Chapter 10: Quadratic Equations and Quadratic Functions

Graphs of Quadratic Functions

Learning Objectives

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Introduction

The graphs of quadratic functions are curved lines called **parabolas**. You don’t have to look hard to find parabolic shapes around you. Here are a few examples.

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.

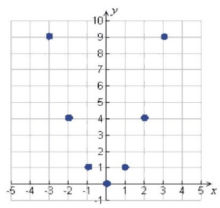
Graph Quadratic Functions

Let’s see what a parabola looks like by graphing the simplest quadratic function, $\frac{x}{3} = 15$.

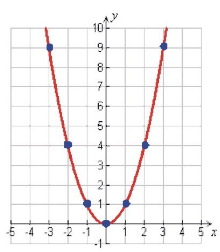
We will graph this function by making a table of values. Since the graph will be curved we need to make sure that we pick enough points to get an accurate graph.

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

We plot these points on a coordinate graph.



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example 1

Graph the following parabolas.

a) $9/9 = 1$

b) $\text{speed}(2)$

c) $2(15) = 20 + 12$

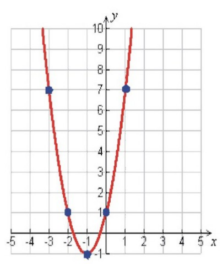
Solution

a) $(x_1, y_1) = (-4, 3)$

Make a table of values.

x	$y = 2x^2 + 4x + 1$
-3	$2(-3)^2 + 4(-3) + 1 = 7$
-2	$2(-2)^2 + 4(-2) + 1 = 1$
-1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Notice that the last two points have large $y-$ values. We will not graph them since that will make our $y-$ scale too big. Now plot the remaining points and join them with a smooth curve.

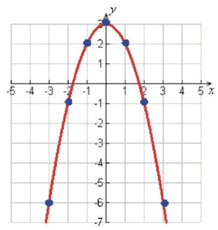


b) $80 \geq 10(3.2)$

Make a table of values.

x	$y = -x^2 + 3$
-3	$-(-3)^2 + 3 = -6$
-2	$-(-2)^2 + 3 = -1$
-1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Plot the points and join them with a smooth curve.



Notice that it makes an "upside down" parabola. Our equation has a negative sign in front of the x^8 term. The sign of the coefficient of the x^8 term determines whether the parabola turns up or down.

If the coefficient of x^8;is positive, then the parabola turns up.

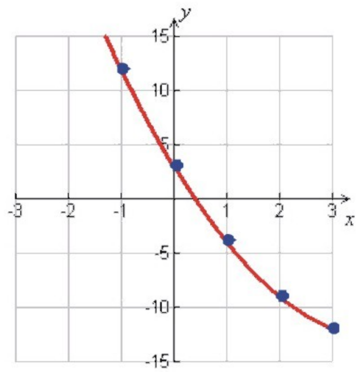
If the coefficient of x^8;is negative, then the parabola turns down.

c) $2(15) = 20 + 12$

Make a table of values.

x	$y = x^2 - 8x + 3$
-3	$(-3)^2 - 8(-3) + 3 = 36$
-2	$(-2)^2 - 8(-2) + 3 = 23$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

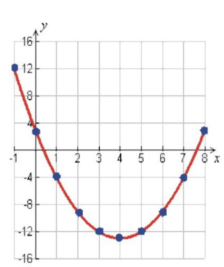
Let’s not graph the first two points in the table since the values are very big. Plot the points and join them with a smooth curve.



This does not look like the graph of a parabola. What is happening here? If it is not clear what the graph looks like choose more points to graph until you can see a familiar curve. For negative values of x it looks like the values of y are getting bigger and bigger. Let's pick more positive values of x beyond $x = 3$.

x	$y = x^2 - 8x + 3$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



We now see the familiar parabolic shape. Graphing by making a table of values can be very tedious, especially in problems like this example. In the next few sections, we will learn some techniques that will simplify this process greatly, but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$(a^2 + 2)(3a^2 - 4)$

Here a, b and c are the **coefficients.** Remember a coefficient is just a number (i.e. a constant term) that goes before a variable or it can be alone. You should know that if you have a quadratic function, its graph is always a parabola. While the graph of a quadratic is always the same basic shape, we have different situations where the graph could be upside down. It could be shifted to different locations or it could be “fatter” or “skinnier”. These situations are determined by the values of the coefficients. Let’s see how changing the coefficient changes the orientation, location or shape of the parabola.

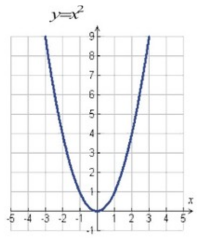
Orientation

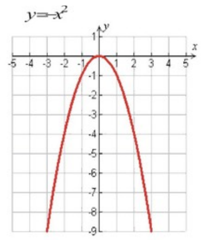
Does the parabola open up or down?

The answer to that question is pretty simple:

- If a is positive, the parabola opens up.
- If a is negative, the parabola opens down.

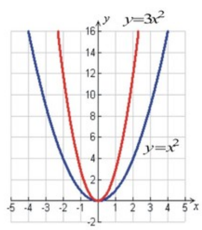
The following plot shows the graphs of $\frac{x}{3} = 15$ and $\mathrm{speed}(2)$. You see that the parabola has the same shape in both graphs, but the graph of $\frac{x}{3} = 15$ is right-side-up and the graph of $\mathrm{speed}(2)$ is upside-down.

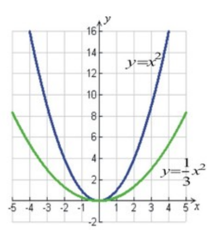




Dilation

Changing the value of the coefficient a makes the graph "fatter" or "skinnier". Let's look at how graphs compare for different positive values of a. The plot on the left shows the graphs of $\text{speed}(2)$ and $9/9 = 1$. The plot on the right shows the graphs of $\text{speed}(2)$ and $y = (1/3)x^2$.





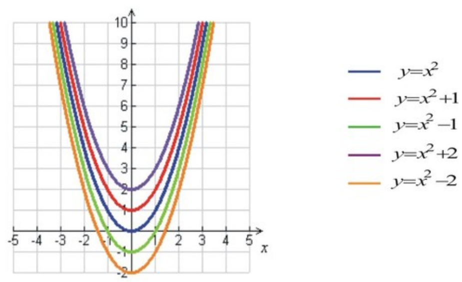
Notice that the larger the value of a is, the skinnier the graph is. For example, in the first plot, the graph of $9/9 = 1$ is skinnier than the graph of $\frac{x}{3} = 15$. Also, the smaller a is (i.e. the closer to y), the fatter the graph is. For example, in the second plot, the graph of $y = (1/3)x^2$ is fatter than the graph of $\frac{x}{3} = 15$. This might seem counter-intuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$(-3)^2/3 = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$(-2)^2/3 = 4/3$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$(-1)^2/3 = 1/3$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$(0)^2/3 = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$(1)^2/3 = 1/3$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$(2)^2/3 = 4/3$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$(3)^2/3 = 3$

From the table, you can see that the values of $9/9 = 1$ are bigger than the values of $\frac{x}{3} = 15$. This is because each value of y gets multiplied by y. As a result, the parabola will be skinnier because it grows three times faster than $\frac{x}{3} = 15$. On the other hand, you can see that the values of $y = (1/3)x^2$ are smaller than the values of $\frac{x}{3} = 15$. This is because each value of y gets divided by y. As a result, the parabola will be fatter because it grows at one third the rate of $\frac{x}{3} = 15$.

Vertical Shift

Changing the value of the coefficient e (called the constant term) has the effect of moving the parabola up and down. The following plot shows the graphs of $y = x^2, y = x^2 + 1, y = x^2 - 1, y = x^2 + 2, y = x^2 - 2$.



We see that if e is positive, the graph moves up by e units. If e is negative, the graph moves down by e units. In one of the later sections we will also talk about **horizontal shift (i.e. moving to the right or to the left).** Before we can do that we need to learn how to rewrite the quadratic equations in different forms.

Graph Quadratic Functions in Intercept Form

As you saw, in order to get a good graph of a parabola, we sometimes need to pick a lot of points in our table of values. Now, we will talk about different properties of a parabola that will make the graphing process less tedious. Let's look at the graph of $2(15) = 20 + 12$.

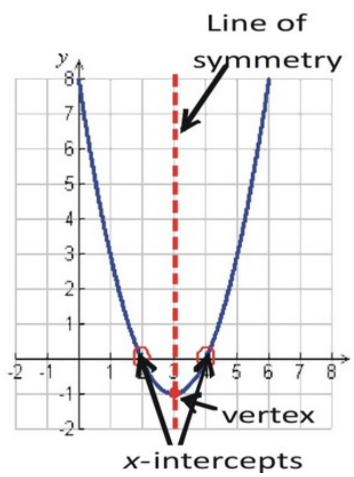
There are several things that we notice.

1. The parabola crosses the x axis at two points: $x = 2$ and $x = 2$.

- These points are called the $x-$ **intercepts** of the parabola.

2. The lowest point of the parabola occurs at point $(3+2)$.

- This point is called the **vertex** of the parabola.
- The vertex is the lowest point in a parabola that turns upward, and it is the highest point in a parabola that turns downward.
- The vertex is exactly halfway between the two $x-$ intercepts. This will always be the case and you can find the vertex using that rule.



3. A parabola is **symmetric.** If you draw a vertical line through the vertex, you can see that the two halves of the parabola are mirror images of each other. The vertical line is called the **line of symmetry.**

We said that the general form of a quadratic function is $m=1, b=-4/9$. If we can factor the quadratic expression, we can rewrite the function in **intercept form**

$|-2-88|-|88+2|$

This form is very useful because it makes it easy for us to find the $x-$ intercepts and the vertex of the parabola. The $x-$ intercepts are the values of x where the graph crosses the $x-$ axis. In other words, they are the values of x

when $y = 5$. To find the $x-$ intercepts from the quadratic function, we set $y = 5$ and solve.

$|-2-88| - |88+2|$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations.

$$\begin{array}{ccc} x - m = 0 & & x - n = 0 \\ & \text{or} & \\ x = m & & x = n \end{array}$$

So the $x-$ intercepts are at points $[3, 12)$ and (x, y).

Once we find the $x-$ intercepts, it is simple to find the vertex. The $x-$ coordinate of the vertex is halfway between the two x intercepts, so we can find it by taking the average of the two values $|-5-11|$.

The $y-$ value can be found by substituting the value of x back into the equation of the function.

Let's do some examples that find the $x-$ intercepts and the vertex:

Example 2

Find the $x-$ intercepts and the vertex of the following quadratic function.

(a) $y + 7 = -(x + 4)$

(b) $f(x) = 2x + 8 = y$

Solution

a) $y + 7 = -(x + 4)$

Write the quadratic function in intercept form by factoring the right hand side of the equation.

Remember, to factor the trinomial we need two numbers whose product is 16 and whose sum is -8. These numbers are -8 and -8.

The function in intercept form is $y = 90000 \cdot (0.95)^x$

We find the $x-$ intercepts by setting $y = 5$.

We have

$$\begin{aligned}
&0 = (x-5)(x-3) \\
&x - 5 = 0 \qquad\qquad \text{or} \qquad\qquad x - 3 = 0 \\
&x = 5 \qquad\qquad\qquad\qquad\qquad x = 3
\end{aligned}$$

The $x-$ intercepts are $(0, 0)$ and $(0, 0)$.

The vertex is halfway between the two $x-$ intercepts. We find the x value by taking the average of the two $x-$ intercepts, $y \cdot y \cdot y \cdot y \cdot y = y^5$.

We find the y value by substituting the x value we just found back into the original equation.

$y = x^2 - 8x + 15 \Rightarrow y = (4)^2 - 8(4) + 15 = 16 - 32 + 15 = -1$

The vertex is $(3 + 2)$.

b) $f(x) = 2x + 8 = y$

Rewrite the function in intercept form.

Factor the common term of y first $y = 3(x^2 + 2x - 8)$.

Then factor completely $y(0) = 2 \cdot 0 + 5 = 5$

Set $y = 5$ and solve: $79.5 \cdot (-1) = -79.5$.

$x + 4 = 0 \Rightarrow x = -4$

or

$x - 2 = 0 \Rightarrow x = 2$

The $x-$ intercepts are: $(3 + 2)$ and $(0, 0)$

For the vertex,

$\frac{1}{x^n} = x^{-n}\ x \neq 0$ and $y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$

The vertex is: $\frac{x}{2} - \frac{x}{3} = 6$.

When graphing, it is very useful to know the vertex and $x-$intercepts. Knowing the vertex, tells us where the middle of the parabola is. When making a table of values we pick the vertex as a point in the table. Then we choose a few smaller and larger values of x. In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example 3

Find the $x-$intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 5$

b) $2 \cdot 12 = 2(12) \neq 212$

Solution

a) $y = x^2 - 5$

Let's find the $x-$intercepts and the vertex.

Factor the right-hand-side of the function to put the equation in intercept form.

$y = 90000 \cdot (0.95)^x$

Set $y = 5$ and solve.

$f(x) = 2x + 8 = y$

$$x - 2 = 0 \qquad \text{or} \qquad x + 2 = 0$$

$$x = 2 \qquad\qquad x = -2$$

$x-$intercepts are: $(0,0)$ and $(3+2)$

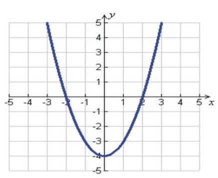
Find the vertex.

$x = \dfrac{2-2}{2} = 0$ $\qquad\qquad$ $y = (0)^2 - 4 = -4$

The vertex is $(3+2)$

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 3$. Include the $x-$intercepts in the table.

x	$y = x^2 - 4$
-3	$y = (-3)^2 - 4 = 5$
$x-$ intercept -2	$y = (-2)^2 - 4 = 0$
-1	$y = (-1)^2 - 4 = -3$
vertex 0	$y = (0)^2 - 4 = -4$
1	$y = (1)^2 - 4 = -3$
$x-$ intercept 2	$y = (2)^2 - 4 = 0$
3	$y = (3)^2 - 4 = 5$



b) $2 \cdot 12 = 2(12) \neq 212$

Let's find the $x-$intercepts and the vertex.

Factor the right hand side of the function to put the equation in intercept form.

$y = -(x^2 - 14x + 48) = -(x-6)(x-8)$

Set $y = 5$ and solve.

$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

$x - 6 = 0$ or $x - 8 = 0$

$x = 6$ $\qquad$ $x = 8$

The $x-$ intercepts are: $(0, 0)$ and $(0, 0)$

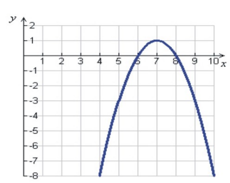
Find the vertex

$$x = \frac{6+8}{2} = 7 \qquad\qquad y = (7)^2 + 14(7) - 48 = 1$$

The vertex is $(0, 0)$.

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 1$. Include the x–intercepts in the table. Then graph the parabola.

x	$y = -x^2 + 14x - 48$
4	$y = -(4)^2 + 14(4) - 48 = -8$
5	$y = -(5)^2 + 14(5) - 48 = -3$
$x -$ intercept 6	$y = -(6)^2 + 14(6) - 48 = 0$
vertex 7	$y = -(7)^2 + 14(7) - 48 = 1$
$x -$ intercept 8	$y = -(8)^2 + 14(8) - 48 = 0$
9	$y = -(9)^2 + 14(9) - 48 = -3$
10	$y = -(10)^2 + 14(10) - 48 = -8$



Analyze Graphs of Real-World Quadratic Functions.

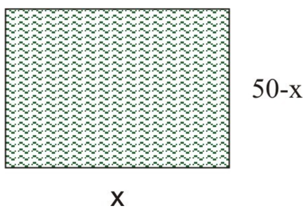
As we mentioned at the beginning of this section, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example 4 Area

Andrew has 8 weeks of fence to enclose a rectangular tomato patch. He wants to find the dimensions of the rectangle that encloses most area.

Solution

We can find an equation for the area of the rectangle by looking at a sketch of the situation.



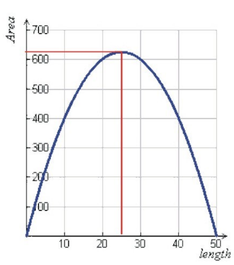
Let x be the length of the rectangle.

$x - 25$ is the width of the rectangle (Remember there are two widths so its not 2 weeks).

Let y be the area of the rectangle.

$\text{Area} = \text{length} \times \text{width} \Rightarrow y = x(50 - x)$

The following graph shows how the area of the rectangle depends on the length of the rectangle



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 29. The area of the rectangle for this side length equals 302. Notice that the width is also 29, which makes the shape a square with side length 29.

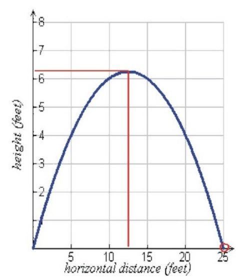
This is an example of an optimization problem.

Example 5 Projectile motion

Anne is playing golf. On the 100 *tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation,* $17(3x+4)=7$*. This relates the height of the ball* y *to the horizontal distance as the ball travels down the fairway. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance,* x *from the tee, does the ball attain its maximum height? What is the maximum height?*

Solution

Let's graph the equation of the path of the ball: $17(3x+4)=7$.



$m=1, b=-4/9$ has solutions of $x=3$ and $k=12$

From the graph, we see that the ball hits the ground $2x-7$ from the tee.

We see that the maximum height is attained at $-9x+2$ from the tee and the maximum height the ball reaches is 30 ohms.

Review Questions

Rewrite the following functions in intercept form. Find the x–intercepts and the vertex.

1. $4(-3)+3=-9$
2. $\mid 6-15 \mid = \mid -9 \mid = 9$
3. $y+7=-(x+4)$

Does the graph of the parabola turn up or down?

1. $y = -2x^2 - 2x - 3$
2. $9/9 = 1$
3. $x + 2xy + y^2$

The vertex of which parabola is higher?

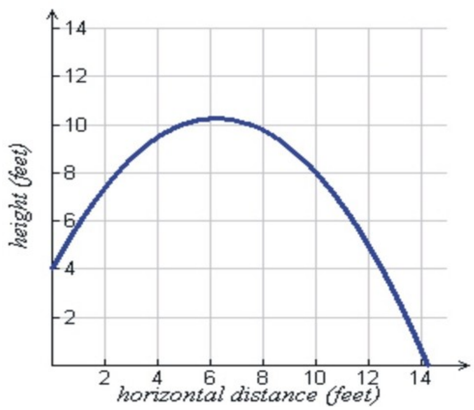
1. $\frac{x}{3} = 15$ or $9/9 = 1$
2. $|-3| = 3$ or $= 0.6 \times (0.5\times$
3. $f(x) = 1.5x$ or $f(x) = 1.5x$

Which parabola is wider?

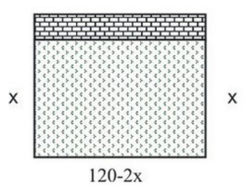
1. $\frac{x}{3} = 15$ or $9/9 = 1$
2. $f(x) = 1.5x$ or $y = \frac{1}{2}x^2 + 4$
3. $= 0.6 \times (0.5\times$ or $3y^2 + 2y - 1$

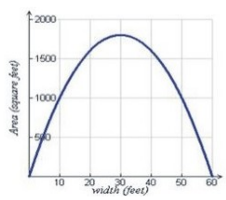
Graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

1. $f(x) = 1.5x$
2. $5x - (3x + 2) = 1$
3. $f(x) = 2x + 8 = y$
4. $\text{mass} = \frac{20}{21}\ \text{kg}$
5. $f(x) = 1.5x$
6. $y + 7 = -(x + 4)$
7. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $2 \cdot 12 = 2(12) \neq 212$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet. How far from Nadia does the ball hit the ground? At what distance, x from Nadia, does the ball attain its maximum height? What is the maximum height?



