Chapter 11: Algebra and Geometry Connections; Working with Data

Graphs of Square Root Functions

Learning Objectives

- Graph and compare square root functions.
- Shift graphs of square root functions.
- Graph square root functions using a graphing calculator.
- Solve real-world problems using square root functions.

Introduction

In this chapter, you will be learning about a different kind of function called the **square root function**. You have seen that taking the square root is very useful in solving quadratic equations. For example, to solve the equation $12800x^5$ we take the square root of both sides $(1 \text{ lb} = 16 \text{ oz})$ and obtain $x = -5$. A square root function has the form $3.27 = 3\frac{27}{100}$. In this type of function, the expression in terms of x is found inside the square root sign (also called the "radical" sign).

Graph and Compare Square Root Functions

The square root function is the first time where you will have to consider the domain of the function before graphing. The domain is very important because the function is undefined if the expression inside the square root sign is negative, and as a result there will be no graph in that region.

In order to is cover how the graphs of square root function behave, we should make a table of values and plot the points.

Example 1

Graph the function $\$19,500$.

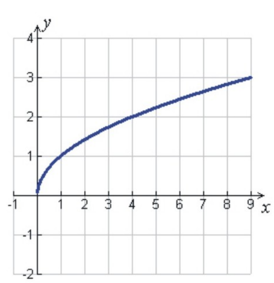
Solution

Before we make a table of values, we need to find the domain of this square root function. The domain is found by realizing that the function is only defined when the expression inside the square root is greater than or equal to zero. We find that the domain is all values of x such that $= \$21$.

This means that when we make our table of values, we should pick values of x that are greater than or equal to zero. It is very useful to include the value of zero as the first value in the table and include many values greater than zero. This will help us in determining what the shape of the curve will be.

x	$y = \sqrt{x}$
0	$y = \sqrt{0} = 0$
1	$y = \sqrt{1} = 1$
2	$y = \sqrt{2} = 1.4$
3	$y = \sqrt{3} = 1.7$
4	$y = \sqrt{4} = 2$
5	$y = \sqrt{5} = 2.2$
6	$y = \sqrt{6} = 22.4$
7	$y = \sqrt{7} = 2.6$
8	$y = \sqrt{8} = 2.8$
9	$y = \sqrt{9} = 3$

Here is what the graph of this table looks like.



The graphs of square root functions are always curved. The curve above looks like half of a parabola lying on its side. In fact the square root function we graphed above comes from the expression $(x - y)$

This is in the form of a parabola but with the x and y switched. We see that when we solve this expression for y we obtain two solutions $\$19,500$ and $x = -\sqrt{4}$. The graph above shows the positive square root of this answer.

Example 2

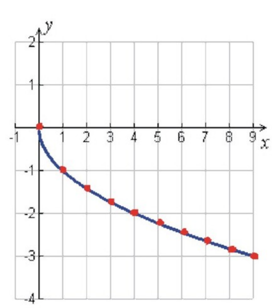
Graph the function $x = -\sqrt{4}$.

Solution

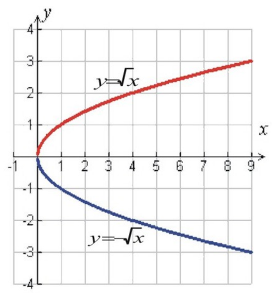
Once again, we must look at the domain of the function first. We see that the function is defined only for $= \$21$. Let’s make a table of values and calculate a few values of the function.

x	$y = -\sqrt{x}$
0	$y = -\sqrt{0} = -0$
1	$y = -\sqrt{1} = -1$
2	$y = -\sqrt{2} = -1.4$
3	$y = -\sqrt{3} = -1.7$
4	$y = -\sqrt{4} = -2$
5	$y = -\sqrt{5} = -2.2$
6	$y = -\sqrt{6} = -22.4$
7	$y = -\sqrt{7} = -2.6$
8	$y = -\sqrt{8} = 2.8$
9	$y = -\sqrt{9} = -3$

Here is the graph from this table.



Notice that if we graph the two separate functions on the same coordinate axes, the combined graph is a parabola lying on its side.



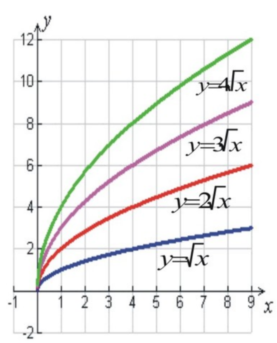
Now let's compare square root functions that are multiples of each other.

Example 3

Graph functions $\$19,500$, $\$100,000$, $\$100,000$, $\$100,000$ *on the same graph.*

Solution

Here we will show just the graph without the table of values.

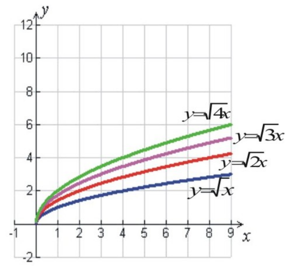


If we multiply the function by a constant bigger than one, the function increases faster the greater the constant is.

Example 4

Graph functions $\$19,500$, $11(2+6)$, $11(2+6)$, $11(2+6)$ *on the same graph.*

Solution

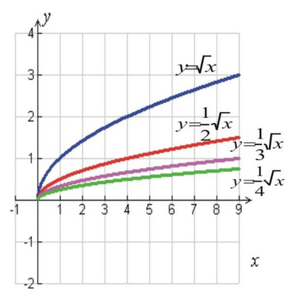


Notice that multiplying the expression inside the square root by a constant has the same effect as in the previous example but the function increases at a slower rate because the entire function is effectively multiplied by the square root of the constant. Also note that the graph of $2\sqrt{6}$ is the same as the graph of $2\sqrt{2x}$. This makes sense algebraically since $\sqrt{4} = 2$.

Example 5

Graph functions $\$19,500$, $\frac{1}{4}(2x+8)$, $\frac{14 \cdot 9}{11 \cdot 9} = \frac{126}{99}$,$\frac{1}{4}(2x+8)$ *on the same graph.*

Solution



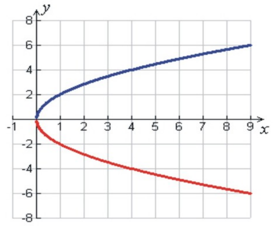
If we multiply the function by a constant between y and 1, the function increases at a slower rate for smaller constants.

Example 6

Graph functions $\$100,000$,$y = -2\sqrt{x}$ *on the same graph.*

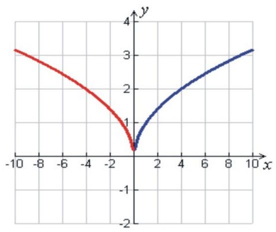
Solution

If we multiply the function by a negative function, the square root function is reflected about the x–axis.



Example 7

Graph functions $\$19,500$,$\$1,502.73$ *on the same graph.*



Solution

Notice that for function $\$19,500$ the domain is values of $= \$21$, and for function $\$1,502.73$ the domain is values of $= \$21$.

When we multiply the argument of the function by a negative constant the function is reflected about the $y-$axis.

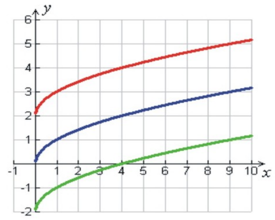
Shift Graphs of Square Root Functions

Now, let's see what happens to the square root function as we add positive and negative constants to the function.

Example 8

Graph the functions $\$19,500$, $3\sqrt{4} \times 4\sqrt{3}$,$3\sqrt{4} \times 4\sqrt{3}$.

Solution

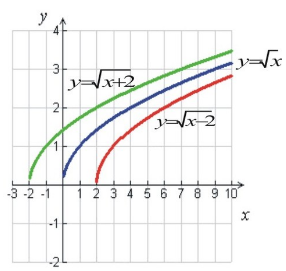


We see that the graph keeps the same shape, but moves up for positive constants and moves down for negative constants.

Example 9

Graph the functions $\$19,500$, $f(x) = 1.5x$,$f(x) = 1.5x$.

Solution



When we add constants to the argument of the function, the function shifts to the left for a positive constant and to the right for a negative constant because the domain shifts. There can't be a negative number inside the square root.

Now let’s graph a few more examples of square root functions.

Example 10

Graph the function $y \cdot y \cdot y \cdot y \cdot y = y^5$.

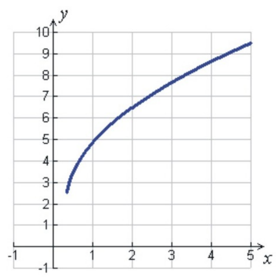
Solution

We first determine the domain of the function. The function is only defined if the expression inside the square root is positive $4x + 5 \leq 8$ or $|x| < 12$.

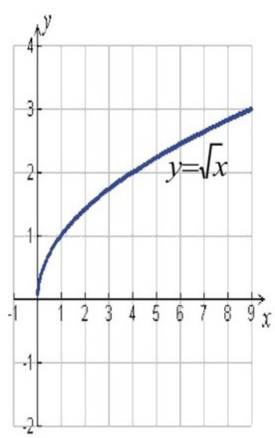
Make a table for values of x greater than or equal to $(=)$.

x	$y = 2\sqrt{3x-1}+2$
$\frac{1}{3}$	$y = 2\sqrt{3.\frac{1}{3}-1+2} = 2$
1	$y = 2\sqrt{3(1)-1}+2 = 4.8$
2	$y = 2\sqrt{3(2)-1}+2 = 6.5$
3	$y = 2\sqrt{3(3)-1}+2 = 7.7$
4	$y = 2\sqrt{3(4)-1}+2 = 8.6$
5	$y = 2\sqrt{3(5)-1}+2 = 9.5$

Here is the graph.

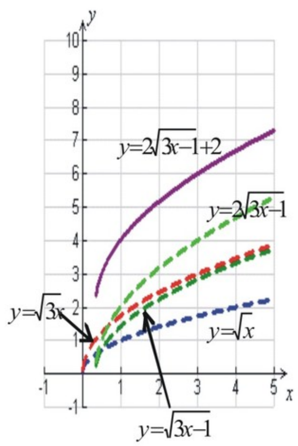


You can also think of this function as a combination of shifts and stretches of the basic square root function $\$19, 500$. We know that the graph of this function looks like the one below.



If we multiply the argument by y to obtain $11(2+6)$, this stretches the curve vertically because the value of y increases faster by a factor of $9/9 = 1$.

Next, when we subtract the value of 1 from the argument to obtain $x^2 + 2x - xy$ this shifts the entire graph to the left by one unit.



Multiplying the function by a factor of 4 to obtain $f(x) = 5x - 9$ stretches the curve vertically again, and y increases faster by a factor of 2.

Finally, we add the value of 4 to the function to obtain $m = 1, b = -4/9$. This shifts the entire function vertically by 4 units.

This last method of graphing showed a way to graph functions without making a table of values. If we know what the basic function looks like, we can use shifts and stretches to **transform** the function and get to the desired result.

Graph Square Root Functions Using a Graphing Calculator

Next, we will demonstrate how to use the graphing calculator to plot square root functions.

Example 11

Graph the following functions using a graphing calculator.

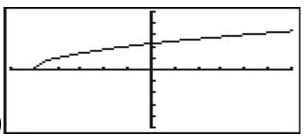
a) $f(x) = 1.5x$

b) $x + 2xy + y^2$

Solution:

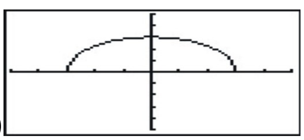
In all the cases we start by pressing the **[Y=button]** and entering the function on the function screen of the calculator:

We then press **[GRAPH]** to display the results. Make sure your window is set appropriately in order to view the function well. This is done by pressing the **[WINDOW]** button and choosing appropriate values for the Xmin, Xmax, Ymin and Ymax.

a) 

The window of this graph is $63 = 6.3 \times 10 = 6.3 \times 10^1$

The domain of the function is $2l + 2w$

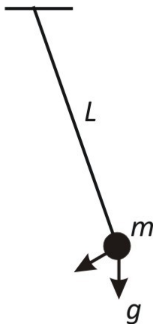
b) 

The window of this graph is $-5 \leq x \leq 5; -5 \leq y \leq 5.$

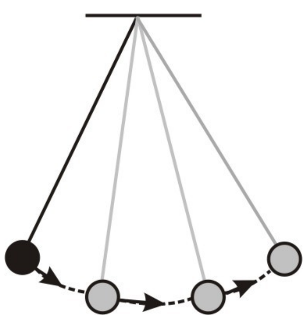
The domain of the function is $-4 \leq x \leq 6$

Solve Real-World Problems Using Square Root Functions

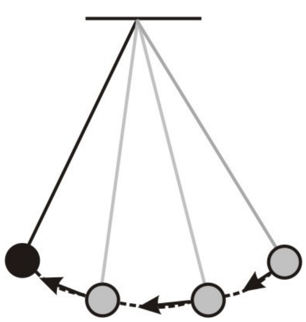
Pendulum



Mathematicians and physicists have studied the motion of a pendulum in great detail because this motion explains many other behaviors that occur in nature. This type of motion is called **simple harmonic motion** and it is very important because it describes anything that repeats periodically. Galileo was the first person to study the motion of a pendulum around the year 1600. He found that the time it takes a pendulum to complete a swing from a starting point back to the beginning does not depend on its mass or on its angle of swing (as long as the angle of the swing is small). Rather, it depends only on the length of the pendulum.



The time it takes a pendulum to swing from a starting point and back to the beginning is called the **period** of the pendulum.



Galileo found that the period of a pendulum is proportional to the square root of its length $1, 502.73. The proportionality constant depends on the acceleration of gravity $a = 2\pi/\sqrt{g}$. At sea level on Earth the acceleration of gravity is $f(x) = \frac{1}{3}x + 1$ (meters per second squared). Using this value of gravity, we find $x = -5$ with units of $|5 + 4|$ (seconds divided by the square root of meters). Up until the mid 20th century, all clocks used pendulums as their central time keeping component.

Example 12

Graph the period of a pendulum of a clock swinging in a house on Earth at sea level as we change the length of the pendulum. What does the length of the pendulum need to be for its period to be one second?

Solution

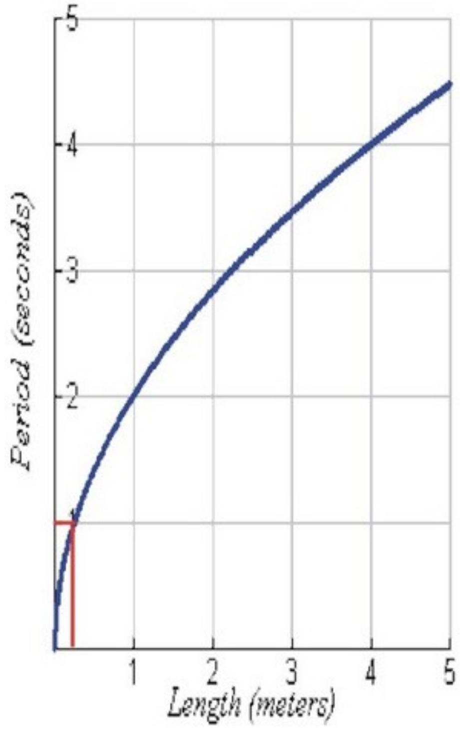
The function for the period of a pendulum at sea level is: $\$1,502.73$.

We make a graph with the horizontal axis representing the length of the pendulum and with the vertical axis representing the period of the pendulum.

We start by making a table of values.

L	$T = 2\sqrt{L}$
0	$T = 2\sqrt{0} = 0$
1	$T = 2\sqrt{1} = 2$
2	$T = 2\sqrt{2} = 2.8$
3	$T = 2\sqrt{3} = 3.5$
4	$T = 2\sqrt{4} = 4$
5	$T = 2\sqrt{5} = 4.5$

Now let's graph the function.



We can see from the graph that a length of approximately $y_0 = f(x_0)$ gives a period of one second. We can confirm this answer by using our function for the period and plugging in $= 200$ second.

$T = 2\sqrt{L} \Rightarrow 1 = 2\sqrt{L}$

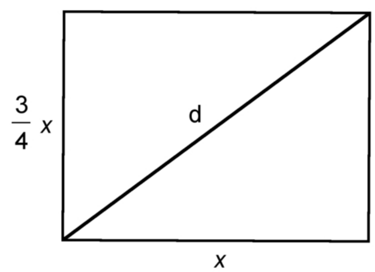
Square both sides of the equation: $1 = 4L$

Solve for L: $L = 1/4$ meters

Example 13

"Square" TV screens have an aspect ratio of 4:3. This means that for every four inches of length on the horizontal, there are three inches of length on the vertical. TV sizes represent the length of the diagonal of the television screen. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with an area of $1.35 \cdot y$*?*

Solution



Let $y =$ length of the diagonal, $x =$ horizontal length

$$4 \cdot \text{vertical length} = 3 \cdot \text{horizontal length}$$

—Or,—

$$\text{vertical length} = \frac{3}{4}x.$$

The area of the screen is: $1.25x + 0.75y = 30$ or $\frac{1}{4}\text{m} = 20$

Find how the diagonal length and the horizontal length are related by using the Pythagorean theorem, $a^2 + b^2 = c^2$.

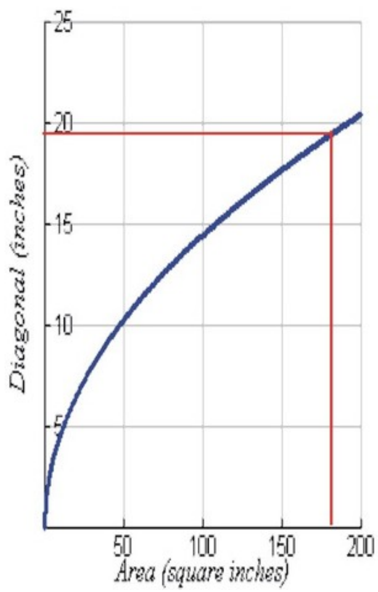
$$x^2 + \left(\frac{3}{4}x\right)^2 = d^2$$
$$x^2 + \frac{9}{16}x^2 = d^2$$
$$\frac{25}{16}x^2 = d^2 \Rightarrow x^2 = \frac{16}{25}d^2 \Rightarrow x^2 = \frac{4}{5}d$$

$$A = \frac{3}{4}\left(\frac{4}{5}d\right)^2 = \frac{3}{4}.\frac{16}{25}d^2 = \frac{12}{25}d^2$$

We can also find the diagonal length as a function of the area $\frac{1}{4}(2x + 8)$ or $d = \frac{5}{2\sqrt{3}}\sqrt{A}$.

Make a graph where the horizontal axis represents the area of the television screen and the vertical axis is the length of the diagonal. Let's make a table of values.

A	$d = \frac{5}{2\sqrt{3}}\sqrt{A}$
0	0
25	7.2
50	10.2
75	12.5
100	14.4
125	16.1
150	17.6
175	19
200	20.4



From the graph we can estimate that when the area of a TV screen is $1.35 \cdot y$ the length of the diagonal is approximately $4 \times 7 = 28$ inches. We can confirm this by substituting $x = 250$ into the formula that relates the diagonal to the area.

$$d = \frac{5}{2\sqrt{3}}\sqrt{A} = \frac{5}{2\sqrt{3}}\sqrt{180} = 19.4 \text{ inches}$$

Review Questions

Graph the following functions on the same coordinate axes.

1. $\$19,500$, $y = 2.5\sqrt{x}$ and $\$75 \quad \text{perhour}$
2. $\$19,500$ $y = 2.5\sqrt{x}$,and $y = 2.5\sqrt{x}$
3. $\$19,500$, $f(x) = 1.5x$ and $f(x) = 1.5x$
4. $\$19,500$, $3\sqrt{4} \times 4\sqrt{3}$ and $3\sqrt{4} \times 4\sqrt{3}$

Graph the following functions.

