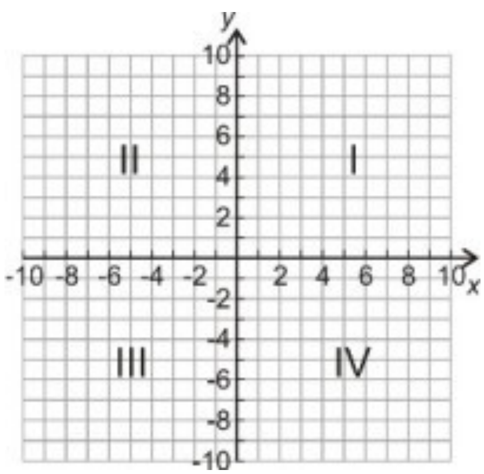
Chapter 4: Graphs of Equations and Functions

The Coordinate Plane

Learning Objectives

- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction



We now make our transition from a number line that stretches in only one dimension (left to right) to something that exists in two dimensions. The **coordinate plane** can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**. The first quadrant (I) contains all the positive numbers from both axes. It is the top right quadrant. The other quadrants are numbered sequentially (II, III, IV) moving counter–clockwise from the first.

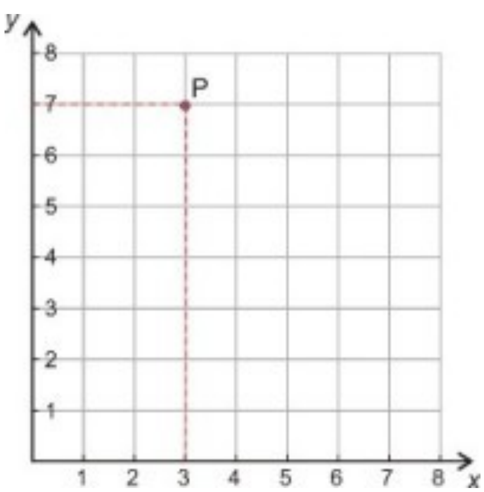
Identify Coordinates of Points

When given a point on a coordinate plane, it is a relatively easy task to determine its **coordinates**. The coordinates of a point are two numbers – written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parenthesis, with the $x-$**coordinate** (also called the **ordinate**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1, 7)$	An ordered pair with an x-value of one and a y-value of seven
$(0, 5)$	an ordered pair with an x-value of zero and a y-value of five
$(-2.5, 4)$	An ordered pair with an x-value of -2.5 and a y-value of four
$(-107.2, -.005)$	An ordered pair with an x-value of -107.2 and a y-value of -0.005.

The first thing to do is realize that identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram to the right.

Imagine you are standing at the origin (the points where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move y units to the **right** (we say this is in the **positive** x direction).

The $x-$coordinate of P is -7.

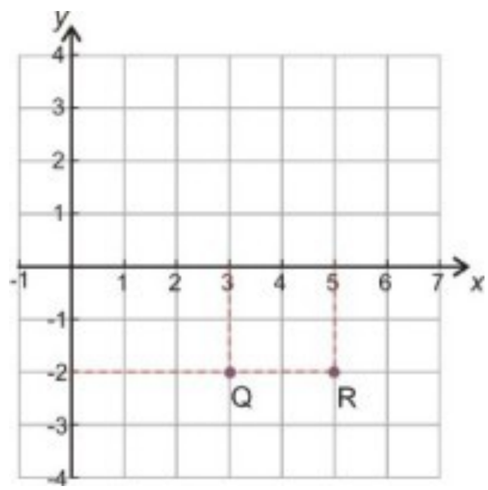
Now if you were standing at the three marker on the $x-$axis, point P would be 7 units above you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is $+7$.

Solution

The coordinates of point P are $(0, 0)$.

Example 2



Find the coordinates of the points labeled $\neq$ and R in the diagram to the right.

In order to get to $\neq$ we move three units to the right, in the positive–x direction, then two units **down**. This time we are moving in the **negative** y direction. The x coordinate of $\neq$ is -7, the y coordinate of $\neq$ is -2.

The coordinates of R are found in a similar way. The x coordinate is -7 (five units in positive x) and the $y-$coordinate is again -2.

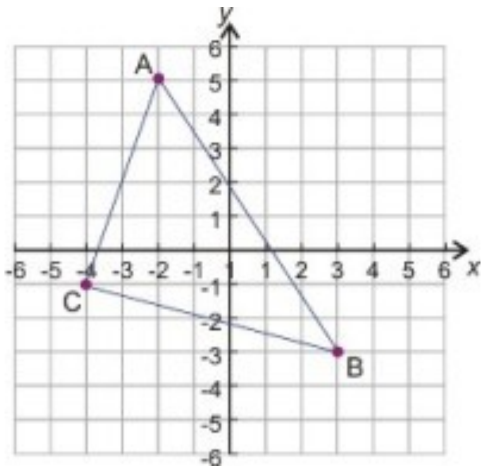
Solution

$\neq (3 + 2)$

$R\,(3 + 2)$

Example 3

Triangle $x + 9$ is shown in the diagram to the right. Find the coordinates of the vertices A, P and C.



Point A:

$x-$coordinate $x = 1$

$y-$coordinate $x + 9$

Point P:

$x-$coordinate $x + 9$

$y-$coordinate $9 > 3$

Point C:

$x-$coordinate $x = 1$

$y-$coordinate $11 \cdot x$

Solution

$A(-2, 5)$
$B(3, -3)$
$C(-4, -1)$

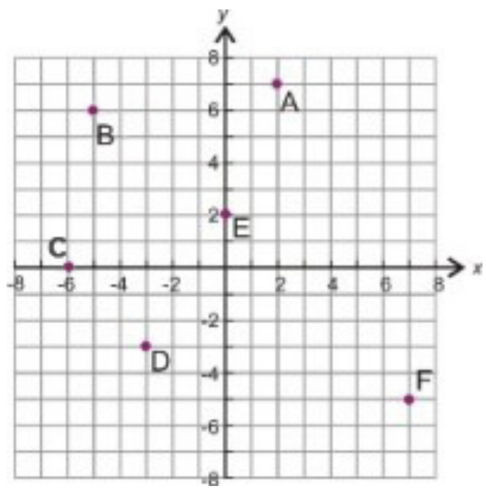
Plot Points in a Coordinate Plane

Plotting points is a simple matter once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$A(2,7)$ $B(-5,6)$ $C(-6,0)$ $D(-3,-3)$ $E(0,2)$ $F(7,-5)$



Point $(3+2)$ is 4 units right, 7 units up. It is in Quadrant I.

Point $5x^2-4y$ is y units left, y units up. It is in Quadrant II.

Point $(-5,-7)$ is y units left, y units up. It is **on the x axis**.

Point $y=x^2-5$ is y units left, y units down. It is in Quadrant III.

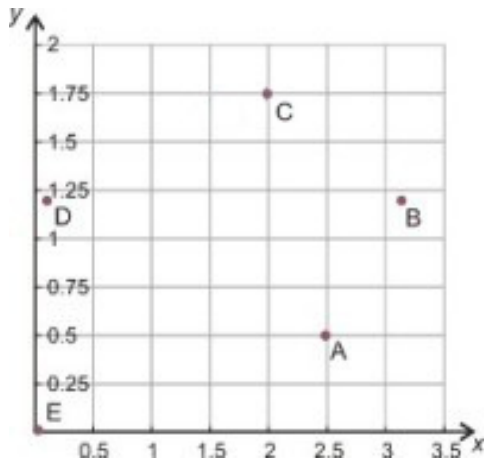
Point $(x-3)$ is 4 units up from the origin. It is **on the y axis**.

Point $5x^2-4y$ is 7 units right, y units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$A(2.5, 0.5)$ $B(\pi, 1.2)$ $C(2, 1.75)$ $D(0.1, 1.2)$ $E(0, 0)$



Choice of axes is always important. In Example Four, it was important to have all four quadrants visible. In this case, all the coordinates are positive. There is no need to show the negative values of x or y. Also, there are no x values bigger than about $-5x$, and -79 is the largest value of y. We will therefore show these points on the following scale $21 + 7 = 28$ and 85.45 cm^2. The points are plotted to the right.

Here are some important points to note about this graph.

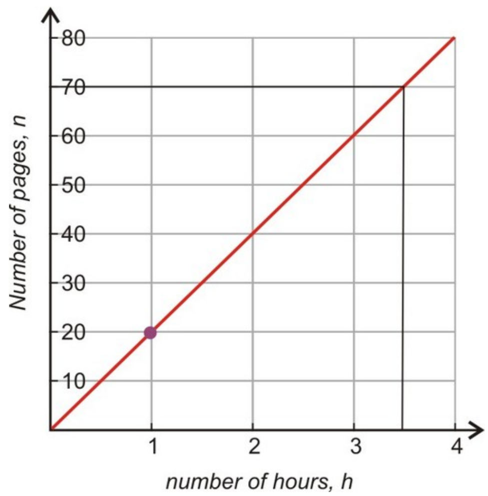
- The tick marks on the axes do not correspond to unit increments (i.e. the numbers do not go up by one).
- The scale on the $x-$axis is different than the scale on the $y-$axis.
- The scale is **chosen** to maximize the clarity of the plotted points.

Graph a Function Given a Table

Once we know how to plot points on a coordinate plane, we can think about how we would go about plotting a relationship between x and y values. So far we have been plotting sets of ordered pairs. This is called a **relation,** and there isn't necessarily a relationship between the x values and y values. In a relation, the set of x values is called the **domain** and the set of y values is called the **range.** If there is a relationship between the x and y values, and each x value corresponds to exactly one y value, then the relation is called a *function.* Remember that a function is a particular way to relate one quantity to another. If you read a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either:

$m = 20 \cdot h \quad n = \text{number of pages; } h = \text{time measured in hours. OR} \ldots$

$h = \dfrac{n}{20}$



So you could use the **function** that related x and h to determine how many pages you could read in $\frac{-3}{2}$ hours, or even to find out how long it took you to read forty-six pages. The graph of this function is shown right, and you can see that if we plot number of pages against number of hours, then we can simply read off the number of pages that you could read in $y-$ hours as seventy pages. You can see that in a similar way it would be possible to estimate how long it would take to read forty-six pages, though the time that was obtained might only be an approximation.

Generally, the graph of a **function** appears as a line or curve that goes through all points that satisfy the relationship that the function describes. If the domain of the function is all real numbers, then we call this a **continuous function.** However, if the domain of the function is a particular set of values (such as whole numbers), then it is called a **discrete function.** The graph will be a series of dots that fall along a line or curve.

In graphing equations, we assume the domain is all real numbers, unless otherwise stated. Often times though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. Sometimes the graph is still shown as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

Example 6

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

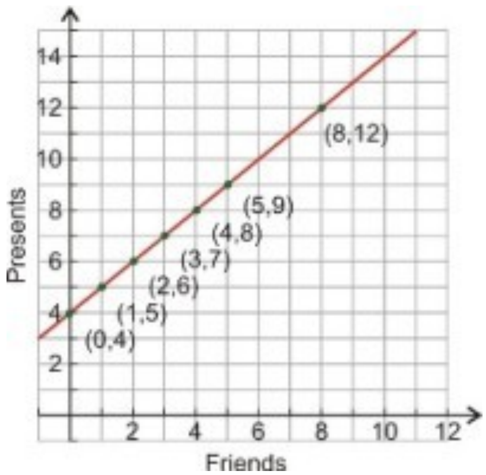
Number of Friends	Number of Presents
y	4
1	y
4	y
y	7
4	y
y	y

The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependant variable is. Clearly in this case, the number of friends can vary **independently** (the domain). The number of presents must **depend** on the number of friends who show up (the range).

We will therefore plot friends on the $x-$axis and presents on the $y-$axis. Let's add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

No. of friends (x)	no. of presents (y)	coordinates (x, y)
y	4	$(0, 0)$
1	y	$(0, 0)$
4	y	$(0, 0)$
y	7	$(0, 0)$
4	y	$(0, 0)$
y	y	$(0, 0)$

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we do not need to worry about anything other than Quadrant I. We need to choose a suitable scale for the x and y axes. We need to consider no more than eight friends (look again at the question to confirm this), but it always pays to allow a **little** extra room on your graph. We also need the y scale to accommodate the presents for eight people. We can see that this is still going to be under 29!



The scale of the graph on the right shows room for up to 12 friends and 16 presents. This will be fine, but there are many other scales that would be equally as good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lay on a straight line, so the function that describes the relationship between x and y will be **linear**. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a **discrete** problem since Sarah can only invite a whole numbers of friends. For instance, it would be impossible for 2.4 friends to show up. Keep in mind that the only permissible points for the function are those points on the line which have integer x and y values.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah would get if eight friends come to her party. Don't forget that x represents the number of friends and y represents the number of presents. If we look at $x = 3$ we can see that the function has a y value of 12.

Solution

If y friends show up, Sarah will receive a total of 12 presents.

Graph a Function Given a Rule

If we are given a rule instead of a table, we can proceed to graph the function in one of two ways. We will use the following example to show each way.

Example 7

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number. Then double it. Then add five to what he got. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the result of applying the rule. He wrote his rule in the form of an equation.

$-2.5, 1.5, 5$

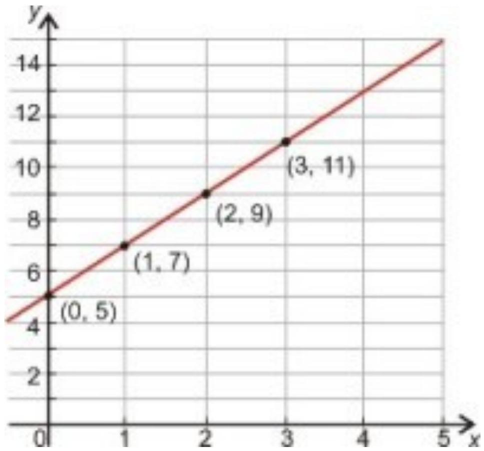
Help him visualize what is going on by graphing the function that this rule describes.

Method One – Construct a Table of Values

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few $y =$ pairs. We will use $y = 5$ and y for x values.

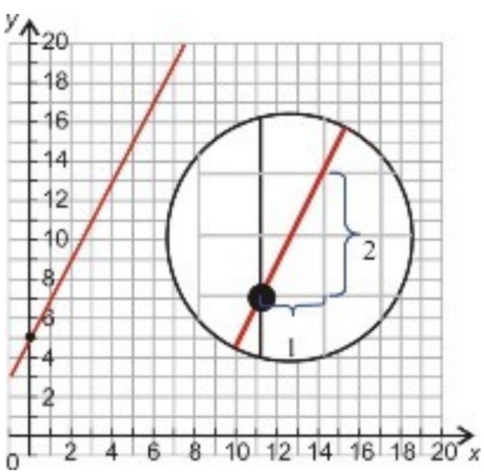
x	y
0	$2 \cdot 0 + 5 = 0 + 5 = 5$
1	$2 \cdot 1 + 5 = 2 + 5 = 7$
2	$2 \cdot 2 + 5 = 4 + 5 = 9$
3	$2 \cdot 3 + 5 = 6 + 5 = 11$

Next, we plot the points and join them with our line.



This method is nice and simple. Plus, with linear relationships there is no need to plot more than two or three points. In this case, the function is continuous because the domain (the number Ali is asked to think of) is all real numbers, even though he may only be thinking of positive whole numbers.

