Vector Autoregressions

- VAR: Vector AutoRegression
  - Nothing to do with VaR: Value at Risk (finance)
- Multivariate autoregression
- Multiple equation model for joint determination of two or more variables
- One of the most commonly used models for applied macroeconometric analysis and forecasting in central banks

Two-Variable VAR

- Two variables: y and x
- Example: output and interest rate
- Two-equation model for the two variables
- One-Step ahead model
- One equation for each variable
- Each equation is an autoregression plus distributed lag, with p lags of each variable

VAR(p) in 2 Variables

$$
\begin{aligned}
y_t &= \mu_1 + \alpha_{11} y_{t-1} + \alpha_{12} y_{t-2} + \cdots + \alpha_{1p} y_{t-p} \\
&\quad + \beta_{11} x_{t-1} + \beta_{12} x_{t-1} + \cdots + \beta_{1p} x_{t-p} + e_{1t}
\end{aligned}
$$

$$
\begin{aligned}
x_t &= \mu_2 + \alpha_{21} y_{t-1} + \alpha_{22} y_{t-2} + \cdots + \alpha_{2p} y_{t-p} \\
&\quad + \beta_{21} x_{t-1} + \beta_{22} x_{t-1} + \cdots + \beta_{2p} x_{t-p} + e_{2t}
\end{aligned}
$$

Multiple Equation System

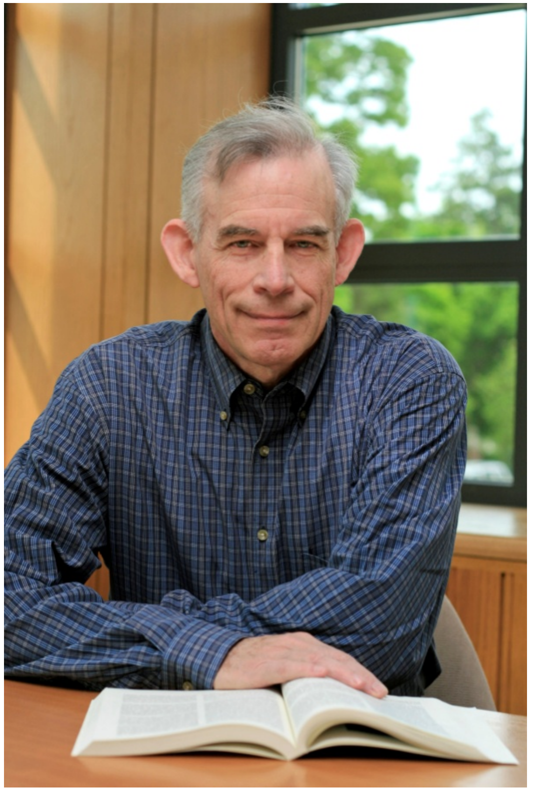
- In general: k variables
- An equation for each variable
- Each equation includes *p* lags of *y* and *p* lags of *x*
- (In principle, the equations could have different # of lags, and different # of lags of each variable, but this is most common specification.)
- There is one error per equation.
  - The errors are (typically) correlated.

Unrestricted VAR

- An unrestricted VAR includes all variables in each equation
- A restricted VAR might include some variables in one equation, other variables in another equation
- Old-fashioned macroeconomic models (so-called simultaneous equations models of the 1950s, 1960s, 1970s) were essentially restricted VARs
  - The restrictions and specifications were derived from simplistic macro theory, e.g. Keynesian consumption functions, investment equations, etc.

VAR Revolution

- Christopher Sims (1942-) of Princeton University
  - 2011 Nobel Prize in Economics
- “Macroeconomics and Reality” (1980)
  - Sims argued that conventional macro models were “incredible” – they were based on ~~non-credible identifying assumptions~~



Sims and VARs

- Sims argued that the conventional models were ~~restricted VARs~~, and the ~~restrictions had no substantive justification~~
  - ~~Based on incomplete and/or non-rigorous theory, or intuition~~
- Sims argued that economists should instead use unrestricted models, e.g. VARs
- He proposed a set of tools for use and evaluation of VARs in practice.

Estimation

- Each equation estimated by OLS

$$y_t = \mu_1 + \alpha_{11} y_{t-1} + \alpha_{12} y_{t-2} + \cdots + \alpha_{1p} y_{t-p}$$
$$+ \beta_{11} x_{t-1} + \beta_{12} x_{t-1} + \cdots + \beta_{1p} x_{t-p} + e_{1t}$$

$$x_t = \mu_2 + \alpha_{21} y_{t-1} + \alpha_{22} y_{t-2} + \cdots + \alpha_{2p} y_{t-p}$$
$$+ \beta_{21} x_{t-1} + \beta_{22} x_{t-1} + \cdots + \beta_{2p} x_{t-p} + e_{2t}$$

Estimation in Stata

- To estimate a VAR in the variables *y* & *x* with lags 1 through p included
  - .varbasic y x, lags(1/p)
- For example, using gdp2013.dta and variables gdp and d.t12 with 3 lags
  - .gen rate=d.t12
  - .varbasic rate gdp, lags(1/3)
- Could also use
  - .var rate gdp, lags(1/3)

