Answers to the Problem Set 3

Econometrics (Ph.D.)

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03 December, 2021

Problem 1

(a) Actual coverage probability is about 0.930 for 10,000 draws of size 50.

(b) Actual coverage probability increases to 0.935 for 1,000 draws of size 50.

Problem 2

The asymptotic distributions for the two estimators can be found as:

i. For the first estimator, it is a simple average. Hence, by the Central Limit Theorem $\sqrt{n}(\hat{\theta}-\theta)$ is distributed as $N(0, Var(X))$, where $Var(X) = \theta^2$ (See Figure 2)

ii. Denoting X_i^2 as Y_i, we know, by the CLT that $\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} Y_i - EY)) \sim N(0, Var(Y))$. Here, EY and $Var(Y)$ is the population mean and variance of Y. Furthermore, the estimator is of the following shape: $\sqrt{\frac{1}{2}Y}$. Its derivative is, thus, $\frac{1}{\sqrt{8Y}}$.

Furthermore, it is easy to show that the standard deviation and the mean of the exponential distribution are both equal to θ (See Figure 2). Using this,

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ \theta^2 &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \theta^2 \\ \theta &= \sqrt{\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} X_i^2} \end{aligned}$$

hence our second estimation. From above, we can also see that $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ or $\frac{1}{n}\sum_{i=1}^{n} Y_i$ is equal to $2\theta^2$. Now it is easy to see that $\sqrt{n}(\tilde{\theta}-\theta)$ is distributed as $N(0, \frac{1}{8\cdot 2\theta^2}\cdot Var(X^2))$ or $N(0, \frac{1}{16\theta^2}\cdot Var(X^2))$. Let me derive $Var(X^2))$ below. Since

$$\begin{aligned} &E((X_i^2 - E(X^2))^2) \\ &= E(X^4) - E(X^2)^2 \end{aligned}$$

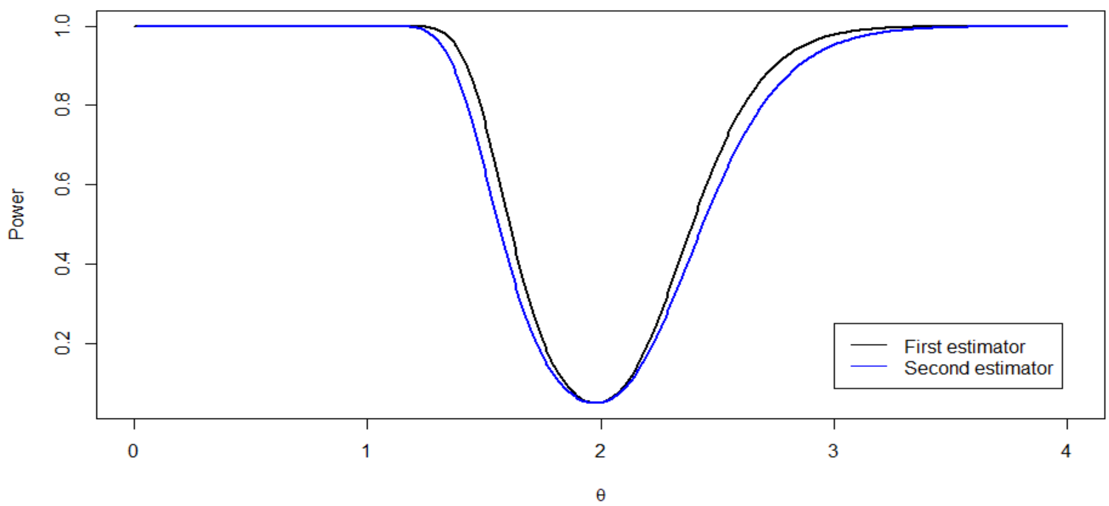
we know from the previous part that $E(X^2)^2 = (2\theta^2)^2$. Now to find $E(X^4)$, let us use integration by parts and the preceding result.

$$
\begin{aligned}
E(X^4) &= \int_0^\infty X_i^4 \frac{1}{\theta} e^{-\frac{X_i}{\theta}} dX_i \\
&= -X_i^4 e^{-\frac{X_i}{\theta}} \Big|_0^\infty + 4 \int_0^\infty X_i^3 e^{-\frac{X_i}{\theta}} dX_i \\
&= 0 + -4\theta X_i^3 e^{-\frac{X_i}{\theta}} \Big|_0^\infty + 4 \cdot 3\theta \int_0^\infty X_i^2 e^{-\frac{X_i}{\theta}} dX_i) \\
&= 12\theta^2 \int_0^\infty X_i^2 \frac{1}{\theta} e^{-\frac{X_i}{\theta}} dX_i) \\
&= 12\theta^2 \cdot 2\theta^2 \\
&= 24\theta^4
\end{aligned}
$$

Therefore, $Var(X^2)) = 24\theta^4 - (2\theta^2)^2 = 20\theta^4$. Thus, the asymptotic distribution of $\sqrt{n}(\tilde{\theta} - \theta)$ is $N(0, \frac{1}{16\theta^2} \cdot 20\theta^4)$ or $N(0, 1.25\theta^2)$

Using the two variance (standard error) estimators just derived and with significance level of 5%, and for the grid of values between 0 and 4 for alternative θs, let us plot the corresponding two power curves for the sample size of 100.

Figure 1: Power curves for the two estimators



The power curve of the second estimator stays always below the power curve of the first estimator. It means that the second estimator is less likely to reject the null hypothesis when it is indeed false. It is due to the higher variance estimator for the second estimator of factor 1.25. In that respect, the second estimator is less desirable as it cannot signal mistakes faster (avoiding Type II error). However, the upside is that it will not reject the true models more than the first estimator, thus lowering the probability of Type I error.

Appendix

Figure 2: Exponential distribution facts

From Variance as Expectation of Square minus Square of Expectation:

$$\operatorname{var}(X) = \mathsf{E}\left(X^2\right) - \left(\mathsf{E}(X)\right)^2$$

From Expectation of Exponential Distribution:

$$\mathsf{E}(X) = \beta$$

The expectation of X^2 is:

$$\begin{aligned}
\mathsf{E}\left(X^2\right) &= \int_{x \,\in\, \Omega_X} x^2 \, f_X(x) \, \mathrm{d}x \\
&= \int_0^\infty x^2 \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= \left[-x^2 \exp\left(-\frac{x}{\beta}\right)\right]_0^\infty + \int_0^\infty 2x \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= 0 + 2\beta \int_0^\infty x \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= 2\beta \, \mathsf{E}(X) \\
&= 2\beta^2
\end{aligned}$$

Thus the variance of X is:

$$\begin{aligned}
\operatorname{var}(X) &= \mathsf{E}\left(X^2\right) - \left(\mathsf{E}(X)\right)^2 \\
&= 2\beta^2 - \beta^2 \\
&= \beta^2
\end{aligned}$$

$\blacksquare$