Chapter 7: Solving Systems of Equations and Inequalities

Linear Systems by Graphing

Learning Objectives

- Determine whether an ordered pair is a solution to a system of equations.
- Solve a system of equations graphically.
- Solve a system of equations graphically with a graphing calculator.
- Solve word problems using systems of equations.

Introduction

In this lesson, we will discover methods to determine if an ordered pair is a solution to a system of two equations. We will then learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we will look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations consists of a set of equations that must be solved together.

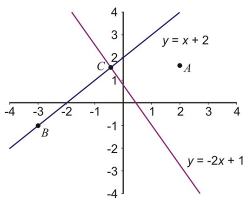
Consider the following system of equations.

$$y = x + 2$$
$$y = -2x + 1$$

Since the two lines are in a system we deal with them together by graphing them on the same coordinate axes. The lines can be graphed using your favorite method. Let's graph by making a table of values for each line.

Line 1 $y = -120$

x	y
0	2
1	3



Line 2 $0.05P \geq 250$

x	y
0	1
1	-1

A solution for a single equation is any point that lies on the line for that equation. A solution for a system of equations is any point that lies on both lines in the system.

For Example

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point P is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In particular, this point marks the intersection of the two lines. It solves both equations, so it solves the system. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered paid that solves both equations.

You can confirm the solution by plugging it into the system of equations, and confirming that the solution works in each equation.

Example 1

Determine which of the points $|\ 3-(-1)\ |=|\ 4\ |=4$ *is a solution to the following system of equations.*

$y=4x-1$
$y=2x+3$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point $(0,0)$

$$\begin{aligned} y &= 4x-1 \\ 3^? &=^? 4(1)-1 \\ 3 &= 3 \end{aligned}$$

The solution checks.

$$\begin{aligned} y &= 2x+3 \\ 3^? &=^? 2(1)+3 \\ 3 &\neq 5 \end{aligned}$$

The solution does not check.

Point $(0,0)$ is on line $y=3x+1$ but it is not on line $-2.5, 1.5, 5$ so it is not a solution to the system.

Point $(0,0)$

$$\begin{aligned} y &= 2x+3 \\ 3^? &=^? 2(1)+3 \\ 3 &\neq 5 \end{aligned}$$

The solution does not check.

Point $(0,0)$ is not on line $y=3x+1$ so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point $(0, 0)$

$$\begin{aligned} y &= 4x - 1 \\ 7^? &=^? 4(2) - 1 \\ 7 &= 7 \end{aligned}$$

The solution checks.

$$\begin{aligned} y &= 4x - 1 \\ 7^? &=^? 4(2) - 1 \\ 7 &= 7 \end{aligned}$$

The solution checks.

Point $(0, 0)$ is a solution to the system since it lies on both lines.

Answer The solution to the system is point $(0, 0)$.

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point which lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions. It is exact only when the x and y values of the solution are integers. However, this method it is a great at offering a visual representation of the system of equations and demonstrates that the solution to a system of equations is the intersection of the two lines in the system.

Example 2

(The equations are in slope-intercept form)

Solve the following system of equations by graphing.

$$\begin{aligned} y &= 3x - 5 \\ y &= -2x + 5 \end{aligned}$$

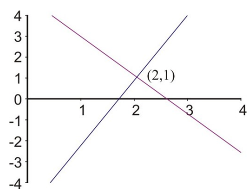
Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, let's make a table of values for each line.

Line 1 $-2.5, 1.5, 5$

x	y
1	-2
2	1



Line 2 $y = 0.8x + 3$

x	y
1	3
2	1

Answer The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point $(0, 0)$. So the solution is $x = 2$, $y = 1$ or $(0, 0)$.

Example 3

(The equations are in standard form)

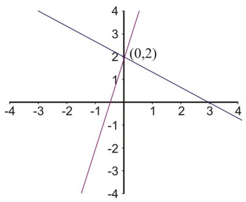
Solve the following system of equations by graphing

$$2x + 3y = 6$$
$$4x - y = -2$$

Solution

Graph both lines on the same coordinate axis using your method of choice.

Here we will graph the lines by finding the $x-$ and $y-$intercepts of each of the lines.



Line 1 $y = 15 + 5x$

$x-$intercept set $y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$ which results in point $(0, 0)$.

$y-$intercept set $x = 0 \Rightarrow 3y = 6 \Rightarrow y = 2$ which results in point $(0, 0)$.

Line 2 $0.05P \geq 250$

$x-$intercept: set $y = 0 \Rightarrow -4x = 2 \Rightarrow x = -\frac{1}{2}$ which results in point $\frac{11x}{2} + 6$.

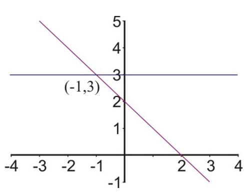
$y-$intercept: set $p - 0.20p = 12$ which results in point $(0, 0)$

Answer The graph shows that the lines intersect at point $(0, 0)$. Therefore, the solution to the system of equations is $t = 19, u = 4$.

Example 4:

Solve the following system by graphing.

$$\begin{aligned} y &= 3 \\ x + y &= 2 \end{aligned}$$



Line 1 $y = 5$ is a horizontal line passing through point $(0, 0)$.

Line 2 $y = -120$

$x-$intercept: $(0, 0)$

$y-$intercept: $(0, 0)$

Answer The graph shows that the solution to this system is $(3 + 2)$
$p - 0.20p = 12$.

These examples are great at demonstrating that the solution to a system of linear equations means the point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can see, however, that determining a solution from a graph would require very careful graphing of the lines, and is really only practical when you are certain that the solution gives integer values for x and y. In most cases, this method can only offer approximate solutions to systems of equations. For exact solutions other methods are necessary.

Solving a System of Equations Using the Graphing Calculator

A graphing calculator can be used to find or check solutions to a system of equations. In this section, you learned that to solve a system graphically, you must graph the two lines on the same coordinate axes and find the point of intersection. You can use a graphing calculator to graph the lines as an alternative to graphing the equations by hand.

Example 6

Solve the following system of equations using a graphing calculator.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

In order to input the equations into the calculator, they must be written in slope-intercept form (i.e., $y = mx + b$ form), or at least you must isolate y.

$$x - 3y = 4 \qquad\qquad y = \frac{1}{3}x - \frac{4}{3}$$

$$\Rightarrow$$

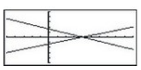
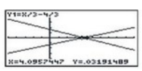
$$2x + 5y = 8 \qquad\qquad y = \frac{-2}{5}x - \frac{8}{5}$$

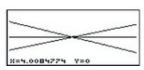
Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$Y_1 = \frac{x}{3} - \frac{4}{3}$$
$$T_2 = -\frac{2x}{3} - \frac{8}{5}$$

Now press **[GRAPH]**. The window below is set to $-5 \leq x \leq 10$ and $-4 \leq x \leq 6$.

The first screen below shows the screen of a TI-83 family graphing calculator with these lines graphed.



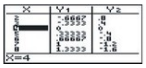
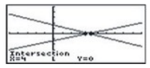
There are a few different ways to find the intersection point.

Option 1 Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be on the bottom of the screen. The second screen above shows the values to be $0.872727272\ldots$ and $Y = .03191489$.

Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $18 - x$ and $V = 6$.

Option 2 Look at the table of values by pressing **[2nd] [GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the values for and are the same. In this case this occurs at $18 - x$ and $V = 6$.

Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by taking smaller values of Table 1.

Option 3 Using the **[2nd] [TRACE]** function gives the screen in the second screen above.

Scroll down and select intersect.

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

The point of intersection is $18 - x$ and $V = 6$.

Notes:

- When you use the "intersect" function, the calculator asks you to select **[FIRSTCURVE]**? and **[SECONDCURVE]**? in case you have more than two graphs on the screen. Likewise, the **[GUESS]**? is requested in case the curves have more than one intersection. With lines you only get one point of intersection, but later in your mathematics studies you will work with curves that have multiple points of intersection.

- Option 3 is the only option on the graphing calculator that gives an exact solution. Using trace and table give you approximate solutions.

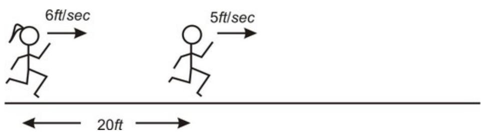
Solve Real-World Problems Using Graphs of Linear Systems

Consider the following problem

Peter and Nadia like to race each other. Peter can run at a speed of $x+9$ *per second and Nadia can run at a speed of* $x+9$ *per second. To be a good sport Nadia likes to give Peter a head start of* $2x-7$*. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?*

Draw a sketch

At time, $b=1$:



Formulas

Let's define two variables in this problem.

122 the time from when Nadia starts running

$y=$ the distance of the runners from the starting point.

Since we have two runners we need to write equations for each of them. This will be the **system of equations** for this problem.

Here we use the formula distance = speed $\times$ time

Nadia's equation $=25\Omega$

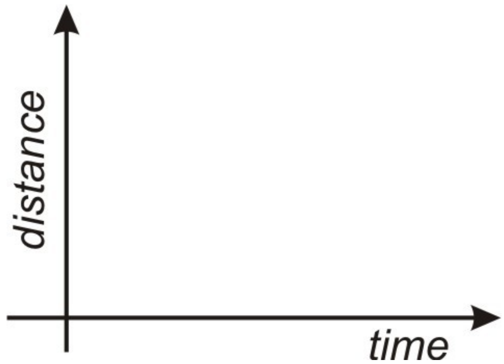
Peter's equation $x-1\leq-5$

(Remember that Peter was already $2x-7$ from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate graph.

Time should be on the horizontal axis since it is the independent variable.

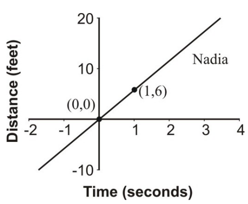
Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines. In this case, we will use the slope-intercept method since it makes more sense physically.

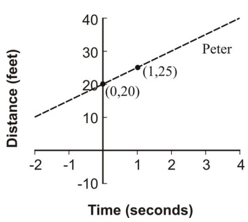
To graph the line that describes Nadia's run, start by graphing the $y-$intercept $(0, 0)$. If you do not see that this is the $y-$intercept, try plugging in the test-value of $x = 3$.

