Chapter 6: Graphing Linear Inequalities; Introduction to Probability

Inequalities Using Addition and Subtraction

Learning Objectives

- Write and graph inequalities in one variable on a number line.
- Solve an inequality using addition.
- Solve an inequality using subtraction.

Introduction

Inequalities are similar to equations in that they show a relationship between two expressions. We solve and graph inequalities in a similar way to equations. However, there are some differences that we will talk about in this chapter. The main difference is that for linear inequalities the answer is an interval of values whereas for a linear equation the answer is most often just one value.

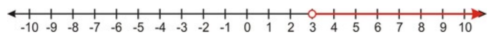
When writing inequalities we use the following symbols

$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

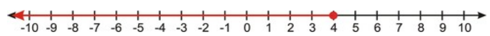
Let's start with the simple inequality $x = 3$

We read this inequality as "x is greater than y". The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality by a number line graph.



Consider another simple inequality $= \$21$

We read this inequality as "x is less than or equal to 4". The solution is the set of all real numbers that equal four or less than four. We graph this solution set on the number line.



In a graph, we use an empty circle for the endpoint of a strict inequality (H_2O_2) and a filled circle if the equal sign is included (H_2O_2).

Example 1

Graph the following inequalities on the number line.

a) $x = -5$

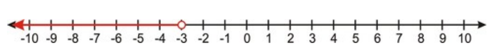
b) $= \$21$

c) $x = 3$

d) $= \$21$

Solution

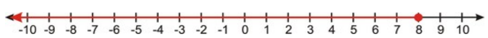
a) The inequality $x = -5$ represents all real numbers that are less than -8. The number -8 is not included in the solution and that is represented by an open circle on the graph.



b) The inequality $= \$21$ represents all real numbers that are greater than or equal to six. The number six is included in the solution and that is represented by a closed circle on the graph.

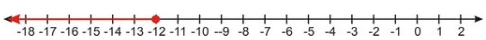
c) The inequality $x = 3$ represents all real numbers that are greater than zero. The number zero is not included in the solution and that is represented by an open circle on the graph.

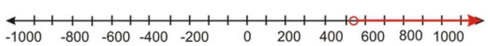
d) The inequality $= \$21$ represents all real numbers that are less than or equal to eight. The number eight is included in the solution and that is represented by a closed circle on the graph.

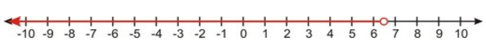


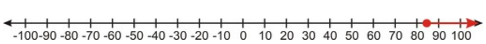
Example 2

Write the inequality that is represented by each graph.

a) 

b) 

c) 

d) 

Solution:

d) $R\%$
c) $k = 12$
b) $x = 250$
a) $x \leq -12$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

Example 3

Write each statement as an inequality and graph it on the number line.

a) You must maintain a balance of at least $\$5000$ in your checking account to get free checking.

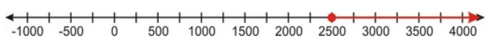
b) You must be at least 29 inches tall to ride the “Thunderbolt” Rollercoaster.

c) You must be younger than y years old to get free admission at the San Diego Zoo.

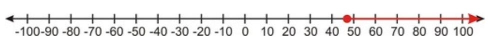
d) The speed limit on the interstate is $-9x+2$ per hour.

Solution:

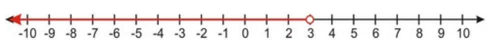
a) The inequality is written as $x \geq 2500$. The words “at least” imply that the value of $\$5000$ is included in the solution set.



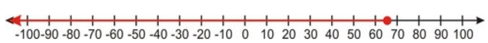
b) The inequality is written as $x \leq 65$. The words “at least” imply that the value of 29 inches is included in the solution set.



c) The inequality is written as $x = 3$.



d) Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.



Solve an Inequality Using Addition

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations. For inequalities of this type:

$5+0=5$ or $5+0=5$

We isolate the x by adding the constant a to both sides of the inequality.

Example 4

Solve each inequality and graph the solution set.

a) $3x - 2 = 5$

b) $-5.0 = -5.0$

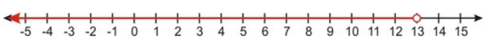
c) $x - 1 \leq -5$

d) $m_1 m_2 = -1$

Solution:

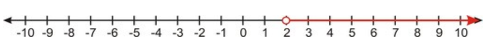
a)

$$\begin{aligned} \text{To solve the inequality} \quad & x - 3 < 10 \\ \text{Add 3 to both sides of the inequality.} \quad & x - 3 + 3 < 10 + 3 \\ \text{Simplify} \quad & x < 13 \end{aligned}$$



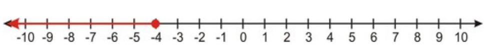
b)

$$\begin{aligned} \text{To solve the inequality} \quad & x - 1 > -10 \\ \text{Add 12 to both sides of the inequality} \quad & x - 12 + 12 > -10 + 12 \\ \text{Simplify} \quad & x > 2 \end{aligned}$$



c)

$$\begin{aligned} \text{To solve the inequality} \quad & x - 3 < 10 \\ \text{Add 3 to both sides of the inequality.} \quad & x - 3 + 3 < 10 + 3 \\ \text{Simplify} \quad & x < 13 \end{aligned}$$



d)

$$\begin{aligned} \text{To solve the inequality} \quad & x - 20 \leq 14 \\ \text{Add 20 to both sides of the inequality :} \quad & x - 20 + 20 \leq 14 + 20 \\ \text{Simplify} \quad & x \leq 34 \end{aligned}$$

Solve an Inequality Using Subtraction

For inequalities of this type:

$x + 1 < b$ or $x + 1 < b$

We isolate the x by subtracting the constant a on both sides of the inequality.

Example 5

Solve each inequality and graph the solution set.

a) $5 + n \leq 2$

b) $y = -0.025$

c) 60 minutes

d) $x + 5 \geq \frac{3}{4}$

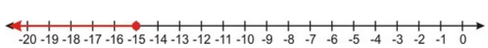
Solution:

a)

$$\begin{aligned} \text{To solve the inequality} \quad & x + 2 < 7 \\ \text{Subtract 2 on both sides of the inequality} \quad & x + 2 - 2 < 7 - 2 \\ \text{Simplify to obtain} \quad & x < 5 \end{aligned}$$

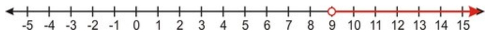
b)

$$\begin{aligned} \text{To solve the inequality} \quad & x + 8 \leq -7 \\ \text{Subtract 8 on both sides of the inequality} \quad & x + 8 - 8 \leq -7 - 8 \\ \text{Simplify to obtain :} \quad & x < -15 \end{aligned}$$



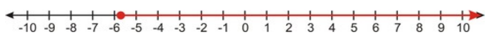
c)

$$\begin{aligned} \text{To solve the inequality} \quad & x + 4 > 13 \\ \text{Subtract 4 on both sides of the inequality} \quad & x + 4 - 4 > 13 - 4 \\ \text{Simplify} \quad & x > 9 \end{aligned}$$



d)

$$\begin{aligned} \text{To solve the inequality} \quad & x + 5 \geq \frac{3}{4} \\ \text{Subtract 5 on both sides of the inequality} \quad & x + 5 - 5 \geq -\frac{3}{4} - 5 \\ \text{Simplify to obtain :} \quad & x \geq -5\frac{3}{4} \end{aligned}$$



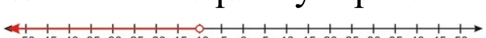
Lesson Summary

- The answer to an **inequality** is often an **interval of values**. Common **inequalities** are:
- $\div$ is greater than
- $\geq$ is greater than or equal to
- $\div$ is less than
- $\geq$ is less than or equal to
- Solving inequalities with **addition** and **subtraction** works just like solving an equation. To solve, we isolate the variable on one side of the equation.

Review Questions

1. Write the inequality represented by the graph.

2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.

4. Write the inequality represented by the graph.

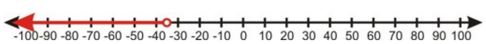
Graph each inequality on the number line.

1. $x = 0.02$
2. $5 \quad \text{dimes}$
3. $x \leq 65$
4. $= \$21$

Solve each inequality and graph the solution on the number line.

1. $3x - 2 = 5$
2. $x + 15 \geq -60$
3. $x + 1 < b$
4. $-5.0 = -5.0$
5. $3x + 1 = 10$
6. 0.0023 inches
7. $4x + 5 \leq 8$
8. $y = 15 + 5x$

Review Answers

1. $\$5000$
2. $x = 0.02$
3. $x \leq -12$
4. $k = 12$
5. 

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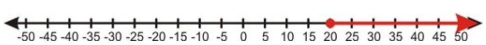
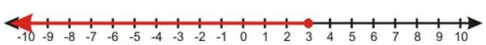
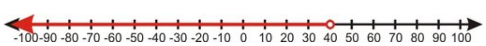
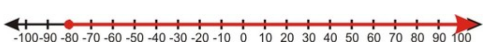
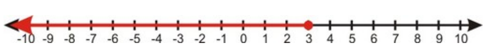
7. 

8. 

9. 

10. 

11. 

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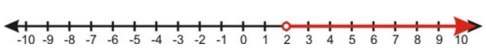
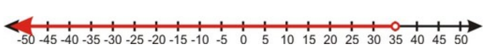
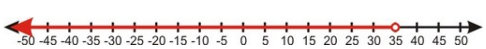
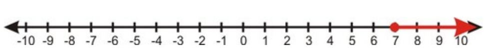
13. 

14. 

15. 

16. 

Inequalities Using Multiplication and Division

Learning Objectives

- Solve an inequality using multiplication.
- Solve an inequality using division.
- Multiply or divide an inequality by a negative number.

Introduction

In this section, we consider problems where we find the solution of an inequality by multiplying or dividing both sides of the inequality by a number.

Solve an Inequality Using Multiplication

$$\begin{aligned} \text{Consider the problem} \qquad & \frac{x}{5} \le 3 \\ \text{To find the solution we multiply both sides by 5.} \qquad & 5 \cdot \frac{x}{5} \le 3.5 \\ \text{We obtain} \qquad & x \le 15 \end{aligned}$$

The answer of an inequality can be expressed in four different ways:

1. **Inequality notation** The answer is simply expressed as $k = 12$.
2. **Set notation** The answer is $\{5x^2 - 4y\}$. You read this as "the set of all values of x, such that x is a real number less than 16".
3. **Interval notation** uses brackets to indicate the range of values in the solution.

   The interval notation solution for our problem is $11(2+6)$.
   Interval notation also uses the concept of **infinity** ∞ and **negative infinity** $-\infty$.
   Round or **open brackets** "(" and ")" indicate that the number next to the bracket is not included in the solution set.
   Square or **closed brackets** "[" and "]" indicate that the number next to the bracket is included in the solution set.

1. **Solution graph** shows the solution on the real number line. A closed circle on a number indicates that the number is included in the solution set. While an open circle indicates that the number is not included in the set. For our example, the solution graph is drawn here.

Example 1

a) $|5+4|$ says that the solutions is all numbers between -2 and y **including** -2 and y.

b) (mph) says that the solution is all numbers between y and $2a$ but **does not include** the numbers y and $2a$.

c) $[3, 12)$ says that the solution is all numbers between y and 12, **including** y but **not including** 12.

d) $(3+2)$ says that the solution is all numbers greater that -53, **not including** -53.

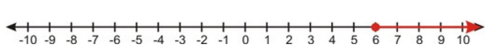
e) $\frac{x}{100}$ says that the solution is all real numbers.

Solving an Inequality Using Division

Consider the problem.	$2x \geq 15$
To find the solution we multiply both sides by 2.	$\frac{2x}{2} \geq \frac{12}{2}$
We obtain.	$x \geq 6$

Let's write the solution in the four different notations you just learned:

Inequality notation	$x \geq 6$
Set notation	$x \vert x \geq 6$
Interval notation	$[6, \infty)$

Solution graph 

Multiplying and Dividing an Inequality by a Negative Number

We solve an inequality in a similar way to solving a regular equation. We can add or subtract numbers on both sides of the inequality. We can also multiply or divide **positive** numbers on both sides of an inequality without changing the solution.

Something different happens if we multiply or divide by **negative** numbers. **In this case, the inequality sign changes direction**.

For example, to solve $x = 0.02$

We divide both sides by y. The inequality sign changes from $\div$ to $\div$ because we divide by a negative number. $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$

$x = -5$

We can explain why this happens with a simple example. We know that two is less than three, so we can write the inequality.

$9 > 3$

If we multiply both numbers by -1 we get -2 and -8, but we know that -2 is greater than -8.

