

ADVANCED MACROECONOMICS

-The Algebra of the RBC-Model-

1 Household problem

The household now **maximizes the utility function**

$$\max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \psi \log(1-l_t) + \theta g_t \right) \quad (1)$$

subject to the budget constraint

$$c_t + i_t = w_t l_t + R_t k_t - T_t \quad \forall t, \quad (2)$$

the **law of motion for capital** with $\gamma_x = 1 + n + g + ng$

$$\gamma_x k_{t+1} = (1 - \delta) k_t + i_t \quad \forall t, \quad (3)$$

an initial value for capital: $k_0 > 0$, and the transversality/No-Ponzi constraint

$$\lim_{t \rightarrow \infty} E_0 \beta^t \lambda_t k_{t+1} = 0. \quad (4)$$

Additionally the household takes into account the **stochastic laws of motion for z_t**

$$A_t = A^* e^{z_t} \quad (5)$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t, \varepsilon_t \sim (0, \sigma_z^2) \quad (6)$$

and g_t

$$g_t = g^* e^{\hat{g}_t} \quad (7)$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^G, \varepsilon_t^G \sim (0, \sigma_G^2), \quad (8)$$

and the **government budget** constraint

$$T_t = g_t \quad \forall t \quad (9)$$

Finally, we have non-negativity constraints

$$k_t \geq 0 \quad (10)$$

$$c_t \geq 0 \quad (11)$$

$$0 \leq l_t \leq 1. \quad (12)$$

Due to our assumptions, we will not have corner solutions and can ignore the non-negativity constraints.

This yields the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \psi \log(1-l_t) + \theta g_t - \lambda_t (c_t + \gamma_x k_{t+1} - (1-\delta) k_t + g_t - w_t l_t - R_t k_t) \right\}$$

with the **First Order Conditions** ($\forall t$)

$$\frac{\partial L}{\partial c_t} = E_t \beta^t (c_t^{-\sigma} - \lambda_t) = 0 \Rightarrow c_t^{-\sigma} = \lambda_t \quad (13)$$

$$\frac{\partial L}{\partial l_t} = E_t \beta^t \left\{ -\psi \frac{1}{1-l_t} + \lambda_t w_t \right\} = 0 \Rightarrow \psi \frac{1}{1-l_t} = \lambda_t w_t \quad (14)$$

$$\begin{aligned} \frac{\partial L}{\partial k_{t+1}} &= E_t \beta^t \left\{ -\lambda_t + \frac{\beta}{\gamma_x} \lambda_{t+1} ((1-\delta) + R_{t+1}) \right\} = 0 \\ &\Rightarrow \lambda_t = \frac{\beta}{\gamma_x} E_t \lambda_{t+1} (R_{t+1} + 1 - \delta) . \end{aligned} \quad (15)$$

and

$$\frac{\partial L}{\partial \lambda_t} = -(c_t + \gamma_x k_{t+1} - (1-\delta)k_t + g_t - w_t l_t - R_t k_t) = 0, \quad (16)$$

that is the budget constraint.

The first three first order conditions **can be combined to yield**

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t c_{t+1}^{-\sigma} (1 - \delta + R_{t+1}) \quad (17)$$

$$\psi \frac{1}{1-l_t} = \frac{w_t}{c_t^\sigma}. \quad (18)$$

2 Firm Problem

The firm problem is again static:

$$\max_{l_t, k_t} A^* e^{z_t} k_t^\alpha l_t^{1-\alpha} - w_t l_t - R_t k_t$$

The resulting first order conditions again state that factors are paid their marginal products

$$w_t = (1 - \alpha) A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^\alpha \quad (19)$$

$$R_t = \alpha A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^{\alpha-1}. \quad (20)$$

3 Market Clearing

Labor Market: labor supply (18) and demand (19) must be equal:

$$(1 - \alpha) A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^\alpha = \psi \frac{1}{1-l_t} c_t^\sigma. \quad (21)$$