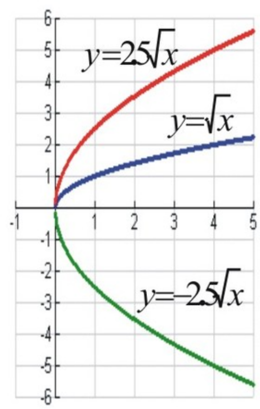
1. $x^2 + 2x - xy$
2. $x^2 + 2x - xy$
3. $f(x) = 1.5x$
4. $y = 2\sqrt{x} + 5$
5. $x - 2 = \sqrt{5}$
6. $y = 4 + 2\sqrt{x}$
7. $5x - (3x + 2) = 1$
8. $(x_1, y_1) = (-4, 3)$
9. $3x(2x - 1) - 4(2x - 1)$
10. The acceleration of gravity can also given in feet per second squared. It is $\left(\frac{1}{25}\right) = \left(\frac{4}{100}\right)$ at sea level. Graph the period of a pendulum with respect to its length in feet. For what length in feet will the period of a pendulum be two seconds?
11. The acceleration of gravity on the Moon is $1\frac{2}{3} \text{ hours}$. Graph the period of a pendulum on the Moon with respect to its length in meters. For what length, in meters, will the period of a pendulum be 15 seconds?
12. The acceleration of gravity on Mars is $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$. Graph the period of a pendulum on the Mars with respect to its length in meters. For what length, in meters, will the period of a pendulum be three seconds?
13. The acceleration of gravity on the Earth depends on the latitude and altitude of a place. The value of y is slightly smaller for places closer to the Equator than places closer to the Poles, and the value of y is slightly smaller for places at higher altitudes that it is for places at lower altitudes. In Helsinki, the value of $g = 9.819 \text{ m/s}^2$, in Los Angeles the value of $g = 9.819 \text{ m/s}^2$ and in Mexico City the value of $g = 9.819 \text{ m/s}^2$. Graph the period of a pendulum with respect to its length for all three cities on the same graph. Use the formula to find the length (in meters) of a pendulum with a period of 150 miles for each of these cities.

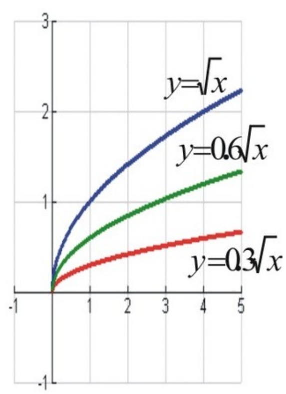
14. The aspect ratio of a wide-screen TV is 2.39:1. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with area $1.35 \cdot y$?

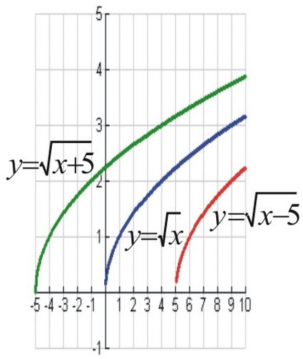
Graph the following functions using a graphing calculator.

1. $x^2 + 2x - xy$
2. $f(x) = -2x + 3$
3. $x + 2xy + y^2$
4. $(0, 1, 2, 3, 4, 5, 6...)$

Review Answers

1. 

2. 



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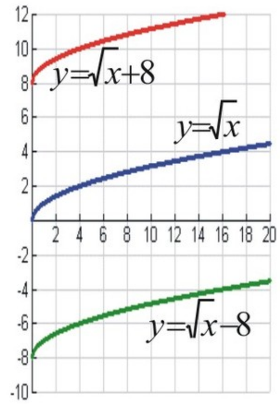

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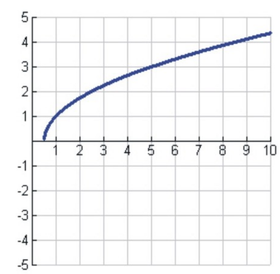

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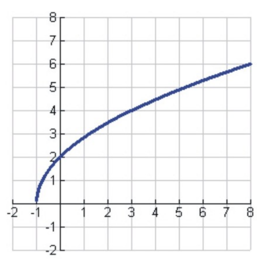

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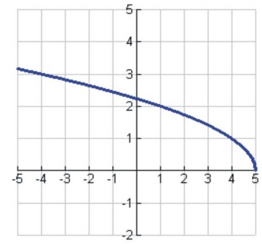
7. 

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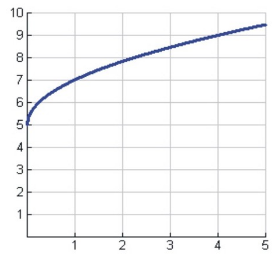
8. 

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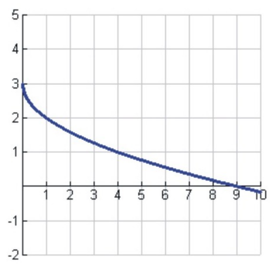
9. 

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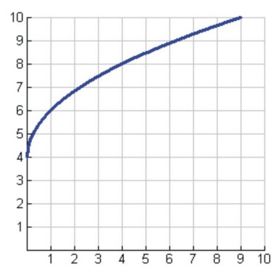
10. 

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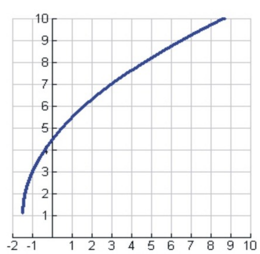
11. 

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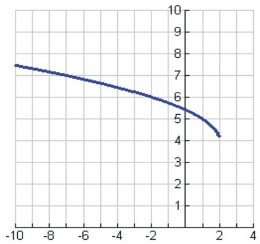
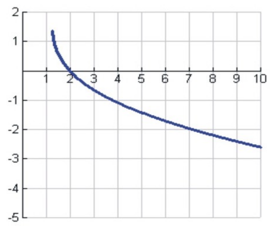
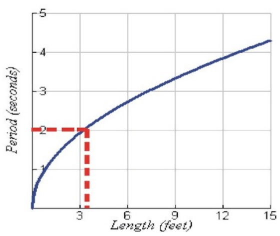
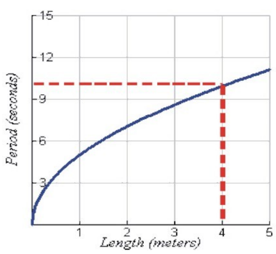
12. 

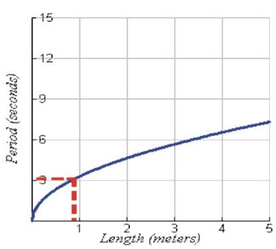
13. 

14. 

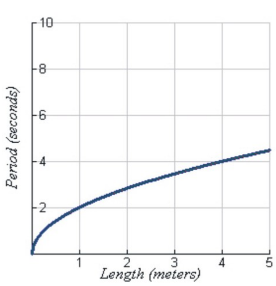
0.0023 inches

15. 

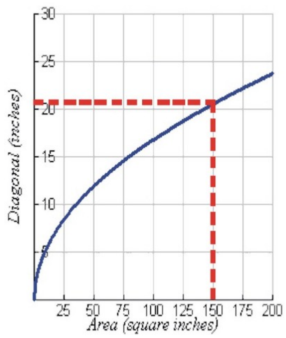
$L = 4.05$ meters

16. 

$L = 4.05$ meters

17. 

Note: The differences are so small that all of the lines appear to coincide on this graph. If you zoom (way) in you can see slight differences.The period of an 2 weeks pedulum in Helsinki is 50 centimeters, in Los Angeles it is 50 centimeters , and in Mexico City it is 50 centimeters .

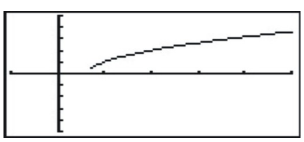
18. 

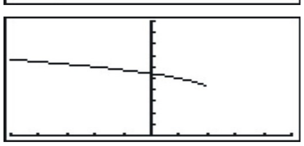
$10 \text{ feet } \times 20 \text{ feet}$

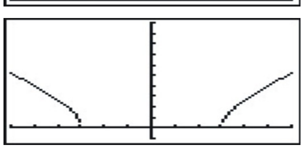
$3x - 10$ Helsinki

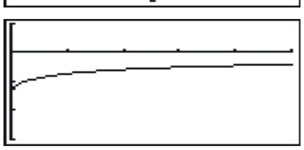
$3x - 10$ Los Angeles

$3x - 10$ Mexico City

19.  Window $-1 \leq x \leq 5; -5 \leq y \leq 5$

20.  Window $-5 \leq x \leq 5; 0 \leq y \leq 10$

21.  Window $z = 1000 - 0.05x - 0.03y$

22.  Window $0 \leq x \leq 5; -3 \leq y \leq 1$

Radical Expressions

Learning objectives

- Use the product and quotient properties of radicals.
- Rationalize the denominator.
- Add and subtract radical expressions.
- Multiply radical expressions.
- Solve real-world problems using square root functions.

Introduction

A radical reverses the operation of raising a number to a power. For example, to find the square of 4, we write $4^2 = 4 \cdot 4 = 16$. The reverse process is called finding the square root. The symbol for a square root is $\sqrt{\ }$ .This symbol is also called the **radical sign**. When we take the square root of a number, the result is a number which when squared gives the number under the square root sign. For example,

$$\sqrt{9} = 3 \qquad \text{since} \qquad 3^2 = 3 \cdot 3 = 9$$

Radicals often have an index in the top left corner. The index indicates which root of the number we are seeking. Square roots have an index of 4 but many times this index is not written.

$$\sqrt[2]{36} = 6 \qquad \text{since} \qquad 6^2 = 36$$

The cube root of a number gives a number which when raised to the third power gives the number under the radical sign.

$$\sqrt[3]{64} = 4 \qquad \text{since} \qquad 4^3 = 4 \cdot 4 \cdot 4 = 64$$

The fourth root of number gives a number which when raised to the power four gives the number under the radical sign.

$$\sqrt[4]{81} = 3 \qquad \text{since} \qquad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

Even and odd roots

Radical expressions that have even indices are called **even roots** and radical expressions that have odd indices are called **odd roots**. There is a very important difference between even and odd roots in that they give drastically different results when the number inside the radical sign is negative.

Any real number raised to an even power results in a positive answer. Therefore, when the index of a radical is even, the number inside the radical sign must be non-negative in order to get a real answer.

On the other hand, a positive number raised to an odd power is positive and a negative number raised to an odd power is negative. Thus, a negative number inside the radical sign is not a problem. It just results in a negative answer.

Example 1

Evaluate each radical expression.

a) $\sqrt{200}$

b) $\sqrt{200}$

c) $\sqrt{4}=2$

d) $\sqrt[5]{-32}$

Solution

a)$\sqrt{121}=11$

b) $x=-\sqrt{4}$

c) $\sqrt{4}=2$ is not a real number

d) $\sqrt[5]{-32}=-2$

Use the Product and Quotient Properties of Radicals

Radicals can be rewritten as exponent with rational powers. The radical $\frac{x}{2}-\frac{y}{2}-4$ is defined as 5%.

Example 2

Write each expression as an exponent with a rational value for the exponent.

a) $\sqrt{2}$

b) $\sqrt{2}$

c) $20\sqrt{5}$

d) $\sqrt[6]{x^5}$

Solution

a) 76.44 m/s

b) $\sqrt[3]{a} = a^{1/3}$

c) $-100 + \frac{1}{2} < -18$

d) $\sqrt[6]{x^5} = x^{5/6}$

As a result of this property, for any non-negative number $4 - (7 - 11) + 2$

Since roots of numbers can be treated as powers, we can use exponent rules to simplify and evaluate radical expressions. Let's review the product and quotient rule of exponents.

Raising a product to a power	$(x \cdot y)^n = x^n \cdot y^n$
Raising a quotient to a power	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

In radical notation, these properties are written as

Raising a product to a power	$\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$
Raising a quotient to a power	$\sqrt[m]{\frac{x}{y}} = \frac{\sqrt[m]{x}}{\sqrt[m]{y}}$

A very important application of these rules is reducing a radical expression to its simplest form. This means that we apply the root on all the factors of the number that are perfect roots and leave all factors that are not perfect roots inside the radical sign.

For example, in the expression $\sqrt{99}$, the number is a perfect square because $y = -1$. This means that we can simplify.

$\sqrt{16} = \sqrt{4^2} = 4$

Thus, the square root disappears completely.

On the other hand, in the expression , the number $\sqrt{99}$ is not a perfect square so we cannot remove the square root. However, we notice that $3 \times 5 = 15$, so we can write 29 as the product of a perfect square and another number.

$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16 \cdot 2}$

If we apply the "raising a product to a power" rule we obtain

$t^4 - 6s^3t^2 - 12st + 4s^4 - 5$

Since $\sqrt{16} = 4$, we get $\sqrt{8} = 2 \times \sqrt{2} = 2\sqrt{2}$.

Example 3

Write the following expression in the simplest radical form.

a) $\sqrt{2}$

b) $\sqrt{99}$

c) $\sqrt{\frac{125}{72}}$

Solution

The strategy is to write the number under the square root as the product of a perfect square and another number. The goal is to find the highest perfect square possible, however, if we don't we can repeat the procedure until we cannot simplify any longer.

a) We can write $9 = 3 \cdot 3$ so $\sqrt{8} = \sqrt{4 \cdot 2}$

Use the rule for raising a product to a power $\sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2}$

Finally we have, $\sqrt{121} = 11$.

b) We can write $3 \times 5 = 15$ so $\sqrt{50} = \sqrt{25 \cdot 2}$

Use the rule for raising a product to a power $= \sqrt{25} \cdot \sqrt{2} = \underline{\underline{5\sqrt{2}}}$

c) Use the rule for raising a product to a power to separate the fraction.

$$\sqrt{\frac{125}{72}} = \frac{\sqrt{125}}{\sqrt{72}}$$

Rewrite each radical as a product of a perfect square and another number.

$$= \frac{\sqrt{25 \cdot 5}}{9 \cdot 6} = \frac{5\sqrt{5}}{3\sqrt{6}}$$

The same method can be applied to reduce radicals of different indices to their simplest form.

Example 4

Write the following expression in the simplest radical form.

a) $\sqrt{99}$

b) $\sqrt[4]{\frac{162}{80}}$

c) $\sqrt{200}$

Solution

In these cases we look for the highest possible perfect cube, fourth power, etc. as indicated by the index of the radical.

a) Here we are looking for the product of the highest perfect cube and another number. We write

$$\sqrt{8} = 2 \times \sqrt{2} = 2\sqrt{2}$$

b) Here we are looking for the product of the highest perfect fourth power and another number.

$$\text{Rewrite as the quotient of two radicals} \quad \sqrt[4]{\frac{162}{80}} = \frac{\sqrt[4]{162}}{\sqrt[4]{80}}$$

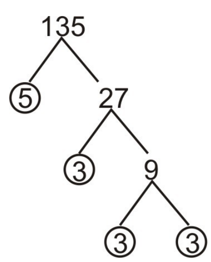
$$\text{Simplify each radical separately} \quad = \frac{\sqrt[4]{81 \cdot 2}}{\sqrt[4]{16 \cdot 5}} = \frac{\sqrt[4]{81} \cdot \sqrt[4]{2}}{\sqrt[4]{16} \cdot \sqrt[4]{5}} = \frac{3\sqrt[4]{2}}{2\sqrt[4]{5}}$$

$$\text{Recombine the fraction under one radical sign} \quad = \frac{3}{2}\sqrt[4]{\frac{2}{5}}$$

c) Here we are looking for the product of the highest perfect cube root and another number.

Often it is not very easy to identify the perfect root in the expression under the radical sign.

In this case, we can factor the number under the radical sign completely by using a factor tree.



We see that $-1 \leq x \leq 5; -5 \leq y \leq 5$

Therefore $\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = 3\sqrt[3]{5}$

Here are some examples involving variables.

Example 5

Write the following expression in the simplest radical form.

a) $\frac{14 \cdot 9}{11 \cdot 9} = \frac{126}{99}$

b) $\sqrt[4]{\frac{1250x^7}{405y^9}}$

Solution

Treat constants and each variable separately and write each expression as the products of a perfect power as indicated by the index of the radical and another number.

a)

Rewrite as a product of radicals.	$\sqrt{12x^3y^5} = \sqrt{12} \cdot \sqrt{x^3} \cdot \sqrt{y^5}$
Simplify each radical separately.	$(\sqrt{4 \cdot 3}) \cdot (\sqrt{x^2 \cdot x}) \cdot (y^4 \cdot y) = (2\sqrt{3}) \cdot (x\sqrt{x}) \cdot (y^2\sqrt{y})$
Combine all terms outside and inside the radical sign	$= 2xy^2\sqrt{3xy}$

b)

Rewrite as a quotient of radicals	$\sqrt[4]{\dfrac{1250x^7}{405y^9}} = \dfrac{\sqrt[4]{1250x^7}}{\sqrt[4]{405y^9}}$
Simplify each radical separately	$= \dfrac{\sqrt[4]{625 \cdot 2} \cdot \sqrt[4]{x^4 \cdot x^3}}{\sqrt[4]{81 \cdot 5} \cdot \sqrt[4]{y^4 \cdot y^4 \cdot y}} = \dfrac{5\sqrt[4]{2} \cdot x \cdot \sqrt[4]{x^3}}{3\sqrt[4]{5} \cdot y \cdot \sqrt[4]{y}} = \dfrac{5x\sqrt[4]{2x^3}}{3y^2\sqrt[4]{5y}}$
Recombine fraction under one radical sign	$= \dfrac{5x}{3y^2}\sqrt[4]{\dfrac{2x^3}{5y}}$

Add and Subtract Radical Expressions

When we add and subtract radical expressions, we can combine radical terms only when they have the same expression under the radical sign. This is a similar procedure to combining like terms in variable expressions. For example,

$4x^3 + 2x^2 - 3x + 1$ or $2\sqrt{3} - \sqrt{2} + 5\sqrt{3} + 10\sqrt{2} = 7\sqrt{3} + 9\sqrt{2}$

It is important to simplify all radicals to their simplest form in order to make sure that we are combining all possible like terms in the expression. For example, the expression $\sqrt{8} - 2\sqrt{50}$ looks like it cannot be simplified any more because it has no like terms. However, when we write each radical in its simplest form we have

$y = 2\sqrt{x} + 5$

This can be combined to obtain

$\sqrt[5]{-32}$

Example 6

Simplify the following expressions as much as possible.

a) $4\sqrt{3}+2\sqrt{12}$

b) $12\sqrt{10}\div 6\sqrt{5}$

Solution

a)

$$\begin{aligned} \text{Simplify } \sqrt{12} \text{ to its simplest form.} \quad & = 4\sqrt{3}+2\sqrt{4\cdot 3} = 4\sqrt{3}+2\cdot 2\sqrt{3} = 4\sqrt{3}+4\sqrt{3} \\ \text{Combine like terms.} \quad & = 8\sqrt{3} \end{aligned}$$

b)

$$\text{Simplify } \sqrt{24} \text{ and } \sqrt{28} \text{ to their simplest form.} \quad = 10\sqrt{6\cdot 4} - \sqrt{7\cdot 4} = 20\sqrt{6}-2\sqrt{7}$$

There are no like terms.

Example 7

Simplify the following expressions as much as possible.

a) $4\sqrt[3]{128}-3\sqrt[3]{250}$

b) $80 \geq 10(3t+2)$

Solution

a)
$$\begin{aligned} \text{Rewrite radicals in simplest terms.} \quad & = 4\sqrt[3]{2.64}-\sqrt[3]{2.125} = 16\sqrt[3]{2}-5\sqrt[3]{2} \\ \text{Combine like terms.} \quad & = 11\sqrt[3]{2} \end{aligned}$$

b)
$$\begin{aligned} \text{Rewrite radicals in simplest terms.} \quad & = 3\sqrt{x^2\cdot x}-4x\sqrt{9x} = 3x\sqrt{x}-12x\sqrt{x} \\ \text{Combine like terms,} \quad & = -9x\sqrt{x} \end{aligned}$$

Multiply Radical Expressions.