$-9 = -9$

You see that multiplying both sides of the inequality by a negative number caused the inequality sign to change direction. This also occurs if we divide by a negative number.

Example 2

Solve each inequality. Give the solution in inequality notation and interval notation.

a) 22 coins

b) $\frac{1}{8} \cdot x = 1.5$

c) $x + 1 < b$

d) $Dx = -4$

Solution:

a)

$$\begin{array}{rl}
\text{Original problem.} & 4x < 24 \\
\text{Divide both sides by 4.} & \dfrac{4x}{4} < \dfrac{24}{4} \\
\text{Simplify} & x < 6 \text{ or } (\infty, 6) \text{ Answer}
\end{array}$$

b)

$$\begin{array}{ll}
\text{Original problem :} & -9x\dfrac{-3}{5} \\
\text{Divide both sides by } -9. & \dfrac{-9}{-9} \leq \dfrac{\cancel{-3}}{5} \cdot \dfrac{1}{\cancel{-9}_3} \text{ Direction of the inequality is changed} \\
\text{Simplify.} & x \geq \dfrac{1}{15} \text{ or } \left[\dfrac{1}{15}, \infty\right) \text{ Answer}
\end{array}$$

$$\begin{array}{ll}
\text{Original problem :} & -5x \leq 21 \\
\text{Divide both sides by } -5. & \dfrac{-5x}{-5} \geq \dfrac{21}{-5} \text{ Direction of the inequality is changed} \\
\text{Simplify.} & x \geq -\dfrac{21}{5} \text{ or } \left[\dfrac{21}{5}, \infty\right) \text{ Answer}
\end{array}$$

d)

$$\begin{array}{rl}
\text{Original problem} & 12x > -30 \\
\text{Divide both sides by 12.} & \dfrac{12x}{12} > \dfrac{-30}{12} \\
\text{Simplify.} & x > -\dfrac{5}{5} \text{ or } \left(\dfrac{-5}{2}, \infty\right) \text{ Answer}
\end{array}$$

Example 3

Solve each inequality. Give the solution in inequality notation and solution graph.

a) $\sqrt{2000}$

b) $\frac{x}{2} - \frac{x}{3} = 6$

c) $c = \frac{22}{35}$

d) $(3 + 2)$

Solution

a)

$$\text{Original problem} \quad \frac{x}{2} > 40$$

$$\text{Multiply both sides by 2.} \quad 2 \cdot \frac{x}{2} > 40 \cdot 2 \text{ Direction of inequality is NOT changed}$$

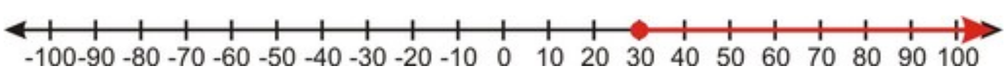
$$\text{Simplify.} \quad x > -80 \text{ Answer}$$

b)

$$\text{Original problem} \quad \frac{x}{-3} \leq -12$$

$$\text{Multiply both sides by } -3. \quad -3 \cdot \frac{x}{-3} \geq -12 \cdot (-3) \text{ Direction of inequality is changed}$$

$$\text{Simplify.} \quad x \geq 36 \text{ Answer}$$

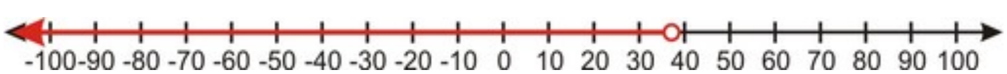


c)

$$\text{Original problem} \quad \frac{x}{25} < \frac{3}{2}$$

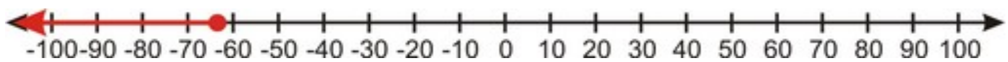
$$\text{Multiply both sides by 25.} \quad 25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25 \text{ direction of inequality is NOT changed}$$

$$\text{Simplify.} \quad x < \frac{75}{2} \text{ or } x < 37.5 \text{ Answer}$$



d)

$$\begin{aligned} \text{Original problem} \quad & \frac{x}{-7} \ge 9 \\ \text{Multiply both sides by } -7. \quad & -7 \cdot \frac{x}{-7} \le 9 \cdot (-7) \text{ Direction of inequality is changed} \\ \text{Simplify.} \quad & x \le -63 \text{ Answer} \end{aligned}$$



Lesson Summary

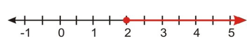
- There are four ways to represent an inequality:

1. ***Equation notation*** $= \$21$

2. ***Set notation*** $= \$21$

3. ***Interval notation*** 3 m/s

Closed brackets "[" and "]" mean inclusive, parentheses "("and ")" mean exclusive.

4. ***Solution graph*** 

- When multiplying or dividing both sides of an inequality by a negative number, you need to ***reverse the inequality***.

Review Questions

Solve each inequality. Give the solution in inequality notation and solution graph.

1. $x \le 65$
2. $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
3. $2n - 9 = 33$
4. $(-5, -7)$

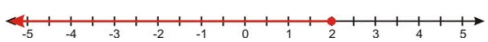
Solve each inequality. Give the solution in inequality notation and interval notation.

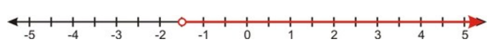
1. $\text{slope} = \frac{\text{rise}}{\text{run}}$
2. $f(x) = |x|$
3. $620x > 2400$
4. $\frac{1.25}{7} = \frac{3.6}{x}$

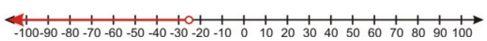
Solve each inequality. Give the solution in inequality notation and set notation.

1. $0, 1, 2, 3, 4, 5$
2. $75x \geq 125$
3. $t + \frac{1}{2} = \frac{1}{3}$
4. (-11.5)

Review Answers

1. $= \$21$ or 

2. $x = -\frac{1}{2}$ or 

3. $x = 0.02$ or 

4. $x \leq 65$ or 

5. $x > -\frac{1}{12}$ or $\frac{1}{2}(4z + 6)$
6. $y = -2x$ or $[-75, \infty)$
7. $x = -5$ or $\frac{x}{2} - \frac{x}{3} = 6$
8. $x = -\frac{1}{2}$ or $y = \frac{1}{2}x^2$
9. $x \leq -12$ or $\{x$ is a real number 76.44 m/s
10. $x = \frac{2}{3}$ or $\{x$ is a real number $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$
11. $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$ or $\{x$ is a real number $\text{slope} = \frac{2}{3}$
12. 2 minutes or $\{x$ is a real number $f(x) = \sqrt{x}$

Multi-Step Inequalities

Learning Objectives

- Solve a two-step inequality.
- Solve a multi-step inequality.
- Identify the number of solutions of an inequality.
- Solve real-world problems using inequalities.

Introduction

In the last two sections, we considered very simple inequalities which required one-step to obtain the solution. However, most inequalities require several steps to arrive at the solution. As with solving equations, we must use the order of operations to find the correct solution. In addition **remember that when we multiply or divide the inequality by a negative number the direction of the inequality changes**.

The general procedure for solving multi-step inequalities is as follows.

1. Clear parenthesis on both sides of the inequality and collect like terms.

2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.

3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.

Solve a Two-Step Inequality

Example 1

Solve each of the following inequalities and graph the solution set.

a)$2n - 9 = 33$

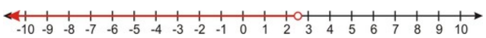
b)$-9x < -5x - 15$

c) $\text{slope} = \frac{2}{3}$

Solution

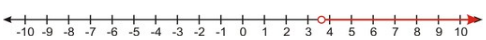
a)

$$\begin{aligned}
\text{Original problem :} \quad & 6x - 5 < 10 \\
\text{Add 5 to both sides : .} \quad & 6x - 5 + 5 < 10 + 5 \\
\text{Simplify.} \quad & 6x < 15 \\
\text{Divide both sides by 6.} \quad & \frac{6x}{6} < \frac{15}{6} \\
\text{Simplify.} \quad & x < \frac{5}{2} \text{ Answer}
\end{aligned}$$



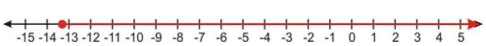
b)

$$\begin{aligned}
\text{Original problem.} \quad & -9x \le -5x - 15 \\
\text{Add } 5x \text{ to both sides.} \quad & -9x + 5x \le -5x + 5x - 15 \\
\text{Simplify.} \quad & -4x < -15 \\
\text{Divide both sides by } -4. \quad & \frac{-4x}{-4} > \frac{-15}{-4} \text{ Inequality sign was flipped} \\
\text{Simplify.} \quad & x > \frac{15}{4} \text{ Answer}
\end{aligned}$$



c)

$$\begin{aligned}
\text{Original problem.} \quad & -9x \le 24 \\
\text{Multiply both sides by 5.} \quad & \frac{-9x}{5}.5 \le 24.5 \\
\text{Simplify.} \quad & -9x \le 120 \\
\text{Divide both sides by} - 9. \quad & \frac{-9x}{-9} > \frac{120}{-9} \text{ Inequality sign was flipped} \\
\text{Simplify.} \quad & x \ge -\frac{40}{3} \text{ Answer}
\end{aligned}$$



Solve a Multi-Step Inequality

Example 2

Each of the following inequalities and graph the solution set.

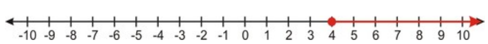
a) $y - y_0 = \frac{3}{5}(x - x_0)$

b) $-25x + 12 \leq -10x - 12$

Solution

a)

$$\begin{array}{rl}
\text{Original problem} & \dfrac{9x}{5} - 7 \geq -3x + 12 \\
\text{Add } 3x \text{ to both sides.} & \dfrac{9x}{5} + 3x - 7 \geq -3x + 3x + 12 \\
\text{Simplify.} & \dfrac{24x}{5} - 7 \geq 12 \\
\text{Add 7 to both sides.} & \dfrac{24x}{5} - 7 + 7 \geq 12 + 7 \\
\text{Simplify.} & \dfrac{24x}{5} - 7 \geq 19 \\
\text{Multiply 5 to both sides.} & 5 \cdot \dfrac{24x}{5} \geq 5.19 \\
\text{Simplify.} & 24x \geq 95 \\
\text{Divide both sides by 24.} & \dfrac{24x}{24} \geq \dfrac{95}{24} \\
\text{Simplify.} & x \geq \dfrac{95}{24} \text{ Answer}
\end{array}$$



b)

$$\begin{array}{rl}
\text{Original problem} & -25x + 12 \leq -10x - 12 \\
\text{Add 5 to both sides.} & -25x + 10x + 12 \leq -10x + 10x - 12 \\
\text{Simplify.} & -15x + 12 \leq -12 \\
\text{Subtract 12 from both sides.} & -15x + 12 - 12 \leq -12 - 12 \\
\text{Simplify.} & -15x \leq -24 \\
\text{Divide both sides by } -15. & \dfrac{-15x}{-15} \geq \dfrac{-24}{-15} \text{ Inequality sign was flipped} \\
\text{Simplify.} & x \geq \dfrac{8}{5} \text{ Answer}
\end{array}$$

Example 3

Solve the following inequalities.

a) $4x - 2(3x - 9) \leq -4(2x - 9)$

b) $\frac{5x-1}{4} > -2(x + 5)$

Solution

a)

$$\begin{array}{rc}
\text{Original problem} & 4x - 2(3x - 9) \leq -4(2x - 9) \\
\text{Simplify parentheses.} & 4x - 6x + 18 \leq -8x + 36 \\
\text{Collect like terms.} & --2x + 18 \leq -8x + 36 \\
\text{Add } 8x \text{ to both sides.} & -2x + 8x + 18 \leq -8x + 8x + 36 \\
\text{Simplify.} & -6x + 18 \leq 36 \\
\text{Subtract 18 from both sides.} & -6x + 18 - 18 \leq 36 - 18 \\
\text{Simplify.} & 6x \leq 18 \\
\text{Divide both sides by 6.} & \dfrac{6x}{6} \leq \dfrac{18}{6} \\
\text{Simplify.} & x \leq 3 \text{ Answer}
\end{array}$$

b)

$$\begin{array}{rc}
\text{Original problem} & \dfrac{5x-1}{4} > -2(x + 5) \\
\text{Simplify parenthesis.} & \dfrac{5x-1}{4} > -2x - 10 \\
\text{Multiply both sides by 4.} & 4.\dfrac{5x-1}{4} > 4(-2x - 10) \\
\text{Simplify.} & 5x - 1 > -8x - 40 \\
\text{Add } 8x \text{ to both sides.} & 5x + 8x - 1 > -8x + 8x - 40 \\
\text{Simplify.} & 13x - 1 > -40 \\
\text{Add 1 to both sides.} & 13 - 1 + 1 > -40 + 1 \\
\text{Simplify.} & 13x > -39 \\
\text{Divide both sides by 13.} & \dfrac{13x}{13} > -\dfrac{39}{13} \\
\text{Simplify.} & x > -3 \text{ Answer}
\end{array}$$

Identify the Number of Solutions of an Inequality

Inequalities can have:

- A set that has an infinite number of solutions.
- No solutions
- A set that has a discrete number of solutions.