The slope tells us that Nadia runs $x + 9$ every one second so another point on the line is $(0, 0)$. Connecting these points gives us Nadia's line.

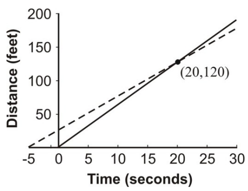


To graph the line that describes Peter's run, again start with the $y-$intercept. In this case, this is the point (mph).

The slope tells us that Peter runs $x + 9$ every one second so another point on the line is (mph). Connecting these points gives us Peter's line.



In order to find when and where Nadia and Peter meet, we will graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet $35\ \text{nickels}$ after Nadia starts running and $8\ \text{weeks}$ from the starting point.

Review Questions

Determine which ordered pair satisfies the system of linear equations.

1. 

    1. $(0, 0)$
    2. $(0, 0)$
    3. $|\frac{x}{5}| \le 6$
2. 

    1. (mph)
    2. $(3 + 2)$
    3. $(0, 0)$
3. 

    1. $(3 + 2)$
    2. $(5 - 11)$
    3. (mph)
4. 

    1. $\frac{11x}{2} + 6$
    2. $(3 + 2)$
    3. $(\frac{1}{2}, 1)$

Solve the following systems using the graphing method.

1. IMAGE NOT AVAILABLE
2. IMAGE NOT AVAILABLE
3. IMAGE NOT AVAILABLE
4. IMAGE NOT AVAILABLE
5. IMAGE NOT AVAILABLE
6. IMAGE NOT AVAILABLE
7. IMAGE NOT AVAILABLE
8. $2x + 4 = 3y$
   $x - 2y + 4 = 0$
9. $y = \frac{x}{2} - 3$
   $2x - 5y = 5$
10. IMAGE NOT AVAILABLE
11. Solve the following problems by using the graphing method.
12. Mary's car is $y = 12x$ old and has a problem. The repair man indicates that it will cost her $\$5000$ to repair her car. She can purchase a different, more efficient car for $\$5000$. Her present car averages about $\$5000$ per year for gas while the new car would average about $\$5000$ per year. Find the number of years for when the total cost of repair would equal the total cost of replacement.
13. Juan is considering two cell phone plans. The first company charges $\$100$ for the phone and $\$12$ per month for the calling plan that Juan wants. The second company charges $\$12$ for the same phone, but charges $\$12$ per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
14. A tortoise and hare decide to race $2x - 7$. The hare, being much faster, decided to give the tortoise a head start of $2x - 7$. The tortoise runs at $A\ (-4, -4)$ and the hare runs at 6 times per second. How long will it be until the hare catches the tortoise?

Review Answers

1. (c)
2. (a)
3. (b)

4. (a)
5. $(0, 0)$
6. $(0, 0)$
7. $(3 + 2)$
8. $(0, 0)$
9. $(3 + 2)$
10. $(0, 0)$
11. $(3 + 2)$
12. $(0, 0)$
13. (mph)
14. $(3 + 2)$
15. $y = -2x$
16. 2.236067977
17. 21° Celsius

Solving Linear Systems by Substitution

Learning Objectives

- Solve systems of equations with two variables by substituting for either variable.
- Manipulate **standard form** equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Introduction

In this lesson, we will learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let’s look again at the problem involving Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of $x + 9$ *per second and Nadia can run at a speed of* $x + 9$ *per second. To be a good sport Nadia likes to give Peter a head start of* $2x - 7$*. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?*

In that example, we came up with two equations.

$$\begin{array}{ll} \text{Nadias equation} & d = 6t \\ \text{Peters equation} & d = 5t + 20 \end{array}$$

We have seen that each relationship produces its own line on a graph, but that to solve the system we find the point at which the lines intersect (Lesson 1). At that point the values for h and t satisfy **both** relationships.

In this simple example, this means that the h in Nadia's equation is the same as the h in Peter's. We can set the two equations equal to each other to solve for t .

$$\begin{array}{ll} 6t = 5t + 20 & \text{Subtract } 5t \text{ from both sides.} \\ t = 20 & \text{Substitute this value for } t \text{ into Nadias equation.} \\ d = 6 \cdot 20 = 120 & \end{array}$$

Even if the equations are not so obvious, we can use simple algebraic manipulation to find an expression for one variable in terms of the other. We can rearrange Peter's equation to isolate t.

$$\begin{array}{ll} d = 5t + 20 & \text{Subtract } 20 \text{ from both sides.} \\ d - 20 = 5t & \text{Divide by } 5. \\ \frac{d - 20}{5} = t & \end{array}$$

We can now ***substitute*** this expression for t into Nadia's equation $(5 - 11)$ to solve it.

$$
\begin{array}{ll}
d = 6\left(\dfrac{d-20}{5}\right) & \text{Multiply both sides by 5.} \\
5d = 6(d-20) & \text{Distribute the 6.} \\
5d = 6d - 120 & \text{Subtract } 6d \text{ from both sides.} \\
-d = -120 & \text{Divide by } -1. \\
d = 120 & \text{Substitute value for } d \text{ into our expression for } t. \\
t = \dfrac{120-20}{5} = \dfrac{100}{5} = 20 &
\end{array}
$$

We find that Nadia and Peter meet 35 nickels **after they start racing, at a distance of** $By = -2$ **away**.

The method we just used is called the **Substitution Method**. In this lesson, you will learn several techniques for isolating variables in a system of equations, and for using the expression you get for solving systems of equations that describe situations like this one.

Example 1

Let us look at an example where the equations are written in **standard form.**

Solve the system

$$
\begin{aligned}
y &= x + 2 \\
y &= -2x + 1
\end{aligned}
$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. It makes sense to use this equation to solve for y.

Solve the second equation for the y variable:

$$
\begin{array}{ll}
-4x + y = 2 & \text{Add } 4x \text{ to both sides.} \\
\qquad y = 2 + 4x &
\end{array}
$$

Substitute this expression into the second equation.

$$\begin{aligned} 2x + 3(2 + 4x) &= 6 && \text{Distribute the 3.} \\ 2x + 6 + 12x &= 6 && \text{Collect like terms.} \\ 14x + 6 &= 6 && \text{Subtract 6 from both sides.} \\ 14x &= 0 \\ x &= 0 \end{aligned}$$

Substitute back into our expression for y.

$y = .25x - 422.1$

As you can see, we end up with the same solution $(x - y) \{ \ x \mid x$ that we found when we graphed these functions (Lesson 7.1). As long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, consider a more complicated example. In the following example the solution gives fractional answers for both x and y, and so would be very difficult to solve by graphing alone!

Example 2

Solve the system

$$\begin{aligned} 2x + 3y &= 3 \\ 2x - 3y &= -1 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. Right now it doesn't matter which equation we use or which variable we solve for.

Solve the first equation for x

$$\begin{aligned} 2x + 3y &= 3 && \text{Subtract } 3y \text{ from both sides.} \\ 2x &= 3 - 3y && \text{Divide both sides by 2.} \\ x &= \frac{3 - 3y}{2} \end{aligned}$$

Substitute this expression into the second equation.

$$\begin{aligned}
\cancel{2}.\frac{1}{\cancel{2}}(3-3y)-3y &= -1 && \text{Cancel the fraction and rewrite terms.}\\
3-3y-3y &= -1 && \text{Collect like terms.}\\
3-6y &= -1 && \text{Subtract 3 from both sides.}\\
-6y &= -4 && \text{Divide by } -6.\\
y &= \frac{2}{3}
\end{aligned}$$

Substitute into the expression and solve for x.

$$x=\frac{1}{2}\left(3-\cancel{3}\frac{2}{\cancel{3}}\right)$$
$$x=\frac{1}{2}$$

So our solution is, $x=\frac{1}{2}, a=\frac{1}{3}$. You can see why the graphical solution $b=\frac{2}{3}$ might be difficult to read accurately.

Solving Real-World Problems Using Linear Systems

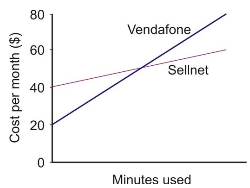
There are many situations where we can use simultaneous equations to help solve real-world problems. We may be considering a purchase. For example, trying to decide whether it is cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but do not know if you would really save any money by buying a new CD every month in that way. One example with which we are all familiar is considering phone contracts. Let’s look at an example of that now.

Example 3

Anne is trying to choose between two phone plans. The first plan, with Vendafone costs $\$12$ *per month, with calls costing an additional* 22 coins *per minute. The second company, Sellnet, charges* $\$12$ *per month, but calls cost only* $3x<5$ *per minute. Which should she choose?*

Anne’s choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the number of minutes is the independent variable, it will be our x. Cost is dependent on minutes. The *cost per month* is the *dependent* variable and will be assigned y.

$$\begin{array}{ll} \text{For Vendafone} & y = 0.25x + 20 \\ \text{For Sellnet} & y = 0.08x + 40 \end{array}$$



By writing the equations in slope-intercept form $20(10) \le 250$ you can visualize the situation in a simple sketched graph, shown right. The line for Vendafone has an intercept of 29 and a slope of $-5x$. The Sellnet line has an intercept of 29 and a slope of $-5x$ (which is roughly a third of the Vendafone line). In order to help Anne decide which to choose, we will determine where the two lines cross, by solving the two equations as a system. Since equation one gives us an expression for $y\ 2 - (3x + 2)$, we can substitute this expression directly into equation two.

$$\begin{aligned} 5x^2 - 4y &= 5(-4)^2 - 4(5) && \text{Substitute } x = -4 \text{ and } y = 5. \\ &= 5(16) - 4(5) && \text{Evaluate the exponent } (-4)^2 = 16. \\ &= 80 - 20 \\ &= 60 \end{aligned}$$

We can now use our sketch, plus this information to provide an answer:

If Anne will use 750 *minutes or less every month, she should choose Vendafone. If she plans on using* 100 *or more minutes, she should choose Sellnet.*

Mixture Problems

Systems of equations crop up frequently when considering chemicals in solutions, and can even be seen in things like mixing nuts and raisins or examining the change in your pocket! Let's look at some examples of these.

Example 4

Nadia empties her purse and finds that it contains only nickels (worth $3x < 5$ each) and dimes (worth $x = 250$ each). If she has a total of 7 coins and they have a combined value of 22 coins, how many of each coin does she have?

