

## OLG with Production

### Assumptions

- Timing:
  - $A_{t-1}, L_{t-1}, K_{t-1}$  are the technology stock, labour force and capital stock at **the end of period  $t-1$** . As such they will be used in production in period  $t$ .
  - This means that in period  $t$ , the population of young agents is  $L_{t-1}$  and the population of old agents is  $L_{t-2}$ .
  - $Y_t, C_t^Y, C_t^O, I_t, S_t$  are the flows of output, consumption, investment, and savings in period  $t$ .
  - $W_t L_{t-1}, r_t K_{t-1}$  are the flows of payment to labour and capital in period  $t$ .

### Aggregate Equations

- Exogenous population and technology processes (where  $L_{t-1}$  is population of young agents in period  $t$ ):

$$\begin{aligned} A_t &= (1+g) A_{t-1} \\ L_t &= (1+n) L_{t-1} \end{aligned}$$

- Expenditure in period  $t$

$$Y_t = C_t^Y + C_t^O + I_t$$

- Income in period  $t$

$$Y_t = W_t L_{t-1} + r_t K_{t-1}$$

- **Equation of motion for capital**

$$K_t = (1-\delta) K_{t-1} + I_t$$

- Production function

$$Y_t = A_{t-1} L_{t-1} \left( \frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^\alpha$$

- **Interest rate**

$$r_t = \frac{\partial Y_t}{\partial K_{t-1}} = \alpha \left( \frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^{\alpha-1}$$

- Wage rate

$$W_t = \frac{\partial Y_t}{\partial L_{t-1}} = A_{t-1} (1-\alpha) \left( \frac{K_{t-1}}{A_{t-1} L_{t-1}} \right)^\alpha$$

- Old spend their savings

$$C_t^O = (1 - \delta) K_{t-1} + r_t K_{t-1}$$

- Young spend and save their wages

$$C_t^Y + S_t = W_t L_{t-1}$$

### Intensive form

- Let

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$$\begin{aligned} C_t^Y &= A_{t-1} L_{t-1} c_t^Y \\ C_t^O &= A_{t-2} L_{t-2} c_t^O \\ I_t &= A_{t-1} L_{t-1} i_t \\ K_t &= A_t L_t k_t \\ Y_t &= A_{t-1} L_{t-1} y_t \\ S_t &= A_{t-1} L_{t-1} s_t \\ W_t &= A_{t-1} w_t \end{aligned}$$

- Then

$$\begin{aligned} y_t &= c_t^Y + \frac{1}{(1+g)(1+n)} c_t^O + i_t \\ y_t &= w_t + r_t k_{t-1} \\ k_t &= \frac{1}{(1+g)(1+n)} [(1-\delta) k_{t-1} + i_t] \\ r_t &= \alpha k_{t-1}^{\alpha-1} \\ w_t &= (1-\alpha) k_{t-1}^\alpha \\ \frac{1}{(1+g)(1+n)} c_t^O &= (1+r_t-\delta) k_{t-1} \\ c_t^Y + s_t &= w_t \end{aligned}$$

### Maximisation Problem

- Solve the young agents' consumption/savings decision problem:

$$U_t = \frac{(A_{t-1} c_t^Y)^{1-\theta} - 1}{1-\theta} + \beta \frac{(A_{t-1} c_{t+1}^O)^{1-\theta} - 1}{1-\theta}$$

- Use

$$c_t^Y = w_t - s_t$$

- And

$$\begin{aligned}
k_t &= \frac{1}{(1+g)(1+n)} [(1-\delta)k_{t-1} + i_t] \\
&= \frac{1}{(1+g)(1+n)} \left[ (1-\delta)k_{t-1} + y_t - c_t^Y - \frac{1}{(1+g)(1+n)} c_t^O \right] \\
&= \frac{1}{(1+g)(1+n)} [(1-\delta)k_{t-1} + k_{t-1}^\alpha - c_t^Y - (1+r_t-\delta)k_{t-1}] \\
&= \frac{1}{(1+g)(1+n)} [k_{t-1}^\alpha - c_t^Y - r_t k_{t-1}] \\
&= \frac{1}{(1+g)(1+n)} [(1-\alpha)k_{t-1}^\alpha - c_t^Y] \\
&= \frac{1}{(1+g)(1+n)} [w_t - c_t^Y] = \frac{s_t}{(1+g)(1+n)}
\end{aligned}$$

i.e.

$$\frac{1}{(1+g)(1+n)} c_{t+1}^O = (1+r_{t+1}-\delta)k_t = (1+r_{t+1}-\delta) \frac{s_t}{(1+g)(1+n)}$$

So

$$c_{t+1}^O = (1+r_{t+1}-\delta)s_t$$

- Therefore write maximisation problem as function of  $s_t$

$$U_t = \frac{(A_{t-1}w_t - A_{t-1}s_t)^{1-\theta} - 1}{1-\theta} + \beta \frac{(A_{t-1}(1+r_{t+1}-\delta)s_t)^{1-\theta} - 1}{1-\theta}$$