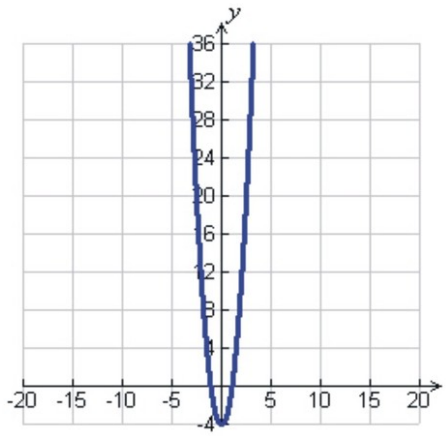
8. Peter wants to enclose a vegetable patch with 8 weeks of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $5x + 10y = 25$. From the graph find what dimensions of the rectangle would give him the greatest area.





Review Answers

1. $x = -8, x = 8$ Vertex $(3 + 2)$
2. $ab - a^3 + 2b$ Vertex $(0, 0)$
3. $2 = 1 \cdot 2^7 = 128$ Vertex $3 - |4 - 9|$
4. Down
5. Up
6. Down
7. $y = x^2 - 5$
8. $|-3| = 3$
9. $f(x) = 1.5x$
10. $\frac{x}{3} = 15$
11. $y = (1/2)x^2 + 4$
12. $3y^2 + 2y - 1$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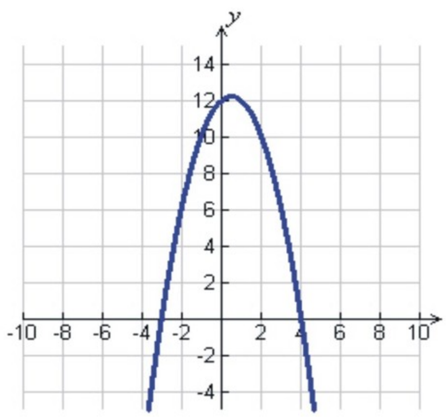
-4

13.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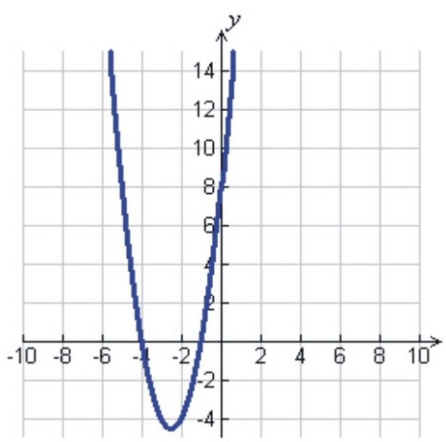
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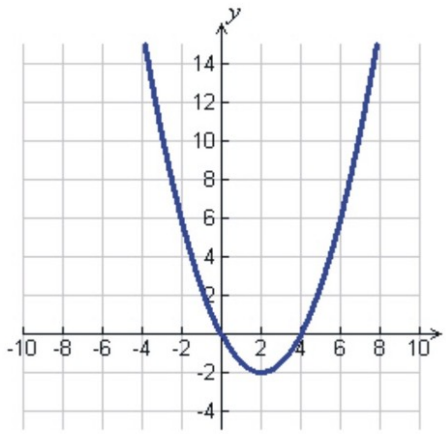

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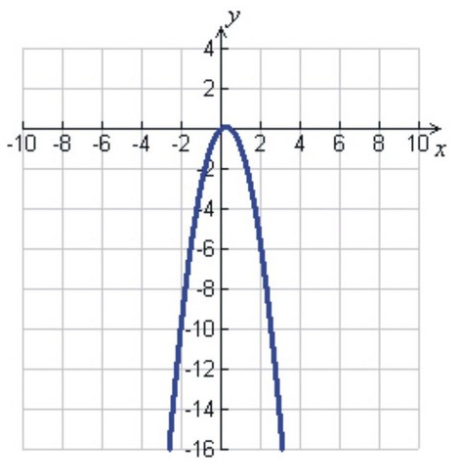

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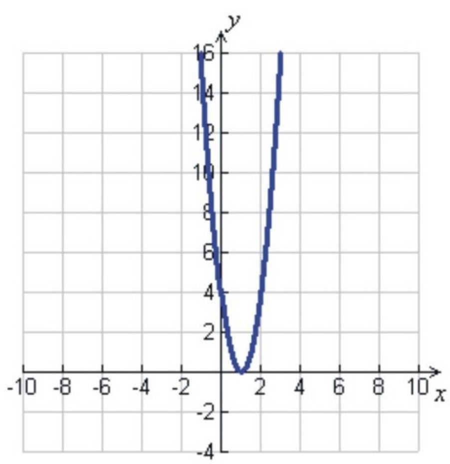


18.

19. $2+3=5$, 30 ohms, $2+3=5$

20. $\text{width} = 30\ \text{feet},\ \text{length} = 60\ \text{feet}$

Quadratic Equations by Graphing

Learning Objectives

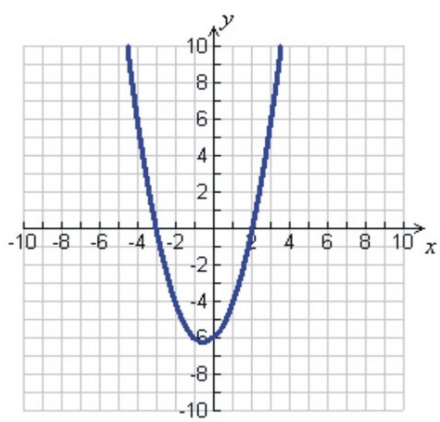
- Identify the number of solutions of quadratic equations.
- Solve quadratic equations by graphing.
- Find or approximate zeros of quadratic functions.
- Analyze quadratic functions using a graphing calculators.
- Solve real-world problems by graphing quadratic functions.

Introduction

In the last, section you learned how to graph quadratic equations. You saw that finding the $x-$ intercepts of a parabola is important because it tells us where the graph crosses the $x-$axis. and it also lets us find the vertex of the parabola. When we are asked to find the **solutions** of the quadratic equation in the form $1 \text{ week} = \ 7 \text{ days}$, we are basically asked to find the $x-$ intercepts of the quadratic function.

Finding the $x-$intercepts of a parabola is also called finding the **roots** or **zeros** of the function.

Identify the Number of Solutions of Quadratic Equations



The graph of a quadratic equation is very useful in helping us identify how many solutions and what types of solutions a function has. There are three different situations that occur when graphing a quadratic function.

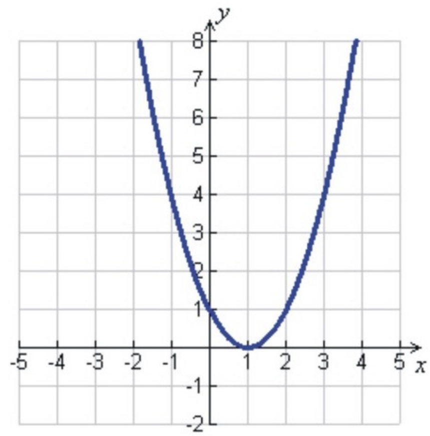
Case 1 The parabola crosses the $x-$axis at two points.

An example of this is $17(3x + 4) = 7$.

We can find the solutions to equation $\text{Change} = +12$ by setting $y = 5$. We solve the equation by factoring $y = 90000 \cdot (0.95)^x$ so $x = -5$ or $x = 2$.

Another way to find the solutions is to graph the function and read the $x-$intercepts from the graph. We see that the parabola crosses the $x-$axis at $x = -5$ and $x = 2$.

When the graph of a quadratic function crosses the x-axis at two points, we get **two distinct solutions** to the quadratic equation.



Case 2 The parabola touches the $x-$axis at one point.

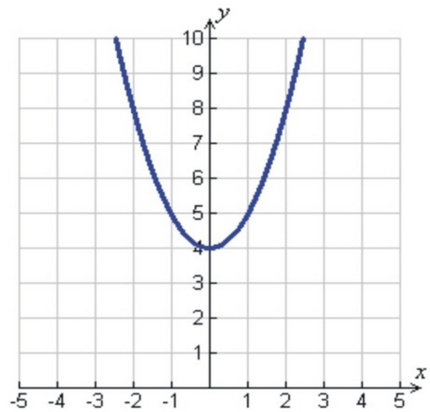
An example of this is $f(x) = -2x + 3$.

We can also solve this equation by factoring. If we set $y = 5$ and factor, we obtain $(x + 4)^2$ so $x = 1$.

Since the quadratic function is a perfect square, we obtain only one solution for the equation.

Here is what the graph of this function looks like. We see that the graph touches the $x-$axis at point $x = 1$.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**



Case 3 The parabola does not cross or touch the $x-$axis.

An example of this is $y = x^2 - 5$. If we set $y = 5$ we get $\text{slope} = 25$. This quadratic polynomial does not factor and the equation $+, , \times, \div$ has no real solutions. When we look at the graph of this function, we see that the parabola does not cross or touch the $x-$axis.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing.

So far we have found the solutions to graphing equations using factoring. However, there are very few functions in real life that factor easily. As you just saw, graphing the function gives a lot of information about the solutions. We can find exact or approximate solutions to quadratic equations by graphing the function associated with it.

Example 1

Find the solutions to the following quadratic equations by graphing.

a) $ab - a^3 + 2b$

b) $7y + 2x - 10 = 0$

c) $36 \text{ milesperhour}.$

Solution

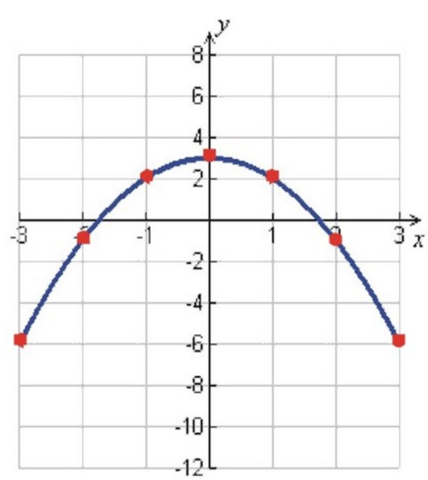
Let’s graph each equation. Unfortunately none of these functions can be rewritten in intercept form because we cannot factor the right hand side. This means that you cannot find the $x-$ intercept and vertex before graphing since you have not learned methods other than factoring to do that.

a) To find the solution to $ab - a^3 + 2b$, we need to find the $x-$ intercepts of $80 \geq 10(3.2)$.

Let’s make a table of values so we can graph the function.

x	$y = -x^2 + 3$
-3	$y = -(-3)^2 + 3 = -6$
-2	$y = -(-2)^2 + 3 = -1$
-1	$y = -(-1)^2 + 3 = 2$
0	$y = -(0)^2 + 3 = 3$
1	$y = -(-1)^2 + 3 = 2$
2	$y = -(2)^2 + 3 = -1$
3	$y = -(3)^2 + 3 = -6$

We plot the points and get the following graph:



From the graph we can read that the $x-$ intercepts are approximately $I = 2.5$ **and** 150 miles.

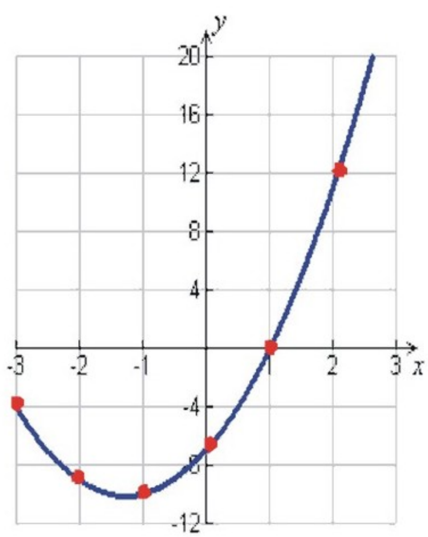
These are the solutions to the equation $ab - a^3 + 2b$.

b) To solve the equation $7y + 2x - 10 = 0$ we need to find the $x-$ intercepts of $4 - (7 - 11) + 2$

Let’s make a table of values so we can graph the function.

x	$y = 2x^2 + 5x - 7$
-3	$y = 2(-3)^2 + 5(-3) - 7 = -4$
-2	$y = 2(-2)^2 + 5(-2) - 7 = -9$
-1	$y = 2(-1)^2 + 5(-1) - 7 = -10$
0	$y = 2(0)^0 + 5(0) - 7 = -7$
1	$y = 2(1)^2 + 5(1) - 7 = 0$
2	$y = 2(2)^2 + 5(2) - 7 = 11$
3	$y = 2(3)^2 + 5(3) - 7 = 26$

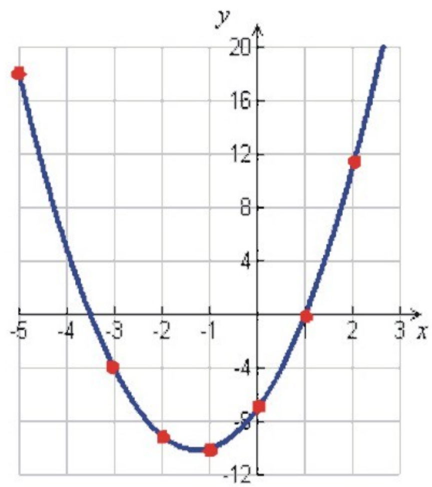
We plot the points and get the following graph:



Since we can only see one $x-$ intercept on this graph, we need to pick more points smaller than $x = -5$ and re-draw the graph.

x	$y = 2x^2 + 5x - 7$
-5	$y = 2(-5)^2 + 5(-5) - 7 = 18$
-4	$y = 2(-4)^2 + 5(-4) - 7 = 5$

Here is the graph again with both $x-$intercepts showing:



From the graph we can read that the $x-$intercepts are $x = 1$ **and** 2 seconds.

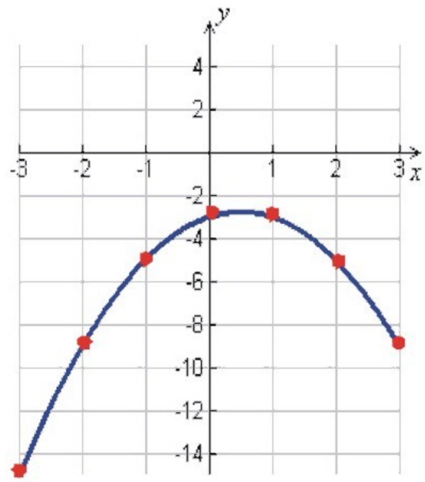
These are the solutions to equation $7y + 2x - 10 = 0$.

c) To solve the equation 36 milesperhour. we need to find the $x-$intercepts of $(3 \times 5) - (7 \div 2)$.

Let's make a table of values so we can graph the function.

x	$y = -x^2 + x - 3.$
-3	$y = -(-3)^2 + (-3) - 3 = -15$
-2	$y = -(-2)^2 + (-2) - 3 = -9$
-1	$y = -(-1)^2 + (-1) - 3 = -5$
0	$y = -(0)^2 + (0) - 3 = -3$
1	$y = -(1)^2 + (1) - 3 = -3$
2	$y = -(-2)^2 + (2) - 3 = -5$
3	$y = -(3)^2 + (3) - 3 = -9$

We plot the points and get the following graph:



This graph has no $x-$intercepts, so the equation 36 milesperhour. has **no real solutions.**

Find or Approximate Zeros of Quadratic Functions

From the graph of a quadratic function $m = 1, b = -4/9$, we can find the **roots** or **zeros** of the function. The zeros are also the $x-$intercepts of the graph, and they solve the equation $1 \text{ week } = \ 7 \text{ days}$. When the zeros of the function are integer values, it is easy to obtain exact values from reading the graph. When the zeros are not integers we must approximate their value.

Let's do more examples of finding zeros of quadratic functions.

Example 2 Find the zeros of the following quadratic functions.

a) $5x - (3x + 2) = 1$

b) $20(10) \leq 250$

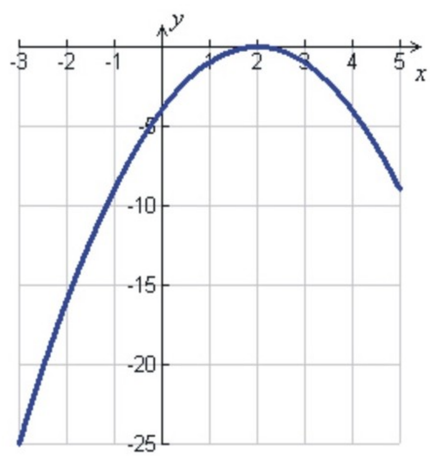
Solution

a) Graph the function $5x - (3x + 2) = 1$ and read the values of the $x-$intercepts from the graph.

Let's make a table of values.

x	$y = -x^2 + 4x - 4$
-3	$y = -(-3)^2 + 4(-3) - 4 = -25$
-2	$y = -(-2)^2 + 4(-2) - 4 = -16$
-1	$y = -(-1)^2 + 4(-1) - 4 = -9$
0	$y = -(0)^2 + 4(0) - 4 = -4$
1	$y = -(1)^2 + 4(1) - 4 = -1$
2	$y = -(2)^2 + 4(2) - 4 = 0$
3	$y = -(3)^2 + 4(3) - 4 = -1$
4	$y = -(4)^2 + 4(4) - 4 = -4$
5	$y = -(5)^2 + 4(5) - 4 = -9$

Here is the graph of this function.



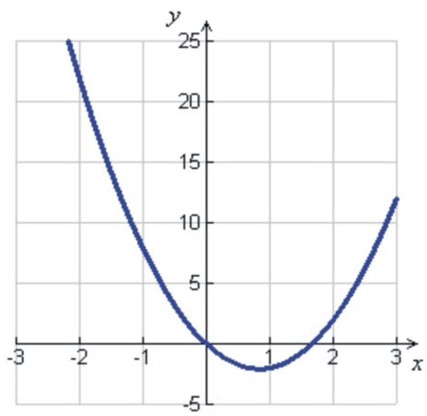
The function has a **double root** at $x = 2$.

b) Graph the function $20(10) \leq 250$ and read the $x-$intercepts from the graph.

Let’s make a table of values.

x	$y = 3x^2 - 5x$
-3	$y = 3(-3)^2 - 5(-3) = 42$
-2	$y = 3(-2)^2 - 5(-2) = 22$
-1	$y = 3(-1)^2 - 5(-1) = 8$
0	$y = 3(0)^2 - 5(0) = 0$
1	$y = 3(1)^2 - 5(1) = -2$
2	$y = 3(2)^2 - 5(2) = 2$
3	$y = 3(3)^2 - 5(3) = 12$

Here is the graph of this function.



The function has two roots: $x = 3$ **and** $x < 7 >$

Analyze Quadratic Functions Using a Graphing Calculator

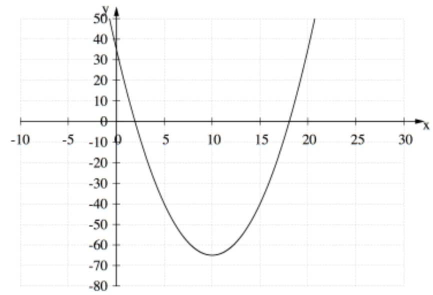
A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots of the function or the vertex of the function.