Example: GDP and Interest Rate

```
. varbasic rate gdp

Vector autoregression

Sample:  1954q1 - 2013q4                          No. of obs      =       240
Log likelihood = -896.4798                        AIC             =  7.553999
FPE            =  6.542089                        HQIC            =  7.612434
Det(Sigma_ml)  =  6.018939                        SBIC            =  7.699025

Equation           Parms      RMSE     R-sq      chi2     P>chi2
----------------------------------------------------------------
rate                 5      .752146   0.1282   35.29972   0.0000
gdp                  5      3.35843   0.1763   51.35579   0.0000
----------------------------------------------------------------
```

Example: GDP and Interest Rate

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
rate						
rate						
L1.	-.0687148	.0650433	-1.06	0.291	-.1961972	.0587677
L2.	.0106681	.0624896	0.17	0.864	-.1118092	.1331454
gdp						
L1.	.0779384	.0140999	5.53	0.000	.0503032	.1055737
L2.	.0037657	.0150392	0.25	0.802	-.0257106	.0332421
_cons	-.2639241	.0718976	-3.67	0.000	-.4048407	-.1230075
gdp						
rate						
L1.	-.6844941	.2904269	-2.36	0.018	-1.25372	-.1152678
L2.	-.7395324	.2790243	-2.65	0.008	-1.28641	-.1926549
gdp						
L1.	.3160889	.0629578	5.02	0.000	.192694	.4394839
L2.	.190147	.0671521	2.83	0.005	.0585313	.3217626
_cons	1.579774	.3210322	4.92	0.000	.950562	2.208985

Order Selection

- A VAR(p) includes *p* lags of each variable in each equation
- In a two-variable system, the number of coefficients in each equation is $1+2p$
  - The total number is $2(1+2p)=2+4p$
- In a *k*-variable system, the number of coefficients in each equation is $1+kp$
  - The total number is $k(1+2p)=k+2kp$
- How should *p* be selected?
- Common approach:
  - Information criterion, primarily AIC

AIC and BIC for VAR Models

$$AIC = -2L + 2\left(k + 2kp\right)$$

$$BIC = -2L + \left(k + 2kp\right)\ln\left(T\right)$$

where *L* is log-likelihood from model

- Select model with smallest AIC (or BIC)

Stata Implementation

- varsoc command
- To calculate information criterion for a VAR in variables *x* and *y* up to a maximum lag of *pmax*:
  - .varsoc x y, maxlag(pmax)
- Produces a convenient table

Example: GDP and Interest Rate

```
. varsoc rate gdp, maxlag(8)

   Selection-order criteria
   Sample:  1955q3 - 2013q4                          Number of obs     =       234
```

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	-908.041				8.18559	7.77813	7.79004	7.80766
1	-882.051	51.979	4	0.000	6.78308	7.59018	7.62591*	7.67878*
2	-876.011	12.081	4	0.017	6.66587*	7.57274*	7.63228	7.7204
3	-874.997	2.0279	4	0.731	6.83834	7.59826	7.68162	7.80499
4	-869.617	10.76*	4	0.029	6.75844	7.58647	7.69364	7.85226
5	-867.435	4.3637	4	0.359	6.86471	7.60201	7.73299	7.92687
6	-865.116	4.6376	4	0.327	6.9647	7.61638	7.77118	8.0003
7	-861.706	6.8213	4	0.146	7.00073	7.62142	7.80003	8.06441
8	-861.084	1.2442	4	0.871	7.20695	7.65029	7.85272	8.15234

```
   Endogenous:  rate gdp
    Exogenous:  _cons
```

Result

- For this example
  - AIC selects *p=3*
  - BIC selects *p=2*
- Notice that the AIC value for p=3 in this table (AIC=7.572) is different from that obtained when we estimated the VAR(3) model (AIC=7.553).
  - This is because for the AIC comparison, all estimates are from a common sample, in this case excluding the first 8 observations since the maximum order is set to 8
- The varsoc command is correct

Let’s look at the VAR(3) estimates again.

Example: GDP and Interest Rate

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
rate						
rate						
L1.	-.0687148	.0650433	-1.06	0.291	-.1961972	.0587677
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Interpretation

- It is difficult to interpret the large number of coefficients in the VAR model
- Main tools for interpretation
  - Impulse responses

Impulse Response Analysis

- VAR(1) with no intercept

$$y_t = \alpha_{11} y_{t-1} + \beta_{11} x_{t-1} + e_{1t}$$
$$x_t = \alpha_{21} y_{t-1} + \beta_{21} x_{t-1} + e_{2t}$$

- The impulse responses are the time-paths of to y and x in response to shocks

Impulse Response Analysis

- The errors may be correlated.
- We "orthogonalize" them

$$
\begin{aligned}
e_{1t} &= u_{1t} \\
e_{2t} &= \rho e_{1t} + u_{2t} \\
&= \rho u_{1t} + u_{2t}
\end{aligned}
$$

Orthogonalized Model

$$y_t = \alpha_{11} y_{t-1} + \beta_{11} x_{t-1} + u_{1t}$$

$$x_t = \alpha_{21} y_{t-1} + \beta_{21} x_{t-1} + \rho u_{1t} + u_{2t}$$

- The shocks u_1 and u_2 are uncorrelated
- The ordering matters
  - The shock to y affects both y and x in period t
  - The shock to x affects only x in period t
- The impulse responses are the time-paths of to y and x in response to the shocks u_1 and u_2
- Imagine y=0 and x=0. Set u_1=1. Trace the history of y and x

