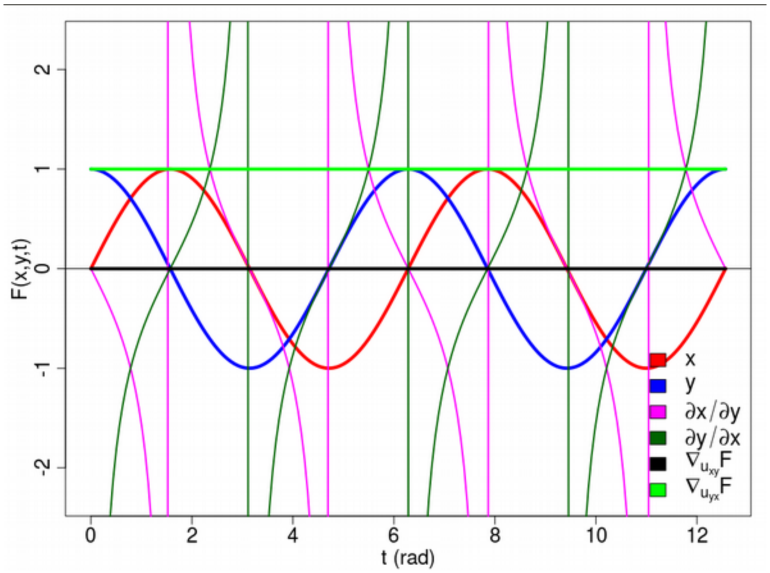
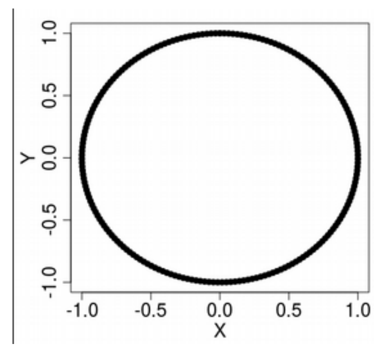
S-Map Coefficients : Analytical & Numerical Comparison

Circle Manifold

Consider the 2-D mapping $F(x,y,t):\begin{cases} x=\sin(t) \\ y=\cos(t)\end{cases}$ where $\dfrac{\partial x}{\partial y}=\dfrac{\cos(t)}{-\sin(t)}=-\cot(t)$ is the Jacobian of x onto y and $\dfrac{\partial y}{\partial x}=\dfrac{-\sin(t)}{\cos(t)}=-\tan(t)$ is the Jacobian of y onto x. This represents an Eulerian reference formed by the orthogonal basis of Cartesian X and Y.

The directional derivative defines the gradient of F at a point (x, y, t) along a unit vector u: $\nabla_u F \equiv \nabla F \cdot \dfrac{u}{|u|}$, in the 2-D case: $\nabla_{u_{xy}} F = \dfrac{\partial F}{\partial x} u_x + \dfrac{\partial F}{\partial y} u_y$. Individual terms can be thought of as Jacobians of the manifold projected on the Eulerian basis unit vectors to yield a Lagrangian reference along the manifold. Applied to F: $\nabla_{u_{xy}} F = \cos(t) u_x - \sin(t) u_y$. We can also consider the orthogonal basis $\nabla_{u_{yx}} F = \sin(t) u_x + \cos(t) u_y$.





The Lagrangian directional derivatives of F are constant, with values of 0, 1. This reflects both the constant curvature of the circle manifold, and the orthogonal relation of x and y.

The directional derivative can be evaluated as $\nabla_u F = \lim_{h\to 0} \dfrac{F(t+hu)-F(t)}{h}$ with the continuous limits yielding $\nabla_{u_{xy}} F = 0$ $\nabla_{u_{yx}} F = 1$ as noted above. To validate numerical approximations when $h > 0$, we expect that the limit values of 0 and 1 (indicating the constant curvature and orthogonal basis) will change accordingly. If F is based on discrete time of 200 points (199 intervals) from 0 to 4π, then $h = 4\pi/199 = 0.0632$. Discrete values reflecting "error" from the nonzero h will be $\nabla_{u_{xy}} F = \cos(t+h) u_x - \sin(t+h) u_y = 0.0632$ and $\nabla_{u_{yx}} F = \sin(t+h) u_x + \cos(t+h) u_y = 0.998$. This approximates the discrete time linear mappings as returned in numerical results.