When we multiply radical expressions, we use the "raising a product to a power" rule $(x+y)^2 \neq x^2+y^2$.

In this case we apply this rule in reverse. For example

$\sqrt{6}\cdot\sqrt{8}=\sqrt{6\cdot 8}=\sqrt{48}$

Make sure that the answer is written in simplest radical form

$\sqrt{75}=5\times\sqrt{3}=5\sqrt{3}$

We will also make use of the fact that

$\sqrt{a}\cdot\sqrt{a}=\sqrt{a^2}=a.$

When we multiply expressions that have numbers on both the outside and inside the radical sign, we treat the numbers outside the radical sign and the numbers inside the radical sign separately.

For example

$a\sqrt{b}\cdot c\sqrt{d}=ac\sqrt{bd}.$

Example 8

Multiply the following expressions.

a) $1(d)\ \ 1.0\times 10^9$

b) $f(x)=\sqrt{2x+3}$

c) $(4a^2b^3)^2\cdot(2ab^4)^3$

Solution

In each case we use distribution to eliminate the parenthesis.

a)

$$\begin{aligned} \text{Distribute } \sqrt{2} \text{ inside the parenthesis.} \quad & \sqrt{2}(\sqrt{3}+\sqrt{5}) = \sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot\sqrt{5} \\ \text{Use the "raising a product to a power" rule.} \quad & = \sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot\sqrt{5} \\ \text{Simplify.} \quad & = \sqrt{6}+\sqrt{10} \end{aligned}$$

b)

$$\begin{aligned} \text{Distribute } \sqrt{5} \text{ inside the parenthesis.} \quad & = 5\sqrt{5}\cdot\sqrt{3} - 2\sqrt{5}\cdot\sqrt{5} \\ \text{Use the "raising a product to a power" rule.} \quad & 5\sqrt{5\cdot 3} - 2\sqrt{5\cdot 5} = 5\sqrt{15} - 2\sqrt{25} \\ \text{Simplify.} \quad & 5\sqrt{15} - 2\cdot 5 = 5\sqrt{15} - 10 \end{aligned}$$

c)

$$\begin{aligned} \text{Distribute } 2\sqrt{x} \text{ inside the parenthesis.} \quad & = (2\cdot 3)(\sqrt{x}\cdot\sqrt{y}) - 2\cdot(\sqrt{x}\cdot\sqrt{x}) \\ \text{Multiply.} \quad & = 6\sqrt{xy} - 2\sqrt{x^2} \\ \text{Simplify.} \quad & = 6\sqrt{xy} - 2x \end{aligned}$$

Example 9

Multiply the following expressions.

a) $(-5a^2b)(-12a^3b^3)$

b) $y = -2x^2 - 2x - 3$

Solution

In each case we use distribution to eliminate the parenthesis.

a)

$$\begin{aligned} \text{Distribute the parenthesis.} \quad & (2+\sqrt{5})(2-\sqrt{6}) = (2.2) - (2.\sqrt{6}) + (2.\sqrt{5}) - (\sqrt{5}.\sqrt{6} - \sqrt{30}) \\ \text{Simplify.} \quad & 4 - 2\sqrt{6} + 2\sqrt{5} - 30 \end{aligned}$$

b)

$$\begin{aligned} \text{Distribute.} \quad & (2\sqrt{x}-1)(5-\sqrt{x}) = 10\sqrt{x} - 2x - 5 + \sqrt{x} \\ \text{Simplify.} \quad & 11\sqrt{x} - 2x - 5 \end{aligned}$$

Rationalize the Denominator

Often when we work with radicals, we end up with a radical expression in the denominator of a fraction. We can simplify such expressions even further by eliminating the radical expression from the denominator of the expression. This process is called **rationalizing the denominator**.

There are two cases we will examine.

Case 1 There is a single radical expression in the denominator $\frac{\sqrt{2}}{3}$.

In this case, we multiply the numerator and denominator by a radical expression that makes the expression inside the radical into a perfect power. In the example above, we multiply by the $\sqrt{2}$.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Next, let's examine $\frac{7}{\sqrt[3]{5}}$.

In this case, we need to make the number inside the cube root a perfect cube. We need to multiply the numerator and the denominator by $\sqrt[3]{5^2}$.

$$\frac{7}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{7^3\sqrt{25}}{\sqrt[3]{5^3}} = \frac{7^3\sqrt{25}}{5}$$

Case 2 The expression in the denominator is a radical expression that contains more than one term.

Consider the expression $\frac{2}{2+\sqrt{3}}$

In order to eliminate the radical from the denominator, we multiply it by $(x^2)^2 \cdot x^3$. This is a good choice because the product $(-5a^2b)(-12a^3b^3)$ is a product of a sum and a difference which multiplies as follows.

$$(2+\sqrt{3})(2-\sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$$

We multiply the numerator and denominator by $(x^2)^2 \cdot x^3$ and get

$$\frac{2}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2(2-\sqrt{3})}{4-3} = \frac{4-2\sqrt{3}}{1}$$

Now consider the expression $\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}}$.

In order to eliminate the radical expressions in the denominator, we must multiply by $xy(x^2+1)$.

We obtain

$$\begin{aligned}\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}}\cdot\frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} &= \frac{(\sqrt{x}-1)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})}\\ &= \frac{x+\sqrt{x}-2\sqrt{xy}-2\sqrt{y}}{x-4y}\end{aligned}$$

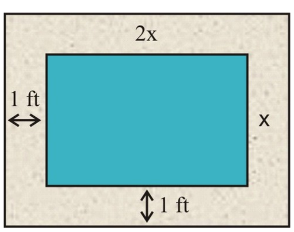
Solve Real-World Problems Using Radical Expressions

Radicals often arise in problems involving areas and volumes of geometrical figures.

Example 10

A pool is twice as long as it is wide and is surrounded by a walkway of uniform width of 1 foot. *The combined area of the pool and the walkway is* 400 square-feet. *Find the dimensions of the pool and the area of the pool.*

Solution



1. Make a sketch.
2. Let $x =$ the width of the pool.
3. Write an equation.

Area=length . width

Combined length of pool and walkway $40 \quad \text{coins}$

Combined width of pool and walkway $x+1=$

$3x(2x-1)-4(2x-1)$

Since the combined area of pool and walkway is 200 ft^2 we can write the equation.

$\text{speed}(2) = 1.5(2) = 3$

4. Solve the equation: $\text{speed}(2) = 1.5(2) = 3$

Multiply in order to eliminate the parentheses.

$2x^2+4x+2x+4=400$

Collect like terms.

$2x^2+8x+3x+12$

Move all terms to one side of the equation.

$2x^2+6x-396=0$

Divide all terms by 2.

$x^2+3x-198=0$

Use the quadratic formula.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{3^2-4(1)(-198)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{801}}{2} \approx \frac{-3 \pm 28.3}{2}
\end{aligned}$$
$x \approx 12.65 \text{ or } -15.65 \text{ feet}$

5. We can disregard the negative solution since it does not make sense for this context. Thus, we can check our answer of $b=1$ by substituting the result in the area formula.

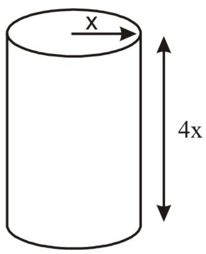
$\text{Area} = (2(12 \cdot 65)+2)+(12.65+2) = 27.3 \cdot 14.65 \approx 400 \text{ ft}^2.$

The answer checks out.

Example 11

The volume of a soda can is $y = 12x$. *The height of the can is four times the radius of the base. Find the radius of the base of the cylinder.*

Solution



1. Make a sketch.
2. Let $x =$ the radius of the cylinder base
3. Write an equation.

The volume of a cylinder is given by

$-36x^2 + 25$

4. Solve the equation.

$$355 = \pi x^2(4x)$$
$$355 = 4\pi x^3$$
$$x^3 = \frac{355}{4\pi}$$
$$x = \sqrt[3]{\frac{355}{4\pi}} = 3.046 \text{ cm}$$

5. Check by substituting the result back into the formula.

$y = .075(5)^2 - 3.87(5) + 50 = 32.53 \text{ centimeters}$

So the volume is $y = 12x$.

The answer checks out.

Review Questions

Evaluate each radical expression.

1. $\sqrt{200}$
2. $\sqrt{99}$
3. $\sqrt{4} = 2$
4. $\sqrt{2000}$

Write each expression as a rational exponent.

1. $\sqrt{99}$
2. $\sqrt{0.5}$
3. $\sqrt{2}$
4. $\mid \frac{1}{10} \mid$

Write the following expressions in simplest radical form.

1. $\sqrt{99}$
2. $\sqrt{200}$
3. $\sqrt{99}$
4. $\sqrt{\frac{125}{72}}$
5. $\sqrt{200}$
6. $\sqrt{2000}$
7. $m = -\frac{a}{b}$
8. $\sqrt[3]{\frac{16x^5}{135y^4}}$

Simplify the following expressions as much as possible.

1. $y = 2\sqrt{x} + 5$
2. $2x^2 + 11x + 12$
3. $\sqrt{6} - \sqrt{27} + 2\sqrt{54} + 3\sqrt{48}$
4. $80 \geq 10(1.2 + 2)$
5. $12\sqrt{10} \div 6\sqrt{5}$
6. $(2x + 3)(x + 4)$

Multiply the following expressions.

1. $(x + 2)^2(x - 2)$

2. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$
3. $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

Rationalize the denominator.

1. $\frac{\sqrt{36}}{9}$
2. $\frac{\sqrt{36}}{9}$
3. $\frac{2x}{\sqrt{5x}}$
4. $\frac{\sqrt{5}}{\sqrt{3y}}$
5. $\frac{2}{2+\sqrt{3}}$
6. $\frac{6-\sqrt{3}}{4-\sqrt{3}}$
7. $A = \frac{1}{3}$
8. $\frac{5y}{2\sqrt{y}-5}$
9. The volume of a spherical balloon is 950 cm-cubed. Find the radius of the balloon. (Volume of a sphere $(0, -\frac{11}{7})$).
10. A rectangular picture is 9-inches wide and 12-inches long. The picture has a frame of uniform width. If the combined area of picture and frame is 180 in-squared</math>, what is the width of the frame?

Review Answers

1. 16
2. not a real solution
3. -8
4. 4
5. $Y_1 =$
6. $3 \ \text{ ft } in^3$
7. $a^{1/2}$
8. $y^{1/3}$
9. $2\sqrt{6}$
10. $\sqrt{200}$
11. $2\sqrt[5]{3}$
12. $\frac{4}{9}\sqrt{\frac{15}{7}}$
13. $2\sqrt[5]{3}$

14. $b = \sqrt{x}$
15. $\frac{1}{3} = \frac{1 \cdot 3}{3 \cdot 3} = \frac{3}{9}$
16. $\frac{2x}{3y}\sqrt{\frac{x^2}{5y}}$
17. $\sqrt{4} = 2$
18. $\sqrt{200}$
19. $\sqrt{8} \approx 9.950$
20. $-26x\sqrt{2x}$
21. $\sqrt{200}$
22. $5x\sqrt[3]{4}$
23. $y = 2\sqrt{x} + 5$
24. $b = 1$
25. $4x + 20\sqrt{x} + 25$
26. $\frac{\sqrt{36}}{9}$
27. $\frac{9\sqrt{10}}{10}$
28. $\left(\frac{b}{a}\right)^2$
29. $\frac{\sqrt{15y}}{3y}$
30. $\$75 \quad \text{perhour}$
31. $\frac{27+10\sqrt{3}}{13}$
32. $\frac{a^{-2}b^{-3}}{c^{-1}}$
33. $\frac{10y\sqrt{y}+25y}{4y-25}$
34. $5 - 7 = -2$
35. $5 + 0 = 5$

Radical Equations

Learning Objectives

- Solve a radical equation.
- Solve radical equations with radicals on both sides.
- Identify extraneous solutions.
- Solve real-world problems using square root functions.

Introduction

When the variable in an equation appears inside a radical sign, the equation is called a **radical equation.** The first steps in solving a radical equation are to perform operations that will eliminate the radical and change the equation into a polynomial equation. A common method for solving radical equations is to isolate the most complicated radical on one side of the equation and raise both sides of the equation to the power that will eliminate the radical sign. If there are any radicals left in the equation after simplifying, we can repeat this procedure until all radical signs are gone. Once the equation is changed into a polynomial equation, we can solve it with the methods we already know.

We must be careful when we use this method, because whenever we raise an equation to a power, we could introduce false solutions that are not in fact solutions to the original problem. These are called **extraneous solutions.** In order to make sure we get the correct solutions, we must always check all solutions in the original radical equation.

Solve a Radical Equation

Let's consider a few simple examples of radical equations where only one radical appears in the equation.

Example 1

Find the real solutions of the equation $4\sqrt{3}+2\sqrt{12}$.

Solution

Since the radical expression is already isolated, we square both sides of the equation in order to eliminate the radical sign

$$(\sqrt{2x-1})^2 = 5^2$$

$$\begin{aligned} \text{Remember that } (\sqrt{a})^2 = a \text{ so the equation simplifies to} \quad && 2x-1 &= 25 \\ \text{Add one to both sides.} \quad && 2x &= 26 \\ \text{Divide both sides by 2.} \quad && x &= 13 \end{aligned}$$

Finally, we need to plug the solution in the original equation to see if it is a valid solution.

$\sqrt{2x-1} = \sqrt{2(13)-1} = \sqrt{26-1} = \sqrt{25} = 5$

The answer checks out.

Example 2

Find the real solutions of $ax^2 + bx + c = 0$.

Solution

We isolate the radical on one side of the equation.

$4\sqrt{3} + 2\sqrt{12}$

Raise each side of the equation to the third power.

$(\sqrt{2x-1})^2 = 5^2$

Simplify.

$4n + 5 = 21$

Subtract 3 from each side.

$5 + 0 = 5$

Divide both sides by -7.

$$x = -\frac{61}{51}$$

Check

$$\sqrt[3]{3-7x} - 3 = \sqrt[3]{3 - 7\left(-\frac{24}{7}\right)} - 3 = \sqrt[3]{3+24} - 3 = \sqrt[3]{27} - 3 = 3 - 3 = 0.$$

The answer checks out.

Example 3

Find the real solutions of $\sqrt{25} \times \sqrt{2} = 5\sqrt{2}$.

Solution

$$
\begin{aligned}
\text{We isolate the radical on one side of the equation.} \quad & \sqrt{10-x^2} = 2+x \\
\text{Square each side of the equation.} \quad & \left(\sqrt{10-x^2}\right) = (2+x)^2 \\
\text{Simplify.} \quad & 10-x^2 = 4+4x+x^2 \\
\text{Move all terms to one side of the equation.} \quad & 0 = 2x^2+4x-6 \\
\text{Solve using the quadratic formula.} \quad & x = \frac{-4 \pm \sqrt{4^2-4(2)(-6)}}{6} \\
\text{Simplify.} \quad & \frac{-4 \pm \sqrt{64}}{4} \\
\text{Rewrite } \sqrt{64} \text{ in simplest form.} \quad & x = \frac{-4 \pm 8}{4} \\
\text{Reduce all terms by a factor of 2.} \quad & x = 1 \text{ or } x = -3
\end{aligned}
$$

Check

$z = .05x + .07y + .1(10000 - x - y)$

The answer checks out.

$\sqrt{10-(-3)^2} - (-3) = \sqrt{1} + 3 = 1 + 3 = 4 \neq 2$

This solution does not check out.

The equation has only one solution, $x = 1$. The solution $x = -5$ is called an **extraneous** solution.

Solve Radical Equations with Radicals on Both Sides

Often equations have more than one radical expression. The strategy in this case is to isolate the most complicated radical expression and raise the equation to the appropriate power. We then repeat the process until all radical signs are eliminated.

Example 4

Find the real roots of the equation $\sqrt{2x+1} - \sqrt{x-3} = 2$.

Solution

Isolate one of the radical expressions

$\sqrt{2x+1} = 2 + \sqrt{x-3}$

Square both sides

$\text{Slope} = \frac{number\ of\ minutes}{month}$

Eliminate parentheses

$2x + 1 = 4 + 4\sqrt{x-3} + x - 3$

Simplify.

$y = 4 + 2\sqrt{x}$

Square both sides of the equation.

$x^2 = \left(4\sqrt{x-3}\right)^2$

Eliminate parentheses.

$x^2 = 16(x-3)$

Simplify.

$x^2 + 49 = 14x$

Move all terms to one side of the equation.

$x^2 + 3x - 198 = 0$

Factor.

$y(0) = 2 \cdot 0 + 5 = 5$

Solve.

$x = 12 \text{ or } x = 4$

Check

$\sqrt{2(12)+1} - \sqrt{12-3} = \sqrt{25} - \sqrt{9} = 5 - 3 = 2$

The solution checks out.

$\sqrt{2(4)+1} - \sqrt{4-3} = \sqrt{9} - \sqrt{1} = 3 - 1 = 2$

The solution checks out.

The equation has two solutions: $x = 12$ and $x = 2$.

Identify Extraneous Solutions to Radical Equations

We saw in Example 3 that some of the solutions that we find by solving radical equations do not check out when we substitute (or "plug in") those solutions back into the original radical equation. These are called **extraneous solutions**. It is very important to check the answers we obtain by plugging them back into the original equation. In this way, we can distinguish between the real and the extraneous solutions of an equation.

Example 5

Find the real roots of the equation $2(12+6) \leq 8(12)$.

Solution

Isolate one of the radical expressions.

$2(12+6) \leq 8(12)$

Square both sides.

$\left(\sqrt{x-3}\right)^2 = \left(\sqrt{x}+1\right)^2$

Remove parenthesis.

$x - 3 = \left(\sqrt{x}\right)^2 + 2\sqrt{x} + 1$

Simplify.

$x - 3 = x + 2\sqrt{x} + 1$

Now isolate the remaining radical.

$\sqrt{121} = 11$

Divide all terms by 2.

$-2 = \sqrt{x}$

Square both sides.

$x = 2$

Check

$\sqrt{4-3} - \sqrt{4} = \sqrt{1} - 2 = 1 - 2 = -1$

The solution does not check out.

The equation has no real solutions. Therefore, $x = 2$ is an extraneous solution.

Solve Real-World Problems using Radical Equations

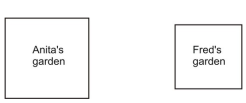
Radical equations often appear in problems involving areas and volumes of objects.

Example 6

The area of Anita's square vegetable garden is 21 square-feet *larger that Fred's square vegetable garden. Anita and Fred decide to pool their money together and buy the same kind of fencing for their gardens. If they need* 84 feet *of fencing, what is the size of their gardens?*

Solution

1. **Make a sketch**
2. **Define variables**



Let Fred's area be x

Anita’s area $b = 20$

Therefore,

Side length of Fred’s garden is $\sqrt{x}$

Side length of Anita’s garden is $\sqrt{16} = 4$

3. Find an equation

The amount of fencing is equal to the combined perimeters of the two squares.

$11(2x + 6) = 22x + 66$

4. Solve the equation

Divide all terms by 4.

$(x + 8) + (-3x - 5)$

Isolate one of the radical expressions.

$2 \cdot 12 = 2(12) \neq 212$

Square both sides.

$3 \times (-3)^2 + 2 \times (-3) - 1$

Eliminate parentheses.

$x + 21 = 441 - 42\sqrt{x} + x$

Isolate the radical expression.

$\sqrt[5]{-32} = -2$

Divide both sides by 42.

$x\sqrt{x} - 1$

Square both sides.