Impulse Responses by Recursion

$$y_1 = \alpha_{11}0 + \beta_{11}0 + 1 = 1$$

$$x_1 = \alpha_{21}0 + \beta_{21}0 + \rho 1 = \rho$$

$$y_2 = \alpha_{11}y_1 + \beta_{11}x_1 = \alpha_{11} + \beta_{11}$$

$$x_2 = \alpha_{21}y_1 + \beta_{21}x_1 = \alpha_{21} + \beta_{21}\rho$$

$$y_3 = \alpha_{11}y_2 + \beta_{11}x_2 = \alpha_{11}\left(\alpha_{11} + \beta_{11}\right) + \beta_{11}\left(\alpha_{21} + \beta_{21}\rho\right)$$

$$x_3 = \alpha_{21}y_2 + \beta_{21}x_2 = \alpha_{21}\left(\alpha_{11} + \beta_{11}\right) + \beta_{21}\left(\alpha_{21} + \beta_{21}\rho\right)$$

Impulse Responses

- The impulse responses are these time-paths of y and x due to the shocks u_1 and u_2
- They are found by this recursion formula
- They are functions of the estimated VAR coefficients

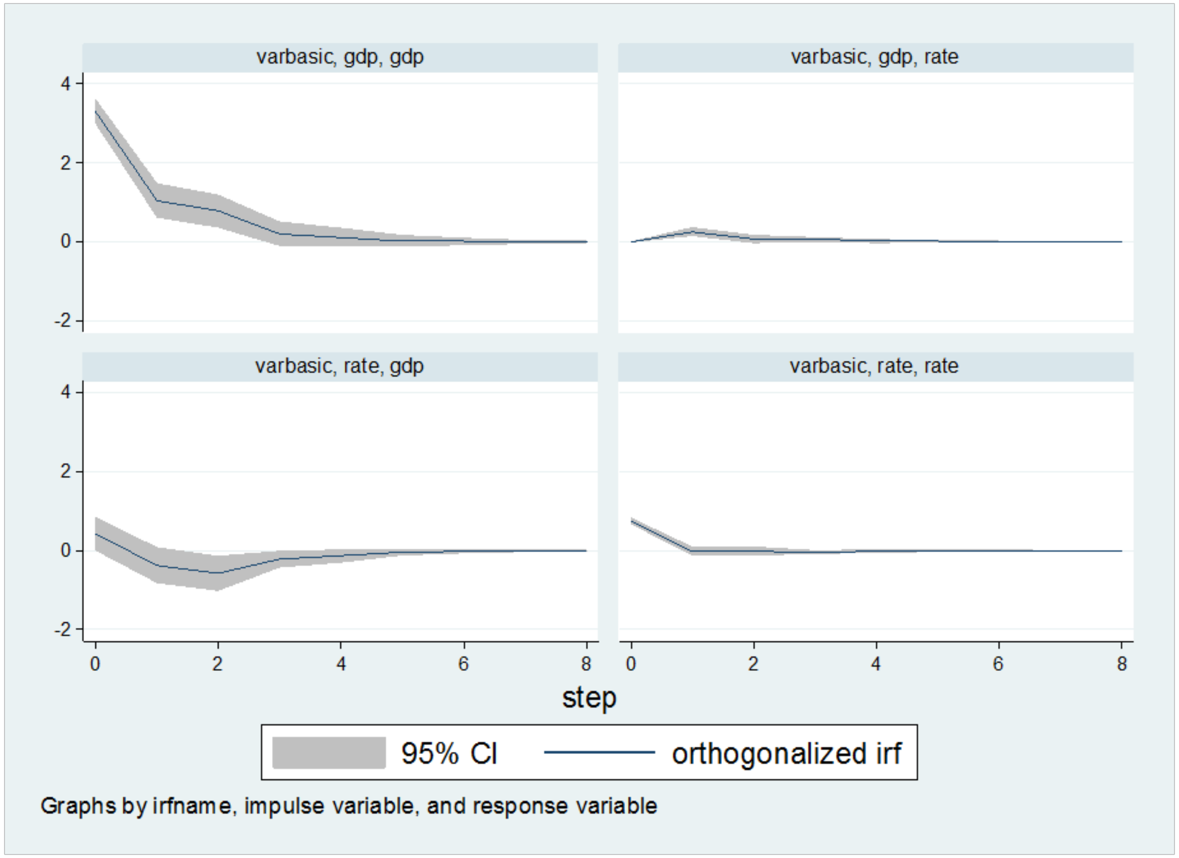
Impact of Shocks on Variables

- In a 2-variable system, there are 4 impulse response functions
  - The effect on y of a shock to y (u_1)
  - The effect on y of a shock to x (u_2)
  - The effect on x of a shock to y (u_1)
  - The effect on x of a shock to x (u_2)
- In a k-variable system, there are k^2 impulse response functions!