Capital market: from (17) and (20) follows

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t c_{t+1}^{-\sigma} \left\{ \alpha A^* e^{z_{t+1}} \left(\frac{k_{t+1}}{l_{t+1}} \right)^{\alpha-1} + (1 - \delta) \right\}. \quad (22)$$

Finally, the goods market equilibrium requires:

$$c_t + \gamma_x k_{t+1} - (1 - \delta)k_t + g_t = A^* e^{z_t} k_t^\alpha l_t^{1-\alpha}. \quad (23)$$

4 Approximating Technology

The Taylor approximation of technology A_t is given by

$$a e^{\hat{x}_t} \approx a + a e^{\hat{x}_t} \Big|_{\hat{x}_t=0} (\hat{x}_t - 0) = a + a \hat{x}_t,$$

where $\hat{x}_t = z_t$ (see Linearization Handout).

5 The Labor FOC

The labor FOC is given by

$$(1 - \alpha) A^* e^{z_t} \left(\frac{k_t}{l_t} \right)^\alpha = \psi \frac{1}{1 - l_t} c_t^\sigma \quad (24)$$

The steady state is

$$(1 - \alpha) A^* \left(\frac{k}{l} \right)^\alpha = \psi \frac{1}{1 - l} c^\sigma. \quad (25)$$

Loglinearizing yields

$$(1 - \alpha) A^* \left(\frac{k}{l} \right)^\alpha (z_t + \alpha \hat{k}_t - \alpha \hat{l}_t) = \psi \frac{1}{1 - l} \sigma c^\sigma \hat{c}_t - \psi c^\sigma \left(\frac{1}{1 - l} \right)^2 (-1) \hat{l}_t. \quad (26)$$

Divide by the steady state

$$z_t + \alpha \hat{k}_t - \alpha \hat{l}_t = \sigma \hat{c}_t + \frac{l}{1 - l} \hat{l}_t. \quad (27)$$

Solving for l yields

$$\hat{l}_t = \left(\frac{l}{1 - l} + \alpha \right)^{-1} (z_t + \alpha \hat{k}_t - \sigma \hat{c}_t). \quad (28)$$

6 The Euler Equation

The Euler Equation is given by

$$c_t^{-\sigma} = \frac{\beta}{\gamma_x} E_t \left\{ c_{t+1}^{-\sigma} \left(\alpha A_{t+1} \left(\frac{k_{t+1}}{l_{t+1}} \right)^{\alpha-1} + (1 - \delta) \right) \right\}. \quad (29)$$

The steady state is

$$1 = \frac{\beta}{\gamma_x} \left\{ \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} + (1 - \delta) \right\}. \quad (30)$$

Loglinearizing yields

$$\begin{aligned} c^{-\sigma} - \sigma c^{-\sigma} \hat{c}_t = E_t & \left[\underbrace{\frac{\beta}{\gamma_x} c^{-\sigma} \left(\alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} + (1 - \delta) \right)}_{c^{-\sigma}} \right. \\ & + \underbrace{\frac{\beta}{\gamma_x} \left\{ \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} + (1 - \delta) \right\}}_1 (-\sigma) c^{-\sigma} \hat{c}_{t+1} \\ & \left. + \frac{\beta}{\gamma_x} c^{-\sigma} \left\{ \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} \right\} (z_{t+1} + (\alpha - 1) \hat{k}_{t+1} - (\alpha - 1) \hat{l}_{t+1}) \right]. \end{aligned}$$

Subtracting the steady state and dividing by $-\sigma c^{-\sigma}$ yields

$$\hat{c}_t = E_t \left[\hat{c}_{t+1} - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (z_{t+1} + (\alpha - 1) \hat{k}_{t+1} - (\alpha - 1) \hat{l}_{t+1}) \right]. \quad (31)$$

7 The Resource Constraint

The resource constraint is given by

$$c_t + \gamma_x k_{t+1} - (1 - \delta) k_t + g_t = A_t k_t^\alpha l_t^{1-\alpha} \quad (32)$$