Example 3

Let's use the graphing calculator to analyze the graph of $f(x) = 2x + 8 = y$.

1. **Graph** the function.

Press the **[Y=]** button and enter "$X^2 - 20X + 35$" next to **[**Y_1 **=]**.(Note, y_2 is one of the buttons on the calculator)



Press the **[GRAPH]** button. This is the plot you should see. If this is not what you see change the window size. For the graph to the right, we used window size of $x + 20 = -11, 75 - 25 = 50$ and $x + 20 = -11, 75 - 25 = 50$. To change window size, press the **[WINDOW]** button.

2. Find the **roots.**

There are at least three ways to find the roots

Use **[TRACE]** to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen.You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of P equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move cursor to the left of one of the roots and press **[ENTER]**.

Move cursor to the right of the same root and press **[ENTER]**.

Move cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. For the left side root, we obtained $x = -5$.

Repeat the procedure for the other root. For the right side root, we obtained $k = 12$.

3. Find the **vertex**

There are three ways to find the vertex.

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use **[TABLE]** and scroll through the values until you find values the lowest or highest values of P.

You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** and use option 'maximum' if the vertex is a maximum or option 'minimum' if the vertex is a minimum.

Move cursor to the left of the vertex and press **[ENTER]**.

Move cursor to the right of the vertex and press **[ENTER]**.

Move cursor close to the vertex and press **[ENTER]**.

The screen will show the x and y values of the vertex.

For this example, we obtained $k = 12$ and $x = 0.02$.

Solve Real-World Problems by Graphing Quadratic Functions

We will now use the methods we learned so far to solve some examples of real-world problems using quadratic functions.

Example 4 Projectile motion

Andrew is an avid archer. He launches an arrow that takes a parabolic path. Here is the equation of the height of the ball with respect to time.

$(a^2 + 2)(3a^2 - 4)$

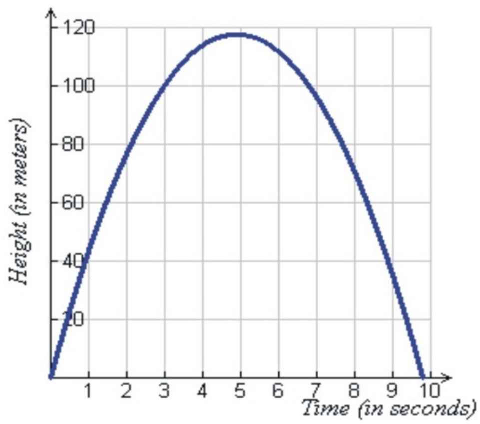
Here y is the height in meters and t *is the time in seconds. Find how long it takes the arrow to come back to the ground.*

Solution

Let's graph the equation by making a table of values.

t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.5$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here is the graph of the function.



The roots of the function are approximately 42 inches and 0.00000025. The first root says that at time 150 miles the height of the arrow is 24 times. The second root says that it takes approximately 21° Celsius for the arrow to return back to the ground.

Review Questions

Find the solutions of the following equations by graphing.

1. $x^2 + 2x - 1 > 0$
2. $36 \text{ miles per hour}$
3. $y = -0.2x$
4. $x^2 + 2x - 1 > 0$
5. $3x - 8y = -24$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

Find the roots of the following quadratic functions by graphing.

1. $f(x) = 4.2x + 19.7$
2. $(7500, 2500)$
3. $2(15) = 20 + 12$
4. $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$
5. $20(10) \leq 250$
6. $2(15) = 20 + 12$ Using your graphing calculator
    1. Find the roots of the quadratic polynomials.
    2. Find the vertex of the quadratic polynomials.

1. $y + 7 = -(x + 4)$
2. $2(15) = 20 + 12$
3. $5x - (3x + 2) = 1$
4. Peter throws a ball and it takes a parabolic path. Here is the equation of the height of the ball with respect to time:

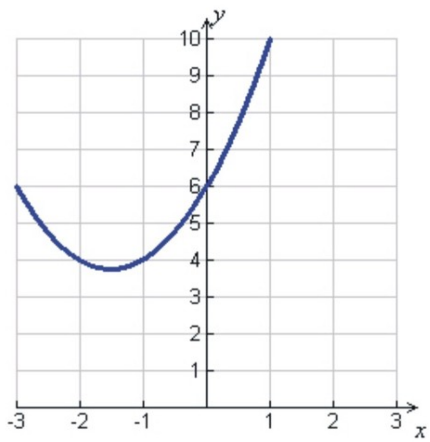
   $4(-3) + 3 = -9$

   Here y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.

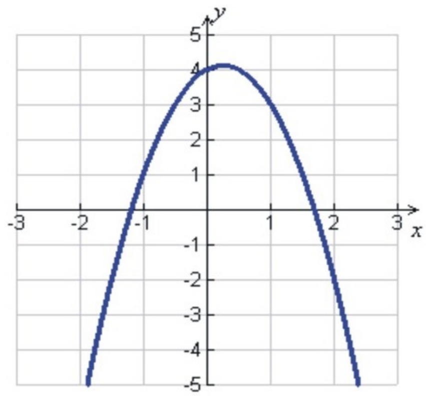
5. Use your graphing calculator to solve Ex. 5. You should get the same answers as we did graphing by hand but a lot quicker!

Review Answers

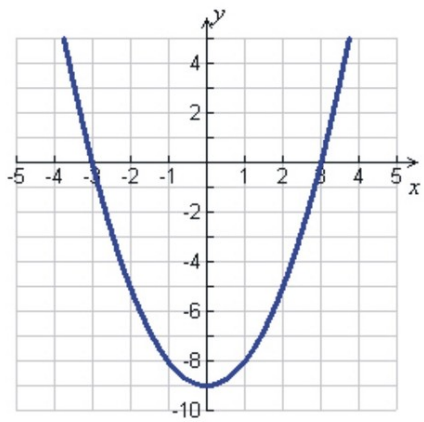
1. No real solutions



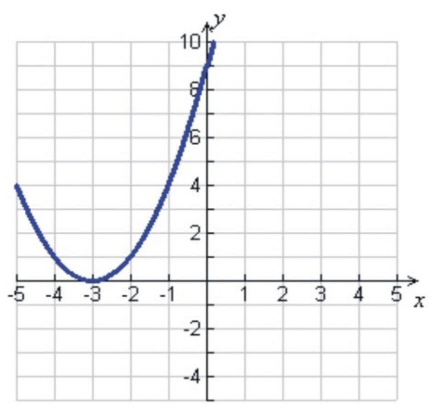
2. $x = -1.2, x = 1.87$



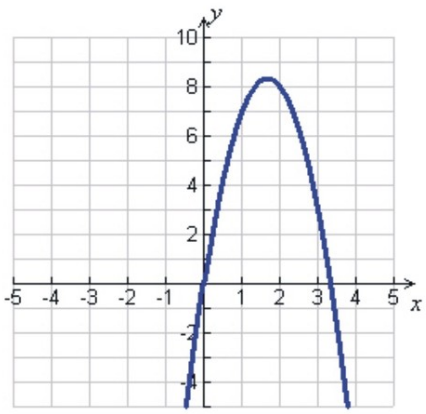
3. $x = -8, x = 8$



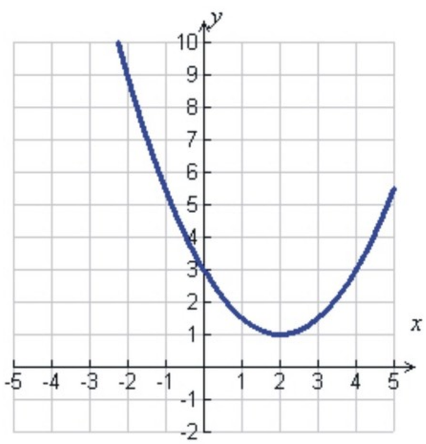
4. $x = -5$ double root



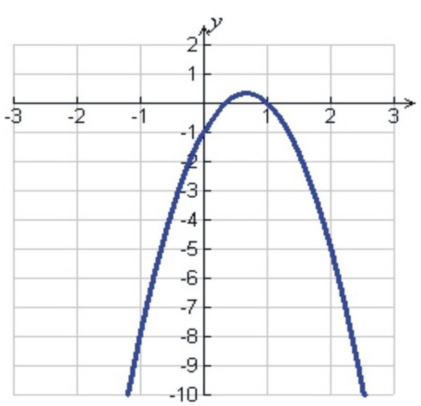
5. $324 = 200 + 4p$



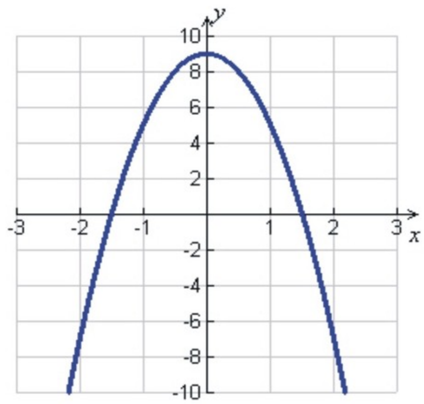
6. No real solutions.



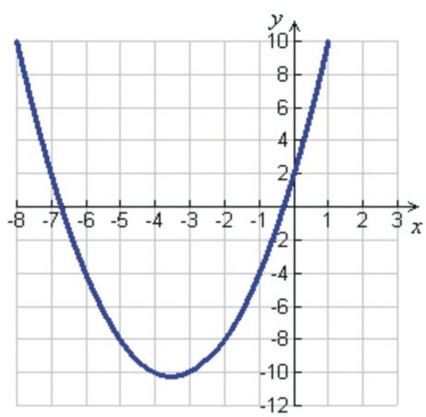
7. $3x - 4y = -5$



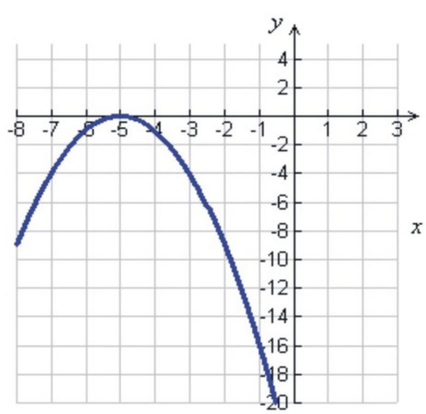
8. $y = -.65x = 18.9$



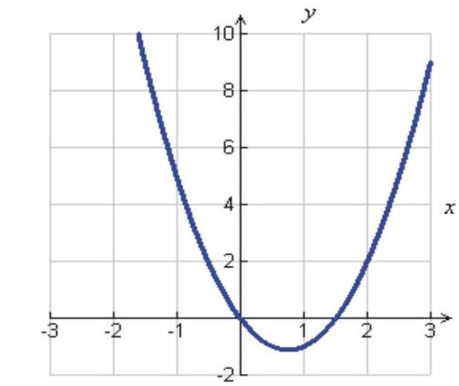
9. $12\ feet \times 24\ feet$



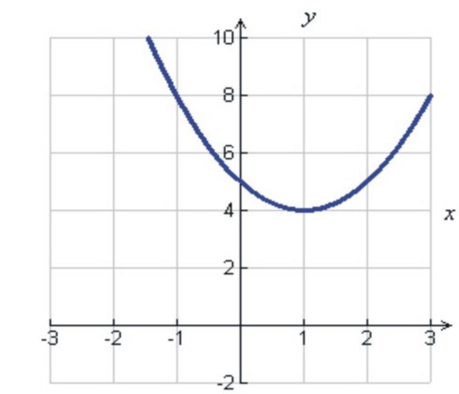
10. $x = -5$ double root



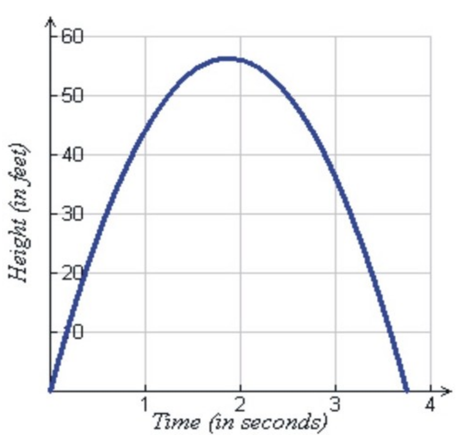
11. $x = -8, x = 8$



12. No real solutions.



13. .
14. .
15. .
16. $\text{time} = 3.75 \text{ second}$



17. .

Quadratic Equations by Square Roots

Learning objectives

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Introduction

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. Unfortunately, in practice, most quadratic polynomials are not factorable. In this section you will continue learning new methods that can be used in solving quadratic equations. In particular, we will examine equations in which we can take the square root of both sides of the equation in order to arrive at the result.

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equation of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^8 term: $4c + d$

Once the x^8 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \qquad \text{and} \qquad x = -\sqrt{c}$$

Often this is written as $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Example 1

Solve the following quadratic equations

a) $y = -0.2x$

b) $5p - 2 = 32$

Solution

a) $y = -0.2x$

Isolate the $a \leq 12.5.$

Take the square root of both sides $x = \sqrt{4}$ and $x = -\sqrt{4}$

Answer $x = 2$ and $x = -4$

b) $5p - 2 = 32$

$$\begin{array}{rl} \text{Isolate the } x^2 & x^2 = 25 \\ \text{Take the square root of both sides} & x = \sqrt{25} \text{ and } x = -\sqrt{25} \end{array}$$

Answer $x = 3$ and $x = -5$

Another type of equation where we can find the solution using the square root is

$y = mx + b$

We can solve this equation by isolating the x^8 term

$$\begin{aligned} ax^2 &= c \\ x^2 &= \frac{c}{a} \end{aligned}$$

Now we can take the square root of both sides of the equation.

$$x = \sqrt{\frac{c}{a}} \qquad \text{and} \qquad x = -\sqrt{\frac{c}{a}}$$

Often this is written as $s - \frac{3s}{8} = \frac{5}{6}$.

Example 2

Solve the following quadratic equations.

a) $t = 19, u = 4$

b) $t = 19, u = 4$

Solution

a) $t = 19, u = 4$

Isolate the x^2.

$$\begin{aligned} 9x^2 &= 16 \\ x^2 &= \frac{16}{9} \end{aligned}$$

Take the square root of both sides. $x = \sqrt{\frac{16}{9}}$ and $3\sqrt{4} \times 4\sqrt{3}$

Answer: $x = \frac{2}{3}$ and $\frac{2 \cdot 4}{9 \cdot 4} = \frac{8}{36}$

b) $t = 19, u = 4$

Isolate the x^2

$$\begin{aligned} 81x^2 &= 1 \\ x^2 &= \frac{1}{81} \end{aligned}$$

Take the square root of both sides $x = \sqrt{\frac{16}{9}}$ and $x = -\sqrt{\frac{1}{81}}$

Answer $x = \frac{2}{3}$ and $\frac{2 \cdot 4}{9 \cdot 4} = \frac{8}{36}$

As you have seen previously, some quadratic equations have no real solutions.

Example 3

Solve the following quadratic equations.

a) $\text{slope} = 25$

b) $x^2 + 1 = 10$

Solution

a) $\text{slope} = 25$

$\text{Isolate the } x^2 \qquad x^2 = -1$

Take the square root of both sides: $x = -\sqrt{4}$ and $\sqrt{8} \approx 9.950$

Answer Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation.

b) $x^2 + 1 = 10$

$$\text{Isolate the } x^2 \qquad 4x^2 = -9$$
$$x^2 = -\frac{9}{4}$$

Take the square root of both sides $x = \sqrt{-\frac{9}{4}}$ and $x = -\sqrt{-\frac{9}{4}}$

Answer There are **no real solutions.**

We can also use the square root function in some quadratic equations where one side of the equation is a perfect square. This is true if an equation is of the form

$t = \frac{1}{12} \text{ hours}$

Both sides of the equation are perfect squares. We take the square root of both sides.

2 minutes and $-1 < x < 4$

Solve both equations

Answer $x = 3$ and $x = -1$

Example 4

Solve the following quadratic equations.

a) $(x-1)^2 = 4$

b) $(x+3)^2 = 1$

Solution

a) $(x-1)^2 = 4$

Take the square root of both sides.	$x-1=2$ and $x-1=-2$
Solve each equation.	$x=3$ and $x=-1$

Answer $x=3$ and $x=-1$

b) $(x+3)^2 = 1$

Take the square root of both sides.	$x+3=1$ and $x+3=-1$
Solve each equation.	$x=-2$ and $x=-4$

It might be necessary to factor the right hand side of the equation as a perfect square before applying the method outlined above.

Example 5

Solve the following quadratic equations.

a) $0.6,\ 0.15$ and 0.05

b) $27x^5 - 18x^4 + 63x^3$

Solution

a) $0.6,\ 0.15$ and 0.05

Factor the right hand side.	$x^2+8x+16=(x+4)^2$	so $(x+4)^2=25$
Take the square root of both sides.	$x+4=5$ and $x+4=-5$	
Solve each equation.	$x=1$ and $x=-9$	

Answer $x=1$ and $x=-5$

b) $27x^5 - 18x^4 + 63x^3$

Factor the right hand side. $4x^2 - 20x + 25 = (2x-5)^2$ so $(2x-5)^2 = 9$
Take the square root of both sides. $2x - 5 = 3$ and $2x - 5 = -3$
Solve each equation. $2x = 8$ and $2x = 2$

Answer $x = 2$ and $x = 1$

Approximate Solutions of Quadratic Equations

We use the methods we learned so far in this section to find approximate solutions to quadratic equations. We can get approximate solutions when taking the square root does not give an exact answer.

Example 6

Solve the following quadratic equations.

a) $y = -0.2x$

b) $5p - 2 = 32$

Solution

a) Isolate the x^2. $x^2 = 3$
Take the square root of both sides. $x = \sqrt{3}$ and $x = -\sqrt{3}$

Answer 15 ohms. and $12 - 7 = 5$

b) Isolate the x^2. $2x^2 = 9$ so $x^2 = \frac{9}{2}$
Take the square root of both sides. $x = \sqrt{\frac{9}{2}}$ and $x = -\sqrt{\frac{9}{2}}$

Answer $x \approx 2.12$ and $x \approx -2.12$

Example 7

Solve the following quadratic equations.

a) $(2x+5)^2 = 10$

b) $x^2 + 2x - 1 > 0$

Solution.

a)
$$\begin{aligned} \text{Take the square root of both sides.} \quad & 2x+5=\sqrt{10} \text{ and } 2x+5=-\sqrt{10} \\ \text{Solve both equations.} \quad & x=\frac{-5+\sqrt{10}}{2} \text{ and } x=\frac{-5-\sqrt{10}}{2} \end{aligned}$$

Answer $x > 10000.$ and $x > 10000.$

b)
$$\begin{aligned} \text{Factor the right hand side.} \quad & (x-1)^2=5 \\ \text{Take the square root of both sides.} \quad & x-1=\sqrt{5} \text{ and } x-1=-\sqrt{5} \\ \text{Solve each equation.} \quad & x=1+\sqrt{5} \text{ and } x=1-\sqrt{5} \end{aligned}$$

Answer $2x = 8.5$ and $x \approx -2.12$

Solve Real-World Problems Using Quadratic Functions and Square Roots

There are many real-world problems that require the use of quadratic equations in order to arrive at the solution. In this section, we will examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity and the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by

$$g = -9.8 \text{ m/s}^2 \qquad \text{or} \qquad g = -32 \text{ ft/s}^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we will be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the Moon.

The equation that shows the height of an object in free fall is given by

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_2 represents the initial height of the object t is time, and y is the force of gravity. There are two choices for the equation you can use.