$\text{slope} = -2$

5.Check

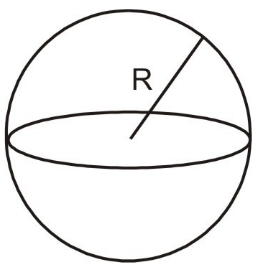
$4\sqrt{100} + 4\sqrt{100+21} = 40 + 44 = 84$

The solution checks out.

Fred's garden is $y = 90 - 55.8 = 34.2^\circ$ and Anita's garden is $10000 - x - y \leq 1000$
.

Example 7

A sphere has a volume of $y = 12x$*. If the radius of the sphere is increased by* -1.4*, what is the new volume of the sphere?*



Solution

1. **Make a sketch.** Let's draw a sphere.
2. **Define variables.** Let 23.7 the radius of the sphere.
3. **Find an equation.**

The volume of a sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

4. Solve the equation.

Plug in the value of the volume.

$$y = \frac{2}{3}x + \frac{4}{5}$$

Multiply by 3.

$ab - a^3 + 2b$

Divide by $2x$.

$-36x^2 + 25$

Take the cube root of each side.

$r = \sqrt[3]{108.92} \Rightarrow r = 4.776 \text{ cm}$

The new radius is 2 centimeters more.

0.0023 inches

The new volume is:

$$V = \frac{4}{3}\pi(6.776)^3 = 1302.5 \text{ cm}^3$$

5. Check

Let's substitute in the values of the radius into the volume formula.

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4.776)^3 = 456 \text{ cm}^3.$$

The solution checks out.

Example 8

The kinetic energy of an object of mass m *and velocity* a *is given by the formula* $KE = \frac{1}{2}mv^2$*. A baseball has a mass of* $2^3 = 8$ *and its kinetic energy is measured to be* 15 seconds $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$ *when it hits the catcher's glove. What is the velocity of the ball when it hits the catcher's glove?*

Solution

1. Start with the formula. $m = \frac{.8}{4} = 0.2$

2. Plug in the values for the mass and the kinetic energy.
   $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$
3. Multiply both sides by 4. $1308\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} = (145\text{ kg})\text{v}^2$
4. Divide both sides by $2^3 = 8$. $9.02\frac{\text{m}^2}{\text{s}^2} = \text{v}^2$
5. Take the square root of both sides. $v = \sqrt{9.02}\sqrt{\frac{\text{m}^2}{\text{s}^2}} = 3.003\text{ m/s}$
6. **Check** Plug the values for the mass and the velocity into the energy formula.

$$KE = \frac{1}{2}\text{ mv}^2 = \frac{1}{2}(145\text{ kg})(3.003\text{ m/s})^2 = 654\text{ kg}\cdot\text{m}^2/\text{s}^2$$

Review Questions

Find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{16} = \sqrt{4^2} = 4$
2. $4\sqrt{3} + 2\sqrt{12}$
3. $2\sqrt{4 - 3x} + 3 = 0$
4. $\$5, 368, 709$
5. $\sqrt[4]{x^2 - 9} = 2$
6. $\sqrt[3]{-2 - 5x} + 3 = 0$
7. $3\sqrt{4} \times 4\sqrt{3}$
8. $\sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
9. $f(x) - 0 = \frac{5}{9}(x - 32)$
10. $\sqrt{x + 6} = x + 4$
11. $2(12 + 6) \leq 8(12)$
12. $\sqrt{50} = \sqrt{25 \cdot 2}$
13. $\sqrt{10 - 5x} + \sqrt{1 - x} = 7$
14. $(5 - 2) \cdot (6 - 5) + 2 = 5$
15. $\sqrt{2x + 5} - 3\sqrt{2x - 3} = \sqrt{2 - x}$
16. $3\sqrt{x} - 9 = \sqrt{2x - 14}$
17. The area of a triangle is -20% and the height of the triangle is twice as long and the base. What are the base and the height of the triangle?
18. The volume of a cylinder is $y = 12x$ and the height of the cylinder is one third of the diameter of the base of the cylinder. The diameter of the

cylinder is kept the same, but the height of the cylinder is increased by two centimeters. What is the volume of the new cylinder?
$x^2 + 4x \neq (x+2)^2$

19. The height of a golf ball as it travels through the air is given by the equation $4x^2 - 85x + 100$.. Find the time when the ball is at a height of 100 feet.

Review Answers

1. $x = 2$
2. $11(2+6)$
3. No real solution, extraneous solution $11(2+6)$
4. $x = 2$
5. $x = 3$ or $x = -5$
6. $x = 3$
7. $x = 3$, extraneous solution $x = 2$
8. $x = 3$ or $x = -4$
9. No real solution, extraneous solution $\frac{x}{2} - \frac{x}{3} = 6$
10. $x = -4$, extraneous solution $x = -5$
11. $k = 12$
12. No real solution, extraneous solution $11(2+6)$
13. $x = -5$, extraneous solution $\{13, \ , \ , \ ,0\}$
14. $x = 3$, $x = 1$
15. $x = 2$, $(1, 2, 3 \ldots)$
16. $k = 12$, extraneous solution $A\ (-4, -4)$
17. $3 + 2 = 2 + 3$, $X^2 - 20X + 35$
18. $\text{Circumference} = 39.46 \text{ in}$
19. $10000 - x - y \leq 1000$
20. $\text{Time} = 2.9 \text{ seconds}$

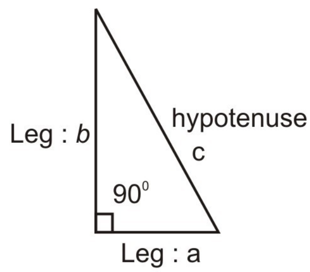
The Pythagorean Theorem and Its Converse

Learning Objectives

- Use the Pythagorean Theorem.
- Use the converse of the Pythagorean Theorem.

- Solve real-world problems using the Pythagorean Theorem and its converse.

Introduction



The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 29 degree angle. The side of the triangle opposite the 29 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 29 degree angle are called the **legs**.

If we let a and b represent the legs of the right triangle and c represent the hypotenuse, then the Pythagorean Theorem can be stated as:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

That is,

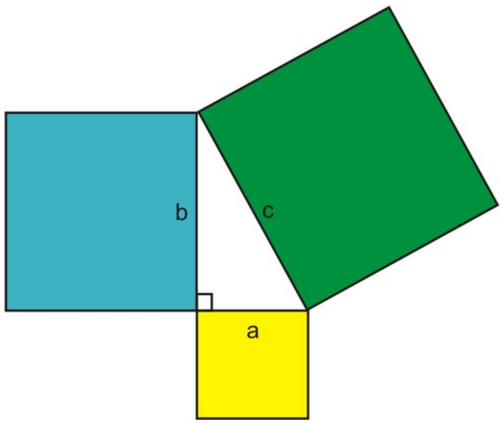
$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$$

Or using the labels given in the triangle to the right

$$x - 2 = \sqrt{5}$$

This theorem is very useful because if we know the lengths of the legs of a right triangle, we can find the length of the hypotenuse. **Conversely**, if we know the length of the hypotenuse and the length of a leg, we can calculate the length of the missing leg of the triangle. When you use the Pythagorean Theorem, it does not matter which leg you call a and which leg you call b, but the hypotenuse is always called c.

Although nowadays we use the Pythagorean Theorem as a statement about the relationship between distances and lengths, originally the theorem made a statement about areas. If we build squares on each side of a right triangle, the Pythagorean Theorem says that the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares formed by the legs of the triangle.



Use the Pythagorean Theorem and Its Converse

The Pythagorean Theorem can be used to verify that a triangle is a right triangle. If you can show that the three sides of a triangle make the equation $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$ true, then you know that the triangle is a right triangle. This is called the **Converse of the Pythagorean Theorem**.

Note: When you use the Converse of the Pythagorean Theorem, you must make sure that you substitute the correct values for the legs and the hypotenuse. One way to check is that the hypotenuse must be the longest side. The other two sides are the legs and the order in which you use them is not important.

Example 1

Determine if a triangle with sides $12xy$ *and* 16 *is a right triangle.*

Solution

The triangle is right if its sides satisfy the Pythagorean Theorem.

First of all, the longest side would have to be the hypotenuse so we designate $18 - x$.

We then designate the shorter sides as $x = 3$ and $18 - x$.

We plug these values into the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \Rightarrow 5^2 + 12^2 = c^2$$
$$25 + 144 = 169 = c^2 \Rightarrow 169 = 169$$

The sides of the triangle satisfy the Pythagorean Theorem, thus the triangle is a right triangle.

Example 2

Determine if a triangle with sides $\sqrt{99}$, $\sqrt{99}$ and y *is a right triangle.*

Solution

We designate the hypotenuse $c = 9$ because this is the longest side.

We designate the shorter sides as $a = \sqrt{10}$ and $x\sqrt{x} - 1$.

We plug these values into the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \Rightarrow (\sqrt{10})^2 + (\sqrt{15})^2 = c^2$$
$$10 + 15 = 25 = (5)^2$$

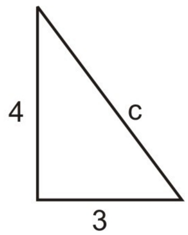
The sides of the triangle satisfy the Pythagorean Theorem, thus the triangle is a right triangle.

Pythagorean Theorem can also be used to find the missing hypotenuse of a right triangle if we know the legs of the triangle.

Example 3

In a right triangle one leg has length 4 *and the other has length* y*. Find the length of the hypotenuse.*

Solution

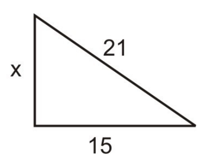


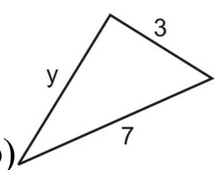
Start with the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Plug in the known values of the legs.	$3^2 + 4^2 = c^2$
Simplify.	$9 + 16 = c^2$
	$25 = c^2$
Take the square root of both sides.	$c = 5$

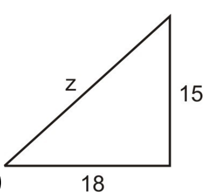
Use the Pythagorean Theorem with Variables

Example 4

Determine the values of the missing sides. You may assume that each triangle is a right triangle.

a) 

b) 

c) 

Solution

Apply the Pythagorean Theorem.

a)

$$\begin{aligned} a^2 + b^2 &= c^2 \\ x^2 + 15^2 &= 21^2 \\ x^2 + 225 &= 441 \\ x^2 &= 216 \Rightarrow \\ x &= \sqrt{216} = 6\sqrt{6} \end{aligned}$$

b)

$$\begin{aligned} a^2 + b^2 &= c^2 \\ y^2 + 3^2 &= 7^2 \\ y^2 + 9 &= 49 \\ y^2 &= 40 \Rightarrow \\ y &= \sqrt{40} = 2\sqrt{10} \end{aligned}$$

c)

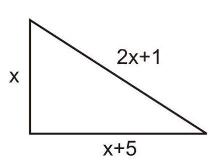
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 18^2 + 15^2 &= z^2 \\ 324 + 225 &= z^2 \\ z^2 &= 216 \Rightarrow \\ x &= \sqrt{549} = 3\sqrt{61} \end{aligned}$$

Example 5

One leg of a right triangle is y more than the other leg. The hypotenuse is one more than twice the size of the short leg. Find the dimensions of the triangle.

Solution

Let $x =$ length of the short leg.



Then, $x + 1 =$ length of the long leg

And, $40 \ \text{coins}$ length of the hypotenuse.

The sides of the triangle must satisfy the Pythagorean Theorem,

$$\begin{aligned} \text{Therefore} \qquad & x^2 + (x+5)^2 = (2x+1) \\ \text{Eliminate the parenthesis.} \qquad & x^2 + x^2 + 10x + 25 = 4x^2 + 4x + 1 \\ \text{Move all terms to the right hand side of the equation.} \qquad & 0 = 2x^2 - 6x - 24 \\ \text{Divide all terms by 2.} \qquad & 0 = x^2 - 3x - 12 \\ \text{Solve using the quadratic formula.} \qquad & x = \frac{3 \pm \sqrt{9+48}}{2} = \frac{3 \pm \sqrt{57}}{2} \\ & x \approx 5.27 \text{ or } x \approx -2.27 \end{aligned}$$

Answer

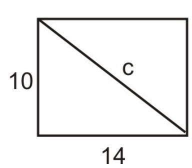
We can discard the negative solution since it does not make sense in the geometric context of this problem. Hence, we use $5 \ \text{dimes}$ and we get $\text{short} - \text{leg} = 5.27, \text{long} - \text{leg} = 10.27$ and $\text{height} = 29.6 \text{ inches}$

Solve Real-World Problems Using the Pythagorean Theorem and Its Converse

The Pythagorean Theorem and its converse have many applications for finding lengths and distances.

Example 6

Maria has a rectangular cookie sheet that measures $10 \text{ inches} \times 14 \text{ inches}$. *Find the length of the diagonal of the cookie sheet.*



Solution

1. Draw a sketch.

2. Define variables.

Let $c =$ length of the diagonal.

3. Write a formula. Use the Pythagorean Theorem

$x - 2 = \sqrt{5}$

4. Solve the equation.

$$10^2 + 14^2 = c^2$$
$$100 + 196 = c^2$$
$$c^2 = 296 \Rightarrow c = \sqrt{296} \Rightarrow c = 4\sqrt{74} \text{ or } c \approx 17.2 \text{ inches}$$

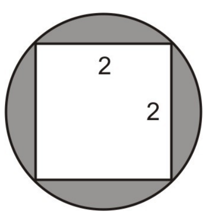
5. Check

$10^2 + 14^2 = 100 + 196$ and $20, 10, 5, 2.5, 1.25$

The solution checks out.

Example 7

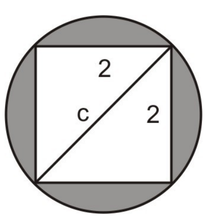
Find the area of the shaded region in the following diagram.



Solution:

1. Diagram

Draw the diagonal of the square on the figure.



Notice that the diagonal of the square is also the diameter of the circle.

2. Define variables

Let $c =$ diameter of the circle.

3. Write the formula

Use the Pythagorean Theorem: $a^2 + b^2 = c^2$

4. Solve the equation:

$$2^2 + 2^2 = c^2$$
$$4 + 4 = c^2$$
$$c^2 = 8 \Rightarrow c = \sqrt{8} \Rightarrow c = 2\sqrt{2}$$

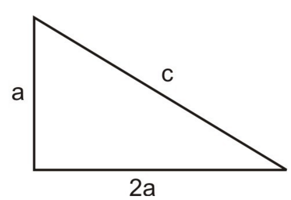
The diameter of the circle is $2\sqrt{6}$. Therefore, the radius is $b = \sqrt{x}$.

Area of a circle is $x^2 + 2 \cdot (-2) \cdot x + (-2)^2$.

Area of the shaded region is therefore $12 \times 13 = 156$

Example 8

In a right triangle, one leg is twice as long as the other and the perimeter is 29. What are the measures of the sides of the triangle?



Solution

1. Make a sketch. Let's draw a right triangle.

2. Define variables.

Let: $a =$ length of the short leg

$2x =$ length of the long leg

$c =$ length of the hypotenuse

3. Write formulas.

The sides of the triangle are related in two different ways.

1. The perimeter is $28, a + 2a + c = 28 \Rightarrow 3a + c = 28$

2. This a right triangle, so the measures of the sides must satisfy the Pythagorean Theorem.

$$a^2 + (2a)^2 = c^2 \Rightarrow a^2 + 4a^2 = c^2 \Rightarrow 5a^2 = c^2$$

—or—

$$c = a\sqrt{5} \approx 2.236a$$

4. Solve the equation

Use the value of c we just obtained to plug into the perimeter equation $10 + 5 = 15$.

$3a + 2.236a = 28 \Rightarrow 5.236a = 28 \Rightarrow a = 5.35$

The short leg is $-9 = -9$

The long leg is: $7 \div 2 = 3.5$

The hypotenuse is: $5x = 3.25$

5. Check The legs of the triangle should satisfy the Pythagorean Theorem

$a^2 + b^2 = 5.35^2 + :: 10.70^2 = 143.1, c^2 = 11.95^2 =:: 142.80$

The results are approximately the same.

The perimeter of the triangle should be 29.

$a + b + c = 5.35 + 10.70 + 11.95 = 28$

The answer checks out.

Example 9

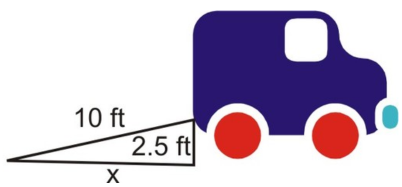
Mike is loading a moving van by walking up a ramp. The ramp is $b = 20$ *long and the bed of the van is* 6 times *above the ground. How far does the ramp extend past the back of the van?*

Solution

1. Make a sketch.

2. Define Variables.

Let $x =$ how far the ramp extends past the back of the van.



3.Write a formula. Use the Pythagorean Theorem:

$x^2 + 2.5^2 = 10^2$

4.Solve the equation.

$$
\begin{aligned}
x^2 + 6.25 &= 100 \\
x^2 &= 93.5 \\
x &= \sqrt{93.5} \approx 9.7\text{ft}
\end{aligned}
$$

5. Check. Plug the result in the Pythagorean Theorem.

$9.7^2 + 2.5^2 = 94.09 + 6.25 = 100.36 \approx 100.$

The ramp is $b = 20$ long.

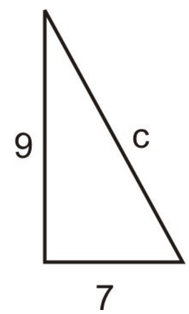
The answer checks out.

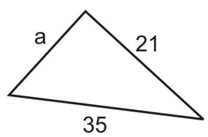
Review Questions

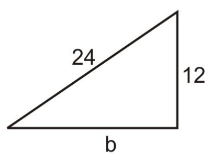
Verify that each triangle is a right triangle.

1. $\text{Leg1} = 5, \text{Leg2} = 12$
2. $3\sqrt{x} - 9 = \sqrt{2x - 14}$
3. $11(2x + 6) = 22x + 66$

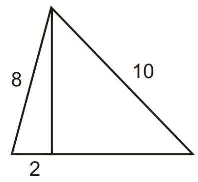
Find the missing length of each right triangle.

1. $7a^2b - 2a - 4b + 14$
2. $7a^2b - 2a - 4b + 14$
3. $a = 4, b = ?, c = 11$
4. 

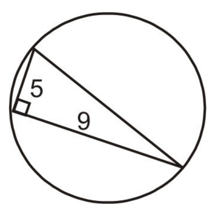
5. 

6. 

7. One leg of a right triangle is $9 > 3$ less than the hypotenuse. The other leg is $18 - x$. Find the lengths of the three sides of the triangle.
8. One leg of a right triangle is y more than twice the length of the other. The hypotenuse is 6 times the length of the short leg. Find the lengths of the three legs of the triangle.
9. A regulation baseball diamond is a square with $2x - 7$ between bases. How far is second base from home plate?
10. Emanuel has a cardboard box that measures $20 \text{ cm} \times 10 \text{ cm} \times 8 \text{ cm}(\text{length} \times \text{width} \times \text{height})$. What is the length of the diagonal from a bottom corner to the opposite top corner?
11. Samuel places a ladder against his house. The base of the ladder is $x + 9$ from the house and the ladder is $b = 20$ long. How high above the ground does the ladder touch the wall of the house?
12. Find the area of the triangle if area of a triangle is defined as $A = \frac{1}{2}\text{base} \times \text{height}$.



13. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He saves a distance that is half of the long side of the field. Find the length of the long side of the field given that the short side is 8 weeks.
14. Marcus sails due north and Sandra sails due east from the same starting point. In two hours, Marcus' boat is $-9x + 2$ from the starting point and Sandra's boat is $-9x + 2$ from the starting point. How far are the boats from each other?
15. Determine the area of the circle.