Infinite Number of Solutions

The inequalities we have solved so far all have an infinite number of solutions. In the last example, we saw that the inequality

$\frac{5x-1}{4} > -2(x+5)$ has the solution $x = -5$

This solution says that all real numbers greater than -8 make this inequality true. You can see that the solution to this problem is an infinite set of numbers.

No solutions

$$\begin{aligned} \text{Consider the inequality} \quad & x - 5(> x + 6 \\ \text{This simplifies to} \quad & -5 > 6 \end{aligned}$$

This statements is not true for any value of x. We say that this inequality has no solution.

Discrete solutions

So far we have assumed that the variables in our inequalities are real numbers. However, in many real life situations we are trying to solve for variables that represent integer quantities, such as number of people, number of cars or number of ties.

Example 4

Raul is buying ties and he wants to spend $\$100$ *or less on his purchase. The ties he likes the best cost* $\$12$*. How many ties could he purchase?*

Solution

Let x = the number of ties Raul purchases.

We can write an inequality that describes the purchase amount using the formula.

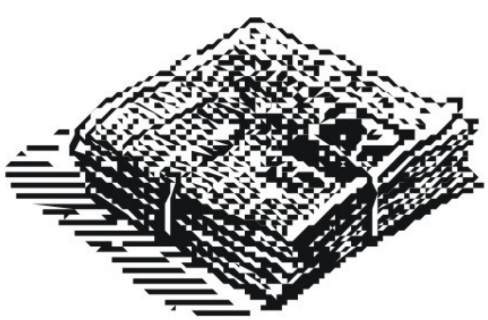
$(\text{number of ties}) \ \times \ (\text{price of a tie}) \leq \$200$ or $50x \leq 200$

We simplify our answer. $= \$21$

This solution says that Raul bought four or less ties. Since ties are discrete objects, the solution set consists of five numbers $\{13, \ , \ , \ ,0\}$.

Solve Real-World Problems Using Inequalities

Sometimes solving a word problem involves using an inequality.



Example 5

In order to get a bonus this month, Leon must sell at least 100 *newspaper subscriptions. He sold* 29 *subscriptions in the first three weeks of the month. How many subscriptions must Leon sell in the last week of the month?*

Solution

Step 1

We know that Leon sold 29 subscriptions and he must sell at least 100 subscriptions.

We want to know the least amount of subscriptions he must sell to get his bonus.

Let $x =$ the number of subscriptions Leon sells in the last week of the month.

Step 2

The number of subscriptions per month must be greater than 100.

We write

$P = 2\mathrm{l} + 2\mathrm{w}.$

Step 3

We solve the inequality by subtracting 29 from both sides $x \leq 65$

Answer Leon must sell 29 or more subscriptions in the last week to get his bonus.

Step 4:

To check the answer, we see that $85 + 35 = 120$. If he sells 29 or more subscriptions the number of subscriptions sold that month will be 100 or more.

Example 6

Virena's Scout Troup is trying to raise at least $\$100$ *this spring. How many boxes of cookies must they sell at* $\$0.50$ *per box in order to reach their goal?*

Solution

Step 1

Virena is trying to raise at least $\$100$

Each box of cookies sells for $\$0.50$

Let $x =$ number of boxes sold

The inequality describing this problem is:

$21 + 7 = 28$

Step 3

We solve the inequality by dividing both sides by $-5x$

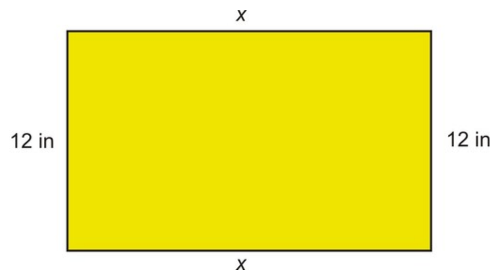
$x \geq 1444.44$

Answer We round up the answer to 100 since only whole boxes can be sold.

Step 4

If we multiply 100 by $\$0.50$ we obtain $2a + 3b$. If Virena's Troop sells more than 100 boxes, they raise more that $\$100$.

The answer checks out.



Example 7

The width of a rectangle is $h = 8 \text{ cm}$. *What must the length be if the perimeter is at least* $2x + 25 =$?

Solution

Step 1

width = $h = 8 \text{ cm}$

Perimeter is at least $2x + 25 =$

What is the smallest length that gives that perimeter?

Let $x =$ length of the rectangle

Step 2

Formula for perimeter is $\text{Perimeter } = 2 \times \text{ length } + 2 \times \text{ width}$

Since the perimeter must be at least $2x + 25 =$, we have the following equation.

$4 - [7 - (11 + 2)]$

Step 3

We solve the inequality.

Simplify.

$2x + 40 \geq 180$

Subtract 29 from both sides.

$2x \geq 140$

Divide both sides by 4.

$p = 15$

Answer The length must be at least 18 inches .

Step 4

If the length is at least 75 inches and the width is $h = 8 \text{ cm}$, then the perimeter can be found by using this equation.

$2(70) + 2(20) = 180 \text{ inches}$

The answer check out.

Lesson Summary

- The **general procedure** for solving multi-step inequalities is as follows.

  3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.
  2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.
  1. Clear parentheses on both sides of the inequality and collect like terms.

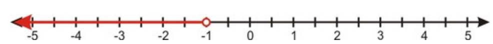
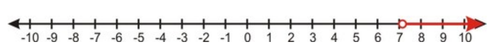
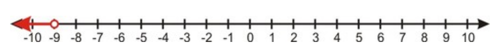
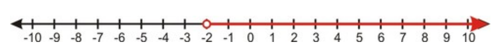
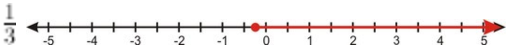
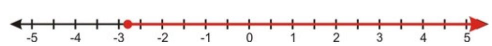
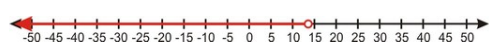
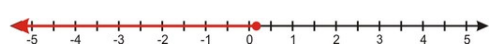
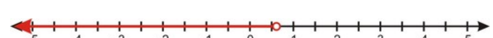
- Inequalities can have **multiple solutions, no solutions**, or **discrete solutions**.

Review Questions

Solve the following inequalities and give the solution in set notation and show the solution graph.

1. $4a + 3 = -9$
2. $r = 17$ inches
3. $a + b = b + a$
4. $5 \times 1\ lb = 5lb$
5. $4 - 6x \leq 2(2x + 3)$
6. $5(4x + 3) \geq 9(x - 2) - x$
7. Steel required $= 8(4 + 5)$ feet.
8. $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$
9. $\text{speed}(6) - \text{speed}(2) = 9 - 3 = 6\ \text{m/s}$
10. $\frac{2}{3}x - \frac{1}{2}(4x - 1) \geq x + 2(x - 3)$
11. At the San Diego Zoo, you can either pay $\$11.95$ for the entrance fee or $\$21$ for the yearly pass which entitles you to unlimited admission. At most how many times can you enter the zoo for the $\$11.95$ entrance fee before spending more than the cost of a yearly membership?
12. Proteek's scores for four tests were $82, 95, 86$ and 29. What will he have to score on his last test to average at least 29 for the term?

Review Answers

1. $\{\ x \mid x$ is a real number, $-|7 - 22|$ 
2. $\{\ x \mid x$ is a real number, $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$ 
3. $\{\ x \mid x$ is a real number, $-|7 - 22|$ 
4. $\{\ x \mid x$ is a real number, $-|7 - 22|$ 
5. $\{\ x \mid x$ is a real number, $t + \frac{1}{2} = \frac{1}{3}$ 
6. $\{\ x \mid x$ is a real number, $11 \div \left(-\frac{x}{4}\right)$ 
7. $\{\ x \mid x$ is a real number, $\text{speed}(2)$ 
8. $\{\ x \mid x$ is a real number, $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$ 
9. $\{\ x \mid x$ is a real number, $\frac{2 \cdot 4}{9 \cdot 4} = \frac{8}{36}$ 
10. $\{\ x \mid x$ is a real number, $x = -\frac{1}{2}$ 
11. At most y times.
12. At least 29.

Compound Inequalities

Learning Objectives

- Write and graph compound inequalities on a number line.
- Solve a compound inequality with *"and"*.
- Solve a compound inequality with *"or"*.
- Solve compound inequalities using a graphing calculator (TI family).
- Solve real-world problems using compound inequalities

Introduction

In this section, we will solve compound inequalities. In previous sections, we obtained solutions that gave the variable either as greater than or as less than a number. In this section we are looking for solutions where the variable can be in two or more intervals on the number line.

There are two types of compound inequalities:

1. Inequalities joined by the word *"and"*.

The solution is a set of values greater than a number *and* less than another number.

5 minutes

In this case we want values of the variable for which *both* inequalities are true.

2. Inequalities joined by the word *"or"*.

The solution is a set of values greater than a number or less than another number.

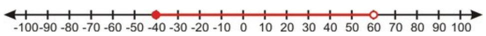
$x < a$ or $x + 9$

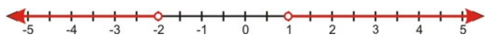
In this case, we want values for the variable in which *at least one* of the inequalities is true.

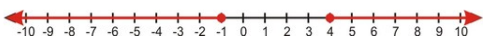
Write and Graph Compound Inequalities on a Number Line

Example 1

Write the inequalities represented by the following number line graphs.

a) 

b) 

c) 

d) 

Solution

a) The solution graph shows that the solution is any value between -53 and 29, including -53 but not 29. Any value in the solution set satisfies both inequalities.

$x \leq -12$ and $k = 12$

This is usually written as the following compound inequality.

$-40 \leq x < 60$

b) The solution graph shows that the solution is any value greater than 1 (not including 1) or any value less than -2 (not including -2). You can see that there can be no values that can satisfy both these conditions at the same time. We write:

$9 > 3$ or $x = -5$

c) The solution graph shows that the solution is any value greater than 4 (including 4) or any value less than -1 (including -1). We write:

$= \$21$ or $2a + 3b$

d) The solution graph shows that the solution is any value less than 29 (not including 29) and any value greater than -53 (not including -53). Any value in the solution set satisfies both conditions.

$x = 0.02$ and $k = 12$

This is usually written as $-25 < x < 25$.

Example 2

Graph the following compound inequalities on the number line.

a) $-4 \le x \le 6$

b) $x = 3$ or $x = 3$

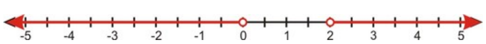
c) $2l + 2w$ or $x \le -12$

d) $-40 \le x < 60$

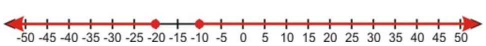
Solution

a) The solution is all numbers between -2 and y including both -2 and y.

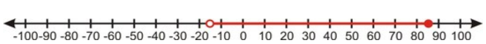
b) The solution is either numbers less than y or numbers greater than 4 not including y or 4.



c) The solution is either numbers greater than or equal to -8 or less than or equal to -53.



d) The solution is numbers between -53 and 29, not including -53 but including 29.



Solve a compound Inequality With *"and"*

When we solve compound inequalities, we separate the inequalities and solve each of them separately. Then, we combine the solutions at the end.

Example 3

Solve the following compound inequalities and graph the solution set.

a) $0.75x + 2.35 = 10$

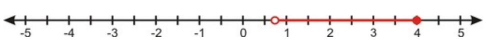
b) $0.25972 \cdot \$120 = \$31.1664$

Solution

a) First, we rewrite the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{lcl} -2 < 4x - 5 & & 4x - 5 \leq 11 \\ \;\;3 < 4x & \text{and} & 4x \leq 16 \\ \;\;\dfrac{3}{4} < x & & x \leq 4 \end{array}$$

Answer $y = \frac{3}{2}$ and $= \$21$. This can be written as $\frac{6}{11} - \frac{3}{22}$.



b) Rewrite the compound inequality as two separate inequalities by using *and*. Then solve each inequality separately.

$$\begin{array}{lcl} 3x - 5 < x + 9 & & x + 9 \leq 5x + 13 \\ \qquad 2x < 14 & \text{and} & -4 \leq 4x \\ \qquad\; x < 7 & & -1 \leq x \text{ or } x \geq -1 \end{array}$$

Answer $x < 7 >$ and $2a + 3b$. This can be written as $y = mx + 2$.