Method Two – Intercept and Slope



One other way to graph this function (and one that we will learn in more detail in the next lesson) is the **slope–intercept method.** To do this, follow the following steps:

1. Find the y value when $x = 3$.

$y(0) = 2 \cdot 0 + 5 = 5$ *So our $y-$intercept is* $(0, 0)$

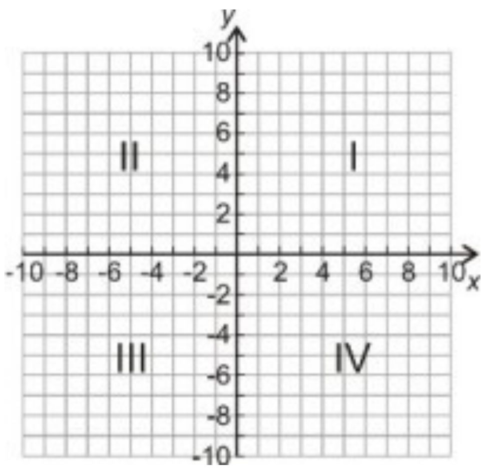
2. Look at the coefficient multiplying the x.

Every time we increase x by one, y increases by two so our slope is -7.

3. Plot the line with the given **slope** that goes through the **intercept.** We start at the point $(0, 0)$ and move over one in the x direction, then up two in the y direction. This gives the slope for our line, which we extend in both directions.

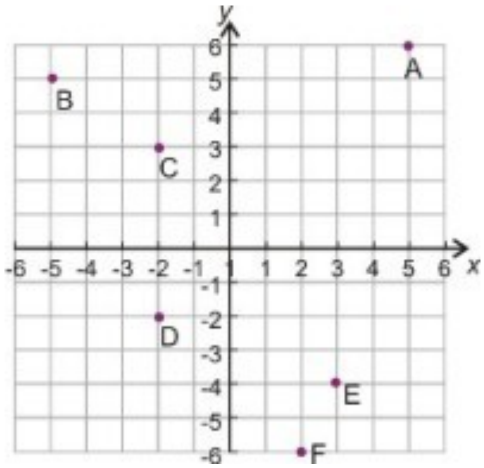
We will properly examine this last method in the next lesson!

Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram at right.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate.** Here is an exaxmple (x, y).
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Review Questions



1. Identify the coordinates of each point, $18 - x$, on the graph to the right.
2. Plot the following points on a graph and identify which quadrant each point lies in:
    1. $(0, 0)$
    2. $(-5, -7)$
    3. $(3 + 2)$
    4. $(-5, -7)$
3. The following three points are three vertices of square $I = 2.5$. Plot them on a graph then determine what the coordinates of the fourth point, P, would be. Plot that point and label it.

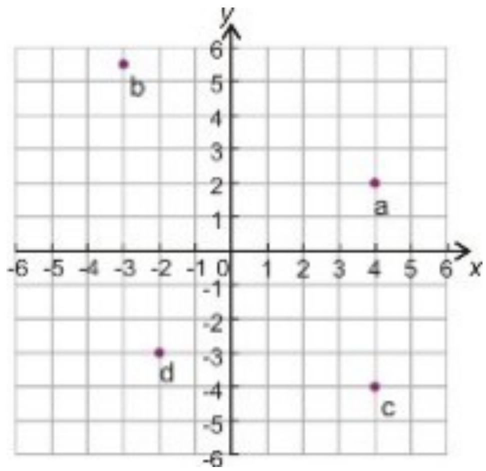
    $A\ (-4, -4)$

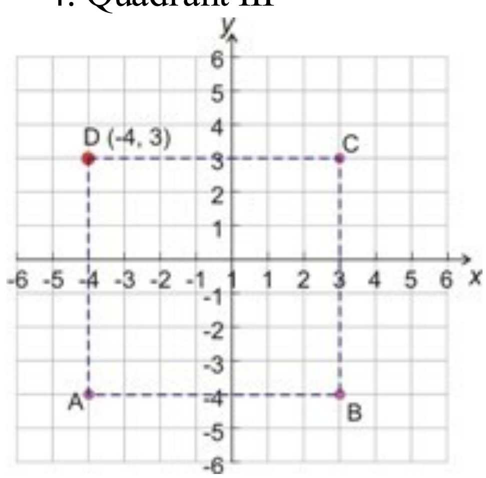
    $|-3| = 3$

    (-11.5)

4. Becky has a large bag of M&Ms that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three M&Ms in return. If x is the number of Starburst that Jaeyun gives Becky, and y is the number of M&Ms he gets in return then complete each of the following.
    1. Write an algebraic rule for y in terms of x
    2. Make a table of values for y with x values of $0, 1, 2, 3, 4, 5$.
    3. Plot the function linking x *and* y on the following scale $0 \leq x \leq 10, 0 \leq y \leq 10$.

Review Answers

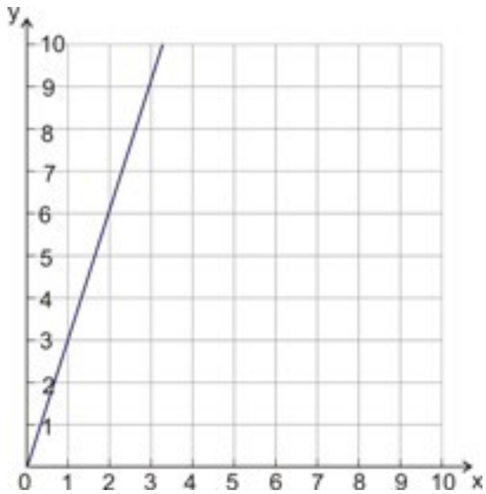
1. $(3+2)\ 5x^2 - 4y\ (-5, -7)\ y = x^2 - 5\ 5x^2 - 4y\ 5x^2 - 4y$
2. 

    1. Quadrant I
    2. Quadrant II
    3. Quadrant IV
    4. Quadrant III
3. 

4.
    1. $1.35 \cdot y$

x	y
0	0
1	3
2	6
3	9
4	12

    2. 5 15

3. 

Graphs of Linear Equations

Learning Objectives

- Graph a linear function using an equation.
- Write equations and graph horizontal and vertical lines.
- Analyze graphs of linear functions and read conversion graphs.

Graph a Linear Equation

At the end of Lesson 4.1 we looked at graphing a function from a rule. A rule is a way of writing the relationship between the two quantities we are graphing. In mathematics, we tend to use the words **formula** and **equation** to describe what we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that you will use frequently in math.

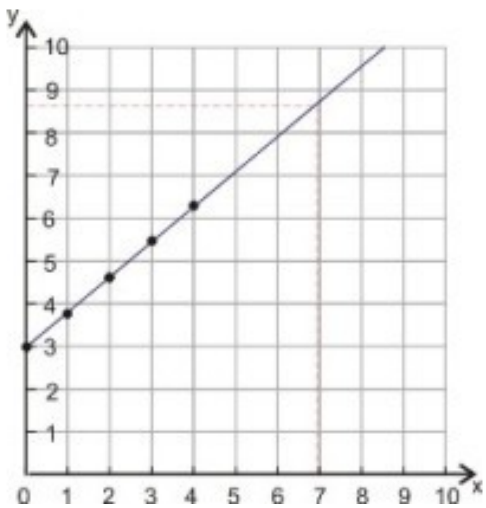
Example 1

A taxi fare costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges x^8 *as a set fee and* $\$0.50$ *per mile traveled. Here is the equation linking the cost in dollars* (y) *to hire a taxi and the distance traveled in miles* (x).

$y = 0.8x + 3$

Graph the equation and use your graph to estimate the cost of a seven mile taxi ride.

We will start by making a table of values. We will take a few values for x $3z^2 - 5w^2$, find the corresponding y values and then plot them. Since the question asks us to find the cost for a seven mile journey, we will choose a scale that will accommodate this.



x	y
0	3
1	3.8
2	4.6
3	5.4
4	6.2

The graph is shown to the right. To find the cost of a seven mile journey we first locate $x = 7$ on the horizontal axis and draw a line up to our graph. Next we draw a horizontal line across to the y axis and read where it hits. It appears to hit around half way between $y = 5$ and $y = 5$. Let's say it is $y-$.

Solution

A seven mile taxi ride would cost approximately $\$0.50$ ($\$0.50$ exactly).

There are a few interesting points that you should notice about this graph and the formula that generated it.

- The graph is a straight line (this means that the equation is **linear**), although the function is **discrete** and will graph as a series of points.
- The graph crosses the $y-$axis at $y = 5$ (look at the equation – you will see a -7 in there!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by $y-$ squares (look at the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by $b = -2$ squares.

Example 2

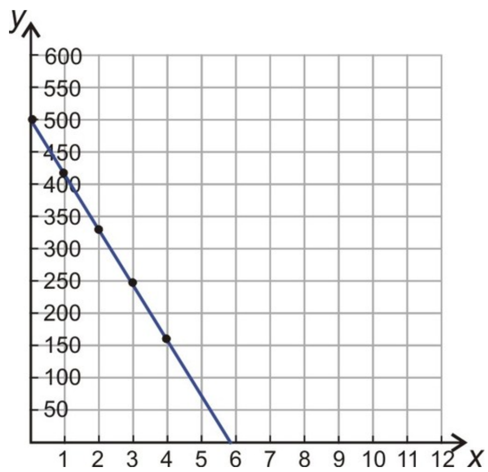
A small business has a debt of $36 \leq 96$ *incurred from start-up costs. It predicts that it can pay off the debt at a rate of* $\$15552$ *per year according to the following equation governing years in business* (x) *and debt measured in thousands of dollars*(y).

$-1, 0, 1, 2, 3, 4, 5$

Graph the above equation and use your graph to predict when the debt will be fully paid.

First, we start with our table of values. We plug in $x-$values and calculate our corresponding $y-$values.

x	y
0	500
1	415
2	330
3	245
4	160



Then we plot our points and draw the line that goes through them.

Take note of the scale that has been chosen. There is no need to have any points above $y = 12x$, but it is still wise to allow a little extra.

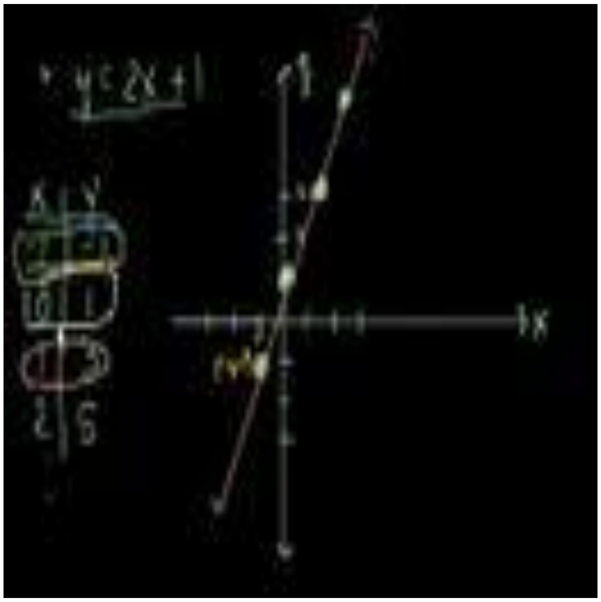
We need to determine how many years (the x value) that it takes the debt (y value) to reach zero. We know that it is greater than four (since at $x = 2$ the y value is still positive), so we need an x scale that goes well past $x = 2$. In this case the x value runs from y to 12, though there are plenty of other choices that would work well.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 5$ (the x axis). It looks as if the line hits pretty close to $x = 3$. So the debt will definitely be paid off in six years.

Solution

The debt will be paid off in six years.

Multimedia Link To see more simple examples of graphing linear equations by hand see the video [Khan Academy Graphing Lines 1]() (9:49)



Graphing linear equations(Watch on Youtube)

Graphs and Equations of Horizontal and Vertical Lines

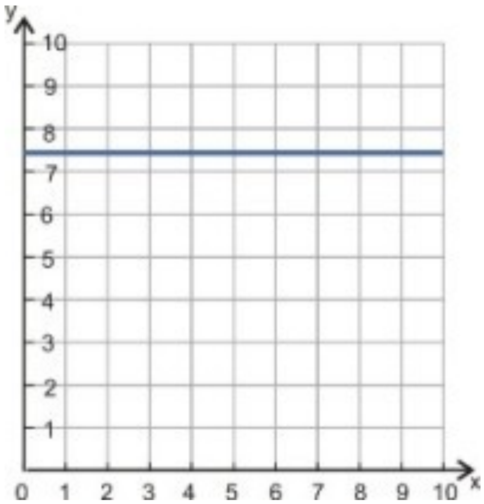
Example 3

"Mad-cabs" have an unusual offer going on. They are charging $\$0.50$ for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) to the length of the journey in miles (x).

To proceed, the first thing we need is an **equation.** You can see from the problem that the cost of a journey does not depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. In fact, any value of x results in the same value of y (-7). Here is the equation.

$$y = 7.5$$

The graph of this function is shown to the right. You can see that the graph $y = 7.5$ is simply a horizontal line.



Any time you see an equation of the form $t = 19, u = 4$ then the graph is a horizontal line that intercepts the $y-$axis at the value of the constant.

Similarly, when you see an equation of the form $x = \text{constant}$ then the graph is a vertical line that intercepts the $x-$axis at the value of the constant. Notice that this a relation, and not a function because each x value (there's only one in this case) corresponds to many (actually an infinite number) y values.

Example 4

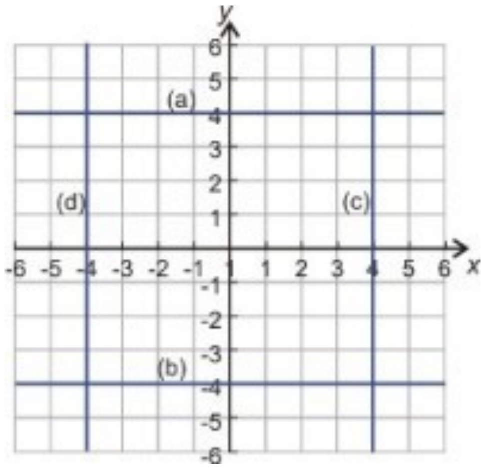
Plot the following graphs.

(a) $y = 5$

(b) $y = -2$

(c) $x = 2$

(d) $x = -4$



(a) $y = 5$ is a horizontal line that crosses the $y-$axis at 4

(b) $y = -2$ is a horizontal line that crosses the $y-$axis at -2

(c) $x = 2$ is a vertical line that crosses the $x-$axis at 4

(d) $x = -4$ is a vertical line that crosses the $x-$axis at -2

Example 5

Find an equation for the $x-$axis and the $y-$axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 3$ and $y = 5$). The following definition could easily work for each axis.

$x-$axis: *A horizontal line crossing the $y-$axis at zero.*

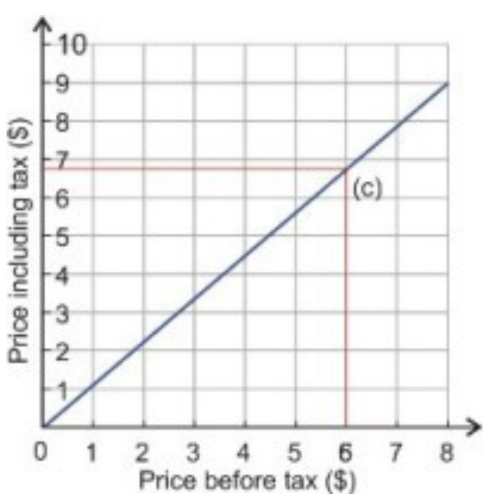
$y-$axis: *A vertical line crossing the $x-$axis at zero.*

So using example 3 as our guide, we could define the $x-$axis as the line $y = 5$ and the $y-$axis as the line $x = 3$.

Analyze Graphs of Linear Functions

We often use line graphs to represent relationships between two linked quantities. It is a useful skill to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about $\$12$, and at the end of the third week it was at about $\$12$. You may also see that in the first five weeks it lost about 25% of its value and that it made about 25% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected for ease of interpretation.

Analyzing line graphs is a part of life – whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many of these graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to the complicated tasks, like reading the graph above. In this section, we will look at reading information from simple linear conversion graphs.



