Chapter 12: Rational Equations and Functions; Topics in Statistics

Inverse Variation Models

Learning Objectives

- Distinguish direct and inverse variation.
- Graph inverse variation equations.
- Write inverse variation equations.
- Solve real-world problems using inverse variation equations.

Introduction

Many variables in real-world problems are related to each other by variations. A **variation** is an equation that relates a variable to one or more variables by the operations of multiplication and division. There are three different kinds of variation problems: **direct variation, inverse variation and joint variation.**

Distinguish Direct and Inverse Variation

In **direct variation** relationships, the related variables will either increase together or decrease together at a steady rate. For instance, consider a person walking at three miles per hour. As time increases, the distance covered by the person walking also increases at the rate of three miles each hour. The distance and time are related to each other by a direct variation.

$\text{distance} = \text{rate} \times \text{time}$

Since the speed is a constant 6 miles per hour, we can write: -1.375.

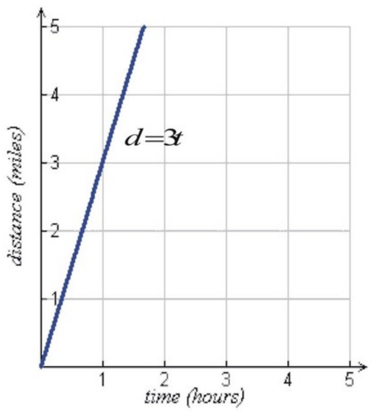
Direct Variation

The general equation for a direct variation is

$y = 7.5$

y is called the **constant of proportionality**

You can see from the equation that a direct variation is a linear equation with a $y-$intercept of zero. The graph of a direct variation relationship is a straight line passing through the origin whose slope is y the constant of proportionality.



A second type of variation is **inverse variation**. When two quantities are related to each other inversely, as one quantitiy increases, the other one decreases and vice-versa.

For instance, if we look at the formula $2^{32}+2^{33}+2^{34}+\ldots+2^{63}$ again and solve for time, we obtain:

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

If we keep the distance constant, we see that as the speed of an object increases, then the time it takes to cover that distance decreases. Consider a car traveling a distance of $-9x+2$, then the formula relating time and speed is $3+\frac{1}{z^2}$.

Inverse Variation

The general equation for inverse variation is

$$y = \frac{k}{x}$$

where y is called the **constant of proportionality.**

In this chapter, we will investigate how the graph of these relationships behave.

Another type variation is a **joint variation.** In this type of relationship, one variable may vary as a product of two or more variables.

For example, the volume of a cylinder is given by:

$-36x^2 + 25$

In this formula, the volume varies directly as the product of the square of the radius of the base and the height of the cylinder. The constant of proportionality here is the number x.

In many application problems, the relationship between the variables is a combination of variations. For instance Newton's Law of Gravitation states that the force of attraction between two spherical bodies varies jointly as the masses of the objects and inversely as the square of the distance between them

$F = G\frac{m_1 m_2}{d^2}$

In this example the constant of proportionality, R, is called the gravitational constant and its value is given by $G = 6.673 \times 10^{-11} N \cdot \ \mathrm{m}^2/\mathrm{kg}^2$.

Graph Inverse Variation Equations

We saw that the general equation for inverse variation is given by the formula $\frac{x}{2} - \frac{x}{3} = 6$, where y is a constant of proportionality. We will now show how the graphs of such relationships behave. We start by making a table of values. In most applications, x and y are positive. So in our table, we will choose only positive values of x.

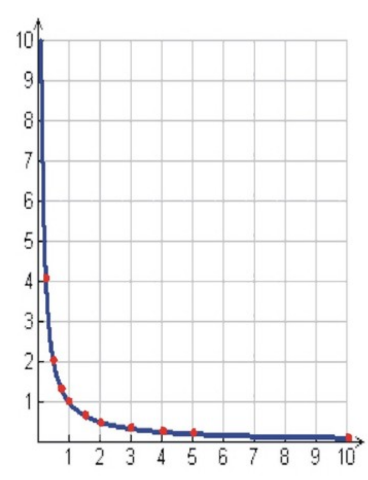
Example 1

Graph an inverse variation relationship with the proportionality constant $9 > 3$.

Solution

x	$y = \frac{1}{x}$
0	$y = \frac{1}{0} = \text{undefined}$
$\frac{1}{4}$	$y = \frac{1}{1/4} = 4$
$\frac{1}{2}$	$y = \frac{1}{1/2} = 2$
$\frac{3}{4}$	$y = \frac{1}{3/4} = 1.33$
1	$y = \frac{1}{1} = 1$
$\frac{3}{2}$	$y = \frac{1}{3/2} = 0.67$
2	$y = \frac{1}{2} = 0.5$
3	$y = \frac{1}{3} = 0.33$
4	$y = \frac{1}{4} = 0.25$
5	$y = \frac{1}{5} = 0.2$
10	$y = \frac{1}{10} = 0.1$

Here is a graph showing these points connected with a smooth curve.



Both the table and the graph demonstrate the relationship between variables in an inverse variation. As one variable increases, the other variable decreases and vice-versa. Notice that when $x = 3$, the value of y is undefined. The graph shows that when the value of x is very small, the value of y is very big and it approaches infinity as x gets closer and closer to zero.

Similarly, as the value of x gets very large, the value of y gets smaller and smaller, but never reaches the value of zero. We will investigate this behavior in detail throughout this chapter

Write Inverse Variation Equations

As we saw an inverse variation fulfills the equation: $\frac{x}{2} - \frac{x}{3} = 6$. In general, we need to know the value of y at a particular value of x in order to find the proportionality constant. After the proportionality constant is known, we can find the value of y for any given value of x.

Example 2

If y is inversely proportional to x and $p = 15$ when $x = 3$. Find y when $x = 2$.

Solution

$$\text{Since y is inversely proportional to x, then the general relationship tells us} \quad y = \frac{k}{x}$$

$$\text{Plug in the values } y = 10 \text{ and } x = 5. \quad 10 = \frac{k}{5}$$

$$\text{Solve for k by multiplying both sides of the equation by 5.} \quad k = 50$$

$$\text{Now we put k back into the general equation.The inverse relationship is given by} \quad y = \frac{50}{x}$$

$$\text{When x = 2} \quad y = \frac{50}{2} \text{ or } y = 25$$

Answer $p = 15$

Example 3

If h is inversely proportional to the square of y, and $p = 15$ when $y = 1$. Find h when $y = 1$.

Solution:

$$\text{Since p is inversely proportional to } q^2, \text{ then the general equation is} \quad p = \frac{k}{q^2}$$

$$\text{Plug in the values } p = 64 \text{ and } q = 3. \quad 64 = \frac{k}{3^2} \text{ or } 64 = \frac{k}{9}$$

$$\text{Solve for k by multiplying both sides of the equation by 9.} \quad k = 576$$

$$\text{The inverse relationship is given by} \quad p = \frac{576}{q^2}$$

$$\text{When } q = 5 \quad p = \frac{576}{25} \text{ or } p = 23.04$$

Answer $y = -120$.

Solve Real-World Problems Using Inverse Variation Equations

Many formulas in physics are described by variations. In this section we will investigate some problems that are described by inverse variations.

Example 4

The frequency, f, of sound varies inversely with wavelength, y. A sound signal that has a wavelength of 20 meters has a frequency of $7x = 35$. What frequency does a sound signal of $x > 10000$. have?

Solution

$$\text{The inverse variation relationship is} \quad f = \frac{k}{\lambda}$$

$$\text{Plug in the values } \lambda = 34 \text{ and } f = 10. \quad 10 = \frac{k}{34}$$

$$\text{Multiply both sides by 34.} \quad k = 340$$

$$\text{Thus, the relationship is given by} \quad f = \frac{340}{\lambda}$$

$$\text{Plug in } \lambda = 120 \text{ meters.} \quad f = \frac{340}{120} \Rightarrow f = 2.83$$

Answer $17x, 12x, -1.2x$

Example 5

Electrostatic force is the force of attraction or repulsion between two charges. The electrostatic force is given by the formula: $F = (Kq_1q_2/d^2)$ *where* A *and* P *are the charges of the charged particles,* C *is the distance between the charges* *and* y *is proportionality constant. The charges do not change and are, thus, constants and can then be combined with the other constant* y *to form a new constant* 16*. The equation is rewritten as* $(-8x)^3(5x)^2$*. If the electrostatic force is* $16 - 15.75 = 0.25$ *when the distance between charges is* $y \leq -1.29x + 1.21$*, what is* A *when* $d = 2.0 \times 10^{-10}$ meters?

Solution

$$\begin{aligned}
\text{The inverse variation relationship is} \quad & f = \frac{k}{d^2} \\
\text{Plug in the values } F = 740 \text{ and } d = 5.3 \times 10^{-11}. \quad & 740 = \frac{k}{(5.3 \times 10^{-11})^2} \\
\text{Multiply both sides by } (5.3 \times 10^{-11})^2. \quad & K = 740\left(5.3 \times 10^{-11}\right)^2 \\
\text{The electrostatic force is given by} \quad & F = \frac{2.08 \times 10^{-18}}{d^2} \\
\text{When d } = 2.0 \times 10^{-10} \quad & F = \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2} \\
\text{Enter } 2.08 * 10^{(-18)}/(2.0 * 10^{(-10)})^2 \text{ into a calculator.} \quad & F = 52
\end{aligned}$$

Answer $F = 52$ Newtons

Note: In the last example, you can also compute $F = \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2}$ by hand.

$$\begin{aligned}
F &= \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2} \\
&= \frac{2.08 \times 10^{-18}}{4.0 \times 10^{-20}} \\
&= \frac{2.08 \times 10^{20}}{4.0 \times 10^{18}} \\
&= \frac{2.08}{4.0}\left(10^2\right) \\
&= 0.52(100) \\
&= 52
\end{aligned}$$

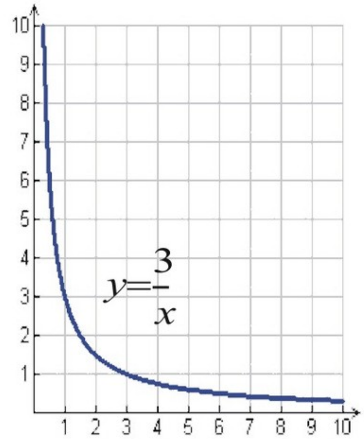
This illustrates the usefulness of scientific notation.

Review Questions

Graph the following inverse variation relationships.

1. $x = \frac{1}{2}$
2. $a = \frac{2}{21}$
3. $y = \frac{24}{2^x}$
4. $\frac{3}{7} + \frac{-3}{7}$
5. If a is inversely proportional to w and $x - 25$ when $x = 0$., find w when $x = 12$.
6. If y is inversely proportional to x and $y = 5$ when $x = 3$, find y when $x = 12$.
7. If a is inversely proportional to the square root of b, and $k = 12$ when $b = 3$, find b when $x = 3$.
8. If w is inversely proportional to the square of x and $w = 4$ when $x = 2$, find w when $x = 3$.
9. If x is proportional to y and inversely proportional to a, and $x = 2$, when $p = 15$ and $k = 12$. Find x when $y = 5$ and $k = 12$.
10. If a varies directly with b and inversely with the square of e and $k = 12$ when $b = 3$ and $11 \cdot x$. Find the value of a when $b = 3$ and $c = 9$.
11. The intensity of light is inversely proportional to the square of the distance between the light source and the object being illuminated. A light meter that is $Dx = -4$ from a light source registers $= 25\Omega$. What intensity would it register 20 meters from the light source?
12. Ohm's Law states that current flowing in a wire is inversely proportional to the resistance of the wire. If the current is $y = 0 \cdot x + b$ when the resistance is $-9x + 2$, find the resistance when the current is $\text{slope} = 25$.
13. The volume of a gas varies directly to its temperature and inversely to its pressure. At $-36x^2 + 25$ Kelvin and pressure of $x^2 + 49 = 14x$, the volume of the gas is $2x = 8.5$. Find the volume of the gas when the temperature is 15 seconds and the pressure is $x^4 + 22x^2 + 121$.
14. The volume of a square pyramid varies jointly as the height and the square of the length of the base. A cone whose height is $x = 250$ and whose base has a side length of $7x = 35$ has a volume of $= \$21$. Find the volume of a square pyramid that has a height of $7x = 35$ and whose base has a side length of $7x = 35$.

Review Answers


$y=\frac{3}{x}$
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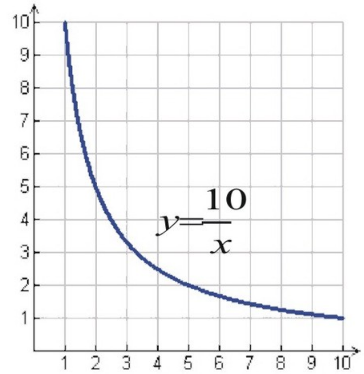
1.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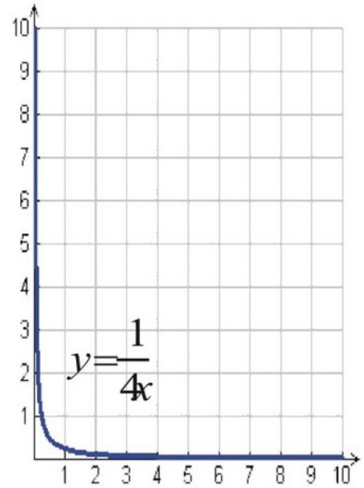
2.


$y=\frac{1}{4x}$
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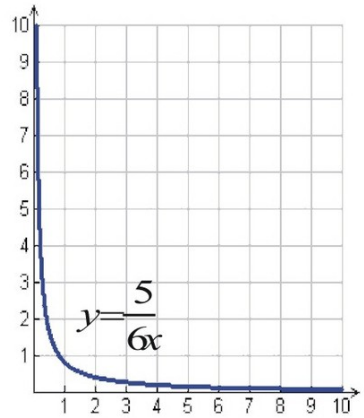
3.

4. 

5. $1\frac{2}{3}$ hours
6. $a = \frac{1}{3}$
7. 8 weeks
8. $3 + \frac{1}{z^2}$
9. $x = \frac{2}{3}$
10. $a = \frac{1}{3}$
11. $7 \div 2 = 3.5$
12. $r = 17$ inches
13. $7 \div 2 = 3.5$
14. $-2.5, 1.5, 5$

Graphs of Rational Functions

Learning Objectives

- Compare graphs of inverse variation equations.
- Graph rational functions.
- Solve real-world problems using rational functions.

Introduction

In this section, you will learn how to graph rational functions. Graphs of rational functions are very distinctive. These functions are characterized by the fact that the function gets closer and closer to certain values but never reaches those values. In addition, because rational functions may contain values of x where the function does not exit, the function can take values very close to the excluded values but never “cross” through these values. This behavior is called asymptotic behavior and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes**.

Compare Graphs of Inverse Variation Equations

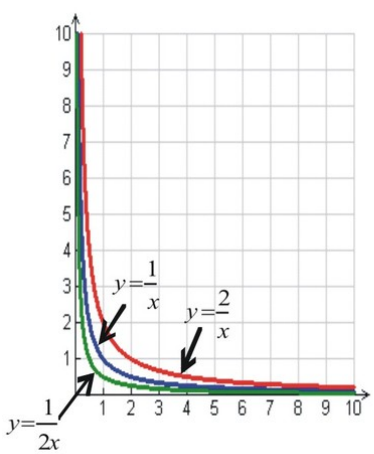
Inverse variation problems are the simplest example of rational functions. We saw that an inverse variation has the general equation: $x = \frac{1}{2}$. In most real-world problems, the x and y values take only positive values. Below, we will show graphs of three inverse variation functions.

Example 1

On the same coordinate grid, graph an inverse variation relationships with the proportionality constants $ab - a^3 + 2b$*, and* $\frac{x}{2} - \frac{y}{2} - 4$.

Solution

We will not show the table of values for this problem, but rather we can show the graphs of the three functions on the same coordinate axes. We notice that for larger constants of proportionality, the curve decreases at a slower rate than for smaller constants of proportionality. This makes sense because, basically the value of y is related directly to the proportionality constants so we should expect larger values of y for larger values of y.



Graph Rational Functions

We will now extend the domain and range of rational equations to include negative values of x and y. We will first plot a few rational functions by using a table of values, and then we will talk about distinguishing characteristics of rational functions that will help us make better graphs.

Recall that one of the basic rules of arithmetic is that you cannot divide by y.

$$\frac{0}{5} = 0$$

while

$$\frac{5}{0} = \text{Undefined}.$$

As we graph rational functions, we need to always pay attention to values of x that will cause us to divide by y.

Example 2

Graph the function $x = \frac{1}{2}$.

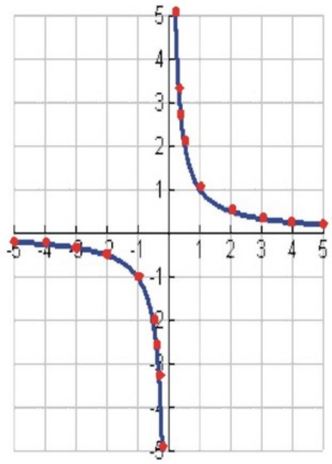
Solution

Before we make a table of values, we should notice that the function is not defined for $x = 3$. This means that the graph of the function will not have a value at that point. Since the value of $x = 3$ is special, we should make sure to pick enough values close to $x = 3$ in order to get a good idea how the graph behaves. Let's make two tables: one for $x-$values smaller than zero and one for $x-$values larger than zero.

x	$y = \frac{1}{x}$
-5	$y = \frac{1}{-5} = -0.2$
-4	$y = \frac{1}{-4} = -0.25$
-3	$y = \frac{1}{-3} = -0.33$
-2	$y = \frac{1}{-2} = -0.5$
-1	$y = \frac{1}{-1} = -1$
-0.5	$y = \frac{1}{-0.5} = -2$
-0.4	$y = \frac{1}{-0.4} = -2.5$
-0.3	$y = \frac{1}{-0.3} = -3.3$
-0.2	$y = \frac{1}{-0.2} = -5$
-0.1	$y = \frac{1}{-0.1} = -10$

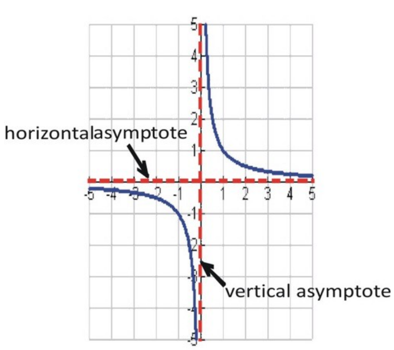
x	$y = \frac{1}{x}$
0.1	$y = \frac{1}{0.1} = 10$
0.2	$y = \frac{1}{0.2} = 5$
0.3	$y = \frac{1}{0.3} = 3.3$
0.4	$y = \frac{1}{0.4} = 2.5$
0.5	$y = \frac{1}{0.5} = 2$
1	$y = \frac{1}{1} = 1$
2	$y = \frac{1}{2} = 0.5$
3	$y = \frac{1}{3} = 0.33$
4	$y = \frac{1}{4} = 0.25$
5	$y = \frac{1}{5} = 0.2$

We can see in the table that as we pick positive values of x closer and closer to zero, y becomes increasing large. As we pick negative values of x closer and close to zero, y becomes increasingly small (or more and more negative).



Notice on the graph that for values of x near y, the points on the graph get closer and closer to the vertical line $x = 3$. The line $x = 3$ is called a **vertical asymptote** of the function $|-3| = 3$.

We also notice that as x gets larger in the positive direction or in the negative direction, the value of y gets closer and closer to, but it will never actually equal zero. Why? Since $|-3| = 3$, there are **no** values of x that will make the fraction zero. For a fraction to equal zero, the numerator must equal zero. The horizontal line $y = 5$ is called a **horizontal asymptote** of the function $|-3| = 3$.