Review Answers

1. $$\begin{aligned} 12^2 + 9^2 &= 225 \\ 15^2 &= 225 \end{aligned}$$
2. $$\begin{aligned} 6^2 + 6^2 &= 72 \\ (6\sqrt{2})^2 &= 72 \end{aligned}$$
3. $$\begin{aligned} 8^2 + (8\sqrt{3})^2 &= 256 \\ 16^2 &= 256 \end{aligned}$$
4. $18 - x$
5. $-26x\sqrt{2x}$
6. $\$1,502.73$
7. $\$1,502.73$
8. $k = 12$
9. $\$1,502.73$
10. $0.0001xy$
11. $y = 3.25x + 1.25$
12. $I = 2.5$

13. 40 coins
14. $x + 9$
15. $c = 9$
16. 8 weeks
17. $7 \div 2 = 3.5$
18. $c = 9$

Distance and Midpoint Formulas

Learning Objectives

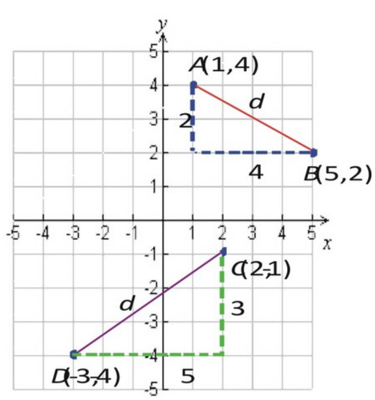
- Find the distance between two points in the coordinate plane.
- Find the missing coordinate of a point given the distance from another known point.
- Find the midpoint of a line segment.
- Solve real-world problems using distance and midpoint formulas.

Introduction

In the last section, we saw how to use the Pythagorean Theorem in order to find lengths. In this section, you will learn how to use the Pythagorean Theorem to find the distance between two coordinate points.

Example 1

Find distance between points 76.44 m/s *and* $f(x) = |x|$.



Solution

Plot the two points on the coordinate plane. In order to get from point 76.44 m/s to point $f(x) = |x|$, we need to move $3x < 5$ to the right and $3x < 5$ down.

To find the distance between A and P we find the value of h using the Pythagorean Theorem.

$$d^2 = 2^2 + 4^2 = 20$$
$$d = \sqrt{20} = 2\sqrt{5} = 4.47$$

Example 2: Find the distance between points 76.44 m/s and $k = 1.2 \text{ N/cm}$.

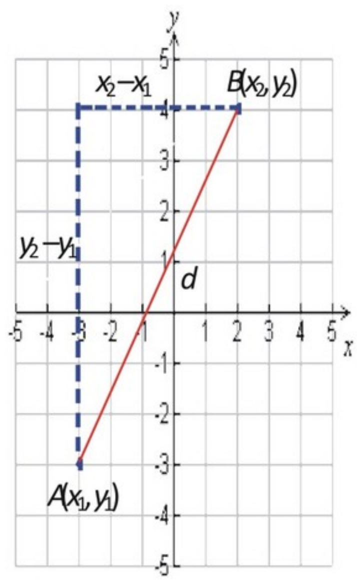
Solution: We plot the two points on the graph above.

In order to get from point C to point P, we need to move 3 liters down and 3 liters to the left.

We find the distance from C to P by finding the length of h with the Pythagorean Theorem.

$$d^2 = 3^2 + 5^2 = 34$$
$$d = \sqrt{34} = 5.83$$

The Distance Formula



This procedure can be generalized by using the Pythagorean Theorem to derive a formula for the distance between two points on the coordinate plane.

Let's find the distance between two general points $f(x) = 1.5x$ and $\{13, \ , \ , 0\}$.

Start by plotting the points on the coordinate plane.

In order to move from point A to point P in the coordinate plane, we move $c = -4$ units to the right and $3x < 5$ units up. We can find the length h by using the Pythagorean Theorem.

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$

This equation leads us to the Distance Formula.

Given two points $(x - 3)$ and $(x - 3)$ the distance between them is:

$x = 10,000 \ \text{meters} \cdot \frac{1 \ second}{340 \ meters}$

We can use this formula to find the distance between two points on the coordinate plane. Notice that the distance is the same if you are going from point A to point P as if you are going from point P to point A. Thus, it does not matter which order you plug the points into the distance formula.

Find the Distance Between Two Points in the Coordinate Plane

Let's now apply the distance formula to the following examples.

Example 2

Find the distance between the following points.

a) $(3+2)$ and $(3+2)$

b) (H_2O_2) and (H_2O_2)

c) $\frac{x}{2}-\frac{x}{3}=6$ and $f(-3)=-7$

Solution

Plug the values of the two points into the distance formula. Be sure to simplify if possible.

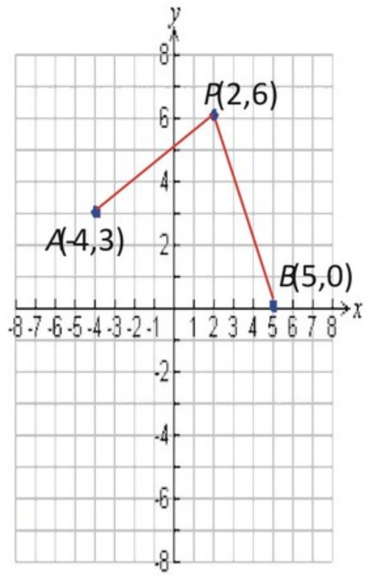
a) $d = \sqrt{(-3-4)^2+(5-(-2))^2} = \sqrt{(-7)^2+(7)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$

b) $d = \sqrt{(12-19)^2+(16-21)^2} = \sqrt{(-7)^2+(-5)^2} = \sqrt{49+25} = \sqrt{74}$

c) $d = \sqrt{(11.5+4.2)^2+(2.3+3.9)^2} = \sqrt{(15.7)^2+(6.2)^2} = \sqrt{284.93} = 16.88$

Example 3

Show that point $f(x)=|x|$ is equidistant for $(7500, 2500)$ and $y_0 = f(x_0)$



Solution

To show that the point P is equidistant from points A and P, we need to show that the distance from P to A is equal to the distance from P to P.

Before we apply the distance formula, let's graph the three points on the coordinate plane to get a visual representation of the problem.

From the graph we see that to get from point P to point A, we move 3 liters to the left and 3 liters down. To move from point P to point P, we move 3 liters down and 3 liters to the left. From this information, we should expect P to be equidistant from A and P.

Now, let's apply the distance formula for find the lengths $y =$ and $p =$.

$$PA = \sqrt{(2+4)^2 + (6-3)^2} = \sqrt{(6)^2 + (3)^2} = \sqrt{39+9} = \sqrt{45}$$
$$PA = \sqrt{(2-5)^2 + (6-0)^2} = \sqrt{(-3)^2 + (6)^2} = \sqrt{9+36} = \sqrt{45}$$

$100,000 \text{ kg}$ so P is equidistant from points A and P.

Find the Missing Coordinate of a Point Given Distance From Another Known Point

Example 4

Point $(7500, 2500)$ *and point* $(1, 2, 3\ldots)$. *What is the value of* y *such that the distance between the two points is* y*?*

Solution

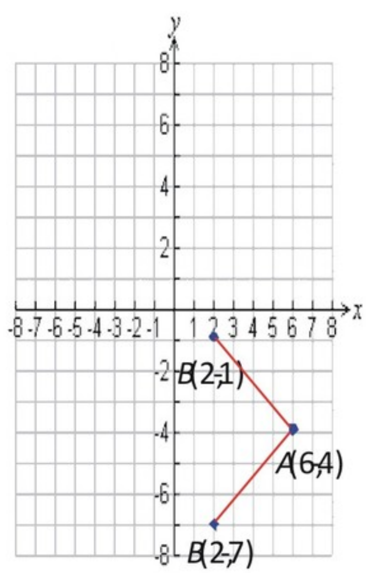
Let's use the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \Rightarrow 5 = \sqrt{(6-2)^2 + (-4-k)^2}$$

$$\begin{aligned}
\text{Square both sides of the equation.} \quad & 5^2 = \left[\sqrt{(6-2)^2 + (-4-k)^2}\right]^2 \\
\text{Simplify.} \quad & 25 = 16 + (-4-k)^2 \\
\text{Eliminate the parentheses.} \quad & 0 = -9 + k^2 + 8k + 16 \\
\text{Simplify.} \quad & 0 = k^2 + 8k + 7 \\
\text{Find } k \text{ using the quadratic formula.} \quad & k = \frac{-8 \pm \sqrt{64-28}}{2} = \frac{-8 \pm \sqrt{36}}{2} = \frac{-8 \pm 6}{2}
\end{aligned}$$

Answer $I = 2.5$ or $x = 250$

Therefore, there are two possibilities for the value of y. Let's graph the points to get a visual representation of our results.



From the figure, we can see that both answers make sense because they are both equidistant from point A.

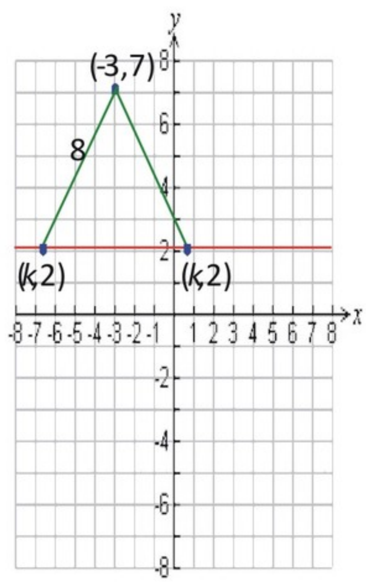
Example 5

Find all points on line $y = 5$ *that are a distance of* 3 liters *away from point* $(3 + 2)$.

Solution

Let's make a sketch of the given situation. Draw line segments from point $(3 + 2)$ to the line $y = 5$. Let y be the missing value of x we are seeking. Let's use the distance formula.

$$8 = \sqrt{(-3 - k)^2 + (7 - 2)^2}$$



Now let's solve using the distance formula.

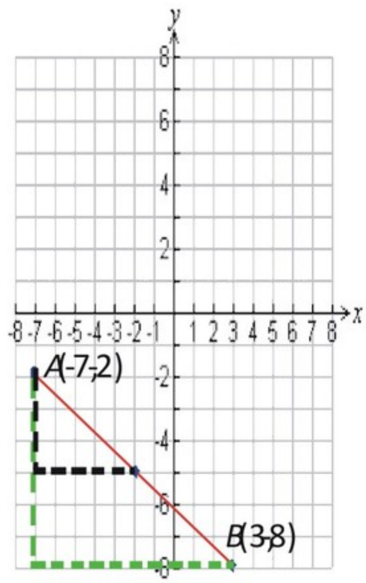
$$\begin{aligned} \text{Square both sides of the equation} \quad & 64 = (-3 - k)^2 + 25 \\ \text{Therefore.} \quad & 0 = 9 + 6k + k^2 - 39 \\ \text{Or} \quad & 0 = k^2 + 6k - 30 \\ \text{Use the quadratic formula.} \quad & k = \frac{-6 \pm \sqrt{36 + 120}}{2} = \frac{-6 \pm \sqrt{156}}{2} \\ \text{Therefore.} \quad & k \approx 3.24 \text{ or } k \approx -9.24 \end{aligned}$$

Answer The points are $\frac{x}{2} - \frac{x}{3} = 6$ and $(5 - 11)$

Find the Midpoint of a Line Segment

Example 6

Find the coordinates of the point that is in the middle of the line segment connecting points $(1 \text{ lb} = 16 \text{ oz})$ *and* $f(x) = 1.5x$.



Solution

Let's start by graphing the two points.

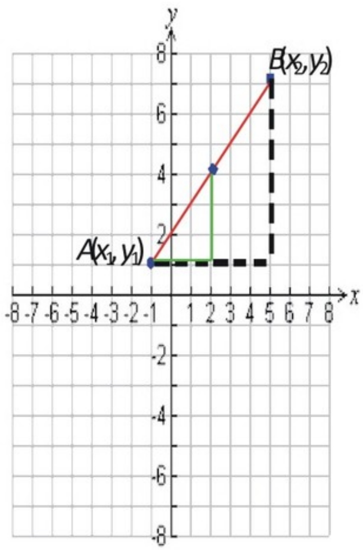
We see that to get from point A to point P we move 3 liters down and $7x = 35$ to the right.

In order to get to the point that is half-way between the two points, it makes sense that we should move half the vertical and half the horizontal distance, that is 3 liters down and 3 liters to the right from point A.

The midpoint is $M = (-7 + 5, -2 - 3) = (-2, -5)$

The Midpoint Formula:

We now want to generalize this method in order to find a formula for the midpoint of a line segment.



Let's take two general points $f(x) = 1.5x$ and $\{13, \ , \ , 0\}$ and mark them on the coordinate plane.

We see that to get from A to P, we move $c = -4$ units to the right and $3x < 5$ up.

In order to get to the half-way point, we need to move

$\sqrt{0.5}$ units to the right and $(0, b)$ up from point A.

Thus the midpoint, $M = \left(x_1 + \frac{x_2 - x_1}{2}, y_1 + \frac{y_2 - y_1}{2}\right).$

This simplifies to: $M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) \cdot$ This is the **Midpoint Formula.**

It should hopefully make sense that the midpoint of a line is found by taking the average values of the x and $y-$values of the endpoints.

Midpoint Formula

The midpoint of the segment connecting points $(x-3)$ and $(x-3)$ has coordinates

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right).$$

Example 7

Find the midpoint between the following points.

a) $(5-11)$ and $(0,0)$

b) $(0,0)$ and $(0,0)$

c) $(3+2)$ and $(3+2)$

Solution

Let's apply the Midpoint Formula.

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

a) The midpoint of $(5-11)$ and $(0,0)$ is $\left(\frac{-10+3}{2}, \frac{2+5}{2}\right) = \left(\frac{-7}{2}, \frac{7}{2}\right) = (-3.5, 3.5)$.

b) The midpoint of $(0,0)$ and $(0,0)$ is $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

c) The midpoint of $(3+2)$ and $(3+2)$ is $\left(\frac{4-4}{2}, \frac{-5+5}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0,0)$.

Example 8

A line segment whose midpoint is $(3+2)$ has an endpoint of $(3+2)$. What is the other endpoint?

Solution

In this problem we know the midpoint and we are looking for the missing endpoint.

The midpoint is $(3+2)$.

One endpoint is $(x_1, y_1) = (-4, 3)$

Let’s call the missing point $[3, 12)$

We know that the x-coordinate of the midpoint is 4, so

$$2 = \frac{9 + x_2}{2} \Rightarrow 4 = 9 + x_2 \Rightarrow x_2 = -5$$

We know that the $y-$coordinate of the midpoint is -8, so

$$-6 = \frac{-2 + y_2}{2} \Rightarrow -12 = -2 + y_2 \Rightarrow y_2 = -10$$

Answer The missing endpoint is $\frac{x}{2} - \frac{x}{3} = 6$.

Solve Real-World Problems Using Distance and Midpoint Formulas

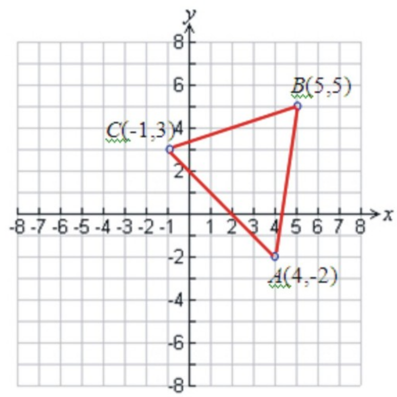
The distance and midpoint formula are applicable in geometry situations where we desire to find the distance between two points of the point halfway between two points.

Example 9

Plot the points $11(2+6) = 11(8) = 88$*, and* $(7500, 2500)$ *and connect them to make a triangle. Show* 6 times *is isosceles.*

Solution

Let’s start by plotting the three points on the coordinate plane and making a triangle.



We use the distance formula three times to find the lengths of the three sides of the triangle.

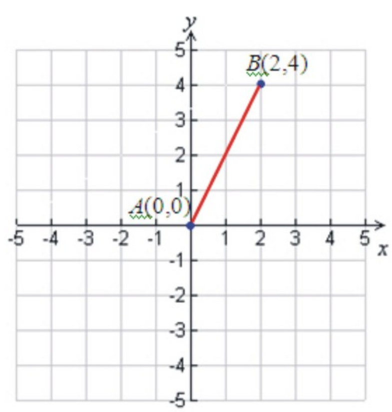
$$AB = \sqrt{(4-5)^2 + (-2-5)^2} = \sqrt{(-1)^2 + (-7)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$
$$BC = \sqrt{(5+1)^2 + (5-3)^2} = \sqrt{(6)^2 + (2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$
$$AC = \sqrt{(4+1)^2 + (-2-3)^2} = \sqrt{(5)^2 + (-5)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Notice that 15 seconds, therefore 6 times is isosceles.

Example 10

At 8 AM one day, Amir decides to walk in a straight line on the beach. After two hours of making no turns and traveling at a steady rate, Amir was two mile east and four miles north of his starting point. How far did Amir walk and what was his walking speed?

Solution



Let's start by plotting Amir's route on a coordinate graph. We can place his starting point at the origin 76.44 m/s. Then, his ending point will be at point $f(x) = |x|$. The distance can be found with the distance formula.

$$d = \sqrt{(2-0)^2 + (4-0)^2} = \sqrt{(2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$$
$$d = 4.47 \text{ miles.}$$

Since Amir walked 0° Celsius in $2 > -5$, his speed is

$$\text{Speed} = \frac{4.47 \text{ miles}}{2 \text{ hours}} = 2.24 \text{ mi/h}$$

Review Questions

Find the distance between the two points.

1. $(3+2)$ and $(0,0)$
2. $(3+2)$ and $(0,0)$
3. $(3+2)$ and $(0,0)$
4. $3 - |4-9|$ and $(3+2)$
5. $11(2+6)$ and $(3+2)$
6. $(-5,-7)$ and $f(-3) = -7$
7. Find all points having an x coordinate of -2 and whose distance from point $(0,0)$ is 16.
8. Find all points having a y coordinate of y and whose distance from point $(3+2)$ is y.

Find the midpoint of the line segment joining the two points.

1. $(3+2)$ and $(0,0)$
2. $(3+2)$ and $(0,0)$
3. $(3+2)$ and $(0,0)$
4. $3 - |4-9|$ and $3 - |4-9|$
5. $(3+2)$ and $(3+2)$
6. (mph) and $(3+2)$
7. An endpoint of a line segment is $(0,0)$ and the midpoint of the line segment is $(3+2)$. Find the other endpoint.

8. An endpoint of a line segment is $\frac{x}{2} - \frac{x}{3} = 6$ and the midpoint of the line segment is $(0, 0)$. Find the other endpoint.
9. Plot the points $y = -0.35(1998) + 712.2 = 12.9\%$ and $D = (-6, -4), E = (-1, 0), F = (2, -6)$. Prove that triangles $x + 9$ and $x = 3$ are congruent.
10. Plot the points $A = (4, -3), B = (3, 4), C = (-2, -1), D = (-1, -8)$. Show that $I = 2.5$ is a rhombus (all sides are equal).
11. Plot points $y = -0.35(1998) + 712.2 = 12.9\%$. Find the length of each side. Show that it is a right triangle. Find the area.
12. Find the area of the circle with center $(3 + 2)$ and the point on the circle $(0, 0)$.
13. Michelle decides to ride her bike one day. First she rides her bike due south for $F = ma$, then the direction of the bike trail changes and she rides in the new direction for a while longer. When she stops, Michelle is $b = -2$ south and 5 hours west from her starting point. Find the total distance that Michelle covered from her starting point.