Example 6

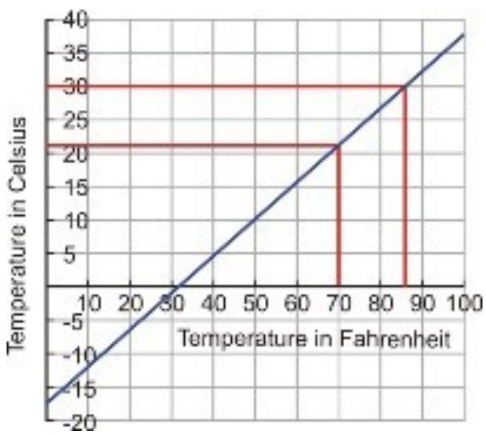
The graph shown at the right shows a chart for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost inclusive of sales tax for a $\$0.50$ pen in the store.

To find the relevant price with tax we find the correct pre-tax price on the $x-$ axis. This is the point $x = 3$.

Draw the line $x = 3$ up until it meets the function, then draw a horizontal line to the $y-$axis. This line hits at $P = 20t;$ (about three fourths of the way from $y = 5$ to $j = 6$).

Solution

The approximate cost including tax is $\$0.50$



Example 7

The chart for converting temperature from Fahrenheit to Celsius is shown to the right. Use the graph to convert the following:

1. -7 *Fahrenheit to Celsius*
2. P *Fahrenheit to Celsius*
3. $y-$ *Celsius to Fahrenheit*
4. P *Celsius to Fahrenheit*

1. To find -7 Fahrenheit we look along the Fahrenheit-axis (in other words the $x-$axis) and draw the line 350 ml up to the function. We then draw a horizontal line to the Celsius-axis ($y-$axis). The horizontal line hits the axis at a little over 29 (12 or $2a$).

Solution

-7 Fahrenheit is approximately equivalent to 21° Celsius

2. To find P Fahrenheit, we are actually looking at the $y-$axis. Don't forget that this axis is simply the line $x = 3$. We just look to see where the line hits the $y-$axis. It hits just below the half way point between -53 and -53.

Solution: P Fahrenheit is approximately equivalent to $75 - 25 = 50$.

3. To find 21° Celsius, we look up the Celsius-axis and draw the line $p = 15$ along to the function. When this horizontal line hits the function, draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 29 .

Solution

21° Celsius is approximately equivalent to $y-$ Fahrenheit.

4. To find 0° Celsius we are looking at the Fahrenheit-axis (the line $y = 5$). We just look to see where the function hits the $x-$axis. It hits just right of 29.

Solution

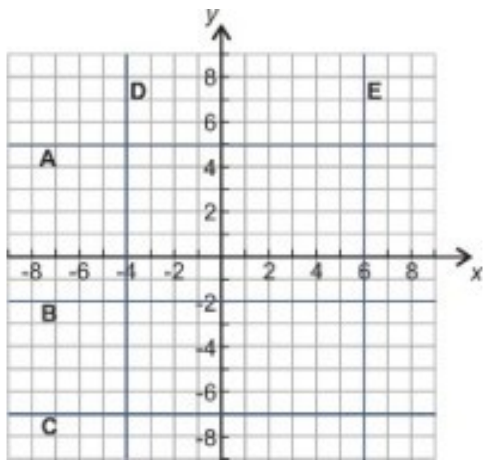
0° Celsius is equivalent to $y-$ Fahrenheit.

Lesson Summary

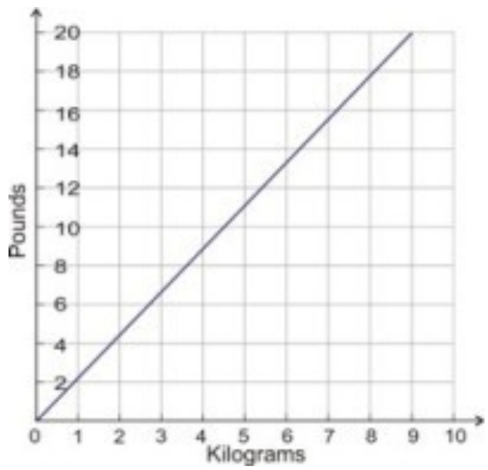
- Equations with the variables y and x can be graphed by making a chart of values that fit the equation and then plotting the values on a coordinate plane. This graph is simply another representation of the equation and can be analyzed to solve problems.
- Horizontal lines are defined by the equation $t = 19, u = 4$ and vertical lines are defined by the equation $x = \text{constant}$.
- Be aware that although we graph the function as a line to make it easier to interpret, the function may actually be discrete.

Review Questions

1. Make a table of values for the following equations and then graph them.
    1. 87.5 grams
    2. $3x^2 + 2x + 1$
    3. $y = 6 - 1.25x$
2. ***"Think of a number. Triple it, and then subtract seven from your answer"***. Make a table of values and plot the function that represents this sentence.

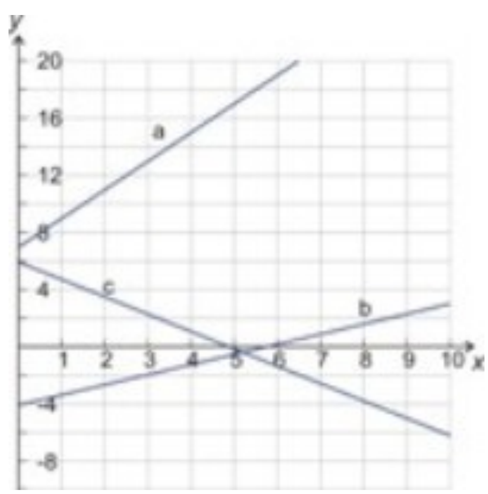


3. Write the equations for the five (A through E) lines plotted in the graph to the right.
4. At the Airport, you can change your money from dollars into Euros. The service costs x^8, and for every additional dollar you get 2.7 Euros. Make a table for this and plot the function on a graph. Use your graph to determine how many Euros you would get if you give the office $\$12$.
5. The graph to below shows a conversion chart for converting between weight in kilograms to weight in pounds. Use it to convert the following measurements.

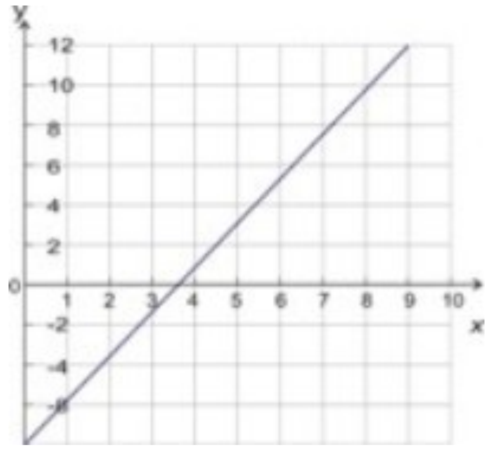


1. 4 kilograms into weight in pounds
2. y kilograms into weight in pounds
3. 12 pounds into weight in kilograms
4. 75 pounds into weight in kilograms

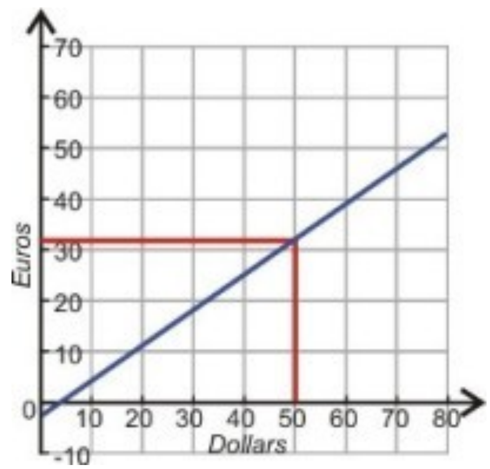
Review Answers

1. 

2. 87.5 grams



3. $A y=5$ $B y=-2$ $C y=-7$ $D x=-4$ $I=2.5$
4. $(3 \cdot 7)+(5 \cdot 7)$



5.
    1. $7 \cdot 5$
    2. $3 \mathrm{~km}$
    3. $5.5 \mathrm{~kg}$
    4. $7.75 \mathrm{~kg}$

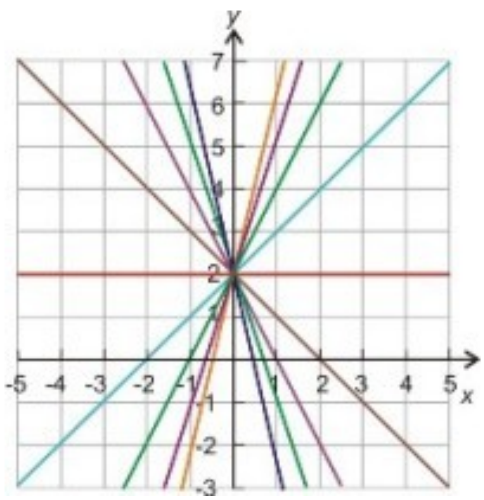
Graphing Using Intercepts

Learning Objectives

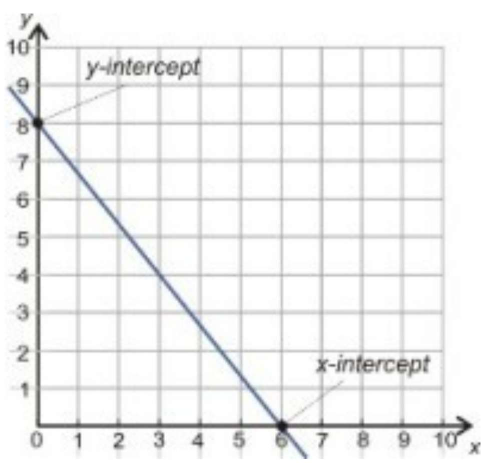
- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.

- Solve real-world problems using intercepts of a graph.

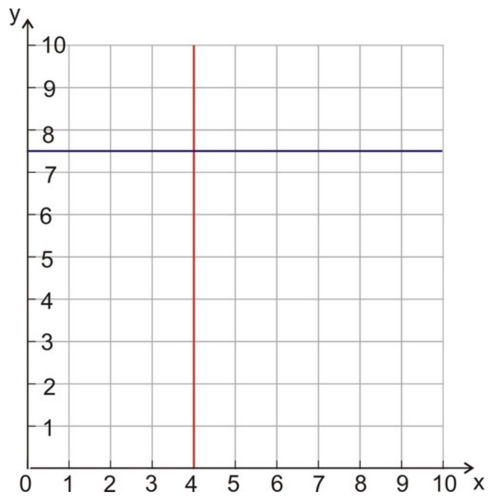
Introduction



Only two distinct points are needed to uniquely define a graph of a line. After all, there are an infinite number of lines that pass through a single point (a few are shown in the graph at right). But if you supplied just one more point, there can only be one line that passes through both points. To plot the line, just plot the two points and use a ruler, edge placed on both points, to trace the graph of the line.



There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we will focus on two points that are rather convenient for graphing: the points where our line crosses the x and y axes, or **intercepts.** We will be finding intercepts algebraically and using them to quickly plot graphs. Similarly, the $x-$intercept occurs at the point where the graph crosses the $x-$axis. The $x-$value in the graph at the right is y.



Look at the graph to the right. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is y.

Similarly the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is y.

Now we know that the x value of all the points on the $y-$axis is zero, and the y value of all the points on the $x-$axis is also zero. So if we were given the coordinates of the two intercepts $(0, 0)$ and $(0, 0)$ we could quickly plot these points and join them with a line to recreate our graph.

Note: Not all lines will have both intercepts but most do. Specifically, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis. For examples of these special case lines, see the graph at right.

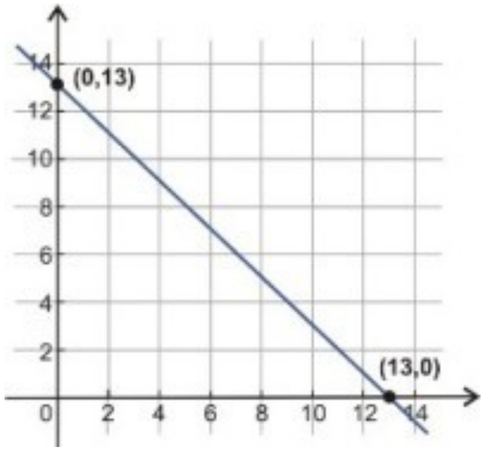
Finding Intercepts by Substitution

Example 1

Find the intercepts of the line $-2.5, 1.5, 5$ *and use them to graph the function.*

The first intercept is easy to find. The $y-$intercept occurs when $x = 3$ Substituting gives:

$$\text{when a is } 750 \qquad \rightarrow \qquad b = 2(750) + 20$$



We know that the $x-$intercept has, by definition, a $y-$value of zero. Finding the corresponding $x-$value is a simple case of substitution:

$$\begin{aligned} 0 &= 13 - x && \text{To isolate } x \text{ subtract } 13 \text{ from both sides.} \\ -13 &= -x && \text{Divide by } -1. \end{aligned}$$

Solution

(mph) is the $x-$intercept.

To draw the graph simply plot these points and join them with a line.

Example 2

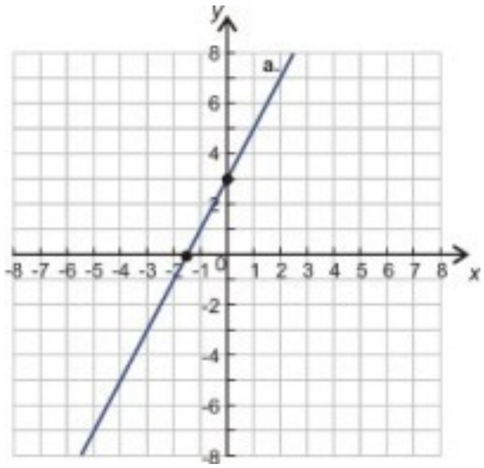
Graph the following functions by finding intercepts.

a. $-2.5, 1.5, 5$

b. 87.5 grams

c. $0, 1, 2, 3, 4, 5$

d. $y = 6 - 1.25x$

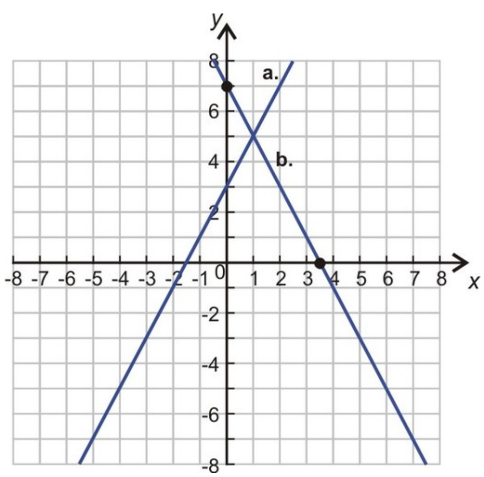


a. Find the $y-$intercept by plugging in $x = 3$.