In steady state we have

$$c + \gamma_x k - (1 - \delta) k + g = A^* k^\alpha l^{1-\alpha}. \quad (33)$$

Loglinearizing the single terms yields

$$c_t \approx c + c \hat{c}_t \quad (34)$$

$$\gamma_x k_{t+1} \approx \gamma_x k + \gamma_x k \hat{k}_{t+1} \quad (35)$$

$$(1 - \delta) k_t \approx (1 - \delta) k + (1 - \delta) k \hat{k}_t \quad (36)$$

$$A_t k_t^\alpha l_t^{1-\alpha} \approx A^* k^\alpha l^{1-\alpha} + A^* k^\alpha l^{1-\alpha} z_t + \alpha A^* k^\alpha l^{1-\alpha} \hat{k}_t + (1 - \alpha) A^* k^\alpha l^{1-\alpha} \hat{l}_t \quad (37)$$

$$g e^{\hat{g}_t} \approx g + g \hat{g}_t. \quad (38)$$

Together

$$\begin{aligned} c + c \hat{c}_t + \gamma_x k + \gamma_x k \hat{k}_{t+1} - (1 - \delta) k - (1 - \delta) k \hat{k}_t + g + g \hat{g}_t = \\ A^* k^\alpha l^{1-\alpha} + A^* k^\alpha l^{1-\alpha} z_t + \alpha A^* k^\alpha l^{1-\alpha} \hat{k}_t + (1 - \alpha) A^* k^\alpha l^{1-\alpha} \hat{l}_t. \end{aligned} \quad (39)$$

Rewrite as

$$\begin{aligned} \underbrace{c + \gamma_x k - (1 - \delta) k + g}_{A^* k^\alpha l^{1-\alpha}} + c \hat{c}_t + \gamma_x k \hat{k}_{t+1} - (1 - \delta) k \hat{k}_t + g \hat{g}_t = \\ A^* k^\alpha l^{1-\alpha} + A^* k^\alpha l^{1-\alpha} \left(z_t + \alpha \hat{k}_t + (1 - \alpha) \hat{l}_t \right). \end{aligned} \quad (40)$$

Solving for \hat{k}_t yields

$$\begin{aligned} c \hat{c}_t &= A^* k^\alpha l^{1-\alpha} z_t + \alpha A^* k^\alpha l^{1-\alpha} \hat{k}_t + (1 - \alpha) A^* k^\alpha l^{1-\alpha} \hat{l}_t - \gamma_x k \hat{k}_{t+1} + (1 - \delta) k \hat{k}_t - g \hat{g}_t \\ c \hat{c}_t &= [\alpha y + (1 - \delta) k] \hat{k}_t + y \left(z_t + (1 - \alpha) \hat{l}_t \right) - \gamma_x k \hat{k}_{t+1} - g \hat{g}_t \\ \hat{k}_{t+1} &= \frac{1}{\gamma_x} \left[\alpha \frac{y}{k} + (1 - \delta) \right] \hat{k}_t + \frac{y}{\gamma_x k} \left(z_t + (1 - \alpha) \hat{l}_t \right) - \frac{c}{\gamma_x k} \hat{c}_t - \frac{g}{\gamma_x k} \hat{g}_t. \end{aligned} \quad (41)$$

8 Eliminating l

Write (28) as

$$\hat{l}_t = \left(\frac{l}{1-l} + \alpha \right)^{-1} \left(z_t + \alpha \hat{k}_t - \sigma \hat{c}_t \right) \equiv \gamma_l \left(z_t + \alpha \hat{k}_t - \sigma \hat{c}_t \right) \quad (42)$$

and plug in into (41)

