ECON501 2016

lab03

SYSTEM OF LINEAR DIFFERENCE EQUATIONS

See chalk board Mathematics for Economics p831 - 835 and p808-812

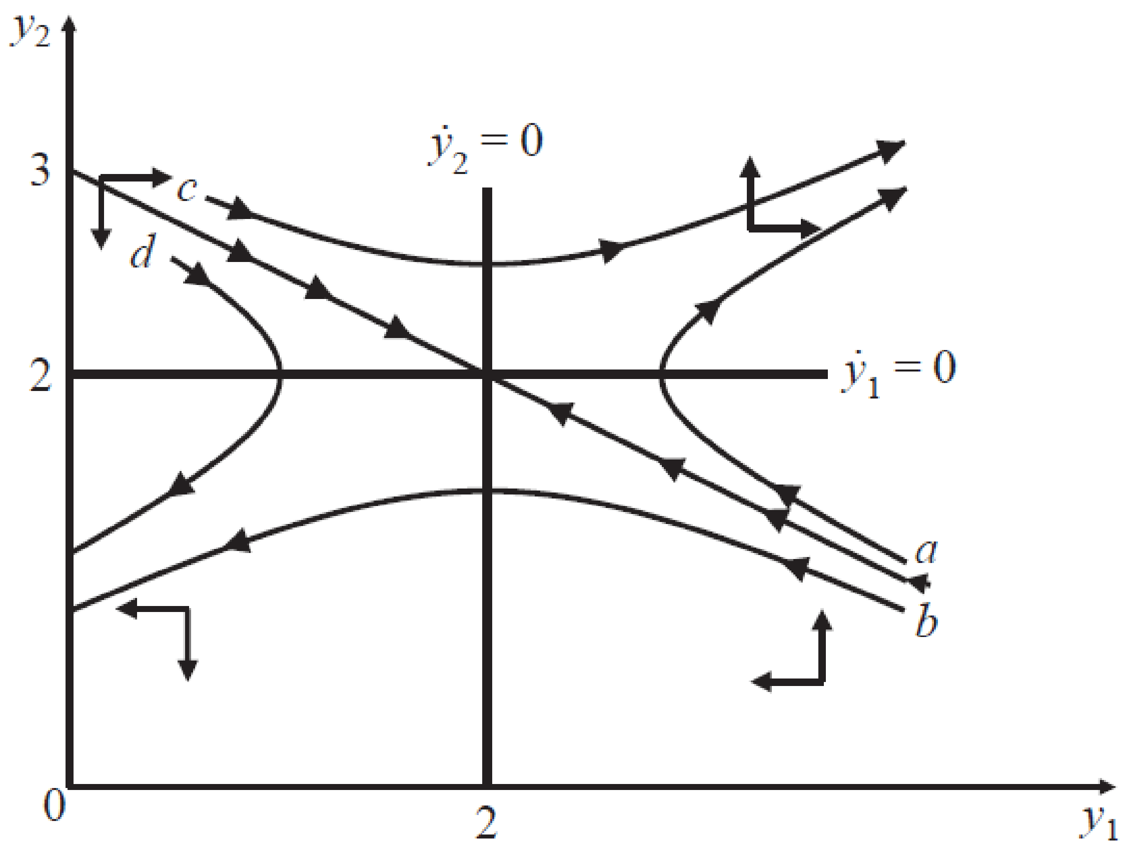


Figure 24.6 Phase diagram for example 24.14 showing representative trajectories: A saddle point

from a steady state which is a **saddle-point equilibrium.** These trajectories also demonstrate the very important property that trajectories must obey the arrows of motion and must be horizontal when they cross the y_2 isocline and vertical when they cross the y_1 isocline.

SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

Systems of nonlinear difference equations cannot be analyzed with the direct methods we have used on linear equations. In this chapter we are concerned with autonomous systems of the form

$$x_t = f(x_{t-1}) \tag{6.1}$$

where x_t is a vector in $\mathbb{R}^n$, typically a low-dimensional Euclidean space, and f is a continuously differentiable map $f: X \to \mathbb{R}^n$, where X is a subset of $\mathbb{R}^n$. Since explicit solutions to (6.1) are available only in rare and very special cases, we must rely on indirect qualitative methods to gather information about the dynamical behavior of nonlinear systems like equation (6.1). We develop graphical methods, called *phase diagrams*, which are useful in the study of low-order equations. The main result of this

6.2 Geometric properties

Scalar systems

The general first-order difference equation may be written

$$x_{t+1} = f(x_t) \tag{6.4}$$

for some real-valued function $f\colon X \to \mathbb{R}$. Once we know f, we plot x_{t+1} as a function of x_t to obtain a phase diagram that can help trace the time path of the state variable. A

LINEARIZATION

Approximate the nonlinear system in the neighborhood of the steady state by the linear system.

$$x_{t+1} = f(x_t) \qquad f: \mathbb{R}^n \supseteq X \to \mathbb{R}^n \qquad f \in C^1 \tag{6.8}$$