$y = 2 \cdot 0 + 3 = 3$ The $y-$ intercept is $(0, 3)$

Find the x–intercept by plugging in $y = 5$.

$$\begin{aligned} 0 &= 2x + 3 && \text{Subtract 3 from both sides.} \\ -3 &= 2x && \text{Divide by 2.} \\ -\frac{3}{2} &= x && \text{The } x - \text{intercept is } (-1.5, 0). \end{aligned}$$

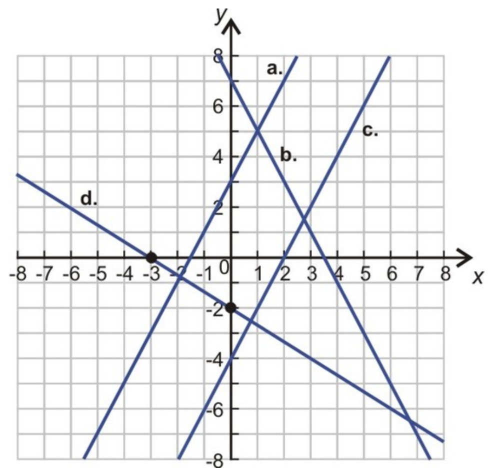


b. Find the $y-$intercept by plugging in $x = 3$.

when a is 750 $\rightarrow$ $b = 2(750) + 20$

Find the x–intercept by plugging in $y = 5$.

$$\begin{aligned} 0 &= 7 - 2x && \text{Subtract 7 from both sides.} \\ -7 &= -2x && \text{Divide by } -2. \\ \frac{7}{2} &= x && \text{The } x-\text{intercept is } (3.5, 0). \end{aligned}$$

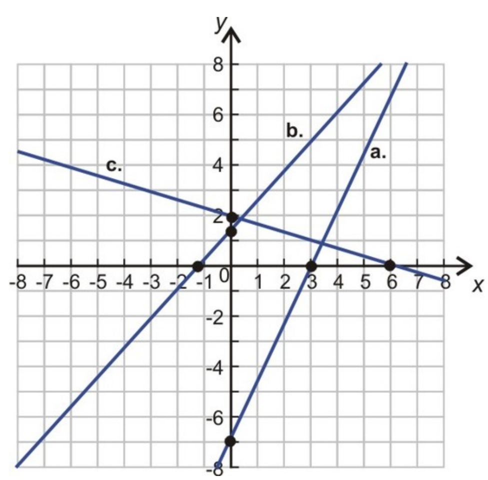


c. Find the $y-$intercept by plugging in $x = 3$.

$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && \text{Divide by } -2. \\ y &= -4 && \text{The } y-\text{intercept is } (0, -4). \end{aligned}$$

Find the $x-$intercept by plugging in $y = 5$.

$$\begin{aligned} 4x - 2 \cdot 0 &= 8 \\ 4x &= 8 && \text{Divide by 4.} \\ x &= 2 && \text{The } x-\text{intercept is } (2, 0). \end{aligned}$$



d. Find the $y-$intercept by plugging in $x = 3$.

$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && \text{Divide by } -2. \\ y &= -4 && \text{The } y-\text{intercept is } (0, -4). \end{aligned}$$

Find the $x-$intercept by plugging in $y = 5$.

$$\begin{aligned} 2x + 3 \cdot 0 &= -6 \\ 2x &= -6 && \text{Divide by } 2. \\ x &= -3 && \text{The } x-\text{intercept is } (-3, 0) \end{aligned}$$

Finding Intercepts for Standard Form Equations Using the Cover-Up Method

Look at the last two equations in Example 2. These equations are written in **standard form**. Standard form equations are always written "**positive coefficient** times x plus (or minus) **positive coefficient** times y equals **value".** Note that the x term *always* has a positive value in front of it while the y value may have a negative term. The equation looks like this:

$$ax + by + c \text{ or } ax - by = c \qquad (a \text{ and } b \text{ are positive numbers})$$

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example 3

Find the intercepts of the following equations.

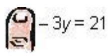
a. $7x - 3y = 21$

b. $12x - 10y = -15$

c. $-2.5, 1.5, 5$

To solve for each intercept, we realize that on the intercepts the value of **either** x **or** y is zero, and so any terms that contain the zero variable effectively disappear. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a. To solve for the $y-$intercept we set $x = 3$ and cover up the x term:

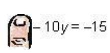


$$
\begin{aligned}
-3y &= 21 \\
y &= -7 \qquad\qquad (0, -7) \text{ is the } y - \text{intercept}
\end{aligned}
$$

Now we solve for the x–intercept:

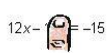
$$
\begin{aligned}
7x &= 21 \\
x &= 3 \qquad\qquad (3, 0) \text{ is the } x - \text{intercept}.
\end{aligned}
$$

b. Solve for the $y-$intercept ($x = 3$) by covering up the x term.



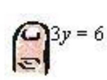
$$
\begin{aligned}
-10y &= -15 \\
y &= -1.5 \qquad\qquad (0, -1.5) \text{ is the } y - \text{intercept}.
\end{aligned}
$$

Solve for the x–intercept ($y = 5$):

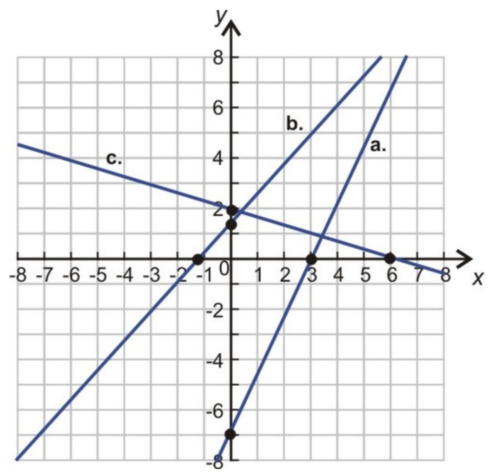


$$
\begin{aligned}
12x &= -15 \\
x &= -\frac{5}{4} \qquad\qquad (-1.25, 0) \text{ is the } x - \text{intercept}.
\end{aligned}
$$

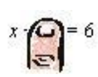
c. Solve for the $y-$intercept ($x = 3$) by covering up the x term:



$$
\begin{aligned}
3y &= 6 \\
y &= 2 \qquad\qquad (0, 2) \text{ is the } y - \text{intercept}.
\end{aligned}
$$



Solve for the $y-$intercept:



$x = 6 \qquad\qquad\qquad (6, 0) \text{ is the } x-\text{intercept}.$

The graph of these functions and the intercepts is shown in the graph on the right.

Solving Real-World Problems Using Intercepts of a Graph

Example 4

The monthly membership cost of a gym is $\$12$ *per month. To attract members, the gym is offering a* $\$100$ *cash rebate if members sign up for a full year. Plot the cost of gym membership over a* 12 *month period. Use the graph to determine the final cost for a* 12 *month membership.*

Let us examine the problem. Clearly the cost is a function of the number of months (not the other way around). Our independent variable is the number of months (the domain will be whole numbers) and this will be our x value. The cost in dollars is the dependent variable and will be our y value. Every month that passes the money paid to the gym goes up by $\$12$. However, we start with a $\$100$ cash gift, so our **initial cost** ($y-$intercept) is $\$100$. This pays for four months $(4 \times \$25 = 100)$ so after four months the cost of membership ($y-$value) is zero.

The $y-$intercept is $11(2+6)$. The $x-$intercept is $(0,0)$.

We plot our points, join them with a straight line and extend that line out all the way to the $x=12$ line. The graph is shown below.

Cost of Gym Membership by Number of Months

To find the cost of a 12 month membership we simply read off the value of the function at the 12 month point. A line drawn up from $x=12$ on the x axis meets the function at a y value of $\$100$.

Solution

The cost of joining the gym for one year is $\$100$.

Example 5

Jesus has $\$12$ to spend on food for a class barbeque. Hot dogs cost $\$0.50$ each (including the bun) and burgers cost $\$0.50$ (including bun and salad). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than $\$12$.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that John buys is x, then the money spent on burgers is $c=9$.

If the number of hot dogs he buys is y then the money spent on hot dogs is $c \neq 0$
.

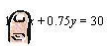
$1.25x + 0.75y$ The total cost of the food.

The total amount of money he has to spend is \$12. If he is to spend it ALL, then we can use the following equation.

$1.25x + 0.75y = 30$

We solve for the intercepts using the cover-up method.

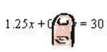
First the $y-$intercept ($x = 3$.



$0.75y = 30$

$y = 40$ $y - \text{intercept}(0, 40)$

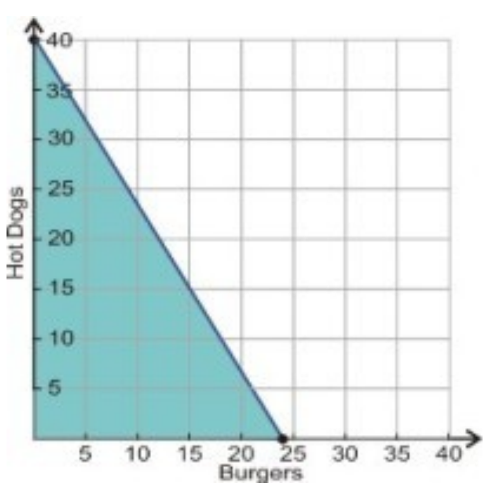
Then the $x-$intercept ($y = 5$)



$1.25x = 30$

$x = 24$ $x - \text{intercept}(24, 0)$

Possible Numbers of Hot Dogs and Hamburgers Purchased for \$12



We can now plot the points and join them to create our graph, shown right.

Here is an alternative to the equation method.

If Jesus were to spend ALL the money on hot dogs, he could buy $\frac{30}{0.75} = 40$ hot dogs. If on the other hand, he were to buy only burgers, he could buy $\frac{30}{0.75} = 40$ burgers. So you can see that we get two intercepts: (y burgers, 29 hot dogs) and ($2a$ burgers, y hot dogs). We would plot these in an identical manner and design our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an **inequality.** He can, in fact, spend any amount up to $\$12$. The only thing he cannot do is spend more than $\$12$. So our graph reflects this. The shaded region shows where Jesus' solutions all lie. We will see inequalities again in Chapter 6.

Lesson Summary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis ($x = 3$) and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis ($y = 5$).
- The $y-$intercept can be found by **substituting** $x = 3$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 5$ into the equation and solving for x.
- A linear equation is in **standard form** if it is written as "positive coefficient times x plus (or minus) positive coefficient times y equals value". Equations in standard form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

Review Questions

1. Find the intercepts for the following equations using substitution.
    1. $-2.5, 1.5, 5$
    2. $y = 0.8x + 3$
    3. $y = 40 + 25x$
    4. 87.5 grams
2. Find the intercepts of the following equations using the cover-up method.
    1. $5x - 6y = 15$
    2. $3x - 4y = -5$

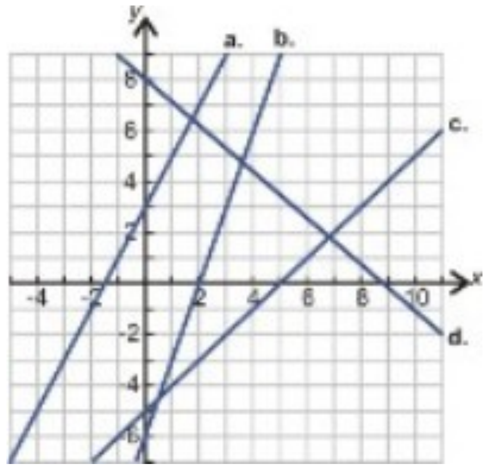
3. $\text{Change} = +12$
4. $5x + 10y = 25$

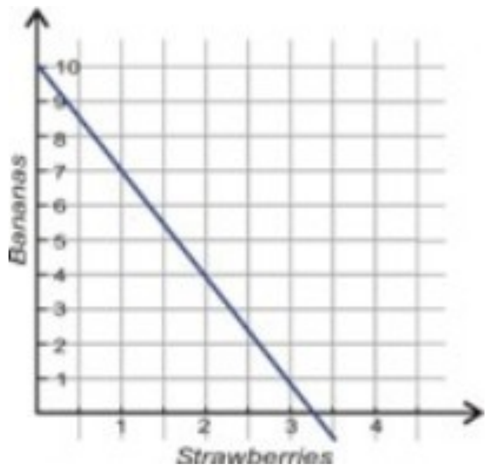
3. Use any method to find the intercepts and then graph the following equations.
    1. $-2.5, 1.5, 5$
    2. $79.5 \cdot (-1) = -79.5$
    3. $y = -120$
    4. $y = -120$
4. At the local grocery store strawberries cost $\$0.50$ per pound and bananas cost $\$0.50$ per pound. If I have $\$12$ to spend between strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly $\$12$.
5. A movie theater charges $\$0.50$ for adult tickets and $\$0.50$ for children. If the theater takes $\$100$ in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
6. Why can't we use the intercept method to graph the following equation?

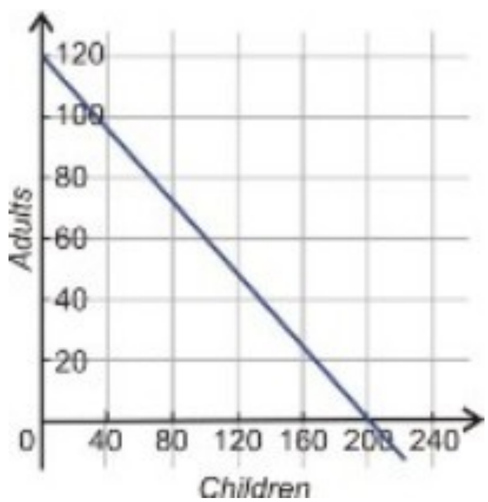
$$80 \geq 10(3(0.4) + 2)$$

Review Answers

1.
    1. $(3 + 2)$, $(0, 0)$
    2. $(0, 0)$, $(0, 0)$
    3. $(5 - 11)$, $(3 + 2)$
    4. $(0, 0)$, $\frac{1}{3} + \frac{1}{4}$
2.
    1. $(-5, -7)$, $(0, 0)$
    2. $(5 - 11)$, $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$
    3. $(0, -\frac{11}{7})$, $(-\frac{11}{2}, 0)$
    4. $(3 + 2)$, $(0, 0)$

3. 

4. 

5. 

6. This equation reduces to $y = 12x$, which passes through $(0, 0)$ and therefore only has **one intercept**. Two intercepts are needed for this method to work.

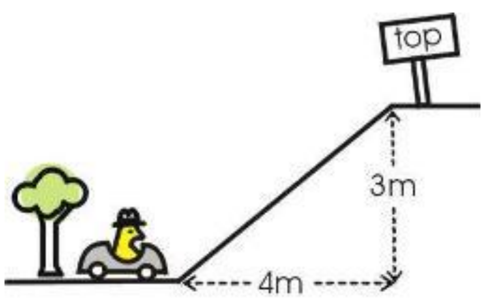
Slope and Rate of Change

Learning Objectives

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Introduction

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, and the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.



$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

This is often reworded to be easier to remember:

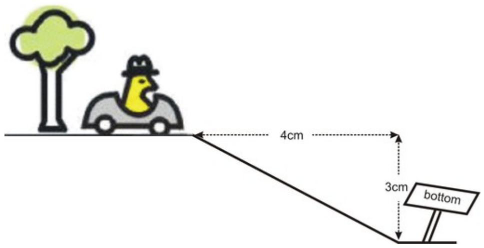
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

Essentially, slope is the change in y if x increases by 1.

In the picture to the right, the slope would be the ratio of the **height** of the hill (the **rise**) to the horizontal **length** of the hill (the **run**).

$$\text{Slope} = \frac{3}{4} = 0.75$$

If the car were driving to the **right** it would **climb** the hill. We say this is a positive slope. Anytime you see the graph of a line that goes up as you move to the right, the slope is **positive**.



If the car were to keep driving after it reached the top of the hill, it may come down again. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**. The picture at right has a **negative slope** of -0.75.

Do not get confused! If the car turns around and drives back down the hill shown, we would still classify the slope as positive. This is because the rise would be -8, but the run would be -2 (think of the $x-$axis – if you move from right to left you are moving in the negative $x-$direction). Our ratio for moving **left** is:

$$\text{Slope} = \frac{-3}{-4} = 0.75 \qquad \text{A negative divided by a negative is a positive.}$$

So as we move from left to right, positive slopes increase while negative slopes decrease.

Find a Positive Slope

We have seen that a function with a positive slope increases in y as we increase x. A simple way to find a value for the slope is to draw a right angled triangle whose hypotenuse runs along the line. It is then a simple matter of measuring the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown right.