$$\begin{aligned} \hat{k}_{t+1} &= \frac{1}{\gamma_x} \left[\alpha \frac{y}{k} + (1 - \delta) \right] \hat{k}_t + \frac{y}{\gamma_x k} z_t + \frac{y}{\gamma_x k} (1 - \alpha) \gamma_l \left(z_t + \alpha \hat{k}_t - \sigma \hat{c}_t \right) - \frac{c}{\gamma_x k} \hat{c}_t - \frac{g}{\gamma_x k} \hat{g}_t \\ \hat{k}_{t+1} &= \frac{1}{\gamma_x} \left(\alpha \frac{y}{k} + (1 - \delta) + \frac{y}{k} (1 - \alpha) \gamma_l \alpha \right) \hat{k}_t + \left(\frac{y}{\gamma_x k} + (1 - \alpha) \gamma_l \frac{y}{\gamma_x k} \right) z_t \\ &\quad - \left(\frac{c}{\gamma_x k} + \frac{y}{\gamma_x k} (1 - \alpha) \gamma_l \sigma \right) \hat{c}_t - \frac{g}{\gamma_x k} \hat{g}_t \\ \hat{k}_{t+1} &= \underbrace{\frac{1}{\gamma_x} \left(\alpha \frac{y}{k} (1 + (1 - \alpha) \gamma_l) + (1 - \delta) \right)}_{\alpha_1} \hat{k}_t + \underbrace{\left(\frac{c}{\gamma_x k} + \frac{y}{\gamma_x k} (1 - \alpha) \gamma_l \sigma \right)}_{\alpha_2} \hat{c}_t \\ &\quad + \underbrace{\left(\frac{y}{\gamma_x k} (1 + (1 - \alpha) \gamma_l) \right)}_{\alpha_3} z_t + \underbrace{- \frac{g}{\gamma_x k}}_{\alpha_4} \hat{g}_t \end{aligned} \quad (43)$$

and into (31)

$$\begin{aligned}
\hat{c}_t &= E_t \left[\hat{c}_{t+1} - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} \left(z_{t+1} + (\alpha-1) \hat{k}_{t+1} - (\alpha-1) \gamma_l \left(z_{t+1} + \alpha \hat{k}_{t+1} - \sigma \hat{c}_{t+1} \right) \right) \right] \\
\hat{c}_t &= E_t \left[\left(1 - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (\alpha-1) \gamma_l \sigma \right) c_{t+1} - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (1 - (\alpha-1) \gamma_l) z_{t+1} \right. \\
&\quad \left. - \frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} ((\alpha-1) - \gamma_l (\alpha-1) \alpha) \hat{k}_{t+1} \right] \\
\hat{c}_t &= E_t \left[\underbrace{-\frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (\alpha-1) (1 - \gamma_l \alpha) \hat{k}_{t+1}}_{\alpha_5} + \underbrace{\left(1 - \frac{\beta}{\gamma_x} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (\alpha-1) \gamma_l \right) c_{t+1}}_{\alpha_6} + \right. \\
&\quad \left. \underbrace{-\frac{\beta}{\gamma_x \sigma} \alpha A^* \left(\frac{k}{l} \right)^{\alpha-1} (1 - (\alpha-1) \gamma_l) z_{t+1}}_{\alpha_7} \right] .
\end{aligned} \tag{44}$$

9 Method of Undetermined Coefficients

Guess a solution of the form

$$\hat{k}_{t+1} = \phi_{kk} \hat{k}_t + \phi_{kz} z_t + \phi_{kg} \hat{g}_t \tag{45}$$

$$\hat{c}_t = \phi_{ck} \hat{k}_t + \phi_{cz} z_t + \phi_{cg} \hat{g}_t . \tag{46}$$

Plugging in Equation (43) yields

$$\begin{aligned}
\hat{k}_{t+1} &= \alpha_1 \hat{k}_t + \alpha_2 \hat{c}_t + \alpha_3 z_t + \alpha_4 \hat{g}_t \\
\phi_{kk} \hat{k}_t + \phi_{kz} z_t + \phi_{kg} \hat{g}_t &= \alpha_1 \hat{k}_t + \alpha_2 \left(\phi_{ck} \hat{k}_t + \phi_{cz} z_t + \phi_{cg} \hat{g}_t \right) + \alpha_3 z_t + \alpha_4 \hat{g}_t .
\end{aligned}$$