- Solving:

$$\begin{aligned}
\frac{\partial U_t}{\partial s_t} &= 0 \implies \\
s_t &= \frac{w_t}{1 + \beta^{-\frac{1}{\theta}} (1+r_{t+1}-\delta)^{\frac{\theta-1}{\theta}}}
\end{aligned}$$

### Problem fully specified

- Given initial state variable  $k_{t-1}$ , we have **8 equations (below) in 8 unknowns**:  $y_t, c_t^Y, c_t^O, i_t, s_t, k_t, r_t, w_t$  (NB  $r_{t+1}$  is a function of  $k_t$  which is already on this list):

$$\begin{aligned}
y_t &= c_t^Y + \frac{1}{(1+g)(1+n)} c_t^O + i_t \\
y_t &= w_t + r_t k_{t-1} \\
k_t &= \frac{1}{(1+g)(1+n)} [(1-\delta) k_{t-1} + i_t] \\
r_t &= \alpha k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) k_{t-1}^\alpha \\
c_t^O &= (1+g)(1+n)(1+r_t-\delta) k_{t-1} \\
w_t &= c_t^Y + s_t \\
s_t &= \frac{w_t}{1 + \beta^{-\frac{1}{\theta}} (1 + r_{t+1} - \delta)^{\frac{\theta-1}{\theta}}}
\end{aligned}$$

### Steady state

- Reduce system to just the 3rd and 8th equations **in only  $k_t$  and  $c_t^Y$**  ( $k_{t-1}$  is known):

$$\begin{aligned}
k_t &= \frac{1}{(1+g)(1+n)} [(1-\delta) k_{t-1} + i_t] \\
&= \frac{1}{(1+g)(1+n)} \left[ (1-\delta) k_{t-1} + y_t - c_t^Y - \frac{1}{(1+g)(1+n)} c_t^O \right] \\
&= \frac{1}{(1+g)(1+n)} \left[ (1+r_t-\delta) k_{t-1} + w_t - c_t^Y - \frac{1}{(1+g)(1+n)} c_t^O \right] \\
&= \frac{1}{(1+g)(1+n)} [(1+r_t-\delta) k_{t-1} + (1-\alpha) k_{t-1}^\alpha - c_t^Y - (1+r_t-\delta) k_{t-1}] \\
&= \frac{1}{(1+g)(1+n)} [(1-\alpha) k_{t-1}^\alpha - c_t^Y]
\end{aligned}$$

$$\begin{aligned}
s_t &= \frac{w_t}{1 + \beta^{-\frac{1}{\theta}} (1 + r_{t+1} - \delta)^{\frac{\theta-1}{\theta}}} \\
w_t - c_t^Y &= \frac{w_t}{1 + \beta^{-\frac{1}{\theta}} (1 + \alpha k_t^{\alpha-1} - \delta)^{\frac{\theta-1}{\theta}}} \\
(1-\alpha) k_{t-1}^\alpha - c_t^Y &= \frac{(1-\alpha) k_{t-1}^\alpha}{1 + \beta^{-\frac{1}{\theta}} (1 + \alpha k_t^{\alpha-1} - \delta)^{\frac{\theta-1}{\theta}}}
\end{aligned}$$

- The **reduced system is two equations in 2 unknowns**: given initial condition  $k_{t-1}$ , there are two processes that relate the choice of  $c_t^Y$  and  $k_t$ : one from

the **equation of motion for capital** and one from the agent's optimisation.

$$\begin{aligned} k_t &= \frac{1}{(1+g)(1+n)} [(1-\alpha)k_{t-1}^\alpha - c_t^Y] \\ (1-\alpha)k_{t-1}^\alpha - c_t^Y &= \frac{(1-\alpha)k_{t-1}^\alpha}{1 + \beta^{-\frac{1}{\theta}} (1 + \alpha k_t^{\alpha-1} - \delta)^{\frac{\theta-1}{\theta}}} \end{aligned}$$

There is a unique choice,  $c_t^Y$ , that satisfies both equations. This is the OLG model in its pure form: all the other equations basically just provide definitions for the other quantities in the model.

- Combine these equations to eliminate  $c_t^Y$ :

$$k_t = \frac{1}{(1+g)(1+n)} \left[ \frac{(1-\alpha)k_{t-1}^\alpha}{1 + \beta^{-\frac{1}{\theta}} (1 + \alpha k_t^{\alpha-1} - \delta)^{\frac{\theta-1}{\theta}}} \right]$$

- Steady state intensive capital stock is the solution to:

$$k = \frac{1}{(1+g)(1+n)} \left[ \frac{(1-\alpha)k^\alpha}{1 + \beta^{-\frac{1}{\theta}} (1 + \alpha k^{\alpha-1} - \delta)^{\frac{\theta-1}{\theta}}} \right]$$

No analytic solution for  $\theta \neq 1$ . Need to use numerical method.

- Other steady state variables:

$$\begin{aligned} c^Y &= (1-\alpha)k^\alpha - (1+g)(1+n)k \\ r &= \alpha k^{\alpha-1} \\ w &= (1-\alpha)k^\alpha \\ s &= w - c^Y \\ c^O &= (1+g)(1+n)(1+r-\delta)k \\ y &= w + rk \\ i &= y - c^Y - \frac{1}{(1+g)(1+n)}c^O \end{aligned}$$