There are already right-triangles drawn for each of the lines. In practice, you would have to do this yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. the coordinates are all integers).

a. The rise shown in this triangle is 4 units, the run is 4 units.

$$\text{Slope} = \frac{4}{2} = 2$$

b. The rise shown in this triangle is 4 units, the run is also 4 units.

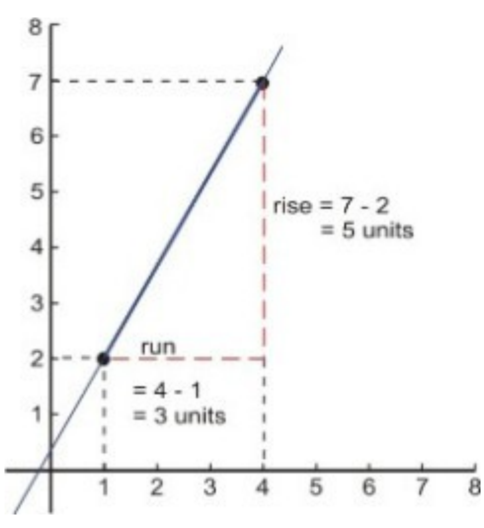
$$\text{Slope} = \frac{4}{4} = 1$$

c. The rise shown in this triangle is 4 units, the run is 4 units.

$$\text{Slope} = \frac{2}{4} = \frac{1}{2}$$

Example 2

Find the slope of the line that passes through the points $(0, 0)$ *and* $(0, 0)$.



We already know how to graph a line if we are given two points. We simply plot the points and connect them with a line. Look at the graph shown at right.

Since we already have coordinates for our right triangle, we can quickly work out that the rise would be y and the run would be y (see diagram). Here is our slope.

$$\text{Slope} = \frac{7-2}{4-1} = \frac{5}{3}$$

If you look closely at the calculations for the slope you will notice that the 7 and 4 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$ coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points $(x-3)$ and $(x-3)$.

Slope between $(x-3)$ and $2.8956 = 2\frac{8956}{10000}$ or $\frac{1.3}{4} = \frac{x}{1.3}$

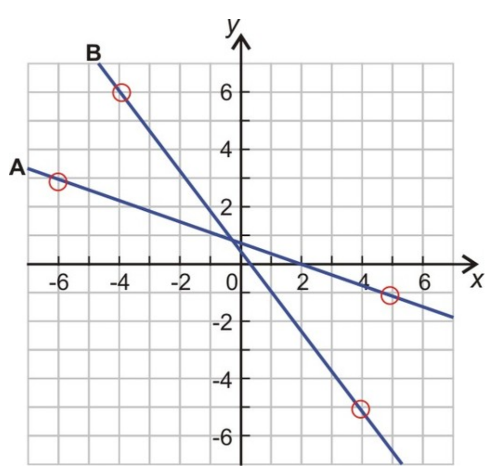
In the second equation, the letter m denotes the slope (you will see this a lot in this chapter) and the Greek letter delta (P) means **change**. So another way to express slope is *change in* y divided by *change in* x. In the next section, you will see that it does not matter which point you choose as point 1 and which you choose as point 4.

Find a Negative Slope

Any function with a negative slope is simply a function that decreases as we increase x. If you think of the function as the incline of a road a negative slope is a road that goes **downhill** as you drive to the **right**.

Example 3

Find the slopes of the lines on the graph to the right.



Look at the lines. Both functions fall (or decrease) as we move from left to right. Both of these lines have a **negative slope**.

Neither line passes through a great number of lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been identified (with rings) and we will use these to determine the slope. We will also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 4.

For line A:

$$(x_1, y_1) = (-6, 3) \ (x_2, y_2) = (5, -1) \qquad (x_1, y_1) = (5, -1) \ (x_2, y_2) = (-6, 3)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (-5)} = \frac{-4}{11} \approx -0.364$$

For line P:

$$(x_1, y_1) = (-4, 6) \ (x_2, y_2) = (4, -5) \qquad (x_1, y_1) = (4, -5) \ (x_2, y_2) = (-4, 6)$$

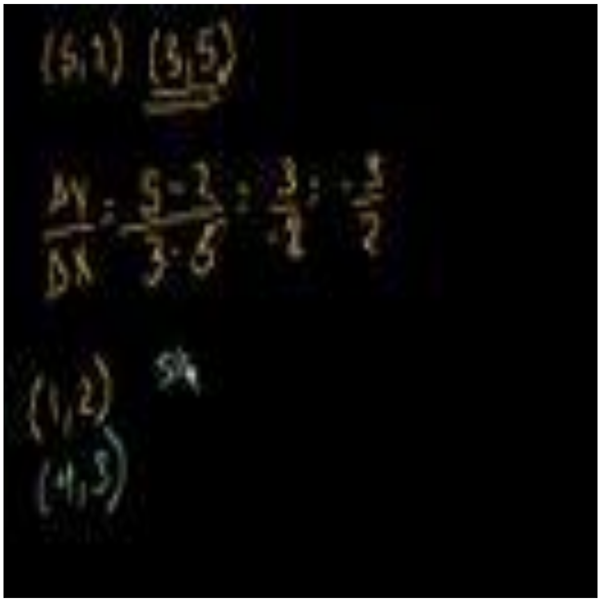
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

You can see that whichever way you select the points, the answers are the same!

Solution

Line A has slope $t = 0.4$. Line P has slope -1.375.

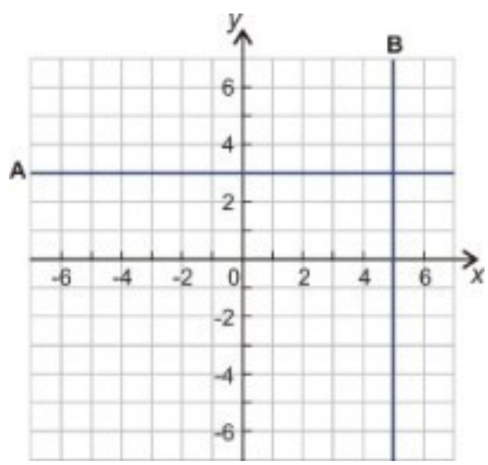
Multimedia Link The series of videos starting at Khan Academy Slope (8:28) models several more examples of finding the slope of a line given two points.



Figuring out the slope of a line([Watch on Youtube]())

Find the Slopes of Horizontal and Vertical lines

Example 4



Determine the slopes of the two lines on the graph at the right.

There are two lines on the graph. A ($y = 5$) and P ($x = 3$).

Let's pick two points on line A. say, $(x_1, y_1) = (-4, 3)$ and $3 \times (5 - 7) \div 2$ and use our equation for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0$$

If you think about it, this makes sense. If there is no change in y as we increase x then there is no slope, or to be correct, a slope of zero. You can see that this must be true for all horizontal lines.

Horizontal lines $|4 - 9| - | - 5|$ all have a slope of y.

Now consider line P. Pick two distinct points on this line and plug them in to the slope equation.

$(x_1, y_1) = (-4, 3)$ and $3 \times (5 - 7) \div 2$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-3)}{(5) - (5)} = \frac{7}{0} \qquad \text{A division by zero!}$$

Divisions by zero lead to infinities. In math we often use the term **undefined** for any division by zero.

Vertical lines $\frac{z}{5} + 1 < z - 20$ all have an infinite (or undefined) slope.

Find a Rate of Change

The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case, the slope refers to a change in one quantity (y) **per** unit change in another quantity (x).

Example 5

Andrea has a part time job at the local grocery store. She saves for her vacation at a rate of $\$12$ *every week. Express this rate as money saved* ***per day*** *and money saved* ***per year***.

Converting rates of change is fairly straight forward so long as you remember the equations for rate (i.e. the equations for slope) and know the conversions. In this case $1 \text{ week} = \ 7 \text{ days}$ and $52 \text{ weeks} = 1 \text{ year}$.

$$\text{rate} = \frac{\$15}{1\text{ week}} \cdot \frac{1\text{ week}}{7\text{ days}} = \frac{\$15}{7\text{ days}} = \frac{15}{7}\text{ dollars per day} \approx \ \$2.14\text{ per day}$$

$$\text{rate} = \frac{\$15}{1\text{ week}} \cdot \frac{52\text{ week}}{1\text{ year}} = \$15 \cdot \frac{52}{\text{year}} = \$780\text{ per year}$$

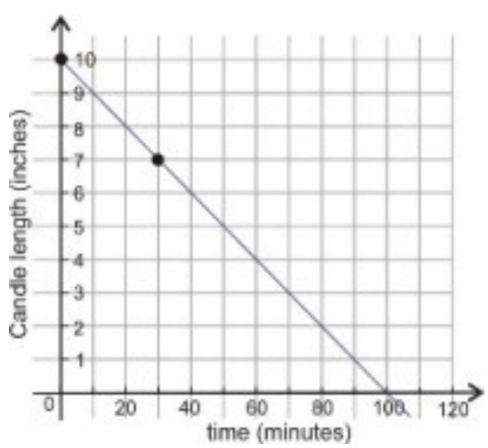
Example 6

A candle has a starting length of 16 *inches. Thirty minutes after lighting it, the length is* $x+1=$ *. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.*

In this case, we will graph the function to visualize what is happening.

We have two points to start with. We know that at the moment the candle is lit $|-5-11|$ the length of the candle is 16 inches. After thirty minutes $f(x)=3.2^x$ the length is $x+1=$. Since the candle length is a function of time we will plot time on the horizontal axis, and candle length on the vertical axis. Here is a graph showing this information.

Candle Length by Burning Time



The rate of change of the candle is simply the slope. Since we have our two points $4-(7-11)+2$ and $4-(7-11)+2$ we can move straight to the formula.

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7\text{ inches}) - (10\text{ inches})}{(30\text{ minutes}) - (0\text{ minutes})} = \frac{-3\text{ inches}}{30\text{ minutes}} = -0.1\text{ inches per minute}$$

The slope is negative. A negative rate of change means that the quantity is decreasing with time.

We can also convert our rate to inches per hour.

$$\text{rate} = \frac{-0.1 \text{ inches}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{-6 \text{ inches}}{1 \text{ hour}} = -6 \text{ inches per hour}$$

To find the point when the candle reaches zero length we can simply read off the graph (100 minutes). We can use the rate equation to verify this algebraically.

$$\text{Length burned } = \text{rate} \times \text{time}$$

$$2 \times 2 \times 9 = 36$$

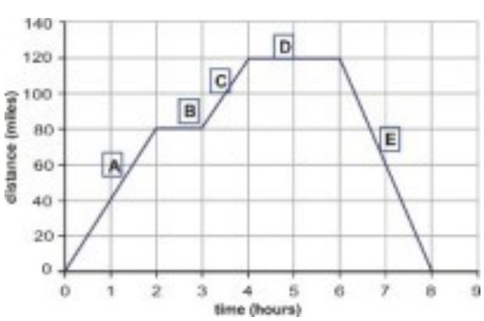
Since the candle length was originally 16 inches this confirms that $4n + 5 = 21$ is the correct amount of time.

Interpret a Graph to Compare Rates of Change

Example 7

> *Examine the graph below. It represents a journey made by a large delivery truck on a particular day. During the day, the truck made two deliveries, each one taking one hour. The driver also took a one hour break for lunch. Identify what is happening at each stage of the journey (stages A through E)*

Truck's Distance from Home by Time



Here is the driver's journey.

A. The truck sets off and travels $-9x + 2$ in $2 > -5$.

B. The truck covers no distance for $b = -2$.

C. The truck covers $(120 - 80) = 40 \text{ miles}$ in $b = -2$

D. the truck covers no distance for $2 > -5$.

E. The truck covers 150 miles in $2 > -5$.

Lets look at the rates of change for each section.

A. $\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40 \text{ miles per hour}$

- The rate of change is a **velocity!** This is a very important concept and one that deserves a special note!

The slope (or rate of change) of a distance-time graph is a velocity.

You may be more familiar with calling **miles per hour** a **speed. Speed** is the **magnitude** of a **velocity,** or, put another way, velocity has a direction, speed does not. This is best illustrated by working through the example.

On the first part of the journey sees the truck travel at a constant velocity of 40 mph for $2 > -5$ covering a distance of $-9x + 2$.

B. Slope = y so **rate of change** = $2^3 = 8$. The truck is stationary for one hour. This could be a lunch break, but as it is only $2 > -5$ since the truck set off it is likely to be the first delivery stop.

C. $\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{(120-80)\ miles}{(4-3)\ hours} = 40 \text{ miles per hour}$. The truck is traveling at 40 mph.

D. Slope = y so **rate of change** = $2^3 = 8$. The truck is stationary for two hours. It is likely that the driver used these $2 > -5$ for a lunch break plus the second delivery stop. At this point the truck is 150 miles from the start position.

E. $\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{(0-120)\ miles}{(8-6)\ hours} = \frac{-120\ miles}{2\ hours} = -60 \text{ miles per hour}$. The truck is traveling at **negative** 40 mph .

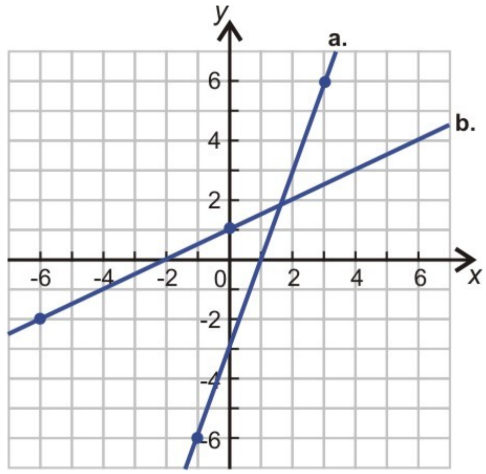
Wait, a negative velocity? Does this mean that the truck is reversing? Well, probably not. What it means is that the distance (and don't forget that is the distance measured from the starting position) is decreasing with time. The truck is simply driving in the opposite direction. In this case, back to where it started from. So, the speed of the truck would be 40 mph, but the velocity (which includes direction) is negative because the truck is getting closer to where it started from. The fact that it no longer has two heavy loads means that it travels faster (40 mph as opposed to 40 mph) covering the 30 ohms return trip in $2 > -5$.

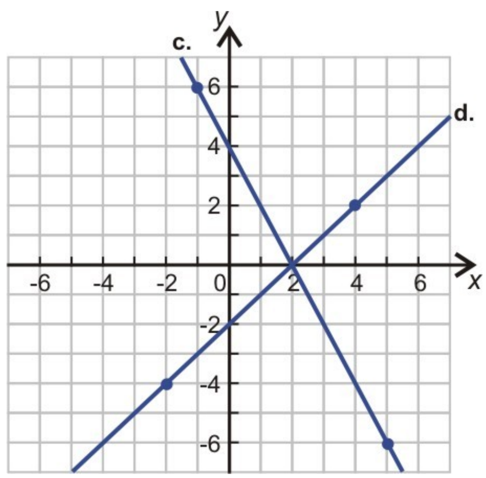
Lesson Summary

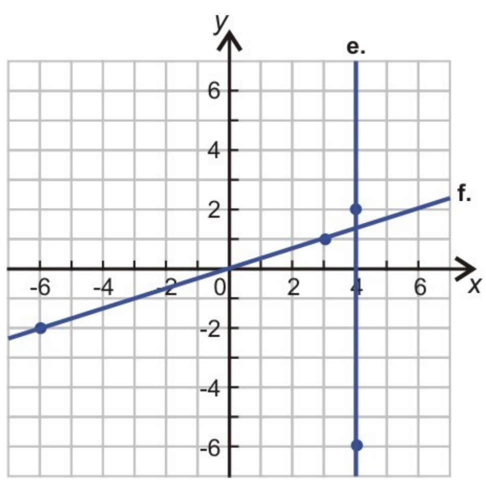
- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- $\frac{15552}{12} = 1296$ or $\frac{1.3}{4} = \frac{x}{1.3}$
- The slope between two points $(x-3)$ and $2.8956 = 2\frac{8956}{10000}$
- **Horizontal lines** ($t = 19, u = 4$) all have a slope of y.
- **Vertical lines** ($x = \text{constant}$) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

Review Questions

1. Use the slope formula to find the slope of the line that passes through each pair of points.
    1. $(3+2)$ and $(0,0)$
    2. $(-5,-7)$ and (mph)
    3. $(3+2)$ and $(3+2)$
    4. $(3+2)$ and $(5-11)$
    5. $(0,0)$ and $(-5,-7)$
    6. $(0,0)$ and $(3+2)$
2. Use the points indicated on each line of the graphs to determine the slopes of the following lines.