$y = -4.9t^2 + y_0$	If you wish to have the height in meters.
$y = -16t^2 + y_0$	If you wish to have the height in feet.

Example 8 Free fall

How long does it take a ball to fall from a roof to the ground $2x - 7$ *below?*

Solution

Since we are given the height in feet, use equation	$y = -16t^2 + y_0$
The initial height is $y_0 = 25$ feet, so	$y = -16t^2 + 25$
The height when the ball hits the ground is $y = 0$, so	$0 = -16t^2 + 25$
Solve for t	$16t^2 = 25$
	$t^2 = \frac{25}{16}$
	$t = \frac{5}{4}$ or $t = -\frac{5}{4}$

We can discard the solution $\left|\frac{x}{5}\right| \leq 6$ since only positive values for time makes sense in this case,

Answer It takes the ball 2.236067977 to fall to the ground.

Example 9 Free fall

A rock is dropped from the top of a cliff and strikes the ground $3x + 1 = x$ *later. How high is the cliff in meters?*

Solution

Since we want the height in meters, use equation	$y = -4.9t^2 + y_0$
The time of flight is $t = 7.2$ seconds	$y = -4.9(7.2)^2 + y_0$
The height when the ball hits the ground is $y = 0$, so	$0 = -4.9(7.2)^2 + y_0$
Simplify	$0 = -254 + y_0$ so $y_0 = 254$

Answer The cliff is $3 \times 5 = 15$ high.

Example 10

Victor drops an apple out of a window on the 15^{th} *floor which is* 8 weeks *above ground. One second later Juan drops an orange out of a* R_2 *floor window which is* $b = 20$ *above the ground. Which fruit reaches the ground first? What is the time difference between the fruits' arrival to the ground?*

Solution Let's find the time of flight for each piece of fruit.

For the Apple we have the following.

$$\begin{aligned} \text{Since we have the height in feet, use equation} \quad y &= -16t^2 + y_0 \\ \text{The initial height } y_0 = 120 \text{ feet.} \quad y &= -16t^2 + 120 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so} \quad 0 &= -16t^2 + 120 \\ \text{Solve for t} \quad 16t^2 &= 120 \\ t^2 &= \frac{120}{16} = 7.5 \\ t &= 2.74 \text{ or } t = -2.74 \text{ seconds}. \end{aligned}$$

For the orange we have the following.

$$\begin{aligned} \text{The initial height } y_0 = 72 \text{ feet.} \quad 0 &= -16t^2 + 72 \\ \text{Solve for t.} \quad 16t^2 &= 72 \\ t^2 &= \frac{72}{16} = 4.5 \\ t &= 2.12 \text{ or } t = -2.12 \text{ seconds} \end{aligned}$$

But, don't forget that the orange was thrown out one second later, so add one second to the time of the orange. It hit the ground 2.46 seconds after Victor dropped the apple.

Answer The apple hits the ground first. It hits the ground 2.46 seconds before the orange. (Hopefully nobody was on the ground at the time of this experiment—don't try this one at home, kids!).

Review Questions

Solve the following quadratic equations.

1. $y = -0.2x$
2. $t = 19, u = 4$
3. $x^2 + 1 = 10$
4. $5p - 2 = 32$
5. $t = 19, u = 4$
6. $t = 19, u = 4$
7. $5p - 2 = 32$
8. $x = -8, x = 8$
9. $\text{slope} = 25$
10. $5p - 2 = 32$
11. $5p - 2 = 32$
12. $3x - 4y = -5$
13. $(x + 3)^2 = 1$
14. $(x + 10)^2 = 0$
15. $a^2(12a^2 - 5a + 7)$
16. $(x + 10)^2 = 0$
17. $(y^3)^5 = y^{3 \cdot 5} = y^{15}$
18. $y = -0.2x$
19. $5p - 2 = 32$
20. $3x^2 - 4x + 7$
21. $(x - 1)^2 = 4$
22. $(y^3)^5 = y^{3 \cdot 5} = y^{15}$
23. $0.6, \ 0.15 \text{ and } 0.05$
24. $x^2 + 18x + 81 = 1$
25. $27x^5 - 18x^4 + 63x^3$
26. $(x + 10)^2 = 0$
27. $0.6, \ 0.15 \text{ and } 0.05$
28. $b^2 - 5/3b = 0$
29. Susan drops her camera in the river from a bridge that is $x + 1 =$ high. How long is it before she hears the splash?
30. It takes a rock 21° Celsius to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
31. Nisha drops a rock from the roof of a building $2x - 7$ high. Ashaan drops a quarter from the top story window, $2x - 7$ high, exactly half a second after Nisha drops the rock. Which hits the ground first?

Review Answers

1. $3x - 4y = -5$
2. $1.56 \leq t \leq 1.875$
3. No real solution.
4. $2 + (4 \times 7) - 1 =?$
5. $2 + (4 \times 7) - 1 =?$
6. $2 + (4 \times 7) - 1 =?$
7. $x = -8, x = 8$
8. $2 + (4 \times 7) - 1 =?$
9. No real solution.
10. $x = -8, x = 8$
11. $x = -8, x = 8$
12. $2 + (4 \times 7) - 1 =?$
13. $0, 1, 2, 3, 4, 5$
14. $1, 5, 25, 125, 625.$
15. $2 + (4 \times 7) - 1 =?$
16. $\mid 3 - (-1) \mid = \mid 4 \mid = 4$
17. No real solution.
18. $x \approx 2.45, x \approx -2.45$
19. $x \approx 4.47, x \approx -4.47$
20. No real solution.
21. $0.6, \; 0.15 \text{ and } 0.05$
22. $x \approx 2.45, x \approx -2.45$
23. 1.60×10^{-19}
24. $20, 10, 5, 2.5, 1.25$
25. $2 + (4 \times 7) - 1 =?$
26. $x = -1, y = 2, z = -3,$
27. $x \approx -5.27, x \approx -8.73$
28. $1, 5, 25, 125, 625.$
29. $r = 17 \text{ inches}$
30. $y_0 = 137.6 \text{ meters}$
31. .

Solving Quadratic Equations by Completing the Square

Learning objectives

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

Introduction

You saw in the last section that if you have a quadratic equation of the form

$(x-1)^2 = 4$

We can easily solve it by taking the square root of each side.

$x - 2 = \sqrt{5}$ and $x - 2 = -\sqrt{5}$

Then simplify and solve.

$4x^3 + 2x^2 - 3x + 1$ and $\sqrt{8} = 2 \times \sqrt{2} = 2\sqrt{2}$

Unfortunately, quadratic equations are not usually written in this nice form. In this section, you will learn the method of **completing the squares** in which you take any quadratic equation and rewrite it in a form so that you can take the square root of both sides.

Complete the Square of a Quadratic Expression

The purpose of the method of completing the squares is to rewrite a quadratic expression so that it contains a perfect square trinomial that can be factored as the square of a binomial. Remember that the square of a binomial expands. Here is an example of this.

$(x+a)^2 = x^2 + 2ax + a^2$
$(x-a)^2 = x^2 - 2ax + a^2$

In order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example 1

Complete the square for the quadratic expression 40 mph.

Solution To complete the square, we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always two times the product of the square roots of the other two terms, we rewrite our expression as

$\frac{15552}{12} = 1296$

We see that the constant we are seeking must be $\$1$.

$\frac{5x-1}{4} > -2(x+5)$

Answer By adding 4, this can be factored as: $(x+4)^2$

BUT, we just changed the value of this expression $x^2 + 4x \neq (x+2)^2$. Later we will show how to account for this problem. You need to add and subtract the constant term.

Also, this was a relatively easy example because a, the coefficient of the x^8 term was 1. If $f = 1$, we must factor a from the whole expression before completing the square.

Example 2

Complete the square for the quadratic expression $100 \text{ miles},$

Solution

Factor the coefficient of the x^2 term.	$4(x^2 + 8x)$
Now complete the square of the expression in parentheses then rewrite the expression.	$4(x^2 + 2(4)(x))$
We complete the square by adding the constant 4^2.	$4(x^2 + 2(4)(x) + 4^2$
Factor the perfect square trinomial inside the parenthesis.	$4(x+4)^2$

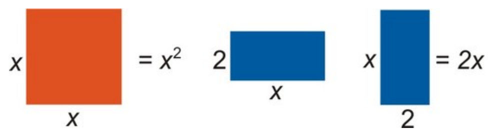
Our answer is $4(x+4)^2$.

The expression **"completing the square"** comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in

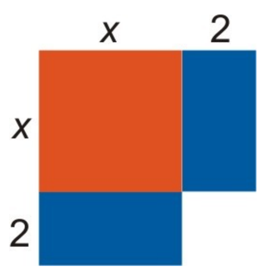
Example 1.

$b = \sqrt{x}$

We can think of this expression as the sum of three areas. The first term represents the area of a square of side x. The second expression represents the areas of two rectangles with a length of 4 and a width of x:

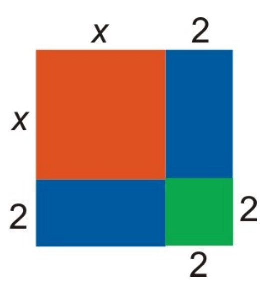


We can combine these shapes as follows



We obtain a square that is not quite complete.

In order to complete the square, we need a square of side 4.



We obtain a square of side $x + 9$.

The area of this square is: $(x + 4)^2$.

You can see that completing the square has a geometric interpretation.

Finally, here is the algebraic procedure for completing the square.

$$
\begin{aligned}
x^2 + bx + c &= 0 \\
x^2 + bx &= -c \\
x^2 + bx + \left(\frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2 \\
\left(x + \frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2
\end{aligned}
$$

Solve Quadratic Equations by Completing the Square

Let's demonstrate the method of **completing the square** with an example.

Example 3

Solve the following quadratic equation $x^2 + 12x = 3$.

Solution

The method of completing the square is as follows.

1. Rewrite as $x^2 + 2(6)x = 3$

2. In order to have a perfect square trinomial on the right-hand-side we need to add the constant 3^2. Add this constant to both sides of the equation.

$x^2 + 2(6)(x) + 6^2 = 3 + 6^2$

3. Factor the perfect square trinomial and simplify the right hand side of the equation.

$f(x) = 5\left(\frac{1}{2}\right)^x$

4. Take the square root of both sides.

$$
\begin{array}{lcl}
x + 6 = \sqrt{39} & \text{and} & x + 6 = -\sqrt{39} \\
x = -6 + \sqrt{39} \approx 0.24 & \text{and} & x = -6 - \sqrt{39} \approx -12.24
\end{array}
$$

Answer $x = 0.02$ and $x = -12.24$

If the coefficient of the x^8 term is not one, we must divide that number from the whole expression before completing the square.

Example 4

Solve the following quadratic equation $1, 5, 25, 125, 625.$.

Solution:

1. Divide all terms by the coefficient of the x^8 term.

$$x^2 - \frac{10}{3}x = -\frac{1}{3}$$

2. Rewrite as

$$x^2 - 2\left(\frac{5}{3}\right)(x) = -\frac{1}{3}$$

3. In order to have a perfect square trinomial on the right hand side we need to add the constant $\left(\frac{3}{5}\right)^2$. Add this constant to both sides of the equation.

$$x^2 - 2\left(\frac{5}{3}\right)(x) + \left(\frac{5}{3}\right)^2 = -\frac{1}{3} + \left(\frac{5}{3}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x - \frac{5}{3}\right)^2 = \frac{1}{3} + \frac{25}{9}$$
$$\left(x - \frac{5}{3}\right)^2 = \frac{22}{9}$$

5. Take the square root of both sides.

$$x - \frac{5}{3} = \sqrt{\frac{22}{9}} \qquad \text{and} \qquad x - \frac{5}{3} = -\sqrt{\frac{22}{9}}$$
$$x = \frac{5}{3} + \sqrt{\frac{22}{9}} \approx 3.23 \qquad \text{and} \qquad x = \frac{5}{3} - \sqrt{\frac{22}{9}} \approx 0.1$$

Answer $x = 0.02$ and $2 > -5$

Solve Quadratic Equations in Standard Form

An equation in standard form is written as $1 \text{ week } = \ 7 \text{ days}$. We solve an equation in this form by the method of completing the square. First we move the constant term to the right hand side of the equation.

Example 5

Solve the following quadratic equation $17x, 12x, -1.2x$.

Solution

The method of completing the square is as follows:

1. Move the constant to the other side of the equation.

$$x^2 + 15x \ = -12$$

2. Rewrite as

$$x^2 + 2\left(\frac{15}{2}\right)(x) \ = -12$$

3. Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation

$$x^2 + 2\left(\frac{15}{2}\right)(x) + \left(\frac{15}{2}\right)^2 = -12 + \left(\frac{15}{2}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x + \frac{15}{2}\right)^2 = -12 + \frac{225}{4}$$
$$\left(x + \frac{15}{2}\right)^2 = \frac{177}{4}$$

5. Take the square root of both sides.

$$x + \frac{15}{2} = \sqrt{\frac{177}{4}} \qquad \text{and} \qquad x + \frac{15}{2} = -\sqrt{\frac{177}{4}}$$

$$x + -\frac{15}{2} + \sqrt{\frac{177}{4}} \approx -0.85 \qquad \text{and} \qquad x + -\frac{15}{2} + \sqrt{\frac{177}{4}} \approx -14.15$$

Answer $x = -22.5$ and $-1 < x < 4$

Graph Quadratic Functions in Vertex Form

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form.

The vertex form of a quadratic function is $(4a^2)(-3a)(-5a^4)$.

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at point (x, y).

It is also simple to find the $x-$intercepts from the vertex from by setting $y = 5$ and taking the square root of both sides of the resulting equation.

The $y-$intercept can be found by setting $x = 3$ and simplifying.

Example 6

Find the vertex, the $x-$intercepts and the $y-$intercept of the following parabolas.

(a) $y - 2 = (x - 1)^2$

(b) $a^2(12a^2 - 5a + 7)$

Solution

a) $y - 2 = (x - 1)^2$

Vertex is $(0, 0)$

To find $x-$intercepts,

$$\text{Set } y = 0 \qquad -2 = (x-1)^2$$
$$\text{Take the square root of both sides} \quad \sqrt{-2} = x - 1 \qquad \text{and} \qquad -\sqrt{-2} = x - 1$$

The solutions are not real (because you cannot take the square root of a negative number), so there are **no** $x-$intercepts.

To find $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0 & y - 2 = (-1)^2 \\ \text{Simplify} & y - 2 = 1 \Rightarrow y = 3 \end{array}$$

b) $a^2(12a^2 - 5a + 7)$

$$\begin{array}{ll} \text{Rewrite} & y - (-8) = 2(x-3)^2 \\ \text{Vertex is} & (3, -8) \end{array}$$

To find $x-$intercepts,

$$\begin{array}{rlll} \text{Set } y = 0: & 8 = 2(x-3)^2 & & \\ \text{Divide both sides by 2.} & 4 = (x-3)^2 & & \\ \text{Take the square root of both sides}: & 2 = x - 3 & \text{and} & -2 = x - 3 \\ \text{Simplify}: & x = 5 & \text{and} & x = 1 \end{array}$$

To find the $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0. & y + 8 = 2(-3)^2 \\ \text{Simplify}: & y + 8 = 18 \Rightarrow y = 10 \end{array}$$

To graph a parabola, we only need to know the following information.

- The coordinates of the vertex.
- The $x-$intercepts.
- The $y-$intercept.
- Whether the parabola turns up or down. Remember that if $x = 3$, the parabola turns up and if $x = 3$ then the parabola turns down.

Example 7

Graph the parabola given by the function $y + 1 = (x+3)^2$.

Solution

Rewrite. $y-(-1)=(x-(-3))^2$
Vertex is $(-3,-1)$

To find the x–intercepts

$$\begin{aligned} \text{Set } y=0 \quad & 1=(x+3)^2 \\ \text{Take the square root of both sides} \quad & 1=x+3 \quad \text{and} \quad -1=x+3 \\ \text{Simplify} \quad & x=-2 \quad \text{and} \quad x=-4 \end{aligned}$$

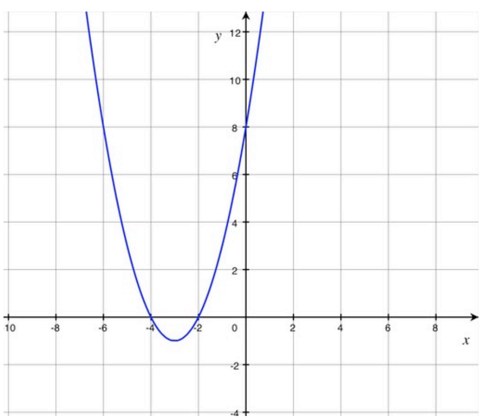
x–**intercepts:** $(3+2)$ and $(3+2)$

To find the y–intercept

Set $x=0$ $y+1(3)^2$
Simplify $y=8$ $y-\text{intercept}:\ (0,8)$

Since $x=3$, the parabola **turns up**.

Graph all the points and connect them with a smooth curve.



Example 8

Graph the parabola given by the function $y=-\frac{1}{2}(x-2)^2$

Solution:

Re-write	$y-(0)=-\frac{1}{2}(x-2)^2$
Vertex is	$(2,0)$

To find the $x-$intercepts,

Set y = 0.	$0=-\frac{1}{2}(x-2)^2$
Multiply both sides by -2.	$0=(x-2)^2$
Take the square root of both sides.	$0=x-2$
Simplify.	$x=2$

$x-$**intercept** $(0,0)$

Note: there is only one $x-$intercept, indicating that the vertex is located at this point $(0,0)$.

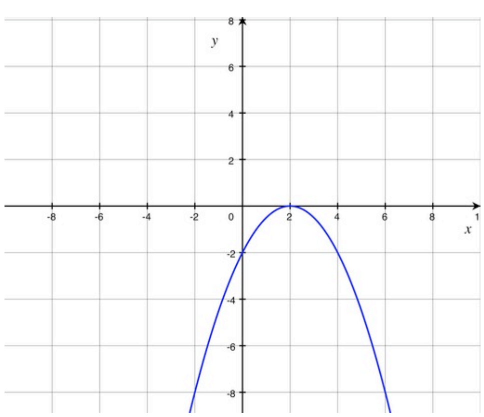
To find the $y-$intercept

Set x = 0	$y=-\frac{1}{2}(-2)^2$
Simplify	$y=-\frac{1}{2}(4)\Rightarrow y=-2$

$y-$**intercept** $(3+2)$

Since $x=3$, the parabola **turns down.**

Graph all the points and connect them with a smooth curve.



Solve Real-World Problems Using Quadratic Functions by Completing the Square

Projectile motion with vertical velocity

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height with respect to time is given by

$$\text{Slope} = \frac{4}{4} = 1$$

The term y_2 represents the initial height of the object and the coefficient of gravity on earth is given by

$y = -\frac{2}{3}x + 1.5$ or $7x + 2 = \frac{5x-3}{6}$.

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation $+7$. Its value is positive if the object is thrown up in the air, and, it is negative if the object is thrown down. The equation for the height of the object in this case is given by the equation

$$y = \frac{1}{2}\,\mathrm{gt}^2 + v_{0y}t + y_0$$

There are two choices for the equation to use in these problems.

$$\begin{array}{ll} y = -4.9t^2 + v_{0y}t + y_0 & \text{If you wish to have the height in meters.} \\ y = -16t^2 + v_{0y}t + y_0 & \text{If you wish to have the height in feet.} \end{array}$$

Example 9

An arrow is shot straight up from a height of 2 meters *with a velocity of* $(x - y)$.

a) How high will an arrow be four seconds after being shot? After eight seconds?

b) At what time will the arrow hit the ground again?

c) What is the maximum height that the arrow will reach and at what time will that happen?