Stata Calculation

- Impulse response automatically calculated with varbasic command
- A kxk matrix of impulse response is created

GDP/Interest Rate Example



Interpretation

- Labeled “Graphs by irfname, impulse variable, and response variable”
  - “Impulse variable” means the source of the shock
  - “Response variable” means the variable being affected
- Upper left: “varbasic, gdp, gdp”
  - Impact of a gdp shock on the time-path of gdp
- Upper right: “varbasic, gdp, rate”
  - Impact of a gdp shock on the time-path of interest rates
- Lower left: “varbasic, rate, gdp,”
  - Impact of an interest rate shock on the time-path of gdp
- Lower fight: “varbasic, rate, rate”
  - Impact of an interest rate shock on the time-path of interest rates
- The impulse response is graphed as a function of forward time periods

Scale

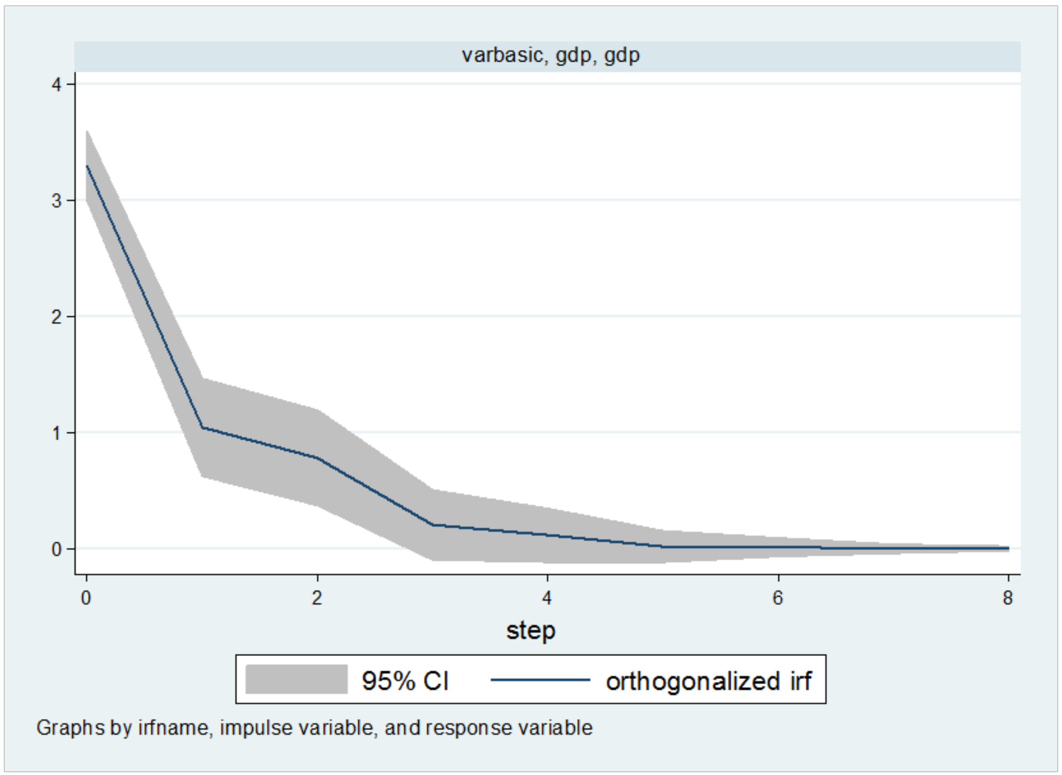
- The graphs are all created on the same scale, so difficult to read
- It may be better to create graphs separate for each impulse response

```
. irf graph oirf, impulse(gdp) response(rate)
```

- This creates the impulse response for the impact of a gdp shock on the time-path of interest rates

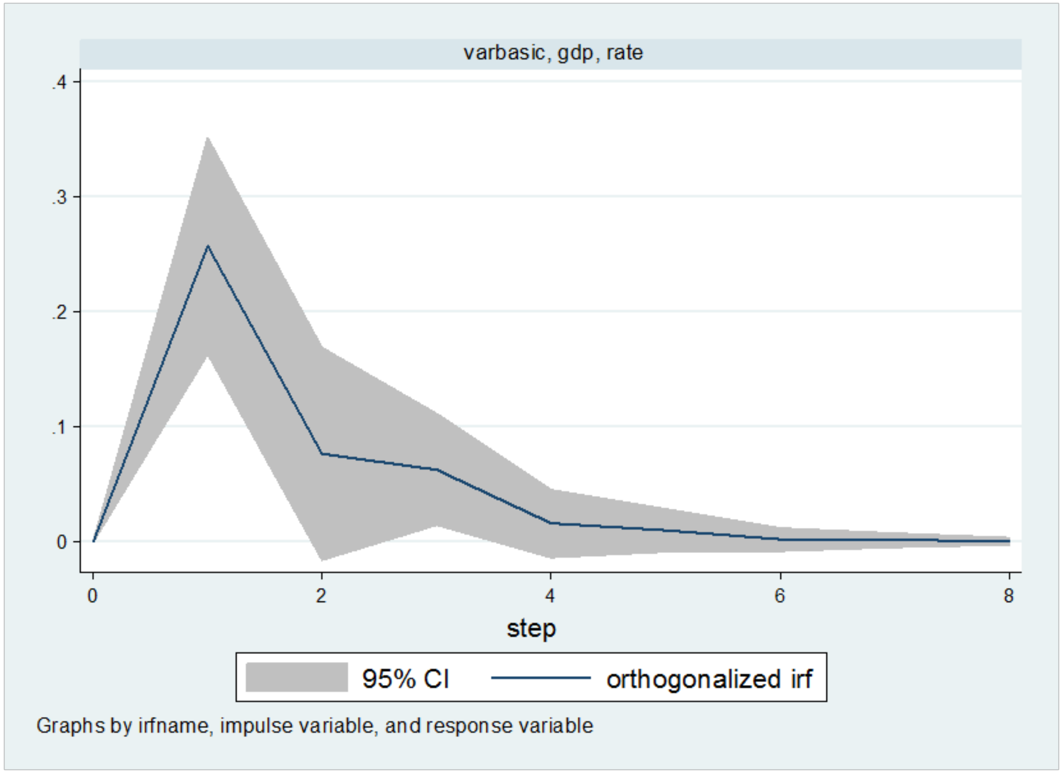
GDP on GDP

```
. irf graph oirf, impulse(gdp) response(gdp)
```