This equation has to hold for all values of the state variables $\hat{g}_t, \hat{k}_t, z_t$. In turn setting all states to 0 except for one, which is set to 1, yields the following equations (this is equivalent to simply comparing the sides):

$$\phi_{kk} = \alpha_1 + \alpha_2 \phi_{ck} \tag{47}$$

$$\phi_{kz} = \alpha_3 + \alpha_2 \phi_{cz} \tag{48}$$

$$\phi_{kg} = \alpha_4 + \alpha_2 \phi_{cg} \tag{49}$$

Equation (44):

$$\begin{aligned}
\hat{c}_t &= E_t \left[\alpha_5 \hat{k}_{t+1} + \alpha_6 \hat{c}_{t+1} + \alpha_7 z_{t+1} \right] \\
\phi_{ck} \hat{k}_t + \phi_{cz} z_t + \phi_{cg} \hat{g}_t &= E_t \left[\alpha_5 \hat{k}_{t+1} + \alpha_6 \left(\phi_{ck} \hat{k}_{t+1} + \phi_{cz} z_{t+1} + \phi_{cg} \hat{g}_{t+1} \right) + \alpha_7 z_{t+1} \right] \\
\phi_{ck} \hat{k}_t + \phi_{cz} z_t + \phi_{cg} \hat{g}_t &= E_t \left[(\alpha_6 \phi_{cz} + \alpha_7) z_{t+1} + (\alpha_6 \phi_{ck} + \alpha_5) \hat{k}_{t+1} + \alpha_6 \phi_{cg} \hat{g}_{t+1} \right]
\end{aligned}$$

Plugging in the laws of motion for z_{t+1} (6) and g_{t+1} (8) and using the expectations operator yields:

$$\begin{aligned}\phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t &= E_t \left[\begin{aligned} &(\alpha_6\phi_{cz} + \alpha_7)(\rho z_t + \varepsilon_{t+1}) \\ &+ (\alpha_6\phi_{ck} + \alpha_5)(\phi_{kk}\hat{k}_t + \phi_{kz}z_t + \phi_{kg}\hat{g}_t) \\ &+ \alpha_6\phi_{cg}(\rho_g g_t + \varepsilon_{t+1}^G) \end{aligned} \right] \\ \phi_{ck}\hat{k}_t + \phi_{cz}z_t + \phi_{cg}\hat{g}_t &= (\rho(\alpha_6\phi_{cz} + \alpha_7) + (\alpha_6\phi_{ck} + \alpha_5)\phi_{kz})z_t \\ &\quad + (\alpha_6\phi_{ck} + \alpha_5)\phi_{kk}\hat{k}_t \\ &\quad + ((\alpha_6\phi_{ck} + \alpha_5)\phi_{kg} + \alpha_6\phi_{cg}\rho_g)\hat{g}_t.\end{aligned}$$

Comparing sides or in turn setting all states to 0 except for one, which is set to 1, yields:

$$\phi_{ck} = (\alpha_6\phi_{ck} + \alpha_5)\phi_{kk} \quad (50)$$

$$\phi_{cz} = \rho(\alpha_6\phi_{cz} + \alpha_7) + (\alpha_6\phi_{ck} + \alpha_5)\phi_{kz} \quad (51)$$

$$\phi_{cg} = (\alpha_6\phi_{ck} + \alpha_5)\phi_{kg} + \alpha_6\phi_{cg}\rho_g. \quad (52)$$