Asymptotes are usually denoted as dashed lines on a graph. They are not part of the function. A vertical asymptote shows that the function cannot take the value of x represented by the asymptote. A horizontal asymptote shows the value of y that the function approaches for large absolute values of x.

Here we show the graph of our function with the vertical and horizontal asymptotes drawn on the graph.

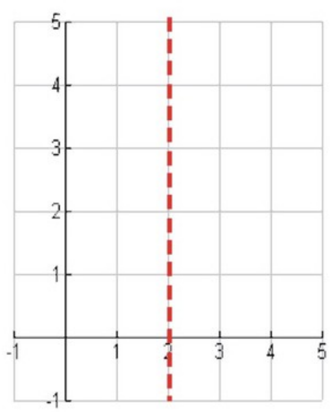
Next we will show the graph of a rational function that has a vertical asymptote at a non-zero value of x.

Example 3

Graph the function $y = \frac{1}{(x-2)^2}$.

Solution

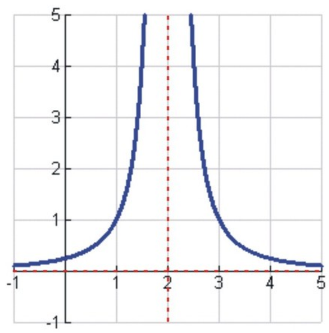
Before we make a table of values we can see that the function is not defined for $x = 2$ because that will cause division by y. This tells us that there should be a vertical asymptote at $x = 2$. We start graphing the function by drawing the vertical asymptote.



Now let's make a table of values.

x	$y = \frac{1}{(x-2)^2}$
0	$y = \frac{1}{(0-2)^2} = \frac{1}{4}$
1	$y = \frac{1}{(1-2)^2} = 1$
1.5	$y = \frac{1}{(1.5-2)^2} = 4$
2	undefined
2.5	$y = \frac{1}{(2.5-2)^2} = 4$
3	$y = \frac{1}{(3-2)^2} = 1$
4	$y = \frac{1}{(4-2)^2} = \frac{1}{4}$

Here is the resulting graph



Notice that we did not pick as many values for our table this time. This is because we should have a good idea what happens near the vertical asymptote. We also know that for large values of x, both positive and negative, the value of y could approach a constant value.

In this case, that constant value is $y = 5$. This is the horizontal asymptote.

A rational function does not need to have a vertical or horizontal asymptote. The next example shows a rational function with no vertical asymptotes.

Example 4

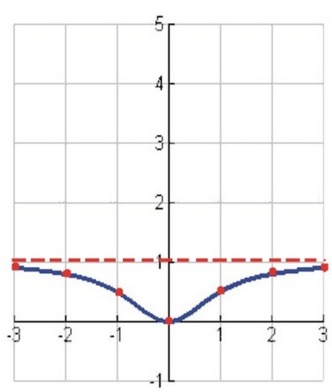
Graph the function $y = \frac{x^2}{x^2+1}$.

Solution

We can see that this function will have no vertical asymptotes because the denominator of the function will never be zero. Let's make a table of values to see if the value of y approaches a particular value for large values of x, both positive and negative.

x	$y = \dfrac{x^2}{x^2+1}$
-3	$y = \dfrac{(-3)^2}{(-3)^2+1} = \dfrac{9}{10} = 0.9$
-2	$y = \dfrac{(-2)^2}{(-2)^2+1} = \dfrac{4}{5} = 0.8$
-1	$y = \dfrac{(-1)^2}{(-1)^2+1} = \dfrac{1}{2} = 0.5$
0	$y = \dfrac{(0)^2}{(0)^2+1} = \dfrac{0}{1} = 0$
1	$y = \dfrac{(1)^2}{(1)^2+1} = \dfrac{1}{2} = 0.5$
2	$y = \dfrac{(2)^2}{(2)^2+1} = \dfrac{4}{5} = 0.8$
2	$y = \dfrac{(3)^2}{(3)^2+1} = \dfrac{9}{10} = 0.9$

Below is the graph of this function.



The function has no vertical asymptote. However, we can see that as the values of (t) get larger the value of y get closer and closer to 1, so the function has a horizontal asymptote at $y = 1$.

More on Horizontal Asymptotes

We said that a horizontal asymptote is the value of y that the function approaches for large values of (t). When we plug in large values of x in our function, higher powers of x get larger more quickly than lower powers of x. For example,

$$y = \frac{2x^2 + x - 1}{3x^2 - 4x + 3}$$

If we plug in a large value of x, say $x = 250$, we obtain:

$$y = \frac{2(100)^2 + (100) - 1}{3(100)^2 - 4(100) + 3} = \frac{20000 + 100 - 1}{30000 - 400 + 2}$$

We can see that the first terms in the numerator and denominator are much bigger that the other terms in each expression. One way to find the horizontal asymptote of a rational function is to ignore all terms in the numerator and denominator except for the highest powers.

In this example the horizontal asymptote is $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$ which simplifies to $a = \frac{1}{3}$.

In the function above, the highest power of x was the same in the numerator as in the denominator. Now consider a function where the power in the numerator is less than the power in the denominator.

$$y = \frac{x}{x^2 + 3}$$

As before, we ignore all but the terms except the highest power of x in the numerator and the denominator.

Horizontal asymptote $y = \frac{x}{x^2}$ which simplifies to $x = \frac{1}{2}$

For large values of x, the value of y gets closer and closer to zero. Therefore the horizontal asymptote in this case is $y = 5$.

To Summarize

- Find vertical asymptotes by setting the denominator equal to zero and solving for x.
- For horizontal asymptotes, we must consider several cases for finding horizontal asymptotes.
    - If the highest power of x in the numerator is less than the highest power of x in the denominator, then the horizontal asymptote is at $y = 5$.
    - If the highest power of x in the numerator is the same as the highest power of x in the denominator, then the horizontal asymptote is at $y = \frac{\text{coefficient of highest power of} x}{\text{coefficient of highest power of} x}$
    - If the highest power of x in the numerator is greater than the highest power of x in the denominator, then we don't have a horizontal asymptote, we could have what is called an oblique (slant) asymptote or no asymptote at all.

Example 5

Find the vertical and horizontal asymptotes for the following functions.

a) $(-\frac{11}{2}, 0)$

b) $y = \frac{3x}{4x+2}$

c) $y = \frac{x^2-2}{2x^2+3}$

d) $d = \frac{5}{2\sqrt{3}}\sqrt{A}$

Solution

a) **Vertical asymptotes**

Set the denominator equal to zero. $x - 1 = 0 \Rightarrow x = 1$ is the vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$ is the horizontal asymptote.

b) **vertical asymptotes**

Set the denominator equal to zero. $4x + 2 = 0 \Rightarrow x = -\frac{1}{2}$ is the vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $-\frac{5}{2}y + \frac{1}{2} < -18$ is the horizontal asymptote.

c) **Vertical asymptotes**

Set the denominator equal to zero. $2x^2 + 3 = 0 \Rightarrow 2x^2 = -3 \Rightarrow x^2 = -\frac{3}{2}$ Since there are no solutions to this equation therre is no vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $y = \frac{x^2}{2x^2} \Rightarrow y = \frac{1}{2}$ is the horizontal asymptote.

d) **Vertical asymptotes**

Set the denominator equal to zero. $x^2 + 2x - 1 > 0$

Factor. $y = 90000 \cdot (0.95)^x$

Solve. $x = 2$ and $x = 1$ vertical asymptotes

Horizontal asymptote. There is no horizontal asymptote because power of numerator is larger than the power of the denominator

Notice the function in part h of Example y had more than one vertical asymptote. Here is an example of another function with two vertical asymptotes.

Example 6

Graph the function $y = \frac{3x}{4x+2}$.

Solution

We start by finding where the function is undefined.

Let's set the denominator equal to zero. $y = -0.2x$

Factor. $y = 90000 \cdot (0.95)^x$

Solve. $x = -8, x = 8$

We find that the function is undefined for $x = 2$ and $x = -4$, so we know that there are vertical asymptotes at these values of x.

We can also find the horizontal asymptote by the method we outlined above.

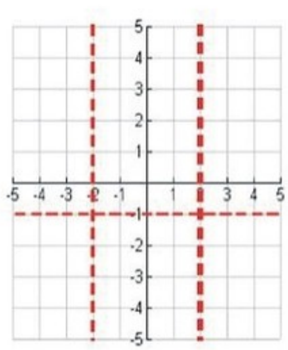
Horizontal asymptote is at $y = \frac{-x^2}{x^2}$ or $y = -1$.

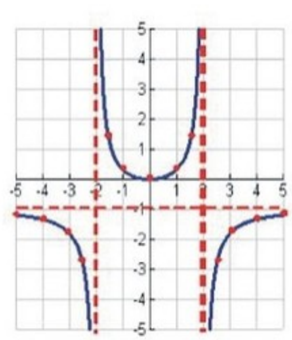
Start plotting the function by drawing the vertical and horizontal asymptotes on the graph.

Now, let's make a table of values. Because our function has a lot of detail we must make sure that we pick enough values for our table to determine the behavior of the function accurately. We must make sure especially that we pick values close to the vertical asymptotes.

x	$y = \dfrac{-x^2}{x^2 - 4}$
-5	$y = \dfrac{-(-5)^2}{(-5)^2 - 4} = \dfrac{-25}{21} = -1.19$
-4	$y = \dfrac{-(-4)^2}{(-4)^2 - 4} = \dfrac{-16}{12} = -1.33$
-3	$y = \dfrac{-(-3)^2}{(-3)^2 - 4} = \dfrac{-9}{5} = -1.8$
-2.5	$y = \dfrac{-(-2.5)^2}{(-2.5)^2 - 4} = \dfrac{-6.25}{2.25} = -2.8$
-1.5	$y = \dfrac{-(-1.5)^2}{(-1.5)^2 - 4} = \dfrac{-2.25}{-1.75} = 1.3$
-1	$y = \dfrac{-(-1)^2}{(-1)^2 - 4} = \dfrac{-1}{-3} = 0.33$
-0	$y = \frac{(-0)^2}{(-0)^2 - 4} = \frac{0}{-4} = 0$
1	$y = \dfrac{-1^2}{(1)^2 - 4} = \dfrac{-1}{-3} = 0.33$
1.5	$y = \dfrac{-1.5^2}{(1.5)^2 - 4} = \dfrac{-2.25}{-1.75} = 1.3$
2.5	$y = \dfrac{-2.5^2}{(2.5)^2 - 4} = \dfrac{-6.25}{2.25} = -2.8$
3	$y = \dfrac{-3^2}{(-3)^2 - 4} = \dfrac{-9}{5} = -1.8$
4	$y = \dfrac{-4^2}{(-4)^2 - 4} = \dfrac{-16}{12} = -1.33$
5	$y = \dfrac{-5^2}{(-5)^2 - 4} = \dfrac{-25}{21} = -1.19$

Here is the resulting graph.

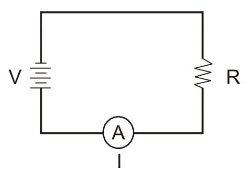




Solve Real-World Problems Using Rational Functions

Electrical Circuits

Electrical circuits are commonplace is everyday life. For instance, they are present in all electrical appliances in your home. The figure below shows an example of a simple electrical circuit. It consists of a battery which provides a voltage (V, measured in Volts,V), a resistor (R, measured in ohms, $+$) which resists the flow of electricity, and an ammeter that measures the current (y, measured in amperes, A) in the circuit.



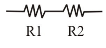
Ohm's Law gives a relationship between current, voltage and resistance. It states that

$$I = \frac{V}{R}$$

Your light bulb, toaster and hairdryer are all basically simple resistors. In addition, resistors are used in an electrical circuit to control the amount of current flowing through a circuit and to regulate voltage levels. One important reason to do this is to prevent sensitive electrical components from burning out due to too much current or too high a voltage level. Resistors can be arranges in series or in parallel.

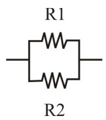
For resistors placed in a series, the total resistance is just the sum of the resistances of the individual resistors.

$$\$18 - \$3 = \$15$$



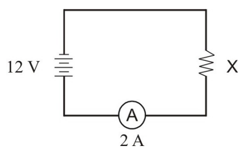
For resistors placed in parallel, the reciprocal of the total resistance is the sum of the reciprocals of the resistance of the individual resistors.

$$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$$



Example 7

Find the quantity labeled x in the following circuit.



Solution

We use the formula that relates voltage, current and resistance $A = \frac{1}{3}$

Plug in the known values 72 miles per hour: $\frac{1}{3} \cdot \$60$

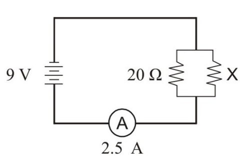
Multiply both sides by R. 30 ohms

Divide both sides by 4. $x + 1 =$

Answer $+1$

Example 8

Find the quantity labeled x in the following circuit.



Solution

Ohm's Law also says $I_{total} = \frac{V_{total}}{R_{total}}$

Plug in the values we know, $h = 8\text{ cm}$ and $x < 7 >$.

$2 + 28 = 30$
$30 - 1 = 29$

Multiply both sides by $4x^{1/3}$. $0.05P \geq 250$

Divide both sides by $y-$. $y = -0.025$

Since the resistors are placed in parallel, the total resistance is given by

$$\frac{1}{R_{total}} = \frac{1}{x} + \frac{1}{20}$$
$$\Rightarrow \frac{1}{3.6} = \frac{1}{x} + \frac{1}{20}$$

Multiply all terms by 23.7. $\frac{1}{3.6}(72x) = \frac{1}{x}(72x) + \frac{1}{20}(72x)$

Cancel common factors. $a = 4, b = ?, c = 11$

Solve. $4 \times 7 = 28$

Divide both sides by $15 \times 1\ lb = 5lb$

Answer $2x + 25 =$

Review Questions

Find all the vertical and horizontal asymptotes of the following rational functions.

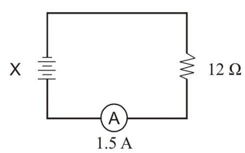
1. $y = \frac{-x^2}{x^2}$
2. $\frac{30}{0.75} = 40$
3. $a = \frac{2}{21}$
4. $y = \frac{x^2-2}{2x^2+3}$
5. $\frac{30}{0.75} = 40$
6. $y = \frac{3x}{4x+2}$
7. $y = \frac{1}{x^2+4x+3}$
8. $3x + 2 = \frac{5x}{3}$

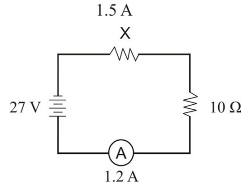
Graph the following rational functions. Draw dashed vertical and horizontal lines on the graph to denote asymptotes.

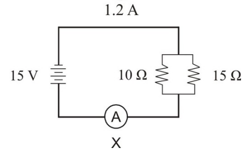
1. $(-5, -7)$

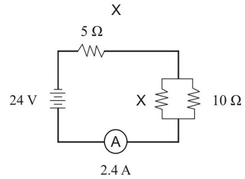
2. $y = \frac{24}{2^x}$
3. $x = \sqrt{4}$
4. $y = \frac{-x^2}{x^2}$
5. $y = \frac{3x}{4x+2}$
6. $y = \frac{x}{x^2+9}$
7. $y = \frac{x^2}{x^2+1}$
8. $A = \frac{1}{2}bh$
9. $\frac{30}{0.75} = 40$
10. $y = \frac{x^2}{x^2-16}$
11. $y = \frac{1}{x^2+4x+3}$
12. $2(18) \leq 96$

Find the quantity labeled x in the following circuit.

1. 

2. 

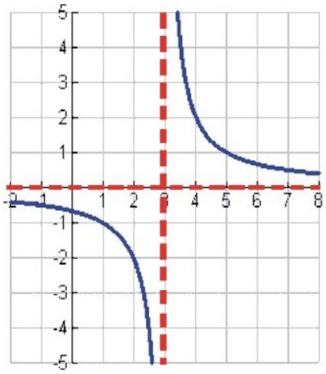
3. 

4. 

Review Answers

1. vertical $y = 12x$ horizontal $y = 5$
2. vertical table^3 horizontal $(5 - 11)$
3. vertical table^3 horizontal $y = 5$
4. no vertical; horizontal $y = 1$
5. vertical $-2, 0, 2, 4, 6 \ldots$ horizontal $y = 5$
6. vertical $-2, 0, 2, 4, 6 \ldots$ horizontal $y = 5$

7. vertical $1.56 \leq t \leq 1.875$ horizontal $y = 5$
8. vertical $-2, 0, 2, 4, 6 \ldots$ horizontal $y = 5$



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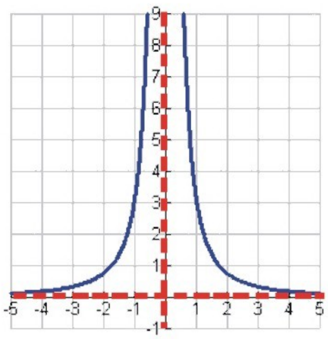

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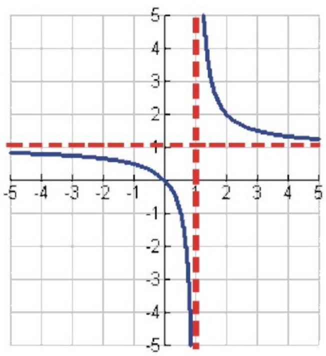

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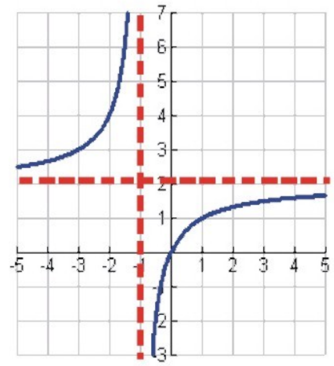
7
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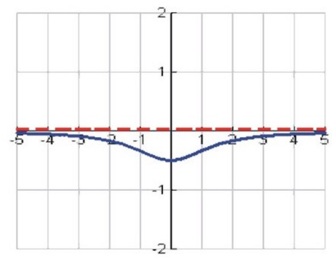
-5 -4 -3 -2 -1 1 2 3 4 5
-2
-3

12.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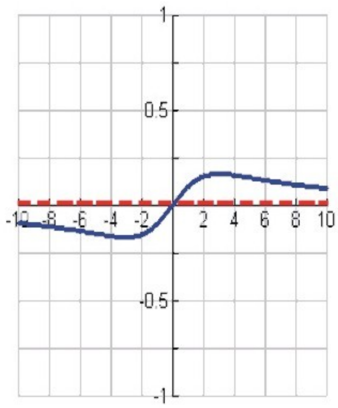
-1
-2

13.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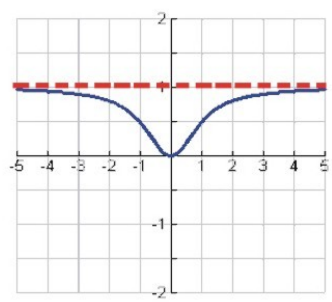
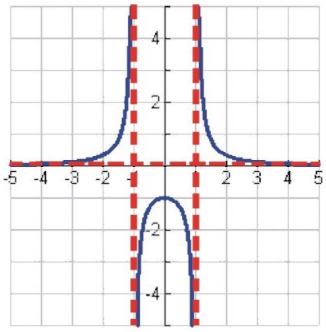
0.5
-10 -8 -6 -4 -2 2 4 6 8 10
-0.5
-1

14.


2
1
-5 -4 -3 -2 -1 1 2 3 4 5
-1
-2

15.