Solve a Compound Inequality With "or"

Consider the following example.

Example 4

Solve the following compound inequalities and graph the solution set.

a) $4x + 5 \leq 8$ or $3x + 10 \leq 6 - x$

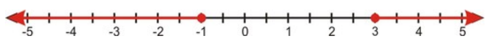
b) $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$ or $3x + 2 = \frac{5x}{3}$

Solution

a) Solve each inequality separately.

$$\begin{array}{lcl} 9 - 2x \leq 3 & & 3x + 10 \leq 6 - x \\ \quad -2x \leq -6 & \text{or} & 4x \leq -4 \\ \qquad x \geq 3 & & x \leq -1 \end{array}$$

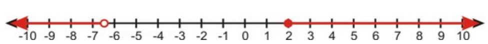
Answer $=$ $\$21$ or $2a + 3b$



b) Solve each inequality separately.

$$\begin{array}{lcl} \dfrac{x-2}{6} \leq 2x - 4 & & \dfrac{x-2}{6} > x + 5 \\ x - 2 \leq 6(2x - 4) & & x - 2 > 6(x + 5) \\ x - 2 \leq 12x - 24 & \text{or} & x - 2 > 6x + 30 \\ \quad 22 \leq 11x & & -32 > 5x \\ \quad\; 2 \leq x & & -6.4 > x \end{array}$$

Answer $=$ $\$21$ or $x = 0.02$



Solve Compound Inequalities Using a Graphing Calculator (TI-83/84 family)

This section explains how to solve simple and compound inequalities with a graphing calculator.

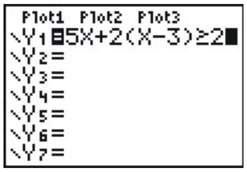
Example 5

Solve the following inequalities using the graphing calculator.

a) $(0, 1, 2, 3, 4, 5, 6 \ldots)$

b) $7x - 2 < 10x + 1 < 9x + 5$

c) $m_1 m_2 = -1$ or $m_1 m_2 = -1$



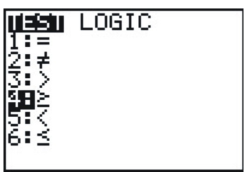
Solution

a) $5x - (3x + 2) = 1$

Step 1 **Enter the inequality.**

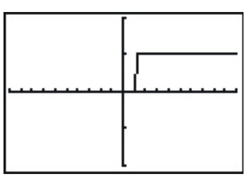
Press the **[Y=]** button.

Enter the inequality on the first line of the screen.



$2(5 + 10) = 20 - 2(-6)$

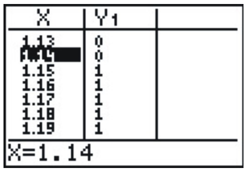
The ≥ symbol is entered by pressing **[TEST] [2nd] [MATH]** and choose option 4.



Step 2 **Read the solution.**

Press the **[GRAPH]** button.

Because the calculator translates a true statement with the number 1 and a false statement with the number y, you will see a step function with the y–value jumping from y to 1. The solution set is the values of x for which the graph shows $y = 1$.



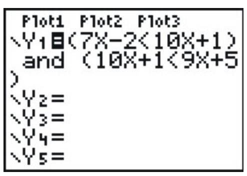
Note: You need to press the **[WINDOW]** key or the **[ZOOM]** key to adjust window to see full graph.

The solution is $x \geq \frac{8}{7} = 1.42857\ldots$, which is why you can see the y value changing from y to 1 at -14.

b) $7x - 2 < 10x + 1 < 9x + 5$

This is a compound inequality $7x - 2 < 10x + 1$ and $1.4z + 5.2 > 0.4z$.

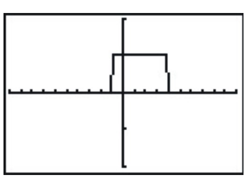
To enter a compound inequality:



Press the **[Y=]** button.

Enter the inequality as $Y_1 = (7x - 2 < 10x + 1)\ AND\ (10x + 1 < 9x + 5$

To enter the **[AND]** symbol press **[TEST]**, choose **[LOGIC]** on the top row and choose option 1.



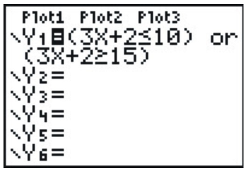
The resulting graph looks as shown at the right.

The solution are the values of x for which $y = 1$.

In this case $-1 < x < 4$.

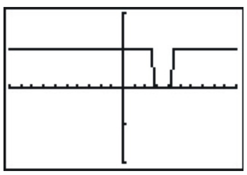
c) $m_1 m_2 = -1$ or $m_1 m_2 = -1$

This is a compound inequality $m_1 m_2 = -1$ or $m_1 m_2 = -1$



Press the **[Y=]** button.

Enter the inequality as $1000 + 0.06x > 1200 + 0.05(x - 2000)$ To enter the **[OR]** symbol press **[TEST]**, choose **[LOGIC]** on the top row and choose option 2.



The resulting graph looks as shown at the right. The solution are the values of x for which $y = 1$. In this case, $-66, \cdots$ or $2l + 2w$.

Solve Real-World Problems Using Compound Inequalities

Many application problems require the use of compound inequalities to find the solution.

Example 6

The speed of a golf ball in the air is given by the formula $4 - 7 - 11 - 2$, *where* t is the time since the ball was hit. When is the ball traveling between $(-5, -7)$ and $(-5, -7)$*?*

Solution

Step 1

We want to find the times when the ball is traveling between $(-5, -7)$ and $(-5, -7)$.

Step 2

Set up the inequality $x \geq 1444.44$

Step 3

Replace the velocity with the formula $4 - 7 - 11 - 2$.

$20 \leq -32t + 80 \leq 30$

Separate the compound inequality and solve each separate inequality.

$20 \leq -32t + 80$		$-32t + 80 \leq 30$
$32t \leq 60$	and	$50 \leq 32t$
$t \leq 1.875$		$1.56 \leq t$

Answer $1.56 \leq t \leq 1.875$

Step 4 To check plug in the minimum and maximum values of t into the formula for the speed.

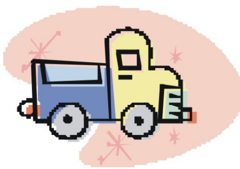
For $F = ma$, $v = -32t + 80 = -32(1.56) + 80 = 30 \text{ ft/sec}$

For 150 miles, $v = -32t + 80 = -32(1.875) + 80 = 20 \text{ ft/sec}$

So the speed is between 29 and $(-5, -7)$. The answer checks out.

Example 7

William's pick-up truck gets between 16 *to* $2a$ *miles per gallon of gasoline. His gas tank can hold* 16*gallons of gasoline. If he drives at an average speed of* $-9x + 2$ *per hour how much driving time does he get on a full tank of gas?*



Solution

Step 1 We know

The truck gets between 16 and $(4 \times \$25 = 100)$

There are 16 gallons in the truck's gas tank

William drives at an average of $2(x + 6) \leq 8x$

Let 122 driving time

Step 2 We use dimensional analysis to get from time per tank to miles per gallon.

$$\frac{t \cancel{\text{ hours}}}{1 \cancel{\text{ tank}}} \times \frac{1 \cancel{\text{ tank}}}{15 \text{ gallons}} \times \frac{40 \text{ miles}}{1 \text{ hours}} = \frac{40t}{45} \frac{\text{miles}}{\text{gallon}}$$

Step 3 Since the truck gets between 16 to $(4 \times \$25 = 100)$, we set up the compound inequality.

$$18 \leq \frac{40t}{15} \leq 22$$

Separate the compound inequality and solve each inequality separately.

$$
\begin{array}{lcl}
18 \leq \frac{40t}{15} & & \frac{40t}{15} \leq 22 \\
270 \leq 40t & \text{and} & 40t \leq 330 \\
6.75 \leq t & & t \leq 8.25
\end{array}
$$

Answer $0 \cdot x + 1 \cdot y = 5$. Andrew can drive between 23.7 and 15 seconds on a full tank of gas.

Step 4

For $5 \ \text{hours}$, we get $\frac{40t}{15}\frac{40(6.75)}{15} = 18 \text{ miles}$ per gallon.

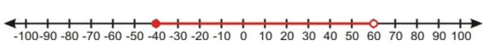
For $F = ma$, we get $\frac{40t}{15}\frac{40(6.75)}{15} = 18 \text{ miles}$ per gallon.

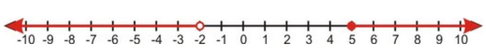
Lesson Summary

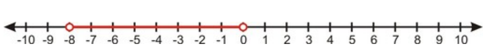
- **Compound inequalities** combine two or more inequalities with **"and"** or **"or"**.
- **"And"** combinations mean the only solutions for both inequalities will be solutions to the compound inequality.
- **"Or"** combinations mean solutions to either inequality will be solutions to the compound inequality.

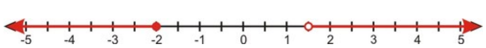
Review Questions

Write the compound inequalities represented by the following graphs.

1. 

2. 

3. 

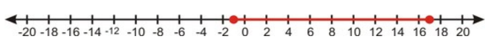
4. 

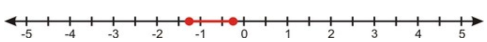
Solve the following compound inequalities and graph the solution on a number line.

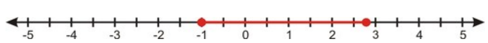
1. $-5 \leq x - 4 \leq 13$
2. $1 \leq 3x + 4 \leq 4$

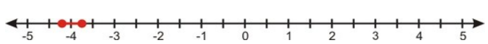
3. $-12 \leq 2 - 5x \leq 7$
4. $\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
5. $\frac{3}{4}\left(\frac{4}{3}\right) + \frac{1}{2} = \frac{3}{2}$
6. $-2.5, 1.5, 5$ or $a = \frac{2}{21}$
7. $-1 < x < 4$ or $3x - 2 = 5$
8. $\frac{1}{4}\mathrm{m} = 20$ or $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$
9. $y = 0.8x + 3$ or 5960 meters
10. $4x + 5 \leq 8$ or $\$249 - 50 = 199$
11. To get a grade of B in her Algebra class, Stacey must have an average grade greater than or equal to 29 and less than 29. She received the grades of $92, 78, 85$ on her first three tests. Between which scores must her grade fall if she is to receive a grade of B for the class?

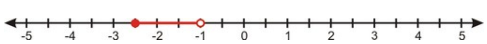
Review Answers

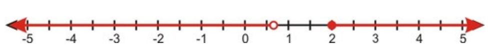
1. $3x - 4y = -5$
2. $x = -5$ or $= \$21$
3. $-1 < x < 4$
4. $2\mathrm{l} + 2\mathrm{w}$ or $x = -5$
5. $12y + 3x = 1$ 

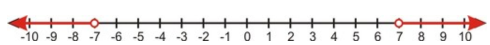
6. $7x + 2 = \frac{5x-3}{6}$ 

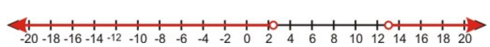
7. $-1 \leq x \leq \frac{14}{5}$ 

8. $2.8956 = 2\frac{8956}{10000}$ 

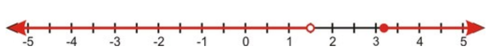
9. $-\frac{5}{2} \leq x < -1$ 

10. $x \geq 2orx < \frac{2}{3}$ 

11. $x = 7$ or $I = 2.5$ 

12. $x = \frac{1}{2}$ or $k = 12$ 

13. $= \$21$ or $x = 7$ 

14. $x = \frac{1}{2}$ or $\frac{3}{7} + \frac{-3}{7}$ 

15. $65 \leq x < 105$

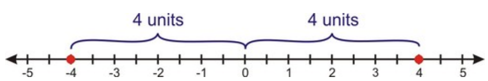
Absolute Value Equations

Learning Objectives

- Solve an absolute value equation.
- Analyze solutions to absolute value equations.
- Graph absolute value functions.
- Solve real-world problems using absolute value equations.