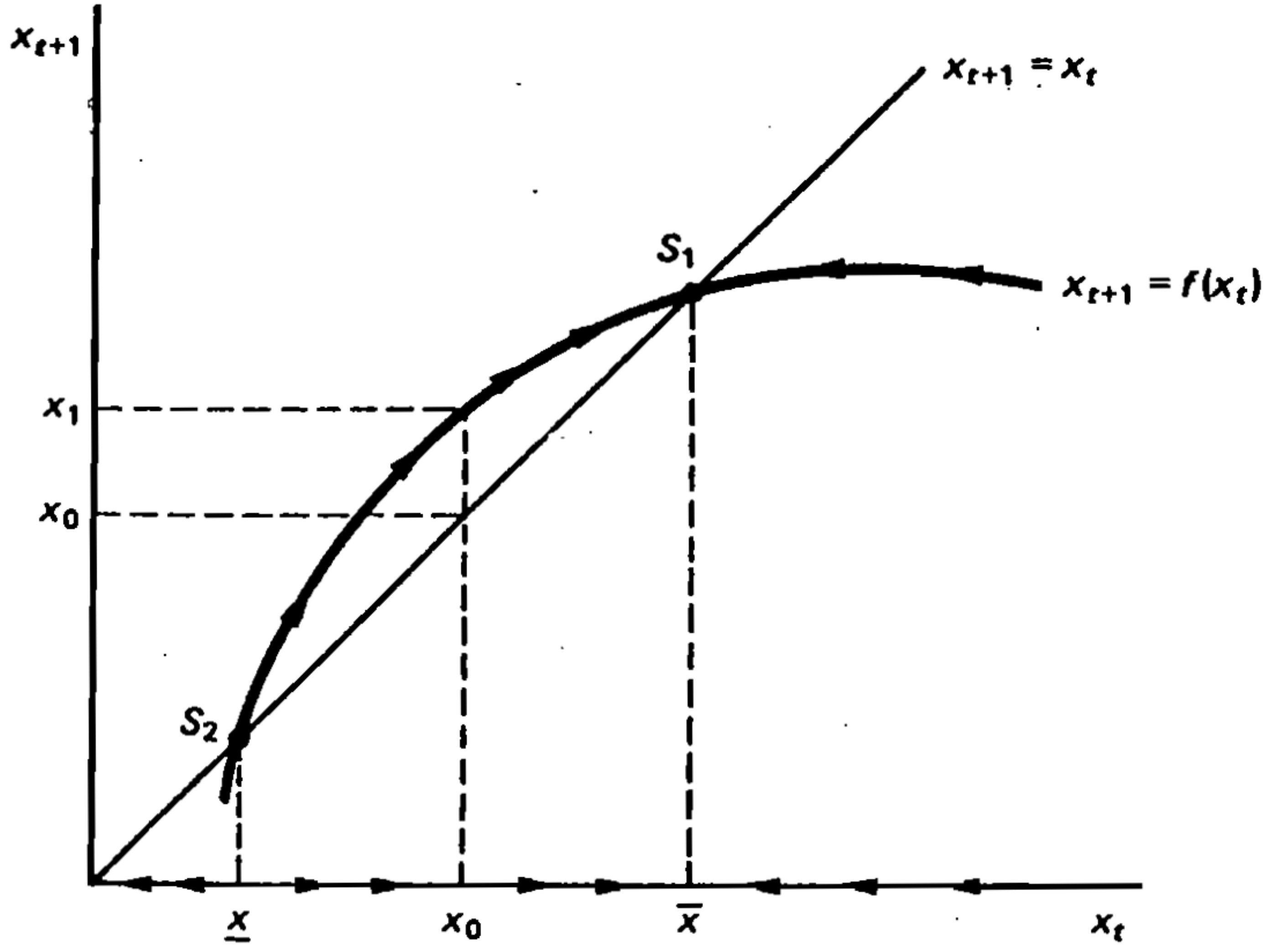
Linearization

Applying Taylor's formula to equation (6.8) we have

$$f(x) = f(\bar{x}) + \mathrm{D}f(\bar{x})(x - \bar{x}) + \mathrm{O}(\|x - \bar{x}\|)$$

where the remainder term $\mathrm{O}(\cdot)$ is small in a well-defined sense. Hence, we expect the linear system

$$x_{t+1} = \bar{x} + \mathrm{D}f(\bar{x})(x_t - \bar{x}) \tag{6.9}$$



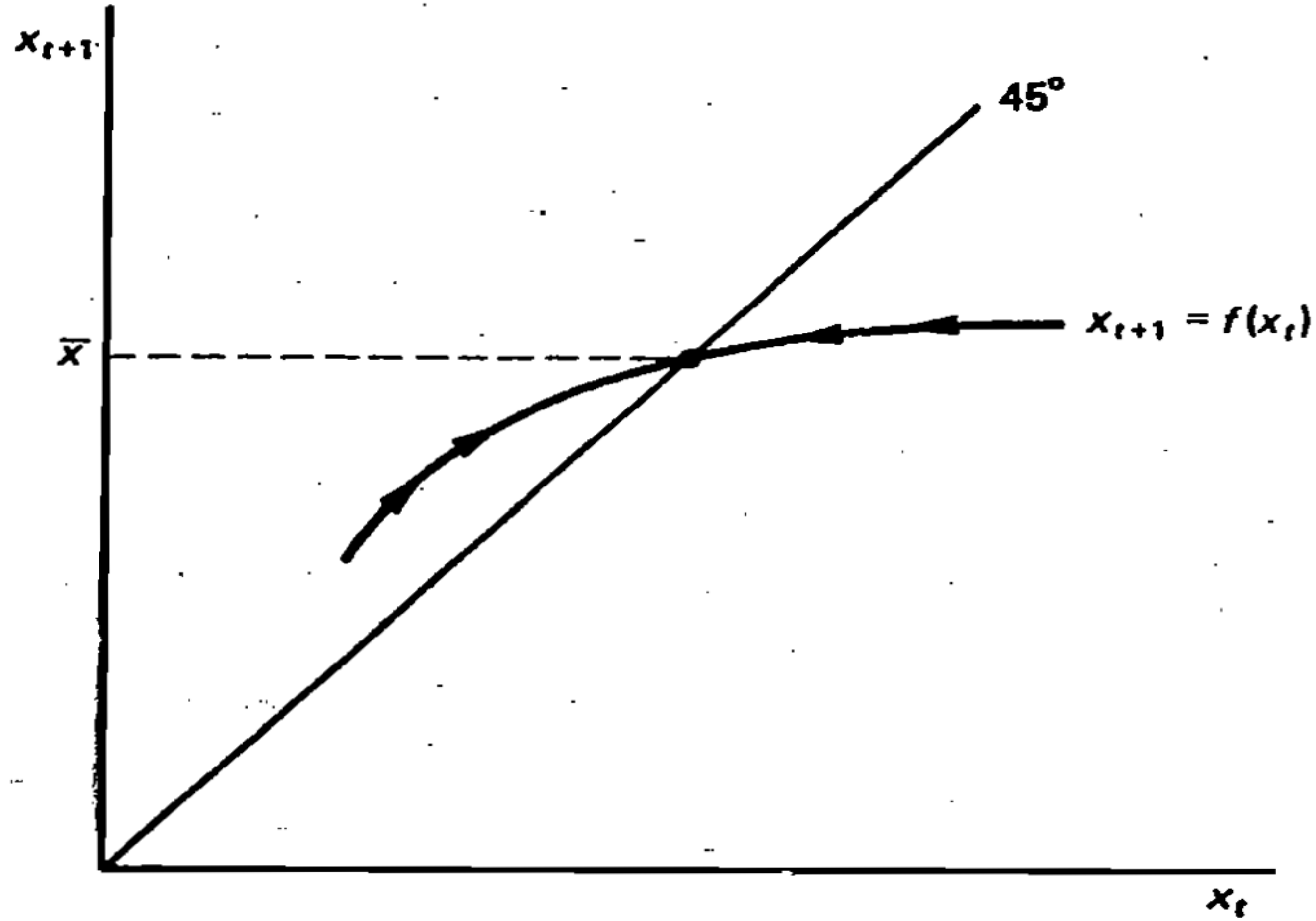
When the phaseline slopes upward, a steady state is asymptotically stable if the phaseline cuts the diagonal from above left, and unstable otherwise. Equivalently, the steady state is asymptotically stable if the slope of the phaseline at the steady state is less than 1 in absolute value, unstable if that slope is greater than 1.