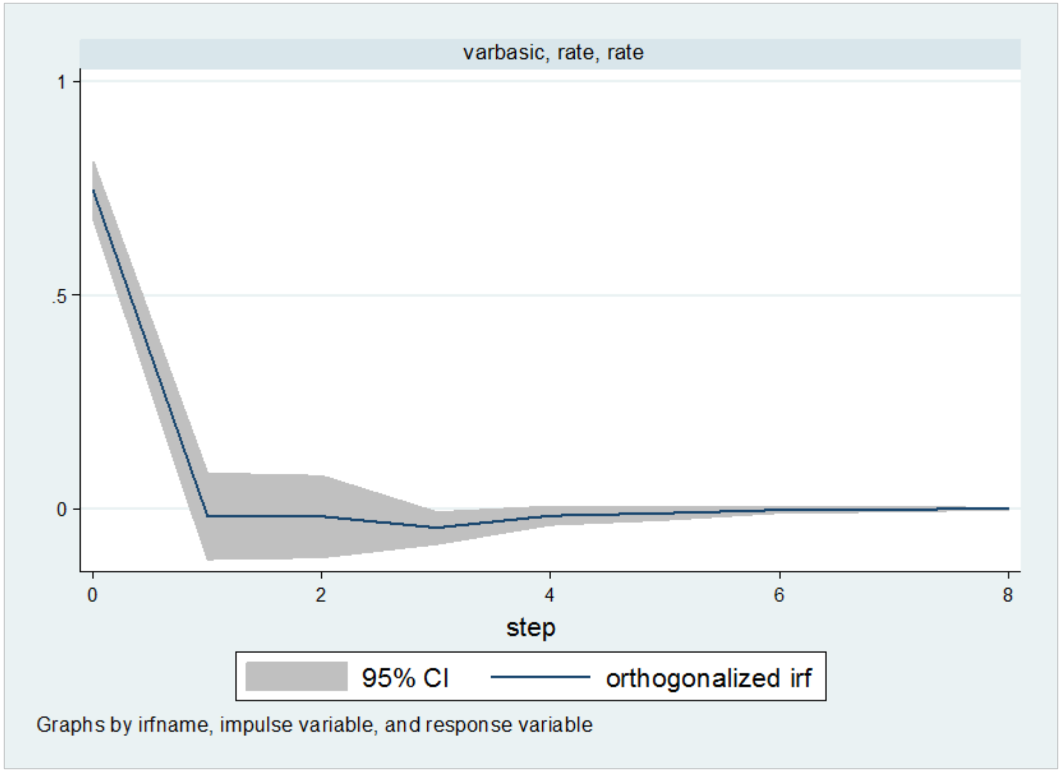
GDP on Interest Rates

```
. irf graph oirf, impulse(gdp) response(rate)
```



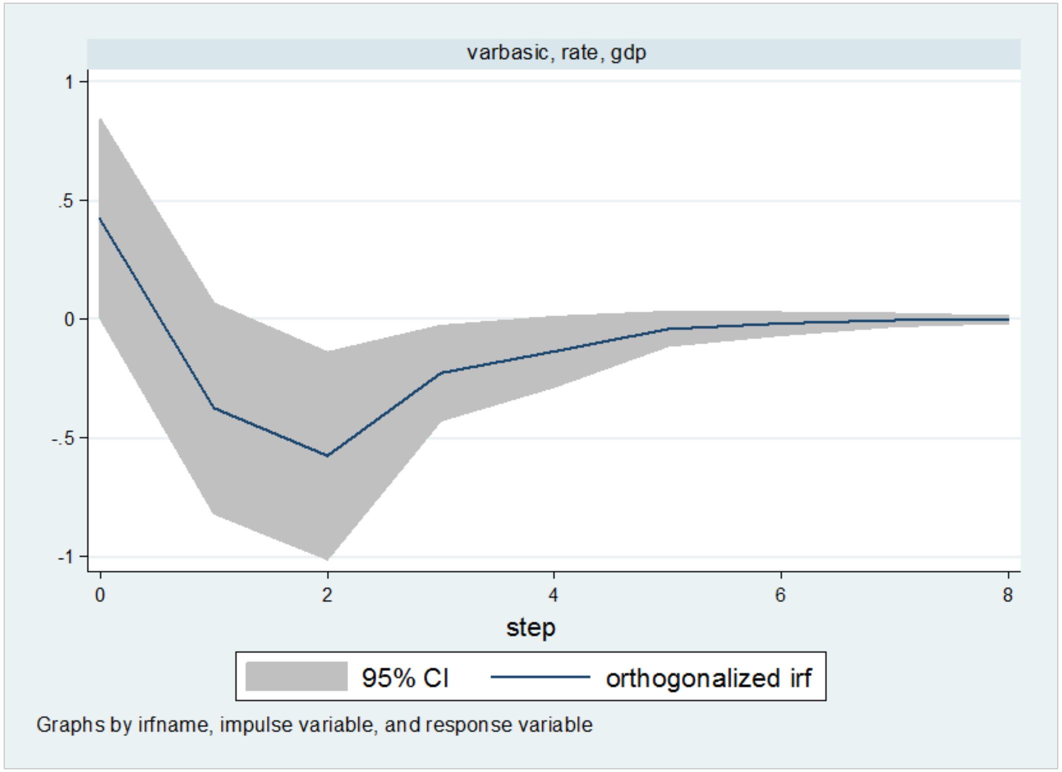
Interest Rates on Interest Rates

```
. irf graph oirf, impulse(rate) response(rate)
```