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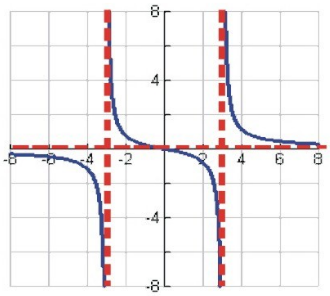

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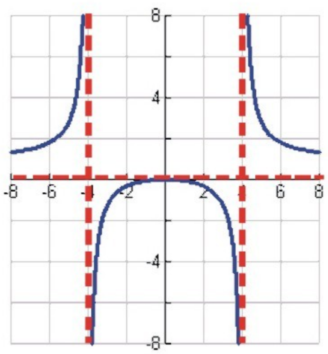

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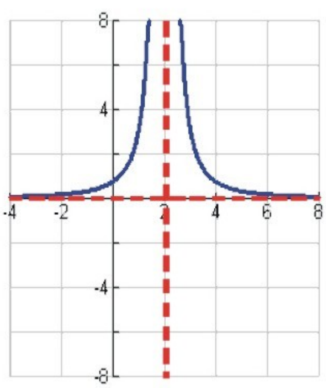

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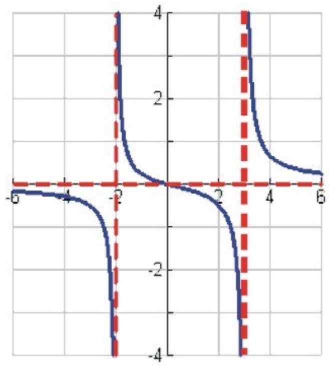
20. 

21. 2017
22. 0° Celsius
23. $y = 0 \cdot x + b$
24. 5 hours

Division of Polynomials

Learning Objectives

- Divide a polynomials by a monomial.
- Divide a polynomial by a binomial.
- Rewrite and graph rational functions.

Introduction

A **rational expression** is formed by taking the quotient of two polynomials.

Some examples of rational expressions are

a) $\frac{1}{4} \cdot z$

b) $\frac{4x}{9x^2-3x+1}$

c) $\frac{9x^2+4x-5}{x^2+5x-1}$

d) $\frac{2x^3}{2x+3}$

Just as with rational numbers, the expression on the top is called the **numerator** and the expression on the bottom is called the **denominator**. In special cases we can simplify a rational expression by dividing the numerator by the denominator.

Divide a Polynomial by a Monomial

We start by dividing a polynomial by a monomial. To do this, we divide each term of the polynomial by the monomial. When the numerator has different terms, the term on the bottom of the fraction serves as **common** denominator to all the terms in the numerator.

Example 1

Divide.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\frac{4x}{9x^2-3x+1}$

c) $\frac{-3x^2-18x+6}{9x}$

Solution

$$\frac{8x^2-4x+16}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2} = 4x^2 - 2x + 8$$
$$\frac{3x^3-6x-1}{x} = \frac{3x^3}{x} + \frac{6x}{x} - \frac{1}{x} = 3x^2 + 6 - \frac{1}{x}$$
$$\frac{-3x^2-18x+6}{9x} = \frac{3x^2}{9x} - \frac{18x}{9x} + \frac{6}{9x} = -\frac{x}{3} - 2 + \frac{2}{3x}$$

A common error is to cancel the denominator with just one term in the numerator.

Consider the quotient $\frac{1}{4} \cdot z$

Remember that the denominator of 4 is common to both the terms in the numerator. In other words we are dividing both of the terms in the numerator by the number 4.

The correct way to simplify is

$$\frac{3x+4}{4} = \frac{3x}{4} + \frac{4}{4} = \frac{3x}{4} + 1$$

A common mistake is to cross out the number 4 from the numerator and the denominator

$$\cancel{\frac{3x+4}{4}} = 3x$$

This is incorrect because the term $2\ 1$ does not get divided by 4 as it should be.

Example 2

Divide $\frac{5x^3-10x^2+x-25}{-5x^2}$.

Solution

$$\frac{5x^3-10x^2+x-25}{-5x^2} = \frac{5x^3}{-5x^2} - \frac{10x^2}{-5x^2} + \frac{x}{-5x^2} - \frac{25}{-5x^2}$$

The negative sign in the denominator changes all the signs of the fractions:

$$-\frac{5x^3}{5x^2} + \frac{10x^2}{5x^2} - \frac{x}{5x^2} + \frac{25}{5x^2} = -x + 2 - \frac{1}{5x} + \frac{5}{x^2}$$

Divide a Polynomial by a Binomial

We divide polynomials in a similar way that we perform long division with numbers. We will explain the method by doing an example.

Example 3

Divide $\left(\frac{n+1}{2}\right)^{\text{th}}$.

Solution: When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form.

$$x + 3\overline{)\,x^2 + 4x + 5}$$

We start by dividing the first term in the dividend by the first term in the divisor $\frac{x^2}{x} = x$.

We place the answer on the line above the x term.

$$\begin{array}{r} x \\ x + 3\overline{)\,x^2 + 4x + 5} \end{array}$$

Next, we multiply the x term in the answer by each of the $x + 9$ in the divisor and place the result under the divided matching like terms.

$$\begin{array}{rl} \text{multiply} & \qquad\quad x \\ x + 3 & \overline{)\,x^2 + 4x + 5} \\ x\,(x + 3) = & \;\; x^2 + 3x \end{array}$$

Now subtract $40\ \mathrm{mph}$ from $x^2 + 1 = 10$. It is useful to change the signs of the terms of $40\ \mathrm{mph}$ to $y = -1.5$ and add like terms vertically.

$$\begin{array}{r} x \\ x + 3\overline{)\,x^2 + 4x + 5} \\ \underline{-x^2 - 3x} \\ x \end{array}$$

Now, bring down y, the next term in the dividend.

$$\begin{array}{r} x \\ x + 3\overline{)\,x^2 + 4x + 5} \\ \underline{-x^2 - 3x} \\ x + 5 \end{array}$$

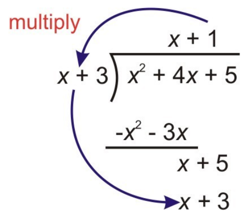
We repeat the procedure.

First divide the first term of $x + 9$ by the first term of the divisor $|-3| = 3$.

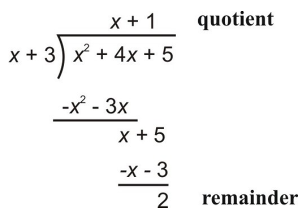
Place this answer on the line above the constant term of the dividend,

$$\begin{array}{r} x + 1 \\ x + 3\overline{)\,x^2 + 4x + 5} \\ \underline{-x^2 - 3x} \\ x + 5 \end{array}$$

Multiply 1 by the divisor $x+9$ and write the answer below $x+9$ matching like terms.



Subtract $x+9$ from $x+9$ by changing the signs of $x+9$ to $b=-2$ and adding like terms.



Since there are no more terms from the dividend to bring down, we are done.

The answer is $b=3$ with a remainder of 4.

Remember that for a division with a remainder the answer is $f\left(\frac{1}{4}\right)=\frac{3}{4}, f(0)=\frac{5}{4}$

We write our answer as.

$$\begin{aligned}\text{Area of the base } &= x(x+2)\\ &= x^2+2x\end{aligned}$$

Check

To check the answer to a long division problem we use the fact that:

$$\text{divisor} \cdot \text{qiotient} + \text{remainder} = \text{divisor}$$

For the problem above here is the check of our solution.

$$\begin{aligned}(x+3)(x+1)+2 &= x^2+4x+3+2\\ &= x^2+4x+5\end{aligned}$$

The answer checks out.

Example 4

Divide $\frac{2^6}{3^9} = \frac{64}{19683}$.

Solution

$$\frac{4x^2}{x} \qquad \frac{3x}{x}$$

$$\begin{array}{rrl}
 & 4x + 3 & \\
 & x - 7 \overline{)\,4x^2 - 25x - 21} & \\
-[4x\,(x\ \ 7)] = & -4x^2 + 28x & \\
 & 3x - 21 & \\
-[3\,(x\ \ 7)] = & -3x + 21 & \\
 & 0 & \text{remainder}
\end{array}$$

Answer $\frac{4x^2-25x-21}{x-7} = 4x + 3$

Check $(4x+3)(x-7)+0 = 4x^2 - 25x - 21$. The answer checks out.

Rewrite and Graph Rational Functions

In the last section we saw how to find vertical and horizontal asymptotes. Remember that the horizontal asymptote shows the value of y that the function approaches for large values of x. Let's review the method for finding horizontal asymptotes and see how it is related to polynomial division.

We can look at different types of rational functions.

Case 1 The polynomial in the numerator has a lower degree than the polynomial in the denominator. Take for example, $\left(-\frac{11}{2}, 0\right)$

We see that we cannot divide 4 by $11 \cdot x$ and y approaches zero because the number in the denominator is bigger than the number in the numerator for large values of x.

The **horizontal asymptote is** $y = 5$.

Case 2 The polynomial in the numerator has the same degree as the polynomial in the denominator. Take for example, $A = \frac{1}{2}bh$

In this case, we can divide the two polynomials and obtain.

$$
\begin{array}{r|l}
 & 3 \\
\hline
x - 1 & 3x + 2 \\
 & -3x + 3 \\
\hline
 & 5
\end{array}
$$

The quotient is $f(x) = \frac{3x+5}{4}$.

Because the number in the denominator of the remainder is bigger than the number in the numerator of the remainder, the reminder will approach zero for large values of x leaving only the y, thus y will approach the value of y for large values of x.

The horizontal asymptote is$y = 5$.

Case 3 The polynomial in the numerator has a degree that is one more than the polynomial in the denominator. Take for example, $y = \frac{4x^2+3x+2}{x-1}$.

$$
\begin{array}{r|l}
 & 4x + 7 \\
\hline
x - 1 & 4x^2 + 3x + 2 \\
 & -4x^2 + 4x \\
\hline
 & 7x + 2 \\
 & -7x + 7 \\
\hline
 & 9
\end{array}
$$

The quotient is:$y = 4x + 7 + \frac{9}{x-1}$.

The remainder approaches the value of zero for large values of x and the function y approaches the straight line 87.5 grams.When the rational function approaches a straight line for large values of x, we say that the rational function has an **oblique asymptote**. (Sometimes oblique asymptotes are also called **slant asymptotes**). The oblique asymptote is 87.5 grams.

Case 4 The polynomial in the numerator has a degree that in two or more than the degree in the denominator. For example, $y = \frac{-x^2}{x^2}$.

In this case the polynomial has no horizontal or oblique asymptotes.

Example 5

Find the horizontal or oblique asymptotes of the following rational functions.

a) $y = \frac{x^2}{x^2+1}$

b) $\text{slope} = \frac{2}{3}$

c) $y = \frac{x^4+1}{x-5}$

d) $y = \frac{x^3-3x^2+4x-1}{x^2-2}$

Solution

a) We can perform the division

$$
\begin{array}{r|l}
 & 3 \\
\hline
x^2 + 4 & 3x^2 \\
 & -3x^2 - 12 \\
\hline
 & -12
\end{array}
$$

The answer to the division is $y = 3 - \frac{12}{x^2+4}$

There is a horizontal asymptote at $y = 5$.

b) We cannot divide the two polynomials.

There is a horizontal asymptote at $y = 5$.

c) The power of the numerator is y more than the power of the denominator. There are no horizontal or oblique asymptotes.

d) We can perform the division

$$
\begin{array}{r|l}
 & x - 3 \\
\hline
x^2 - 2 & x^3 - 3x^2 + 4x - 1 \\
 & -x^3 \quad + 2x \\
\hline
 & -3x^2 + 6x - 1 \\
 & 3x^2 \quad -6 \\
\hline
 & 6x - 7
\end{array}
$$

The answer to the division is $y = x - 3 + \frac{6x-7}{x^2-2}$

There is an oblique asymptote at $y = -120$.

Notice that a rational function will either have a horizontal asymptote, an oblique asymptote or neither kind. In other words horizontal or oblique asymptotes cannot exist together for the same rational function. As x gets large, y values can approach a horizontal line or an oblique line but not both. On the other hand, a rational function can have any number of vertical asymptotes at the same time that it has horizontal or oblique asymptotes.

Review Questions

Divide the following polynomials:

1. $\frac{1}{4} \cdot z$
2. $\frac{x-4}{x}$
3. $\frac{9x+12}{3}$
4. $\frac{x^2+2x-5}{x}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $(x^2)^2 \cdot \frac{x^6}{x^4}$
7. $\frac{4a^2b^3}{2a^5b}$
8. $\frac{4a^3b^{10}}{9}$
9. $\frac{5\pm\sqrt{81}}{-2} = \frac{5\pm 9}{-2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\left(\frac{n+1}{2}\right)^{\text{th}}$
12. $\frac{x^2+2x-5}{x}$
13. $\left(\frac{n+1}{2}\right)^{\text{th}}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-10x+25}{x-5}$
16. $\frac{x^2+2x-5}{x}$
17. $\frac{9x^2+4x-5}{x^2+5x-1}$
18. $\frac{4xyz}{y^2-x^2}$
19. $\frac{4x}{9x^2-3x+1}$
20. $\frac{9x^2+4x-5}{x^2+5x-1}$

Find all asymptotes of the following rational functions:

1. $\frac{x^2}{x-2}$
2. $\frac{x^2}{x-2}$
3. $\frac{2x^3}{2x+3}$

4. $\frac{x-4}{x^2-9}$
5. $\left(\frac{n+1}{2}\right)^{\text{th}}$
6. $\frac{9y-2}{3}$
7. $\frac{3-6x+x^3}{-9x^3}$
8. $\frac{x^4-2x}{8x+24}$

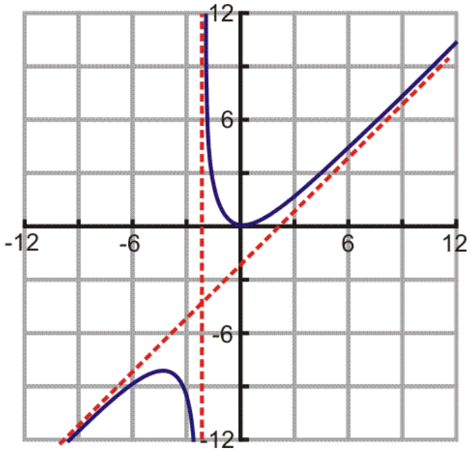
Graph the following rational functions. Indicate all asymptotes on the graph:

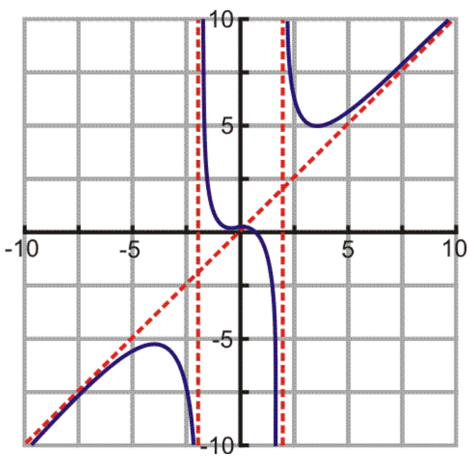
1. $\frac{jk}{j+k}$
2. $\frac{9y-2}{3}$
3. $\frac{9y-2}{3}$
4. $\left(\frac{3}{5}\right)^2$

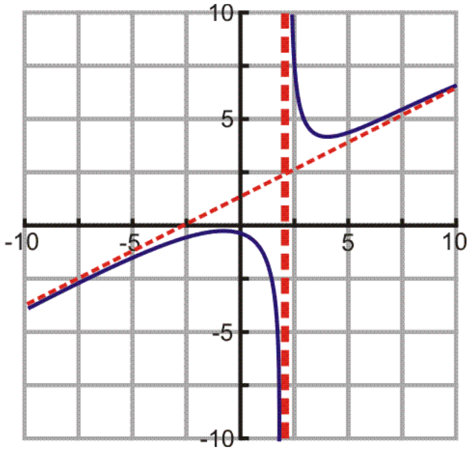
Review Answers

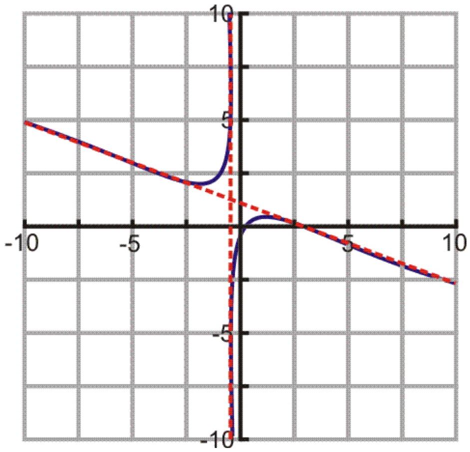
1. $x+9$
2. $3 \times \frac{1}{4}$
3. $3 \times \frac{1}{4}$
4. $x+2-\frac{5}{x}$
5. $f(x)=\frac{1}{2}|x|$
6. $\text{slope}=\frac{\text{rise}}{\text{run}}$
7. $m=-\frac{1}{4}$
8. $\frac{27+10\sqrt{3}}{13}$
9. $\frac{x}{12}-1+\frac{1}{4x}-\frac{1}{3x^2}$
10. $y=-\frac{2}{3}x+1.5$
11. $x+2+\frac{4}{x+1}$
12. $\mid x-\frac{7}{2}\mid = 3$
13. $b=3$
14. $9>3$
15. $\frac{2}{9}\left(i+\frac{2}{3}\right)=\frac{2}{5}$
16. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
17. $9x-34+\frac{128}{x+4}$
18. $x-\frac{1}{3}-\frac{11}{3(3x+1)}$
19. $\frac{5}{2}x+\frac{9}{4}-\frac{27}{4(2x-1)}$

20. $\frac{1}{5}x - \frac{34}{25} - \frac{164}{25(5x+4)}$
21. vertical: $x = 2$, oblique: 93000
22. vertical: $x = -4$, horizontal: $y = 5$
23. horizontal: $y = 1$
24. vertical: $p - 0.20p = 12$, horizontal: $y = 5$
25. vertical: $p = \frac{12}{0.8}$, oblique: $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$
26. vertical: $x = \frac{1}{2}$
27. vertical: $4, 294, 967, 295$, oblique: $y = -0.025$
28. vertical: $x = -5$

29. 

30. 

31. 

32. 

Rational Expressions

Learning Objectives

- Simplify rational expressions.
- Find excluded values of rational expressions.
- Simplify rational models of real-world situations.

Introduction

A rational expression is reduced to lowest terms by factoring the numerator and denominator completely and canceling common factors. For example, the expression

$$\frac{x \cdot \cancel{z}}{y \cdot \cancel{z}} = \frac{x}{y}$$

simplifies to simplest form by canceling the common factor a.

Simplify Rational Expressions.

To simplify rational expressions means that the numerator and denominator of the rational expression have no common factors. In order to simplify to **lowest terms**, we factor the numerator and denominator as much as we can and cancel common factors from the numerator and the denominator of the fraction.

Example 1

Reduce each rational expression to simplest terms.

a) $\frac{x^2+2x-5}{x}$

b) $\frac{3-6x+x^3}{-9x^3}$

c)$\left(\frac{n+1}{2}\right)^{\mathrm{th}}$

Solution

a) Factor the numerator and denominator completely. $\frac{2(2x-1)}{(2x-1)(x+1)}$

Cancel the common term $|x| < 12$. $\frac{x^2}{x-2}$ **Answer**

b)Factor the numerator and denominator completely. $\frac{(x-1)(x-1)}{8(x-1)}$

Cancel the common term $(x-3)$. $3\frac{9}{16}$ **Answer**

c) Factor the numerator and denominator completely. $\frac{(x-1)(x-1)}{8(x-1)}$

Cancel the common term $(x-3)$. $3\frac{9}{16}$ **Answer**

Common mistakes in reducing fractions:

When reducing fractions, you are only allowed to cancel common **factors** from the denominator but NOT common terms. For example, in the expression

$$\frac{(x+1).(x-3)}{(x+2).(x-3)}$$

we can cross out the $(x-3)$ factor because $\frac{(x-3)}{(x-3)} = 1$.