Since we have two types of coins, let's call the number of nickels x and the number of dimes will be our y. We are given two key pieces of information to make our equations, the number of coins and their value.

Number of coins equation	$x + y = 7$	(number of nickels) + (number of dimes)
The value equation	$5x + 10y = 55$	Since nickels are worth five cents and dimes ten cents

We can quickly rearrange the first equation to isolate x.

Image courtesy of Kevin@flickr.com/creativecommons

$x = 7 - y$	Now substitute into equation two.
$5(7 - y) + 10y = 55$	Distribute the 5.
$35 - 5y + 10y = 55$	Collect like terms.
$35 + 5y = 55$	Subtract 35 from both sides.
$5y = 20$	Divide by 5.
$y = 4$	Substitute back into equation one.
$+4 = 7$	Subtract 4 from both sides.
$x = 3$	

Solution

Nadia has y nickels and 4 dimes.

Sometimes the question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product**. Three examples are below.

Type of Mixture	First Equation	Second Equation
Coins (items with $\$$ value)	Total number of items $(x-y)$	Total value (item value x no. of items)
Chemical solutions	Total solution volume $11(2+6)$	Amount of solute (vol x concentration)
Density of two substances	Total amount or volume of mix	Total mass (volume x density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of solute in the individual parts and in the final mixture. A solute is simply the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine. Even if the chemical is more exotic, we are still interested in the **total amount** of that chemical in each part. To find this, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 350 ml *of copper-sulfate solution with a* 75% *concentration. In order to do this, he wishes to use a high concentration solution* $[3, 12)$ *and dilute it with a low concentration solution* $|25|$. *How much of each solution should he use?*

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (h). We will also convert the percentages (25%, 75% and 5%) into decimals ($0.6,\ 0.15 \text{ and } 0.05$). The two pieces of critical information we need is the final volume $(60, 720)$ and the final amount of solute (75% of $3x + 150 = 300$). Our equations will look like this.

$$\begin{array}{rr} \text{Volume equation} & x + y = 500 \\ \text{Solute equation} & 0.6x + 0.05y = 75 \end{array}$$

You should see that to isolate a variable for substitution it would be easier to start with equation one.

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$$\begin{aligned}
x + y &= 500 && \text{Subtract } y \text{ from both sides.}\\
x &= 500 - y && \text{Now substitute into equation two.}\\
0.6(500 - y) + 0.05y &= 75 && \text{Distribute the 6.}\\
300 - 0.6y + 0.05y &= 75 && \text{Collect like terms.}\\
300 - 0.55y &= 75 && \text{Subtract 300 from both sides.}\\
-0.55y &= -225 && \text{Divide both sides by } -0.55.\\
y &= 409 \text{ ml} && \text{Substitute back into equation for } x.\\
x &= 500 - 409 = 91 \text{ ml}
\end{aligned}$$

Solution

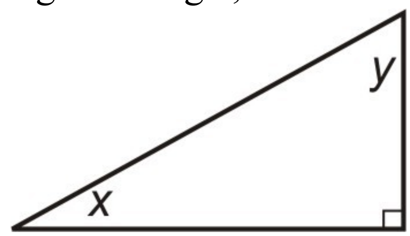
The chemist should mix $x = 7$ of the 25% solution with 350 ml of the 5% solution.

Review Questions

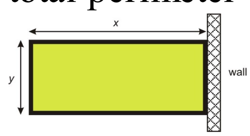
1. Solve the system:

2. solve the system.

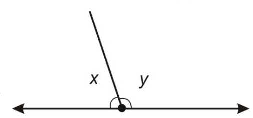
3. Of the two non-right angles in a right angled triangle, one measures twice that of the other. What are the angles?



4. The sum of two numbers is 75. They differ by 11. What are the numbers?
5. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is $1.2\ \mathrm{Amps}$. If the field has a total perimeter of $1.2\ \mathrm{Amps}$, what are the dimensions of the field?



6. A ray cuts a line forming two angles. The difference between the two angles is -8. What does each angle measure?



7. I have $\$12$ and wish to buy five pounds of mixed nuts for a party. Peanuts cost $\$0.50$ per pound. Cashews cost $\$0.50$ per pound. How many pounds of each should I buy?
8. A chemistry experiment calls for one liter of sulfuric acid at a 75% concentration, but the supply room only stocks sulfuric acid in concentrations of 75% and in 25%. How many liters of each should be mixed to give the acid needed for the experiment?
9. Bachelle wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is $11(2+6)$ and the density of silver is $11(2+6)$. The jeweler told her that the volume of silver used was $1\ \mathrm{km}$ and the volume of gold used was 13.21 . Find the combined density of her bracelet.

Review Answers

1. $y = 12x\ j = 6$
2. $5.5\ \mathrm{kg}\ y = -1$
3. $x = 30^\circ,\ y = -2$
4. $-5x$ and $-5x$
5. $y = -0.2x + 7\ 3x^2 + 2x + 1$
6. $x = 30^\circ,\ y = -2$

7. 87.5 grams of peanuts, $y = 4x + b$ of cashews
8. 15 ohms. of 75%, 15 ohms. of 25%
9. $11(2+6)$

Solving Linear Systems by Elimination through Addition or Subtraction

Learning Objectives

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve real-world problems using linear systems by elimination.

Introduction

In this lesson, we will look at using simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y) this method is often referred to as **solving by elimination**. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Example 1

If one apple plus one banana costs $\$0.50$ *and one apple plus two bananas costs* $\$0.50$, *how much does it cost for one banana? One apple?*

It shouldn't take too long to discover that each banana costs $\$0.50$. You can see this by looking at the difference between the two situations. Algebraically, using a and b as the cost for apples and bananas, we get the following equations.

$$\begin{aligned} a + b &= 1.25 \\ a + 2b &= 2.00 \end{aligned}$$

If you look at the difference between the two equations you see that the difference in items purchased is one banana, and the difference in money paid is $7x = 35$. So one banana costs $7x = 35$.

$(a + 2b) - (a + b) = 2.00 - 1.25 \Rightarrow b = 0.75$

To find out how much one apple costs, we subtract $\$0.50$ from the cost of one apple and one banana. So an apple costs 22 coins.

$a + 0.75 = 1.25 \Rightarrow a = 1.25 - 0.75 \Rightarrow a = 0.50$

To solve systems using addition and subtraction, we will be using exactly this idea. By looking at the sum or difference of the two equations, we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method requires us to combine two equations in such a way that the resulting equation has only one variable. We can then use simple linear algebra methods of solving for that variable. If required, we can always substitute the value we get for that variable back in either one of the original equations to solve for the remaining unknown variable.

Example 2

Solve the system by addition:

$3x + 2y = 11$
$5x - 2y = 13$

We will add everything on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right.

$(3x + 2y) + (5x - 2y) = 11 + 13 \Rightarrow 8x = 24 \Rightarrow x = 3$

A simpler way to visualize this is to keep the equations as they appear above, and to add in columns. However, just like adding units tens and hundreds, you MUST keep $+1$ and y's in their own columns. You may also wish to use terms like "$2y$" as a placeholder!

$$\begin{aligned} 3x + 2y &= 11 \\ +(3x - 2y) &= 13 \\ \hline 8x + 0y &= 24 \end{aligned}$$

Again we get 22 coins or $x = 3$.

To find a value for y we simply substitute our value for x back in.

Substitute $x = 3$ into the second equation.

$$\begin{aligned} 5 \cdot 3 - 2y &= 13 && \text{Since } 5 \times 3 = 15, \text{ we subtract } 15 \text{ from both sides.} \\ -2y &= -2 && \text{Divide by 2 to get the value for } y. \\ y &= 1 \end{aligned}$$

The first example has a solution at $x = 3$ and $y = 1$. You should see that the method of addition works when the coefficients of one of the variables are opposites. In this case it is the coefficients of y that are opposites, being -7 in the first equation and -2 in the second.

There are other, similar, methods we can use when the coefficients are not opposites, but for now let's look at another example that can be solved with the method of addition.

Example 3

Dagwood21/www.flickr.com/creativecommons

Andrew is paddling his canoe down a fast moving river. Paddling downstream he travels at -1.375 *per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at* 15 ohms. *per hour. If he paddles equally hard in both directions, calculate, in miles per hour, the speed of the river and the speed Andrew would travel in calm water.*

Step One First, we convert our problem into equations. We have two unknowns to solve for, so we will call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew's speed is boosted by the river current, so his total speed is the canoe speed plus the speed of the river $(x-3)$. Upstream, his speed is hindered by the speed of the river. His speed upstream is $(x-y)$.

$$\begin{aligned} \text{Downstream Equation} \qquad & x+y=7 \\ \text{Upstream Equation} \qquad & x-y=1.5 \end{aligned}$$

Step Two Next, we are going to eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is $+1$ in the first equation and -1 in the second. Clearly $(4+5)-(5+2)$, so this is the variable we will eliminate. To do this we add equation 1 to equation 4. We must be careful to collect like terms, and that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$2-(3(2)+2) \qquad 3(2) \text{ is the same as } 3\times 2$$

Or, using the column method we used in example one.

$$\begin{aligned} x+y&=7 \\ +\,(x-y)&=1.5 \\ \hline 2x+0y&=8.5 \end{aligned}$$

Again you see we get $2x=8.5$, or $x=0.02$. To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, we’ll plug it into the first equation.

Substitute $x=3$ into the second equation:

$$\begin{aligned} 4.25+y&=7 \qquad \text{Subtract 4.25 from both sides.} \\ y&=2.75 \end{aligned}$$

Solution

Andrew paddles at 0° Celsius *per hour. The river moves at* 0° Celsius *per hour*.

Solving Linear Systems Using Subtraction of Equations

Another, very similar method for solving systems is subtraction. In this instance, you are looking to have identical coefficients for x or y (including the sign) and then subtract one equation from the other. If you look at Example one you can see that the coefficient for x in both equations is $+1$. You could have also used the method of subtraction.

$$(x + y) - (x - y) = 200 - 80 \Rightarrow 2y = 120 \Rightarrow y = 60$$

or

$$\begin{array}{r} x + y = 200 \\ + (x - y) = -80 \\ \hline 0x + 2y = 120 \end{array}$$

So again we get $p = 15$, from which we can determine x. The method of subtraction looks equally straightforward, and it is so long as you remember the following:

1. Always put the equation you are subtracting in parentheses, and distribute the negative.
2. Don’t forget to subtract the numbers on the right hand side.
3. Always remember that subtracting a negative is the same as adding a positive.