use a first-order Taylor approximation to f in the neighborhood of $\overline{x}$

$$x_{t+1} = \overline{x} + f'(\overline{x})(x_t - \overline{x}) \tag{6.5}$$

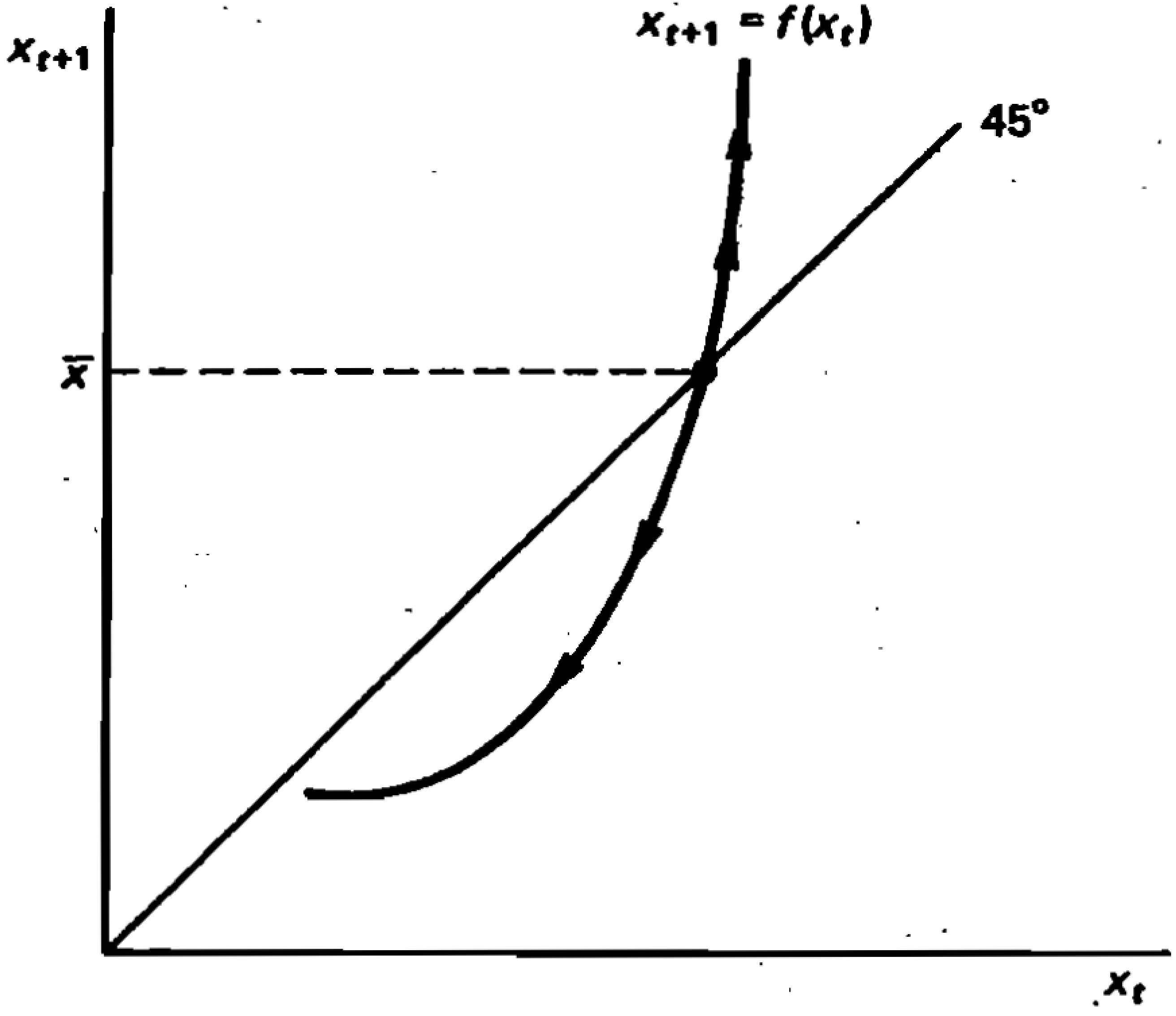
which describes the tangent to the phaseline at the steady state. As we shall see later,