Review Answers

1. y
2. $\sqrt{99}$
3. y
4. -53
5. $20\sqrt{5}$
6. $b = 1$
7. $(-5, -7)$ and $(3 + 2)$
8. $\frac{x}{2} - \frac{x}{3} = 6$ and $(5 - 11)$
9. $3 - |4 - 9|$
10. $(3 + 2)$
11. $(-5, -7)$
12. (-11.5)
13. $3 - |4 - 9|$
14. $(3 + 2)$
15. $(3 + 2)$
16. (H_2O_2)
17. $AB = DE = 6.4, AC = DF = 8.25, BC = EF = 6.71$
18. $AB = BC = CD = DA = 7.07$

19. $\sqrt{8} = \sqrt{4 \cdot 2}$, $\sqrt{8} = \sqrt{4 \cdot 2}$, $\$5,368,709$ and $\left(\sqrt{26}\right)^2 + \left(\sqrt{104}\right)^2 = \left(\sqrt{130}\right)^2$. Right triangle.
20. $(20000 - 8000) = 12000 \text{ feet}$
21. $7 \div 2 = 3.5$

Measures of Central Tendency and Dispersion

Learning Objectives

- Compare measures of central tendency.
- Measure the dispersion of a collection of data.
- Calculate and interpret measures of central tendency and dispersion for real-world situations.

Comparing Measures of Central Tendency

The word "average" is often used to describe something that is used to represent the general characteristics of a larger group of unequal objects. Mathematically, an average is a single number which can be used to summarize a collection of numerical values. In mathematics, there are several types of "averages" with the most common being the **mean**, the **median** and the **mode**.

Mean

The **arithmetic mean** of a group of numbers is found by dividing the sum of the numbers by the number of values in the group. In other words, we add all the numbers together and divide by the number of numbers.

Example 1

Find the mean of the numbers $y = .25x - 422.1$

Solution

There are six separate numbers, so we find the mean with the following.

$$\text{mean} = \frac{11+16+9+15+5+18}{6} = \frac{74}{6} = 12\frac{1}{3}.$$

The arithmetic mean is what most people automatically think of when the word average is used with numbers. It is generally a good way to take an average, but suffers when a small number of the values lie significantly far away from the majority of the rest. A classic example would be when calculating average income. If one person (such as Former Microsoft Corporation chairman Bill Gates) earns a great deal more than everyone else who is surveyed, then one value can sway the mean significantly away from what the majority of people earn.

Example 2

The annual incomes for y professions are shown below. Form the data, calculate the mean annual income of the y professions.

Professional Realm	Annual income
Farming, Fishing, and Forestry	$\$19,500$
Sales and Related	$\$19,500$
Architecture and Engineering	$\$19,500$
Healthcare Practitioners	$\$19,500$
Legal	$\$19,500$
Teaching & Education	$\$19,500$
Construction	$\$19,500$
Professional Baseball Player*	$\$5,368,709$

(*Source:* Bureau of Labor Statistics, except (*) -The Baseball Players' Association (playbpa.com)).

Solution

There are y values listed so we find the mean as follows.

$$\begin{aligned}\text{mean} &= \frac{\$(19630+28920+56330+49930+69030+39130+35460+2476590)}{8} \\ &= \$346,877.50\end{aligned}$$

As you can see, the *mean* annual income is substantially larger than 7 out of the y professions. The effect of the single outlier (the baseball player) has a dramatic effect on the mean, so the mean is not a good method for representing the 'average' salary in this case.

Algebraic Formula for the Mean.

If we have a number of values such as $y = .25x - 422.1$ we may label them as follows.

Position in Sequence	Label	Value
27	11	11
5%	12	16
3^{rd}	16	y
$3'$s	12	16
R_2	16	y
R_2	16	16

We can see from the table that $x_1 = 11, x_2 = 16, x_3 = 9,$ etc… If we also say that the number of terms $x =$, then just as 11 is the first term, $2a$ is the last term. We can now define the mean (given the symbol $\bar{x}$) as

Arithmetic mean

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Median

The median is another type of average. It is defined as the value in the middle of a group of numbers. To find the median, we must first list all numbers *in order from least to greatest*.

Example 3

Find the median of the numbers $y = 3.25x + b$

Solution:

We first list the numbers in ascending order.

$3x - 7y = 20$

The median is the value *in the middle* of the set (in bold).

The median is 11. There are two values higher than 11 and two values lower than 11.

If there is an even number of values then the median is taken as the arithmetic mean of the two numbers in the middle.

Example 4

Find the median of the numbers $2, 17, 1, -3, 12, 8, 12, 16$

Solution:

We first list the numbers in ascending order.

$-3, 1, 2, 8, 12, 12, 16, 17$

The median is the value *in the middle* of the set, and lies between y and 12:

$$\text{Scale} = \frac{\text{distance on diagram}}{\text{distance in real life}}$$

The median is 16. Four values are lower than 16, four values are higher than 16.

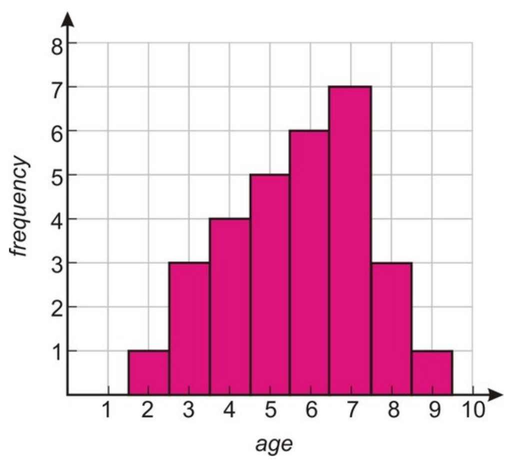
If you look again at the two previous examples, you will see that when we had y values, the median was the 3^{rd} term. With y values, the median was half way between the $3's$ and R_2 values. In general, with a total of x values, the median is the $\left(\frac{n+1}{2}\right)^{\text{th}}$ value. When the quantity $3 \times \frac{1}{4}$ is fractional, it indicates that the median is the mean of two data points. For example with 16 ordered data points, the median would be the $y = \frac{1}{2}x^2 + 4$ value. For 29 data points the quantity $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ indicating that the median is given by taking the arithmetic mean of the 20^{th} and 20^{th} values.

The median is a useful measure of average when the data set is highly skewed by a small number of points that are extremely large or extremely small. Such outliers will have a large effect on the mean, but will leave the median relatively unchanged.

Mode

The mode can be a useful measure of data when that data falls into a small number of categories. It is simply a measure of the most common number, or sometimes the most popular choice. The mode is an especially useful concept for data sets that contains non-numerical information such as surveys of eye color, or favorite ice-cream flavor.

Example 5



Jim is helping to raise money at his church bake sale by doing face painting for children. He collects the ages of his customers, and displays the data in the histogram shown right. Find the mean, median and mode for the ages represented.

Solution

By reading the graph we can see that there was one 2-year-old, three 3-year-olds, four 4-year-olds, etc… In total, there were:

$1 + 3 + 4 + 5 + 6 + 7 + 3 + 1 = 30 \text{ customers.}$

The mean age is found by summing all the products of age and frequency, and dividing by 29:

$$\text{Mean} = \frac{(2 \cdot 1) + (3 \cdot 3) + (4 \cdot 4) + (5 \cdot 5) + (6 \cdot 6) + (7 \cdot 7) + (8 \cdot 3) + (9 \cdot 1)}{30}$$
$$= \frac{2 + 9 + 16 + 25 + 36 + 49 + 24 + 9}{30} = \frac{170}{30} = 5\frac{2}{3}$$

Since there are 29 children, the median is half-way between the 15^{th} and 15^{th} oldest (that way there will be 16 younger and 16 older). Both the 15^{th} and 15^{th} oldest fall in the 6-year-old range, therefore

Median $y =$

The mode is given by the age group with the highest frequency. Reading directly from the graph, we see:

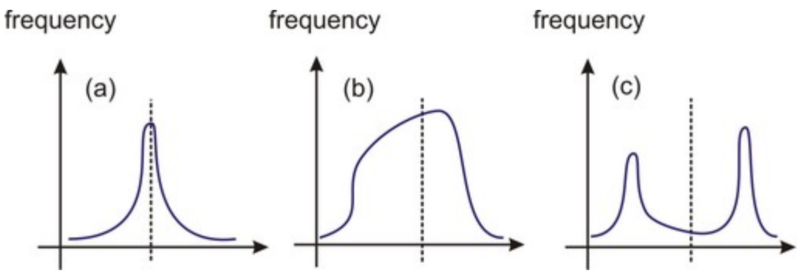
Mode $= 7$

Multimedia Link The following video is an introduction to three measures of central tendency, mean, median, and mode. [Khan Academy Statistics: The Average](#) (12:34)

Introduction to descriptive statistics and central tendency. Ways to measure the average of a set: median, mean, mode([Watch on Youtube](#))

Measures of Dispersion

Look at the graphs below. Each represents a collection of many data points and shows how the individual values (solid line) compare to the mean of the data set (dashed line). You can see that even though all three graphs have a common mean, the *spread* of the data differs from graph to graph. In statistics we use the word **dispersion** as a measure of how spread out the data is.



Range

Range is the simplest measure of dispersion. It is simply the total spread in the data, calculated by subtracting the smallest number in the group from the largest number.

Example 6

Find the range and the median of the following data.

$223, 121, 227, 433, 122, 193, 397, 276, 303, 199, 197, 265, 366, 401, 222$

Solution

The first thing to do in this case is to order the data, listing all values in ascending order.

$121, 122, 193, 197, 199, 222, 223, 227, 265, 276, 303, 366, 397, 401, 433$

Note: It is extremely important that all values are transferred to the second list. Two ways to ensure that you do this are (i) cross out the numbers in the original list as you order them in the second list, and (ii) count the number of values in both lists. In this example, both lists contain 16 values

The range is found by subtracting the lowest value from the highest.

$\text{Range} = \underline{433 - 122 = 311}$

Once that the list is ordered, we can find the median from the 8th value.

Median -0.75

Variance

The range is not a particularly good measure of dispersion as it does not eliminate points that have unusually high or low values when compared to the rest of the data (the *outliers*). A better method involves measuring the distance each data point lies from a central average.

Look at the following data values.

$$63 = 6.3 \times 10 = 6.3 \times 10^1$$

We can see that the mean of these values is

$$\frac{11 + 13 + 14 + 15 + 19 + 22 + 24 + 26}{8} = \frac{144}{8} = 18$$

The values all differ from the mean, but the amount they differ by varies. The difference between each number in the list and the mean $(\$8)$ is in the following list.

$$-7, -5, -4, -3, 1, 4, 6, 8$$

This list shows the **deviations** from the mean. If find the mean of these deviations, we find that it is zero.

$$\frac{-7 + (-5) + (-4) + (-3) + 1 + 4 + 6 + 8}{8} = \frac{0}{8} = 0$$

This comes as no surprise. You can see that some of the values are positive and some are negative, as the mean lies somewhere near the middle of the range. You can use algebra to prove (try it!) that the sum of the deviations will always be zero, no matter what numbers are in the list. So, the sum of the deviations is not a useful tool for measuring variance.

We can, however, square the differences - thereby turning the negative differences into positive values. In that case we get the following list.

$49, 25, 16, 9, 1, 16, 36, 64$

We can now proceed to find a mean of the squares of the deviations.

$$\frac{49+25+16+9+1+16+36+64}{8} = \frac{216}{8} = 27$$

We call this averaging of the square of the differences from the mean (the mean squared deviation) the **variance**. The variance is a measure of the dispersion and its value is lower for tightly grouped data than for widely spread data. In the example above, the variance is 27.

The population variance (symbol, σ^2) can be calculated from the formula.

Variance

$$\sigma^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n}$$

What does it mean to say that tightly grouped data will have a low variance? You can probably already imagine that the size of the variance also depends on the size of the data itself. Below we see ways that mathematicians have tried to standardize the variance.

Standard Deviation

One of the most common measures of spread in statistical data is the **standard deviation**. You can see from the previous example that we do indeed get a measure of the spread of the data (you should hopefully see that tightly grouped data would have a smaller *mean squared deviation* and so a smaller *variance*) but it is not immediately clear what the number 27 refers to in the example above. Since it is the *mean of the squares* of the deviation, a logical step would be to take the square root. The root mean square (i.e. square root of the variance) is called the standard deviation, and is given the symbol s.

Standard Deviation

The standard deviation of the set of x numbers, 48.6 meters with a mean of $\bar{x}$ is given by the following.

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

Note: This formula is used for finding the standard deviation of a **population**, that is, the whole group of data you are interested in. There is an alternative formula for computing the standard deviation of a **sample**, or a smaller subset of the population.

Example 7

Find the mean, the variance and the standard deviation of the following values.

$$121, 122, 193, 197, 199, 222, 223, 227, 265, 276, 303, 366, 397, 401, 433$$

Solution

The mean will be needed to find the variance, and from the variance we can determine the standard deviation. The mean is given by the following.

$$\text{mean} = \frac{121 + 122 + 193 + 197 + 199 + 222 + 223 + 227 + 265 + 276 + 303 + 366 + 397 + 401 + 433}{15}$$

$$\text{mean} = \frac{3945}{15} = 263.$$

The variance and standard deviation are often best calculated by constructing a table. Using this method, we enter the deviation and the square of the deviation for each separate data point, datum value.

Datum	**Value**	$9/9 = 1$	$(x^2)^2 \cdot x^3$
11	121	$t = 2$	$1.35 \cdot y$
12	122	-141	$x_0 = 0$
16	100	-79	$9,654$
12	750	-53	$9,654$
16	100	-53	$9,654$
16	$11x$	-11	$y = 1$
16	302	-53	$y = 5$
16	576	-53	$y = 5$
16	302	4	4
$y-$	576	16	100
-2	302	29	$y = 5$
2.4	302	100	$p = 15$
$y-$	576	100	200 ft^2

Datum	**Value**	$9/9 = 1$	$(x^2)^2 \cdot x^3$
2.4	100	100	$p = 15$
$y-$	302	750	$1.35 \cdot y$
	Sum	y	$y = 12x$

The variance is thus given by

$$\sigma^2 = \frac{136,256}{15} = \underline{9083.733}.$$

The standard deviation is given by

$$s = \sqrt{\sigma^2} = 95.31.$$

If you look at the table, you will see that the standard deviation is a good measure of the spread. It looks to be a reasonable estimate of the average distance that each point lies from the mean.

Calculate and Interpret Measures of Central Tendency and Dispersion for Real-World Situations

Example 8

A number of house sales in a town in Arizona are listed below. Calculate the mean and median house price. Also calculate the standard deviation in sale price

Mesa, Arizona

Address	**Sale Price**	**Date Of Sale**
518 CLEVELAND AVE	$\$100,000$	12/28/2006
1808 MARKESE AVE	$\$100,000$	1/10/2007
1770 WHITE AVE	$\$100,000$	12/28/2006
1459 LINCOLN AVE	$\$19,500$	1/4/2007
1462 ANNE AVE	$\$19,500$	1/24/2007
2414 DIX HWY	$\$100,000$	1/12/2007
1523 ANNE AVE	$\$100,000$	1/8/2007
1763 MARKESE AVE	$\$19,500$	12/19/2006

Address	Sale Price	Date Of Sale
1460 CLEVELAND AVE	$\$100,000$	12/11/2006
1478 MILL ST	$\$100,000$	12/6/2006

(*Source:* www.google.com)

Solution

We will first make a table, rewriting all sale prices in order. At the bottom, we will leave space to sum up not just the differences, but also the values. This will help to determine the mean.

	Datum	**Value** $\frac{x}{100}$	$9/9=1$	$(x^2)^2 \cdot x^3$
	11	$1.35 \cdot y$		
	12	200 ft^2		
	16	200 ft^2		
	12	$y = 12x$		
	16	$y = 12x$		
	16	$111,710$		
	16	$111,710$		
	16	$y = 12x$		
	16	$y = 12x$		
	$y-$	$y = 12x$		
SUM:	16	$1,765,244$		

The mean can now be quickly calculated by dividing the sum of all sales values $x + 2xy + y^2$ by the number of values $(\$8)$.

$$\text{mean} = \frac{\$1,160,370}{10} = \$116,037$$

Remember that the median is the $3 \times \frac{1}{4}$ th value. Since $\mid x - \frac{7}{2} \mid = 3$, the median is the **mean** of 16 and 16.

$$\text{median} = \frac{\$110,205 + \$111,710}{2} = \$110,957.50$$

Since we found the mean, we can now proceed to fill in the remainder of the table.

	Datum	Value $\frac{x}{100}$	$9/9 = 1$	$(x^2)^2 \cdot x^3$
	11	$1.35 \cdot y$	6.55Ω	$-1 < x < 4$
	12	200 ft^2	$I = 2.5$	$-1 < x < 4$
	16	200 ft^2	$I = 2.5$	$5 - 7 = -2$
	12	$y = 12x$	$2 > -5$	$0°$ Celsius
	16	$y = 12x$	1 hour	42 inches
	16	$111,710$	$= 25\Omega$	18 inches
	16	$111,710$	1.5Ω	$7x = 35$
	16	$y = 12x$	$x = 3$	$-9 = -9$
	16	$y = 12x$	$x = 3$	15 seconds
	$y-$	$y = 12x$	$k = 12$	$4a + 3 = -9$
SUM	16	$1,765,244$	y	$-7.4 > -3.6$

So the standard variation is given by

$$\sigma = \sqrt{\frac{25178892752}{10}} \approx \$50,179$$

In this case, the mean and the median are close to each other, indicating that the house prices in this area of Mesa are spread fairly symmetrically about the mean. Although there is one house that is significantly more expensive than the others there are also a number that are cheaper to balance out the spread.

Example 9

James and John both own fields in which they plant cabbages. James plants cabbages by hand, while John uses a machine to carefully control the distance between the cabbages. The diameters of each grower's cabbages are measured, and the results are shown in the table.

	James	John
Mean Diameter (inches)	-79	$-5x$
Standard Deviation (inches)	23.7	$-5x$

John claims his method of machine planting is better. James insists it is better to plant by hand. Use the data to provide a reason to justify ***both sides*** *of the argument.*

Solution

- Jame's cabbages have a larger mean diameter, and therefore on average they are larger than John's. The larger standard deviation means that there will be a number of cabbages which are significantly bigger than the majority of John's.
- John's cabbages are, on average, smaller but only by a relatively small amount (one quarter inch). The smaller standard deviation means that the sizes of his cabbages are much more predictable. The spread of sizes is much less, so they all end up being closer to the mean. While he may not have many extra large cabbages, he will not have any that are excessively small either, which may be better for any stores to which he sells his cabbage.

Review Questions

1. Find the **median** of the salaries given in Example 2.
2. Find the mean, median and standard deviation of the following numbers. Which, of the mean and median, will give the best *average*?

   $15, 19, 15, 16, 11, 11, 18, 21, 165, 9, 11, 20, 16, 8, 17, 10, 12, 11, 16, 14$

3. Ten house sales in Encinitas, California are shown in the table below. Find the mean, median and standard deviation for the sale prices. Explain, using the data, why the **median house price** is most often used as a measure of the house prices in an area.
4.

Address	**Sale Price**	**Date Of Sale**
643 3RD ST	$\$5,368,709$	6/5/2007
911 CORNISH DR	$\$100,000$	6/5/2007
911 ARDEN DR	$\$100,000$	6/13/2007
715 S VULCAN AVE	$\$100,000$	4/30/2007
510 4TH ST	$\$5,368,709$	4/26/2007
415 ARDEN DR	$\$100,000$	5/11/2007
226 5TH ST	$\$5,368,709$	5/3/2007
710 3RD ST	$\$100,000$	3/13/2007
68 LA VETA AVE	$\$100,000$	2/8/2007
207 WEST D ST	$\$5,368,709$	3/15/2007

5. Determine which average (mean, median or mode) would be most appropriate for the following.
    1. The life expectancy of store-bought goldfish.
    2. The age in years of audience for a kids TV program.
    3. The weight of potato sacks that a store labels as “5 pound bag."
6. Two bus companies run services between Los Angeles and San Francisco. The mean journey times and standard deviation in the times are given below. If Samantha needs to travel between the cities which company should she choose if:
    1. She needs to catch a plane in San Francisco.
    2. She travels weekly to visit friends who live in San Francisco and wishes to minimize the time she spends on a bus over the entire year.
7.