Example 4

Peter examines the coins in the fountain at the mall. He counts 18 inches*, all of which are either pennies or nickels. The total value of the coins is* $\$0.50$*. How many of each coin did he see?*

We have two types of coins. Let’s call the number of pennies x and the number of nickels y. The total value of pennies is just x, since they are worth one cent each. The total value of nickels is $2y$. We are given two key pieces of information to make our equations. The number of coins and their value.

$$\begin{aligned} \text{Number of Coins Equation} \quad & x + y = 107 \quad \text{(number of pennies)} + \text{(number of nickels)} \\ \text{The Value Equation:} \quad & x + 5y = 347 \quad \text{pennies are worth } 1c, \text{nickels are worth } 5c. \end{aligned}$$

We will jump straight to the subtraction of the two equations.

$$\begin{aligned} x + y &= 107 \\ +(x + 5y) &= -347 \\ \hline 4y &= -240 \end{aligned}$$

Let's substitute this value back into the first equation.

$$\begin{aligned} x + 60 &= 107 \qquad \text{Subtract } 60 \text{ from both sides.} \\ x &= 47 \end{aligned}$$

So Peter saw 27 pennies (worth $x < 7 >$) and 29 nickels (worth $\$0.50$) for a total of $\$0.50$.

We have now learned three techniques for solving systems of equations.

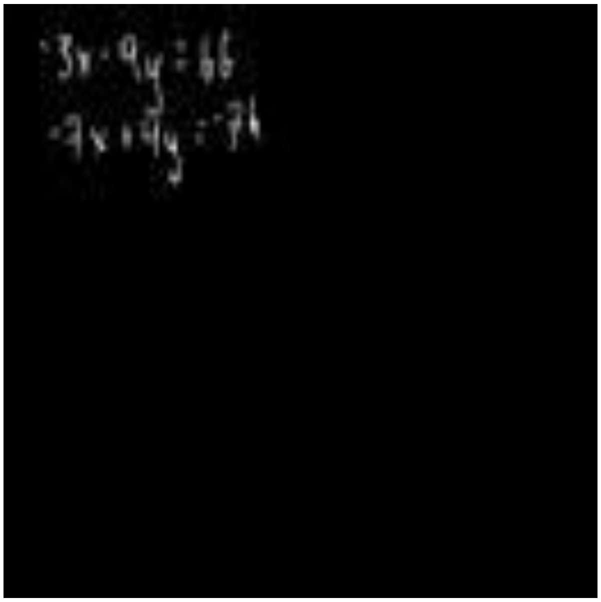
1. Graphing
2. Substitution
3. Elimination

You should be starting to gain an understanding of which method to use when given a particular problem. For example, **graphing** is a good technique for seeing what the equations are doing, and when one service is less expensive than another. Graphing alone may not be ideal when an exact numerical solution is needed.

Similarly, **substitution** is a good technique when one of the coefficients in your equation is $+1$ or -1.

Addition or **subtraction** is ideal when the coefficient of one of the variables matches the coefficient of the same variable in the other equation. In the next lesson, we will learn the last technique for solving systems of equations exactly, when none of the coefficients match and the coefficient is not one.

Multimedia Link The following video contains three examples of solving systems of equations using multiplication and addition and subtraction as well as multiplication (which is the next topic). [Khan Academy Systems of Equations](#) (9:57)



systems of equations([Watch on Youtube](#))

Review Questions

1. Solve the system:

2. Solve the system

3. Solve the system

4. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $\$0.50$. Peter also buys three candy bars, but can only

afford one additional fruit roll-up. His purchase costs $\$0.50$. What is the cost of each candy bar and each fruit roll-up?

5. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at $2 + 3 = 5$ per hour. Another, identical plane, moving in the opposite direction has a ground-speed of $2 + 3 = 5$ per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
6. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a $F = ma$ journey costs $\$11.95$ and a 5 hours journey costs $\$37.71$, calculate:
   1. the pick-up fee
   2. the per-mile rate
   3. the cost of a seven mile trip
7. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a seven minute call costs $\$0.50$ and a 12 minute call costs $\$0.50$, find each rate.
8. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $\$12$ per hour, and the builder earns $\$12$ per hour. Together they were paid $2a + 3b$, but the plumber earned $2a + 3b$ more than the builder. How many hours did each work?
9. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 30 ohms per week. In his first week, he sold eight warranties and earned $\$100$. In his second week, he managed to sell 16 warranties and earned $\$100$. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

Review Answers

1. $y = 12x$ $y = -2$
2. $y = 12x$ $y = -2$
3. $x = 7.5, y = -1$
4. Candy bars cost 22 coins each and fruit roll-ups cost 22 coins each.
5. The wind speed is 40 mph

6.
    1. 75%
    2. $\$0.50$
    3. $\$0.50$
7. $7x = 35$ per minute for the first $4y = 5$ 22 coins per minute additional
8. The plumber works 15 seconds, the builder works $2 > -5$
9. Paul earns a base of $\$0.50$ per hour

Solving Systems of Equations by Multiplication

Learning objectives

- Solve a linear system by multiplying one equation.
- Solve a linear system of equations by multiplying both equations.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Introduction

We have now learned three techniques for solving systems of equations.

- Graphing, Substitution and Elimination (through addition and subtraction).

Each one of these methods has both strengths and weaknesses.

- **Graphing** is a good technique for seeing what the equations are doing, and when one service is less expensive than another, but graphing alone to find a solution can be imprecise and may not be good enough when an exact numerical solution is needed.

- **Substitution** is a good technique when one of the coefficients in an equation is $+1$ or -1, but can lead to more complicated formulas when there are no unity coefficients.

- **Addition** or **Subtraction** is ideal when the coefficients of either x or y match in both equations, but so far we have not been able to use it when coefficients do not match.

In this lesson, we will again look at the method of elimination that we learned in Lesson 7.3. However, the equations we will be working with will be more complicated and one can not simply add or subtract to eliminate one variable. Instead, we will first have to multiply equations to ensure that the coefficients of one of the variables are matched in the two equations.

Quick Review: Multiplying Equations

Consider the following questions

1. *If 16 apples cost x^8, how much would 29 apples cost?*
2. *If 3 bananas plus 4 carrots cost x^8, how much would y bananas plus 4 carrots cost?*

You can look at the first equation, and it should be obvious that each apple costs $\$0.50$. 29 apples should cost $\$11.95$.

Looking at the second equation, it is not clear what the individual price is for either bananas or carrots. Yet we know that the answer to question two is $\$0.50$. How?

If we look again at question one, we see that we can write the equation $3x - 10$ (a being the cost of one apple).

To find the cost of 29, we can either solve for a them multiply by 29, or we can multiply both sides of the equation by three.

$$30a = 15$$
$$a = \frac{1}{2} \text{ or } 0.5$$

Now look at the second question. We could write the equation $3b + 2c = 4$.

We see that we need to solve for $5x^2 - 4y$ which is simply two times the quantity $5x^2 - 4y$!

Algebraically, we are simply multiplying the entire equation by two.

$$\begin{aligned} 2(3b+2c) &= 2 \cdot 4 && \text{Distribute and multiply.} \\ 6b+4c &= 8 \end{aligned}$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

We can multiply every term in an equation by a fixed number (a **scalar**), it is clear that we could use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match. In the simplest case, the coefficient as a variable in one equation will be a multiple of the coefficient in the other equation.

Example 1

Solve the system.

$$\left(x - \frac{3}{5}\right) = 7$$

It is quite simple to see that by multiplying the second equation by two the coefficients of y will be -7 and -2, allowing us to complete the solution by addition.

Take two times equation two and add it to equation one. Then divide both sides by 75 to find x.

$$\begin{aligned} 10x - 4y &= 22 \\ +\,(7x + 4y) &= 17 \end{aligned}$$

$$\begin{aligned} 17y &= 34 \\ x &= 2 \end{aligned}$$

Now simply substitute this value for x back into equation one.

$$\begin{aligned} 7 \cdot 2 + 4y &= 17 && \text{Since } 7 \times 2 = 14, \text{ subtract } 14 \text{ from both sides.} \\ 4y &= 3 && \text{Divide by } 4. \\ y &= 0.75 \end{aligned}$$

Example 2

Anne is rowing her boat along a river. Rowing downstream, it takes her two minutes to cover 1.2 Amps. Rowing upstream, it takes her eight minutes to travel the same 1.2 Amps. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Step One First we convert our problem into equations. We need to know that *distance traveled* is equal to *speed* x time. We have two unknowns, so we will call the speed of the river x, and the speed that Anne rows at y. When traveling downstream her total speed is the boat speed plus the speed of the river $(x-3)$. Upstream her speed is hindered by the speed of the river. Her speed upstream is $(x-y)$.

$$\begin{aligned} \text{Downstream Equation} & \qquad 2(x+y)=400 \\ \text{Upstream Equation} & \qquad 8(x-y)=400 \end{aligned}$$

Distributing gives us the following system.

$$2x+2y=400$$
$$8x-8y=400$$

Right now, we cannot use the method of elimination as none of the coefficients match. But, if we were to multiply the top equation by four, then the coefficients of y would be -7 and -8. Let's do that.

$$\begin{aligned} 8x-8y &= 1,600 \\ +(8x-8y) &= 400 \\ \hline 16x &= 2,000 \end{aligned}$$

Now we divide by 16 to obtain $x=250$.

Substitute this value back into the first equation.

$$\begin{aligned} 2(125+y) &= 400 && \text{Divide both sides by 2.} \\ 125+y &= 200 && \text{Subtract 125 from both sides.} \\ y &= 75 \end{aligned}$$

Solution

Anne rows at $By = -2$ per minute, and the river flows at 75 yards per minute.

Solve a Linear System by Multiplying Both Equations

It is a straightforward jump to see what would happen if we have no matching coefficients and no coefficients that are simple multiples of others. Just think about the following fraction sum.

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

This is an example of finding a **lowest common denominator**. In a similar way, we can always find a lowest common multiple of two numbers (the **lowest common multiple** of 4 and y is y). This way we can always find a way to multiply equations such that two coefficients match.