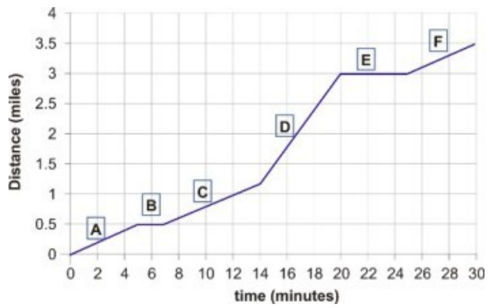




3. The graph below is a distance-time graph for Mark's three and a half mile cycle ride to school. During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He

stopped once at a traffic light and at one point he stopped to mend a tire puncture. Identify each section of the graph accordingly.

Andrew's Distance from Home by Time



Review Answers

1.
    1. -1.4
    2. 23.7
    3. $= 15$
    4. undefined
    5. 1
    6. $= 15$
2.
    1. y
    2. $y-$
    3. -2
    4. 1
    5. undefined
    6. $\frac{2}{3}$
3.
    1. A. uphill
    2. B. stopped (traffic light)
    3. C. uphill
    4. D. downhill
    5. E. stopped (puncture)
    6. F. uphill

Graphs Using Slope-Intercept Form

Learning Objectives

- Identify the slope and $y-$intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Identify Slope and y-intercept

One of the most common ways of writing linear equations prior to graphing them is called **slope-intercept form.** We have actually seen several slope-intercept equations so far. They take the following form:

$y = mx + b$ where m is the slope and the point $(0, b)$ is the $y-$intercept.

We know that the $y-$intercept is the point at which the line passes through the $y-$axis. The slope is a measure of the steepness of the line. Hopefully, you can see that if we know **one point** on a line and the slope of that line, we know what the line is. Being able to quickly identify the $y-$intercept and slope will aid us in graphing linear functions.

Example 1

Identify the slope and $y-$intercept of the following equations.

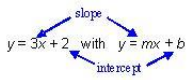
a) $-2.5, 1.5, 5$

b) $y = 0.8x + 3$

c) $y = -5d$

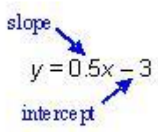
d) $y = -2$

Solution

a) 

Comparing, we see that 1 hour *and* $c = 9$.

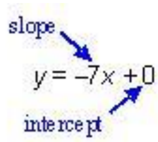
$-2.5, 1.5, 5$ **has a slope of** y and a $y-$**intercept of** $(0, 0)$

b) 

has a slope of $y-$ *and a* y*-intercept of* $(3 + 2)$.

Note that the $y-$intercept is **negative**. The b term includes the sign of the operator in front of the number. Just remember that $y = 0.8x + 3$ is identical to $4(-3) + 3 = -9$ and is in th eform $y = mx + b$.

c) **At first glance, this does not appear to fit the slope-intercept form. To illustrate how we deal with this, let us rewrite the equation.**



. *We now see that we get a slope of* -7 *and a* $y-$*intercept of* $(0, 0)$.

Note that the slope is negative. The $(0, 0)$ intercept means that the line passes through origin.

d) **Rewrite as** $-2.5, 1.5, 5$, **giving us a slope of 0 and an intercept of** $(3 + 2)$.

Remember:

- When 1 hour the slope is negative.

For example, $y = 0.8x + 3$ has a slope of -8.

- When $b = 3$ the intercept is below the x axis.

For example, $-2.5, 1.5, 5$ has a $y-$intercept of $(3 + 2)$.

- When 1 hour the slope is zero and we have a horizontal line.

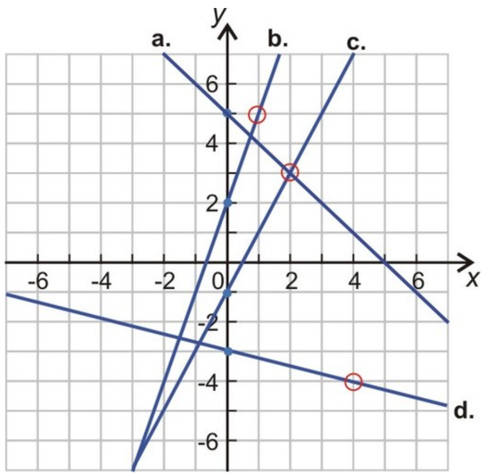
For example, $y = 5$ can be written as $-2.5, 1.5, 5$.

- When $b = 3$ the graph passes through the origin.

For example, $1.35 \cdot y$ can be written as $40 \quad \text{coins}$.

Example 2

Identify the slope and y-intercept of the lines on the graph shown to the right.



The intercepts have been marked, as have a number of lattice points that lines pass through.

a. The $y-$intercept is $(0, 0)$. The line also passes through $(0, 0)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$$

b. The $y-$intercept is $(0, 0)$. The line also passes through $(0, 0)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$$

c. The $y-$intercept is $(3 + 2)$. The line also passes through $(0, 0)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$$

d. The $y-$intercept is $(3+2)$. The line also passes through $(3+2)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{-1}{4} = \frac{-1}{4} \text{ or } -0.25$$

Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line it is easy to graph it. Just remember what slope means. Let's look back at this example from Lesson 4.1.

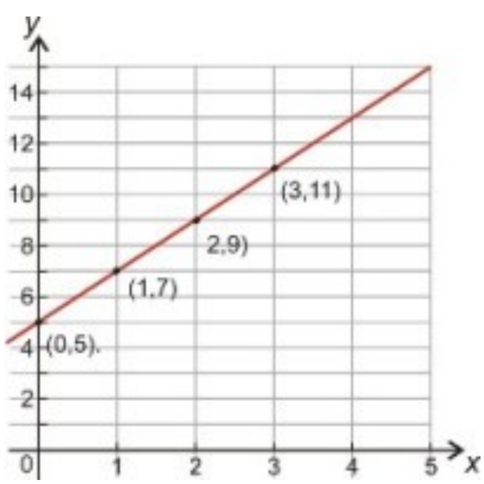
Example 3

Ahiga is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number. Then double it. Then add five to what he got. Ahiga has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the result of applying the rule. His rule is:

$-2.5, 1.5, 5$

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed the following table of values.



x	y
0	$2.0 + 5 = 0 + 5 = 5$
1	$2.1 + 5 = 2 + 5 = 7$
2	$2.2 + 5 = 4 + 5 = 9$
3	$2.3 + 5 = 6 + 5 = 11$

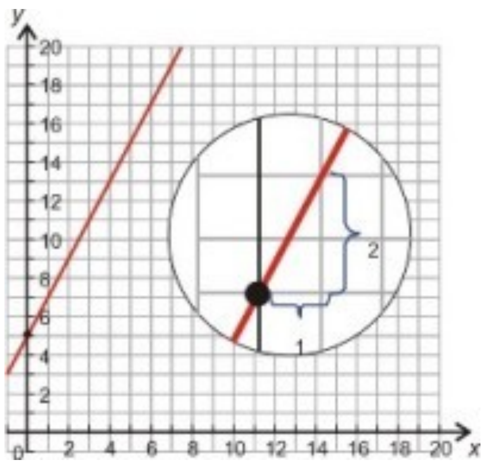
The first entry gave us our y intercept $(0, 0)$. The other points helped us graph the line.

We can now use our equation for slope, and two of the given points.

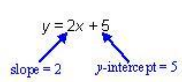
Slope between $3 \times (5 - 7) \div 2$ and $4 - (7 - 11) + 2$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 5}{3 - 0} = \frac{6}{3} = 2$$

Thus confirming that the slope, $= 12\times$.



An easier way to graph this function is the slope–intercept method. We can now do this quickly, by identifying the intercept and the slope.

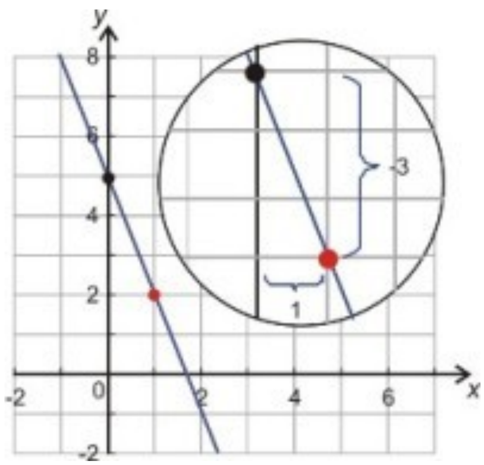


Look at the graph we drew, the line intersects the $y-$axis at y, and every time we move to the right by one unit, we move up by two units.

So what about plotting a function with a negative slope? Just remember that a negative slope means the function decreases as we increase x.

Example 4

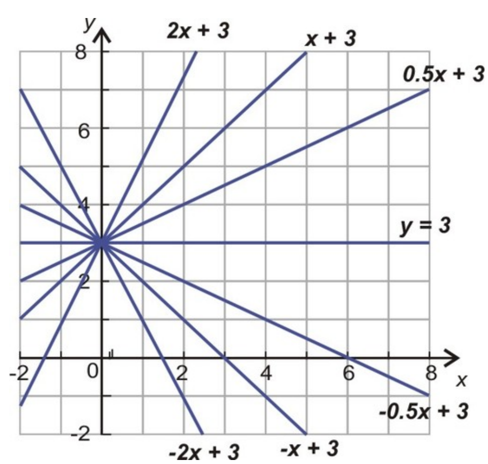
Graph the following function. $y = 0.8x + 3$



- Identify $y-$intercept $b = 3$
- Plot intercept $(0, 0)$
- Identify slope $F = ma$
- Draw a line through the intercept that has a slope of -8.

To do this last part remember that $\text{slope} = \frac{\text{rise}}{\text{run}}$ so for every unit we move to the right the function increases by -8 (in other words, for every square we move right, the function comes **down** by y).

Changing the Slope of a Line



Look at the graph on the right. It shows a number of lines with different slopes, but all with the same $y-$intercept $(0, 0)$.

You can see all the positive slopes increase as we move from left to right while all functions with negative slopes fall as we move from left to right.

Notice that the higher the value of the slope, the steeper the graph.

The graph of $-2.5, 1.5, 5$ appears as the mirror image of $y = 0.8x + 3$. The two slopes are equal but opposite.

Fractional Slopes and *Rise Over Run*

Look at the graph of $y = 0.8x + 3$. As we increase the x value by 1, the y value increases by $y-$. If we increase the x value by 4, then the y value increases by 1. In fact, if you express any slope as a fraction, you can determine how to plot the graph by looking at the numerator for the *rise* (keep any negative sign included in this term) and the denominator for the *run*.

Example 5

Find integer values for the ***rise*** *and* ***run*** *of following slopes then graph lines with corresponding slopes.*

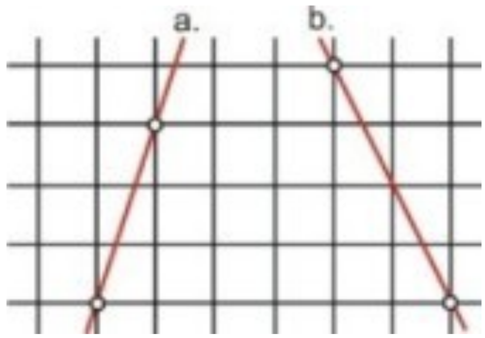
a. 1 hour

b. $m = -2$

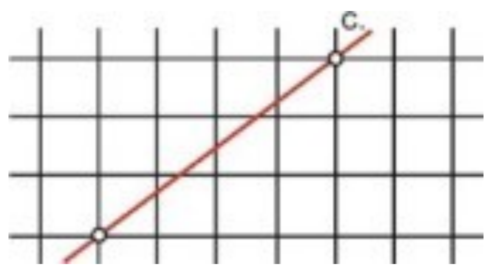
c. 150 miles

d. 2.46 seconds

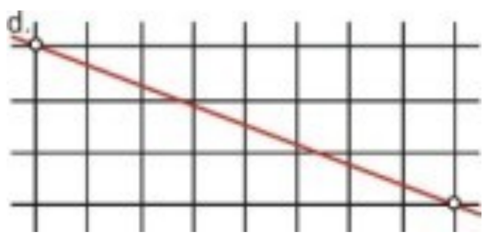
Solution:



a. $3=\frac{3}{1}$ As we move **across** 1 unit we move **up** by 3



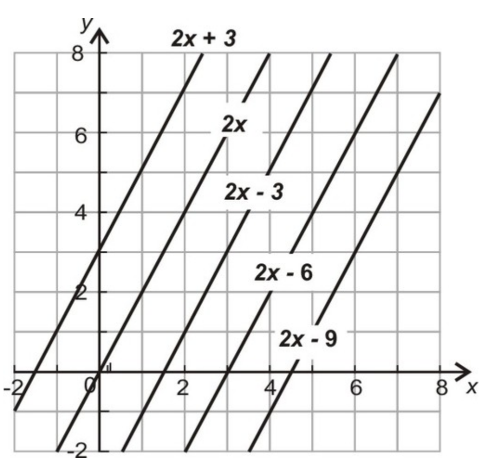
b. $-2=\frac{-2}{1}$ As we move **across** 1 unit we move **down** by 2



c. $0.75=\frac{3}{4}$ As we move **across** 4 units we move **up** by 3

d. $-0.375=\frac{-3}{8}$ As we move **across** 8 units we move **down** by 3

Changing the Intercept of a Line



When we take an equation (such as $1.35 \cdot y$) and change the y intercept (leaving the slope intact) we see the following pattern in the graph on the right.

Notice that changing the intercept simply translates the graph up or down. Take a point on the graph of $1.35 \cdot y$, such as $(0, 0)$. The corresponding point on $-2.5, 1.5, 5$ would be $(0, 0)$. Similarly the corresponding point on the $-2.5, 1.5, 5$ line would be $(3 + 2)$.

Will These Lines Ever Cross?

To answer that question, let us take two of the equations $1.35 \cdot y$ and $-2.5, 1.5, 5$ and solve for values of x and y that satisfy both equations. This will give us the (x, y) coordinates of the point of intersection.

$$\begin{array}{lcl} 2x = 2x + 3c & & \text{Subtract } 2x \text{ from both sides.} \\ 0 = 0 + 3 & \text{or} & 0 = 3 \text{ This statement is FALSE!} \end{array}$$

When we get a false statement like this, it means that there are **no** ($y =$) values that satisfy both equations simultaneously. The lines will **never** cross, and so they **must be parallel**.

Identify Parallel Lines

In the previous section, when we changed the intercept but left the slope the same, the new line was parallel to the original line. This would be true whatever the slope of the original line, as changing the intercept on a $y = mx + b$ graph does nothing to the slope. This idea can be summed up as follows.

Any two lines with identical slopes are parallel.