We now have 6 equations in six unknown coefficients ϕ . Hence, we need to solve this system. Use (47) and (50)

$$\phi_{kk} = \alpha_1 + \alpha_2\phi_{ck} \Rightarrow \phi_{ck} = \frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} \quad (53)$$

$$\phi_{ck} = (\alpha_6\phi_{ck} + \alpha_5)\phi_{kk}. \quad (50)$$

Set (53) and (50) equal and transform to get quadratic equation

$$\begin{aligned}\frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} &= \left(\alpha_6 \left(\frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} \right) + \alpha_5 \right) \phi_{kk} = \frac{\alpha_6}{\alpha_2} \phi_{kk}^2 + \left(\alpha_5 - \frac{\alpha_6\alpha_1}{\alpha_2} \right) \phi_{kk} \\ \frac{\alpha_6}{\alpha_2} \phi_{kk}^2 + \left(\alpha_5 - \frac{\alpha_6\alpha_1}{\alpha_2} - \frac{1}{\alpha_2} \right) \phi_{kk} + \frac{\alpha_1}{\alpha_2} &= 0 \\ \phi_{kk}^2 + \left(\frac{\alpha_5\alpha_2}{\alpha_6} - \frac{\alpha_6\alpha_1}{\alpha_6} - \frac{1}{\alpha_6} \right) \phi_{kk} + \frac{\alpha_1}{\alpha_6} &= 0.\end{aligned}$$

The solution for ϕ_{kk} is given as

$$\phi_{kk} = \left(\frac{1 - \alpha_5\alpha_2 + \alpha_6\alpha_1}{2\alpha_6} \right) \pm \sqrt{\left(\frac{1 - \alpha_5\alpha_2 + \alpha_6\alpha_1}{2\alpha_6} \right)^2 - \frac{\alpha_1}{\alpha_6}}. \quad (54)$$

ϕ_{ck} then follows from:

$$\phi_{ck} = \frac{\phi_{kk}}{\alpha_2} - \frac{\alpha_1}{\alpha_2}. \quad (53)$$

Second, use

$$\phi_{kz} = \alpha_3 + \alpha_2\phi_{cz} \quad (48)$$

$$\phi_{cz} = \rho(\alpha_6\phi_{cz} + \alpha_7) + (\alpha_6\phi_{ck} + \alpha_5)\phi_{kz} \quad (51)$$

to get

$$\phi_{kz} = \alpha_3 + \alpha_2\phi_{cz} \quad (55)$$

$$\phi_{cz} = \rho(\alpha_6\phi_{cz} + \alpha_7) + (\alpha_6\phi_{ck} + \alpha_5)\phi_{kz} \quad (56)$$

$$\phi_{cz} = \alpha_6\rho\phi_{cz} + \rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_6\phi_{ck}\alpha_2\phi_{cz} + \alpha_3\alpha_5 + \alpha_5\alpha_2\phi_{cz} \quad (57)$$

$$\phi_{cz}(1 - \alpha_6\rho - \alpha_6\phi_{ck}\alpha_2 - \alpha_5\alpha_2) = \rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_3\alpha_5 \quad (58)$$

$$\phi_{cz} = \frac{\rho\alpha_7 + \alpha_6\phi_{ck}\alpha_3 + \alpha_3\alpha_5}{(1 - \alpha_6\rho - \alpha_6\phi_{ck}\alpha_2 - \alpha_5\alpha_2)}. \quad (59)$$

Given ϕ_{cz} , ϕ_{kz} can be computed from

$$\phi_{kz} = \alpha_3 + \alpha_2 \phi_{cz} . \quad (48)$$

Finally, plug

$$\phi_{kg} = \alpha_4 + \alpha_2 \phi_{cg} \quad (49)$$

into

$$\phi_{cg} = (\alpha_6 \phi_{ck} + \alpha_5) \phi_{kg} + \alpha_6 \phi_{cg} \rho_g \quad (52)$$

to get

$$\begin{aligned} \phi_{cg} &= (\alpha_6 \phi_{ck} + \alpha_5) (\alpha_4 + \alpha_2 \phi_{cg}) + \alpha_6 \phi_{cg} \rho_g \\ \phi_{cg} &= \frac{(\alpha_6 \phi_{ck} + \alpha_5) \alpha_4}{1 - (\alpha_6 \phi_{ck} + \alpha_5) \alpha_2 - \alpha_6 \rho_g} . \end{aligned} \quad (60)$$