Example 3

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of $y = mx + b$ in five days. Anne travels $2 + 3 = 5$ from Norfolk, Virginia, and it takes her three days. If Anne pays $\$100$ and Andrew pays $\$5000$, what does I-Haul charge

a) per day?

b) per mile traveled?

First, we will set-up our equations. Again we have two unknowns, the **daily rate** (we will call this x), and the **rate per mile** (let's call this y).

$$\begin{array}{lc} \text{Annes equation} & 3x + 880y = 840 \\ \text{Andrews Equation} & 5x + 2060y = 1845 \end{array}$$

We cannot simply multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y) we see that they both have a common multiple of 16 (in fact 16 is the **lowest common multiple**). So this time we need to multiply both equations:

Multiply Anne's equation by five:

$15x + 4400y = 4200$

Multiply Andrew's equation by three:

$15x + 4400y = 4200$

Subtract:

$$\begin{aligned} 15x + 4400y &= 4200 \\ -(15x + 6180y) &= 5535 \\ \hline -1780y &= -1335 \end{aligned}$$

Divide both sides by $= 25\Omega$

$P = 20t;$

Substitute this back into Anne's equation.

$$\begin{aligned} 3x + 880(0.75) &= 840 \quad \text{Since } 880 \times 0.75 = 660, \text{ subtract } 660 \text{ from both sides.} \\ 3x &= 180 \quad \text{Divide both sides by 3.} \\ x &= 60 \end{aligned}$$

Solution

I-Haul charges $\$12$ per day plus $\$0.50$ per mile.

Comparing Methods for Solving Linear Systems

Now that we have covered the major methods for solving linear equations, let’s review. For simplicity, we will look at the four methods (we will consider **addition** and **subtraction** one method) in table form. This should help you decide which method would be better for a given situation.

Method:	**Best used when you…**	**Advantages:**	**Comment:**
Graphing	…don't need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don't have to do any computation.	Can lead to imprecise answers with non-integer solutions.
Substitution	…have an explicit equation for one variable (e.g. $y = 15 + 5x$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, thus it can lead to more complicated formulas
Elimination by Addition or Subtraction	…have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	…do not have any variables defined explicitly or any matching coefficients.	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

The table above is only a guide. You may like to use the graphical method for every system in order to better understand what is happening, or you may prefer to use the multiplication method even when a substitution would work equally well.

Example 4

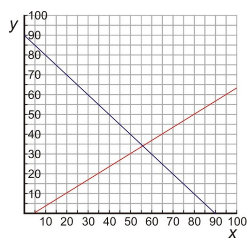
Two angles are ***complementary*** *when the sum of their angles is* $y-$*. Angles* A *and* P *are complementary angles, and twice the measure of angle* A *is* P *more than three times the measure of angle* P*. Find the measure of each angle.*

First, we write out our two equations. We will use x to be the measure of Angle A and y to be the measure of Angle P. We get the following system

$$x + y = 90$$
$$2x = 3y + 9$$

The first method we will use to solve this system is the graphical method. For this we need to convert the two equations to $y = mx + b$ form

$$x + y = 90 \qquad \Rightarrow \qquad y = -x + 90$$
$$2x = 3y + 9 \qquad \Rightarrow \qquad y = \frac{2}{3}x - 3$$



The first line has a slope of -1 and a $y-$intercept of 29.

The second has a slope of $\frac{2}{3}$ and a $y-$intercept of -8.

In the graph, it appears that the lines cross at around $k = 12$, $p = 15$ but it is difficult to tell exactly! Graphing by hand is not the best method if you need to know the answer with more accuracy!

Next, we will try to solve by substitution. Let's look again at the system:

$$x + y = 90$$
$$2x = 3y + 9$$

We have already seen that we can solve for y with either equation in trying to solve the system graphically.

Solve the first equation for y.

$-2.5, 1.5, 5$

Substitute into the second equation

$$\begin{aligned} 2x &= 3(90 - x) + 9 && \text{Distribute the 3.} \\ 2x &= 270 - 3x + 9 && \text{Add } 3x \text{ to both sides.} \\ 5x &= 270 + 9 = 279 && \text{Divide by 5.} \\ x &= 55.8^\circ \end{aligned}$$

Substitute back into our expression for y.

$y = 90 - 55.8 = 34.2^\circ$

Solution

Angle A measures 1 km. Angle P measures 1 km

Finally, we will examine the method of elimination by multiplication.

Rearrange equation one to standard form

$x + y = 90 \Rightarrow 2x + 2y = 180$

Multiply equation two by two.

$2x = 3y + 9 \Rightarrow 2x - 3y = 9$

Subtract.

$$\begin{array}{r} 2x + 2y = 180 \\ -\,(2x - 2y) = -9 \\ \hline 5y = 171 \end{array}$$

Divide both sides by y

$y = 34.2$

Substitute this value into the very first equation.

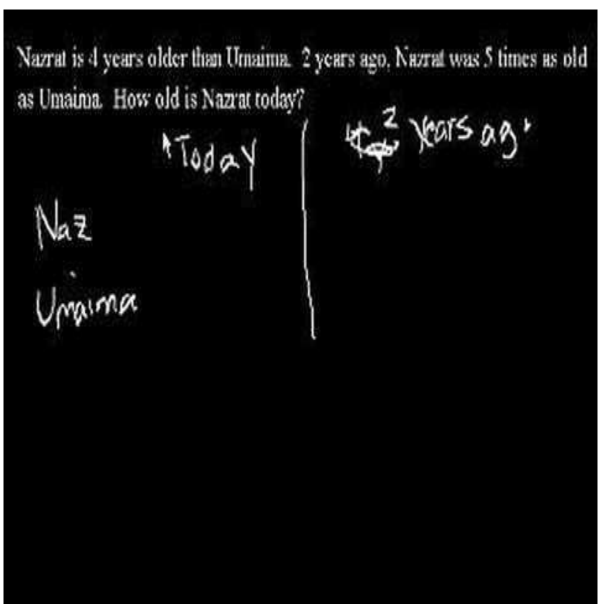
$$x + 34.2 = 90 \qquad \text{Subtract 34.2 from both sides.}$$
$$x = 55.8^\circ$$

Solution

Angle A *measures* 1 km. *Angle* P *measures* 1 km.

Even though this system looked ideal for substitution, the method of multiplication worked well, also. Once the algebraic manipulation was performed on the equations, it was a quick solution. You will need to decide yourself which method to use in each case you see from now on. Try to master all techniques, and recognize which technique will be most efficient for each system you are asked to solve.

Multimedia Link For even more practice, we have this video. One common type of problem involving systems of equations (especially on standardized tests) is "age problems." In the following video the narrator shows two examples of age problems, one involving a single person and one involving two people. Khan Academy Age Problems (7:13)



Age word problems(Watch on Youtube)

Review Questions

1. Solve the following systems using multiplication.
   1. IMAGE NOT AVAILABLE
   2. IMAGE NOT AVAILABLE
   3. IMAGE NOT AVAILABLE
   4. IMAGE NOT AVAILABLE
   5. IMAGE NOT AVAILABLE
   6. IMAGE NOT AVAILABLE
2. Solve the following systems using any method.
   1. IMAGE NOT AVAILABLE
   2. IMAGE NOT AVAILABLE
   3. IMAGE NOT AVAILABLE
   4. IMAGE NOT AVAILABLE
   5. IMAGE NOT AVAILABLE
   6. IMAGE NOT AVAILABLE
3. Supplementary angles are two angles whose sum is -53. Angles A and P are supplementary angles. The measure of Angle A is -8 less than twice the measure of Angle P. Find the measure of each angle.
4. A farmer has fertilizer in 5% and 75% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 75% solution?
5. A $By = -2$ pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?
6. Mr. Stein invested a total of $\$100,000$ in two companies for a year. Company A's stock showed a 75% annual gain, while Company B showed a 5% loss for the year. Mr. Stein made an 5% return on his investment over the year. How much money did he invest in each company?
7. A baker sells plain cakes for x^8 or decorated cakes for $\$21$. On a busy Saturday the baker started with 100 cakes, and sold all but three. His takings for the day were $\$991$. How many plain cakes did he sell that day, and how many were decorated before they were sold?
8. Twice John's age plus five times Claire's age is 302. Nine times John's age minus three times Claire's age is also 302. How old are John and

Claire?

Review Answers

1.
    1. $x = 3$, $y = -1.5$
    2. $x = 3$, $y = 5$
    3. $x = 0.02$, $y = 34.2$
    4. $x = \frac{2}{3}$, $a = \frac{1}{3}$
    5. $\frac{2 \cdot 4}{9 \cdot 4} = \frac{8}{36}$, $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
    6. $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$, $\frac{1}{3} \cdot \$60$
2.
    1. $x = -5$, $y = -2$
    2. $x = 1$, $y = 5$
    3. $x = 2$, $y = 5$
    4. $x = -4$, $y = 5$
    5. $\frac{3}{7} + \frac{-3}{7}$, $\frac{1}{3} \cdot \$60$
    6. $\frac{1 \cdot 11}{9 \cdot 11} = \frac{11}{99}$, $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
3. $-9 = -9$, $x < 7 >$
4. $x < 7 >$ of 5%, $7x = 35$ of 75%
5. 51 yards and 51 yards
6. $\$19,500$ in Company A, $\$19,500$ in Company B
7. 75 plain, 29 decorated
8. John is 29, Claire is 29

Special Types of Linear Systems

Learning Objectives

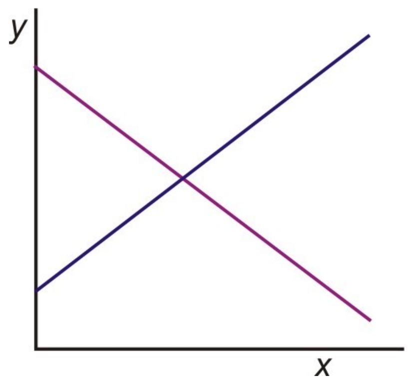
- Identify and understand what is meant by an **inconsistent linear system.**
- Identify and understand what is meant by a **consistent linear system.**
- Identify and understand what is meant by a **dependent linear system.**

Introduction

As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Determining the Type of System Graphically

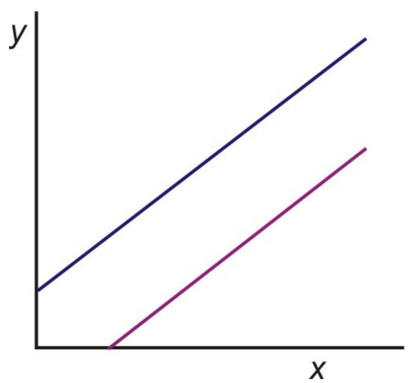
If we graph two lines on the same coordinate plane, three different situations may occur.