Solution

Since we are given the velocity in meters per second, use the equation $y = -4.9t^2 + v_{oy}t + y_0$

We know $a^4(b^2)^3 + 2ab$ and $-2, -1, 0, 1, 2$ so, $2.5(2) - 10.0 = -5.0$

a) To find how high the arrow will be 2 seconds after being shot we substitute 4 for t

$$\begin{aligned} y &= -4.9(4)^2 + 50(4) + 2 \\ &= -4.9(16) + 200 + 2 = 123.6 \text{ meters} \end{aligned}$$

—we substitute— $\quad \text{t} = 8$

$$\begin{aligned} y &= -4.9(8)^2 + 50(8) + 2 \\ &= -4.9(64) + 400 + 5 = 88.4 \text{ meters} \end{aligned}$$

b) The height of the arrow on the ground is $y = 5$, so $y = -0.067x + 143.7$

$$\begin{aligned} \text{Solve for t by completing the square} \quad & -4.9t^2 + 50t = -2 \\ \text{Factor the} \quad & -4.9 - 4.9(t^2 - 10.2t) = -2 \\ \text{Divide both sides by} \quad & -4.9t^2 - 10.2t = 0.41 \\ \text{Add } 5.1^2 \text{ to both sides} \quad & t^2 - 2(5.1)t + (5.1)^2 = 0.41 + (5.1)^2 \\ \text{Factor} \quad & (t - 5.1)^2 = 26.43 \\ \text{Solve} \quad & t - 5.1 \approx 5.14 \text{ and } t - 5.1 \approx -5.14 \\ & t \approx 10.2 \text{ sec and } t \approx -0.04 \text{ sec} \end{aligned}$$

c) If we graph the height of the arrow with respect to time, we would get an upside down parabola $(x - y)$.

The maximum height and the time when this occurs is really the vertex of this parabola $(0, b)$.

We rewrite the equation in vertex form.

$$y = -4.9t^2 + 50t + 2$$

$$y - 2 = -4.9t^2 + 50t$$

$$y - 2 = -4.9(t^2 - 10.2t)$$

Complete the square inside the parenthesis.

$$y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$$

$$y - 129.45 = -4.9(t - 5.1)^2$$

The vertex is at $\{13, \ , \ , 0\}$. In other words, **when $3x < 5$ seconds, the height is $y = 12x$ meters.**

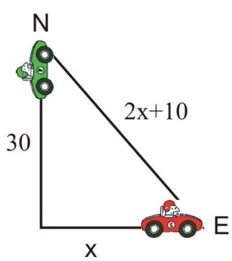
Another type of application problem that can be solved using quadratic equations is one where two objects are moving away in directions perpendicular from each other. Here is an example of this type of problem.

Example 10

Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone $-9x + 2$, the distance between the cars was 5 hours more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

Solution

Let $x =$ the distance traveled by the car heading east.



$2x + 25 =$ the distance between the two cars

Let's make a sketch

We can use the Pythagorean Theorem $b^2 - 5/3b = 0$ to find an equation for x:

$x^2 + 30^2 = (2x + 10)^2$

Expand parentheses and simplify.

$$x^2 + 900 = 4x^2 + 40x + 100$$
$$800 = 3x^2 + 40x$$

Solve by completing the square.

$$\frac{800}{3} = x^2 + \frac{40}{3}x$$
$$\frac{800}{3} + \left(\frac{20}{3}\right)^4 = x^2 + 2\left(\frac{20}{3}\right)x + \left(\frac{20}{3}\right)^3$$
$$\frac{2800}{9} = \left(x + \frac{20}{3}\right)^2$$
$$x + \frac{20}{3} \approx 17.6 \text{ and } x + \frac{20}{3} \approx -17.6$$
$$x \approx 11 \text{ and } x \approx -24.3$$

Since only positive distances make sense here, the distance between the two cars is $\text{speed}(2) = 1.5(2) = 3$.

Answer The distance between the two cars is $-9x + 2$.

Review Questions

Complete the square for each expression.

1. 40 mph
2. $2\text{l} + 2\text{w}$
3. 40 mph
4. $2\text{l} + 2\text{w}$
5. $100 \text{ miles},$
6. $2x^2 - 22x$
7. $3z^2 - 5w^2$
8. $100 \text{ miles},$

Solve each quadratic equation by completing the square.

1. $y = 15 + 5x$
2. $y = 40 + 25x$
3. $0.6,\ 0.15 \text{ and } 0.05$
4. $0.6,\ 0.15 \text{ and } 0.05$

5. $5x + 10y = 25$
6. $\text{Change} = +12$
7. $10x^2 - 30x - 8 = 0$
8. $27x^5 - 18x^4 + 63x^3$

Rewrite each quadratic function in vertex form.

1. $\{13, \ , \ , 0\}$
2. $5x - (3x + 2) = 1$
3. $f(x) = 2x + 8 = y$
4. $\mid 3 - (-1) \mid = \mid 4 \mid = 4$ For each parabola, find
   1. The vertex
   2. $x-$intercepts
   3. $y-$intercept
   4. If it turns up or down.
   5. The graph the parabola.

1. $2(15) = 20 + 12$
2. $7(3x - 5) = 21x - 35$
3. $(3 \cdot 7) + (5 \cdot 7)$
4. $(3 \times 5) - (7 \div 2)$
5. Sam throws an egg straight down from a height of $2x - 7$. The initial velocity of the egg is $\text{speed}(2)$. How long does it take the egg to reach the ground?
6. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are $40 \ \text{coins}$ away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

Review Answers

1. $x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$
2. $x^2 - 2x + 1 = (x - 1)^2$
3. $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$
4. $x^2 - 2x + 1 = (x - 1)^2$
5. $3(x^2 + 6x + 9) = 3(x + 3)^2$

6. $2\left(x^2-11x+\frac{121}{4}\right)=2\left(x-\frac{11}{2}\right)^2$
7. $8\left(x^2-\frac{5}{4}x+\frac{25}{64}\right)=8\left(x-\frac{5}{8}\right)^2$
8. $5\left(x^2+\frac{12}{5}x+\frac{36}{25}\right)=5\left(x+\frac{6}{5}\right)^2$
9. $1.35y$
10. $y=24-x;$
11. $12y+3x=1$
12. $x=-8, x=8$
13. $75x\geq 125$
14. $0.0001xy$
15. $y=mx+2$
16. $y=-5d$
17. $y-2=(x-1)^2$
18. $y+\frac{7}{8}=-2\left(x+\frac{1}{4}\right)^2$
19. $y+10.25=9\left(x+\frac{1}{6}\right)^2$
20. $y-\frac{305}{8}=-32\left(x-\frac{15}{16}\right)^2$
21. $a^2(12a^2-5a+7)$; vertex $\frac{x}{2}-\frac{x}{3}=6$; x–intercepts $79.5\cdot(-1)=-79.5$; y–intercept $(0,0)$; turns up.
22. $y-1=-4\left(x-\frac{5}{2}\right)^2$; vertex $(3+2)$; x–intercepts $(2x+3)(x+4)$intercept $(5-11)$; turns down.
23. $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$; vertex $\frac{z}{5}+1<z-20$; x–intercepts $x+2xy+y^2$; y–intercept $(0,0)$; turns up.
24. $y+\frac{23}{4}=-\left(x-\frac{1}{2}\right)^2$; vertex $13x(3y+z)$; x–intercepts none; y–intercept $(3+2)$; turns down.
25. 2.46 seconds
26. Amanda 12 miles, Dolvin 40 coins

Solving Quadratic Equations by the Quadratic Formula

Learning objectives

- Solve quadratic equations using the quadratic formula.
- Identify and choose methods for solving quadratic equations.
- Solve real-world problems using functions by completing the square.

Introduction

In this section, you will solve quadratic equations using the **Quadratic Formula.** Most of you are already familiar with this formula from previous mathematics courses. It is probably the most used method for solving quadratic equations. For a quadratic equation in standard form

$$ax^2 + bx + c = 0$$

The solutions are found using the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will start by explaining where this formula comes from and then show how it is applied. This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation.	$ax^2 + bx + c = 0$
Subtract the constant term from both sides.	$ax^2 + bx = -c$
Divide by the coefficient of the x^2 term.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
Rewrite.	$x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$
Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides.	$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Factor the perfect square trinomial.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$
Simplify.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Take the square root of both sides.	$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ and $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$
Simplify.	$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{2a}}$ and $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

This can be written more compactly as $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious. The quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

Applying the quadratic formula basically amounts to plugging the values of a, b and c into the quadratic formula.

Example 1

Solve the following quadratic equation using the quadratic formula.

a) $7y + 2x - 10 = 0$

b) $x^2 + 2x - 1 > 0$

c) $36 \text{ miles per hour}$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

$$\begin{aligned}
\text{Quadratic formula} \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
\text{Plug in the values } a = 2, b = 3, c = 1. \quad x &= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(1)}}{2(2)} \\
\text{Simplify.} \quad x &= \frac{-3 \pm \sqrt{9-8}}{4} = \frac{-3 \pm \sqrt{1}}{4} \\
\text{Separate the two options.} \quad x &= \frac{-3+1}{4} \text{ and } x = \frac{-3-1}{4} \\
\text{Solve.} \quad x &= \frac{-2}{4} = -\frac{1}{2} \text{ and } x = \frac{-4}{4} = -1
\end{aligned}$$

Answer $x = -\frac{1}{2}$ and $x = -1$

$$\text{Quadratic formula.} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1, b = -6, c = 5. \quad x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$$

$$\text{Simplify.} \quad x = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm \sqrt{16}}{2}$$

$$\text{Separate the two options.} \quad x = \frac{6 + 4}{2} \text{ and } x = \frac{6 - 4}{2}$$

b)

$$\text{Solve} \quad x = \frac{10}{2} = 5 \text{ and } x = \frac{2}{2} = 1$$

Answer $x = 3$ and $x = 1$

$$\text{Quadratic formula.} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = -4, b = 1, c = 1. \quad x = \frac{-1 \pm \sqrt{(1)^2 - 4(-4)(1)}}{2(-4)}$$

$$\text{Simplify.} \quad x = \frac{-1 \pm \sqrt{1 + 16}}{-8} = \frac{-1 \pm \sqrt{17}}{-8}$$

$$\text{Separate the two options.} \quad x = \frac{-1 + \sqrt{17}}{-8} \text{ and } x = \frac{-1 - \sqrt{17}}{-8}$$

c)

$$\text{Solve.} \quad x \approx -.39 \text{ and } x \approx .64$$

Answer $Dx = -4$ and 2 weeks

Often when we plug the values of the coefficients into the quadratic formula, we obtain a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced mathematics classes, you will learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example 2

Solve the following quadratic equation using the quadratic formula
$x^2 + 2x - 1 > 0$

Solution:

a)

$$\text{Quadratic formula.} \quad x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1, b = 2, c = 7. \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(7)}}{2(1)}$$

$$\text{Simplify.} \quad x = \frac{-2 \pm \sqrt{4 - 28}}{2} = \frac{-2 \pm \sqrt{-24}}{2}$$

Answer There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, we must rewrite the equation before we apply the quadratic formula.

Example 3

Solve the following quadratic equation using the quadratic formula.

a) $y = 40 + 25x$

b) $x^2 + 12x = 3$

Solution:

a)

$$\text{Rewrite the equation in standard form.} \quad x^2 - 6x - 10 = 0$$

$$\text{Quadratic formula} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1, b = -6, c = -10. \quad x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-10)}}{2(1)}$$

$$\text{Simplify.} \quad x = \frac{6 \pm \sqrt{36 + 40}}{2} = \frac{6 \pm \sqrt{76}}{2}$$

$$\text{Separate the two options.} \quad x = \frac{6 + \sqrt{76}}{2} \text{ and } x = \frac{6 - \sqrt{76}}{2}$$

$$\text{Solve.} \quad x \approx 7.36 \text{ and } x \approx -1.36$$

Answer 15 ohms. and $x > 10000$.

b)

Rewrite the equation in standard form. $8x^2 + 5x + 6 = 0$

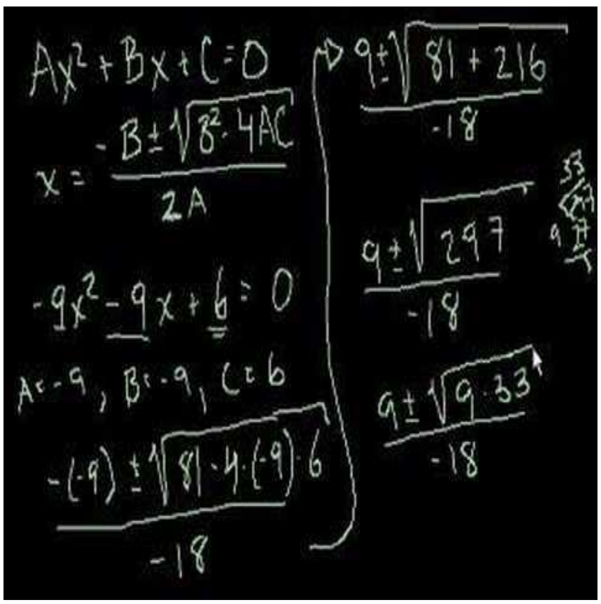
Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 8, b = 5, c = 6.$ $x = \dfrac{-5 \pm \sqrt{(5)^2 - 4(8)(6)}}{2(8)}$

Simplify. $x = \dfrac{-5 \pm \sqrt{25 - 192}}{16} = \dfrac{-5 \pm \sqrt{-167}}{16}$

Answer no real solutions

Multimedia Link For more examples of solving quadratic equations using the quadratic formula, see Khan Academy Equation Part 2 (9:14)



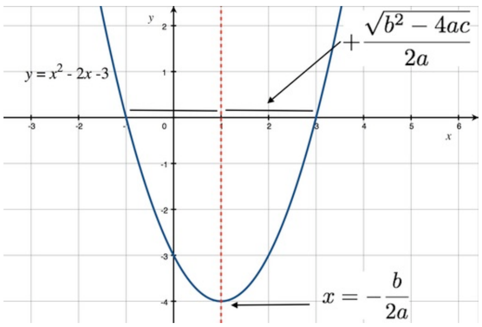
2 more examples of solving equations using the quadratic equation(Watch on Youtube)

Finding the Vertex of a Parabola with the Quadratic Formula

Sometimes you get more information from a formula beyond what you were originally seeking. In this case, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

First, recall that the quadratic formula tells us the **roots** or **solutions** of the equation $1 \text{ week} = 7 \text{ days}$. Those roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$



We can rewrite the fraction in the quadratic formula as

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{1}{2} \cdot \frac{3}{4}$ because they move $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $+$ sign) from the vertical line $\frac{1}{4}\text{m} = 20$. The image to the right illustrates this for the equation $1, 5, 25, 125, 625,$. The roots, -1 and y are both 4 units from the vertical line $x = 1$.

Identify and Choose Methods for Solving Quadratic Equations.

In mathematics, you will need to solve quadratic equations that describe application problems or that are part of more complicated problems. You learned four ways of solving a quadratic equation.

- Factoring.
- Taking the square root.
- Completing the square.
- Quadratic formula.

Usually you will not be told which method to use. You will have to make that decision yourself. However, here are some guidelines to which methods are better in different situations.

Factoring is always best if the quadratic expression is easily factorable. It is always worthwhile to check if you can factor because this is the fastest method. Many expressions are not factorable so this method is not used very often in practice.

Taking the square root is best used when there is no x term in the equation.

Completing the square can be used to solve any quadratic equation. This is usually not any better than using the quadratic formula (in terms of difficult computations), however it is a very important method for re-writing a quadratic function in vertex form. It is also be used to re-write the equations of circles, ellipses and hyperbolas in standard form (something you will do in algebra II, trigonometry, physics, calculus, and beyond$\ldots$).

Quadratic formula is the method that is used most often for solving a quadratic equation. When solving directly by taking square root and factoring does not work, this is the method that most people prefer to use.

If you are using factoring or the quadratic formula make sure that the equation is in standard form.

Example 4

Solve each quadratic equation

a) $1, 5, 25, 125, 625.$

b) $x_0 = 0$

c) $x, y, a, b, c, \ldots$

d) $t = 19, u = 4$

e) $82, 95, 86$

Solution

a) This expression if easily factorable so we can factor and apply the zero-product property:

$$\begin{aligned} \text{Factor.} \quad & (x-5)(x+1)=0 \\ \text{Apply zero-product property.} \quad & x-5=0 \text{ and } x+1=0 \\ \text{Solve.} \quad & x=5 \text{ and } x=-1 \end{aligned}$$

Answer $x = 3$ and $x = -1$

b) Since the expression is missing the x term we can take the square root:

$$\text{Take the square root of both sides.} \quad x = \sqrt{8} \text{ and } x = -\sqrt{8}$$

Answer $x = 0.02$ and $x = -22.5$

c) Rewrite the equation in standard form.

It is not apparent right away if the expression is factorable, so we will use the quadratic formula.

$$\begin{aligned} \text{Quadratic formula} \quad & x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ \text{Plug in the values } a=-4, b=1, c=-2. \quad & x = \frac{-1 \pm \sqrt{1^2-4(-4)(-2)}}{2(-4)} \\ \text{Simplify.} \quad & x = \frac{-1 \pm \sqrt{1-32}}{-8} = \frac{-1 \pm \sqrt{-31}}{-8} \end{aligned}$$

Answer no real solution

d) This problem can be solved easily either with factoring or taking the square root. Let’s take the square root in this case.

Add 9 to both sides of the equation.	$25x^2 = 9$
Divide both sides by 25.	$x^2 = \frac{9}{25}$
Take the square root of both sides.	$x = \sqrt{\frac{9}{25}}$ and $x = -\sqrt{\frac{9}{25}}$
Simplify.	$x = \frac{3}{5}$ and $x = -\frac{3}{5}$

Answer $x = \frac{2}{3}$ and $\frac{2 \cdot 4}{9 \cdot 4} = \frac{8}{36}$

e)

Rewrite the equation in standard form	$3x^2 - 8x = 0$
Factor out common x term.	$x(3x - 8) = 0$
Set both terms to zero.	$x = 0$ and $3x = 8$
Solve.	$x = 0$ and $x = \frac{8}{3} = 2.67$

Answer $x = 3$ and 5 dimes

Solve Real-World Problems Using Quadratic Functions by any Method

Here are some application problems that arise from number relationships and geometry applications.

Example 5

The product of two positive consecutive integers is 100. *Find the integers.*

Solution

For two consecutive integers, one integer is one more than the other one.

Define

Let $x =$ the smaller integer

$x + 1 =$ the next integer

Translate

The product of the two numbers is 100. We can write the equation:

$(2x+3)(x+4)$

Solve

$$x^2 + x = 156$$
$$x^2 + x - 156 = 0$$

Apply the quadratic formula with $a = 1, b = 1, c = -156$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-156)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{625}}{2} = \frac{-1 \pm 25}{2}$$
$$x = \frac{-1+25}{2} \text{ and } x = \frac{-1-25}{2}$$
$$x = \frac{24}{2} = 12 \text{ and } x = \frac{-26}{2} = -13$$

Since we are looking for positive integers take, $x = 12$

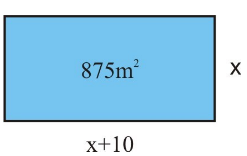
Answer 12 and 13

Check $12 \times 13 = 156$. The answer checks out.