Lesson Summary

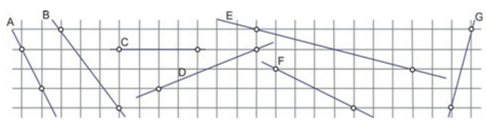
- A common form of a line (linear equation) is **slope-intercept form:**

  $y = mx + b$ where m is the slope and the point $(0, b)$ is the $y-$intercept

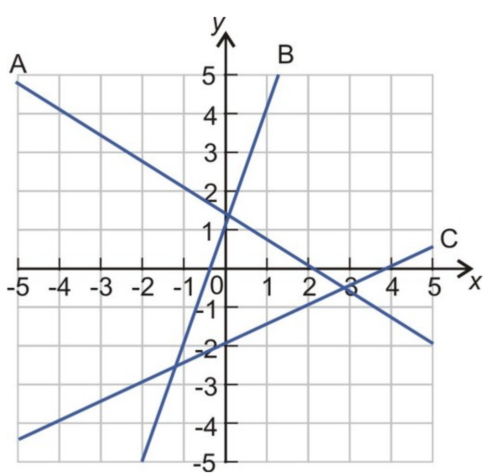
- Graphing a line in slope-intercept form is a matter of first plotting the $y-$intercept $(0, b)$, then plotting more points by moving a step to the right (adding 1 to x) and moving the value of the slope vertically (adding m to y) before plotting each subsequent point.
- Any two lines with identical slopes are **parallel.**

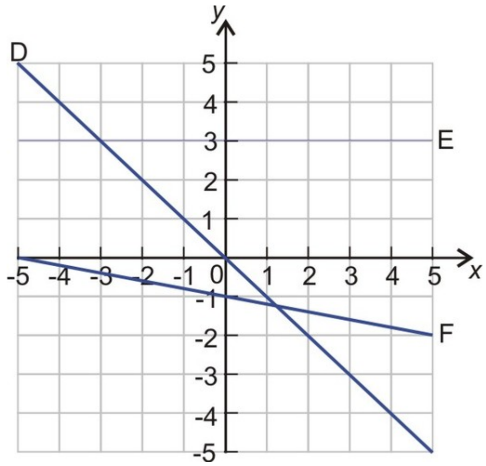
Review Questions

1. Identify the slope and $y-$intercept for the following equations.
    1. $-2.5, 1.5, 5$
    2. $y = -0.2x + 7$
    3. 93000
    4. $P = 20t;$
2. Identify the slope of the following lines.



3. Identify the slope and $y-$intercept for the following functions.

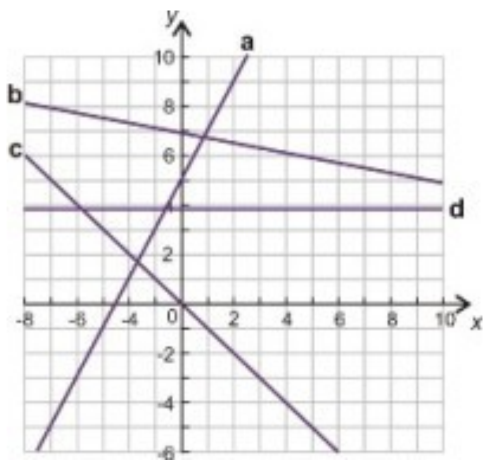




4. Plot the following functions on a graph.
    1. $-2.5, 1.5, 5$
    2. $y = -0.2x + 7$
    3. 2 weeks
    4. $P = 20t;$
5. Which two of the following lines are parallel?
    1. $-2.5, 1.5, 5$
    2. $y = -0.2x + 7$
    3. 2 weeks
    4. $P = 20t;$
    5. $\frac{3x+4}{3} - 5x = 6$
    6. $y = 0.8x + 3$
    7. $3y + 5 = -2y$
    8. $y = 0.8x + 3$

Review Answers

1.
    1. $80 \geq 10(3.2)$
    2. $2(15) = 20 + 12$
    3. $80 \geq 10(3.2)$
    4. $80 \geq 10(3t + 2)$
2.
    1. A. $m = -2$
    2. B. $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
    3. C. 1 hour

    4. D. $\frac{1}{3} \cdot \$60$
    5. E. $2 - 3 = -1$
    6. F. $r \cdot w = 15$
    7. G. $= 12\times$
3.
    1. A. $y = -\frac{2}{3}x + 1.5$
    2. B: $y = 3x + 1$
    3. C: $y = 0.8x + 3$
    4. D: 2 weeks
    5. E: $y = 5$
    6. F: $y = -0.2x - 1$
4. 

5. *b* and *e*

Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges $\$11.95$ for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths of the amount of strawberries, you would pay two-fifths of $\$11.95$ for your strawberries.

$$\frac{2}{5} \times \$12.50 = \$5.00$$

Similarly, if you bought 16 pounds of strawberries (twice the amount) you would pay $2 \times \$12.50$ and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as:

$$y = -5d$$

y is called the **constant of proportionality**.

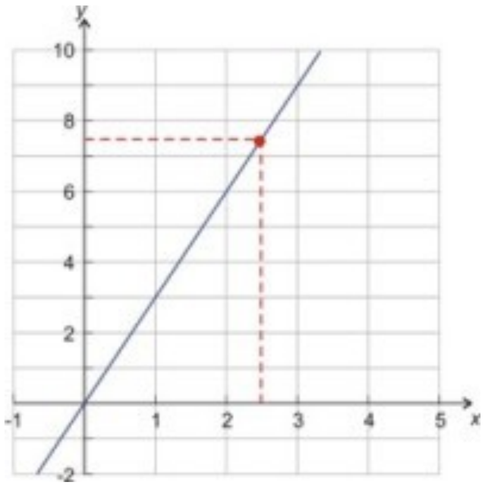
If we were to graph this function you can see that it passes through the origin, because $y = 5$, when $x = 3$ whatever the value of y. So we know that a direct variation, when graphed, has a single intercept at $(0, 0)$.

Example 1

If y varies directly with x according to the relationship $y = -5d$, and $y = 7.5$ when $x = -5$, determine the constant of proportionality, y.

We can solve for the constant of proportionality using substitution.

Substitute $x = -5$ and $y = 7.5$ into the equation $y = -5d$



$7.5 = k(2.5)$ Divide both sides by 2.5.

$$\frac{7.5}{2.5} = k = 3$$

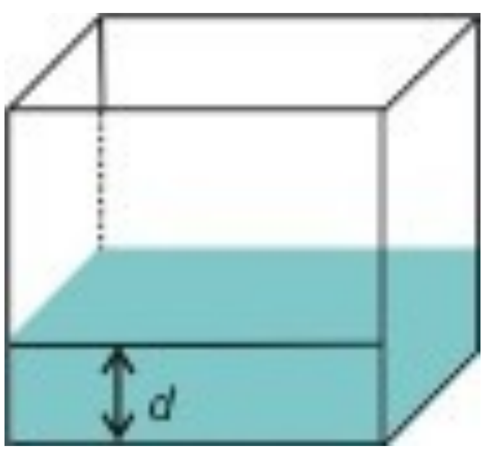
Solution

The constant of proportionality, $x = 7$.

We can graph the relationship quickly, using the intercept $(0, 0)$ and the point $9/9 = 1$. The graph is shown right. It is a straight line with a slope $y =$.

The graph of a direct variation has a slope that is equal to the constant of proportionality, y.

Example 2



The volume of water in a fish-tank, V, varies directly with depth, h. If there are 16 gallons in the tank when the depth is eight inches, calculate how much water is in the tank when the depth is 29 inches.

This is a good example of a direct variation, but for this problem we will need to determine the equation of the variation ourselves. Since the volume, V, depends on depth, h, we will use the previous equation to create new one that is better suited to the content of the new problem.

$$\begin{aligned} &y = k \cdot x \quad \text{In place of } y \text{ we will use } V \text{ and in place of } x \text{ we will use d.} \\ &V = k \cdot d \end{aligned}$$

We know that when the depth is y inches, the volume is 16 gallons. Now we can substitute those values into our equation.

Substitute 6 times and $x = 3$:

$$\begin{aligned} V &= k \cdot d \\ 15 &= k(8) \qquad \text{Divide both sides by8.} \\ \frac{15}{8} &= k = 1.875 \end{aligned}$$

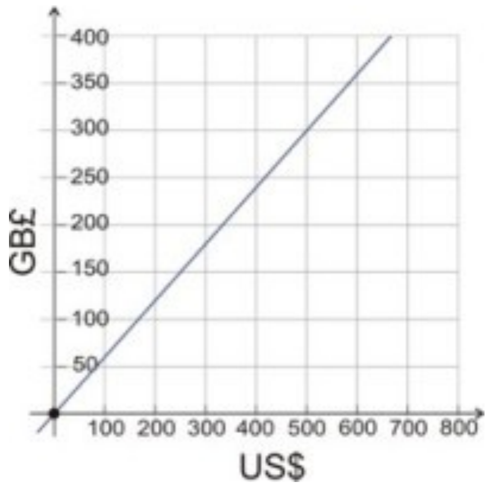
Now to find the volume of water at the final depth we use 150 miles and substitute for our new h.

$$\begin{aligned} V &= k \cdot d \\ V &= 1.875 \times 20 \\ V &= 37.5 \end{aligned}$$

Solution

At a depth of 29 inches, the volume of water in the tank is 23.7 gallons.

Example 3



The graph shown to the right shows a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine the following.

(i) The number of pounds you could buy for $\$100$.

(ii) The number of dollars it would cost to buy £302.

(iii) The exchange rate in pounds per dollar.

(iv) Is the function continuous or discrete?

Solution

In order to solve (i) and (ii) we could simply read off the graph: it looks as if at $x = 250$ the graph is about one fifth of the way between £302 and £302. So $\$100$ would buy £302. Similarly, the line $y = 12x$ would appear to intersect the graph about a third of the way between $\$100$ and $\$100$. We would probably round this to $\$100$. So it would cost approximately $\$100$ to buy £302.

To solve for the exchange rate we should note that as this is a direct variation, because the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value. Looking closely at the graph, it is clear that there is one lattice point that the line passes through $\frac{x}{2} - \frac{y}{2} - 4$. This will give us the most accurate estimate for the slope (exchange rate).

$$y = k \cdot x \Rightarrow k = \frac{y}{x}$$

$$\text{rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, y. Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

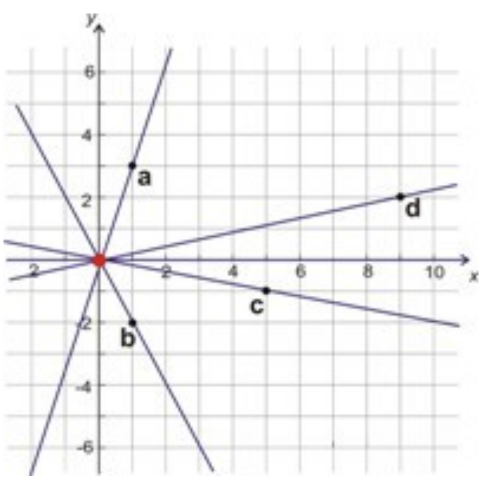
Plot the following direct relations on the same graph.

a. $1.35 \cdot y$

b. $y = -2x$

c. $y = -0.2x$

d. $p = \frac{12}{0.8}$



Solution

a. The line passes through $(0, 0)$. All these functions will pass through this point. It is plotted in red. This function has a slope of y. When we move across by one unit, the function increases by three units.

b. The line has a slope of -2. When we move across the graph by one unit the function **falls** by two units.

c. The line has a slope of $= 15$. As a fraction this is equal to $-\frac{8}{9}$ When we move across by five units, the function **falls** by one unit.

d. The line passes through $(0, 0)$ and has a slope of $\frac{2}{3}$. When we move across the graph by y units, the function increases by two units.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time that we have one quantity that doubles when another related quantity doubles, we say that they follow a direct variation.

Newton's Second Law

In 1687, Sir Isaac Newton published the famous *Principea Mathematica*. It contained, among other things, his Second Law of Motion. This law is often written as:

$5x = 3.25$

A force of A (Newtons) applied to a mass of m (kilograms) results in acceleration of a $(\mathrm{meterspersecond}^2)$.

Example 5

If a 750 Newton force causes a heavily loaded shopping cart to accelerate down the aisle with an acceleration of $\frac{30}{0.75} = 40$, calculate

(i) The mass of the shopping cart.

(ii) The force needed to accelerate the same cart at $6\ \mathrm{m/s}^2$.

Solution

(i) This question is basically asking us to solve for the constant of proportionality. Let us compare the two formulas.

$$\begin{array}{ll} y = k \cdot x & \text{The direct variation equation} \\ F = m \cdot a & \text{Newton's Second law} \end{array}$$

We see that the two equations have the same form; y is analogous to force and x analogous to acceleration.

We can solve for m (the mass) by substituting our given values for force and acceleration:

Substitute 30 ohms, $x = -5$

$$\begin{array}{ll} 175 = m(2.5) & \text{Divide both sides by } 2.5. \\ 70 = m & \end{array}$$

The mass of the shopping cart is $c \neq 0$.

(ii) Once we have solved for the mass we simply substitute that value, plus our required acceleration back into the formula $5x = 3.25$ and solve for A:

Substitute $I = 2.5$, $x = 3$

$$a + 6 = 2 + 6 = 8$$

The force needed to accelerate the cart at $\frac{5}{16} + \frac{5}{12}$ is 302 Newtons.

Ohm's Law

The electrical current, y (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$$\begin{array}{llll} \text{Guess} & 5 & 4(5) + 16 = 36 & \text{This is the right age.} \end{array}$$

The resistance is considered to be a constant for all values of V and y.

Example 6

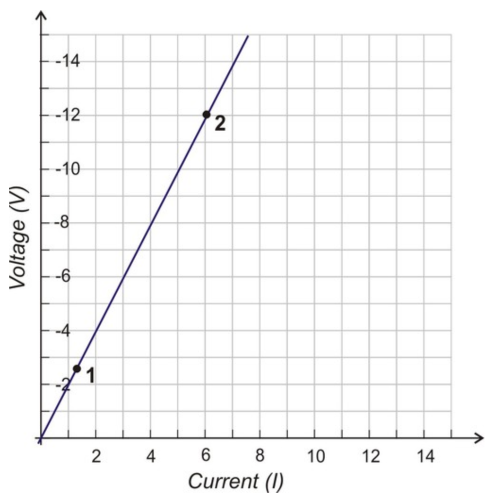
A certain electronics component was found to pass a current of -8 *amps at a voltage of* $y-$ *volts. When the voltage was increased to* -53 *volts the current was found to be* $y-$ *amps.*

a) Does the component obey Ohms law?

b) What would the current be at y *volts?*

Solution

a) Ohm's law is a simple direct proportionality law. Since the resistance R is constant, it acts as our constant of proportionality. In order to know if the component obeys Ohm's law we need to know if it follows a direct proportionality rule. In other words **is** V **directly proportional to** y?



Method One – Graph It

If we plot our two points on a graph and join them with a line, does the line pass through $(0, 0)$?

Point 1 $9 > 3$, $I = 2.5$ our point is $\frac{x}{2} - \frac{y}{2} - 4$

Point 2 $h = 8 \text{ cm}$, $I = 2.5$ our point is (mph)

Plotting the points and joining them gives the following graph.

The graph does appear to pass through the origin, so…

Yes, the component obeys Ohms law.

Method Two – Solve for R

We can quickly determine the value of R in each case. It is the ratio of the voltage to the resistance.

Case 1 $R = \frac{V}{I} = \frac{2.6}{1.3} = 2 \text{ Ohms}$

Case 2 $R = \frac{V}{I} = \frac{12}{6} = 2 \text{ Ohms}$

> The values for R agree! This means that the line that joins point 1 to the origin is the same as the line that joins point 4 to the origin. **The component obeys Ohms law**.

b) To find the current at y volts, simply substitute the values for V and R into 5 minutes

Substitute $V = 6$, $V = 6$

- In physics, it is customary to plot voltage on the horizontal axis as this is most often the independent variable. In that situation, the slope gives the **conductance,** x. However, by plotting the current on the horizontal axis, the **slope** is equal to the **resistance,** R.

$$
\begin{aligned}
6 &= I(2) && \text{Divide both sides by 2.}\\
3 &= I
\end{aligned}
$$

Solution

The current through the component at a voltage of y volts is y amps.