Case 1: The two lines intersect at a single point; hence the lines are not parallel.

If these lines were to represent a system of equations, the system would exactly one solution, where the lines cross.

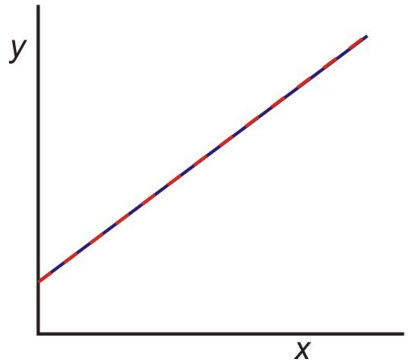
A system with exactly one solution is called a **consistent system.**



Case 2: The two lines do not intersect. The two lines are parallel.

If the lines represent a system of equations, then the system has no solutions.

A system with no solutions is called an **inconsistent system.**



Case 3: The two lines are identical. They intersect at all points on the line.

If this were a system of equations it would have an **infinite number** of solutions. Reason being, the two equations are really the same.

Such a system is called a **dependent system.**

To identify a system as **consistent**, **inconsistent**, or **dependent**, we can graph the two lines on the same graph and match the system with one of the three cases we discussed.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$intercepts of the two lines. To do this we must remember that:

- Lines that intersect have different slopes.
- Lines that are parallel have the same slope but different $y-$intercepts.
- Lines that have the same slope and the same $y-$intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$y = x + 2$
$y = -2x + 1$

Solution

The equations are already in slope-intercept form. The slope of the first equation is y and the slope of the second equation is -2. Since the slopes are different, the lines must intersect at a single point. Therefore, the system has exactly one solution. This is a **consistent system**.

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{ccccc} 2x - 5y = 2 & & -5y = -2x + 2 & & y = \frac{2}{5}x - \frac{2}{5} \\ & \Rightarrow & & \Rightarrow & \\ 4x + y = 5 & & y = -4x + 5 & & y = -4x + 5 \end{array}$$

The slopes of the two equations are different. Therefore, the lines must cross at a single point, and the system has exactly one solution. This is a **consistent system**.

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{lcl}3x = 5 - 4y & & 4y = -3x + 5 & & y = \dfrac{-3}{4}x - \dfrac{5}{4} \\ & \Rightarrow & & \Rightarrow & \\ 6x + 8y = 7 & & 8y = -6x + 7 & & y = \dfrac{-3}{4}x + \dfrac{7}{8}\end{array}$$

The slopes of the two equations are the same but the $y-$intercepts are different, therefore the lines never cross and the system has no solutions. This is an **inconsistent system**.

Example 4

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{lcl}x + y = 3 & & y = -x + 3 & & y = -x + 3 \\ & \Rightarrow & & \Rightarrow & \\ 3x + 3y = 9 & & 3y = -3x + 9 & & y = -x + 3\end{array}$$

The lines are identical. Therefore the system has an infinite number of solutions. It is a **dependent system.**

Determining the Type of System Algebraically

A third option for identifying systems as **consistent**, inconsistent or dependent is to solve the system algebraically using any method and use the result as a guide.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x - 3y &= 3 \\ 2x + y &= 9 \end{aligned}$$

Solution

Let’s solve this system using the substitution method.

Solve the second equation for the y variable.

$$2x + y = 9 \Rightarrow y = -2x + 9$$

Substitute for y in the first equation.

$$\begin{aligned} 10x - 3y &= 3 \\ 10x - 3(-2x + 9) &= 3 \\ 10x + 6x - 27 &= 3 \\ 16x &= 30 \\ x &= \frac{15}{8} \end{aligned}$$

Substitute the value of x back into the second equation and solve for y.

$$y = -\frac{7}{4}(7) + 41 = -\frac{49}{4} + 41 = -\frac{49}{4} + \frac{164}{4} = \frac{115}{4} = 28\frac{3}{4}$$

Answer The solution to the system is $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$. The system is consistent since it has only one solution.

Another method to determine if the system of equations is an inconsistent, consistent or dependent system is to solve them algebraically using the elimination or substitution method.

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

Solution

Let's solve this system by the method of multiplication.

Multiply the first equation by y.

$$\begin{array}{lcr} 3(3x-2y=4) & & 9x-6y=12 \\ & \Rightarrow & \\ 9x-6y=1 & & 9x-6y=1 \end{array}$$

Add the two equations.

$$\begin{aligned} 9x-6y&=12 \\ 9x-6y&=1 \end{aligned}$$

$$0=13 \quad \text{This Statement is not true}$$

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is not true, then the system is **inconsistent**.

Example 7

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x-3y&=3 \\ 2x+y&=9 \end{aligned}$$

Solution

Let's solve this system by substitution.

Solve the first equation for y.

$$2x+y=9 \Rightarrow y=-2x+9$$

Substitute this expression for y in the second equation.

$$\begin{aligned} 12x+3y&=9 \\ 12x+3(-4x+3)&=9 \\ 12x-12x+9&=9 \\ 9&=9 \end{aligned}$$

This is always a **true** statement.

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is always true, then the system is dependent.

A second glance at the system in this example reveals that the second equation is three times the first equation so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Let's clarify this statement. An infinite number of solutions does not mean that any ordered pair (x, y) satisfies the system of equations. Only ordered pairs that solve the equation in the system are also solutions to the system.

For example, $(0, 0)$ is not a solution to the system because when we plug it into the equations it does not check out.

$$\begin{aligned} 4x + y &= 3 \\ 4(1) + 2 &\neq 3 \end{aligned}$$

To find which ordered pair satisfies this system, we can pick any value for x and find the corresponding value for y.

For $x = 1$, $2(5 + 10) = 20 - 2(-6)$

For $x = 2$, $6(\$10) + 6(\$20) = \$180$

Let's summarize our finding for determining the type of system algebraically.

- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will always give a FALSE statement (for example $9 > 3$).
- A **dependent system** will always give a TRUE statement (such as $9 > 3$ or $9 > 3$).

Applications

In this section, we will look at a few application problems and see how consistent, inconsistent and dependent systems might arise in practice.

Example 8

A movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of $\$12$ *and then rent each movie for* x^8 *or they can choose not to pay the membership fee and rent each movie for* $\$0.50$*. How many movies would you have to rent before membership becomes the cheaper option?*

Solution

Let's translate this problem into algebra. Since there are two different options to consider, we will write two different equations and form a system.

The choices are membership option and no membership option.

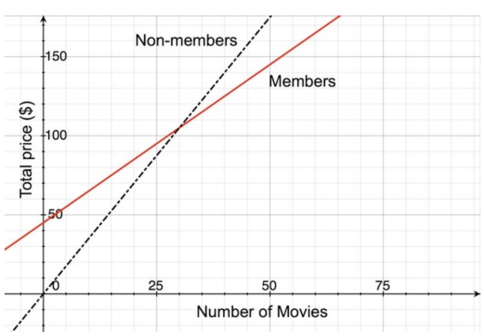
Our variables are

The number of movies you rent, let's call this x.

The total cost for renting movies, let's call this y.

	flat fee	**rental fee**	**total**
membership	$\$12$	$2x$	$y = 15 + 5x$
no membership	x^8	$9 > 3$	$y = -2x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent movies. For the membership option, the rental fee is $2x$ since you would pay x^8 for each movie you rented. For the no membership option, the rental fee is $9 > 3$ since you would pay $\$0.50$ for each movie you rented.



Our system of equations is

$y = 45 + 2x$
$y = 3.50x$

The graph of our system of equations is shown to the right.

This system can be solved easily with the method of substitution since each equation is already solved for y. Substitute the second equation into the first one

$$\begin{aligned} y &= 45 + 2x \\ & \qquad \Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies} \\ y &= 3.50x \end{aligned}$$

Answer You would have to rent 29 movies per year before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Let's examine a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (but the $y-$intercept is different). Let's change the previous problem so that this is the case.

Example 9

Two movie rental stores are in competition. Movie House charges an annual membership of $\$12$ *and charges* x^8 *per movie rental. Flicks for Cheap charges an annual membership of* $\$12$ *and charges* x^8 *per movie rental. After how many movie rentals would Movie House become the better option?*

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per move as Flicks for Cheap.

The lines that describe each option have different $y-$intercepts, namely 29 for Movie House and 16 for Flicks for Cheap. They have the same slope, three dollars per movie. This means that the lines are parallel and the system is inconsistent.

Let's see how this works algebraically:

Our variables are:

The number of movies you rent, let's call this x.

The total cost for renting movies, let's call this y.

	flat fee	**rental fee**	**total**
Movie House	$\$12$	$2\ 1$	$y = 15 + 5x$
Flicks for Cheap	$\$12$	$2\ 1$	$y = 15 + 5x$

The system of equations that describes this problem is

$$y = 45 + 2x$$
$$y = 3.50x$$

Let's solve this system by substituting the second equation into the first equation.

$$y = 30 + 3x$$
$$\Rightarrow 15 + 3x = 30 + 3x \Rightarrow 15 = 30$$
$$y = 15 + 3x$$

This statement is always false.

Answer This means that the system is inconsistent.

Example 10

Peter buys two apples and three bananas for x^8. *Nadia buys four apples and six bananas for* x^8 *from the same store. How much does one banana and one apple costs?*

Solution

We must write two equations, one for Peter's purchase and one for Nadia's purchase.

Let's define our variables as

a is the cost of one apple

b is the cost of one banana

	Cost of Apples	**Cost of Bananas**	**Total Cost**
Peter	$2a$	75	$-4 \leq x \leq 6$
Nadia	$2a$	75	$-4 \leq x \leq 6$

The system of equations that describes this problem is:

$$2a + 3b = 4$$
$$4a + 6b = 8$$

Let's solve this system by multiplying the first equation by -2 and adding the two equations.

$$\begin{aligned} -2(2a + 3b &= 4) \\ 4a + 6b &= 8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -4a - 6b &= -8 \\ 4a + 6b &= 8 \\ \hline 0 + 0 &= 0 \end{aligned}$$

This statement is always true. This means that the system is **dependent**.