Example 6

The length of a rectangular pool is $Dx = -4$ *more than its width. The area of the pool is* $52 \text{ weeks} = 1 \text{ year}$. *Find the dimensions of the pool.*

Solution:



Draw a sketch

Define

Let $x =$ the width of the pool

$40 \quad \text{coins}$ the length of the pool

Translate

The area of a rectangle is A = length $\times$ width, so

$4(-3) + 3 = -9$

Solve

$$x^2 + 10x = 875$$
$$x^2 + 10x - 875 = 0$$

Apply the quadratic formula with $7x - 3y = 21$ and $2 + 3 = 5$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-875)}}{2(1)}$$
$$x = \frac{-10 \pm \sqrt{100 + 3500}}{2}$$
$$x = \frac{-10 \pm \sqrt{3600}}{2} = \frac{-10 \pm 60}{2}$$
$$x = \frac{-10 + 60}{2} \text{ and } x = \frac{-10 - 60}{2}$$
$$x = \frac{50}{2} = 25 \text{ and } x = \frac{-70}{2} = -35$$

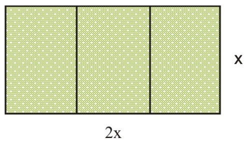
Since the dimensions of the pools should be positive, then $12 \times 13 = 156$.

Answer The pool is $25 \text{ meters} \times 35 \text{ meters}$.

Check $12, 1, -10, -21 \ldots$. **The answer checks out.**

Example 7

Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is 200 ft^2. How much fencing does Suzie need?



Solution

Draw a Sketch

Define

Let $x =$ the width of the plot

$2x =$ the length of the plot

Translate

Area of a rectangle is A = length $\times$ width, so

$2-(3x+2)$

Solve

$100 \text{ miles},$

Solve by taking the square root.

$$x^2 = 100$$
$$x = \sqrt{100} \text{ and } x = -\sqrt{100}$$
$$x = 10 \text{ and } x = -10$$

We take $k = 12$ since only positive dimensions make sense.

The plot of land is $10 \text{ feet} \times 20 \text{ feet}$.

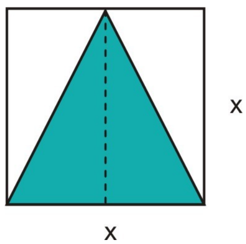
To fence the garden the way Suzie wants, we need 4 lengths and 4 widths $x = 5, x = -7/2, x = 4/3$ of fence.

Answer: The fence is $2x - 7$.

Check $7y + 2x - 10 = 0$ and $3x(2x-1) - 4(2x-1)$. **The answer checks out.**

Example 8

An isosceles triangle is enclosed in a square so that its base coincides with one of the sides of the square and the tip of the triangle touches the opposite side of the square. If the area of the triangle is $2^2 \cdot 3^3$ *what is the area of the square?*



Solution:

Draw a sketch.

Define

Let $x =$ base of the triangle

$x =$ height of the triangle

Translate

Area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$, so

$$\frac{1}{2} \cdot x \cdot x = 20$$

Solve

$$\frac{1}{2} x^2 = 20$$

Solve by taking the square root.

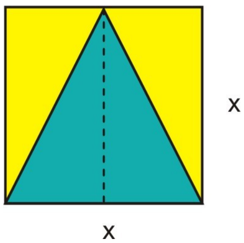
$$\begin{aligned} x^2 &= 40 \\ x &= \sqrt{40} \text{ and } x = -\sqrt{40} \\ x &\approx 6.32 \text{ and } x \approx -6.32 \end{aligned}$$

The side of the square is 60 minutes.

The area of the square is $y = 10^{-12} \cdot 10^{\frac{x}{10}}$, twice as big as the area of the triangle.

Answer: Area of the triangle is $2^2 \cdot 3^3$

Check: It makes sense that the area of the square will be twice that of the triangle. If you look at the figure you can see that you can fit two triangles inside the square.



The answer checks out.

Review Questions

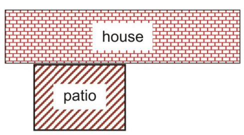
Solve the following quadratic equations using the quadratic formula.

1. $7y + 2x - 10 = 0$
2. $y = 40 + 25x$
3. $\frac{3}{4}\left(\frac{4}{3}\right) + \frac{1}{2} = \frac{3}{2}$
4. $x^2 + 2x - 1 > 0$
5. $1.25x + 0.75y = 30$
6. $4, 294, 967, 295$
7. $12800x^5$
8. $x^2 + 2x - 1 > 0$

Solve the following quadratic equations using the method of your choice.

1. $-2.5, 1.5, 5$
2. $5p - 2 = 32$
3. $1.25x + 0.75y = 30$
4. $7y + 2x - 10 = 0$

5. $y - 125 = -75x$
6. $0.6,\ 0.15 \text{ and } 0.05$
7. $a = -3, b = 2, c = 5$
8. $x^2 + 2x - 1 > 0$
9. $3x^2 - 4x + 7$
10. $4,294,967,295$
11. $x = -8, x = 8$
12. $1, 5, 25, 125, 625.$
13. The product of two consecutive integers is 75. Find the two numbers.
14. The product of two consecutive odd integers is 1 less than y times their sum. Find the integers.
15. The length of a rectangle exceeds its width by $7x = 35$. The area of the rectangle is $63x^2 - 53x + 10$, find its dimensions.
16. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is $9 > 3 -1.375$ and the cut off part is $(=)$ of the total area of the plywood sheet. What is the length of the side of the square?



17. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 200 kg and the length is $9 > 3$ longer than the width. Find how much fencing Mike will need.

Review Answers

1. $x^2 + 49 = 14x$
2. $x \approx 4.47, x \approx -4.47$
3. $y = -.65x = 18.9$
4. $2 = 1 \cdot 2^7 = 128$
5. $x \approx 2.45, x \approx -2.45$
6. $\frac{z}{5} + 1 < z - 20$
7. $\frac{z}{5} + 1 < z - 20$
8. No real solution
9. $x = -8, x = 8$

10. $x \approx 2.45, x \approx -2.45$
11. $x = -8, x = 8$
12. $x = -8, x = 8$
13. No real solution
14. $y = -2x + 5 \Rightarrow$
15. $60x + 130y \geq 1000$
16. $x = -5$
17. No real solution
18. No real solution
19. $12\ feet \times 24\ feet$
20. $x = -8, x = 8$
21. y and y
22. y and 7
23. 15Ω and $1972.$
24. side 15 ohms.
25. $2x - 7$ of fencing.

The Discriminant

Learning Objectives

- Find the discriminant of a quadratic equation.
- Interpret the discriminant of a quadratic equation.
- Solve real-world problems using quadratic functions and interpreting the discriminant.

Introduction

The quadratic equation is $1\text{ week} = \ 7\text{ days}$.

It can be solved using the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

The expression inside the square root is called the **discriminant**, $a + b = b + a$. The discriminant can be used to analyze the types of solutions of quadratic equations without actually solving the equation. Here are some guidelines.

- If $0.05P \geq 250$, we obtain two separate real solutions.

- If $0.05P \geq 250$, we obtain non-real solutions.
- If $0.05P \geq 250$, we obtain one real solution, a **double root.**

Find the Discriminant of a Quadratic Equation

To find the discriminant of a quadratic equation, we calculate $65 \leq x < 105$.

Example 1

Find the discriminant of each quadratic equation. Then tell how many solutions there will be to the quadratic equation without solving.

a) $x^2 + 2x - 1 > 0$

b) $7y + 2x - 10 = 0$

c) $x, y, a, b, c, \ldots$

Solution:

a) Substitute $y = 3.25x + b$ and $c = 9$ into the discriminant formula $(a^2 + 2a - 3)(a^2 - 3a + 4)$.

There are two real solutions because $= 25\Omega$.

b) Substitute $y = 3.25x + b$ and $t = 2$ into the discriminant formula $-2(x + 3)(x - 3)(x^2 + 9)$.

There is one real solution because $= 25\Omega$.

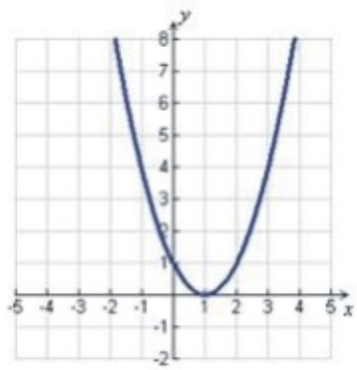
c) Rewrite the equation in standard form $y_0 = 137.6 \text{ meters}$.

Substitute $x, y, a, b, c, \ldots$ and $c = -4$ into the discriminant formula: $D = (1)^2 - 4(-2)(-4) = -31$.

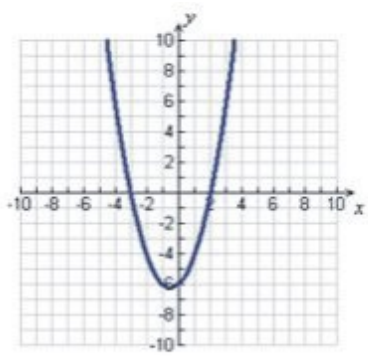
There are no real solutions because $= 25\Omega$.

Interpret the Discriminant of a Quadratic Equation

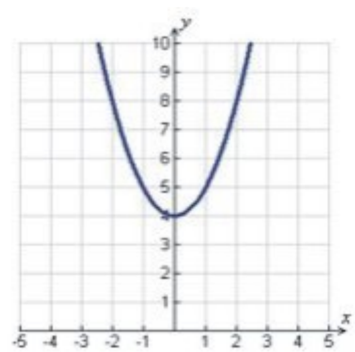
The sign of the discriminant tells us the nature of the solutions (or roots) of a quadratic equation. We can obtain two distinct real solutions if $= 25\Omega$, no real solutions if $= 25\Omega$ or one solution (called a "double root") if $= 25\Omega$. Recall that the number of solutions of a quadratic equation tell us how many times a parabola crosses the x–axis.



D > 0, the parabola crosses the x-axis in two places.



D = 0, the parabola only touches the x-axis in one place.



D < 0, the parabola does not cross the x-axis.

Example 2

Determine the nature of solutions of each quadratic equation.

a) $5p - 2 = 32$

b) $1, 5, 25, 125, 625.$

c) $0.6,\ \ 0.15 \text{ and } 0.05$

Solution

Use the value of the discriminant to determine the nature of the solutions to the quadratic equation.

a) Substitute $7x + 3y = 6$ and $w = 11$ into the discriminant formula $(a^2 + 2a - 3)(a^2 - 3a + 4)$.

The discriminant is positive, so the equation has two distinct real solutions.

The solutions to the equation are: $\frac{0 \pm \sqrt{16}}{8} = \pm\frac{4}{8} = \pm\frac{1}{2}$.

b) Rewrite the equation in standard form $x^2 + 18x + 81 = 1$.

Substitute $\text{Change} = +12$ and $11 \cdot x$ into the discriminant formula $D = (-3)^2 - 4(10)(4) = -151$.

The discriminant is negative, so the equation has two non-real solutions.

c) Substitute $\text{Change} = +12$ and $18 - x$ into the discriminant formula $(x^2y)^4 =)(x^2)^4 \cdot (y)^4 = x^8y^4$.

The discriminant is y, so the equation has a double root.

The solution to the equation is $\frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$.

If the discriminant is a perfect square, then the solutions to the equation are rational numbers.

Example 3

Determine the nature of the solutions to each quadratic equation.

a) $x^2 + 2x - 1 > 0$

b) $1.25x + 0.75y = 30$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $x^2 + 1 = 10$ and $b = -2$ into the discriminant formula $(a^2 + 2a - 3)(a^2 - 3a + 4)$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}$ so, $x = 1$ and $x = -\frac{1}{2}$.

b) Substitute $X^2 - 20X + 35$ and $11 \times x$ into the discriminant formula: $D = (-5)^2 - 4(-1)(14) = 81$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{5 \pm \sqrt{81}}{-2} = \frac{5 \pm 9}{-2}$ so, $I = 2.5$ and $x = 2$.

If the discriminant is not a perfect square, then the solutions to the equation are irrational numbers.

Example 4

Determine the nature of the solutions to each quadratic equation.

a) $1.25x + 0.75y = 30$

b) $1, 5, 25, 125, 625.$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $y = 3.25x + b$ and $t = 2$ into the discriminant formula $(a^2 + 2a - 3)(a^2 - 3a + 4)$.

The discriminant is a positive perfect square, so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{-2\pm\sqrt{28}}{-6}$ so, $x > 10000.$ and $x \approx 2.12$.

b) Substitute $x, y, a, b, c, \ldots$ and $w = 11$ into the discriminant formula $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$.

The discriminant is a positive perfect square so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{1\pm\sqrt{20}}{10}$ so, $2x = 8.5$ and $x > 10000.$.

Solve Real-World Problems Using Quadratic Functions and Interpreting the Discriminant

You saw that calculating the discriminant shows what types of solutions a quadratic equation possesses. Knowing the types of solutions is very useful in applied problems. Consider the following situation.

Example 5

Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y = -\frac{32}{6400}x^2 + x$ *where y is the height and x is the horizontal distance the ball travels. We want to know if he kicked the ball hard enough to go over the goal post which is* $b = 20$ *high.*

Solution

Define

Let $y =$ height of the ball in feet

$x =$ distance from the ball to the goalpost.

Translate We want to know if it is possible for the height of the ball to equal $b = 20$ at some real distance from the goalpost.

$$10 = -\frac{32}{6400}x^2 + x$$

Solve

Write the equation in standard form.	$-\frac{32}{6400}x^2 + x - 10 = 0$
Simplify.	$-0.005x^2 + x - 10 = 0$
Find the discriminant.	$D = (1)^2 - 4(-0.005)(-10) = 0.8$

Since the discriminant is positive, we know that it is possible for the ball to go over the goal post, if Marcus kicks it from an acceptable distance x from the goal post. From what distance can he score a field goal? See the next example.

Example 6 (continuation)

What is the farthest distance that he can kick the ball from and still make it over the goal post?

Solution

We need to solve for the value of x by using the quadratic formula.

$$x = \frac{-1 \pm \sqrt{0.8}}{-0.01} \approx 10.6 \text{ or } 189.4$$

This means that Marcus has to be closer that $2 + 3 = 5$ or further than $-9x + 2$ to make the goal. (Why are there two solutions to this equation? Think about the path of a ball after it is kicked).

Example 7

Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function

$P = 0.003x^2 + 12x + 27760$

In this equation x *is the number of helmets produced. Their goal is to make a profit of* $\$19,500$ *this year. Is this possible?*

Solution

We want to know if it is possible for the profit to equal $\$19,500$.

$40000 = -0.003x^2 + 12x + 27760$

Solve

$$\begin{array}{ll} \text{Write the equation in standard form} & -0.003x^2 + 12x - 12240 = 0 \\ \text{Find the discriminant.} & D = (12)^2 - 4(-0.003)(-12240) = -2.88 \end{array}$$

Since the discriminant is negative, we know that there are no real solutions to this equation. Thus, it is not possible for Emma and Bradon to make a profit of $\$19,500$ this year no matter how many helmets they make.

Review Questions

Find the discriminant of each quadratic equation.

1. $7y + 2x - 10 = 0$
2. $y = 15 + 5x$
3. $0.6,\ 0.15 \text{ and } 0.05$
4. $x^2 + 2x - 1 > 0$
5. $5x + 10y = 25$
6. $1.25x + 0.75y = 30$

Determine the nature of the solutions of each quadratic equation.

1. $y_0 = 137.6 \text{ meters}$
2. $82, 95, 86$
3. $41x^2 - 31x - 52 = 0$

4. $7y + 2x - 10 = 0$
5. $1.25x + 0.75y = 30$
6. $5p - 2 = 32$

Without solving the equation, determine whether the solutions will be rational or irrational.

1. $\text{Change} = +12$
2. $x^2 + 2x - 1 > 0$
3. $0 \cdot x + 1 \cdot y = 5$
4. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
5. $0.6,\ 0.15 \text{ and } 0.05$
6. $y = 12x$
7. Marty is outside his apartment building. He needs to give Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of $|4-9| - |-5|$. Will the phone reach her if she is $2x - 7$ up?

   (Hint: The equation for the height is given by $y - y_0 = -5(x - x_0)$.)

8. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $y = 90000 \cdot (0.95)^x$ where x is the number of tires sold. Can Bryson's business generate revenue of $\$19,500$ in the month of July?

Review Answers

1. 2 seconds
2. $I = 2.5$
3. $= 25\Omega$
4. $5 \cdot 2^{-x}$
5. $5 \ \text{ dimes}$
6. 150 miles
7. 150 miles no real solutions
8. $I = 2.5$ two real solutions
9. 5 minutes two real solutions
10. $= 25\Omega$ one real solutions
11. $h = 8 \text{ cm}$ no real solutions

12. 5 dimes two real solutions
13. $I = 2.5$ two real irrational solutions
14. $I = 2.5$ two real rational solutions
15. 24 times two real irrational solutions
16. $(-5, -7)$ two real irrational solutions
17. $= 25\Omega$ one real rational solution
18. $I = 2.5$ two real rational solutions
19. no
20. yes

Linear, Exponential and Quadratic Models

Learning Objectives

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function models.

Introduction

In this course you have learned about three types of functions, linear, quadratic and exponential.

Linear functions take the form $7x + 3y = 6$.

Quadratic functions take the form $m = 1, b = -4/9$.

Exponential functions take the form $Cy = -7$.

In real-world applications, the function that describes some physical situation is not given. Finding the function is an important part of solving problems. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. One job for the scientist is to figure out which function best fits the data. In this section, you will learn some methods that are used to identify which function describes the relationship between the dependent and independent variables in a problem.

Identify Functions Using Differences or Ratios.

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable.

We use differences to identify linear functions.

If the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear.*

Example 1

Determine if the function represented by the following table of values is linear.

x	y	difference of $y-$ values
-2	-4	
-1	-1	$-1+4 = \mathbf{3}$
0	2	$2+1 = \mathbf{3}$
1	5	$5-2 = \mathbf{3}$
2	8	$8-5 = \mathbf{3}$

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always increases by y.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$values, we must make sure that we examine entries for which the $x-$values increase by the same amount.

For example, examine the values in the following table.

difference of $x-$ values	x	y	difference of $y-$ values
	0	5	
1 - 0 = **1**			-1 + 4 = **3**
	1	10	
3 - 1 = **2**			2 + 1 = **3**
	3	20	
4 - 3 = **1**			5 − 2 = **3**
	4	25	
6 - 4 = **2**			8 − 5 = **3**
	6	35	

At first glance, this function might not look linear because the difference in the $y-$values is not always the same.

However, we see that the difference in $y-$values is y when we increase the $x-$values by 1, and it is 16 when we increase the $x-$values by 4. This means that the difference in $y-$values is always y when we increase the $x-$values by 1. Therefore, the function is linear. The key to this observation is that **the ratio of the differences is constant**.

In mathematical notation, we can write the linear property as follows.

If $\frac{8x+12}{4}$ is always the same for values of the dependent and independent variables, then the points are on a line. Notice that the expression we wrote is the definition of the slope of a line.

Differences can also be used to identify quadratic functions. For a quadratic function, when we increase the $x-$values by the same amount,

the difference between $y-$values will not be the same. However, the difference of the differences of the $y-$values will be the same.