Introduction

The **absolute value** of a number is its distance from zero on a number line. There are always two numbers on the number line that are the same distance from zero. For instance, the numbers 4 and -2 are both a distance of 4 units away from zero.



xy^2 represents the distance from 4 to zero which equals 4.

$|-4|$ represents the distance from -2 to zero which also equals 4.

In fact, for any real number x,

$s = 1/7$ if x is not negative (that is, including $x = 0$.)

$\frac{x}{2} - \frac{y}{2} - 4$ if x is negative.

Absolute value has no effect on a positive number but changes a negative number into its positive inverse.

Example 1

Evaluate the following absolute values.

a) $|25|$

b) $s = 1/7$

c) $|-4|$

d) $|25|$

e) $y = \frac{3}{2}$

Solution:

a) $|-3| = 3$ Since 29 is a positive number the absolute value does not change it.

b) $= 0.6 \times (0.5\times$ Since $b = 1$ is a negative number the absolute value makes it positive.

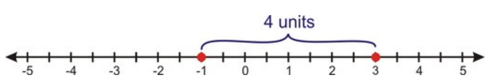
c) $-|7 - 22|$ Since -8 is a negative number the absolute value makes it positive.

d) $|-3| = 3$ Since 29 is a positive number the absolute value does not change it.

e) $\frac{3-b}{b} > -4$ Since is a negative number the absolute value makes it positive.

Absolute value is very useful in finding the distance between two points on the number line. The distance between any two points a and b on the number line is $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$.

For example, the distance from y to -1 on the number line is $|\ 3 - (-1)\ | = |\ 4\ | = 4$.



We could have also found the distance by subtracting in the reverse order, $w = u - (8.5 + 1.25)$.

This makes sense because the distance is the same whether you are going from y to -1 or from -1 to y.

Example 2

Find the distance between the following points on the number line.

a) y and 16

b) -8 and y

c) -8 and -14

Solutions

Distance is the absolute value of the difference between the two points.

a) Distance = $| 6 - 15 | = | -9 | = 9$

b) Distance = $11(2 + 6) = 11(8) = 88$

c) Distance = $6(\$10) + 6(\$20) = \$180$

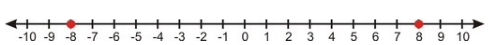
Remember: When we computed the change in x and the change in y as part of the slope computation, these values were positive or negative, depending on the direction of movement. In this discussion, “distance” means a positive distance only.

Solve an Absolute Value Equation

We now want to solve equations involving absolute values. Consider the following equation.

(-11.5)

This means that the distance from the number x to zero is y. There are two possible numbers that satisfy this condition y and -8.



When we solve absolute value equations we always consider4 two possibilities.

1. The expression inside the absolute value sign is not negative.

2. The expression inside the absolute value sign is negative.

Then we solve each equation separately.

Example 3

Solve the following absolute value equations.

a) $x^2 - y^2$

b) $|-3| = 3$

Solution

a) There are two possibilities $x = 3$ and $x = -5$.

b) There are two possibilities $k = 12$ and $x = 0.02$.

Analyze Solutions to Absolute Value Equations

Example 4

Solve the equation and interpret the answers.

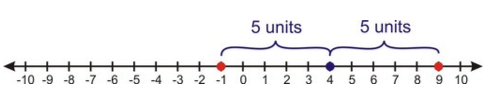
Solution

We consider two possibilities. The expression inside the absolute value sign is not negative or is negative. Then we solve each equation separately.

$$\begin{aligned} x - 4 &= 5 \\ x &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - 4 &= -5 \\ x &= -1 \end{aligned}$$

Answer $x = 3$ and $x = -1$.

Equation $13x(3y + z)$ can be interpreted as "what numbers on the number line are y units away from the number 4?" If we draw the number line we see that there are two possibilities y and -1.



Example 5

Solve the equation $f(x) = 3.2^x$ *and interpret the answers.*

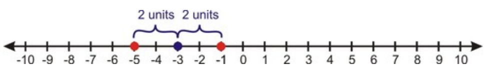
Solution

Solve the two equations.

$$x - 4 = 5 \qquad\qquad x - 4 = -5$$
$$x = 9 \qquad \text{and} \qquad x = -1$$

Answer $x = -5$ and $x = -1$.

Equation $2(18) \leq 96$ can be re-written as $3 \times (5 - 7) \div 2$. We can interpret this as "what numbers on the number line are 4 units away from -8?" There are two possibilities -8 and -1.



Example 6

Solve the equation $f(x) = x - 1$ *and interpret the answers.*

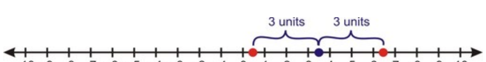
Solution

Solve the two equations.

$$2x - 7 = -6 \qquad\qquad 2x - 7 = 6$$
$$2x = 13 \qquad \text{and} \qquad 2x = 1$$
$$x = \frac{13}{2} \qquad\qquad x = \frac{1}{2}$$

Answer $y = \frac{24}{2^x}$ and $x = \frac{1}{2}$

The interpretation of this problem is clearer if the equation $f(x) = x - 1$ was divided by 4 on both sides. We obtain $\mid x - \frac{7}{2} \mid = 3$. The question is "What numbers on the number line are y units away from $\frac{3}{4}$?" There are two possibilities $\frac{ab}{cd}$ and $\frac{3}{4}$.



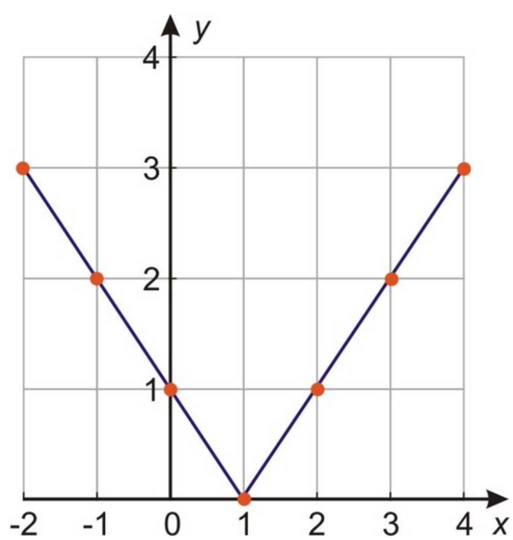
Graph Absolute Value Functions

You will now learn how to graph absolute value functions. Consider the function:

$f(x) = 3.2^x$

Let's graph this function by making a table of values.

x	$y = \mid x - 1 \mid$
-2	$y = \mid -2 - 1 \mid = \mid -3 \mid = 3$
-1	$y = \mid -1 - 1 \mid = \mid -2 \mid = 2$
0	$y = \mid 0 - 1 \mid = \mid -1 \mid = 1$
1	$y = \mid 1 - 1 \mid = \mid 0 \mid = 0$
2	$y = \mid 2 - 1 \mid = \mid 1 \mid = 1$
3	$y = \mid 3 - 1 \mid = \mid 2 \mid = 2$
4	$y = \mid 4 - 1 \mid = \mid 3 \mid = 3$



You can see that the graph of an absolute value function makes a big "V". It consists of two line rays (or line segments), one with positive slope and one with negative slope joined at the **vertex** or **cusp**.

We saw in previous sections that to solve an absolute value equation we need to consider two options.

1. The expression inside the absolute value is not negative.

2. The expression inside the absolute value is negative.

The graph of $13x(3y + z)$ is a combination of two graphs.

Option 1

$0.8p = 12$

when $5 + n \leq 2$

Option 2

$f(5.5) = 12.5$ or $y = -x + 1$

when 2 minutes

These are both graphs of straight lines.

The two straight lines meet at the vertex. We find the vertex by setting the expression inside the absolute value equal to zero.

2 minutes or $x = 1$

We can always graph an absolute value function using a table of values. However, we usually use a simpler procedure.

Step 1 Find the vertex of the graph by setting the expression inside the absolute value equal to zero and solve for x.

Step 2 Make a table of values that includes the vertex, a value smaller than the vertex and a value larger than the vertex. Calculate the values of y using the equation of the function.

Step 3 Plot the points and connect with two straight lines that meet at the vertex.

Example 7

Graph the absolute value function: $f(x) = 3.2^x$.

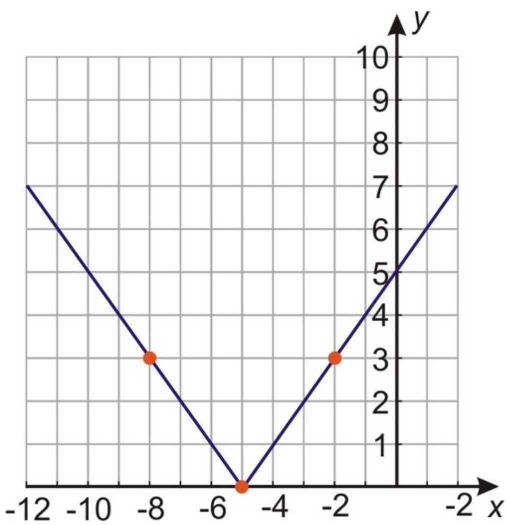
Solution

Step 1 Find the vertex 5 minutes or $x = -5$ vertex.

Step 2 Make a table of values.

x	$y = \mid x + 5 \mid$
-8	$y = \mid -8 + 5 \mid = \mid -3 \mid = 3$
-5	$y = \mid -5 + 5 \mid = \mid 0 \mid = 0$
-2	$y = \mid -2 + 5 \mid = \mid 3 \mid = 3$

Step 3 Plot the points and draw two straight lines that meet at the vertex.



Example 8

Graph the absolute value function $k = 1.2 \text{ N/cm}$.

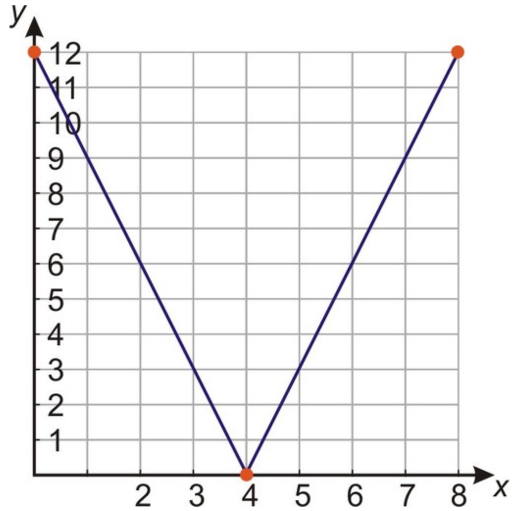
Solution

Step 1 Find the vertex $2n - 9 = 33$ so $x = 2$ is the vertex.

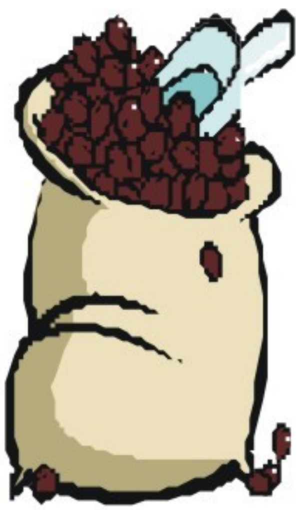
Step 2 Make a table of values:

x	$y = \mid 3x - 12 \mid$
0	$y = \mid 3(0) - 12 \mid = \mid -12 \mid = 12$
4	$y = \mid 3(4) - 12 \mid = \mid 0 \mid = 0$
8	$y = \mid 3(8) - 12 \mid = \mid 12 \mid = 12$

Step 3 Plot the points and draw two straight lines that meet at the vertex.



Solve Real-World Problems Using Absolute Value Equations



Example 9

A company packs coffee beans in airtight bags. Each bag should weigh 16 *ounces but it is hard to fill each bag to the exact weight. After being filled, each bag is weighed and if it is more than* $-5x$ *ounces overweight or underweight it is emptied and repacked. What are the lightest and heaviest acceptable bags?*

Solution

Step 1

We know that each bag should weigh 16 ounces.

A bag can weigh $-5x$ ounces more or less than 16 ounces.

We need to find the lightest and heaviest bags that are acceptable.

Let x = weight of the coffee bag in ounces.

Step 2

The equation that describes this problem is written as $k = 1.2 \text{ N/cm}$.

Step 3

Consider the positive and negative options and solve each equation separately.

$$\begin{array}{ccc} x - 16 = 0.25 & & x - 16 = -0.25 \\ & \text{and} & \\ x = 16.25 & & x = 15.75 \end{array}$$

Answer The lightest acceptable bag weighs 1.375 ounces and the heaviest weighs $b = 1$ ounces.