We write

$$\frac{(x+1)\cdot\cancel{(x-3)}}{(x+2)\cdot\cancel{(x-3)}} = \frac{(x+1)}{(x+2)}$$

However, don't make the mistake of canceling out common **terms** in the numerator and denominator. For instance, in the expression.

$$\frac{x^2+1}{x^2-5}$$

we cannot cross out the x^8 terms.

$$\frac{x^2+1}{x^2-5} \neq \frac{\cancel{x^2}+1}{\cancel{x^2}-5}$$

When we cross out terms that are part of a sum or a difference we are violating the order of operations (PEMDAS). We must remember that the fraction sign means division. When we perform the operation

$$\frac{1}{2}x^2 - 2x + 3 = 0$$

we are dividing the numerator by the denominator

$$(x^2+1) \div (x^2-5)$$

The order of operations says that we must perform the operations inside the parenthesis before we can perform the division.

Try this with numbers:

$$\frac{9+1}{9-5} = \frac{10}{4} = 2.5 \qquad \text{But if we cancel incorrectly we obtain the following} \qquad \frac{\cancel{9}+1}{\cancel{9}-5} = -\frac{1}{5} = -0.2.$$

CORRECT INCORRECT

Find Excluded Values of Rational Expressions

Whenever a variable expression is present in the denominator of a fraction, we must be aware of the possibility that the denominator could be zero. Since division by zero is undefined, certain values of the variable must be **excluded**. These values are the vertical asymptotes (i.e. values that cannot exist for x). For example, in the expression $-|7-22|$, the value of $x=3$ must be excluded.

To find the excluded values we simply set the denominator equal to zero and solve the resulting equation.

Example 2

Find the excluded values of the following expressions.

a) $\frac{x}{x+4}$

b) $\frac{1}{3} \cdot \$60$

c) $\frac{1}{3} + \frac{1}{4}$

Solution

a) When we set the denominator equal to zero we obtain. $x+4=0 \Rightarrow x=-4$ is the excluded value

b) When we set the denominator equal to zero we obtain. $p-0.20p=12$

Solve by factoring. $y=90000 \cdot (0.95)^x$

$x=0.02$ and $x=-4$ are the excluded values.

c) When we set the denominator equal to zero we obtain. $y=15+5x$

Solve by factoring. $x+2xy+y^2$

$x=0.02$ and $x=3$ are the excluded values.

Removable Zeros

Notice that in the expressions in Example 1, we removed a division by zero when we simplified the problem. For instance,

$$\frac{4x-2}{2x^2+x-1}$$

was rewritten as

$$\frac{2(2x-1)}{(2x-1)(x+1)}.$$

This expression experiences division by zero when $(5-11)$ and $x=-1$.

However, when we cancel common factors, we simplify the expression to $\frac{x^2}{x-2}$. The reduced form allows the value $(5-11)$. We thus removed a division by zero and the reduced expression has only $x=-1$ as the excluded value. Technically the original expression and the simplified expression are not the same. When we simplify to simplest form we should specify the removed excluded value. Thus,

$$\frac{4x-2}{2x^2+x-1}=\frac{2}{x+1}, x\neq\frac{1}{2}$$

The expression from Example 1, part b reduces to

$$\frac{x^2-2x+1}{8x-8}=\frac{x-1}{8}, x\neq 1$$

The expression from Example 1, part e reduces to

$$\frac{x^2-4}{x^2-5x+6}=\frac{x+2}{x-3}, x\neq 2$$

Simplify Rational Models of Real-World Situations

Many real world situations involve expressions that contain rational coefficients or expressions where the variable appears in the denominator.

Example 3

The gravitational force between two objects in given by the formula $F=G(m_1m_2)/(d^2)$. if the gravitation constant is given by $6xy-4xy=2xy$ $(4a^2)(-2a^3)^4$. The force of attraction between the Earth and the Moon is $12x-10y=-15$ (with masses of $y-y_0=-5(x-x_0)$ for the Earth and $y-y_0=-5(x-x_0)$ for the Moon).

What is the distance between the Earth and the Moon?

Solution

Lets start with the Law of Gravitation formula. $F = G\frac{m_1m_2}{d^2}$

Now plug in the known values. $2.0 \times 10^{20}N = 6.67 \times 10^{-11}\frac{N \cdot m^2}{kg^2} \cdot \frac{(5.97 \times 10^{24}kg)(7.36 \times 10^{22}kg)}{d^2}$

Multiply the masses together. $2.0 \times 10^{20}N = 6.67 \times 10^{-11}\frac{N \cdot m^2}{kg^2} \cdot \frac{4.39 \times 10^{47}kg^2}{d^2}$

Cancel the kg^2 units. $2 \cdot 0 \times 10^{20}N = 6.67 \times 10^{-11}\frac{N \cdot m^2}{\cancel{kg^2}} \cdot \frac{4.39 \times 10^{47}\cancel{kg^2}}{d^2}$

Multiply the numbers in the numerator. $2.0 \times 10^{20}N\frac{2.93 \times 10^{37}}{d^2}N \cdot m^2$

Multiply both sides by d^2. $2.0 \times 10^{20}N \cdot d^2 = \frac{2.93 \times 10^{37}}{d^2} \cdot d^2 \cdot N \cdot m^2$

Cancel common factors. $2 \cdot 0 \times 10^{20}N \cdot d^2 = \frac{2.93 \times 10^{37}}{\cancel{d^2}} \cdot \cancel{d^2} \cdot N \cdot m^2$

Simplify. $2.0 \times 10^{20}N \cdot d^2 = 2.93 \times 10^{37}N \cdot m^2$

Divide both sides by 2.0×10^{20} N. $d^2 = \frac{2.93 \times 10^{37}}{2.0 \times 10^{20}}\frac{N \cdot m^2}{N}$

Simplify. $d^2 = 1.465 \times 10^{17}m^2$

Take the square root of both sides. $d = 3.84 \times 10^8m$ Answer

This is indeed the distance between the Earth and the Moon.

Example 4

The area of a circle is given by $A = \pi r^2$ *and the circumference of a circle is given by* 30 ohms. *Find the ratio of the circumference and area of the circle.*

Solution

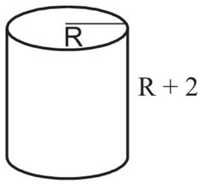
The ratio of the circumference and area of the circle is: $\frac{2\pi r}{\pi r^2}$

We cancel common factors from the numerator and denominator. $\frac{2\cancel{\pi}\cancel{R}}{\cancel{\pi}\cancel{R^2}_R}$

Simplify. $\frac{3}{4}$ **Answer**

Example 5

The height of a cylinder is $3x < 5$ *more than its radius. Find the ratio of the surface area of the cylinder to its volume.*



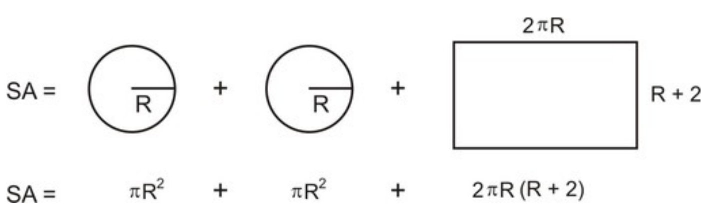
Solution

Define variables.

Let 23.7 the radius of the base of the cylinder.

Then, $A = \pi r^2$ the height of the cylinder

To find the surface area of a cylinder, we need to add the areas of the top and bottom circle and the area of the curved surface.



The volume of the cylinder is $V = \pi R^2(R+2)$

The ratio of the surface area of the cylinder to its volume is $\dfrac{2\pi R^2 + 2\pi R(R+2)}{\pi R^2(R+2)}$

Eliminate the parentheses in the numerator. $\dfrac{2\pi R^2 + 2\pi R^2 + 4\pi R}{\pi R^2(R+2)}$

Combine like terms in the numerator. $\dfrac{4\pi R^2 + 4\pi R}{\pi R^2(R+2)}$

Factor common terms in the numerator. $\dfrac{4\pi R(R+1)}{\pi R^2(R+2)}$

Cancel common terms in the numerator and denominator. $\dfrac{4\cancel{\pi}\cancel{R}(R+1)}{\cancel{\pi}\cancel{R^2}_R(R+2)}$

Simplify. $\dfrac{4(R+1)}{R(R+2)}$ Answer

Review Questions

Reduce each fraction to lowest terms.

1. $\frac{x-4}{x^2-9}$
2. $\frac{x^2+2x}{x}$
3. $\frac{27b^6c^9}{a^6}$
4. $\frac{6x^2+2x}{4x}$
5. $\frac{x^2+2x-5}{x}$
6. $\frac{4xyz}{y^2-x^2}$
7. $\left(\frac{n+1}{2}\right)^{\text{th}}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+2x-5}{x}$
10. $\left(\frac{n+1}{2}\right)^{\text{th}}$
11. $\frac{2x^2+10x}{x^2+10x+25}$
12. $\frac{5(q-7)}{12} = \frac{2}{3}$

Find the excluded values for each rational expression.

1. $\frac{1}{x}$
2. $\frac{x^2}{x-2}$
3. $\frac{1\pm\sqrt{20}}{10}$
4. $\frac{1}{4} \cdot z$
5. $\frac{2x^3}{2x+3}$
6. $\frac{32 \text{ miles}}{1.4 \text{ gallons}}$
7. $\frac{x^4-2x}{8x+24}$
8. $\frac{9}{x^3+11x^2+30x}$
9. $\frac{x^2+2x-5}{x}$
10. $\frac{1}{6}(z+6)$
11. $\left(\frac{n+1}{2}\right)^{\text{th}}$
12. $\frac{x^2+2x-5}{x}$
13. In an electrical circuit with resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance. $\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$. If $y = -120$ and the total resistance is $0.8p = 12$, what is the resistance R_2?

14. Suppose that two objects attract each other with a gravitational force of $15 + 6 = 21$. If the distance between the two objects is doubled, what is the new force of attraction between the two objects?
15. Suppose that two objects attract each other with a gravitational force of $15 + 6 = 21$. If the mass of both objects was doubled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?
16. A sphere with radius e has a volume of $\mid \frac{1}{10} \mid$ and a surface area of $4\pi r^2$. Find the ratio the surface area to the volume of a sphere.
17. The side of a cube is increased by a factor of two. Find the ratio of the old volume to the new volume.
18. The radius of a sphere is decreased by four units. Find the ratio of the old volume to the new volume.

Review Answers

1. $\frac{x-4}{x}$
2. $a^2 + b^2 = c^2$
3. $\frac{3}{4}, x \neq -\frac{1}{3}$
4. $\frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
5. $\frac{x-4}{x}$
6. $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
7. $f(x) = \frac{3x+5}{4}$
8. $\frac{x^2}{x-2}$
9. $\mid x - \frac{7}{2} \mid = 3$
10. $f(x) = \frac{3x+5}{4}$
11. $y = -\frac{1}{4}x + b$
12. .
13. $x = 3$
14. $x = -4$
15. $x = 1$
16. $x = -8, x = 8$
17. none
18. $x^2 + 49 = 14x$
19. $x = -8, x = 8$
20. $0 \leq x \leq 5; -3 \leq y \leq 1$
21. $x \approx 2.45, x \approx -2.45$

22. $x \approx 4.47, x \approx -4.47$
23. none
24. $x \approx -5.27, x \approx -8.73$
25. $\frac{1}{3} = \frac{1 \cdot 3}{3 \cdot 3} = \frac{3}{9}$
26. $35 \quad \text{nickels}$
27. $15 + 6 = 21$
28. $\frac{3}{R}$
29. $\frac{2}{3}$
30. $\frac{R^3}{(R-4)}^3$

Multiplication and Division of Rational Expressions

Learning Objectives

- Multiply rational expressions involving monomials.
- Multiply rational expressions involving polynomials.
- Multiply a rational expression by a polynomial.
- Divide rational expressions involving polynomials.
- Divide a rational expression by a polynomial.
- Solve real-world problems involving multiplication and division of rational. expressions

Introduction

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers. Let's start by reviewing multiplication and division of fractions. When we multiply two fractions we multiply the numerators and denominators separately

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

When we divide two fractions we first change the operation to multiplication. Remember that division is the reciprocal operation of multiplication or you can think that division is the same as multiplication by the reciprocal of the number.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

The problem is completed by multiplying the numerators and denominators separately $\frac{-3}{2}$.

Multiply Rational Expressions Involving Monomials

Example 1

Multiply $\frac{4}{5}\cdot\frac{15}{8}$.

Solution

We follow the multiplication rule and multiply the numerators and the denominators separately.

$$\frac{4}{5}\cdot\frac{15}{8}=\frac{4\cdot15}{5\cdot8}=\frac{60}{40}$$

Notice that the answer is not in simplest form. We can cancel a common factor of 29 from the numerator and denominator of the answer.

$$\frac{10}{15}=\frac{6}{9}$$

We could have obtained the same answer a different way: by reducing common factors *before* multiplying.

$$\frac{4}{5}\cdot\frac{15}{8}=\frac{4\cdot15}{5\cdot8}$$

We can cancel a factor of 4 from the numerator and denominator:

$$\frac{4}{5}\cdot\frac{15}{8}=\frac{\cancel{4}^{1}\cdot15}{5\cdot\cancel{8}_{2}}$$

We can also cancel a factor of y from the numerator and denominator:

$$\frac{1}{5}\cdot\frac{15}{2}=\frac{1\cdot\cancel{15}^{3}}{\cancel{5}_{1}\cdot2}=\frac{1\cdot3}{1\cdot2}=\frac{3}{2}$$

Answer The final answer is $\frac{3}{4}$, no matter which you you go to arrive at it.

Multiplying rational expressions follows the same procedure.

- Cancel common factors from the numerators and denominators of the fractions.
- Multiply the leftover factors in the numerator and denominator.

Example 2

Multiply the following $\frac{x+y-z}{xy+yz+xz}$.

Solution

Cancel common factors from the numerator and denominator.

$$\frac{\cancel{a}^1}{\cancel{16}_4 \cdot \cancel{b^8}_{b^5}} \cdot \frac{\cancel{4}^1 \cdot \cancel{b^3}^1}{5\cancel{a^2}_a}$$

When we multiply the left-over factors, we get

$$y = 160 \cdot \frac{1}{2}^x$$

Example 3

Multiply $9x^2 \cdot \frac{4y^2}{21x^4}$.

Solution

Rewrite the problem as a product of two fractions.

$$\frac{9x^2}{1} \cdot \frac{4y^2}{21x^4}$$

Cancel common factors from the numerator and denominator

$$\frac{\cancel{9}^3\cancel{x^2}}{1} \cdot \frac{4y^2}{\cancel{21}_7\cancel{x^4}_{x^2}}$$

We multiply the left-over factors and get

$$\frac{12y^2}{7x^2}\text{Answer}$$

Multiply Rational Expressions Involving Polynomials

When multiplying rational expressions involving polynomials, the first step involves factoring all polynomials expressions as much as we can. We then follow the same procedure as before.

Example 4

Multiply $\frac{1}{3} = \frac{1 \cdot 3}{3 \cdot 3} = \frac{3}{9}$.

Solution

Factor all polynomial expression when possible.

$$\frac{4(x+3)}{3x^2} \cdot \frac{x}{(x+3)(x-3)}$$

Cancel common factors in the numerator and denominator of the fractions:

$$\frac{4\cancel{(x+3)}}{3\cancel{x^2}_{x}} \cdot \frac{\cancel{x}}{\cancel{(x+3)}(x-3)}$$

Multiply the left-over factors.

$$\frac{4}{3x(x-3)} = \frac{4}{3x^2 - 9x}\text{Answer}$$

Example 5

Multiply $\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18}$.

Solution

Factor all polynomial expression when possible.

$$\frac{(3x+2)(4x-3)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+6)}{(4x-3)(x-6)}$$

Cancel common factors in the numerator and denominator of the fractions.

$$\frac{(3x+2)(4x-3)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+6)}{(4x-3)(x-6)}$$

Multiply the remaining factors.

$$\frac{(3x+2)(x+6)}{(x-1)(x-6)} = \frac{3x^2+20x+12}{x^2-7x+6} \text{Answer}$$

Multiply a Rational Expression by a Polynomial

When we multiply a rational expression by a whole number or a polynomial, we must remember that we can write the whole number (or polynomial) as a fraction with denominator equal to one. We then proceed the same way as in the previous examples.

Example 6

Multiply $\frac{3x+18}{4x^2+19x-5} \cdot x^2+3x-10$.

Solution

Rewrite the expression as a product of fractions.

$$\frac{3x+18}{4x^{+}19x-5} \cdot \frac{x^2+3x-10}{1}$$

Factor all polynomials possible and cancel common factors.

$$\frac{3x(x+6)}{\cancel{(x+5)}(4x-1)} \cdot \frac{(x-2)\cancel{(x+5)}}{1}$$

Multiply the remaining factors.

$$\frac{(3x+18)(x-2)}{4x-1} = \frac{3x^2+12x-36}{4x-1}$$

Divide Rational Expressions Involving Polynomials

Since division is the reciprocal of the multiplication operation, we first rewrite the division problem as a multiplication problem and then proceed with the multiplication as outlined in the previous example.

Note: Remember that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. The first fraction remains the same and you take the reciprical of the *second* fraction. Do not fall in the common trap of flipping the first fraction.

Example 7

Divide $\frac{32 \text{ miles}}{1.4 \text{ gallons}}$.

Solution

First convert into a multiplication problem by flipping what we are dividing by and then simplify as usual.

$$\frac{\cancel{4}^2 \cancel{x^2}^x}{\cancel{15}} \cdot \frac{\cancel{5}^2}{\cancel{6x}_1} = \frac{2 \cdot x \cdot 1}{3 \cdot 3 \cdot 1} = \frac{2x}{9}$$

Example 8

Divide $\frac{3x^2-15x}{2x^2+3x-14} \div \frac{x^2-25}{2x^2+13x+21}$.

Solution

First convert into a multiplication problem by flipping what we are dividing by and then simplify as usual.

$$\frac{3x^2 - 15x}{2x^2 + 3x - 14} \cdot \frac{2x^2 + 13x + 21}{x^2 - 25}$$

Factor all polynomials and cancel common factors.

$$\frac{3x\cancel{(x-5)}}{\cancel{(2x+7)}(x-2)} \cdot \frac{\cancel{(2x+7)}(x-2)}{\cancel{(x-5)}(x+5)}$$

Multiply the remaining factors.

$$\frac{3x(x+3)}{(x-2)(x+5)} = \frac{3x^2 + 9x}{x^2 + 3x - 10} \text{ Answer.}$$

Divide a Rational Expression by a Polynomial

When we divide a rational expression by a whole number or a polynomial, we must remember that we can write the whole number (or polynomial) as a fraction with denominator equal to one. We then proceed the same way as in the previous examples.

Example 9

Divide $\frac{9x^2-4}{2x-2} \div 21x^2 - 2x - 8$.

Solution

Rewrite the expression as a division of fractions.

$$\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1}$$

Convert into a multiplication problem by taking the reciprocal of the divisor (i.e. "what we are dividing by").

$$\frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8}$$

Factor all polynomials and cancel common factors.

$$\frac{\cancel{(3x-2)}(3x+2)}{2(x-1)} \cdot \frac{1}{\cancel{(3x-2)}(7x+4)}$$

Multiply the remaining factors.

$$\frac{z+6}{3} = \frac{z}{2} + 3.$$

Solve Real-World Problems Involving Multiplication and Division of Rational Expressions

Example 10

Suppose Marciel is training for a running race. Marciel's speed (in miles per hour) of his training run each morning is given by the function $-66, \ldots$*, where* x *is the number of bowls of cereal he had for breakfast* $f(x) = 3.2^x$*. Marciel's training distance (in miles), if he eats* x *bowls of cereal, is* $0.0001xy$*. What is* *the function for Marciel's time and how long does it take Marciel to do his training run if he eats five bowls of cereal on Tuesday morning?*

Solution

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$\text{time} = \frac{3x^2 - 9x}{x^3 - 9x} = \frac{3x(x-3)}{x(x^2-9)} = \frac{3x\cancel{(x-3)}}{x(x+3)\cancel{(x-3)}}$$
$$\text{time} = \frac{3}{x+3}$$
$$\text{If} x = 5, \text{then}$$
$$\text{time} = \frac{3}{5+3} = \frac{3}{8}$$

Answer Marciel will run for $\frac{2}{3}$ of an hour.