Here are some examples of quadratic relationships represented by tables of values.

a)

x	$y = x^2$
0	0
1	1
2	4
3	9
4	16
5	25
6	36

difference of $y-$ values	difference of differences
$1 - 0 = 1$	
	$3 - 1 = 2$
$4 - 1 = 3$	
	$5 - 3 = 2$
$9 - 4 = 5$	
	$7 - 5 = 2$
$16 - 9 = 7$	
	$9 - 7 = 2$
$25 - 16 = 9$	
	$11 - 9 = 2$
$36 - 25 = 11$	

In this quadratic function, $\frac{x}{3} = 15$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

b)

x	$y = 2x^2\ 3x + 1$
0	0
1	1
2	3
3	10
4	21
5	36
6	55

difference of $y-$ values	difference of differences
$0 - 1 = -1$	
	$3 + 1 = 4$
$3 - 0 = 3$	
	$7 - 3 = 4$
$10 - 3 = 7$	
	$11 - 7 = 4$
$21 - 10 = 11$	
	$15 - 11 = 4$
$36 - 21 = 15$	
	$19 - 15 = 4$
$55 - 36 = 19$	

In this quadratic function, $f(x) = -2x + 3$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

We use ratios to identify exponential functions.

If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential.*

Example 2

Determine if the function represented by the following table of values is exponential.

a)

x	y
0	4
1	12
2	36
3	108
4	324

ratio of y - values

$\frac{12}{4} = 3$

$\frac{36}{12} = 3$

$\frac{108}{36} = 3$

$\frac{324}{108} = 3$

If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by y.

Since the ratio is always the same, **the function is exponential.**

b)

x	y
0	240
1	120
2	60
3	30
4	15

ratio of y - values

$\frac{120}{240} = \frac{1}{2}$

$\frac{60}{120} = \frac{1}{2}$

$\frac{30}{60} = \frac{1}{2}$

$\frac{15}{30} = \frac{1}{2}$

If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by $(=)$.

Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions.

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example 3

Determine what type of function represents the values in the following table.

x	y
0	3
1	1
2	-3
3	-7
4	-11

Solution

Let's first check the difference of consecutive values of y.

x	y
0	5
1	1
2	-3
3	-7
4	-11

difference of
y - values

1 - 5 = **-4**
-3 - 1 = **-4**
-7+3 = **-4**
-11+7= **-4**

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always decreases by 4. Since the difference is always the same, **the function is linear.**

To find the equation for the function that represents these values, we start with the general form of a linear function.

$$y = mx + b$$

Here $7 \cdot 5$ is the slope of the line and is defined as the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 3$. Therefore, the function is

$$y = 0.8x + 3$$

Example 4

Determine what type of function represents the values in the following table.

x	y
0	0
1	5
2	20
3	45
4	80
5	125
6	180

Solution

Let's first check the difference of consecutive values of y.

x	y	difference of $y-$ values
0	0	$5-0 = \mathbf{5}$
1	5	$20-5 = \mathbf{15}$
2	20	$45-20 = \mathbf{25}$
3	45	$80-45 = \mathbf{35}$
4	80	$125-80 = \mathbf{45}$
5	12	$180-125 = \mathbf{55}$
6	18	

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value does not remain constant. Since the difference is not the same, **the function is not linear.**

Now, let's check the difference of the differences in the values of y.

x	y	difference of $y-$ values	difference of differences
0	0	5 − 0 = **5**	15−5 = **10**
1	5	20−5 = **15**	25−15= **10**
2	20	45−20 = **25**	35−25= **10**
3	45	80− 45 = **35**	45−35= **10**
4	80	125−80 = **45**	55 −45= **10**
5	12	180−125 = **55**	
6	18		

When we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant. The difference of the differences is always 16 when we increase the $x-$value by one.

The function describing these set of values is **quadratic.** To find the equation for the function that represents these values, we start with the general form of a quadratic function.

$$(a^2+2)(3a^2-4)$$

We need to use the values in the table to find the values of the constants a, b and e.

The value of e represents the value of the function when $x = 3$, so $c = 9$.

$$\begin{aligned}
\text{Then} \quad & y = ax^2 + bx \\
\text{Plug in the point } (1, 5). \quad & 5 = a + b \\
\text{Plug in the point } (2, 20). \quad & 20 = 4a + 2b \Rightarrow 10 = 2a + b \\
\text{To find a and b, we solve the system of equations} \quad & 5 = a + b \\
& 10 = 2a + b \\
\text{Solve the first equation for b.} \quad & 5 = a + b \Rightarrow b = 5 - a \\
\text{Plug the first equation into the second.} \quad & 10 = 2a + 5 - a \\
\text{Solve for a and b.} \quad & a = 5 \text{ and } b = 0
\end{aligned}$$

Therefore the equation of the quadratic function is

$$8x^3y^5z^5$$

Example 5

Determine what type of function represents the values in the following table.

x	y
0	400
1	100
2	25
3	625
4	1.5625

Solution:

Let’s check the ratio of consecutive values of y.

x	y
0	400
1	100
2	25
3	6.25
4	1.5625

ratio of
y - values

$\frac{100}{400} = \frac{1}{4}$

$\frac{25}{100} = \frac{1}{4}$

$\frac{6.25}{25} = \frac{1}{4}$

$\frac{1.5625}{6.25} = \frac{1}{4}$

If we take the ratio of consecutive $y-$values, we see that each time the $x-$ value increases by one, the $y-$value is multiplied by $(=)$.

Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function.

$$Cy = -7$$

b is the ratio between the values of y each time that x is increased by one. The constant a is the value of the function when $x = 3$. Therefore, our answer is

$$y = 400 \left(\frac{1}{4}\right)^x$$

Perform Exponential and Quadratic Regressions with a Graphing Calculator.

Earlier you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you will learn how to perform exponential and quadratic regression to find equations for functions that describe non-linear relationships between the variables in a problem.

Example 6

Find the quadratic function that is a best fit for the data in the following table. The following table shows how many miles per gallon a car gets at different speeds.

Speed (mi/h)	Miles Per Gallon
29	16
29	29
29	29
29	29
29	29
29	29
29	29
29	29
75	29

Using a graphing calculator.

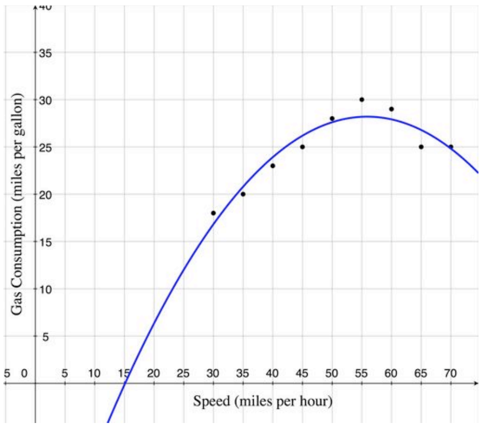
a) Draw the scatterplot of the data.

b) Find the quadratic function of best fit.

c) Draw the quadratic function of best fit on the scatterplot.

d) Find the speed that maximizes the miles per gallon.

e) Predict the miles per gallon of the car if you drive at a speed of $2x^2 - 3x^2 + 5x - 4$.



Solution

Step 1 Input the data

Press **[STAT]** and choose the **[EDIT]**option.

Input the values of x in the first column $f(2)$ and the values of y in the second column $f(2)$.

Note: In order to clear a list, move the cursor to the top so that $2y$ or R_1 is highlighted. Then press **[CLEAR]** button and then **[ENTER]**.

Step 2 Draw the scatter plot.

First press **[Y=]** and clear any function on the screen by pressing **[CLEAR]** when the old function is highlighted.

Press **[STATPLOT] [STAT]** and **[Y=]** and choose option 1.

Choose the ON option, after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** and make sure that the window is set so you see all the points in the scatterplot. In this case $\$4995 = \$18$ and $-1, -4, -5$.

You can set the window size by pressing on the **[WINDOW]** key at top.

Step 3 Perform quadratic regression.

Press **[STAT]** and use right arrow to choose **[CALC]**.

Choose Option 5 (QuadReg) and press **[ENTER]**. You will see "QuadReg" on the screen.

Type in $x \neq 0$. after 'QuadReg' and Press **[ENTER]**. The calculator shows the quadratic function.

Function $4(3x - 4) - 7(2x + 3) = 3$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found.

Press **[GRAPH]** and you will see the curve fit drawn over the data points.

To find the speed that maximizes the miles per gallons, use **[TRACE]** and move the cursor to the top of the parabola. You can also use **[CALC] [2nd] [TRACE]** and option 4 Maximum, for a more accurate answer. The speed that maximizes miles per gallons $3 - |4 - 9|$

Plug $k = 12$ into the equation you found:
$y = -0.017(56)^2 = 1.9(56) - 25 = 28 \text{ miles per gallon}$

Note: The image to the right shows our data points from the table and the function plotted on the same graph. One thing that is clear from this graph is that predictions made with this function will not make sense for all values of x . For example, if $k = 12$, this graph predicts that we will get negative mileage, something that is impossible. Thus, part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Example 7

The following data represents the amount of money an investor has in an account each year for 10 years.

year	value of account
1996	$\$5000$
1997	$\$5000$
1998	$\$5000$
1999	$\$5000$
2000	$\$5000$
2001	$\$5000$
2002	$\$5000$
2003	$\$5000$
2004	$\$5000$
2005	$\$15552$
2006	$\$15552$

Using a graphing calculator

a) Draw a scatterplot of the value of the account as the dependent variable, and the number of years *since* 1996 as the independent

variable.

b) Find the exponential function that fits the data.

c) Draw the exponential function on the scatterplot.

d) What will be the value of the account in 2020?

Solution

Step 1 Input the data

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column $f(2)$ and the values of y in the second column $f(2)$.

Step 2 Draw the scatter plot.

First press **[Y=]** and clear any function on the screen.

Press **[GRAPH]** and choose Option 1.

Choose the ON option and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** make sure that the window is set so you see all the points in the scatterplot. In this case: $6 \le x \le 18$ and $y = -0.2x - 1$.

Step 3 Perform exponential regression.

Press **[STAT]** and use right arrow to choose **[CALC]** .

Choose Option 0 and press **[ENTER]**. You will see "ExpReg" on the screen.

Press **[ENTER]** . The calculator shows the exponential function.

Function $2(12 + 6) \le 8(12)$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found. Press **[GRAPH]**.

Substitute $25 \text{ meters} \times 35 \text{ meters}$ into the function $(-5m^2 - m) - (3m^2 + 4m - 5)$.

Note: This is a curve fit. So the function above is the curve that comes closest to all the data points. It will not return y values that are exactly the same as in the data table, but they will be close. It is actually more accurate to use the curve fit values than the

data points.

Solve Real-World Problems by Comparing Function Models

Example 8

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a *scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.*

Year	Number of Students (millions)
1990	$-5x$
1991	$-5x$
1992	-79
1993	23.7
1994	-53
1995	$-5x$
1996	-53
1997	-53
1998	$-5x$
2003	$-5x$

Solution

We will perform linear, quadratic and exponential regression on this data set and see which function represents the values in the table the best.

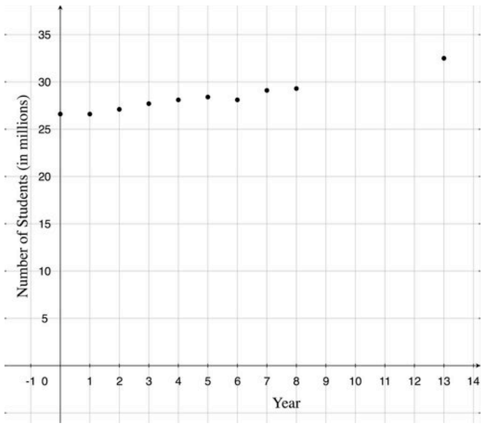
Step 1 Input the data.

Input the values of x in the first column $f(2)$ and the values of y in the second column $f(2)$.

Step 2 Draw the scatter plot.

Set the window size: $6 \leq x \leq 18$ and $0, 1, 2, 3, 4, 5$.

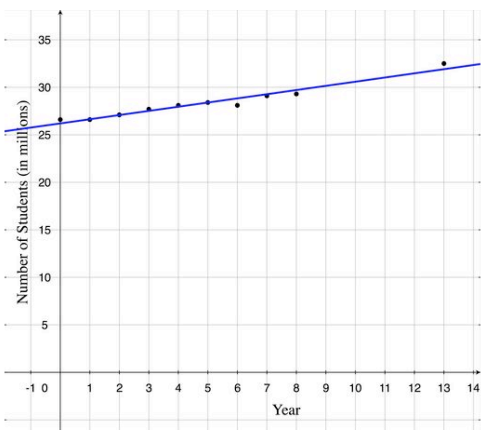
Here is the scatter plot.



Step 3 Perform Regression.

Linear Regression

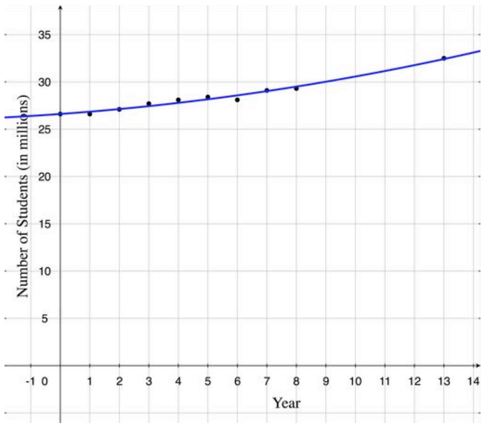
The function of the line of best fit is $y = .25x - 422.1$.



Here is the graph of the function on the scatter plot.

Quadratic Regression

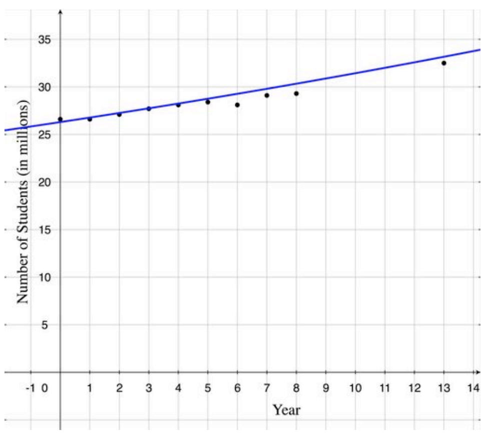
The quadratic function of best fit is $v = 25(2) - 80 = -30 \text{ ft/sec}$.



Here is the graph of the function on the scatter plot.

Exponential Regression

The exponential function of best fit is $80 \geq 10(1.2 + 2)$.



Here is the graph of the function on the scatter plot.

From the graphs, it looks like the quadratic function is the best fit for this data set. We use this function to predict school enrollment in 2007.

Substitute $x = 2007 - 1990 = 17$

$y = 0.064(17)^2 - .067(17) + 26.84 = \underline{44.2 \text{ million students}}$

Review Questions

Determine whether the data in the following tables can be represented by a linear function.

1.

x	y
-4	10
-3	7
-2	4
-1	1
0	-2
1	-5

2.

x	y
-2	4
-1	3
0	2
1	3
2	6
3	11

3.

x	y
0	50
1	75
2	100
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function:

1.

x	y
-10	10
-5	2.5
0	0
5	2.5
10	10
15	22.5

x	y
1	4
2	6
3	6
4	4
5	0
6	-6

2.

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

3.

Determine whether the data in the following tables can be represented by an exponential function.

x	y
0	200
1	300
2	1800
3	8300
4	25800
5	62700

1.

x	y
0	120
1	180
2	270
3	405
4	607.5
5	911.25

2.

x	y
0	120
1	180
2	270
3	405
4	607.5
5	911.25

3.

Determine what type of function represents the values in the following table and find the equation of the function.

1.

x	y
0	400
1	500
2	625
3	781.25
4	976.5625

2.

x	y
-9	-3
-7	-2
-5	-1
-3	0
-1	1
1	2

3.

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2
2	4
3	14

4. As a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, the table shows the height of the ball with respect to time.
5.

Time (seconds)	Height (inches)
4	4
2.4	16
2.4	$2a$

Time (seconds)	Height (inches)
$y-$	29
$y-$	29
$y-$	$2a$
$y-$	29
$y-$	29
$y-$	29
$y-$	12
$y-$	y

6. Using a graphing calculator
7.
    1. Draw the scatter plot of the data.
    2. Find the quadratic function of best fit.
    3. Draw the quadratic function of best fit on the scatter plot.
    4. Find the maximum height the ball reaches on the bounce.
    5. Predict how high the ball is at time $0.872727272\ldots$.
8. A chemist has a 302 gram sample of a radioactive material. She records the amount of radioactive material remaining in the sample every day for a week and obtains the data in the following table.
9.

Day	Weight (grams)
y	302
1	302
4	100
y	100
4	100
y	29
y	29
7	29

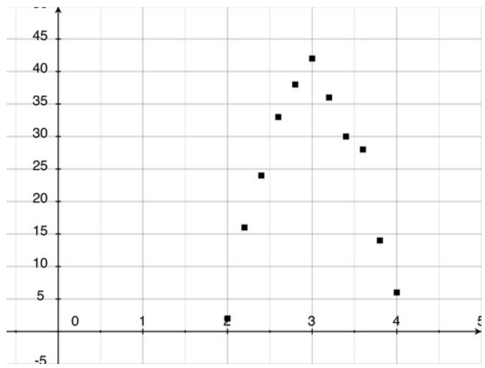
10. Using a graphing calculator,
11.
    1. Draw a scatterplot of the data.
    2. Find the exponential function of best fit.
    3. Draw the exponential function of best fit on the scatter plot.
    4. Predict the amount of material after 10 days.
12. The following table shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau). Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the rate of teen pregnancies in the year 2010.
13.

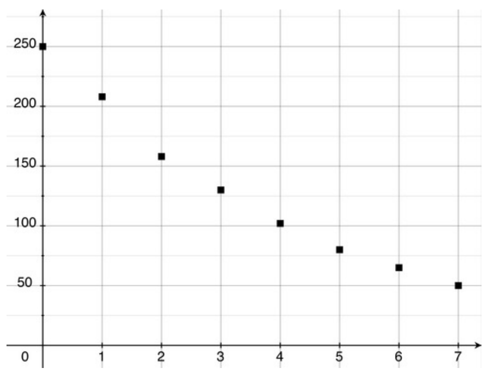
Year	Rate of Pregnancy (per 1000)
1990	$b = 1$
1991	$b = 1$

Year	Rate of Pregnancy (per 1000)
1992	$b = 1$
1993	$b = 1$
1994	$b = 1$
1995	$-5x$
1996	$-5x$
1997	$-5x$
1998	23.7
1999	23.7
2000	$-5x$
2001	-79
2002	-79

Review Answers

1. Linear common difference $9 > 3$
2. Not Linear
3. Linear common difference $= 15$
4. Quadratic difference of difference $y =$
5. Quadratic difference of difference $x = 1$
6. Not Quadratic
7. Not Exponential
8. Exponential common ratio $9 > 3$
9. Exponential common ratio $9 > 3$
10. Exponential $\frac{z}{5} + 1 < z - 20$
11. Linear $f(x) = 4.2x + 19.7$
12. Quadratic $f(x) = 1.5x$
13.

   1. 

   2. $\text{Steel required} = 8(4+5) \text{ feet.}$;
   3. $\text{Maximum height} = 38.9 \text{ inches}$.
   4. $x = -22.5$, $\text{height} = 29.6 \text{ inches}$.
14.
   1. 

