium will be autarky. Hence, the money will have no value. If $\tau = \frac{1}{2}w_1$, then the young is indifferent between saving or not and the money will have no value either and the autarky will prevail. Only if $\tau < \frac{1}{2}w_1$, then the money will have positive value and the corresponding equilibrium will be attained.

Regarding the condition $r_t = n$, whenever $r_t < n$, the young will be better off by saving (in form of money, of course), because the yield (n) will be higher than his/her willingness to pay (r_t). $r_t < n$ is likely to result when the young has much more current consumption than when it is old. If the tax is so high, then the condition can reverse to $r_t > n$ and the old will refuse to trade with the young resulting in autarky.

3. It is clear that $1 + r_t = \frac{p_{t-1}}{p_t}$ and p_1 can be derived easily as $p_1 = \frac{m}{(1+n)(\frac{1}{2}w_1-\tau)}$ as in Section 2.1. This yield the equilibrium. If the p_1 is different, then the backward bending offer curve (or richer young and poorer old and the young is more willing to trade) tells that the other starting point will lead eventually away from the equilibrium towards an autarky.

4. As it is already driven above, an stationary equilibrium exists with $c_t^t = \frac{1}{2}w_1$ and $c_{t+1}^t = \frac{1}{2}(1+n)w_1$, $p_1 = \frac{m}{(1+n)(\frac{1}{2}w_1-\tau)}$ and $1+r_t = 1+n = \frac{p_{t-1}}{p_t}$.

   Thus,

$$p_t = \frac{1}{(1+n)^t} \frac{m}{(\frac{1}{2}w_1 - \tau)}$$

   The higher the tax, the higher the price will be for the given money supply.

Question 3: OLG models in application

The problem is formulated as below:

$$v(a,t) = log(c_{t_0}) + \beta v(a,t+1)$$

s.t

$$\begin{aligned}
&a_{t+1} + c_{t+1} = (1+r)a_t + y_t \\
&0 \leq t \leq T \\
&a_t > 0, \forall t \\
&v(a, T+1) = 0
\end{aligned}$$

v - is the value function for given a at time t.

Since the agent knows that he will die after age T, he will consume everything he/she has in the last period. That is

$$c_T = (1+r)a_T + y_T$$

The agent knows this already in the beginning of her life. Hence, there is no uncertainty and he/she can plan it perfectly. Given the log utility, the agent will aim to smooth its consumption over time which will also be affected by his/her time preference (β) and the return on the asset. Therefore, the agent's consumption plan will be governed by the Euler equation.

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}}$$

Once we know the last value of the asset, namely a_T, the last period consumption will be derived easily. From that we can compute the consumption in all periods using Euler equation. Once we know the consumption plan, it is just another repeated use of the budget constraints to derive the amount of the asset (or savings from each period) in each period. Therefore, the key computational problem is to find such an a_T which gives the highest life-time utility or value.

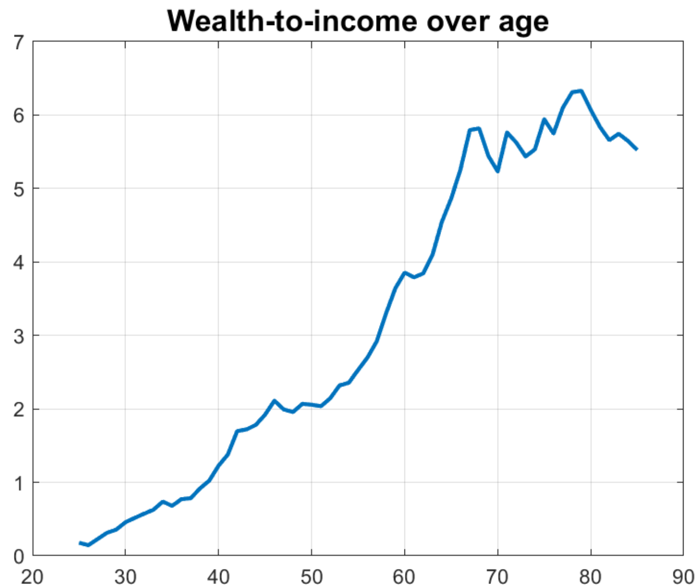
My approach for selecting a_T for the given set of parameters was to choose a_T such that a_{T-1} is just $1+r$ times lower than it. Although it may not be a sufficient for finding the optimal a_T, it is necessary condition. Using this logic, I searched for the optimal a_T using my UDF "min_guess" and others, and the Matlab built-in "fminbnd" function. The code should definitely be improved in the direction of searching within the grid of a_T.