(a) Monotone convergence, $f'(\overline{x}) \in (0, 1)$;



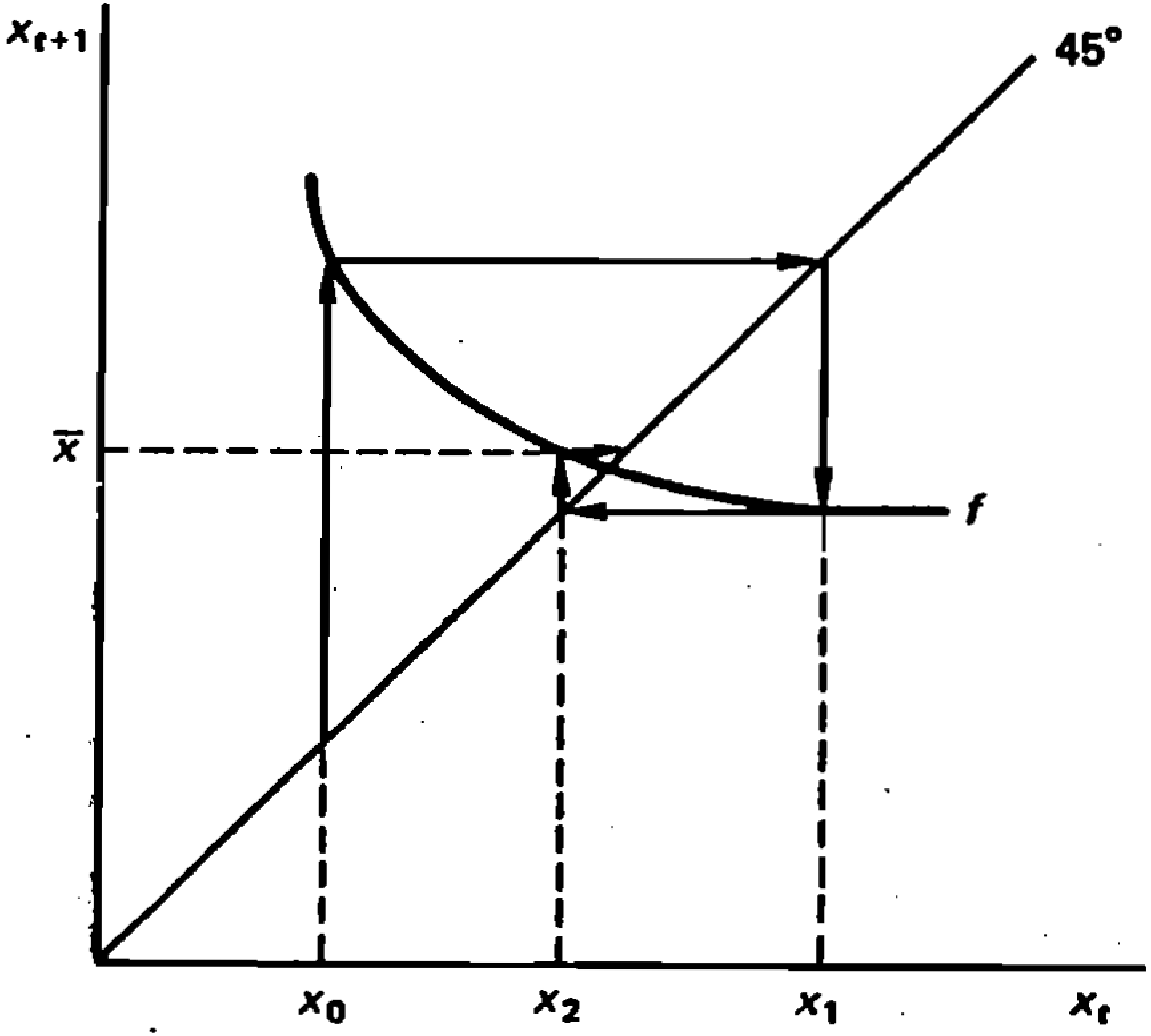
(a)

(b) monotone divergence, $f'(\bar{x}) > 1$;