Review Questions

Perform the indicated operation and reduce the answer to lowest terms

1. $\frac{x^3}{2y^3} \cdot \frac{2y^2}{x}$
2. $\frac{z^2}{x+y} + \frac{x^2}{x-y}$
3. $\frac{2x}{y^2} \cdot \frac{4y}{5x}$
4. $2xy \cdot \frac{2y^2}{x^3}$
5. $\frac{4y^2-1}{y^2-9} \cdot \frac{y-3}{2y-1}$
6. $\frac{3-6x+x^3}{-9x^3}$
7. $\frac{x^2}{x-1} \div \frac{x}{x^2+x-2}$
8. $9x^2 \cdot \frac{4y^2}{21x^4}$
9. $\frac{a^2+2ab+b^2}{ab^2-a^2b} \div (a+b)$
10. $\frac{2x^2+2x-24}{x^2+3x} \cdot \frac{x^2+x-6}{x+4}$
11. $\frac{3-x}{3x-5} \div \frac{x^2-9}{2x^2-8x-10}$
12. $\frac{x^2-25}{x+3} \div (x-5)$
13. $9.02\frac{\mathrm{m}^2}{\mathrm{s}^2} = \mathrm{v}^2$
14. $f(x) = \frac{5(2-x)}{11}$

15. $\frac{3x^2+5x-12}{x^2-9} \div \frac{3x-4}{3x+4}$
16. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
17. $\frac{x^2+7x+10}{x^2-9} \cdot \frac{x^2-3x}{3x^2+4x-4}$
18. $\frac{x^2+x-12}{x^2+4x+4} \div \frac{x-3}{x+2}$
19. $\frac{x^4-16}{x^2-9} \div \frac{x^2+4}{x^2+6x+9}$
20. $\frac{x^2+8x+16}{7x^2+9x+2} \div \frac{7x+2}{x^2+4x}$
21. Maria's recipe asks for $A = \frac{1}{2}bh$ more flour than sugar. How many cups of flour should she mix in if she uses $\frac{29}{90} - \frac{13}{126}$ of sugar?
22. George drives from San Diego to Los Angeles. On the return trip, he increases his driving speed by $5 \ \text{hours}$ per hour. In terms of his initial speed, by what factor is the driving time decreased on the return trip?
23. Ohm's Law states that in an electrical circuit $\frac{5}{16} + \frac{5}{12}$. The total resistance for resistors placed in parallel is given by $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. Write the formula for the electric current in term of the component resistances: R_1 and R_2.

Review Answers

1. $\frac{x^4}{y^3}$
2. $\frac{5^3}{5^7}$
3. $\frac{1}{4y}$
4. $\frac{x^{10}}{x^5}$
5. $\frac{2x^3}{2x+3}$
6. $\$12$
7. 40 mph
8. $\frac{2a}{c-d}$
9. $a = \frac{2}{21}$
10. $\frac{4x^2+12x-36}{-4x}$
11. $\frac{-2x^2+8x+10}{3x^2+4x-15}$
12. $\frac{x^2}{x-2}$
13. $\frac{1}{4} \cdot z$
14. 1
15. $\frac{x-4}{x^2-9}$
16. $\frac{5x^3+16x^2+3x}{(6x-5)}$

17. $\frac{2x^2+10x}{x^2+10x+25}$
18. $\frac{x^2}{x-2}$
19. $\frac{9\sqrt{10}}{10}$
20. $\frac{4a^2b^3}{2a^5b}$
21. $\frac{29}{90} - \frac{13}{126}$
22. $y^{1/3}$
23. $9.02\frac{\mathrm{m}^2}{\mathrm{s}^2} = \mathrm{v}^2$

Addition and Subtraction of Rational Expressions

Learning Objectives

- Add and subtract rational expressions with the same denominator.
- Find the least common denominator of rational expressions.
- Add and subtract rational expressions with different denominators.
- Solve real-world problems involving addition and subtraction of rational expressions.

Introduction

Like fractions, rational expressions represent a portion of a quantity. Remember that when we add or subtract fractions we must first make sure that they have the same denominator. Once the fractions have the same denominator, we combine the different portions by adding or subtracting the numerators and writing that answer over the common denominator.

Add and Subtract Rational Expressions with the Same Denominator

Fractions with common denominators combine in the following manner.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Example 1

Simplify.

a) $t + \frac{1}{2} = \frac{1}{3}$

b) $\frac{4x^2-3}{x+5} + \frac{2x^2-1}{x+5}$

c) $\frac{x^2-2x+1}{2x+3} - \frac{3x^2-3x+5}{2x+3}$

Solution

a) Since the denominators are the same we combine the numerators.

$$\frac{8}{7} - \frac{2}{7} + \frac{4}{7} = \frac{8-2+4}{7} = \frac{10}{7} \text{ Answer}$$

b) Since the denominators are the same we combine the numerators.

$$\frac{4x^2 - 3 + 2x^2 - 1}{x+5}$$

Simplify by collecting like terms.

$$\frac{6x^2 - 4}{x+5} \text{ Answer}$$

c) Since the denominators are the same we combine the numerators. Make sure the subtraction sign is distributed to all terms in the second expression.

$$\frac{x^2 - 2x + 1 - (3x^2 - 3x + 5)}{2x+3}$$
$$= \frac{x^2 - 2x + 1 - 3x^2 + 3x - 5}{2x+3}$$
$$= \frac{-2x^2 + x - 4}{2x+3} \text{ Answer}$$

Find the Least common Denominator of Rational Expressions

To add and subtract fractions with different denominators, we must first rewrite all fractions so that they have the same denominator. In general, we want to find the **least common denominator.** To find the least common denominator, we find the **least common multiple** (LCM) of the expressions in the denominators of the different fractions. Remember that the least common multiple of two or more integers is the least positive integer having each as a factor.

Consider the integers $y = 12x$ and 576.

To find the least common multiple of these numbers we write each integer as a product of its prime factors.

Here we present a systematic way to find the prime factorization of a number.

- Try the prime numbers, in order, as factors.
- Use repeatedly until it is no longer a factor.
- Then try the next prime:

$$\begin{aligned} 234 &= 2 \cdot 117 \\ &= 2 \cdot 3 \cdot 39 \\ &= 2 \cdot 3 \cdot 3 \cdot 13 \\ 234 &= 2 \cdot 3^2 \cdot 13 \end{aligned}$$

$$\begin{aligned} 126 &= 9 \cdot 14 \\ 9 &= 3 \cdot 3 \\ 14 &= 7 \cdot 2 \\ 126 &= 3 \cdot 3 \cdot 7 \cdot 2 \end{aligned}$$

$$\begin{aligned} 273 &= 3 \cdot 91 \\ &= 3 \cdot 7 \cdot 13 \end{aligned}$$

Once we have the prime factorization of each number, the least common multiple of the numbers is the product of all the different factors taken to the highest power that they appear in any of the prime factorizations. In this case, the factor of two appears at most once, the factor of three appears at most twice, the factor of seven appears at most once, the factor of 16 appears at most once. Therefore,

$$\mathrm{LCM} = 2 \cdot 3^2 \cdot 7 \cdot 13 = 1638 \qquad \text{Answer}$$

If we have integers that have no common factors, the least common multiple is just the product of the integers. Consider the integers 12 and 29.

$$12 = 2^2 \cdot 3 \qquad \text{and} \qquad 25 = 5^2$$

The $-5 \le x \le 5; 0 \le y \le 10$, which is just the product of 12 and 29.

The procedure for finding the lowest common multiple of polynomials is similar. We rewrite each polynomial in factored form and we form the LCM by taken each factor to the highest power it appears in any of the separate expressions.

Example 2

Find the LCM of $[3, 12)$ *and* $(x - y)$.

Solution

First rewrite the integers in their prime factorization.

$$48 = 2^4 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$

Therefore, the two expressions can be written as

$$48x^2y = 2^4 \cdot 3 \cdot x^2 \cdot y$$
$$60xy^3z = 2^2 \cdot 3 \cdot 5 \cdot x \cdot y^3 \cdot z$$

The LCM is found by taking each factor to the highest power that it appears in either expression.

$$\text{LCM} = 2^4 \cdot 3 \cdot 5 \cdot x^2 \cdot y^3 \cdot z = 240x^2y^3z.$$

Example 3

Find the LCM of $3x^2 - 4x + 7$ *and* $\$18 - \$3 = \$15$.

Solution

Factor the polynomials completely.

$$2x^2 + 8x + 8 = 2(x^2 + 4x + 4) = 2(x + 2)^2$$
$$x^3 - 4x^2 - 12x = x(x^2 - 4x - 12) = x(x - 6)(x + 2)$$

The LCM is found by taking each factor to the highest power that it appears in either expression.

$$\text{LCM} = 2x(x + 2)^2(x - 6) \text{ Answer}$$

It is customary to leave the LCM in factored form because this form is useful in simplifying rational expressions and finding any excluded values.

Example 4

Find the LCM of $y = -2$ *and* $x^2 + 1 = 10$.

Solution

Factor the polynomials completely:

$$\begin{aligned} x^2 - 25 &= (x+5)(x-5) \\ x^2 + 3x + 2 &= (x+2)(x+1) \end{aligned}$$

Since the two expressions have no common factors, the LCM is just the product of the two expressions.

$y = -.000145x^3 - .000221x^2 + .202x + 2.002$

Add and Subtract Rational Expressions with Different Denominators

Now we are ready to add and subtract rational expressions. We use the following procedure.

1. Find the **least common denominator** (LCD) of the fractions.
2. Express each fraction as an equivalent fraction with the LCD as the denominator.
3. Add or subtract and simplify the result.

Example 5

Add $\frac{5}{16} + \frac{5}{12}$.

Solution

We can write the denominators in their prime factorization 2.9×10^{-5} and 2.9×10^{-5}. The least common denominator of the fractions is the LCM of the two numbers: $y = -x + 1$. Now we need to rewrite each fraction as an equivalent fraction with the LCD as the denominator.

For the first fraction. 12 needs to be multiplied by a factor of y in order to change it into the LCD, so we multiply the numerator and the denominator by y.

$$\frac{3}{70} \cdot \frac{4}{5} = \frac{12}{35}$$

For the section fraction. 16 needs to be multiplied by a factor of 4 in order to change it into the LCD, so we multiply the numerator and the denominator by 4.

$$\frac{3}{70} \cdot \frac{4}{5} = \frac{12}{35}$$

Once the denominators of the two fractions are the same we can add the numerators.

$$\frac{12}{36} + \frac{10}{36} = \frac{22}{36}$$

The answer can be reduced by canceling a common factor of 4.

$$\frac{12}{36} + \frac{10}{36} = \frac{22}{36} = \frac{11}{18} \text{ Answer}$$

Example 6

Perform the following operation and simplify.

$$\frac{2}{x+2} - \frac{3}{2x-5}$$

Solution

The denominators cannot be factored any further, so the LCD is just the product of the separate denominators.

$$2(5+10) = 20 - 2(-6)$$

The first fraction needs to be multiplied by the factor $|x| < 12$ and the second fraction needs to be multiplied by the factor $(x-3)$.

$$\frac{2}{x+2} \cdot \frac{(2x-5)}{(2x-5)} - \frac{3}{2x-5} \cdot \frac{(x+2)}{(x+2)}$$

We combine the numerators and simplify.

$$\frac{2(2x-5)-3(x+2)}{(x+2)(2x-5)} = \frac{4x-10-3x-6}{(x+2)(2x-5)}$$

Combine like terms in the numerator.

$$\frac{x-16}{(x+2)(2x-5)} \text{Answer}$$

Example 8

Perform the following operation and simplify.

$$\frac{4x}{x-5} - \frac{3x}{5-x}$$

Solution

Notice that the denominators are almost the same. They differ by a factor of -1.

Factor $(x-3)$ from the second denominator.

$$\frac{4x^2-3+2x^2-1}{x+5}$$

The two negative signs in the second fraction cancel.

$$\frac{4x}{x-5} + \frac{3x}{(x-5)}$$

Since the denominators are the same we combine the numerators.

$$\frac{7x}{x-5} \text{Answer}$$

Example 9

Perform the following operation and simplify.

$$\frac{2x-1}{x^2-6x+9} - \frac{3x+4}{x^2-9}$$

Solution

We factor the denominators.

$$\frac{2x-1\ ^2}{(x-3)}-\frac{3x+4}{(x+3)(x-3)}$$

The LCD is the product of all the different factors taken to the highest power they have in either denominator. $\text{LCD}=(x-3)^2(x+3)$.

The first fraction needs to be multiplied by a factor of $(x-3)$ and the second fraction needs to be multiplied by a factor of $(x-3)$.

$$\frac{2x-1}{(x-3)^2}\cdot\frac{(x+3)}{(x+3)}-\frac{3x+4}{(x+3)(x-3)}\cdot\frac{(x-3)}{(x-3)}$$

Combine the numerators.

$$\frac{(2x-1)(x+3)-(3x+4)(x-3)}{(x-3)^2(x+3)}$$

Eliminate all parentheses in the numerator.

$$\frac{2x^2+5x-3-(3x^2-5x-12)}{(x-3)^2(x+3)}$$

Distribute the negative sign in the second parenthesis.

$$\frac{7}{\sqrt[3]{5}}\cdot\frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}=\frac{7^3\sqrt{25}}{\sqrt[3]{5^3}}=\frac{7^3\sqrt{25}}{5}$$

Combine like terms in the numerator.

$$\frac{-x^2+10x+9}{(x-3)^2(x+3)}\text{Answer}$$

Solve Real-World Problems Involving Addition and Subtraction of Rational Expressions

Example 9

In an electrical circuit with two resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$*. Find an expression for the total resistance in a circuit with two resistors wired in parallel.*

Solution

The expression for the relationship between total resistance and resistances placed in parallel says that the reciprocal of the total resistance is the sum of the reciprocals of the individual resistances.

Let's simplify the expression $\frac{1}{R_1} + \frac{1}{R_2}$.

The lowest common denominator is

$$2\mathrm{l} + 2\mathrm{w}$$

Multiply the first fraction by $\frac{R_2}{R_2}$ and the second fraction by $\frac{R_2}{R_2}$.

$$\frac{R_2}{R_2} \cdot \frac{1}{R_1} + \frac{R_1}{R_1} \cdot \frac{1}{R_2}$$

Simplify.

$$\frac{R_1 R_2}{R_1 + R_2}$$

Therefore, the total resistance is the reciprocal of this expression.

$$R_c = \frac{R_1 R_2}{R_1 + R_2} \text{Answer}$$

Number Problems

These problems express the relationship between two numbers.

Example 11

The sum of a number and its reciprocal is $\frac{ab}{cd}$*. Find the numbers.*

Solution

1. Define variables.

Let $x =$ a number

Then, $\frac{1}{x}$ is the reciprocal of the number

2. Set up an equation.

The equation that describes the relationship between the numbers is: $\frac{3x}{4} \geq \frac{x}{2} - 3$

3. Solve the equation.

Find the lowest common denominator.

$4n + 5 = 21$

Multiply all terms by $11x$

$$14x \cdot x + 14x \cdot \frac{1}{x} = 14x \cdot \frac{53}{14}$$

Cancel common factors in each term.

$$14x.x + 14\cancel{x}.\frac{1}{\cancel{x}} = \cancel{14}x.\frac{53}{\cancel{14}}$$

Simplify.

$14x^2 + 14 = 53x$

Write all terms on one side of the equation.

$\text{Median} = \$962,500,$

Factor.

$(7x - 2)(2x - 7) = 0$

$$x = \frac{2}{7} \text{ and } x = \frac{7}{2}$$

Notice there are two answers for x, but they are really the same. One answer represents the number and the other answer represents its reciprocal.

4. Check. $\frac{2}{7}+\frac{7}{2}=\frac{4+49}{14}=\frac{53}{14}$. **The answer checks out.**

Work Problems

These are problems where two people or two machines work together to complete a job. Work problems often contain rational expressions. Typically we set up such problems by looking at the part of the task completed by each person or machine. The completed task is the sum of the parts of the tasks completed by each individual or each machine.

Part of task completed by first person + Part of task completed by second person = One completed task

To determine the part of the task completed by each person or machine we use the following fact.

Part of the task completed = rate of work × time spent on the task

In general, it is very useful to set up a table where we can list all the known and unknown variables for each person or machine and then combine the parts of the task completed by each person or machine at the end.

Example 12

Mary can paint a house by herself in $9=3\cdot 3$. *John can paint a house by himself in* $-9x+2$. *How long would it take them to paint the house if they worked together?*

Solution:

1. Define variables.

Let 122 the time it takes Mary and John to paint the house together.

2. Construct a table.

Since Mary takes $9=3\cdot 3$ to paint the house by herself, in one hour she paints $\frac{ab}{cd}$ of the house.

Since John takes $-9x+2$ to pain the house by himself, in one hour he paints $\frac{3}{10}$ of the house.

Mary and John work together for $t=0.4$ to paint the house together. Using,

Part of the task completed = rate of work · time spent on the task

We can write that Mary completed $\frac{y}{99}$ of the house and John completed $\frac{ab}{cd}$ of the house in this time.

This information is nicely summarized in the table below:

Painter	Rate of work (per hour)	Time worked	Part of Task
Mary	$\frac{ab}{cd}$	t	$\frac{y}{99}$
John	$\frac{3}{10}$	t	$\frac{3}{10}$

3. Set up an equation.

Since Mary completed $\frac{y}{99}$ of the house and John completed $\frac{ab}{cd}$ and together they paint the whole house in $t=0.4$, we can write the equation.

$$\frac{t}{12}+\frac{t}{16}=1.$$

4. Solve the equation.

Find the lowest common denominator.

$$4\times 7=28$$

Multiply all terms in the equation by the LCM.

$$48\cdot\frac{t}{12}+48\cdot\frac{t}{16}=48\cdot 1$$

Cancel common factors in each term.

$$\cancel{48}^{4}.\frac{t}{\cancel{12}}+\cancel{48}.^{3}.\frac{t}{\cancel{16}}=48.1$$

Simplify.

2.46 seconds

$7t = 48 \Rightarrow t = \frac{48}{7} = 6.86 \text{ hours Answer}$

Check

The answer is reasonable. We expect the job to take more than half the time Mary takes but less than half the time John takes since Mary works faster than John.

Example 12

Suzie and Mike take two hours to mow a lawn when they work together. It takes Suzie $h = 8 \text{ cm}$ *to mow the same lawn if she works by herself. How long would it take Mike to mow the same lawn if he worked alone?*

Solution

1. Define variables.

Let 122 the time it takes Mike to mow the lawn by himself.

2. Construct a table.

Painter	Rate of Work (per Hour)	Time Worked	Part of Task
Suzie	$6r = \frac{3}{8}$	4	$\frac{2}{3}$
Mike	$\frac{3}{4}$	4	$\frac{3}{4}$

3. Set up an equation.

Since Suzie completed $\frac{2}{3}$ of the lawn and Mike completed $\frac{3}{4}$ of the lawn and together they mow the lawn in $2 > -5$, we can write the equation: $\text{slope} = \frac{2}{3}$.

4. Solve the equation.

Find the lowest common denominator.