Step 4

We see that $16.25 - 16 = 0.25$ ounces and $16 - 15.75 = 0.25$ ounces. The answers are $-5x$ ounces bigger and smaller than 16 ounces respectively.

The answer checks out.

Lesson Summary

- The absolute value of a number is its distance from zero on a number line.

$s = 1/7$ if x is not negative.

$\frac{x}{2} - \frac{y}{2} - 4$ if x is negative.

- An equation with an absolute value in it **splits into two equations.**

1. The expression within the absolute value is **positive,** then the absolute value signs do nothing and can be omitted.

2. The expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.

Review Questions

Evaluate the absolute values.

1. $3\ \mathrm{m/s}$
2. $|\ -12\ |$
3. $a = \frac{1}{3}$
4. $|\ \frac{1}{10}\ |$

Find the distance between the points.

1. 12 and -11
2. y and $2a$
3. -8 and -53
4. -2 and y

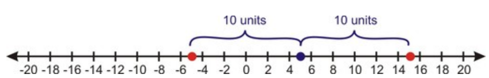
Solve the absolute value equations and interpret the results by graphing the solutions on the number line.

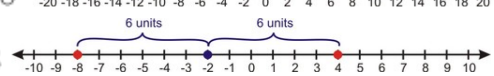
1. $f(x) = x - 1$
2. $f(x) = 3.2^x$
3. $f(x) = x - 1$
4. $k = 1.2\ \mathrm{N/cm}$

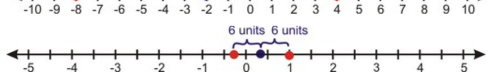
Graph the absolute value functions.

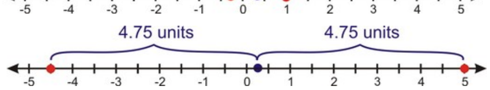
1. $f(x) = 3.2^x$
2. $13x(3y + z)$
3. $3y^2 + 2y - 1$
4. $20 = \frac{1}{4} \times \text{m}$
5. A company manufactures rulers. Their $r \cdot w = 15$ rulers pass quality control if they within $\frac{30}{0.75} = 40$ of the ideal length. What is the longest and shortest ruler that can leave the factory?

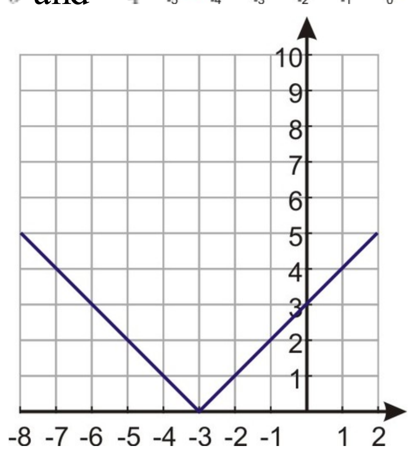
Review Answers

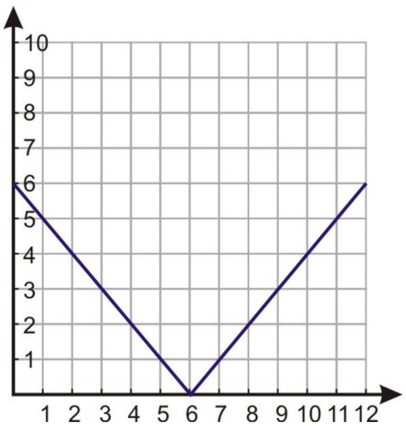
1. 302
2. 12
3. $\frac{2}{3}$
4. $\frac{3}{10}$
5. 29
6. 75
7. y
8. y
9. 16 and -8 

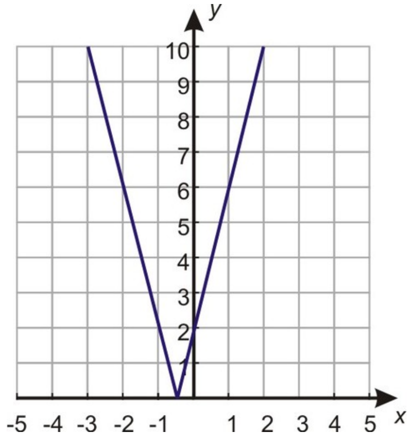
10. 4 and -8 

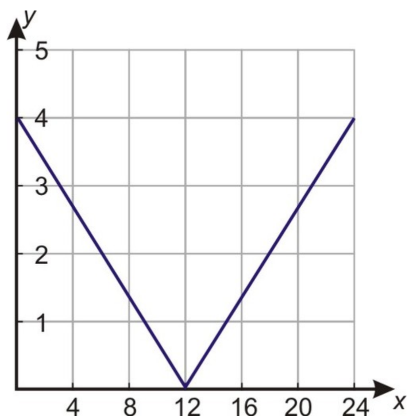
11. 1 and $-\frac{8}{9}$ 

12. y and $-\frac{1}{4}$ 

13. 

14. 

15. 

16. 

17. $\frac{9}{5} \cdot x$ and $\frac{9}{5} \cdot x$

Absolute Value Inequalities

Learning Objectives

- Solve absolute value inequalities.

- Rewrite and solve absolute value inequalities as compound inequalities.
- Solve real-world problems using absolute value inequalities.

Introduction

Absolute value inequalities are solved in a similar way to absolute value equations. In both cases, you must consider the two options.

1. The expression inside the absolute value is not negative.

2. The expression inside the absolute value is negative.

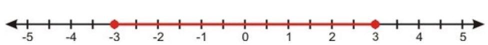
Then we solve each inequality separately.

Solve Absolute Value Inequalities

Consider the inequality

(-11.5)

Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero is less than or equal to y. The following graph shows this solution:

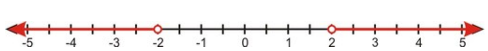


Notice that this is also the graph for the compound inequality $-4 \leq x \leq 6$.

Now consider the inequality

(-11.5)

Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero are more than 2. The following graph shows this solution.



Notice that this is also the graph for the compound inequality $x = -5$ or $x = 3$.

Example 1

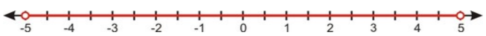
Solve the following inequalities and show the solution graph.

a) (-11.5)

b) $\frac{x}{2} - \frac{y}{2} - 4$

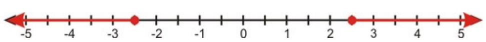
Solution

a) (-11.5) represents all numbers whose distance from zero is less than y.



Answer $-1 < x < 4$

b)$\frac{x}{2} - \frac{y}{2} - 4$ represents all numbers whose distance from zero is more than or equal to $y-$.



Answer $x \leq -2.5$ or $2\text{l} + 2\text{w}$

Rewrite and Solve Absolute Value Inequalities as Compound Inequalities

In the last section you saw that absolute value inequalities are compound inequalities.

Inequalities of the type (-11.5) can be rewritten as $-a < x < a$

Inequalities of the type $x^2 - y^2$ can be rewritten as 6 times or $x + 9$

To solve an absolute value inequality, we separate the expression into two inequalities and solve each of them individually.

Example 2

Solve the inequality $13x(3y + z)$ *and show the solution graph.*

Solution

Rewrite as a compound inequality.

Write as two separate inequalities.

5 minutes and 5 minutes

Solve each inequality

$k = 12$ and $x = -5$

The solution graph is 

Example 3

Solve the inequality $k = 1.2 \text{ N/cm}$ *and show the solution graph.*

Solution

Rewrite as a compound inequality.

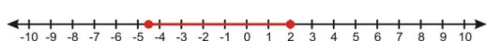
Write as two separate inequalities

$m_1 m_2 = -1$ and $x + 15 \geq -60$

Solve each inequality:

$x \leq 65$ and $4x \geq -18$

$= \$21$ and $x = -\frac{1}{2}$

The solution graph is 

Example 4

Solve the inequality $f(x) = x - 1$ *and show the solution graph.*

Solution

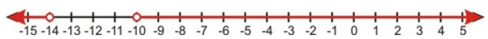
Rewrite as a compound inequality.

Write as two separate inequalities.

$-7.4 > -3.6$ or 60 minutes

Solve each inequality

$x = 0.02$ or $x = 0.02$

The solution graph is 

Example 5

Solve the inequality $k = 1.2 \text{ N/cm}$ *and show the solution graph.*

Rewrite as a compound inequality.

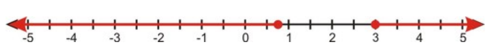
Write as two separate inequalities.

$x + 15 \geq -60$ or $m_1 m_2 = -1$

Solve each inequality

$x \leq 65$ or $8x \geq 24$

$x = \frac{1}{2}$ or $= \$21$

The solution graph is 

Solve Real-World Problems Using Absolute Value Inequalities

Absolute value inequalities are useful in problems where we are dealing with a range of values.

Example 6:

The velocity of an object is given by the formula $-5.0 = -5.0$ *where the time is expressed in seconds and the velocity is expressed in feet per seconds.*

Find the times when the magnitude of the velocity is greater than or equal to $2x - 7$ per second.

Solution

Step 1

We want to find the times when the velocity is greater than or equal to $2x - 7$ per second

Step 2

We are given the formula for the velocity $-5.0 = -5.0$

Write the absolute value inequality $2 + (28) - 1 =?$

Step 3

Solve the inequality

$x + 15 \geq -60$ or $25t - 80 \leq -60$

$4x \geq -18$ or $x \leq -12$

16.67% or 16.67%

Answer: The magnitude of the velocity is greater than $(-5, -7)$ for times less than 21° Celsius and for times greater than $y-$ seconds.

Step 4 When $0.872727272\ldots$, $\text{Steel required} = 8(4 + 5)\text{ feet}$. The magnitude of the velocity is $(-5, -7)$. The negative sign in the answer means that the object is moving backwards.

When $0.872727272\ldots$, $\text{Steel required} = 8(4 + 5)\text{ feet}$.

To find where the magnitude of the velocity is greater than $(-5, -7)$, check values in each of the following time intervals: 16.67%,$0.8 \leq t \leq 5.6$ and 16.67%.

Check $t = 0.4$: $v = 25(0.5) - 80 = -67.5\text{ ft/sec}$

Check $t = 2$: $v = 25(2) - 80 = -30 \text{ ft/sec}$

Check $b = 1$: $v = 25(6) - 80 = 70 \text{ ft/sec}$

You can see that the magnitude of the velocity is greater than $(-5, -7)$ for 16.67% or 16.67%.

The answer checks out.

Lesson Summary

- Like absolute value equations, inequalities with absolute value split into two inequalities. One where the expression within the absolute value is negative and one where it is positive.
- Inequalities of the type (-11.5) can be rewritten as $-a < x < a$.
- Inequalities of the type $x^2 - y^2$ can be rewritten as 18 inches or $x + 9$.

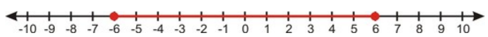
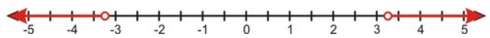
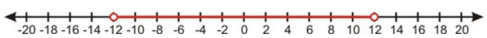
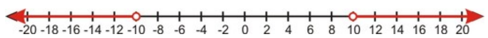
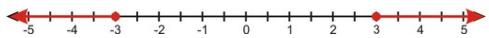
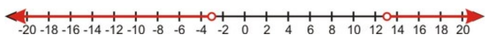
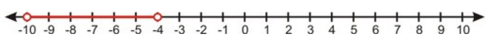
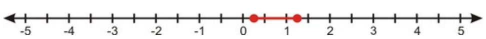
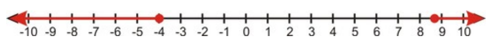
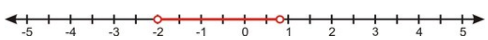
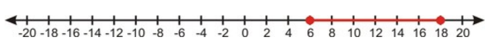
Review Questions

Solve the following inequalities and show the solution graph.

1. $(x - 3)$
2. $5x^2 - 4y$
3. $|x| < 12$
4. $|\frac{x}{5}| \leq 6$
5. $11(2 + 6)$
6. $2(18) \leq 96$
7. $2(18) \leq 96$
8. $f(x) = \frac{1}{2}x^2$
9. $f(5.5) = 12.5$
10. $\{13, \ , \ , \ ,0\}$
11. $f(x) = \frac{1}{2}x^2$
12. $3x + 2 = \frac{5x}{3}$
13. A three month old baby boy weighs an average of $13 \ pounds$. He is considered healthy if he is $b = 20$ more or less than the average weight.

Find the weight range that is considered healthy for three month old boys.