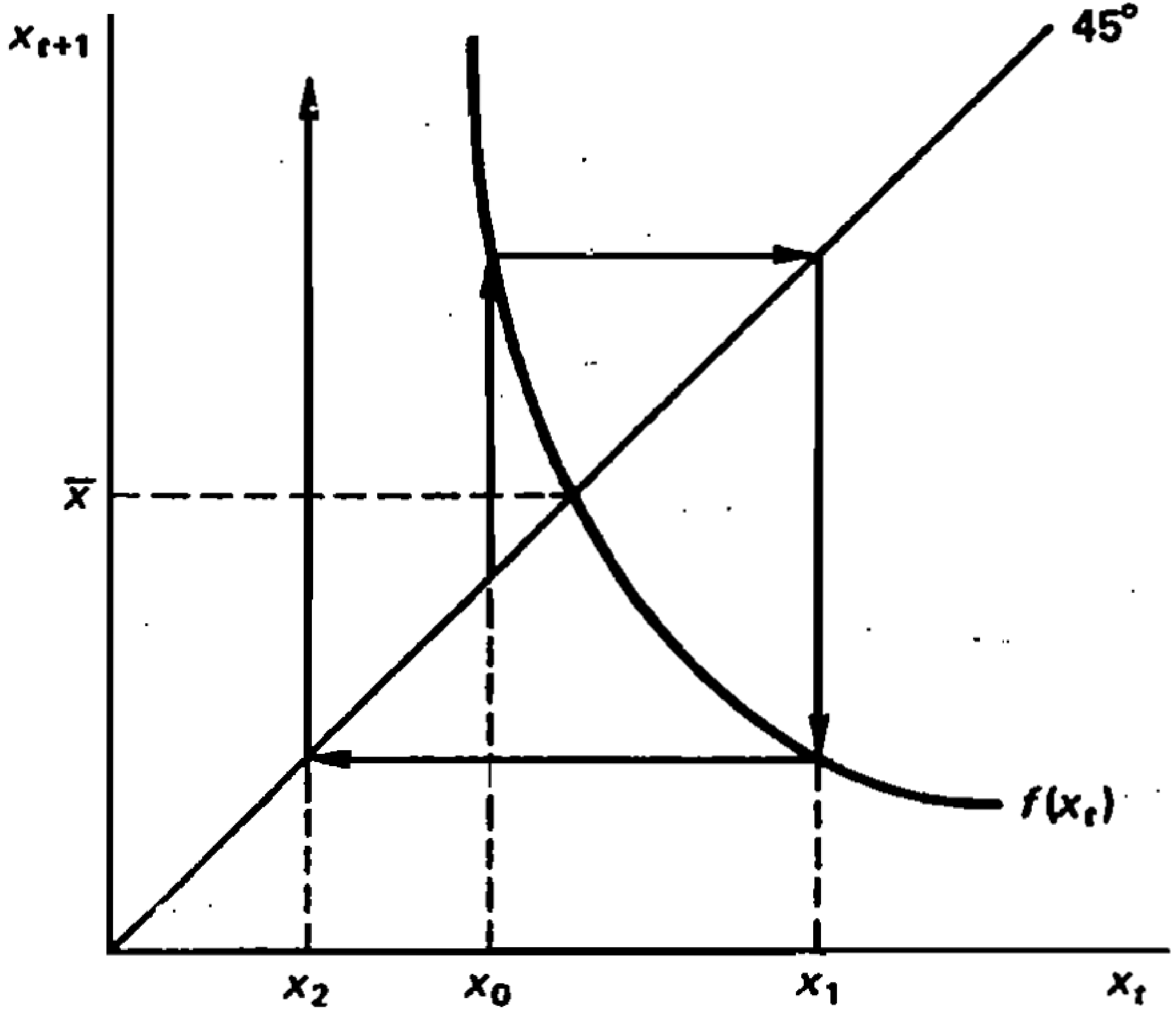
(b)

(c) damped oscillations, $f'(\bar{x}) \in (-1, 0)$;



(c)

(d) explosive oscillations, $f'(\bar{x}) < -1$.



(d)

Steady states $(\bar{x}, \bar{y})$ of the dynamical system (6.6) are solutions to the system of equations

$$x = f(x,y) \qquad y = g(x,y) \tag{6.7}$$

in which each equation describes one phaseline. Let us approximate the nonlinear system (6.6) in the neighborhood of the steady state by the linear system

$$x_{t+1} = \bar{x} + \frac{\partial f(x_t,y_t)}{\partial x_t}(x_t-\bar{x}) + \frac{\partial f(x_t,y_t)}{\partial y_t}(y_t-\bar{y}) \tag{6.6a$'$}$$

$$y_{t+1} = \bar{y} + \frac{\partial g(x_t,y_t)}{\partial x_t}(x_t-\bar{x}) + \frac{\partial g(x_t,y_t)}{\partial y_t}(y_t-\bar{y}) \tag{6.6b$'$}$$

where all the partial derivatives are evaluated at the steady state. Note that the phaselines of the associated linear system will be tangent to those of the original system at the steady state. In some neighborhood of $(\bar{x},\bar{y})$, then, the phase diagram for equations (6.6a), (6.6b) can be approximated by that for equations (6.6a)′, (6.6b)′. Chapter 4 outlines how to draw phase diagrams for linear systems.

6.4 Stability of planar systems

Planar autonomous systems of difference equations are quite common in dynamic economics. In this section we examine stability conditions for such systems from a geometric viewpoint.

We start with two continuously differentiable maps, $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$, and a nonlinear system of the form

$$x_{t+1} = f(x_t, y_t) \tag{6.11}$$

$$y_{t+1} = g(x_t, y_t) \tag{6.12}$$

We know that the stability type of a steady state $s = (x,y)$ depends on the eigenvalues of the Jacobian matrix of partial derivatives

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

The eigenvalues of J are obtained by solving the following equation:

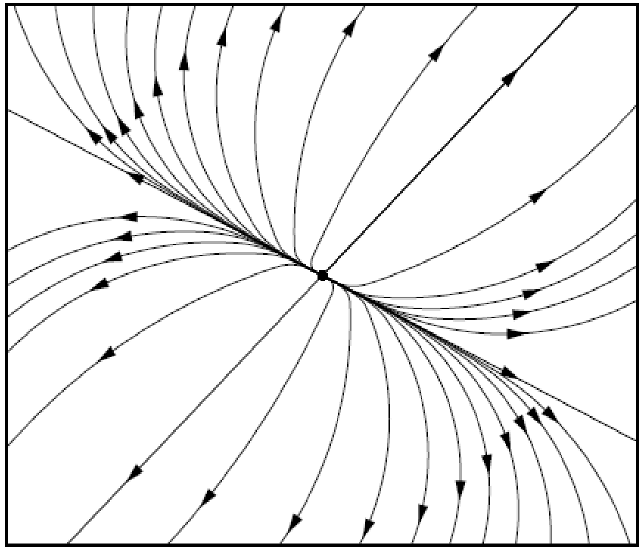
$$
\begin{aligned}
p(\lambda) = |J - \lambda I| &= \begin{vmatrix} f_x - \lambda & f_y \\ g_x & g_y - \lambda \end{vmatrix} \\
&= (f_x - \lambda)(g_y - \lambda) - f_y g_x \\
&= \lambda^2 - (f_x + g_y)\lambda + f_x g_y - f_y g_x = 0 \\
&= \lambda^2 - (\operatorname{tr} J)\lambda + \det J = 0
\end{aligned}
$$