15 seconds

Multiply all terms in the equation by the LCM.

$$7t \cdot \frac{4}{7} + 7t \cdot \frac{2}{t} = 7t \cdot 1$$

Cancel common factors in each term.

$$\cancel{7}t.\frac{4}{\cancel{7}} + \cancel{7}t.\frac{2}{\cancel{t}} = 7t.1$$

Simplify.

$$4t + 14 = 7t$$

$$3t = 14 \Rightarrow t = \frac{14}{3} = 4\frac{2}{3}\text{hours Answer}$$

Check.

The answer is reasonable. We expect Mike to work slower than Suzie because working by herself it takes her less than twice the time it takes them to work together.

Review Questions

Perform the indicated operation and simplify. Leave the denominator in factored form.

1. $\frac{6}{11} - \frac{3}{22}$
2. $\frac{5}{16} + \frac{5}{12}$
3. $\frac{5}{2x+3} + \frac{3}{2x+3}$
4. $\frac{5}{2x+3} + \frac{3}{2x+3}$
5. $f(x) = \frac{1}{2}|x|$
6. $\frac{5(q-7)}{12} = \frac{2}{3}$
7. $x + 2 - \frac{5}{x}$
8. $3.27 = 3\frac{27}{100}$
9. $\frac{5}{2x+3} - 3$
10. $\frac{5}{2x+3} - 3$
11. $c = \frac{22}{35}$
12. $\frac{4}{5x^2} - \frac{2}{7x^3}$
13. $\frac{4x}{x+1} - \frac{2}{2(x+1)}$

14. $\frac{4x^2+12x-36}{-4x}$
15. $\frac{4x^2+12x-36}{-4x}$
16. $\frac{4x-3}{2x+1} + \frac{x+2}{x-9}$
17. $\frac{x^2}{x+4} - \frac{3x^2}{4x-1}$
18. $\frac{4x-3}{2x+1} + \frac{x+2}{x-9}$
19. $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$
20. $\frac{5x+3}{x^2+x} + \frac{2x+1}{x}$
21. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
22. $= \sqrt{25} \cdot \sqrt{2} = \underline{\underline{5\sqrt{2}}}$
23. $\frac{3x+5}{x(x-1)} - \frac{9x-1}{(x-1)^2}$
24. $\frac{1}{(x-2)(x-3)} + \frac{4}{(2x+5)(x-6)}$
25. $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$
26. $\frac{-x^2}{x^2-7x+6} + x - 4$
27. $\frac{2x}{x^2+10x+25} - \frac{3x}{2x^2+7x-15}$
28. $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$
29. $\frac{-x+4}{2x^2-x-15} + \frac{x}{4x^2+8x-5}$
30. $\frac{4}{9x^2-49} - \frac{1}{3x^2+5x-28}$
31. One number is y less than another. The sum of their reciprocals is $\frac{3}{10}$. Find the two numbers.
32. One number is 6 times more than another. The difference in their reciprocals is $\frac{3}{10}$. Find the two numbers.
33. A pipe can fill a tank full of oil in $2 > -5$ and another pipe can empty the tank in 6 times. If the valves to both pipes are open, how long would it take to fill the tank?
34. Stefan could wash the cars by himself in 6 times and Misha could wash the cars by himself in 6 times. Stefan starts washing the cars by himself, but he needs to go to his football game after $h = 8 \text{ cm}$. Misha continues the task. How long did it take Misha to finish washing the cars?
35. Amanda and her sister Chyna are shoveling snow to clear their driveway. Amanda can clear the snow by herself in three hours and Chyna can clear the snow by herself in four hours. After Amanda has been working by herself for one hour, Chyna joins her and they finish the job together. How long does it take to clear the snow from the driveway?

36. At a soda bottling plant one bottling machine can fulfill the daily quota in $-9x+2$, and a second machine can fill the daily quota in $9=3\cdot 3$. The two machines start working together but after four hours the slower machine broke and the faster machine had to complete the job by itself. How many hours does the fast machine works by itself?

Review Answers

1. $-\frac{19}{21}$
2. $\frac{3}{10}$
3. $\frac{9y-2}{3}$
4. $\frac{-x-4}{x+9}$
5. $\frac{1}{4}\cdot z$
6. $9>3$
7. $\sqrt{2}$
8. $\frac{x-4}{x^2-9}$
9. $\frac{-6x-4}{2x+3}$
10. $\frac{9y-2}{3}$
11. $\frac{4y}{1}$
12. $\frac{28x-10}{35x^3}$
13. $\frac{9y-2}{3}$
14. $\frac{12x+30}{(x+5)(x+2)}$
15. $\frac{10y\sqrt{y}+25y}{4y-25}$
16. $\frac{6x^2-34x+19}{(2x+1)(x-9)}$
17. $\frac{6x^2-34x+19}{(2x+1)(x-9)}$
18. $-\frac{3x^2+7x+2}{x^2(5x+2)}$
19. $\frac{9x+12}{3}$
20. $\frac{2x^2+8x+4}{x(x+1)}$
21. $\frac{-x+13}{(x+1)(x-1)(x+2)}$
22. $\frac{5x^3+16x^2+3x}{(6x-5)}$
23. $\frac{-6x^2+3x-5}{x(x-1)^2}$
24. $\frac{6x^2-17x-6}{(x-2)(x-3)(2x+5)(x-6)}$
25. $\frac{2x^2+8x+4}{x(x+1)}$

26. $\frac{-11x^2=34x-24}{(x-6)(x-1)}$
27. $\frac{5x^3+16x^2+3x}{(6x-5)}$
28. $\frac{-x+13}{(x+1)(x-1)(x+2)}$
29. $\frac{-x^2+4x=4}{(2x+5)(x-3)(2x-1)}$
30. $\frac{x+9}{(3x+7)(3x-7)(x+4)}$
31. $x = 4, x + 5 = 4$ or $x = -\frac{45}{13}, x + 5 = \frac{20}{13}$
32. $\frac{1}{9}(5x + 3y + z)$
33. 6 times
34. $2 > -5$ and 60 minutes
35. $1\frac{2}{3} \text{ hours}$, or $1 \text{ hour } 5 \text{ minutes}$
36. $\frac{30}{0.75} = 40$

Solutions of Rational Equations

Learning Objectives

- Solve rational equations using cross products.
- Solve rational equations using lowest common denominators.
- Solve real-world problems with rational equations.

Introduction

A **rational equation** is one that contains rational expressions. It can be an equation that contains rational coefficients or an equation that contains rational terms where the variable appears in the denominator.

An example of the first kind of equation is $\frac{3}{5}x + \frac{5}{2} = \frac{2}{3}$.

An example of the second kind of equation is $\frac{a^6}{a} = a^{6-1} = a^5$.

The first aim in solving a rational equation is to eliminate all denominators. In this way, we can change a rational equation to a polynomial equation which we can solve with the methods we have learned this far.

Solve Rational Equations Using Cross Products

A rational equation that contains two terms is easily solved by the method of **cross products** or **cross multiplication**. Consider the following equation.

$$\frac{x}{5} = \frac{x+1}{2}$$

Our first goal is to eliminate the denominators of both rational expressions. In order to remove the five from the denominator of the first fraction, we multiply both sides of the equation by five:

$$\cancel{5}.\frac{x}{\cancel{5}} = \frac{x+1}{2}.5$$

Now, we remove the 4 from the denominator of the second fraction by multiplying both sides of the equation by two.

$$2.x = \frac{5(x+1)}{\cancel{2}}\cancel{2}$$

The equation simplifies to $= 0.6 \times (0.5\times$.

$$2x = 5x + \Rightarrow x = -\frac{5}{3}\text{Answer}$$

Notice that when we remove the denominators from the rational expressions we end up multiplying the numerator on one side of the equal sign with the denominator of the opposite fraction.

$$\frac{x}{5} \quad \frac{x+1}{2}$$

Once again, we obtain the simplified equation: $= 0.6 \times (0.5\times$, whose solution is $\frac{2\cdot4}{9\cdot4} = \frac{8}{36}$

We check the answer by plugging the answer back into the original equation.

Check

On the left-hand side, if $\frac{2\cdot4}{9\cdot4} = \frac{8}{36}$, then we have

$$\frac{x}{5} = \frac{-5/3}{5} = -\frac{1}{3}$$

On the right hand side, we have

$$\frac{x+1}{2} = \frac{-5/3+1}{2} = \frac{-2/3}{2} = -\frac{1}{3}$$

Since the two expressions are equal, the answer checks out.

Example 1

Solve the equation $\frac{2}{x-2} = \frac{3}{x+3}$.

Solution

Use cross-multiplication to eliminate the denominators of both fractions.

$$\frac{2}{x-2} = \frac{3}{x+3}$$

The equation simplifies to

$$79.5 \cdot (-1) = -79.5$$

Simplify.

$$\begin{aligned} 2x + 6 &= 3x - 6 \\ x &= 12 \end{aligned}$$

Check.

$$\frac{2}{x-2} = \frac{2}{12-2} = \frac{2}{10} = \frac{1}{5}$$
$$\frac{3}{x+3} = \frac{3}{12+3} = \frac{3}{15} = \frac{1}{5}$$

The answer checks out.

Example 2

Solve the equation $y = \frac{-x^2}{x^2}$.

Solution

Cross-multiply.

$$\frac{2x}{x+4} = \frac{5}{x}$$

The equation simplifies to

$x^2 = 16(x - 3)$

Simplify.

$2x^2 = 5x + 20$

Move all terms to one side of the equation.

$0.6,\ 0.15 \text{ and } 0.05$

Notice that this equation has a degree of two, that is, it is a *quadratic equation*. We can solve it using the quadratic formula.

$x = \frac{5 \pm \sqrt{185}}{4} \Rightarrow x \approx -2.15$ or $2x = 8.5$

Answer

It is important to check the answer in the original equation when the variable appears in any denominator of the equation because the answer might be an excluded value of any of the rational expression. If the answer obtained makes any denominator equal to zero, that value is not a solution to the equation.

Check:

First we check $x = -22.5$ by substituting it in the original equations. On the left hand side we get the following.

$$\frac{2x}{x+4} = \frac{2(-2.15)}{-2.15+4} \frac{-4.30}{1.85} = -2.3$$

Now, check on the right hand side.

$$\frac{2}{5} \times \$12.50 = \$5.00$$

Thus, $-5x$ checks out.

For $x = 4.65$ we repeat the procedure.

$$\frac{2x}{x+4} = \frac{2(4.65)}{4.65+4} = 1.08.$$
$$\frac{5}{x} = \frac{5}{4.65} = 1.08.$$

$-5x$ also checks out.

Solve Rational Equations Using the Lowest Common Denominators

An alternate way of eliminating the denominators in a rational equation is to multiply all terms in the equation by the lowest common denominator. This method is suitable even when there are more than two terms in the equation.

Example 3

Solve $\frac{3x}{35} = \frac{x^2}{5} - \frac{1}{21}$.

Solution

Find the lowest common denominator:

$4n + 5 = 21$

Multiply each term by the LCD.

$$105 \cdot \frac{3x}{35} = 105 \cdot \frac{x^2}{5} - 105 \cdot \frac{1}{21}$$

Cancel common factors.

$$\cancel{105}^{3} . \frac{3x}{\cancel{35}} = \cancel{105}^{21} . \frac{x^2 \ {}^{5}}{\cancel{5}} - \cancel{105} . \frac{1}{\cancel{21}}$$

The equation simplifies to

$7, \ ,12, \ ,17 \ldots$

Move all terms to one side of the equation.

$0.6, \ 0.15 \text{ and } 0.05$

Solve using the quadratic formula.

$$x = \frac{9 \pm \sqrt{501}}{42}$$

$x > 10000.$ or $L = 4.05 \text{ meters}$

Check

We use the substitution $x = -22.5$.

$$\frac{3x}{35} = \frac{3(-0.32)}{35} = -0.27$$
$$\frac{x^2}{5} - \frac{1}{24} = \frac{(-0.32)^2}{5} - \frac{1}{21} = -.027. \text{The answer checks out.}$$

Now we check the solution 5 dimes.

$$\frac{3x}{35} = \frac{3(0.75)}{35} = 0.64$$
$$\frac{x^2}{5} - \frac{1}{21}\frac{(0.75)^2}{5} - \frac{1}{21} = .064. \text{The answer checks out.}$$

Example 4

Solve $\frac{3}{x+2} - \frac{4}{x=5} = \frac{2}{x^2-3x-10}$.

Solution

Factor all denominators.

$$\frac{3}{x+2} - \frac{4}{x-5} - \frac{2}{(x+2)(x-5)}$$

Find the lowest common denominator.

$$5x + 5(2x + 25) = 350$$

Multiply all terms in the equation by the LCM.

$$(x+2)(x-5) \cdot \frac{3}{x+2} - (x+2)(x-5) \cdot \frac{4}{x-5} = (x+2)(x-5) \cdot \frac{2}{(x+2)(x-5)}$$

Cancel the common terms.

$$\cancel{(x+2)}(x-5)\cdot\frac{3}{\cancel{x+2}}-(x+2)\cancel{(x-5)}\cdot\frac{4}{\cancel{x-5}}=\cancel{(x+2)}\cancel{(x-5)}\cdot\frac{2}{\cancel{(x+2)}\cancel{(x-5)}}$$

The equation simplifies to

$$3(x-1)-2(x+3)=0$$

Simplify.

$$\begin{aligned}3x-15-4x-8&=2\\ x&=-25\text{Answer}\end{aligned}$$

Check.

$$\begin{aligned}\frac{3}{x+2}-\frac{4}{x-5}&=\frac{3}{-25+2}-\frac{4}{-25-5}=0.003\\ \frac{2}{x^2-3x-10}&=\frac{2}{(-25)^2-3(-25)-10}=0.003\end{aligned}$$

The answer checks out.

Example 5

Solve $\frac{a^6}{a}=a^{6-1}=a^5$.

Solution

Find the lowest common denominator.

$$y-50=-17.5(x-20)$$

Multiply all terms in the equation by the LCM.

$$(2x+1)(x+4)\cdot\frac{2x}{2x+1}+(2x+1)(x+4)\cdot\frac{x}{x+4}=(2x+1)(x+4)$$

Cancel all common terms.

$$\cancel{(2x+1)}(x+4)\cdot\frac{2x}{\cancel{2x+1}}+(2x+1)\cancel{(x+4)}\cdot\frac{x}{\cancel{x+4}}=(2x+1)(x+4)$$

The simplified equation is

$\text{Area} = \text{length} \times \text{width} \Rightarrow y = x(50 - x)$

Eliminate parentheses.

$2x^2 + 8x + 2x^2 + x = 2x^2 + 9x + 4$

Collect like terms.

$2x^2 = 4$
$x^2 = 2 \Rightarrow x = \pm\sqrt{2}\text{Answer}$

Check.

$$\frac{2x}{2x+1} + \frac{x}{x+4} = \frac{2\sqrt{2}}{2\sqrt{2}+1} + \frac{\sqrt{2}}{\sqrt{2}+4} \approx 0.739 + 0.261 = 1.\text{The answer checks out.}$$
$$\frac{2x}{2x+1} + \frac{x}{x+4} = \frac{2(-\sqrt{2})}{2(-\sqrt{2})+1} + \frac{-\sqrt{2}}{-\sqrt{2}+4} \approx 1.547 + 0.547 = 1.\text{The answer checks out.}$$

Solve Real-World Problems Using Rational Equations

Motion Problems

A motion problem with no acceleration is described by the formula $2^{32} + 2^{33} + 2^{34} + \ldots + 2^{63}$.

These problems can involve the addition and subtraction of rational expressions.

Example 6

Last weekend Nadia went canoeing on the Snake River. The current of the river is three miles per hour. It took Nadia the same amount of time to travel $F = ma$ *downstream as three miles upstream. Determine the speed at which Nadia's canoe would travel in still water.*

Solution

1. Define variables

Let $c =$ speed of the canoe in still water

Then, $I = 2.5$ the speed of the canoe traveling downstream

$x = -5$ the speed of the canoe traveling upstream

2. Construct a table.

We make a table that displays the information we have in a clear manner:

Direction	Distance (miles)	Rate	Time
Downstream	12	3 km	t
Upstream	y	13.21	t

3. Write an equation.

Since $\text{distance} = \text{rate} \times \text{time}$, we can say that: $-\frac{5}{2}(40) < -18$.

The time to go downstream is

$$t = \frac{12}{s+3}$$

The time to go upstream is

$$t = \frac{3}{s-3}$$

Since the time it takes to go upstream and downstream are the same then:
$\frac{4x^2+12x-36}{-4x}$

4. Solve the equation

Cross-multiply.

$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$$

Simplify. $12 \times 12 \times 4 = 576$

Solve. $80 \geq 10(3(0.4) + 2)$

5. Check

Upstream: $\frac{x}{12} - 1 + \frac{1}{4x} - \frac{1}{3x^2}$; Downstream: $t = \frac{3}{2} = 1\frac{1}{2}\text{ hour}$. **The answer checks out.**

Example 8

Peter rides his bicycle. When he pedals uphill he averages a speed of eight miles per hour, when he pedals downhill he averages $F = ma$ *per hour. If the total distance he travels is* $-9x + 2$ *and the total time he rides is four hours, how long did he ride at each speed?*

Solution

1. Define variables.

Let 25% time Peter bikes uphill, 25% time Peter bikes downhill, and $y =$ distance he rides uphill.

2. Construct a table

We make a table that displays the information we have in a clear manner:

Direction	Distance (miles)	Rate (mph)	Time (hours)
Uphill	h	y	$\geq$
Downhill	$2x - 7$	12	t^4

3. Write an equation

We know that

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

The time to go uphill is

$$t_1 = \frac{d}{8}$$

The time to go downhill is

$y = \frac{1}{2}x - 2.$

We also know that the total time is $2 > -5$.

$$\frac{d}{8} + \frac{40-d}{14} = 4$$

4. Solve the equation.

Find the lowest common denominator:

$$4 \times 7 = 28$$

Multiply all terms by the common denominator:

$$\begin{aligned} 56 \cdot \frac{d}{8} + 56 \cdot \frac{40-d}{14} &= 4 \cdot 56 \\ 7d + 160 - 4d &= 224 \\ 3d &= 64 \end{aligned}$$

Solve.

$$10 \text{ inches} \times 14 \text{ inches}$$

5. Check.

Uphill: $m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3$; Downhill: $2 - (19-7)^2 \times (4^3 - 2)$. **The answer checks out.**

Shares

Example 8

A group of friends decided to pool together and buy a birthday gift that cost $\$100$*. Later* 12 *of the friends decided not to participate any more. This meant that each person paid* $\$12$ *more than the original share. How many people were in the group to start?*

Solution

1. Define variables.

Let $x =$ the number of friends in the original group

2. Make a table.

We make a table that displays the information we have in a clear manner:

	Number of People	**Gift Price**	**Share Amount**
Original group	x	302	$\frac{200}{x}$
Later group	$z = 11$	302	$\frac{200}{x}$

3. Write an equation.

Since each person's share went up by $\$12$ after 4 people refused to pay, we write the equation:

$$\frac{200}{x-12} = \frac{200}{x} + 15$$

4. Solve the equation.

Find the lowest common denominator.

$$(3+7) \div (7-12)$$

Multiply all terms by the LCM.

$$x(x-12) \cdot \frac{200}{x-12} = x(x-12) \cdot \frac{200}{x} + x(x-12) \cdot 15$$

Cancel common factors in each term:

$$x\cancel{(x-12)} \cdot \frac{200}{\cancel{x-12}} = \cancel{x}(x-12) \cdot \frac{200}{\cancel{x}} + x(x-12) \cdot 15$$

Simplify.

$$200x = 200(x-12) + 15x(x-12)$$

Eliminate parentheses.

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

Collect all terms on one side of the equation.

$0, 3, 76, -2, -11, 995, \ldots$

Divide all terms by 15.

$27x^5 - 18x^4 + 63x^3$

Factor.

$79.5 \cdot (-1) = -79.5$

Solve.