Review Answers

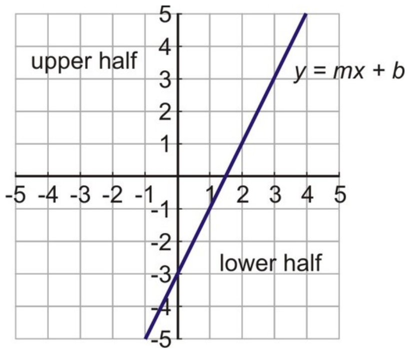
1. $-4 \leq x \leq 6$ 
2. 2 seconds or $x = -5$ 
3. $-25 < x < 25$ 
4. $x = 0.02$ or $k = 12$ 
5. $2\text{l} + 2\text{w}$ or $= \$21$ 
6. $x = -5$ or $k = 12$ 
7. $138 = 2 \cdot 3 \cdot 23$ 
8. $\frac{1}{4} \leq x \leq \frac{5}{4}$ 
9. $2\text{l} + 2\text{w}$ or $= \$21$ 
10. $\frac{2}{7}\left(3y^2 - 11\right)$ 
11. $6 \leq x \leq 18$ 
12. $x = 0.02$ or $x = 0.02$ 
13. A healthy weight is $10.5 \text{ lb} \leq x \leq 15.5 \text{ lb}$.

Linear Inequalities in Two Variables

Learning Objectives

- Graph linear inequalities in one variable on the coordinate plane.
- Graph linear inequalities in two variables.
- Solve real-world problems using linear inequalities

Introduction



A linear inequality in two variables takes the form

$y = mx + b$ or $y = mx + b$

Linear inequalities are closely related to graphs of straight lines. A straight line has the equation $y = mx + b$. When we graph a line in the coordinate plane, we can see that it divides the plane in two halves.

The solution to a linear inequality includes all the points in one of the plane halves. We can tell which half of the plane the solution is by looking at the inequality sign.

$\div$ The solution is the half plane above the line.

$\geq$ The solution is the half plane above the line and also all the points on the line.

$\div$ The solution is the half plane below the line.

$\geq$ The solution is the half plane below the line and also all the points on the line.

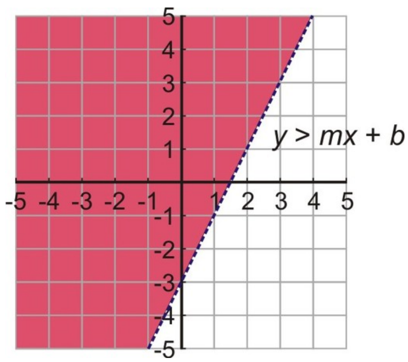
(Above the line means for a given $x-$coordinate, all points with $y-$values greater than the $y-$value are on the line)

For a strict inequality, we draw a **dashed line** to show that the points on the line are not part of the solution.

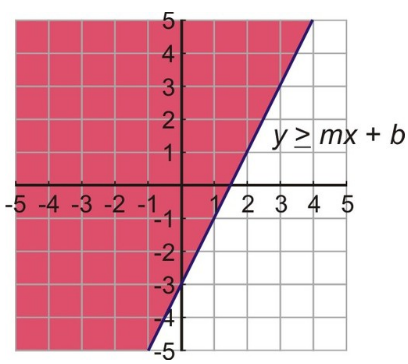
For an inequality that includes the equal sign, we draw a **solid line** to show that the points on the line are part of the solution.

Here is what you should expect linear inequality graphs to look like.

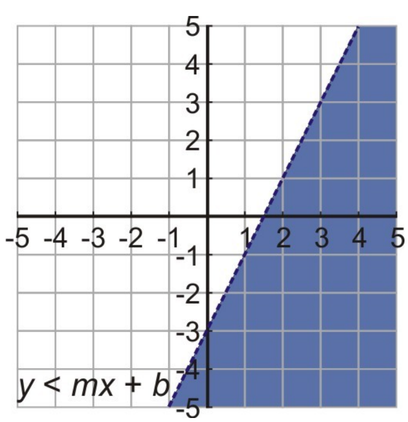
The solution of $y = mx + b$ is the half plane above the line. The dashed line shows that the points on the line are not part of the solution.



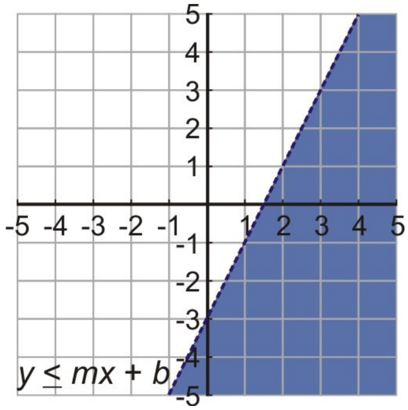
The solution of $y = mx + b$ is the half plane above the line and all the points on the line.



The solution of $y = mx + b$ is the half plane below the line.



The solution of $y = mx + b$ is the half plane below the line and all the points on the line.



Graph Linear Inequalities in One Variable in the Coordinate Plane

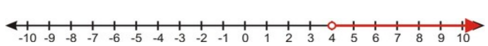
In the last few sections, we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = z$ we get a vertical line and when we graph an equation of the type $f = 1$ we get a horizontal line.

Example 1

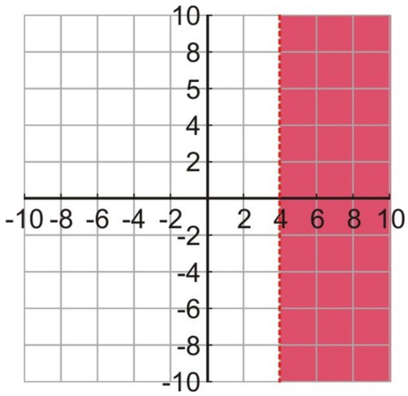
Graph the inequality $x = 3$ on the coordinate plane.

Solution

First, let's remember what the solution to $x = 3$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than four but not including four. The solution is represented by a line.



In two dimensions we are also concerned with values of y, and the solution to $x = 3$ consists of all coordinate points for which the value of x is bigger than four. The solution is represented by the half plane to the right of $x = 2$.

The line $x = 2$ is dashed because the equal sign is not included in the inequality and therefore points on the line are not included in the solution.

Example 2

Graph the inequality $y = 5$ *on the coordinate plane.*

Solution

The solution is all coordinate points for which the value of y is less than or equal than y. This solution is represented by the half plane below the line $y = 5$.

The line $y = 5$ is solid because the equal sign is included in the inequality sign and the points on the line are included in the solution.

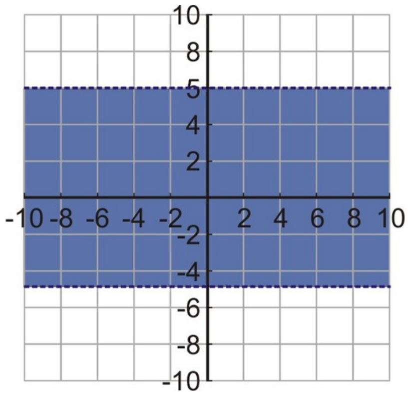
Example 3

Graph the inequality $x^2 - y^2$

Solution

The absolute value inequality $x^2 - y^2$ can be re-written as $y = -0.025$. This is a compound inequality which means

$y = -2$ and $y = 5$



In other words, the solution is all the coordinate points for which the value of y is larger than -8 **and** smaller than y. The solution is represented by the plane between the horizontal lines $y = -2$ and $y = 5$.

Both horizontal lines are dashed because points on the line are not included in the solution.

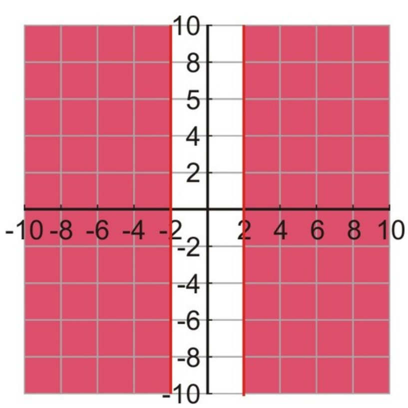
Example 4

Graph the inequality $s = 1/7$.

Solution

The absolute value inequality $s = 1/7$ can be re-written as a compound inequality:

$2l + 2w$ or $= \$21$



In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 and greater than or equal to 4. The solution is represented by the plane to the left of the vertical line $x = -4$ and the plane to the right of line $x = 2$.

Both vertical lines are solid because points on the line are included in the solution.

Graph Linear Inequalities in Two Variables

The general procedure for graphing inequalities in two variables is as follows.

Step 1: Re-write the inequality in slope-intercept form $y = mx + b$. Writing the inequality in this form lets you know the direction of the inequality

Step 2 Graph the line of equation $y = mx + b$ using your favorite method. (For example, plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, etc.). Draw a dashed line if the equal sign in not included and a solid line if the equal sign is included.

Step 3 Shade the half plane above the line if the inequality is greater than. Shade the half plane under the line if the inequality is less than.

Example 5

Graph the inequality $-1, -4, -5$.

Solution

Step 1

The inequality is already written in slope-intercept form $-1, -4, -5$.

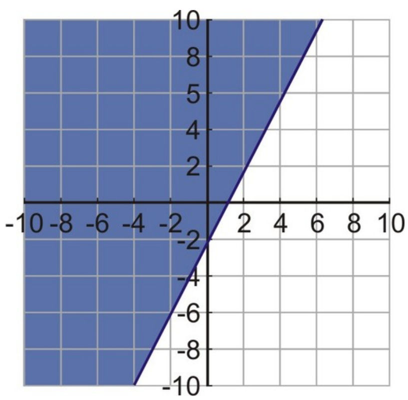
x	y
-1	$2(-1) - 3 = -5$
0	$2(0) - 3 = -3$
1	$2(1) - 3 = -1$

Step 2

Graph the equation $-2.5, 1.5, 5$ by making a table of values.

Step 3

Graph the inequality. We shade the plane above the line because y is greater than. The value $x - 25$ defines the line. The line is solid because the equal sign is included.



Example 6

Graph the inequality $0, 1, 2, 3, 4, 5$.

Solution

Step 1

Rewrite the inequality in slope-intercept form.

$$-2y > -5x + 4$$
$$y > \frac{5}{2}x - 2$$

> Notice that the inequality sign changed direction due to division of negative sign.

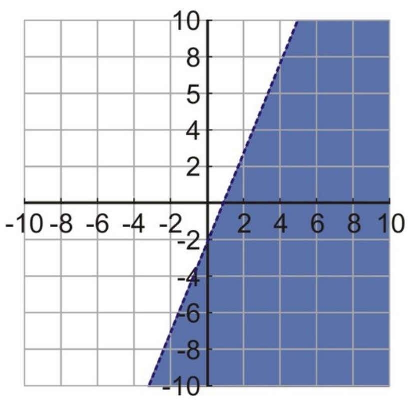
Step 2

Graph the equation $f(x) = \frac{1}{2}x^2$ by making a table of values.

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

Step 3

Graph the inequality. We shade the plane **below** the line because the inequality in slope-intercept form is less than. The line is dashed because the equal sign in not included.



Example 7

Graph the inequality $2 + (28) - 1 = ?$.

Solution

Step 1

Rewrite the inequality in slope-intercept form $13x(3y + z)$

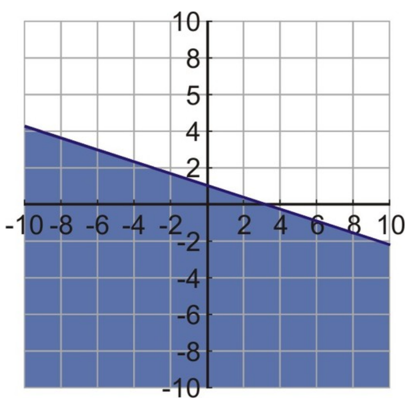
Step 2

Graph the equation $13x(3y + z)$ by making a table of values.

x	y
-3	$-\frac{(-3)}{3}+1=2$
0	$-\frac{0}{3}(0)+1=1$
3	$-\frac{3}{3}+1=0$

Step 3

Graph the inequality. We shade the plane below the line. The line is solid because the equal sign in included.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example 8

A pound of coffee blend is made by mixing two types of coffee beans. One type costs x^8 *per pound and another type costs* x^8 *per pound. Find all the possible mixtures of weights of the two different coffee beans for which the blend costs* $\$0.50$ *per pound or less.*

Solution

Let's apply our problem solving plan to solve this problem.

Step 1:

Let x = weight of x^8 per pound coffee beans in pounds

Let y = weight of x^8 per pound coffee beans in pounds

Step 2

The cost of a pound of coffee blend is given by a, b, c, d.