Eigenvalues are thus roots of the *characteristic polynomial* $p(\lambda)$. Rearranging, we obtain

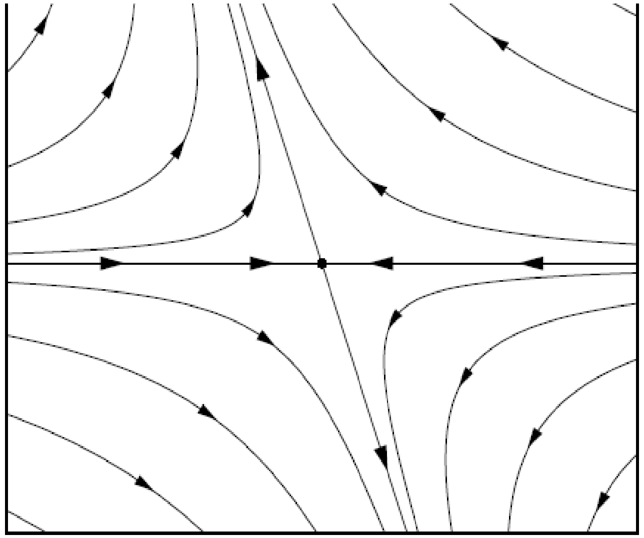
$$p(\lambda) = \lambda^2 - (\operatorname{tr} J)\lambda + \det J = 0 \tag{6.13}$$

where $\operatorname{tr} J$ and $\det J$ are the trace and determinant of the Jacobian matrix.

In particular, s is a *sink* if both eigenvalues have modulus less than 1, a *source* if both have modulus greater than 1, and a *saddle* if one eigenvalue is inside the unit circle in the complex plane and the other is outside.



Reverse all the arrows in the left figure. Paths go in toward (0, 0)



$0 < s_2 < s_1$
Source : Unstable

$s_1 < s_2 < 0$
Sink : Stable

$s_2 < 0 < s_1$
Saddle : Unstable

Figure 3.6: **Real roots s_1 and s_2**. The paths of the point $(y(t), y'(t))$ lead out when roots are positive and lead in when roots are negative. With $s_2 < 0 < s_1$, the s_2-line leads in but all other paths eventually go out near the s_1-line : *The picture shows a saddle point.*

7 The structure of growth models

7.2 Descriptive growth

We reproduce from chapter 1 the discrete variant of the model due to Solow (1956), which satisfies an autonomous nonlinear scalar first-order difference equation of the form

$$k_{t+1} = h(k_t) \tag{7.1a}$$

where

$$h(k) = \frac{(1 - \delta)k + sf(k)}{1 + n} \tag{7.1b}$$

In equation (7.1b), $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 map inheriting all its properties from the continuously differentiable neoclassical production function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ and from the three parameters $n > -1$, $\delta \in [0,1]$ and $s \in [0,1]$, representing respectively the rates of population growth, depreciation, and saving. The production function maps capital per worker into output per worker. It is increasing, strictly concave, and satisfies two assumptions

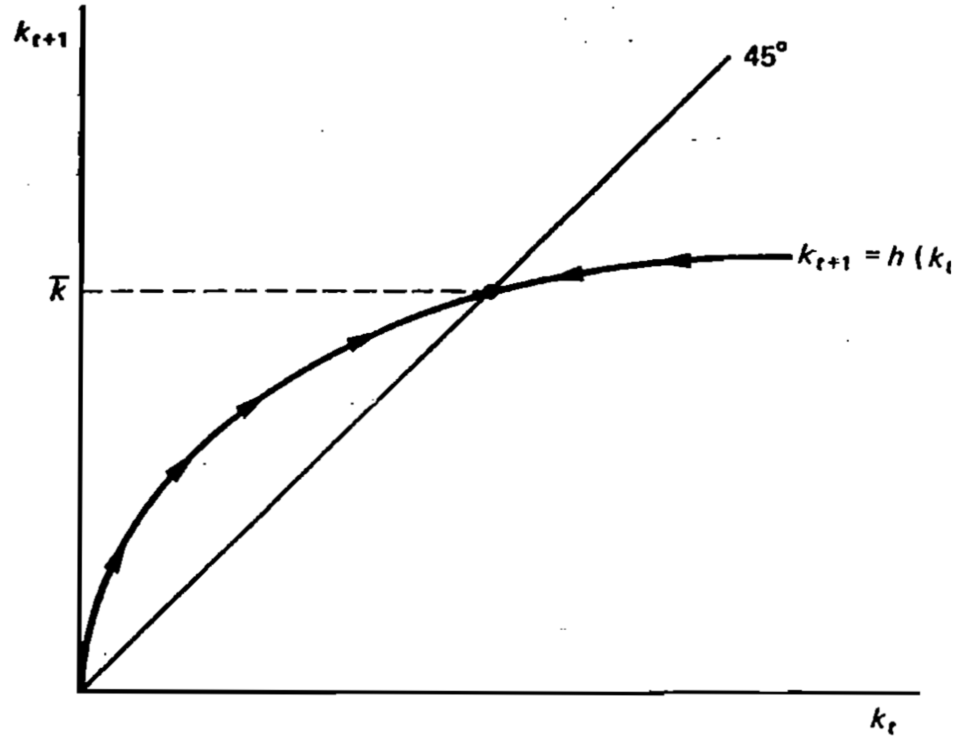
$$f(0) = 0 \tag{7.2a}$$

$$0 \leqslant \lim_{k\to\infty} f'(k) < \frac{\delta + n}{s} < \lim_{k\to 0} f'(k) \leqslant +\infty \tag{7.2b}$$

find that h is concave because f is, $h(0) = 0$ because $f(0) = 0$, and

$$\lim_{k \to 0} h'(k) = (1 + n)^{-1}[1 - \delta + s \lim_{k \to 0} f'(k)] > 1$$

$$\lim_{k_t \to \infty} \left(\frac{k_{t+1}}{k_t} \right) < 1 \qquad (7.4b)$$



The descriptive growth model.