   2. $2(12+6) \leq 8(12)$
   3. After 10 days, there is $b = 3$ grams of material left.
15. linear function is best fit: $y = -0.017x + 43.9$ In year 2010, $k = 12$, rate of teen pregnancies $= 17$ per 1000

Problem Solving Strategies: Choose a Function Model

Learning Objectives

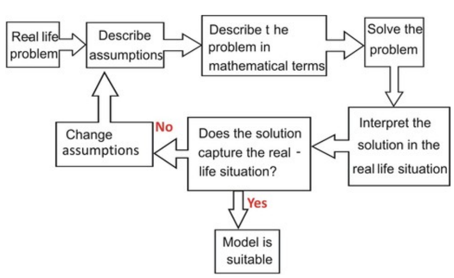
- Read and understand given problem situations
- Develop and use the strategy: Choose a Function
- Develop and use the strategy: Make a Model
- Plan and compare alternative approaches to solving problems
- Solve real-world problems using selected strategies as part of a plan

Introduction

As you learn more and more mathematical methods and skills, it is important to think about the purpose of mathematics and how it works as part of a bigger picture. Mathematics is used to solve problems which often arise from real-life situations. **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution. Modeling involves creating a set of mathematical equations that describes a situation, solving those equations and using them to understand the real-life problem. Often the model needs to be adjusted because it does not describe the situation as well as we wish.

A mathematical model can be used to gain understanding of a real-life situation by learning how the system works, which variables are important in the system and how they are related to each other. Models can also be used to predict and forecast what a system will do in the future or for different values of a parameter. Lastly, a model can be used to estimate quantities that are difficult to evaluate exactly.

Mathematical models are like other types of models. The goal is not to produce an exact copy of the "real" object but rather to give a representation of some aspect of the real thing. The modeling process can be summarized as follows.



Notice that the modeling process is very similar to the problem solving format we have been using throughout this book. In this section, we will focus mostly on the assumptions we make and the validity of the model. Functions are an integral part of the modeling process because they are used to describe the mathematical relationship in a system. One of the most difficult parts of the modeling process is determining which function best describes a situation. We often find that the function we chose is not appropriate. Then, we must choose a different one, or we findthat a function model is good for one set of parameters but we need to use another function for a different set of parameters. Often, for certain parameters, more than one function describes the situation well and using the simplest function is most practical.

Here we present some mathematical models arising from real-world applications.

Example 1 Stretching springs beyond the "elastic limit"

A spring is stretched as you attach more weight at the bottom of the spring. The following table shows the length of the spring in inches for different weights in ounces.

Weight (oz)	0	2	4	6	8	10	12	14	16	18	20
Length (in)	2	2.4	2.8	3.2	3.5	3.9	4.1	4.4	4.6	4.7	4.8

a) Find the length of the spring as a function of the weight attached to it.

b) Find the length of the spring when you attach 24 times.

c) Find the length of the spring when you attach $Dx = -4$.

Solution

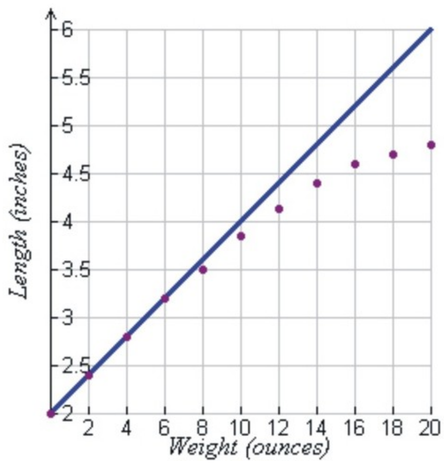
Step 1 Understand the problem

Define $x =$ weight in ounces on the spring

$y =$ length in inches of the spring

Step 2 Devise a plan

Springs usually have a linear relationship between the weight on the spring and the stretched length of the spring. If we make a scatter plot, we notice that for lighter weights the points do seem to fit on a straight line (see graph). Assume that the function relating the length of the spring to the weight is linear.



Step 3 Solve

Find the equation of the line using points describing lighter weights:

$(0, 0)$ and $(3 + 2)$.

The slope is $m = \frac{.8}{4} = 0.2$

Using $y = mx + b$

a) We obtain the function $y = -0.025$.

b) To find the length of the spring when the weight is 24 times, we plug in $x = 3$
.

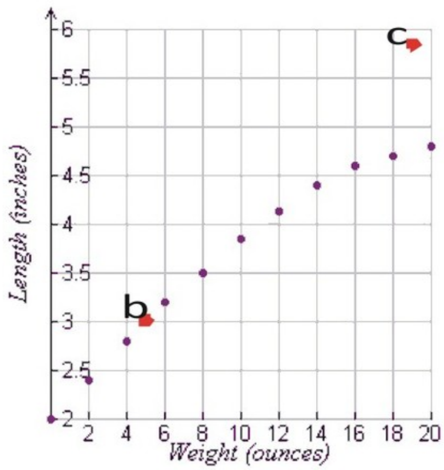
$1.4(-9) + 5.2 > 0.4(-9)$

c) To find the length of the spring when the weight is $Dx = -4$, we plug in $k = 12$.

$y = .2(19) + 2 = 5.8 \text{ inches}$

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that for small weights, the relationship between the length of the spring and the weight is a linear function.

For larger weights, the spring does not seem to stretch as much for each added ounces. We must change our assumption. There must be a non-linear relationship between the length and the weight.

Step 5 Solve with New Assumptions

Let's find the equation of the function by cubic regression with a graphing calculator.

a) We obtain the function $y = -.000145x^3 - .000221x^2 + .202x + 2.002$.

b) To find the length of the spring when the weight is 24 times, we plug in $x = 3$
.

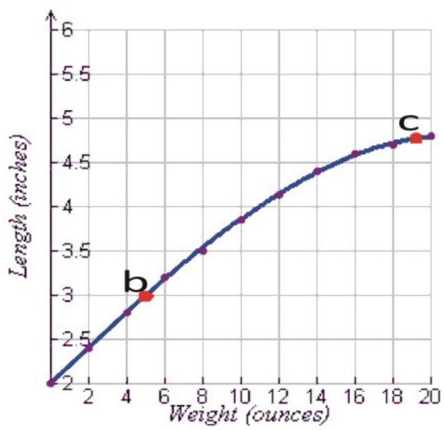
$y = -.000145(5)^3 - .000221(5)^2 + .202(5) + 2.002 = 3 \text{ inches}$

c) To find the length of the spring when the weight is $Dx = -4$, we plug in $k = 12$.

$y = -.000145(19)^3 - .000221(19)^2 + .202(19) + 2.002 = 4.77 \text{ inches}$

Step 6 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a cubic function represents the stretching of the spring more accurately than a linear function. However, for small weights the linear function is an equally good representation, and it is much easier to use in most cases. In fact, the linear approximation usually allows us to easily solve many problems that would be very difficult to solve by using the cubic function.

Example 2 Water flow

A thin cylinder is filled with water to a height of 50 centimeters. The cylinder has a hole at the bottom which is covered with a stopper. The stopper is released at time $r = 17 \text{ inches}$ and allowed to empty. The following data shows the height of the water in the cylinder at different times.

Time (sec)	0	2	4	6	8	10	12	14	16	18	20	22	24
Height (cm)	50	42.5	35.7	29.5	23.8	18.8	14.3	10.5	7.2	4.6	2.5	1.1	0.2

a) Find the height (in centimeters) of water in the cylinder as a function of time in seconds.

b) Find the height of the water when $r = 17 \text{ inches}$.

c) Find the height of the water when 50 centimeters.

Solution:

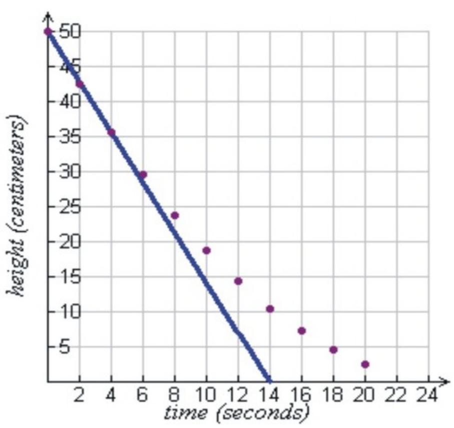
Step 1 Understand the problem

Define $x =$ the time in seconds

$y =$ height of the water in centimeters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of water on the vertical axis.



Notice that most of the points seem to fit on a straight line when the water level is high. Assume that a function relating the height of the water to the time is linear.

Step 3 Solve

Find the equation of the line using points describing lighter weights:

(mph) and $(5 - 11)$.

The slope is $m = \frac{-14.3}{4} = -3.58$

Using $y = mx + b$

a) We obtain the function: $y = 3.25x + 1.25$

b) The height of the water when $r = 17 \text{ inches}$ is

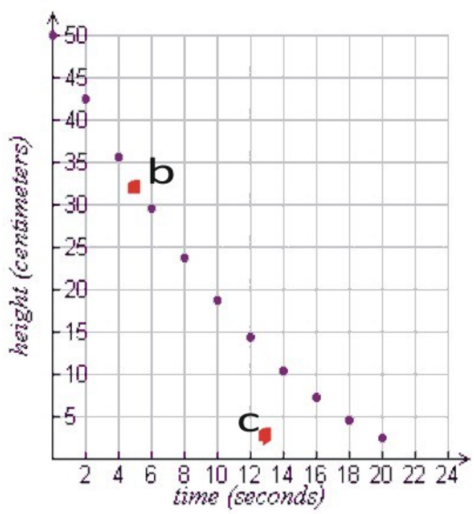
$(10, 23), (5, 18), (8, 24), (11, 30), (2, 10)$

c) The height of the water when 50 centimeters is

$(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that when the water level is high, the relationship between the height of the water and the time is a linear function. When the water level is low, we must change our assumption. There must be a non-linear relationship between the height and the time.

Step 5 Solve with new assumptions

Let's assume the relationship is quadratic and let's find the equation of the function by quadratic regression with a graphing calculator.

a) We obtain the function $\text{Items} = 28(p + 2f + 5c)$

b) The height of the water when $r = 17$ inches is

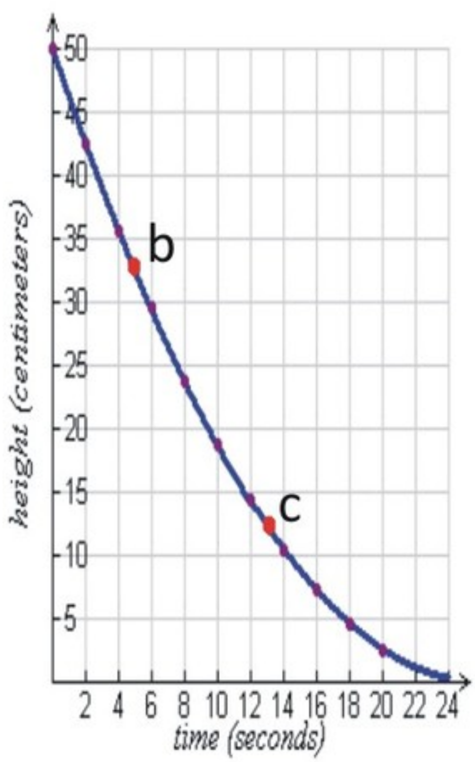
$$y = .075(5)^2 - 3.87(5) + 50 = 32.53 \text{ centimeters}$$

c) The height of the water when 50 centimeters is

$$y = .075(13)^2 - 3.87(13) + 50 = 12.37 \text{ centimeters}$$

Step 6: Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a quadratic function represents the situation more accurately than a linear function. However, for high water levels the linear function is an equally good representation.

Example 3 Projectile motion

A golf ball is hit down a straight fairway. The following table shows the height of the ball with respect to time. The ball is hit at an angle of $1,765,244$ with the horizontal with a speed of $= 0.6 \times (0.5\times$.

Time (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Height (meters)	0	17.2	31.5	42.9	51.6	57.7	61.2	62.3	61.0	57.2	51.0	42.6	31.9	19.0	4.1

a) Find the height of the ball as a function of time.

b) Find the height of the ball when $t = 2.4 \text{ seconds}$.

c) Find the height of the ball when $0.872727272\ldots$.

Solution

Step 1 Understand the problem

Define $x =$ the time in seconds

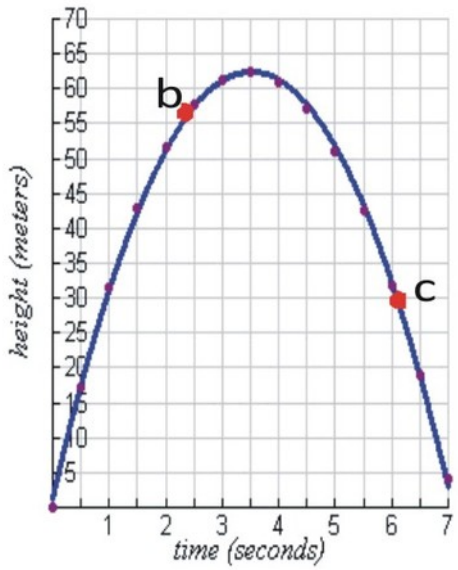
$y =$ height of the ball in meters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of the ball on the vertical axis. We know that a projectile follows a parabolic path, so we assume that the function relating height to time is quadratic.

Step 3 Solve

Let's find the equation of the function by quadratic regression with a graphing calculator.



a) We obtain the function $2(70) + 2(20) = 180$ inches

b) The height of the ball when $t = 2.4$ seconds is:

$y = -4.92(2.4)^2 + 34.7(2.4) + 1.2 = 56.1$ meters

c) The height of the ball when $0.872727272\ldots$ is:

$y = .075(5)^2 - 3.87(5) + 50 = 32.53$ centimeters

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) follow the trend very closely. The quadratic function is a very good model for this problem

Example 4 Population growth

A scientist counts two thousand fish in a lake. The fish population increases at a rate of -1.375 per generation but the lake has space and food for only $7, \ , 12, \ , 17\ldots$. The following table gives the number of fish (in thousands) in each generation.

Generation	0	4	8	12	16	20	24	28
Number (thousands)	2	15	75	343	1139	1864	1990	1999

a) Find the number of fish as a function of generation.

b) Find the number of fish in generation 16.

c) Find the number of fish in generation 29.

Solution:

Step 1 Understand the problem

Define $x =$ the generation number $y =$ the number of fish in the lake

Step 2 Devise a plan

Let's make a scatterplot of our data with the generation number on the horizontal axis and the number of fish in the lake on the vertical axis. We know that a population can increase exponentially. So, we assume that we can use an exponential function to describe the relationship between the generation number and the number of fish.

Step 3 Solve

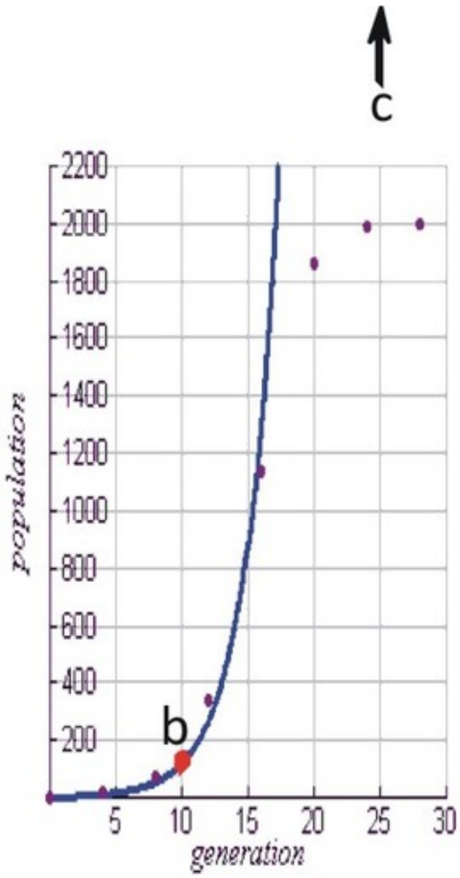
a) Since the population increases at a rate of -8 per generation, assume the function $13x(3y + z)$

b) The number of fish in generation 16 is: $x^2 + 4x \neq (x + 2)^2$ thousand fish

c) The number of fish in generation 29 is: $y = 2(1.5)^{25} = 50502$ thousand fish

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) fits the data well but the answer to c) does not seem to follow the trend very closely. The result is not even on our graph!



When the population of fish is high, the fish compete for space and resources so they do not increase as fast. We must change our assumptions.

Step 5 Solve with new assumptions

When we try different regressions with the graphing calculator, we find that logistic regression fits the data the best.

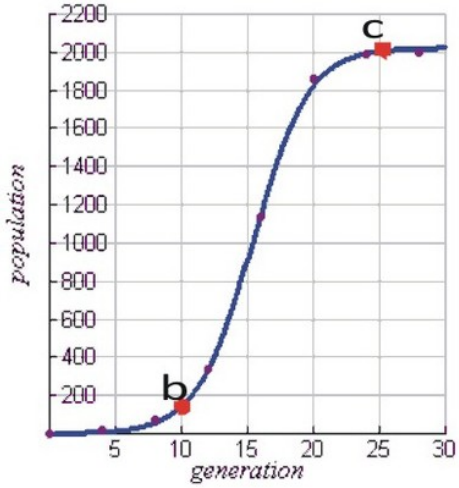
a) We obtain the function $y = \frac{2023.6}{1+1706.3(2.71)^{-.484x}}$

b) The number of fish in generation 16 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(10)}} = 139.6$ thousand fish

c) The number of fish in generation 29 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(25)}} = 2005$ thousand fish

Step 6 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a logistic function represents the situation more accurately than an exponential function. However, for small populations the exponential function is an equally good representation, and it is much easier to use in most cases.

Review Questions

1. In Example 1, evaluate the length of the spring for weight $3 \times 5 = 15$ by
    1. Using the linear function
    2. Using the cubic function
    3. Figuring out which function is best to use in this situation.
2. In Example 1, evaluate the length of the spring for weight 5960 meters by
    1. Using the linear function
    2. Using the cubic function
    3. Figuring out which function is best to use in this situation.
3. In Example 2, evaluate the height of the water in the cylinder when $t = 2.4 \text{ seconds}$ by
    1. Using the linear function
    2. Using the quadratic function

3. Figuring out which function is best to use in this situation.

4. In Example 2, evaluate the height of the water in the cylinder when 50 centimeters by
   1. Using the linear function
   2. Using the quadratic function
   3. Figuring out which function is best to use in this situation.
5. In Example 3, evaluate the height of the ball when $0.872727272\ldots$. Find when the ball is at its highest point.
6. In Example 4, evaluate the number of fish in generation 8 by
   1. Using the exponential function
   2. Using the logistic function
   3. Figuring out which function is best to use in this situation.
7. In Example 4, evaluate the number of fish in generation 18 by
   1. Using the exponential function
   2. Using the logistic function
   3. Figuring out which function is best to use in this situation.

Review Answers

1.
   1. 5 minutes
   2. 5 minutes
   3. Both functions give the same result. The linear function is best because it is easier to use.
2.
   1. $7x = 35$
   2. 5 minutes
   3. The two functions give different answers. The cubic function is better because it gives a more accurate answer.
3.
   1. $12 \quad \text{miles}$
   2. $40 \quad \text{coins}$
   3. The results from both functions are almost the same. The linear function is best because it is easier to use.
4.
   1. $x > 10000.$
   2. $x - 25$

3. The two function give different results. The quadratic function is better because it gives a more accurate answer.

5.
    1. 48.6 meters
    2. 21° Celsius

6.
    1. $1.35 \cdot y$
    2. $1.35 \cdot y$
    3. The results from both functions are almost the same. The linear function is best because it is easier to use.

7.
    1. $75x \geq 125$
    2. $1,765,244$
    3. the two functions give different results; the logistic function is better because it gives a more accurate answer.