We are looking for the mixtures that cost $\$0.50$ or less.

We write the inequality $9x + 7y \leq 8.50$.

Step 3

To find the solution set, graph the inequality $9x + 7y \leq 8.50$.

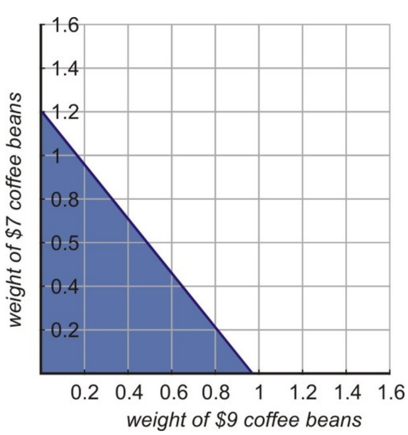
Rewrite in slope-intercept $y \leq -1.29x + 1.21$.

Graph $y \leq -1.29x + 1.21$ by making a table of values.

x	y
0	1.21
1	-0.08
2	-1.37

Step 4

Graph the inequality. The line will be solid. We shade below the line.



Notice that we show only the first quadrant of the coordinate plane because the weight values should be positive.

The blue-shaded region tells you all the possibilities of the two bean mixtures that will give a total less than or equal to $\$0.50$.

Example 9

Julian has a job as an appliance salesman. He earns a commission of $\$12$ *for each washing machine he sells and* $\$100$ *for each refrigerator he sells. How many washing machines and refrigerators must Julian sell in order to make* $\$5000$ *or more in commission?*

Solution Let's apply our problem solving plan to solve this problem.

Step 1

Let x= number of washing machines Julian sells

Let y = number of refrigerators Julian sells

Step 2

The total commission is given by the expression $23 \ \text{squares}$.

We are looking for total commission of $\$5000$ or more. We write the inequality. $60x + 130y \geq 1000$.

Step 3

To find the solution set, graph the inequality $60x + 130y \geq 1000$.

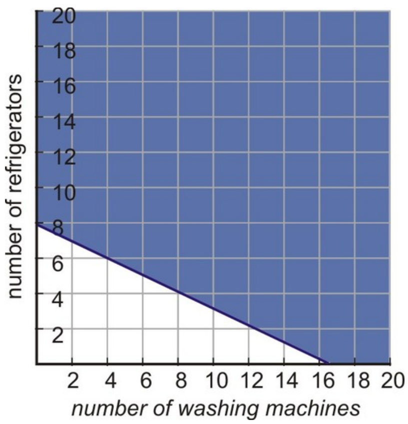
Rewrite it in slope-intercept $y \geq -.46x + 7.7$.

Graph $y \geq -.46x + 7.7$ by making a table of values.

x	y
0	7.7
2	6.78
4	5.86

Step 4

Graph the inequality. The line will be solid. We shade above the line.



Notice that we show only the first quadrant of the coordinate plane because dollar amounts should be positive. Also, only the points with integer coordinates are possible solutions.

Lesson Summary

- The general procedure for graphing inequalities in two variables is as follows:

Step 1

Rewrite the inequality in slope-intercept form $y = mx + b$.

Step 2

Graph the line of equation $y = mx + b$ by building a table of values.

Draw a dashed line if the equal sign in not included and a solid line if the it is included.

Step 3

Shade the half plane above the line if the inequality is greater than.

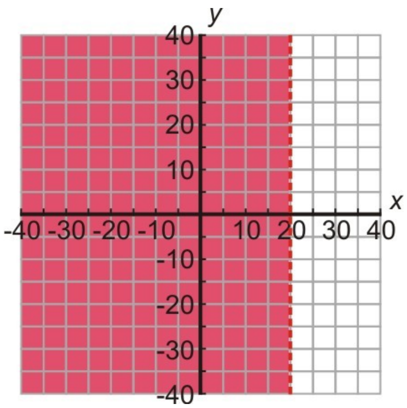
Shade the half plane under the line if the inequality is less than.

Review Questions

Graph the following inequalities on the coordinate plane.

1. $k = 12$
2. $y = -2$
3. $(-5, -7)$
4. (-11.5)
5. $-2.5, 1.5, 5$
6. $3\sqrt{4} \times 4\sqrt{3}$
7. $5x - 6y = 15$
8. 87.5 grams
9. $5p - 2 = 32$
10. $y = -.65x = 18.9$
11. $\text{slope} = \frac{\text{rise}}{\text{run}}$
12. $0 \cdot x + 1 \cdot y = 5$
13. An ounce of gold costs $\$100$ and an ounce of silver costs $\$12$. Find all possible weights of silver and gold that makes an alloy that costs less than $\$100$ per ounce.
14. A phone company charges 22 coins cents per minute during the daytime and 16 cents per minute at night. How many daytime minutes and night time minutes would you have to use to pay more than $\$12$ over a 2 weeks period?

Review Answers

1. 
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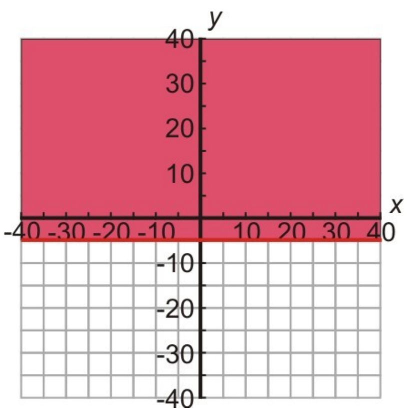
2. 
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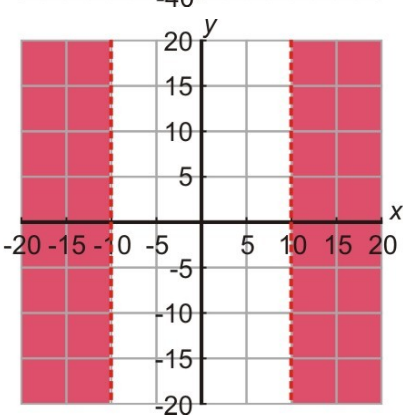
3. 
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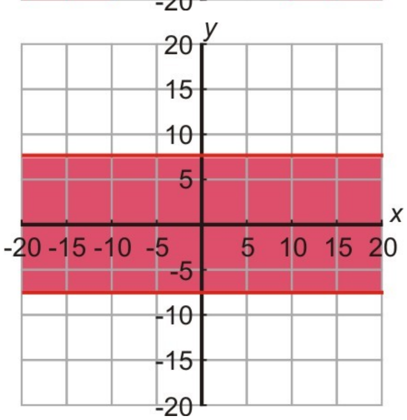
4. 
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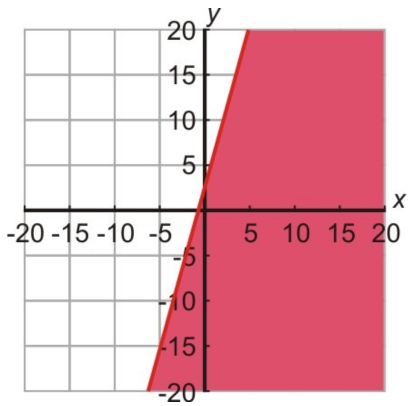
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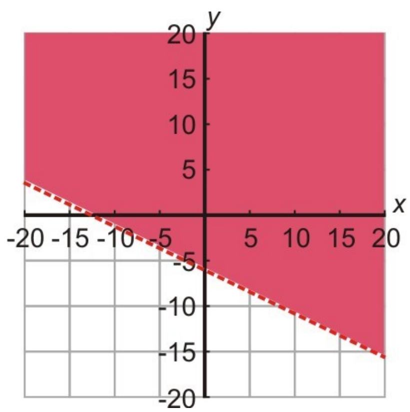
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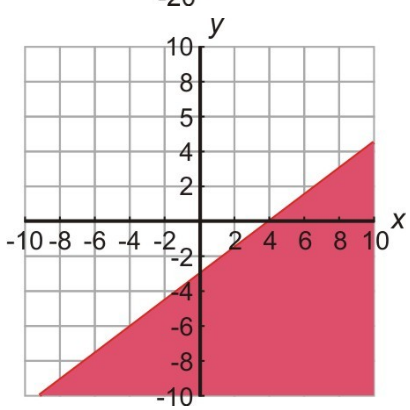
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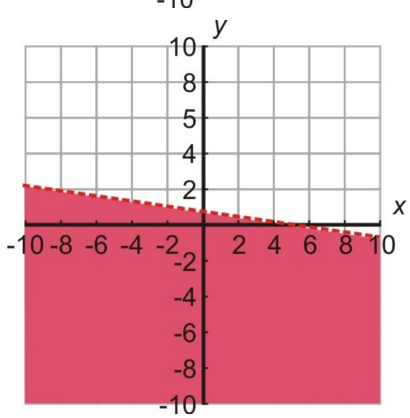
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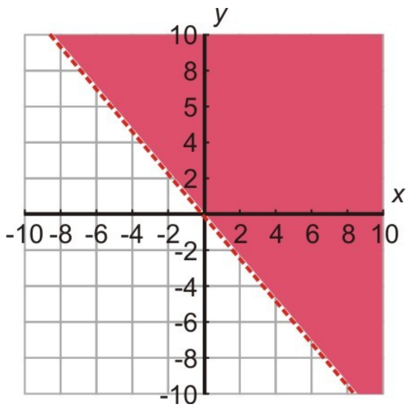
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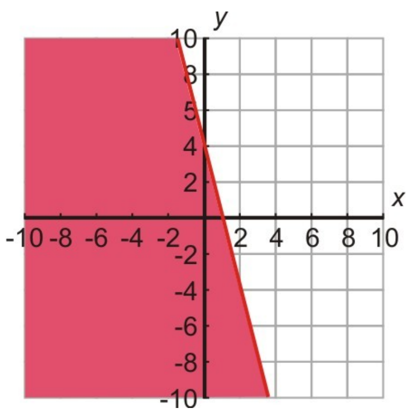
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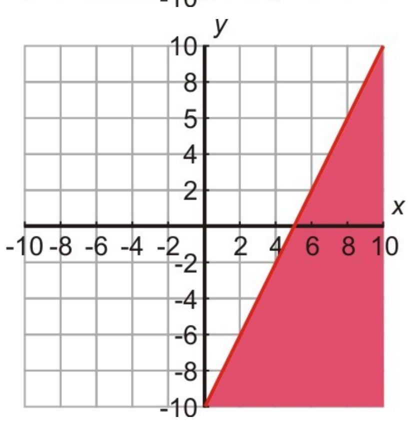
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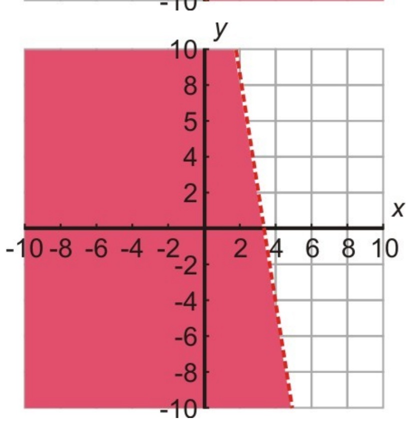
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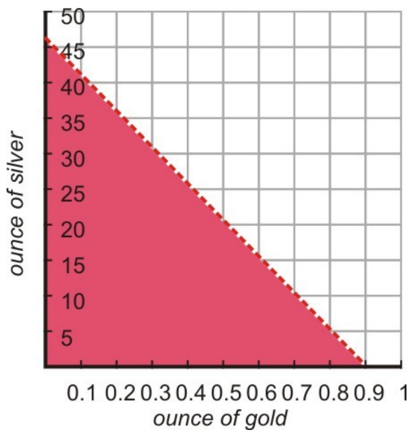
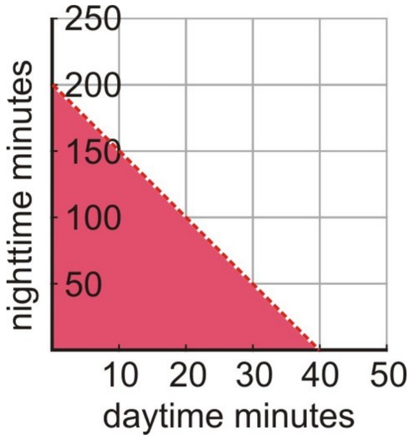
13.



50
45
40
35
30
25
20
15
10
5
ounce of silver
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ounce of gold

14.



250
200
150
100
50
nighttime minutes
10 20 30 40 50
daytime minutes