# Dynamic Macroeconomics
# Problem Set 1

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# Theory exercise 1

Let $T$ have a contraction with a discount parameter $0 < \rho < 1$ as follows.

$$d(Tx_{k+1}, Tx_k) \leq \rho d(x_{k+1}, x_k), \forall k \geq 0$$

Then

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \text{ (by repeated applications of the triangle inequality)} \\
&\leq \rho^{n-m-1} d(x_{m+1}, x_m) + \rho^{n-m-2} d(x_{m+1}, x_m) + d(x_{m+1}, x_m) \\
&= (\rho^{n-m-1} + \rho^{n-m-2} + ...1) d(x_{m+1}, x_m) \\
&= \frac{1-\rho^{n-m}}{1-\rho} d(x_{m+1}, x_m) \\
&= \rho^m \frac{1-\rho^{n-m}}{1-\rho} d(x_1, x_0)
\end{aligned}$$

In short, $d(x_n, x_m) \leq \rho^m \frac{1-\rho^{n-m}}{1-\rho} d(x_1, x_0)$. Since $0 < \rho < 1$

$$\lim_{n\to\infty, m\to\infty} \frac{\rho^m(1-\rho^{n-m})}{1-\rho} d(x_1, x_0) = 0, n > m$$

Hence, $\forall \varepsilon, \exists N(\varepsilon)$ such that $\forall m > N(\varepsilon)$, $\frac{\rho^m(1-\rho^{n-m})}{1-\rho} d(x_1, x_0) < \varepsilon$ and so, $\forall n > m > N(\varepsilon), d(x_n, x_m) < \varepsilon$ must be true. Therefore, $T^n x = x_n$ is a Cauchy sequence.

# Theory exercise 2

In general,

$$\begin{aligned} c_t &= y_t - k_{t+1} + (1-\delta)k_t \\ &= y_t - k_{t+1} \end{aligned}$$

$$V(k_t) = \max_{k_{t+1}} \{U(y_t - k_{t+1}) + \beta E_t V(k_{t+1})\}$$

$$\frac{\partial V(k_t)}{\partial k_{t+1}} = U_c \cdot (-1) + \beta \frac{\partial E_t V(k_{t+1})}{\partial k_{t+1}} = 0$$

Assume log felicity, $U(c) = ln(c)$, note $y_t = z_t k_t^{\alpha}$ and $V(k) = A + Bln(k) + Cln(z)$.
Then

$$\begin{aligned} \frac{\partial V(k_t)}{\partial k_{t+1}} = \frac{1}{c_t} \cdot (-1) + \beta \frac{1}{k_{t+1}} &= \frac{1}{z_t k_t^{\alpha} - k_{t+1}} \cdot (-1) + \beta \frac{B}{k_{t+1}} = 0 \\ \Rightarrow \frac{1}{z_t k_t^{\alpha} - k_{t+1}} &= \beta \frac{B}{k_{t+1}} \\ \Rightarrow k_{t+1} &= \frac{\beta B}{1+\beta B} z_t k_t^{\alpha} \text{ - policy function} \end{aligned}$$

Now replace $V(k_{t+1})$ by its guess and then $k_{t+1}$ by the above found policy.

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \{ln(z_t k_t^{\alpha} - k_{t+1}) + \beta E_t V(k_{t+1})\} \\ &= ln(z_t k_t^{\alpha} - k_{t+1}) + \beta E_t(A + Bln(k_{t+1}) + CE_t ln(z_{t+1})) \\ &= ln(z_t k_t^{\alpha} - \frac{\beta B}{1+\beta B} z_t k_t^{\alpha}) + \beta(A + Bln(\frac{\beta B}{1+\beta B} z_t k_t^{\alpha}) + C(\rho ln(z_t) + \mu(1-\rho))) \\ &= ln(\frac{1}{1+\beta B}) + \beta A + \beta B ln(\frac{\beta B}{1+\beta B}) + \beta C\mu(1-\rho) + (\alpha + \beta B\alpha) ln(k_t) + (1 + \beta B + C\rho) ln(z_t) \end{aligned}$$

This expression should equal to

$$A + Bln(k_t) + Cln(z_t)$$

according to the guess. Matching the coefficients of the common terms of the two expressions, we can find the undetermined coefficients $A$, $B$ and $C$ as follows.

$$\alpha + \beta B\alpha = B \Rightarrow B = \frac{\alpha}{1-\beta\alpha}$$

$$1 + \beta B + C\rho = C \Rightarrow C = \frac{1 + \beta B}{1 - \rho}$$

$$\begin{aligned} ln(\frac{1}{1+\beta B}) + \beta A + \beta B ln(\frac{\beta B}{1+\beta B}) + \beta C\mu(1-\rho) = A \\ \Rightarrow A = \frac{1}{1-\beta}(ln(\frac{1}{1+\beta B}) + \beta B ln(\frac{\beta B}{1+\beta B}) + \beta C\mu(1-\rho)) \end{aligned}$$

All $A, B, C$ coefficients are determined by model parameters, hence the solution is found.

# Exercise 1a

Please find the code in the accompanying code files (”Ex1_a.jl” and ”VFI_update_spline.jl”) in the folder ”codes”.