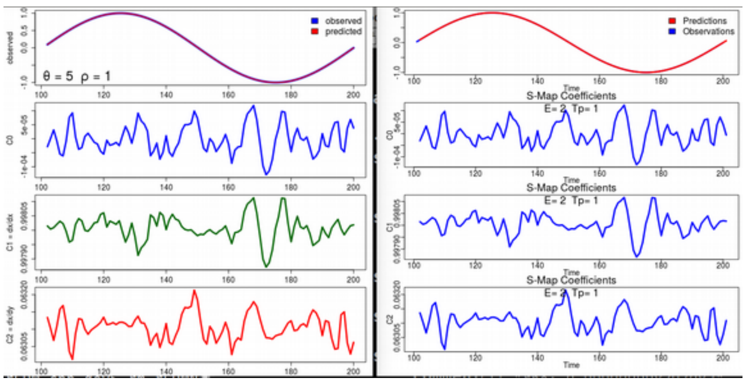
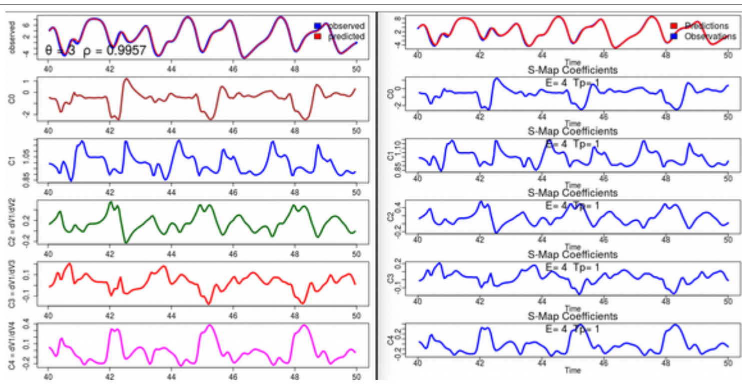
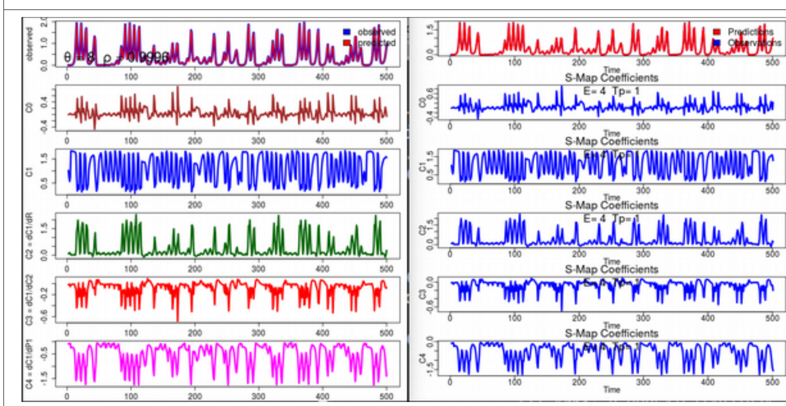
S-Map Coefficients

S-map is a linear regression ($Ac=b$) of state space coordinates localised with a distance dependent decay kernel $w(d)=e^{-\theta\, d/d_m}$ with localisation parameter θ. Localised observation (library) vectors of dimension E at k_{nn} nearest neighbors form the matrix $A_{knn,E}$, vector b_E is assigned target library coordinates to determine regression coefficients c_E. State space estimates from observation vector O

are $\hat{O} = c_0 + \sum_{i=1}^{E} c_i O_i$. As interpreted in Deyle et al.[1], the Lagrangian basis of directional derivatives can be approximated with S-map coefficients:

"Here we note simply that, in multivariate embeddings (i.e. native embeddings using causal variables rather than lags of a single variable), the S-map coefficients approximate the Jacobian or interaction elements at successive points along the attractor. That is, S-maps generate the relevant Jacobian elements that define the interaction strengths, and as required do so sequentially (S = 'Sequential Jacobian') as the system travels along its attractor."

rEDM S-map coefficients

	Circle (200 points) Left: rEDM 0.7.4 block_lnlp() Right: rEDM 1.0.1 SMap() Non-constant numerical values (σ ~ 3.9e-05, 4.4e-05) are presumed numerical rounding inaccuracies propagated through the SVD. Mean values $\nabla_{u_{xy}} F = 0.0631$ $\nabla_{u_{xy}} F = 0.998$ are consistent with the discrete time Δ*h* projections.
	Lorenz '96 5-D (4-D embedding) Left: rEDM 0.7.4 block_lnlp() Right: rEDM 1.0.1 SMap()
	5 Species model (Deyle et al. 2016) Left: rEDM 0.7.4 block_lnlp() Right: rEDM 1.0.1 SMap()

1Deyle ER, May RM, Munch SB, Sugihara G. 2016.Tracking and forecasting ecosystem interactions in real time. Proc. R. Soc. B 283: 20152258.