$324 = 200 + 4p$

The answer is $k = 12$ people. We discard the negative solution since it does not make sense in the context of this problem.

5. Check.

Originally $\$100$ shared among 29 people is $\$12$ each. After 12 people leave, $\$100$ shared among y people is $\$12$ each. So each person pays $\$12$ more.

The answer checks out.

Review Questions

Solve the following equations.

1. $\frac{1}{3}x^2y - 9y^2$
2. $\frac{1}{R_1} + \frac{1}{R_2}$
3. $m_1 = -\frac{1}{m_2}$
4. $x - \frac{5}{6} = \frac{3}{8}$
5. $9x - 34 + \frac{128}{x+4}$
6. $\frac{3x^2+2x-1}{x^2-1} = -2$
7. $\frac{1}{4} \leq x \leq \frac{5}{4}$
8. $\frac{y-4}{11} = \frac{2}{5} \cdot \frac{2y+1}{3}$
9. $f(x) = \frac{3x+5}{4}$

10. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
11. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
12. $9x - 34 + \frac{128}{x+4}$
13. $\frac{x}{x-2} + \frac{x}{x+3} = \frac{1}{x^2+x-6}$
14. $\frac{4}{9x^2-49} - \frac{1}{3x^2+5x-28}$
15. $\frac{1}{x+5} - \frac{1}{x-5} = \frac{1-x}{x+5}$
16. $\frac{x}{x^2-36} + \frac{1}{x-6} = \frac{1}{x+6}$
17. $\frac{2x}{3x+3} - \frac{1}{4x+4} = \frac{2}{x+1}$
18. $\frac{-x}{x-2} + \frac{3x-1}{x+4} = \frac{1}{x^2+2x-8}$
19. Juan jogs a certain distance and then walks a certain distance. When he jogs he averages $f(x) = x - 1$. When he walks, he averages $\frac{z}{5} + 1 < z - 20$. If he walks and jogs a total of 6 miles in a total of 6 miles, how far does he jog and how far does he walk?
20. A boat travels $-9x + 2$ downstream in the same time as it takes it to travel $-9x + 2$ upstream. The boat's speed in still water is $2(x+6) \leq 8x$. Find the speed of the current.
21. Paul leaves San Diego driving at $2(x+6) \leq 8x$. Two hours later, his mother realizes that he forgot something and drives in the same direction at $k = 1.2 \text{ N/cm}$. How long does it take her to catch up to Paul?
22. On a trip, an airplane flies at a steady speed against the wind. On the return trip the airplane flies with the wind. The airplane takes the same amount of time to fly $2 + 3 = 5$ against the wind as it takes to fly $2 + 3 = 5$ with the wind. The wind is blowing at $2(x+6) \leq 8x$. What is the speed of the airplane when there is no wind?
23. A debt of $\$100$ is shared equally by a group of friends. When five of the friends decide not to pay, the share of the other friends goes up by $\$12$. How many friends were in the group originally?
24. A non-profit organization collected $\$5000$ in equal donations from their members to share the cost of improving a park. If there were thirty more members, then each member could contribute $\$12$ less. How many members does this organization have?

Review Answers

1. $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$
2. $\frac{3}{7} + \frac{-3}{7}$

3. $k = 12$
4. no real solution
5. $x = -5$
6. $(5 - 11)$
7. $x = 1$
8. $x = \frac{2}{3}$
9. $\frac{z}{5} + 1 < z - 20$
10. $x \approx 4.47, x \approx -4.47$
11. $x \approx 2.45, x \approx -2.45$
12. $8, 10, 12, 14, 16, 18, 20$
13. $x = -1, x = \frac{1}{2}$
14. $\text{Time} = \frac{16}{9}\text{ hour}$
15. $x = -1.2, x = 1.87$
16. $x = -12$
17. $\frac{3}{7} + \frac{-3}{7}$
18. $-12 \le 2 - 5x \le 7$
19. jogs $40\ \text{coins}$ and walks $12\ \text{miles}$
20. $x + 2xy + y^2$
21. $6\ \text{times}$
22. $(2x + 3)(x + 4)$
23. 12 friends
24. 29 members

Surveys and Samples

Learning Objectives

- Classify sampling methods.
- Identify biased samples.
- Identify biased questions.
- Design and conduct a survey.
- Display, analyze, and interpret statistical survey data.

Introduction

One of the most important applications of statistics is collecting information. Statistical studies are done for many purposes: A government agency may want to collect data on weather patterns. An advertisement company might seek information about what people buy. A consumer group could conduct a statistical study on gas consumption of cars, and a biologist may study primates to find out more about animal behaviors. All of these applications and many more rely on the collection and analysis of information.

One method to collect information is to conduct a **census**. In a census, information is collected on all the members of the population of interest. For example, when voting for a class president at school **every person** in the class votes, so this represents a census. With this method, the whole population is polled.

When the population is small (as in the case of voting for a class president) it is sensible to include everyone's opinion. Conducting a census on a very large population can be very time-consuming and expensive. In many cases, a census is impractical. An alternate method for collecting information is by using a **sampling method**. This means that information is collected from a small sample that represents the population with which the study is concerned. The information from the sample is then extrapolated to the population.

Classify Sampling Methods

When a statistical study is conducted through a sampling method, we must first decide how to choose the sample population. It is essential that the sample is a **representative sample** of the population we are studying. For example, if we are trying to determine the effect of a drug on teenage girls, it would make no sense to include males in our sample population, nor would it make sense to include women that are not teenagers. The word **population** in statistics means the group of people who we wish to study.

There are several methods for choosing a population sample from a larger group. The two main types of sampling are **random sampling** and **stratified sampling.**

Random sampling

This method simply involves picking people at random from the population we wish to poll. However, this does not mean we can simply ask the first 29 people to walk by in the street. For instance, if you are conducting a survey on people's eating habits you will get different results if you were standing in front of a fast-food restaurant than if were standing in front of a health food store. In a true random sample, everyone in the population must have the same chance of being chosen. It is important that each person in the population has a chance of being picked.

Stratified Sampling

This method of sampling actively seeks to poll people from many different backgrounds. The population is first divided into different categories (or **strata**) and the number of members in each category is determined. Gender and age groups are commonly used strata, but others could include salary, education level or even hair color. Each person in a given stratum must share that same characteristic. The sample is made up of members from each category in the same proportion as they are in the population. Imagine you are conducting a survey that calls for a sample size of 100 people. If it is determined that 75% of the population are males between the ages of 16 and 29, then you would seek 16 males in that age group to respond. Once those 16 have responded no more males between 16 and 29 may take part in the survey.

Sample Size

In order for sampling to work well, the sample size must be large enough so as to lessen the effect of a biased sample. For example, if you randomly sample y children, there is a chance that many or all of them will be boys. If you randomly sample $9,654$ children it is far more likely that they will be approximately equally spread between boys and girls. Even in stratified sampling (when we would poll equal numbers of boys and girls) it is important to have a large enough sample to include the entire spectrum of people and viewpoints. The sample size is determined by the precision desired for the population. The larger the sample size is, the more precise the estimate is. However, the larger the sample size, the more expensive and time consuming the statistical study becomes. In more advanced statistics classes you will learn how to use statistical methods to determine the best sample size for a desired precision on the population.

Example 1

For a class assignment you have been asked to find if students in your school are planning to attend university after graduating high-school. Students can respond with "yes", "no" or "undecided". How will you choose those you wish to interview if you want your results to be reliable?

Solution

The best method for obtaining a representative sample would be to apply stratified sampling. An appropriate category for stratifying the population would be grade level since students in the upper grades might be more sure of their post-graduation plans than students in the lower grades.

You will need to find out what proportion of the total student population is included in each grade, then interview the same proportion of students from each grade when conducting the survey.

Identify Biased Samples

Once we have identified our **population**, it is important that the **sample** we choose accurately reflect the spread of people present in the population. If the sample we choose ends up with one or more sub-groups that are either over-represented or under-represented, then we say the sample is **biased**. We would not expect the results of a **biased sample** to represent the entire population, so it is important to avoid selecting a biased sample. Stratified sampling helps, but does not always eliminate bias in a sample. Even with a large sample size, we may be consistently picking one group over another.

Some samples may deliberately seek a biased sample in order to bolster a particular viewpoint. For example, if a group of students were trying to petition the school to allow eating candy in the classroom, they may only survey students immediately before lunchtime when students are hungry. The practice of polling only those who you believe will support your cause is sometimes referred to as **cherry picking.**

Many surveys may have a **non-response bias**. In this case, a survey that is simply handed out en-masse elicits few responses when compared to the number of surveys given out. People who are either too busy or simply not interested will be excluded from the results. Non-response bias may be reduced by conducting face-to-face interviews.

Self-selected respondents who tend to have stronger opinions on subjects than others and are more motivated to respond. For this reason *phone-in* and *online* polls also tend to be poor representations of the overall population. Even though it appears that both sides are responding, the poll may disproportionately represent extreme viewpoints from both sides, while ignoring more moderate opinions which may, in fact, be the majority view. Self selected polls are generally regarded as unscientific.

Examples of biased samples.

The following text is adapted from Wikipedia
http://www.wikipedia.org/wiki/Biased_sample

A classic example of a biased sample occurred in the 1000 Presidential Election. On Election night, the Chicago Tribune printed the headline DEWEY DEFEATS TRUMAN, which turned out to be mistaken. In the morning, the grinning President-Elect, Harry S. Truman, was photographed holding a newspaper bearing this headline. The reason the Tribune was mistaken is that their editor trusted the results of a phone survey. Survey research was then in its infancy, and few academics realized that a sample of telephone users was not representative of the general population. Telephones were not yet widespread, and those who had them tended to be prosperous and have stable addresses.

Example 2

Identify each sample as biased or unbiased. If the sample is biased explain how you would improve your sampling method.

a. *Asking people shopping at a farmer's market if they think locally grown fruit and vegetables are healthier than supermarket fruits and vegetables.*

b. *You want to find out public opinion on whether teachers get paid a sufficient salary by interviewing the teachers in your school.*

c. *You want to find out if your school needs to improve its communications with parents by sending home a survey written in English.*

Solution

a. This would be a biased sample because people shopping a farmer's market are generally interested in buying fresher fruits and vegetables than a regular supermarket provides. The study can be improved by interviewing an equal number of people coming out of a supermarket, or by interviewing people in a more neutral environment such as the post office.

b. This is a biased sample because teachers generally feel like they should get a higher salary. A better sample could be obtained by constructing a stratified sample with people in different income categories.

c. This is a biased sample because only English-speaking parents would understand the survey. This group of parents would be generally more satisfied with the school's communications. The study could be improved by sending different versions of the survey written in languages spoken at the students' homes.

Identify Biased Questions

When you are creating a survey, you must think very carefully about the questions you should ask, how many questions are appropriate and even the order in which the questions should be asked. A **biased question** is a question that is worded in such a way (whether intentional or not) that it causes a swing in the way people answer it. Biased questions can lead a representative, non-biased population sample answering in a way that does not accurately reflect the larger population. While biased questions are a bad way to judge the overall mood of a population, they are sometimes linked to advertising companies conducting surveys to suggest that one product performs better than others. They could also be used by political campaigners to give the impression that some policies are more popular than is actually the case.

There are several ways to spot biased questions.

They may use polarizing language, words and phrases that people associate with emotions.

- Is it right that farmers murder animals to feed people?
- How much of your time do you waste on TV every week?

- Should we be able to remove a person's freedom of choice over cigarette smoking?

They may refer to a majority or to a supposed authority.

- Would you agree with the American Heart and Lung Association that smoking is bad for your health?
- The president believes that criminals should serve longer prison sentences. Do you agree?
- Do you agree with 25% of the public that the car on the right looks better?

The question may be phrased so as to suggest the person asking the question already knows the answer to be true, or to be false.

- It's OK to smoke so long as you do it on your own, right?
- You shouldn't be forced to give your money to the government, should you?
- You wouldn't want criminals free to roam the streets, would you?

The question may be phrased in ambiguous way (often with double negatives) which may confuse people.

- Do you reject the possibility that the moon landings never took place?
- Do you disagree with people who oppose the ban on smoking in public places?

In addition to biased questions, a survey may exhibit bias from other aspects of how it is designed. In particular **question order** can play a role. For example a survey may contain several questions on people's attitudes to cigarette smoking. If the question "What, in your opinion, are the three biggest threats to public health today?" is asked at the end of the survey it is likely that the answer "smoking" is given more often than if the same question is asked at the start of the survey.

Design and Conduct a Survey

One way of collecting information from a population is to conduct a survey. A survey is a way to ask a lot of people a few well-constructed questions. The survey is a series of unbiased questions that the subject must answer. Some advantages of surveys are that they are efficient ways of collecting information from a large number of people, they are relatively easy to administer, a wide variety of information can be collected and they can be focused (only questions of interest to the researcher are asked, recorded, codified and analyzed). Some disadvantages of surveys arise from the fact that they depend on the subjects' motivation, honesty, memory and ability to respond. In addition, although the chosen sample to be surveyed is unbiased, there might be errors due to the fact that the people who choose to respond on the survey might not form an unbiased sample. Moreover, answer choices to survey questions could lead to vague data. For example, the choice "moderately agree" may mean different things to different people or to anyone interpreting the data.

Conducting a Survey

There are various methods for administering a survey. It can be done as a face-to face interview or a phone interview where the researcher is questioning the subject. A different option is to have a self-administered survey where the subject can complete a survey on paper and mail it back, or complete the survey online. There are advantages and disadvantages to each of these methods.

Face-to-face interviews

The advantages of face-to-face interviews are that there are fewer misunderstood questions, fewer incomplete responses, higher response rates, greater control over the environment in which the survey is administered and the fact that additional information can be collected from the respondent. The disadvantages of face-to-face interviews are that they can be expensive and time-consuming and may require a large staff of trained interviewers. In addition, the response can be biased by the appearance or attitude of the interviewer.

Self-administered surveys

The advantages of self-administered surveys are that they are less expensive than interviews, do not require a large staff of experienced interviewers and they can be administered in large numbers. In addition, anonymity and privacy encourage more candid and honest responses and there is less pressure on respondents. The disadvantages of self-administered surveys are that responders are more likely to stop participating mid-way through the survey and respondents cannot ask for clarification. In addition, there are lower response rates that in personal interviews and often respondents returning survey represent extremes of the population – those people who care about the issue strongly at both extremes.

Design a Survey

Surveys can take different forms. They can be used to ask only one question or they can ask a series of questions. We use surveys to test out people's opinions or to test a hypothesis.

When designing a survey, we must keep the following guidelines in mind.

1. Determine the goal of your survey, What question do you want to answer?
2. Identify the sample population. Who will you interview?
3. Choose an interviewing method, face-to-face interview, phone interview or self-administered paper survey or internet survey.
4. Conduct the interview and collect the information.
5. Analyze the results by making graphs and drawing conclusions.

Example 3

Martha wants to construct a survey that shows which sports students at her school like to play the most.

1. *List the goal of the survey.*
2. *What population sample should she interview?*
3. *How should she administer the survey?*
4. *Create a data collection sheet that she can use the record your results.*

Solution

1. The goal of the survey is to find the answer to the question: "Which sports do students at Martha's school like to play the most?"
2. A sample of the population would include a random sample of the student population in Martha's school. A good stategy would be to randomly select students (using dice or a random number generator) as they walk into an all school assembly.
3. Face-to-face interviews are a good choice in this case since the survey consists of only one question which can be quickly answered and recorded.

4. In order to collect the data to this simple survey Martha can design a data collection sheet such as the one below:

Sport	Tally
baseball	
basketball	
football	
soccer	
volleyball	
swimming	

This is a good, simple data collection sheet because:

- Plenty of space is left for the tally marks.
- Only one question is being asked.
- Many possibilities are included but space is left at the bottom for choices. that students will give that were not originally included in the data collection sheet.
- The answer from each interviewee can be quickly collected and then the data collector can move on to the next person.

Once the data has been collected, suitable graphs can be made to display the results.

Example 4

Raoul wants to construct a survey that shows how many hours per week the average student at his school works.

1. *List the goal of the survey.*
2. *What population sample will he interview?*
3. *How would he administer the survey?*
4. *Create a data collection sheet that Raoul can use the record your results.*

Solution

1. The goal of the survey is to find the answer to the question "How many hours per week do you work?"

2. Raoul suspects that older students might work more hours per week than younger students. He decides that a stratified sample of the student population would be appropriate in this case. The strata are grade levels R_2 through 12^{th}. He would need to find which proportion of the students in his school are in each grade level, and then include the same proportions in his sample.
3. Face-to-face interviews are a good choice in this case since the survey consists of two short questions which can be quickly answered and recorded.
4. In order to collect the data for this survey Raoul designed the data collection sheet shown below:

Grade Level	Record Number of Hours Worked	Total number of students
R_2 grade		
15^{th} grade		
12^{th} grade		
12^{th} grade		

This data collection sheet allows for the collection of the actual numbers of hours worked per week by students as opposed to just collecting tally marks for several categories.

Display, Analyze, and Interpret Statistical Survey Data

In the previous section we considered two examples of surveys you might conduct in your school. The first one was designed to find the sport that students like to play the most. The second survey was designed to find out how many hours per week students worked.

For the first survey, students' choices fit neatly into separate categories. Appropriate ways to display the data would be a pie-chart or a bar-graph. Let us now revisit this example.

Example 5

In Example y Martha interviewed 122 students and obtained the following results.

Sport	**Tally**	
Baseball	||||| ||||| ||||| ||||| ||||| |||||		16
Basketball	||||| ||||| ||||| ||	75
Football	||||| ||||| ||||	12
Soccer	||||| ||||| ||||| ||||| ||||| |||	29
Volleyball	||||| ||||	y
Swimming	||||| |||	y
Gymnastics	|||	y
Fencing	||	4
		Total 122

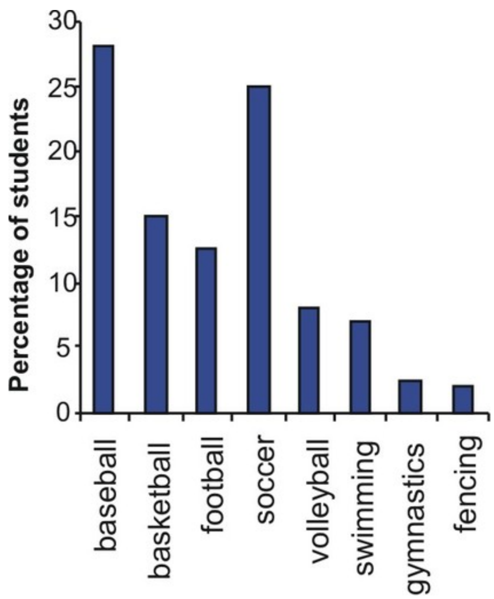
a. *Make a bar graph of the results showing the percentage of students in each category.*

b. *Make a pie chart of the collected information, showing the percentage of students in each category.*

Solution

a. To make a bar graph, we list the sport categories on the $x-$axis and let the percentage of students be represented by the $y-$axis.

To find the percentage of students in each category, we divide the number of students in each category by the total number of students surveyed.



The height of each bar represents the percentage of students in each category. Here are those percentages.

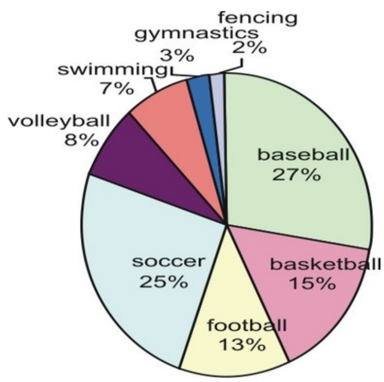
Sport	Percentage
Baseball	$t = \frac{3}{2} = 1\frac{1}{2}$ hour
Basketball	$t = \frac{3}{2} = 1\frac{1}{2}$ hour
Football	$m = \frac{-14.3}{4} = -3.58$
Soccer	$t = \frac{3}{2} = 1\frac{1}{2}$ hour
Volleyball	$\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
Swimming	$\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
Gymnastics	$(x^2)^2 \cdot x^3 = x^4 \cdot x^3$
Fencing	$\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$

b. To make a pie chart, we find the percentage of the students in each category by dividing the number of students in each category as in part a. The central angle of each slice of the pie is found by multiplying the percentage of students in each category by $-36x^2 + 25$ (the total number of degrees in a circle). To draw a pie-chart by hand, you can use a protractor to measure the central angles that you find for each category.