Interest Rates on GDP

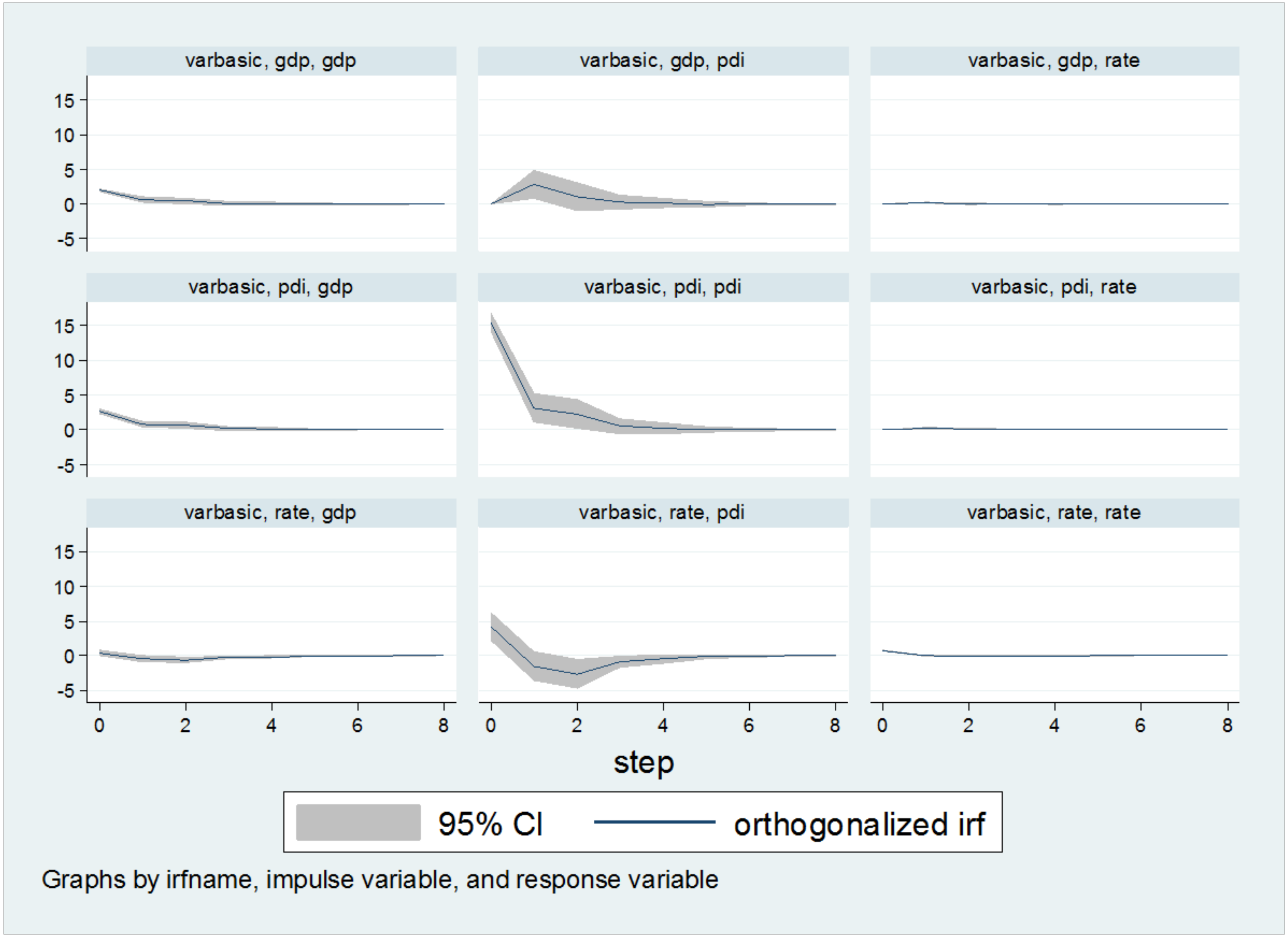
```
. irf graph oirf, impulse(rate) response(gdp)
```