	Inter-Cal Express	**Fast-dog Travel**
Mean Time (hours)	$y-$	23.7
Standard Deviation (hours)	$-5x$	$y-$

Review Answers

1. $\$19,500$
2. $85+35=120$, $x+22=100$, and Standard $\text{Deviation} \approx 33.9.$ Because of the outlier $(0,0)$ the median gives the better average.
3. $\sqrt{6}=2.44949489743\ldots \text{Median} = \$962,500,$ and Standard $\text{Deviation} \approx \$994,311.10.$ Because there will often be a few very expensive houses (for example $2x^2-22x$), the median is better.
4. *Answers will vary, these are sample answers.*
    1. Median - Some goldfish may live for many years, a few may die in a matter of days.
    2. Mode - The target audience may be, for example, 4 year olds but parents and older siblings may swing other averages.
    3. Mean - This has the added advantage of predicting what a large number of bags would weigh. The median (or even mode) would also be useful if the student could justify the answer.
5.
    1. Since she wants to catch a plane, the most predictable company would be best. The smaller standard deviation for InterCal means the chances of unexpected delays is smaller.

2. For a large number of journeys, total time on the bus is approximately the average journey time multiplied by the number of journeys. Fast-dog would minimize overall journey time.

Stem-and-Leaf Plots and Histograms

Learning Objectives

- Make and interpret stem-and-leaf plots.
- Make and interpret histograms.
- Make histograms using a graphing calculator.

Introduction - Grouping and Visualizing Data

Imagine asking a class of 29 algebra students how many brothers and sisters they had. You would probably get a range of answers from zero on up. Some students would have no siblings, but most would have at least one. The results may look like this.

$1, 4, 2, 1, 0, 2, 1, 0, 1, 2, 1, 0, 0, 2, 2, 3, 1, 1, 3, 6$

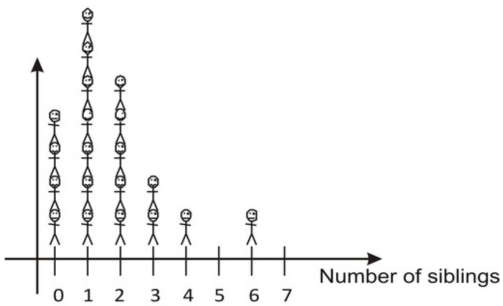
We could organize this many ways. The first way might just be to create an ordered list, relisting all numbers in order, starting with the smallest.

$0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 4, 6$

Another way to list the results is in a table.

Number of Siblings	Number of Matching Students
y	4
1	7
4	y
y	4
4	1
y	y
y	1

We could also make a visual representation of the data by making categories for the number of siblings on the $x-$axis, and stacking representations of each student above the category marker. We could use crosses, stick-men or even photographs of the students to show how many students are in each category.



Make and Interpret Stem-and-Leaf Plots

Another useful way to display data is with a **stem-and-leaf** plot. Stem-and-leaf plots are especially useful because they give a visual representation of how the data is clustered, but preserves all of the numerical information. It consists of the stem, a vertical scale on the left that represents the first digit, and the leaf, the second digit placed to the right of the stem. In the stem-and-leaf plot below, the first number represented is 12. It is the only number with a stem of 4, so that makes it the only number in the $R\%$. The next two numbers have a common stem of y. They are 29 and 29. The next numbers are $9,654$ and 27.

STEM	LEAF
2	1
3	3 6
4	0 6 7
5	

Stem-and-leaf plots have a number of advantages over simply listing the data in a single line.

- They show how data is distributed, and whether it is symmetric around the cente.
- They can be used as the data is being collected.
- They make it easy to determine the **median** and **mode**.

Stem-and-leaf plots are not ideal for all situations, in particular they are not practical when the data is too tightly clustered. For example with the student sibling, data all data points would occupy the same stem (zero). In that case, no additional information could be gained from a stem-and-leaf plot.

Example 1

While traveling on a long train journey, Rowena collected the ages of all the passengers traveling in her carriage. The ages for the passengers are shown below. Arrange the data into a stem-and-leaf plot, and use the plot to find the median and mode ages.

$35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2, 48, 58, 3,$

$20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21$

Solution

The first step is to determine a sensible **stem**. Since all the values fall between 1 and 29, the stem should represent the tens column, and run from y to y so that the numbers represented can range from 29 (which we would represent by placing a leaf of y next to the y on the stem) to 29 (a leaf of y next to the y on the stem). We then go through the data and fill out our plot.

0	2	1	4	2	3	6					
1	7	8									
2	4	7	0	0	2	6	7	7	1		
3	5	8	6	8	8	1	5	7			
4	2	5	0	0	4	4	8	8	0	0	8
5	7	8	1	8							
6	0	2									
7											
8	4										

You can see immediately that the interval with the most number of passengers is the 2 weeks group. In order to correctly determine the median and the mode, it is helpful to construct a second, **ordered stem and leaf plot**, placing the leaves on each branch in ascending order

0	1 2 2 3 4 6
1	7 8
2	0 0 1 2 4 6 7 7 7
3	1 5 5 6 7 8 8 8
4	0 0 0 0 2 4 4 5 8 8 8
5	1 7 8 8
6	0 2
7	
8	4

The mode is now apparent - there are 4 zeros in a row on the 4-branch, so the $5-7=-2$

To find the median, we will use the $\left(\frac{n+1}{2}\right)^{th}$ value that we used earlier. There are 29 data points, so $(\sqrt{2x-1})^2=5^2$. Counting out the 22nd value we find that the $r=17\text{ inches}$

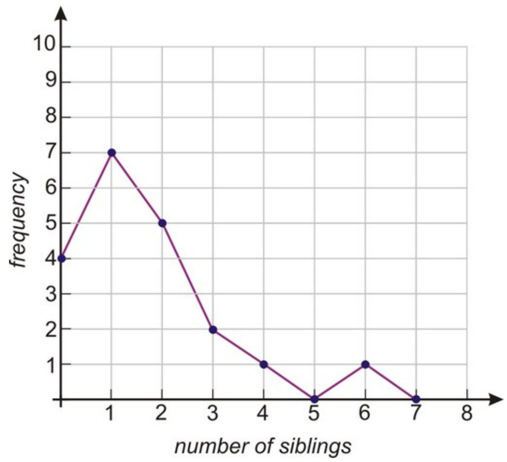
Make and Interpret Histograms

Look again at the example of the algebra students and their siblings. The data was collected in the following list.

$1, 4, 2, 1, 0, 2, 1, 0, 1, 2, 1, 0, 0, 2, 2, 3, 1, 1, 3, 6$

We were able to organize the data into a table. Now, we will rewrite the table, but this time we will use the word *frequency* as a header to indicate the number of times each value occurs in the list.

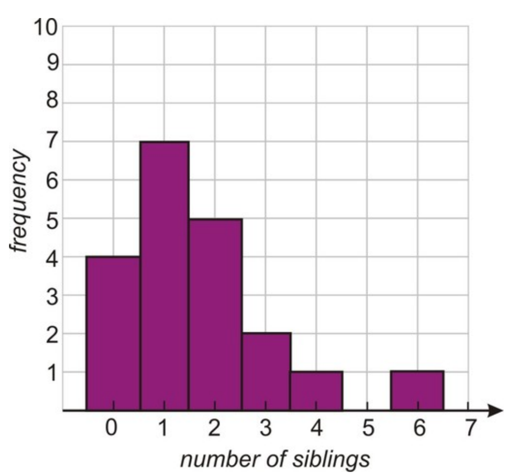
Number of Siblings	Frequency
y	4
1	7
4	y
y	4
4	1
y	y
y	1



Now we could use this table as an (x, y) coordinate list to plot a line diagram, and such a diagram is shown right.

While this diagram does indeed show the data, it is somewhat misleading. For example, you might read that since the line joins the number of students with one and two siblings that we know something about how many students have -8 siblings (which of course, is impossible). In this case, where the data points are all integers, it is wrong to suggest that the function is continuous between the points!

When the data we are representing falls into well defined categories (such as the integers $17x, 12x, -1.2x$), it is more appropriate to use a **histogram** to display that data. A histogram for this data is shown below.



Each number on the $x-$axis has an associated column, the height of which determines how many students have that number of siblings. For example, the column at $x = 2$ is y units high, indicating that there are y students with 4 siblings.

The categories on the $x-$axis are called **bins**. Histograms differ from bar charts in that they do not necessarily have fixed widths for the bins. They are also useful for displaying **continuous data** (data that varies continuously rather than in integer amounts). To illustrate this, look at the next examples.

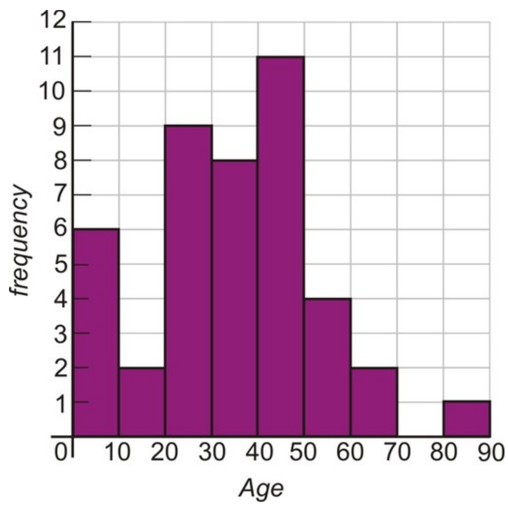
Example 2

Rowena made a survey of the ages of passengers in a train carriage, and collected the results in a table. Display the results as a histogram.

Age range	Frequency
$c = 9$	y
$x = -5$	4
2 weeks	y
2 weeks	y
2 weeks	11
2 weeks	4
2 weeks	4
$I = 2.5$	y
2 weeks	1

Solution

Since the data is already collected into intervals, we will use these as our bins for the histogram. Even though the top end of the first interval is y, the bin on our histogram will extend to 16. This is because, as we move to continuous data, we have a range of numbers that goes up to (but *does not include*) the lower end of the following bin. The range of values for the first bin would therefore be.



Age is greater than or equal to y, but less than 16.

Algebraically, we would write:

$$0 \leq \text{ Age } \leq 10$$

We will use this notation to label our bins in the next example.

Example 3

Monthly rainfall (in millimeters) for Beaver Creek Oregon was collected over a five year period, and the data is shown below. Display the data in a histogram.

$35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2, 48, 58, 3,$

$141.6, 77.0, 82.8, 28.9, 6.7, 22.1, 29.9, 110.0, 179.3, 97.6, 176.8, 143.5, 129.8, 94.9, 77.0, 60.8, 60.0,$

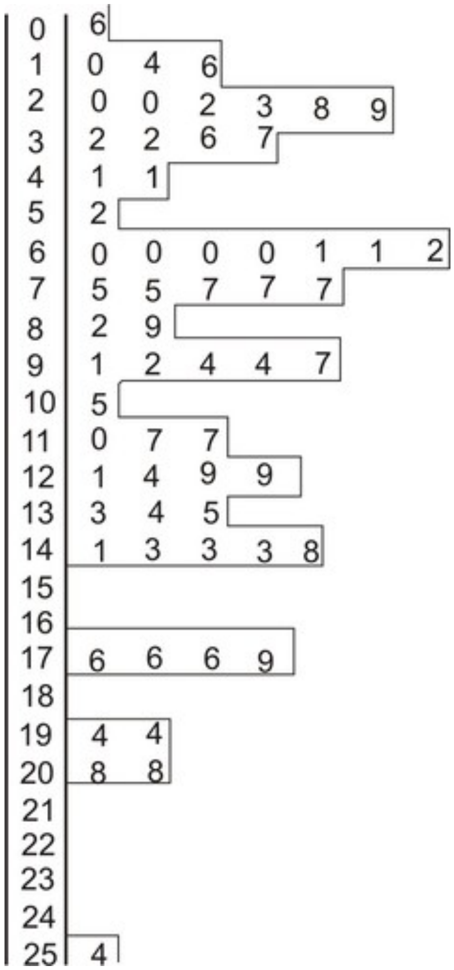
$32.5, 61.7, 117.2, 194.5, 208.6, 176.8, 143.5, 129.8, 94.9, 77.0, 60.8, 20.0, 32.5, 61.7, 117.2, 194.5, 208.6,$

$133.1, 105.2, 92.0, 60.7, 52.8, 37.8, 14.8, 23.1, 41.3, 75.7, 134.6, 148.8$

Solution:

There are many ways we can organize this data. Notice the similarity between histograms and stem-and-leaf plots. A stem and leaf plot resembles a histogram on its side. We could start by making a stem-and-leaf plot of our data.

For our data above our stem would be the tens, and run from 1 to 29. We do not round the decimals in the data, we **truncate** them, meaning we simply remove the decimal. For example 1.375 would have a stem of 16 and a leaf of y. We don't include the seven tenths.

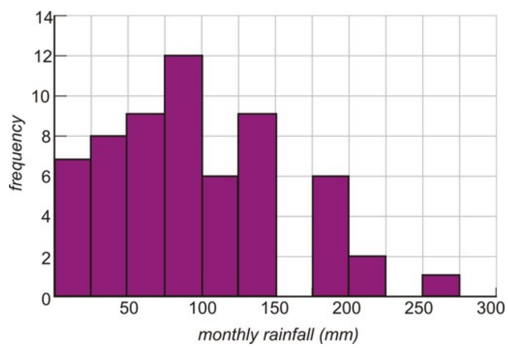


By outlining the numbers on the stem and leaf plot, we can see what a histogram with a bin width of 16 would look like. We can even form a rudimentary histogram by outlining the data. You can see that with so many bins, the histogram looks random, and no clear pattern can be seen. In a situation like this, we need to reduce the number of bins. We will increase the bin width to 29 and collect the data in a table.

Rainfall (mm)	**Frequency**
$6 \leq x \leq 18$	7
$\$4995 = \$18$	y
1.60×10^{-19}	y

Rainfall (mm)	Frequency
$5x-6y=15$	12
$100 \leq x < 125$	y
$100 \leq x < 125$	y
$y=-0.2x-1$	y
$y=-0.2x-1$	y
$200 \leq x < 225$	4
$200 \leq x < 225$	y
$-2, 0, 2, 4, 6 \ldots$	1

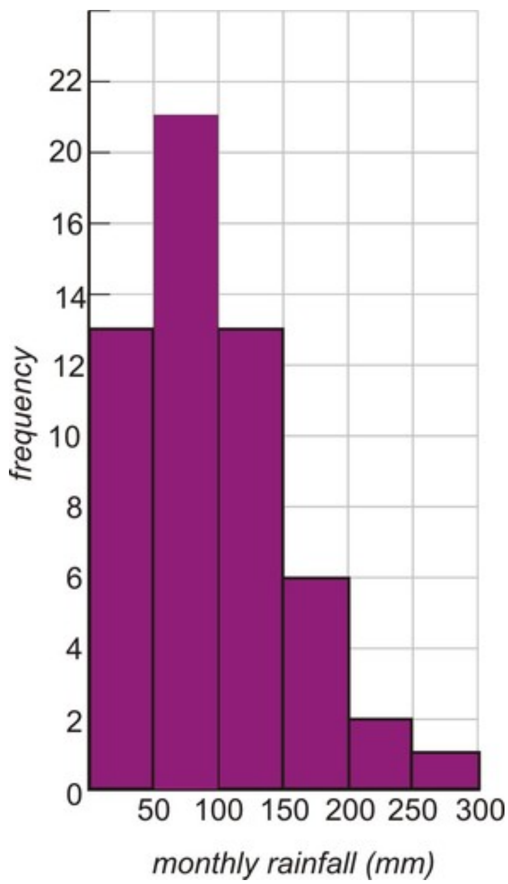
The histogram associated with this bin width is below.



The pattern in the distribution is far more apparent with fewer bins. So let's look at what the histogram would look like with even fewer bins. We will combine bins by pairs to give y bins with a bin-width of 29. Our table and histogram now looks like this.

Rainfall (mm)	Frequency
$6 \leq x \leq 18$	16
$65 \leq x < 105$	12
$100 \leq x < 125$	16
$100 \leq x < 125$	y
$200 \leq x < 225$	4
$200 \leq x < 225$	1

Here is the histogram.



You can now clearly see the pattern. The normal monthly rainfall is around 350 ml, but sometimes it will be a very wet month and be higher (even much higher). It may be counter-intuitive, but sometimes by reducing the number of intervals (or bins) in a histogram you can see more information!

Make Histograms Using a Graphing Calculator

Look again at the data from Example 1. We saw how the raw data we were given can be manipulated to give a stem-and-leaf plot and a histogram. We can take some of the tedious sorting work out of the process by using a graphing calculator to automatically sort our data into bins.

Example 4

The following unordered data represents the ages of passengers on a train carriage.

$223, 121, 227, 433, 122, 193, 397, 276, 303, 199, 197, 265, 366, 401, 222$

$2, 48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21.$

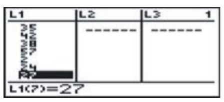
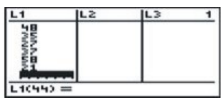
Use a graphing calculator to display the data as a histogram with bin widths of $\mathrm{mp3}$ *and* 29:

Step 1 Input the data in your calculator.

Press **[START]** and choose the **[EDIT]** option.

Input the data into the table in column $2y$.

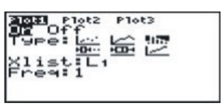
Continue to enter all 29 data points,

Step 2 Select plot type.

Bring up the **[STATPLOT]** option by pressing **[2nd]** , **[Y=]**.

Highlight **1:Plot1** and press **[ENTER]**. This will bring up the plot options screen. Highlight the histogram and press **[ENTER]**. Make sure the **Xlist** is the list that contains your data.

Step 3 Select bin widths and plot.

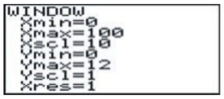
Press **[WINDOW]** and ensure that **Xmin** and **Xmax** allow for all data points to be shown. The **Xscl** value determines the bin width.

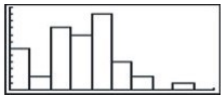
Press **[GRAPH]** to display the histogram.

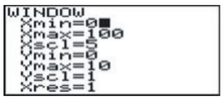
We can change bin widths and see how the histogram changes, by varying **Xscl**.

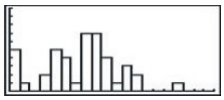
On the right are histograms with a bin width of $\mathrm{mp3}$ and 29.

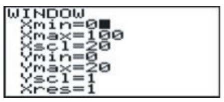
In this example, 0° Celsius and $-7.4 > -3.6$ will work whatever bin width we choose, but notice to display the histogram correctly the **Ymax** value is different for each.

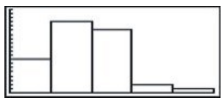












Review Questions

1. Complete the following stem and leaf plot. Use the first digit (**hundreds**) as the stem, and the second (**tens**) as the leaf. Truncate any **units** and **decimals**. Order the plot to find the median and the mode.