Looking at the problem again, we see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for x^8 it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price $(\$8)$. Since the second equation does not give any new information, it is not possible to find out the price of each piece of fruit.

Answer The two equations describe the same line. This means that the system is dependent.

Review Questions

1. Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.
    1. IMAGE NOT AVAILABLE
    2. IMAGE NOT AVAILABLE
    3. IMAGE NOT AVAILABLE
    4. IMAGE NOT AVAILABLE
    5. IMAGE NOT AVAILABLE
    6. IMAGE NOT AVAILABLE
2. Find the solution of each system of equations using the method of your choice. Please state whether the system is inconsistent or dependent.
    1. IMAGE NOT AVAILABLE
    2. IMAGE NOT AVAILABLE
    3. IMAGE NOT AVAILABLE
    4. IMAGE NOT AVAILABLE
    5. IMAGE NOT AVAILABLE
    6. IMAGE NOT AVAILABLE
    7. IMAGE NOT AVAILABLE
    8. IMAGE NOT AVAILABLE
    9. IMAGE NOT AVAILABLE
3. A movie house charges $\$0.50$ for children and $\$0.50$ for adults. On a certain day, 1000 people enter the movie house and $\$2,200$ is collected.

How many children and how many adults attended?

4. Andrew placed two orders with an internet clothing store. The first order was for thirteen ties and four pairs of suspenders, and totaled $\$100$. The second order was for six ties and two pairs of suspenders, and totaled $\$100$. The bill does not list the per-item price but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
5. An airplane took four hours to fly $12 - 7 = 5$ in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane's speed in still air and the jet-stream's speed?
6. Nadia told Peter that she went to the farmer's market and she bought two apples and one banana and that it cost her $\$0.50$. She thought that Peter might like some fruit so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so will only pay her for four apples. Nadia told him that the second time she paid $\$0.50$ for fruit. Please help Peter figure out how much to pay Nadia paid for four apples.

Review Answers

1.
   1. consistent
   2. dependent
   3. consistent
   4. dependent
   5. inconsistent
   6. consistent
2.
   1. $y = 12x$ $y = 5$
   2. $y = 12x$ $y = -2$
   3. $y = 12x$ $y = -2$
   4. inconsistent
   5. 85.45 cm^2 $y = 5$
   6. dependent
   7. $\frac{3}{7} + \frac{-3}{7}$ $a = \frac{1}{3}$
   8. $x = 0,\ p = 15$
   9. dependent

3. 302 children, 302 Adults
4. $\text{Ties} = \$23, \text{suspenders} = \$47$
5. $\text{Airplane speed} = 540 \text{ mph}, \text{ jet} - \text{streamspeed} = 60 \text{ mph}$
6. This represents an inconsistent system. Someone is trying to overcharge! It is not possible to determine the price of apples alone.

Systems of Linear Inequalities

Learning Objectives

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Introduction

In the last chapter, you learned how to graph a linear inequality in two variables. To do that you graphed the equation of the straight line on the coordinate plane. The line was solid for signs where the equal sign is included. The line was dashed for $\div$ or $\div$ where signs the equal sign is not included. Then you shaded above the line (if $= 7$ or 15^{th}) or below the line (if $= 7$ or 15^{th}).

In this section, we will learn how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes. Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example 1

Solve the following system.

$$3x + 2y = 11$$
$$5x - 2y = 13$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection regions of the solution.

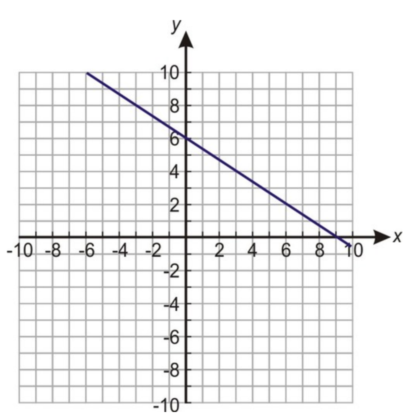
Let's rewrite each equation in slope-intercept form. This form is useful for graphing but also in deciding which region of the coordinate plane to shade. Our system becomes

$$\begin{matrix} 3y \le -2x + 18 & & y \le \dfrac{-2}{3}x + 6 \\ & \Rightarrow & \\ -4y \le -x + 12 & & y \ge \dfrac{x}{4} - 3 \end{matrix}$$

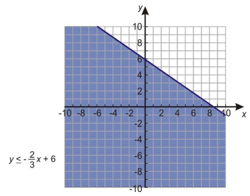
Notice that the inequality sign in the second equation changed because we divided by a negative number.

For this first example, we will graph each inequality separately and then combine the results.

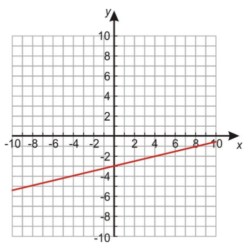
We graph the equation of the line in the first inequality and draw the following graph.



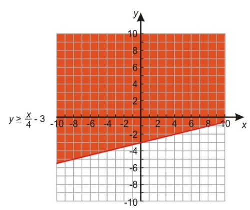
The line is solid because the equal sign is included in the inequality. Since the inequality is less than or equal to, we shade below the line.



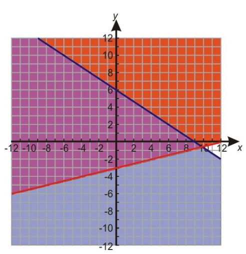
We graph the second equation in the inequality and obtain the following graph.



The line is solid again because the equal sign is included in the inequality. We now shade above because y is **greater** than or equal.



When we combine the graphs, we see that the blue and red shaded regions overlap. This overlap is where both inequalities work. Thus the purple region denotes the solution of the system.



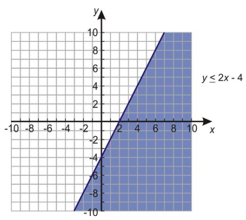
The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

Example 2

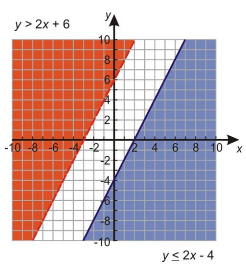
There are also situations where a system of inequalities has no solution. For example, let's solve this system.

$$y \leq 2x - 4$$
$$y > 2x + 6$$

Solution: We start by graphing the first line. The line will be solid because the equal sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equal sign is not included in the inequality. We must shade upward because y is greater than.

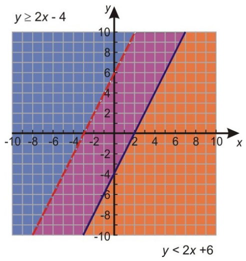


This graph shows no overlapping between the two shaded regions. We know that the lines will never intersect because they are parallel . The slope equals two for both lines. The regions will never overlap even if we extend the lines further.

This is an example of a system of inequalities with no solution.

For a system of inequalities, we can still obtain a solution even if the lines are parallel. Let's change the system of inequalities in Example 2 so the inequality signs for the two expressions are reversed.

$$y \leq 2x - 4$$
$$y > 2x + 6$$



The procedure for solving this system is almost identical to the previous one, except we shade upward for the first inequality and we shade downward for the second inequality. Here is the result.

In this case, the shaded regions do overlap and the system of inequalities has the solution denoted by the purple region.

Graph a System of More Than Two Linear Inequalities

In the previous section, we saw how to find the solution to a system of two linear inequalities. The solutions for these kinds of systems are always unbounded. In other words, the region where the shadings overlap continues infinitely in at least one direction. We can obtain **bounded** solutions by solving systems that contain more than two inequalities. In such cases the solution region will be bounded on four sides.

Let's examine such a solution by solving the following example.

Example 3

Find the solution to the following system of inequalities.

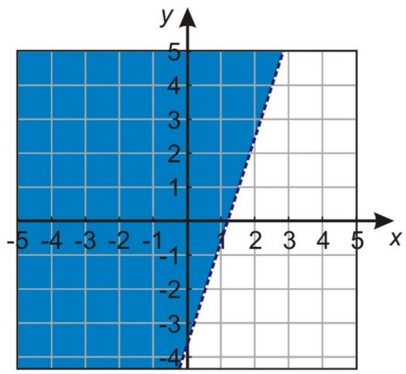
$$\begin{aligned} 3x - y &< 4 \\ 4y + 9x &< 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution

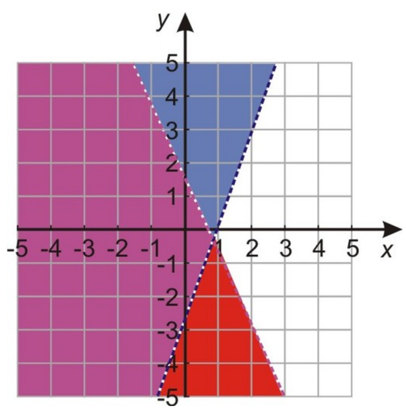
Let's start by writing our equation in slope-intercept form.

$y > 3x - 4$
$y < -\frac{9}{4}x + 2$
$x \geq 0$
$y \geq 0$

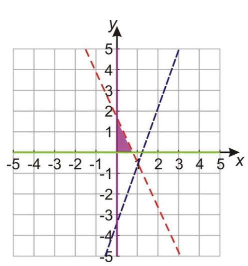
Now we can graph each line and shade appropriately. First we graph $-2.5, 1.5, 5$.



Next we graph $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$



Finally we graph and $=$ $\$21$ and $y = 5$, and the intersecting region is shown in the following figure.



The solution is **bounded** because there are lines on all sides of the solution region. In other words the solution region is a bounded geometric figure, in this case a triangle.

Write a System of Linear Inequalities

There are many interesting application problems that involve the use of system of linear inequalities. However, before we fully solve application problems, let's see how we can translate some simple word problems into algebraic equations.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend works in a certain region of the restaurant. The restaurant is also known for its great views but you have to sit in a certain area of the restaurant that offers these view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best views and be served by your friend.

Typically, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints.

Example 4

Write a system of linear inequalities that represents the following conditions.