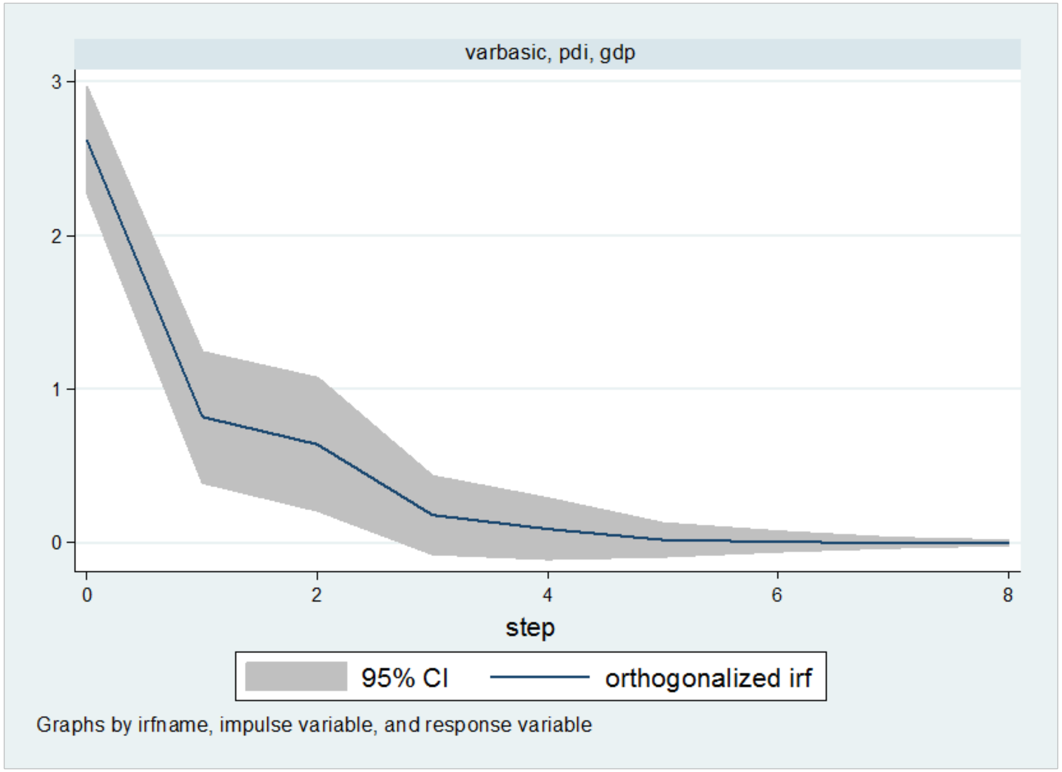
3-variable system

- Interest Rate Change (12-month T-Bill)
- Investment Growth Rate
- GDP Growth Rate