A NONLINEAR SYSTEM IN OPTIMAL GROWTH MODEL

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t \tag{7.7a}$$

$$\beta u'(c_{t+1}) = \frac{u'(c_t)}{1 - \delta + f'(k_{t+1})} \tag{7.7b}$$

First we seek positive steady states $(\bar{k}, \bar{c})$, that is, nontrivial solutions to the system of equations

$$c = f(k) - \delta k \tag{7.8a}$$

$$f'(k) = \rho + \delta \tag{7.8b}$$

depreciation rate $\delta \geq 0$ and the *rate of time preference* $\rho = 1/\beta - 1 > 0$.

A NONLINEAR SYSTEM IN SOLOW MODEL

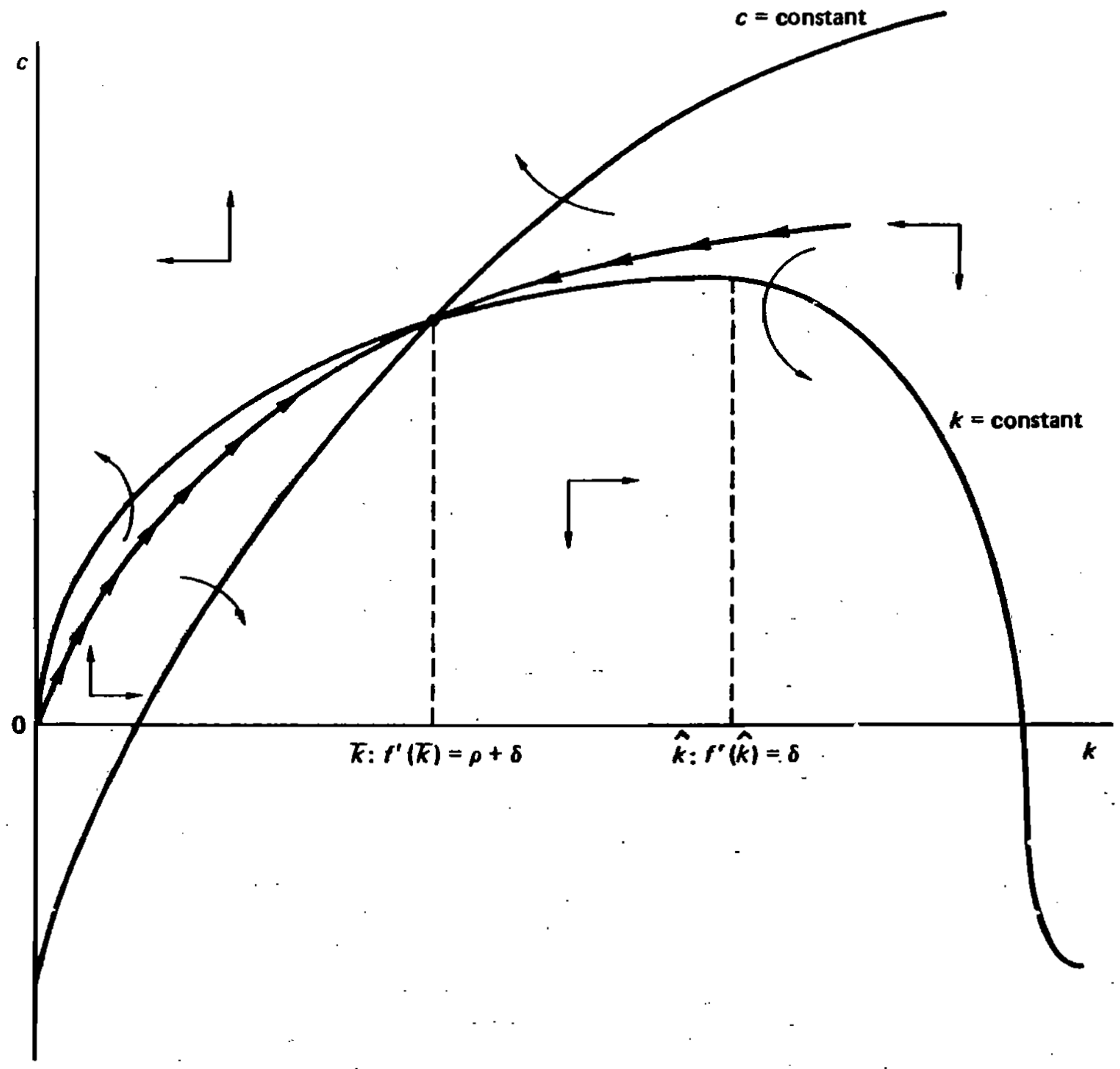


Figure 7.2 The optimal growth model.