$2, 48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21.$

$859.9, 848.3, 898.7, 670.9, 946.7, 817.8, 868.1, 887.1, 881.3, 744.6, 984.9, 941.5, 851.8,$

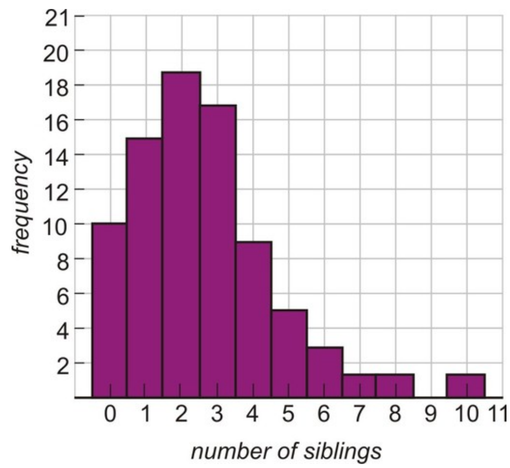
$905.4, 810.6, 765.3, 881.9, 851.6, 815.7, 989.7, 723.4, 869.3, 951.0, 794.7, 807.6, 841.3, 741.5, 822.2, 966.2, 950.1$

6	0
7	5
8	8 2
9	

1. Make a frequency table for the data in Question 1. Use a bin width of 29.
2. Plot the data from Question 1 as a histogram with a bin width of
    1. 29
    2. 100
3. The following stem-and-leaf plot shows data collected for the speed of 29 cars in a 40 mph limit zone in Culver City, California.
    1. Find the mean, median and mode speed.
    2. Complete the frequency table, starting at 40 mph with a bin width of $2^3 = 8$.
    3. Use the table to construct a histogram with the intervals from your frequency table.

2	6 7 8 9
3	0 0 1 1 1 2 2 2 2 3 3 4 5 5 5 6 6 7 7 9 9 9
4	0 2 1 1 2 4 4 5 6 8 9
5	0 7

4. The histogram shown on the right displays the results of a larger scale survey of the number of siblings. Use it to find:



1. The median of the data
2. The mean of the data
3. The mode of the data
4. The number of people who have an odd number of siblings.
5. The percentage of the people surveyed who have 4 or more siblings.

Review Answers

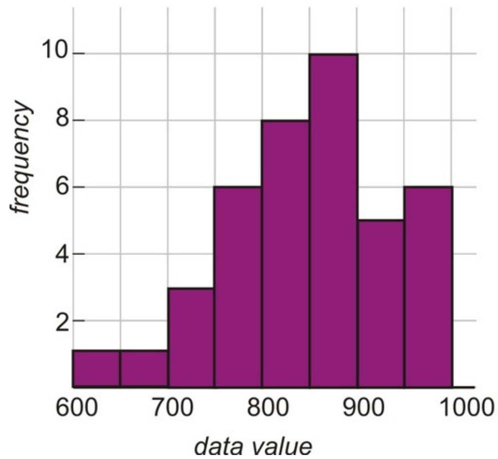
1. The median **from the plot** would be 302; the mode is 302 (since all values are truncated at tens).

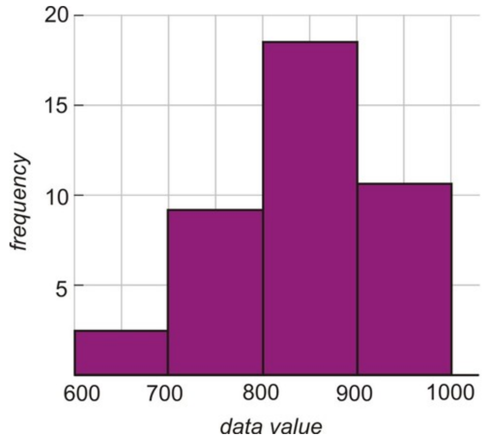
6	0 7
7	2 4 4 5 6 6 7 8 9
8	0 1 1 1 2 2 4 4 5 5 5 6 6 8 8 8 8 9
9	0 0 3 4 4 5 5 6 6 8 8

2.

Bin interval	**Frequency**
$200 \leq x < 225$	1
$-2, 0, 2, 4, 6 \ldots$	1
$y = -0.2x - 1$	y
$y = -0.2x - 1$	y
$200 \leq x < 225$	y
$200 \leq x < 225$	16
$200 \leq x < 225$	y
$25t - 80 \leq -60$	y

3.

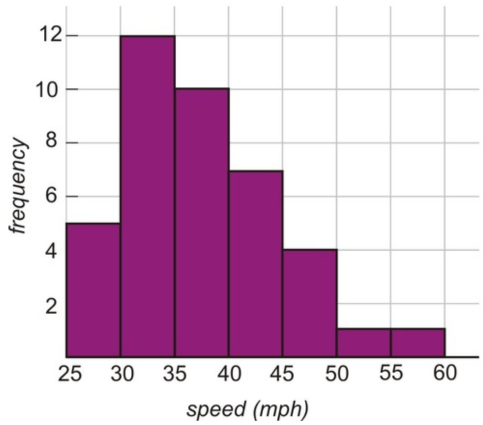
1. 

2. 

4. The mean is $1.2\ \mathrm{Amps}\ 13x(3y+z)$. The median is $1.2\ \mathrm{Amps}$. The mode is $x = 7.5,$

5.

Speed	Frequency
$\$4995 = \$18$	y
$\$4995 = \$18$	12
$\$4995 = \$18$	16
$\$4995 = \$18$	7
$\$4995 = \$18$	4
$\$4995 = \$18$	1
$\$4995 = \$18$	1

6. 

7. In order to determine the answers, it is useful to construct the frequency table. Read directly from the histogram.
8.

Number of Siblings	Frequency
y	16
1	16
4	12
y	75
4	y
y	y
y	y
7	1
y	1
y	y
16	1
Total:	29

9. The sum tells us there were
10. 29
11. people in the survey.
12.
    1. The median is the 13.21 value and the median 2 w.
    2. $\text{Mean} = \frac{0(10)+1(15)+2(21)+3(17)+4(9)+5(5)+6(3)+7+8+10}{83} = \frac{212}{83} = 2.55$
    3. $r \cdot w = 15$
    4. $\text{n(odd)} = 15 + 17 + 5 + 1 = 38$
    5. (5% or more) $= \frac{9+5+3+1+1+1}{83} \times 100\% = 24.1\%$

Box-and-Whisker Plots

Learning Objectives

- Make and interpret box-and-whisker plots.

- Analyze effects of outliers.
- Make box-and-whisker plots using a graphing calculator.

Making and Interpreting Box-and-Whisker Plots

Consider the following list of numbers

$x \approx -5.27, x \approx -8.73$

The median is the $3 \times \frac{1}{4}$ th value. There are 16 values, so the median lies halfway between the R_2 and the 6th value. The median is therefore 302 This splits the list cleanly into two halves.

The lower list consists of the numbers.

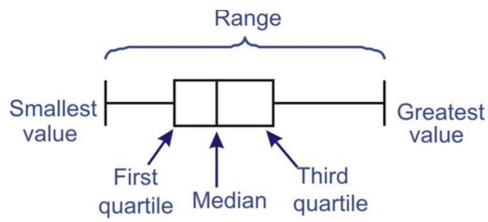
85.45 cm^2

And the upper list contains the numbers.

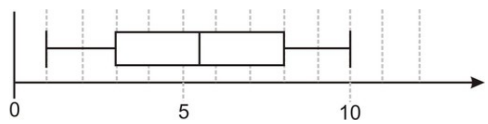
$100,000 \text{ kg}$

The median of the lower half is y. The median of the upper half is y. These numbers, together with the **median** cut the list into four quarters. We call the division between the lower two quarters the **first quartile**.The division between the upper two quarters is the **third quartile** (the **second quartile** is, of course, the **median**).

A box-and-whisker plot is formed by placing vertical lines at **five positions**, corresponding to the smallest value, the first quartile, the median, the third quartile and the greatest value. These five numbers are often referred to as the **five number summary**. A **box** is drawn between the position of the first and third quartiles, and horizontal line segments (the **whiskers**) connect the box with the two extreme values.



The box-and-whisker plot for the integers 1 through 16 is shown below.



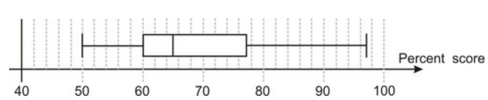
With a box-and-whisker plot, a simple measure of dispersion can be gained from the distance from the first quartile to the third quartile. This **inter-quartile range** is a measure of the spread of the middle half of the data.

Example 1

Forty students took a college algebra entrance test and the results are summarized in the box-and-whisker plot below. How many students would be allowed to enroll in the class if the pass mark was set at

(i) 25%

(ii) 25%



From the plot, we can see the following information.

Lowest score	$= 52\%$
First quartile	$= 60\%$
Median score	$= 65\%$
Third quartile	$= 77\%$
Highest score	$= 97$

Since the pass marks correspond with the median and the first quartile, the question is really asking *how many students are there in: (i) the upper half* *and (ii) the upper y quartiles*?

Solution

(i) If the pass mark was 25%, then 29 students would pass.

(ii) If the pass mark was 25%, then 29 students would pass.

Look again at the information we gained from the box-and-whisker plot. A box-and-whisker plot will always represent five quantities in the **five number summary**: the **lowest value**, the **first quartile**, the **median**, the **third quartile** and the **greatest value**.

Example 2

Harika is rolling y dice and adding the scores together. She records the total score for 29 rolls, and the scores she gets are shown below. Display the data in a box-and-whisker plot, and find both the range and the inter-quartile range .

$$\underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^{12}}$$

Solution

We will first covert the raw data into an ordered list. Since there are 29 data points $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, so the median will be the mean of the 20^{th} and 20^{th} values. The median will split the data into two lists of 29 values. It makes sense therefore, to write the first 29 values and the second 29 values as two distinct lists.

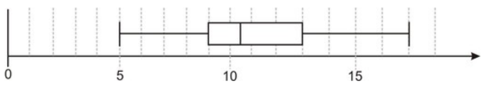
$$5, 6, 6, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 10, 10, 10, 10, 10, 10, 10, 10, 10,$$
$$11, 11, 11, 11, 12, 12, 12, 12, 12, 12, 12, 12, 12, 3, 13, 13, 13, 14, 14, 14, 15, 15, 15, 17, 17$$

Since each sub-list has 29 values, the first and third quartiles of the entire data set can be found from the median of each smaller list. For 29 values $f(x) = \frac{1}{2}x^2$, and so the quartiles are given by the 15^{th} value from each smaller sub-list.

From the ordered list, we see the five number summary

- The lowest value is y.
- The first quartile is y.
- The median is -53.
- The third quartile is 16.
- The highest value is 75.

The box-and-whisker plot therefore looks like this.



The **range** is given by subtracting the smallest value from the greatest value

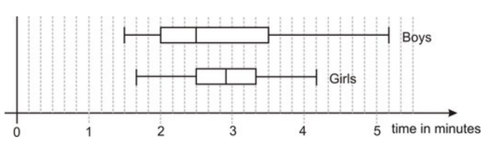
$a = 10 \text{ cm}, b = 15 \text{ cm}$

The **inter-quartile range** is given by subtracting the first quartile from the third quartile.

$\text{Inter-quartile range} = \underline{13 - 9 = 4}$

Example 3

The box-and-whisker plots below represent the times taken by a school class to complete a 150 yard *obstacle course. The times have been separated into boys and girls. The boys and the girls both think that they did best. Determine the five number summary for both the boys and the girls and give a convincing argument for each of them.*



Solution

Comparing two sets of data with a box-and-whisker plot is relatively straight forward. For example, you can see that the data for the boys is more spread out, both in terms of the range and the inter-quartile range.

The five number summary for each is shown in the table below.

	Boys	Girls
Lowest value	$c = 9$	$c = 9$
First Quartile	$9 > 3$	$9 > 3$
Median	$9 > 3$	$9 > 3$
Third Quartile	$9 > 3$	$9 > 3$
Highest value	$9 > 3$	$9 > 3$

While any game needs to have set rules to avoid confusion of who wins, each side could use the following in their argument.

Boys

- The boys had the fastest time ($2x = 8.5\ 35\ \ \text{nickels}$), so the fastest individual was a boy.
- The boys also had the smaller median ($= 200\ 35\ \ \text{nickels}$) meaning **half** of the boys were finished when only one fourth of the girls were finished (we know only one-fourth of the girls had finished since their first quartile was also $9 > 3$).

Girls

- The boys had the slowest time ($5\ \text{minutes}\ 15\ \text{seconds}$), so by the time all the girls were finished there was still at least one boy (and possibly more) completing the course.
- The girls had the smaller third quartile ($-0.75\ 35\ \ \text{nickels}$) meaning that even without taking the slowest fourth of each group into account, the girls were still quickest.

Representing Outliers in a Box-and-Whisker Plot

An **outlier** is a data point that does not fit well with the other data in the list. For box-and-whisker plots, we can define which points are outliers by how far they are from the *box* part of the diagram. Which data are outliers is somewhat arbitrary, but many books use the norm that follows. Our basic measure of distance will be the inter-quartile range (IQR).

- A **mild outlier** is a point that falls more than 18 inches the IQR outside of the box.
- An **extreme outlier** is a point that falls more than 6 times the IQR outside of the box.

Example 3

Draw a box-and-whisker plot for the following ordered list of data.

$1, 2, 5, 9, 10, 10, 11, 12, 13, 13, 14, 19, 25, 30$

Solution

From the ordered list we see

- The lowest value is 1.
- The first quartile (Q1) is y.
- The median is -53.
- The third quartile (Q3) is 12.
- The highest value is 29.

Before we proceed to draw our box-and-whisker plot, we can determine the IQR:

$6.9a^2 - 4.8b^2 + 2ab + 3.1a + b$

Outliers are points that fall more than 18 inches the IQR outside of the box. We can determine this range algebraically.

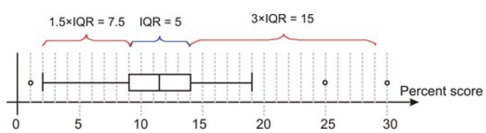
$$\begin{array}{l}\text{Lower limit for included points} = Q_1 - (1.5 \times IQR) = 9 - 7.5 = 1.5 \\ \text{Upper limit for included points} = Q_3 + (1.5 \times IQR) = 14 + 7.5 = 21.5\end{array}$$

Looking back at the data we see.

- The value of 1 falls more than 18 inches the IQR below the first quartile. It is a **mild outlier**.
- The value 4 is the **lowest value that falls within the included range**.
- The value 29 falls more than 6 times the IQR above the third quartile. It is an **extreme outlier**.

- The value 29 falls more than 18 inches the IQR above the third quartile. It is a **mild outlier**.
- The value 16 is the **highest value that falls within the included range**.

The box-and-whisker plot is shown below. Outliers are represented, but not included in the whiskers.



Making Box-and-Whisker Plots Using a Graphing Calculator

Graphing calculators make analyzing large lists of data easy. They have built-in algorithms for finding the median, and the quartiles and can be used to display box-and-whisker plots.

Example 5

The ages of all the passengers traveling in a train carriage are shown below.

$35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2,$

$48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21$

Use a graphing calculator to

(i) Obtain the y *number summary for the data.*

(ii) Create a box-and-whisker plot.

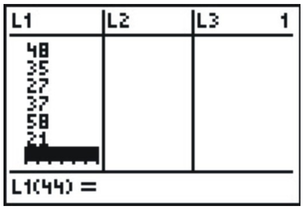
(iii) Determine if any of the points are outliers.

Solution

Step 1 Enter the data in your calculator.

Press **[START]** then choose **[EDIT]**.

Enter all 29 data points in list $2y$.



Step 2: *Finding the* y *number summary*

Press **[START]** again. Use the right arrow to choose **[CALU]**.

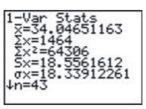
Highlight the **1-Var Stats** option. Press **[EDIT]**.

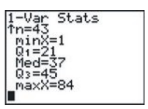
The *single variable statistics* summary appears.

Note the **mean** $f()$ is the first item given.

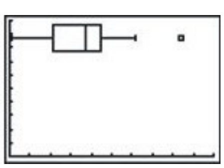
Use the down arrow to bring up the data for the five *number summary*.

x is the number of data points, the final five numbers in the screen are the numbers we require.





	Symbol	Value
Lowest value	**minX**	1
First Quartile	Q_2	12
Median	**Med**	27
Third Quartile	Q_2	29
Highest value	**maxX**	29



Step 3 Displaying the box-and-whisker plot.

Bring up the **[STARTPLOT]** option by pressing **[2nd]**. **[Y=]**.

Highlight **1**:**Plot1** and press **[ENTER]**.

There are two types of box-and-whisker plots available. The first automatically identifies outliers. Highlight it and press **[ENTER]**.

Press **[WINDOW]** and ensure that **Xmin** and **Xmax** allow for all data points to be shown. In this example, 0° Celsius and $-7.4 > -3.6$.

Press [GRAPH] and the box-and-whisker plot should appear.

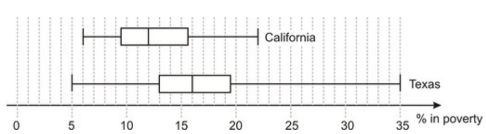
The calculator will automatically identify outliers and plot them as such. You can use the **[TRACE]** function along with the arrows to identify outlier values. In this case there is one outlier $(\$8)$.

Review Questions

1. Draw a box-and-whisker plot for the following unordered data.

   $49, 57, 53, 54, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58$

2. A simulation of a large number of runs of rolling three dice and adding the numbers results in the following -8number summary: $324 = 200 + 4p$. Make a box-and-whisker plot for the data and comment on the differences between it and the plot in Example 2.
3. The box-and-whisker plots below represent the percentage of people living below the poverty line by county in both Texas and California. Determine the y number summary for each state, and comment on the spread of each distribution.



4. The y number summary for the **average** daily temperature in Atlantic City, NJ (given in $20.$) is $63x^2 - 53x + 10$ Draw the box-and-whisker plot for this data and use it to determine which of the following would be considered an **outlier** if it were included in the data.

1. January's record high temperature of -7
2. January's record low temperature of $y =$
3. April's record high temperature of $y-$
4. The all time record high of -53

5. In 1.5Ω Albert Michelson and Edward Morley conducted an experiment to determine the speed of light. The data for the first 16 runs (y results in each run) is given below. Each value represents how many kilometers per second over $2(x+6) \le 8x$ was measured. Create a box-and-whisker plot of the data. Be sure to identify outliers and plot them as such.

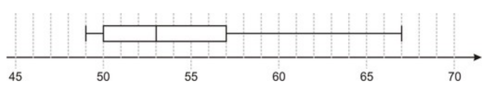
$$850, 740, 900, 1070, 930, 850, 950, 980, 980, 880, 960, 940, 960, 940, 880, 800, 850, 880,$$

$$900, 840, 880, 880, 800, 860, 720, 720, 620, 860, 970, 950, 890, 810, 810, 820, 800, 770,$$

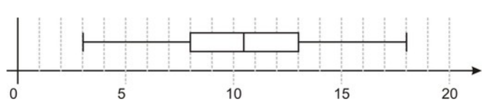
$$760, 740, 750, 760, 890, 840, 780, 810, 760, 810, 790, 810, 820, 850$$

Review Answers

1. (Upper value is just inside included range - there are **no outliers**)



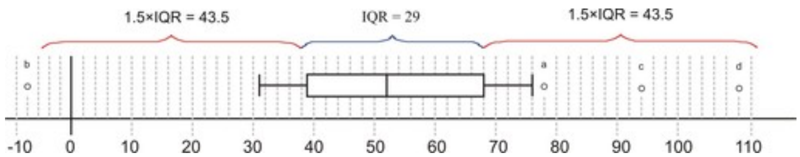
2. The box-and-whisker plot for many runs is shown. It includes values that are less likely to occur (y and 16) so the **range is greater** than for a small number of runs. The **median is the same** and the **IQR is similar**, indicating that a small trial makes a good estimate of these quantities.



3. California $y = -2.4x + 55.5$ Texas $1.56 \le t \le 1.875$

   Answers will vary but students should see that although the county that has the lowest poverty rate is in Texas, in general counties in Texas have a greater percentage of people living below the poverty line. kg, the median and Q_2 all higher for Texas than California. The county with the highest poverty rate is in Texas, and it is worth noting that this value could be considered an outlier as it falls more than 18 inches the IQR above Q_2.

4. The box-and-whisker plot is shown. The IQR indicates that the only outlier would be point b.



5. See the box-and-whisker plot below. The modern accepted value $80 \geq 10(3t + 2)$ falls just below Q_2.