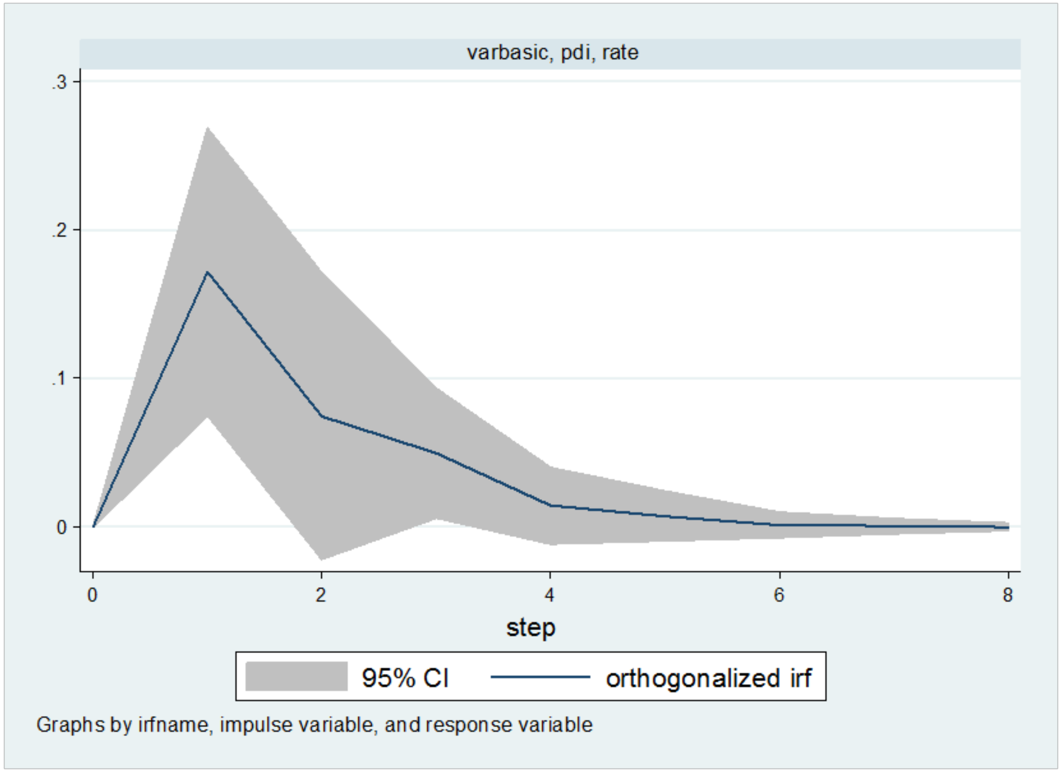
GDP/Investment/Interest Rate



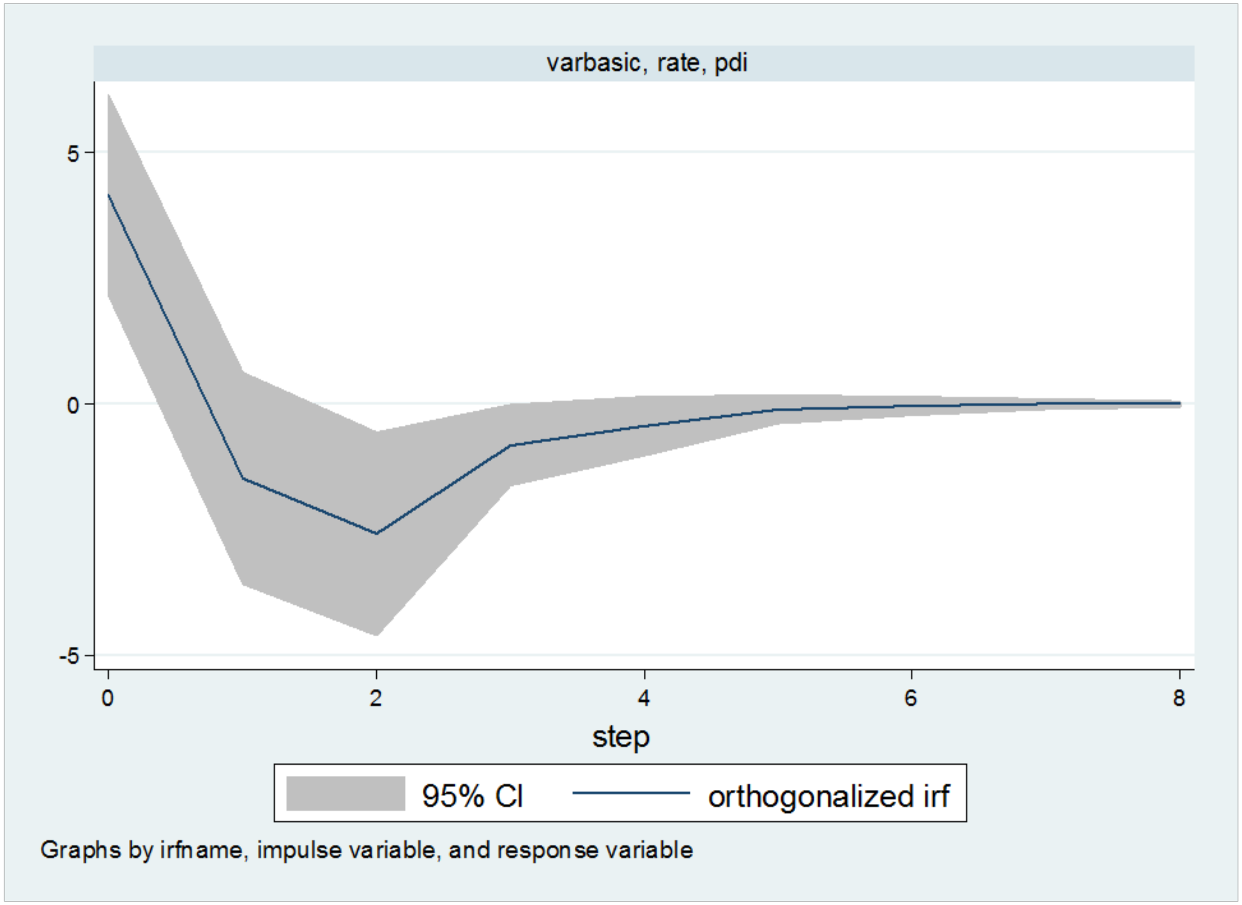
Investment Shock on GDP



Investment Shock on Interest Rate



Interest Rate Shock on Investment



High Dimensional Estimation

- What if you have a situation where the number of regressions p exceeds the number of observations n ?
- Classic example: gene array data
  - Goal: Determine which gene causes cancer
  - Number of regressors p = number of genes (5000)
  - Number of observations p = 50 (or similar)

LASSO

- One solution is LASSO estimation
- Similar idea: LARS, SCAD, Elastic Net
- Idea: Minimize the sum-of-squared errors subject to a penalty based on the sum of the absolute value of the coefficients

$$x_t = \mu_2 + \alpha_{21} y_{t-1} + \alpha_{22} y_{t-2} + \cdots + \alpha_{2p} y_{t-p}$$
$$+ \beta_{21} x_{t-1} + \beta_{22} x_{t-1} + \cdots + \beta_{2p} x_{t-p} + e_{2t}$$

LASSO

Model

$$y_t = \mu + \beta_1 x_{1t} + \beta_2 x_{2t} + \cdots + \beta_p x_{pt} + e_t$$

Minimize sum-of-squared errors plus penalty

$$\sum_{t=1}^{T} \left(y_t - \mu + \beta_1 x_{1t} + \beta_2 x_{2t} + \cdots + \beta_p x_{pt} \right)^2$$

$$+ \lambda \sum_{j=1}^{p} \left| \beta_j \right|$$

The penalty changes the problem.

Most coefficient estimates are zero.

LASSO and Forecasting

- Lasso very popular in high-dimensional statistics
- I haven’t yet seen Lasso being discussed in economic forecasting
- It is just a matter of time
- Not programmed in Stata
- If interested, I recommend the R package

Software after UW??

- You are unlikely to have access to Stata outside a university environment
  - Some corporations may have a few licenses
  - Non-academic price is expensive
- Excel widely available
  - Often used for regression analysis in corporations
  - Highly limited & clumsy
- R is a viable option
  - Free, open-source
  - Continuously updated
  - Popular among statisticians
  - http://www.r-project.org/
  - A different style; may need to do more programming
  - Documentation sometimes limited