The findings from the data and the model, and the answers to the question follow.

Note: Computations on the SCF data are saved in "compute_medians.py" and the optimization and plotting operations are in "main.m" file.

1. The ratio of the median wealth over the median income for a US citizen increases over time until about 70 years old and then stabilizes (Figure 1). That is due to the accumulation of wealth over time which (the ratio) further boosted by the slight decline of income as one nears the age of 85.

Figure 1: Median wealth and income ratio over age in the U.S.A



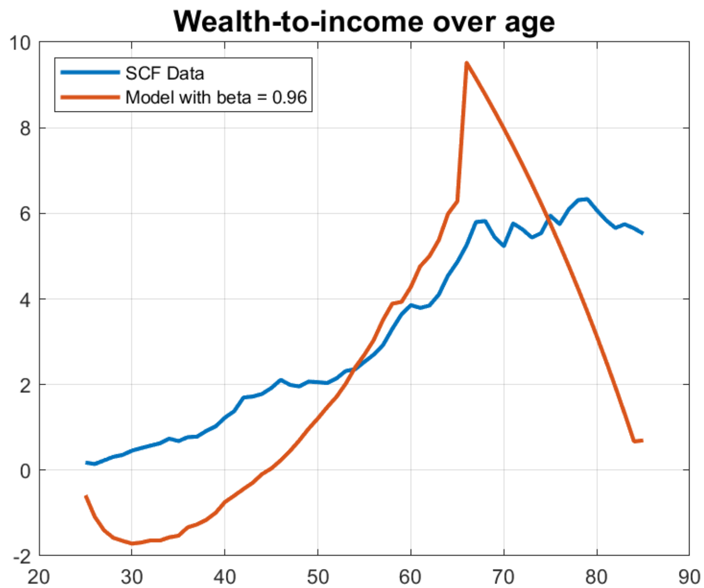
Note:
This values of wealth and income are the averages of medians over 5 different imputations in the SCF data. It was further smoothed by centered moving average of window of 5 over age

2. While the empirical wealth-to-income ratio increases and then stabilizes, the model predicts very sharp rise and then decline in the wealth/income ratio from and around the starting period of the retirement (Figure 2). This stark contrast is mostly due to the assumption of the model that the model household will exhaust all the resources before dying whereas in real life, the people may have bequest motives.

3. As the shape of the two curves are significantly different to the end of the curves, it is tricky to find the most similar model curve. Nevertheless, I computed the betas from the equally spaced grid with 0.01 spaced grid between 0.9 and 1. The beta that yields the smallest Euclidean distance from the empirical curve was 0.960 (Figure 3).

   It is useful to visualize the distance between the two curves with respect to various betas. In the Figure 3, I show the values.

Figure 2: Wealth-income data and Model prediction ($\beta = 0.96$)



Note: As the nonnegativity constraint for the asset was not handled, the ratio went below zero.

Figure 3: Wealth-income data and Model prediction ($\beta = 0.9600$)

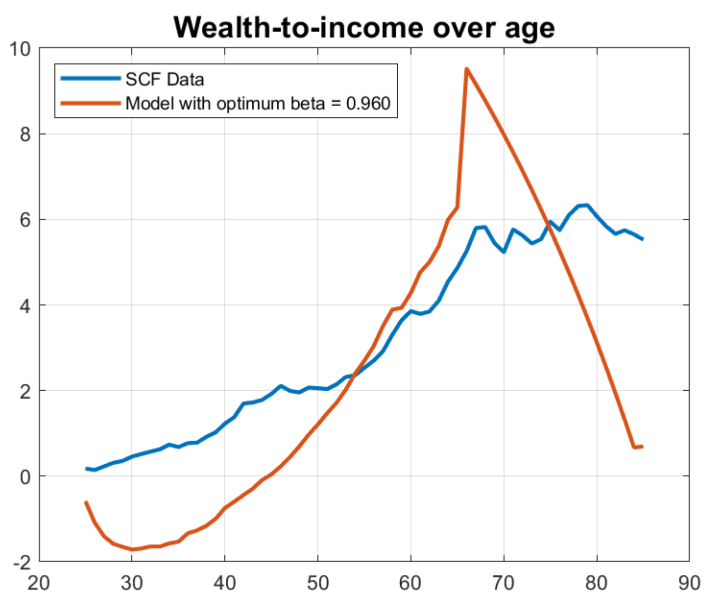


Figure 4: The distance between model and empirical values for various beta values