# Answers to the Problem Set 3

Econometrics (Ph.D.)

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## Problem 1

(a) Actual coverage probability is about 0.930 for 10,000 draws of size 50.

(b) Actual coverage probability increases to 0.935 for 1,000 draws of size 50.

## Problem 2

The asymptotic variances for the two estimators can be found as:

i. For the first estimator, it is a simple average. Hence, by the Central Limit Theorem $\sqrt{n}(\hat{\theta} - \theta)$ is distributed as $N(0, Var(X))$.

ii. Denoting $X_i^2$ as $Y_i$, we know, by the CLT that $\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} Y_i - EY)) \sim N(0, Var(Y))$. Here, $EY$ and $Var(Y)$ is the population mean and variance of $Y$. Furthermore, the estimator is of the following shape: $\sqrt{\frac{1}{2}Y}$. Its derivative is, thus, $\frac{1}{\sqrt{8Y}}$.

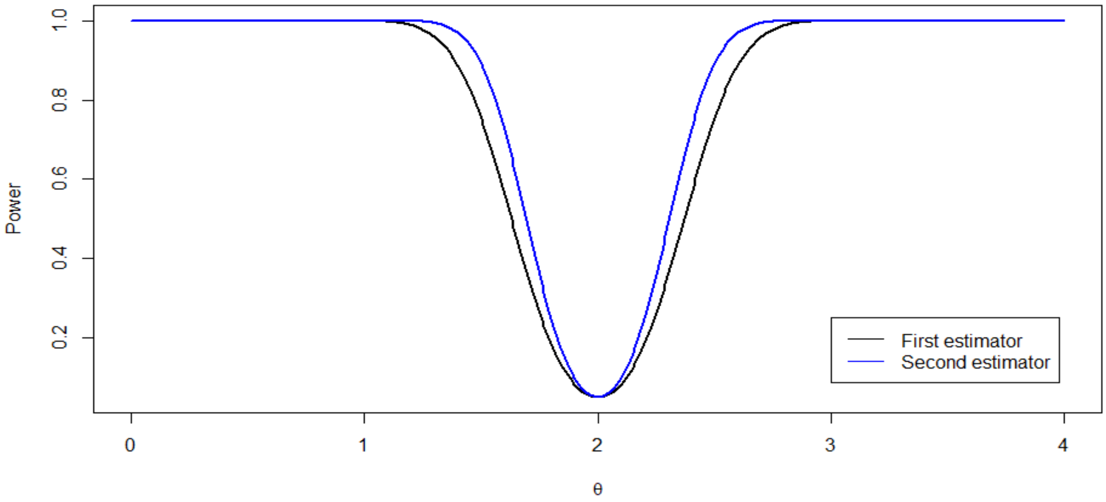
   Furthermore, it is easy to show that the standard deviation and the mean of the exponential distribution are both equal to $\theta$ (See, for example, Figure 2).

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ \theta^2 &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \theta^2 \\ \theta &= \sqrt{\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} X_i^2} \end{aligned}$$

   hence our second estimation. From above, we can also see that $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ or $Y$ is equal to $2\theta^2$. Now it is easy to see that $\sqrt{n}(\tilde{\theta} - \theta)$ is distributed as $N(0, \frac{1}{8 \cdot 2\theta^2} \cdot Var(X^2))$ or $N(0, \frac{1}{16\theta^2} \cdot Var(X^2))$.

Using the two standard error (variance) estimators just derived and with significance level of 5%, let us plot the corresponding two power curves for the sample size of 100.

Figure 1: Power curves for two types of estimators




The power curve of the second estimator stays always above the power curve of the first estimator. It means that the second estimator is more likely to reject the null hypothesis when it is indeed false. In that respect, the second estimator is more desirable as it signals mistake faster and, thus, have higher power. However, the downside is that it will reject the true models more often than the first estimator.

# Appendix

Figure 2: Exponential distribution facts

From Variance as Expectation of Square minus Square of Expectation:

$$\operatorname{var}(X) = \mathsf{E}\left(X^2\right) - \left(\mathsf{E}(X)\right)^2$$

From Expectation of Exponential Distribution:

$$\mathsf{E}(X) = \beta$$

The expectation of $X^2$ is:

$$\begin{aligned}
\mathsf{E}\left(X^2\right) &= \int_{x \in \Omega_X} x^2 f_X(x) \, \mathrm{d}x \\
&= \int_0^\infty x^2 \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= \left[-x^2 \exp\left(-\frac{x}{\beta}\right)\right]_0^\infty + \int_0^\infty 2x \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= 0 + 2\beta \int_0^\infty x \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) \mathrm{d}x \\
&= 2\beta \, \mathsf{E}(X) \\
&= 2\beta^2
\end{aligned}$$

Thus the variance of $X$ is:

$$\begin{aligned}
\operatorname{var}(X) &= \mathsf{E}\left(X^2\right) - \left(\mathsf{E}(X)\right)^2 \\
&= 2\beta^2 - \beta^2 \\
&= \beta^2
\end{aligned}$$

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