To make *sure* that $(\bar{k},\bar{c})$ is a saddlepoint we have to evaluate at that point the Jacobian matrix of partial derivatives for the dynamical system (7.7a) and (7.7b). Eliminating k_{t+1} from equation (7.7b), we obtain

$$k_{t+1} = x(k_t,c_t) \tag{7.12a}$$

$$\beta u'(c_{t+1}) = \frac{u'(c_t)}{1 - \delta + f'[x(k_t,c_t)]} \tag{7.12b}$$

where we defined

$$x(k,c) = f(k) + (1 - \delta)k - c \tag{7.12c}$$

At $(\bar{k},\bar{c})$ we compute the Jacobian matrix

$$J = \begin{pmatrix} x_k & x_c \\ \dfrac{\partial c_{t+1}}{\partial c_t} & \dfrac{\partial c_{t+1}}{\partial c_t} \end{pmatrix} = \begin{pmatrix} 1/\beta & -1 \\ \dfrac{f''(\bar{k})}{A(\bar{c})} & 1 - \dfrac{\beta f''(\bar{k})}{A(\bar{c})} \end{pmatrix} \tag{7.13}$$

which we express in terms of the positive function

$$A(c) = -\frac{u''(c)}{u'(c)}$$

This matrix has trace

$$T = 1 + \frac{1}{\beta} - \frac{\beta f''}{A} \geqslant 1 + \frac{1}{\beta} > 2 \tag{7.14a}$$

and determinant

$$D = \frac{1}{\beta} > 1 \tag{7.14b}$$

Hence, $T^2 - 4D \geqslant (1 + 1/\beta)^2 - 4/\beta = (1 - 1/\beta)^2 > 0$, and the characteristic polynomial $p(\lambda) = \lambda^2 - T\lambda + D$ has two positive real roots $\lambda_2 > \lambda_1 > 0$ whose sum exceeds 2 and whose product exceeds 1. Because

$$p(1) = 1 - T + D = \beta f''/A < 0$$

the two real eigenvalues lie on either side of unity, which means that $0 < \lambda_1 < 1 < \lambda_2$ and $(\bar{k},\bar{c})$ really is a saddle. One shows easily that the saddlepath slopes upward in the state space, and its slope near the steady state lies somewhere in the interval $(\rho,1+\rho)$.

Equations (7.8b) and (7.5) reveal how closely optimal growth theory agrees with the Solow model on the long-run determinants of income per worker. Both affirm that steady state income is positively associated with the productivity of inputs, the durability of capital, and some parameter that depends on consumer patience. The Ramsey model, however, states that $(\bar{k},\bar{c})$ is independent of the instantaneous utility function u.

A NONLINEAR SYSTEM IN OPTIMAL GROWTH MODEL

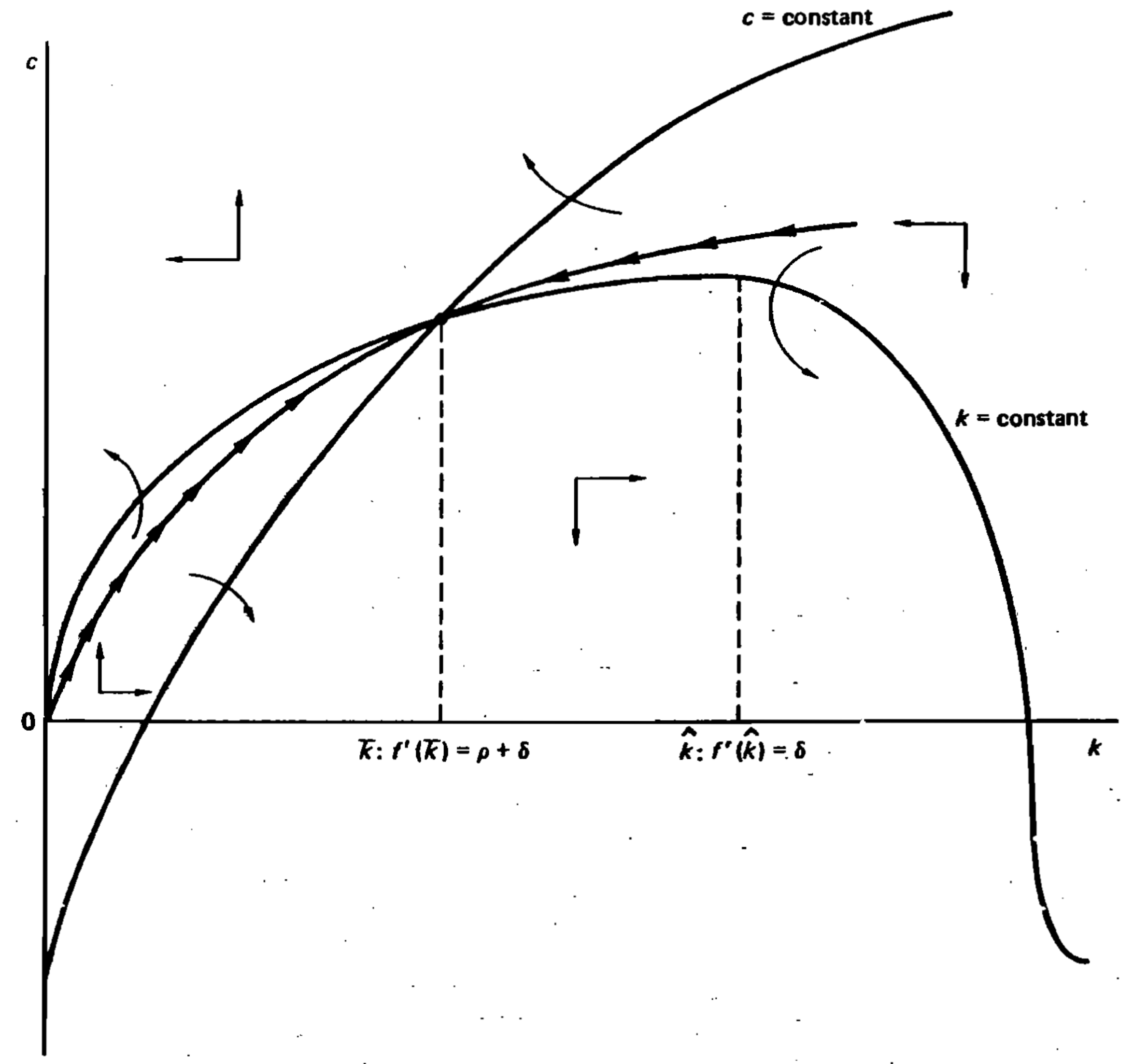


Figure 7.2 The optimal growth model.