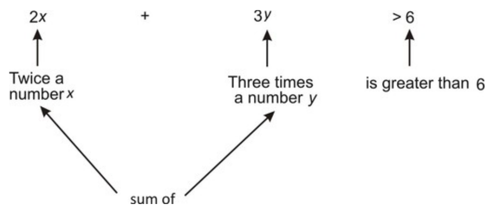
The sum of twice a number x and three times another number y is greater than y, and y is less than three times x.

Solution

Let's take each statement in turn and write it algebraically:

1.The sum of twice a number x and three times another number y is greater than y.

This can be written as



2. y is less than three times x.

This can be written as

$$\begin{array}{ll} y < & 3x \\ \nearrow & \nwarrow \\ y \text{ is less than} & 3 \text{ times } x \end{array}$$

The system of inequalities arising from these statements is

$$\begin{aligned} 2x + 3y &> 6 \\ y &< 3x \end{aligned}$$

This system of inequalities can be solved using the method outlined earlier in this section. We will not solve this system because we want to concentrate on learning how to write a system of inequalities from a word problem.

Solve Real-World Problems Using Systems of Linear Inequalities

As we mentioned before, there are many interesting application problems that require the use of systems of linear inequalities. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the best possible value under those conditions. A typical example would be taking the limitations of materials and labor, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real life systems can have dozens or hundreds of variables, or more. In this section, we will only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the $x,y-$plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the systems of equations that give the solutions to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

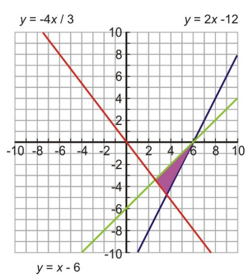
Example 5

Find the maximum and minimum value of $y = 15 + 5x$ *given the constraints.*

$$\begin{aligned} 2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\leq 6 \end{aligned}$$

Solution

Step 1: Find the solution to this system of linear inequalities by graphing and shading appropriately. To graph we must rewrite the equations in slope-intercept form.



$$y \geq 2x - 12$$
$$y \geq -\frac{4}{3}x$$
$$y \geq x - 6$$

These three linear inequalities are called the **constraints**.

The solution is the shaded region in the graph. This is called the **feasibility region**. That means all possible solutions occur in that region. However in order to find the optimal solution we must go to the next steps.

Step 2

Next, we want to find the corner points. In order to find them exactly, we must form three systems of linear equations and solve them algebraically.

System 1:

$$y = 2x - 12$$
$$y = -\frac{4}{3}x$$

Substitute the first equation into the second equation:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$

$$y = 2x - 12 \Rightarrow y - 2(3.6) - 12 \Rightarrow y = -4.8$$

The intersection point of lines is $3 - |4 - 9|$

System 2:

$$x - 3y = 4$$
$$2x + 5y = 8$$

Substitute the first equation into the second equation.

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$

$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0$$

The intersection point of lines is $(0, 0)$.

System 3:

$$y = -\frac{4}{3}x$$
$$y = x - 6$$

Substitute the first equation into the second equation.

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$

$$\text{Percent Equation: Rate} \times \text{Total} = \text{Part}$$

The intersection point of lines is $f(x) = x - 1$.

So the corner points are $3 - |4 - 9|$, $(0, 0)$ and $f(x) = x - 1$.

Step 3

Somebody really smart proved that, for linear systems like this, the maximum and minimum values of the optimization equation will always be on the corners of the feasibility region. So, to find the solution to this exercise, we need to plug these three points into $y = 15 + 5x$.

$$\begin{array}{rl} (3.6, -4.8) & z = 2(3.6) + 5(-4.8) = -16.8 \\ (6, 0) & z = 2(6) + 5(0) = 12 \\ (2.57, -3.43) & z = 2(2.57) + 5(-3.43) = -12.01 \end{array}$$

The highest value of 12 occurs at point $(0, 0)$ and the lowest value of $= 200$ occurs at $3 - |4 - 9|$.

In the previous example, we learned how to apply the method of linear programming out of context of an application problem. In the next example, we will look at a real-life application.

Example 6

You have $\$19,500$ to invest, and three different funds from which to choose. The municipal bond fund has a 5% return, the local bank's CDs have a $+7$ return, and a high-risk account has an expected 75% return. To minimize risk, you decide not to invest any more than $\$2,200$ in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. Assuming the year-end yields are as expected, what are the optimal investment amounts?

Solution

Let's define some *variables*.

x is the amount of money invested in the municipal bond at 5% return.

y is the amount of money invested in the bank's CD at $+7$ return.

$3x - 4y = -5$ is the amount of money invested in the high-risk account at 75% return.

a is the total interest returned from all the investments or $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of a.

Now, let's write inequalities for the *constraints*.

You decide not to invest more than $\$5000$ in the high-risk account.

$10000 - x - y \leq 1000$

You need to invest at least three times as much in the municipal bonds as in the bank CDs.

$-66, \ldots$

Also we write expressions for the fact that we invest more than zero dollars in each account.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 10000 - x - y &\geq 0 \end{aligned}$$

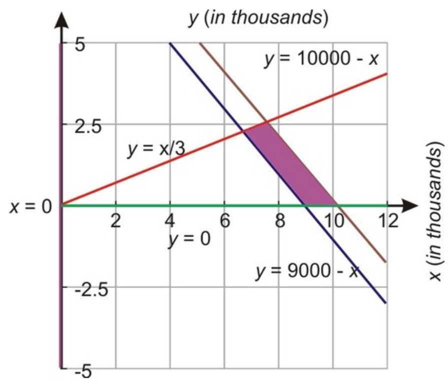
To summarize, we must maximize the expression $z = 1000 - .05x - .03y$.

Using the constraints,

$$\begin{array}{lcl} 10000 - x - y \leq 1000 & & y \geq 9000 - x \\ 3y \leq x & & y \leq \frac{x}{3} \\ x \geq 0 & \text{Lets rewrite each in slope-intercept form.} & x \geq 0 \\ y \geq 0 & & y \geq 0 \\ 10000 - x - y \geq 0 & & y \leq 10000 - x \end{array}$$

Step 1 Find the solution region to the set of inequalities by graphing each line and shading appropriately.

The following figure shows the overlapping region.



The purple region is the feasibility region where all the possible solutions can occur.

Step 2 Next, we need to find the corner points of the shaded solution region. Notice that there are four intersection points. To find them we must pair up the relevant equations and solve the resulting system.

System 1:

$$y = -\frac{x}{3}x$$
$$y = 10000 - x$$

Substitute the first equation into the second equation.

$$\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow x = 7500$$
$$y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500$$

The intersection point is $(7500, 2500)$.

System 2:

$$y = -\frac{x}{3}x$$
$$y = 9000 - x$$

Substitute the first equation into the second equation.

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$

$$\frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is $(7500, 2500)$.

System 3:

$$2x + 2y = 400$$
$$8x - 8y = 400$$

The intersection point is $|-3| = 3$.

System 4:

$$3x + 2y = 11$$
$$5x - 2y = 13$$

The intersection point is $(60, 720)$.

Step 3: In order to find the maximum value for a, we need to plug all intersection points into a and take the largest number.

$(7500, 2500)$	$z = 1000 - 0.05(7500) - 0.03(2500) = 550$
$(6750, 2250)$	$z = 1000 - 0.05(6750) - 0.03(2250) = 595$
$(10000, 0)$	$z = 1000 - 0.05(10000) - 0.03(0) = 500$
$(9000, 0)$	$z = 1000 - 0.05(9000) - 0.03(0) = 550$

Answer

The maximum return on the investment of $\$100$ occurs at point $(7500, 2500)$. This means that

$\$2,200$ is invested in the municipal bonds.

$\$2,200$ is invested in the bank CDs.

$\$2,200$ is invested in the high-risk account.

Review Questions

Find the solution region of the following systems of inequalities

1. IMAGE NOT AVAILABLE
2. IMAGE NOT AVAILABLE
3. IMAGE NOT AVAILABLE
4. IMAGE NOT AVAILABLE
5. IMAGE NOT AVAILABLE
6. IMAGE NOT AVAILABLE

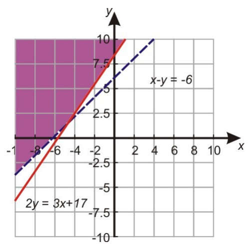
Solve the following linear programming problems:

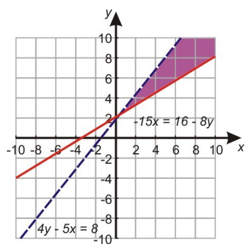
1. Given the following constraints find the maximum and minimum values of IMAGE NOT AVAILABLE
2. In Andrew's Furniture Shop, he assembles both bookcases and TV cabinets. Each type of furniture takes him about the same time to assemble. He figures he has time to make at most 16 pieces of furniture by this Saturday.

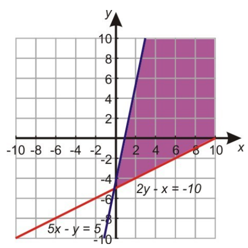
   The materials for each bookcase cost him $\$12$ and the materials for each TV stand costs him $\$12$. He has $\$100$ to spend on materials. Andrew makes a profit of $\$12$ on each bookcase and a profit of $\$100$ for each TV stand.

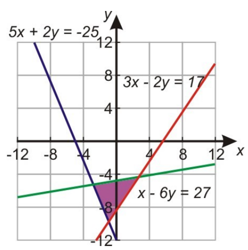
Find how many of each piece of furniture Andrew should make so that he maximizes his profit.

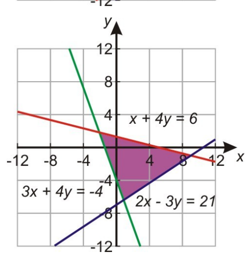
Review Answers

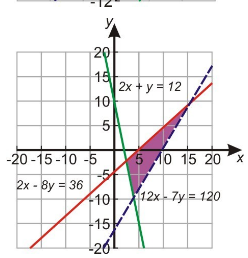
1. 

2. 

3. 

4. 

5. 

6. 

7. Maximum of $x = 3$ at point $(0, 0)$, minimum of $x = 0.02$ at point $(-5, -7)$
8. Maximum profit of $\$2,200$ by making y bookcases and y TV stands.