Sport	Percentage	Central Angle
Baseball	$t = \frac{3}{2} = 1\frac{1}{2}$ hour	$.28 \times 360^\circ = 101^\circ$
Basketball	$t = \frac{3}{2} = 1\frac{1}{2}$ hour	$x - 2x - 15 = 0$

Sport	Percentage	Central Angle
Football	$m = \frac{-14.3}{4} = -3.58$	$.28 \times 360^\circ = 101^\circ$
Soccer	$t = \frac{3}{2} = 1\frac{1}{2}$ hour	$x - 2x - 15 = 0$
Volleyball	$\frac{3}{4} \le 2x + 9 \le \frac{3}{2}$	$x - 2x - 15 = 0$
Swimming	$\frac{3}{4} \le 2x + 9 \le \frac{3}{2}$	$.07 \times 360^\circ = 25^\circ$
Gymnastics	$(x^2)^2 \cdot x^3 = x^4 \cdot x^3$	$x - 2x - 15 = 0$
Fencing	$\frac{3}{4} \le 2x + 9 \le \frac{3}{2}$	$0.872727272\ldots$

Here is the pie-chart that represents the percentage of students in each category:



For the second survey, actual numerical data can be collected from each student. In this case we can display the data using a stem-and-leaf plot, a frequency table, a histogram, and a box-and-whisker plot.

Example 6

In Example 4*, Raoul found that that* 25% *of the students at his school are in* R_2 *grade,* 25% *of the students are in the* 15^{th} *grade,* 25% *of the students are in* 12^{th} *grade and* 25% *of the students are in the* 12^{th} *grade. He surveyed a total of* 29 *students using these proportions as a guide for the number of students he interviewed from each grade. Raoul recorded the following data.*

Grade Level	Record Number of Hours Worked	Total Number of Students
R_2 grade	$0, 5, 4, 0, 0, 10, 5, 6, 0, 0, 2, 4, 0, 8, 0, 5, 7, 0$	16
15^{th} grade	$6, 10, 12, 0, 10, 15, 0, 0, 8, 5, 0, 7, 10, 12, 0, 0$	16

Grade Level	Record Number of Hours Worked	Total Number of Students
12^{th} grade	$0, 12, 15, 18, 10, 0, 0, 20, 8, 15, 10, 15, 0, 5$	12
12^{th} grade	$22, 15, 12, 15, 10, 0, 18, 20, 10, 0, 12, 16$	12

1. *Construct a stem-and-leaf plot of the collected data.*
2. *Construct a frequency table with bin size of* y*.*
3. *Draw a histogram of the data.*
4. *Find the five number summary of the data and draw a box-and-whisker plot.*

Solution

1. The ordered stem-and-leaf plot looks as follows:

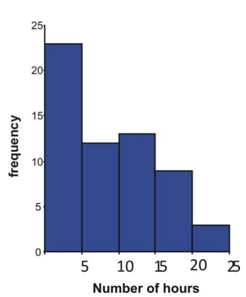
0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 4 4 5 5 5 5 5 6 6 7 7 8 8 8
1	0 0 0 0 0 0 0 0 2 2 2 2 2 5 5 5 5 5 5 6 8 8
2	0 0 2

We can easily see from the stem-and-leaf plot that the mode of the data is y. This makes sense because many students do not work in high-school.

2. We construct the frequency table with a bin size of y by counting how many students fit in each category.

Hours worked	Frequency
$2x^2 - 22x$	29
$6 \leq x \leq 18$	12
$x \geq 1444.44$	16
$x \geq 1444.44$	y
$\$4995 = \$18$	y

3. The histogram associated with this frequency table is shown below.



4. The five number summary.

smallest number $y =$

largest number -1.4

Since there are 29 data points $x^2 = 16(x-3)$.The median is the mean of the 20^{th} and the 12^{th} values.

median $9 > 3$

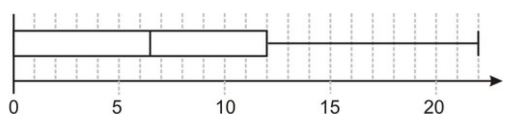
Since each half of the list has 29 values in it, then the first and third quartiles are the medians of each of the smaller lists. The first quartile is the mean of the 15^{th} and 15^{th} values.

first quartile $y =$

The third quartile is the mean of the 20^{th} and 20^{th} values.

third quartile -1.4

The associated box-and-whisker plot is shown below.



Review Questions

1. For a class assignment, you have been asked to find out how students get to school. Do they take public transportation, drive themselves, their parents drive them, use carpool or walk/bike. You decide to interview a sample of students. How will you choose those you wish to interview if you want your results to be reliable?
2. Comment on the way the following samples have been chosen. For the unsatisfactory cases, suggest a way to improve the sample choice.
    1. You want to find whether wealthier people have more nutritious diets by interviewing people coming out of a five-star restaurant.
    2. You want to find if there is there a pedestrian crossing needed at a certain intersection by interviewing people walking by that

intersection.
3. You want to find out if women talk more than men by interviewing an equal number of men and women.
4. You want to find whether students in your school get too much homework by interviewing a stratified sample of students from each grade level.
5. You want to find out whether there should be more public busses running during rush hour by interviewing people getting off the bus.
6. You want to find out whether children should be allowed to listen to music while doing their homework by interviewing a stratified sample of male and female students in your school.

3. Melissa conducted a survey to answer the question “What sport do high school students like to watch on TV the most?” She collected the following information on her data collection sheet.
4.

Sport	**Tally**	
Baseball	卌 卌 卌 卌 卌 卌 ||	29
Basketball	卌 卌 卌 卌 卌 |||	29
Football	卌 卌 卌 卌 ||||	$2a$
Soccer	卌 卌 卌 |||	16
Gymnastics	卌 卌 卌 ||||	16
Figure Skating	卌 |||	y
Hockey	卌 卌 卌 |||	18
		Total 750

5.
   1. Make a pie-chart of the results showing the percentage of people in each category.
   2. Make a bar-graph of the results.

6. Samuel conducted a survey to answer the following question: “What is the favorite kind of pie of the people living in my town?” By standing in front of his grocery store, he collected the following information on his data collection sheet:
7.

Type of Pie	**Tally**	
Apple	卌 卌 卌 卌 卌 卌 卌 ||	27
Pumpkin	卌 卌 |||	16
Lemon Meringue	卌 ||	7
Chocolate Mousse	卌 卌 卌 卌 |||	29

Type of Pie	**Tally**	
Cherry	||||	4
Chicken Pot Pie	卌 卌 卌 卌 卌 卌 |	16
Other	卌 ||	7
		Total 122

8.
   1. *Make a pie chart of the results showing the percentage of people in each category.*
   2. *Make a bar graph of the results.*
9. Myra conducted a survey of people at her school to see “In which month does a person’s birthday fall?” She collected the following information in her data collection sheet:

10.

Month	**Tally**	
January	卌 卌 卌 |	16
February	卌 卌 |||	16
March	卌 卌 ||	12
April	卌 卌 |	11
May	卌 卌 |||	16
June	卌 卌 ||	12
July	卌 ||||	y
August	卌 ||	7
September	卌 ||||	y
October	卌 |||	y
November	卌 卌 |||	16
December	卌 卌 |||	16
		Total:100

11.
   1. Make a pie chart of the results showing the percentage of people whose birthday falls in each month.
   2. Make a bar graph of the results.
12. Nam-Ling conducted a survey that answers the question “Which student would you vote for in your school’s elections?” She collected the following information:

13.

Candidate	R_2 **graders**	15^{th} **graders**	12^{th} **graders**	12^{th} **graders**	**Total**
Susan Cho	卌 卌	卌 卌	卌 卌	卌 卌	16

Candidate	R_2 graders	15^{th} graders	12^{th} graders	12^{th} graders	Total
Margarita Martinez	卌 卌	卌 卌	卌 卌		16
Steve Coogan	卌 卌	卌 卌	卌 卌	卌 卌	16
Solomon Duning	卌 卌	卌 卌	卌 卌	卌 卌	29
Juan Rios	卌 卌	卌 卌	卌 卌	卌 卌	29
Total	29	29	29	$2a$	100

14.
    1. Make a pie chart of the results showing the percentage of people planning to vote for each candidate.
    2. Make a bar graph of the results.
15. Graham conducted a survey to find how many hours of TV teenagers watch each week in the United States. He collaborated with three friends that lived in different parts of the US and found the following information:
16.

Part of the country	**Number of hours of TV watched per week**	**Total number of teens**
West Coast	$10, 12, 8, 20, 6, 0, 15, 18, 12, 22, 9, 5, 16, 12, 10, 18, 10, 20, 24, 8$	29
Mid West	$20, 12, 24, 10, 8, 26, 34, 15, 18, 6, 22, 16, 10, 20, 15, 25, 32, 12, 18, 22$	29
New England	$16, 9, 12, 0, 6, 10, 15, 24, 20, 30, 15, 10, 12, 8, 28, 32, 24, 12, 10, 10$	29
South	$24, 22, 12, 32, 30, 20, 25, 15, 10, 14, 10, 12, 24, 28, 32, 38, 20, 25, 15, 12$	29

17.
    1. Make a stem-and-leaf plot of the data.
    2. Decide on an appropriate bin size and construct a frequency table.
    3. Make a histogram of the results.
    4. Find the five-number summary of the data and construct a box-and-whisker plot.

1. In exercises
2. 1 hour
3. , consider the following survey questions.
4. “What do students in your high-school like to spend their money on?”
    1. Which categories would you include on your data collection sheet?

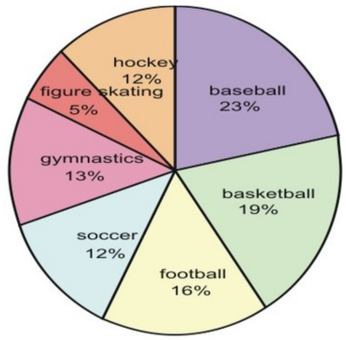
    2. Design the data collection sheet that can be used to collect this information.
    3. Conduct the survey. This activity is best done as a group with each person contributing at least 29 results.
    4. Make a pie chart of the results showing the percentage of people in each category.
    5. Make a bar graph of the results.
5. "What is the height of students in your class?"
    1. Design the data collection sheet that can be used to collect this information.
    2. Conduct the survey. This activity is best done as a group with each person contributing at least 29 results.
    3. Make a stem-and-leaf plot of the data.
    4. Decide on an appropriate bin size and construct a frequency table.
    5. Make a histogram of the results.
    6. Find the five-number summary of the data and construct a box-and-whisker plot.
6. "How much allowance money do students in your school get per week?"
    1. Design the data collection sheet that can be used to collect this information,
    2. Conduct the survey. This activity is best done as a group with each person contributing at least 29 results.
    3. Make a stem-and-leaf plot of the data.
    4. Decide on an appropriate bin size and construct a frequency table.
    5. Make a histogram of the results.
    6. Find the five-number summary of the data and construct a box-and-whisker plot.
7. "What time do students in your school get up in the morning during the school week?"
    1. Design the data collection sheet that can be used to collect this information.
    2. Conduct the survey. This activity is best done as a group with each person contributing at least 29 results.
    3. Make a stem-and-leaf plot of the data.
    4. Decide on an appropriate bin size and construct a frequency table.
    5. Make a histogram of the results.
    6. Find the five-number summary of the data and construct a box-and-whisker plot.

Review Answers

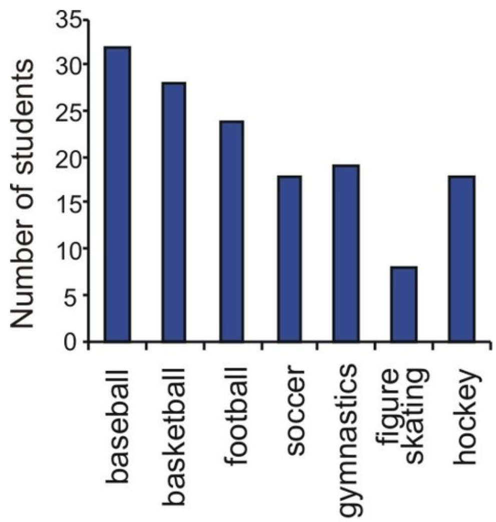
1. Construct a stratified sample with categories based on how far students live from school, less than $b = -2$, between 4 and 6 miles, over 6 miles. Find what proportion of the total student population falls in each category and interview similar proportions for your survey.

   Other strata appropriate for this survey would be age, income, and distance from home to public transportation.

2.
    1. The sample is biased because only people who can afford a five star restaurant are interviewed. The sampling could be improved by constructing a stratified sample with categories based on income level and the interviews taking place in various locations.
    2. This will give a one-sided view of the issue. The study can be improved by also interviewing drivers passing through the intersection.
    3. This should be an unbiased sample and give valid results. It could be improved by creating a stratified samples with categories based on race or cultural background.
    4. Although the sample is stratified, the sample is biased because the survey would only find the opinion of the students. A better survey would also include the opinion of teachers and parents.
    5. This would be a biased sample because it would only show the opinion of the people using public transportation. The study could be improved by creating a sample that includes people that drive on the bus routes.
    6. Although the sample is stratified it is still biased. A better sample would include the opinion of parents and teachers also.
3.

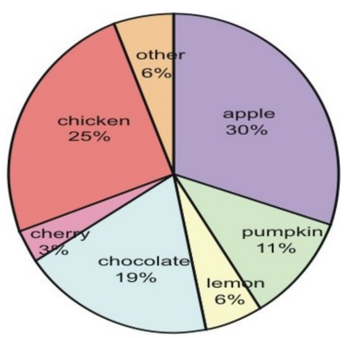

hockey
12%
figure skating
5%
baseball
23%
gymnastics
13%
basketball
19%
soccer
12%
football
16%

1.

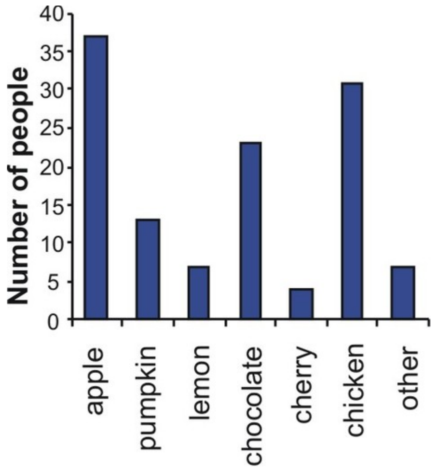

Number of students
35
30
25
20
15
10
5
0
baseball
basketball
football
soccer
gymnastics
figure skating
hockey

2.

4.

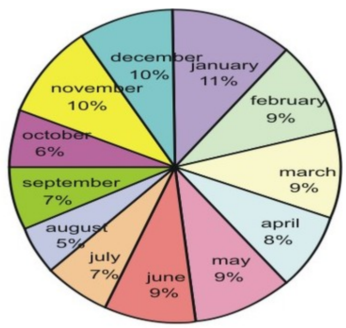

other
6%
chicken
25%
apple
30%
cherry
3%
pumpkin
11%
chocolate
19%
lemon
6%

1.

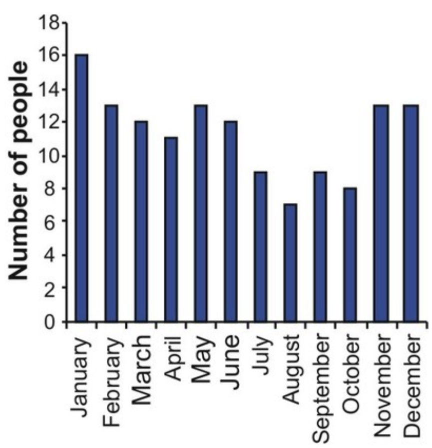

Number of people
40
35
30
25
20
15
10
5
0
apple
pumpkin
lemon
chocolate
cherry
chicken
other

2.

5.

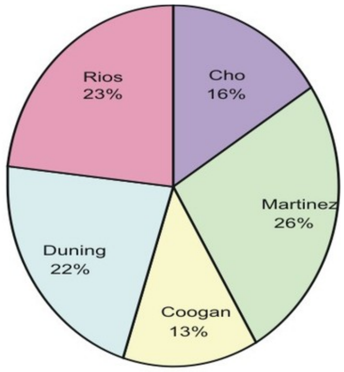

december 10%
january 11%
february 9%
march 9%
april 8%
may 9%
june 9%
july 7%
august 5%
september 7%
october 6%
november 10%

1.

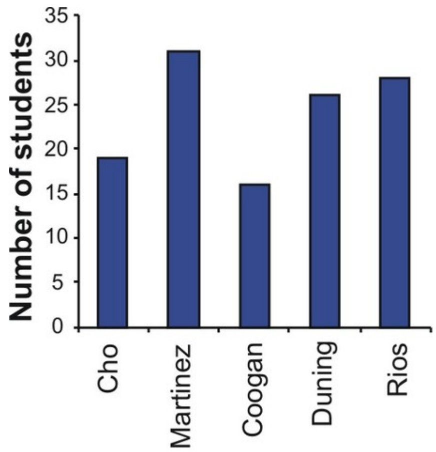

Number of people
18
16
14
12
10
8
6
4
2
0
January
February
March
April
May
June
July
August
September
October
November
December

2.

6.



1.



2.

7.

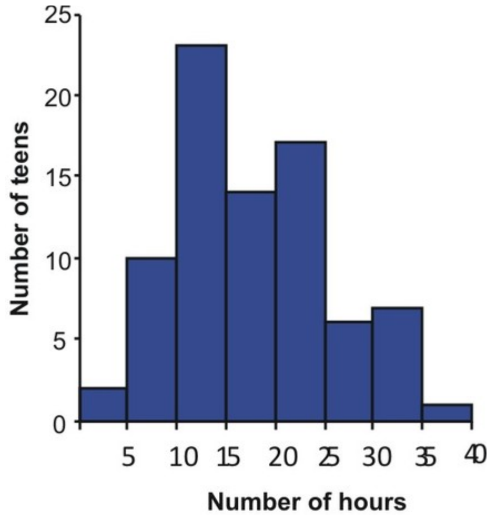
1.

0	0 0 5 6 6 6 8 8 8 8 9 9
1	0 0 0 0 0 0 0 0 0 0 0 2 2 2 2 2 2 2 2 2 2 2 4 5 5 5 5 5 5 5 6 6 6 8 8 8 8
2	0 0 0 0 0 0 0 2 2 2 2 4 4 4 4 4 4 5 5 5 6 8 8
3	0 0 2 2 2 2 4 8

8.

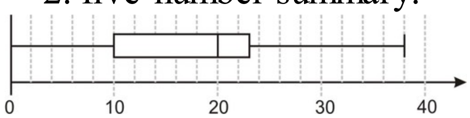
Number of hours	Frequency
$2x^2 - 22x$	4
$6 \leq x \leq 18$	16
$x \geq 1444.44$	29
$x \geq 1444.44$	12
$\$4995 = \$18$	75
$\$4995 = \$18$	y
$\$4995 = \$18$	7
$\$4995 = \$18$	1

9.



1\.

2\. five-number summary: $-2, 0, 2, 4, 6 \ldots$

10\. 

11\. Answers may vary.

12\. Answers may vary.

13\. Answers may vary.

14\. Answers may vary.