Lesson Summary

- If a variable y varies ***directly*** with variable x, then we write the relationship as

$y = -5d$

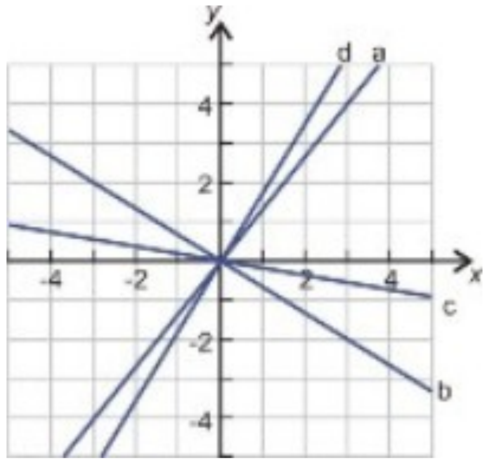
Where y is a constant called the **constant of proportionality.**

- **Direct variation** is very common in many areas of science.

Review Questions

1. Plot the following direct variations on the same graph.
   1. $p = \frac{12}{0.8}$
   2. $\frac{1.25}{7} = \frac{3.6}{x}$
   3. $\frac{1.25}{7} = \frac{3.6}{x}$
   4. $100 \text{ miles},$
2. Dasan's mom takes him to the video arcade for his birthday. In the first 16 minutes, he spends $\$0.50$ playing games. If his allowance for the day is $\$11.95$, how long can he keep playing games before his money is gone?
3. The current standard for low-flow showerheads heads is $y-$ gallons per minute. Calculate how long it would take to fill a 29 gallon bathtub using such a showerhead to supply the water.
4. Amen is using a hose to fill his new swimming pool for the first time. He starts the hose at 16 P.M. and leaves it running all night. At y AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
5. Land in Wisconsin is for sale to property investors. A 302 acre lot is listed for sale for $36 \leq 96$. Assuming the same price per acre, how much would a 29 acre lot sell for?
6. The force (A) needed to stretch a spring by a distance x is given by the equation 2 seconds, where y is the spring constant (measured in Newtons per centimeter, $\frac{x}{3} = 15$. If a 12 Newton force stretches a certain spring by 93000, calculate:
   1. The spring constant, y
   2. The force needed to stretch the spring by 2017 .
   3. The distance the spring would stretch with a 29 Newton force.

Review Answers

1. 

2. 27 minutes 150 miles
3. $3 \times 5 = 15$
4. $18 - x$ Midday
5. $\$21, 302$
6.
    1. $k = 1.2 \text{ N/cm}$
    2. 2.46 seconds
    3. 15 ohms.

Linear Function Graphs

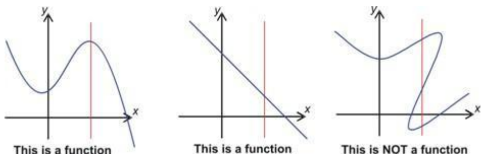
Learning Objectives

- Recognize and use function notation.
- Graph a linear function.
- Change slope and intercepts of function graphs.
- Analyze graphs of real-world functions.

Introduction – Functions

So far we have used the term **function** to describe many of the equations we have been graphing, but the concept of a function is extremely important in mathematics. Not all equations are functions. In order to be a function, the relationship between two variables, x and y, must map each $x-$value to **exactly one** $y-$value.

Visually this means the graph of y versus x must pass the **vertical line test** meaning that a vertical line drawn through the graph of the function must never intersect the graph in more than one place.



Use Function Notation

When we write functions we often use the notation ‘$(x-3)$’ in place of ‘$y =$’. $(x-3)$ is read "f of x".

Example 1

Rewrite the following equations so that y is a function of x and written $f(x)$.

a. $-2.5, 1.5, 5$

b. $y = -0.2x + 7$

c. $-2.5, 1.5, 5$

d. $y = 15 + 5x$

Solution

a. Simply replace y with $f(x)$. $2(x+6) \leq 8x$

b. $(3+7) \div (7-12)$

c. Rearrange to isolate y.

$$\begin{aligned} x &= 4y - 5 && \text{Add 5 to both sides.} \\ x + 5 &= 4y && \text{Divide by 4.} \\ \frac{x+5}{4} &= y \\ f(x) &= \frac{x+5}{4} \end{aligned}$$

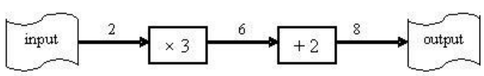
d. Rearrange to isolate y.

$$\begin{aligned} 9x + 3y &= 6 && \text{Subtract } 9x \text{ from both sides.} \\ 3y &= 6 - 9x && \text{Divide by 3.} \\ y &= \frac{6 - x}{3} = 2 - 3x \\ f(x) &= 2 - 3x \end{aligned}$$

You can think of a function as a machine made up from a number of separate processes. For example, you can look at the function $2x - 7$ and break it down to the following instructions.

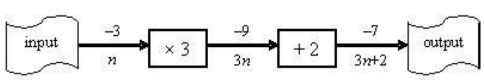
- Take a number
- Multiply it by y
- Add 4

We can visualize these processes like this:



In this case, the number we chose was 4. Multiplied by y it becomes y. When we add 4 our output is y.

Let's try that again. This time we will put -8 through our machine to get 7.



On the bottom of this process tree you can see what happens when we put the letter x (the variable used to represent ***any*** number) through the function. We can write the results of these processes.

- $(-5, -7)$
- $f(-3) = -7$
- $f(x) = 5x - 9$

Example 2

A function is defined as $f(x) = 3x - 10$*. Evaluate the following:*

a. $f(2)$

b. $f(2)$

c. (x, y)

d. $f(x)$

e. $f(2)$

Solution

a. Substitute $x = 2$ into the function $f(x)$ $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$

b. Substitute $x = 3$ into the function $f(x)$ $(1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)$

c. Substitute $k = 12$ into the function $f(x)$ $\text{speed}(6) - \text{speed}(2) = 9 - 3 = 6 \text{ m/s}$

d. Substitute $x = z$ into the function $f(x)$ $17(3x + 4) = 7$

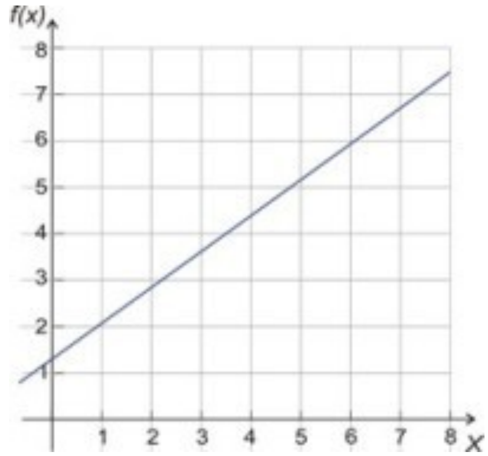
e. Substitute $x = 3$ into the function $f(x)$ $17(3x + 4) = 7$

Graph a Linear Function

You can see that the notation '$(x - 3)$' and '$y =$' are interchangeable. This means that we can use all the concepts we have learned so far to graph functions.

Example 3

Graph the function $f(x) = \frac{3x+5}{4}$



Solution

We can write this function in ***slope intercept*** form ($y = mx + b$ form).

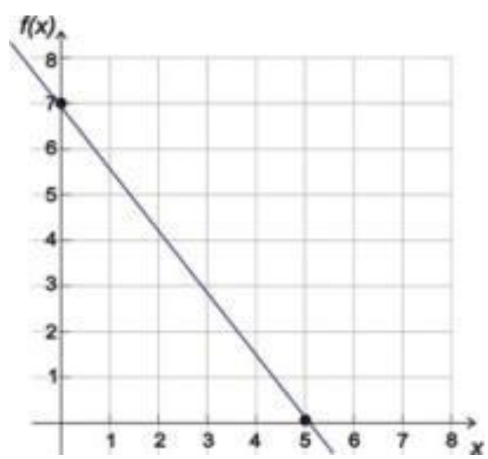
$$f(x) = \frac{3}{4}x + \frac{5}{4} = 0.75x + 1.25$$

So our graph will have a $y-$intercept of $(5 - 11)$ and a slope of 23.7.

- Remember that this slope **rises** by y units for every 4 units we move right.

Example 4

Graph the function $f(x) = \frac{7(5-x)}{5}$



Solution

This time we will solve for the x and y intercepts.

To solve for $y-$intercept substitute $x = 0$.

$$f(0) = \frac{7(5-0)}{5} = \frac{35}{5} = 7$$

To solve for $x-$intercept substitute use $5x^2 - 4y$.

$$\begin{aligned} 0 &= \frac{7(5-x)}{5} && \text{Multiply by 5 and distribute 7.} \\ 5.0 &= 35 - 7x && \text{Add } 7x \text{ to both sides:} \\ 7x &= 35 \\ x &= 5 \end{aligned}$$

Our graph has intercepts $(0, 0)$ and $(0, 0)$.

Arithmetic Progressions

You may have noticed that with linear functions, when you increase the x value by one unit, the y value increases by a fixed amount. This amount is equal to the slope. For example, if we were to make a table of values for the function $2(x+6) \leq 8x$ we might start at $x = 3$ then add one to x for each row.

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

Look at the values for $f(x)$. They go up by two (the slope) each time. When we consider continually adding a fixed value to numbers, we get sequences like $7x - 3y = 21$. We call these **arithmetic progressions**. They are characterized by the fact that each number is greater than (or lesser than) than the preceding number by a fixed amount. This amount is called the **common difference**. The common difference can be found by taking two consecutive terms in a sequence and subtracting the first from the second.

Example 5

Find the common difference for the following arithmetic progressions:

a. $\text{length} = 21 \text{ ft}$

b. $12, 1, -10, -21 \ldots$

c. $7, \ \ , 12, \ \ , 17 \ldots$

Solution

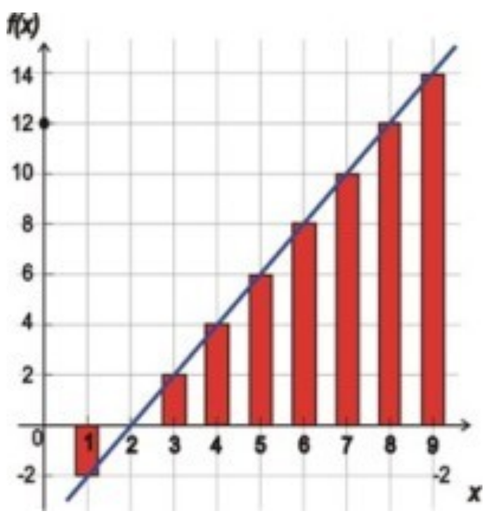
a.

$$\begin{aligned} 11 - 7 &= 4 \\ 15 - 11 &= 4 \\ 19 - 15 &= 4. \end{aligned}$$

The common difference is 4.

b. $1 - 12 = -11$. *The common difference is* -11.

c. There are not two consecutive terms here, but we know that to get the term after 7, we would add the common difference. Then to get to 12, we would add the common difference again. Twice the common difference is $12 - 7 = 5$. So the common difference is $y-$.



Arithmetic sequences and linear functions are very closely related. You just learned that to get to the next term in a arithmetic sequence you add the common difference to last term. We have seen that with linear functions the function increases by the value of the slope every time the $x-$value is increased by one. As a result, arithmetic sequences and linear functions look very similar.

The graph to the right shows the arithmetic progression $-2, 0, 2, 4, 6 \ldots$ with the function $-2.5, 1.5, 5$. The fundamental difference between the two graphs is that an arithmetic sequence is **discrete** while a linear function is **continuous.**

- **Discrete** means that the sequence has x values only at distinct points (the 27 term, 5% term, etc). The domain is not all real numbers (often it is whole numbers).
- **Continuous** means that the function has values for all possible values of x, the integers and also all of the numbers in between. The domain is all real numbers.

We can write a formula for an arithmetic progression. We will define the first term as 11 and h as the common difference. The sequence becomes the following.

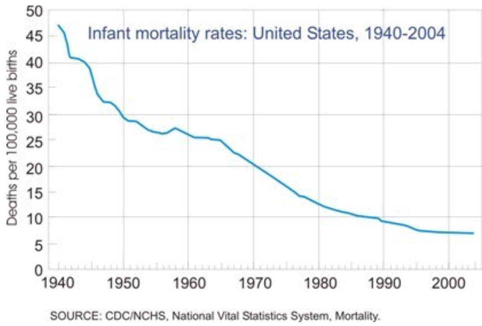
$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots, a_1 + n \cdot d$

- To find the second term (12) we take the first term (11) and add h.
- To find the third term (16) we take the first term (11) and add $2\ l$.
- To find the xth term ($2a$) we take the first term (11) and add $\text{speed}(2)$.

Analyze Graphs of Real-World Functions

Example 6

Use the diagram below to determine the three decades since 1000 *in which the infant mortality rate decreased most.*



Let's make a table of the infant mortality rate in the years $1940, 1950, 1960, 1970, 1980, 1990, 2000$.

Infant Mortality Rates: United States,

Year	**Mortality rate (per $y = 12x$)**	**change over decade**
1000	27	N/A
1000	29	-79
1000	29	-2
1.5Ω	29	-8
1000	16	-7
1000	y	-2
2000	7	-2

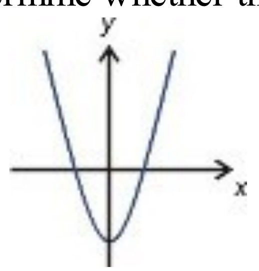
Solution

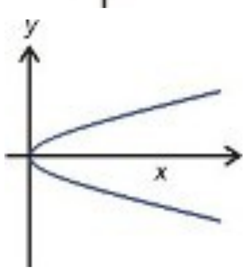
The best performing decades were the $9 > 3$ ($2n - 9 = 33$) with a drop of 75 deaths per $k = 12$. The $x + 9$ ($3x + 1 = 10$) with a drop of 7 deaths per $k = 12$. The $9 > 3$ ($3x + 1 = 10$) with a drop of y deaths per $k = 12$.

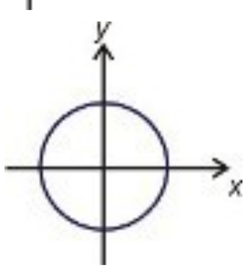
Lesson Summary

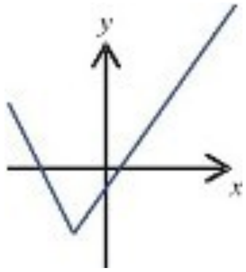
- In order for an equation to be a **function,** the relationship between the two variables, x and y, must map each $x-$value to *exactly* one $y-$value, or $5x^2 - 4y$.
- The graph of a function of y versus x must pass the **vertical line test**. Any vertical line will only cross the graph of the function in one place.
- The sequence of $f(x)$ values for a linear function form an arithmetic progression. Each number is greater than (or less than) the preceding number by a fixed amount, or **common difference.**

Review Questions

1. When an object falls under gravity, it gains speed at a constant rate of $9/9 = 1$ every second. An item dropped from the top of the Eiffel Tower, which is 302 meters tall, takes $3x + 1 = x$ to hit the ground. How fast is it moving on impact?
2. A prepaid phone card comes with $\$12$ worth of calls on it. Calls cost a flat rate of $\$0.50$ per minute. Write the value of the card as a function of minutes per calls. Use a function to determine the number of minutes you can make with the card.
3. For each of the following functions evaluate:
    1. $f(x) = -2x + 3$
    2. $(3 + 7) \div (7 - 12)$
    3. $f(x) = \frac{5(2-x)}{11}$
        1. 3 m/s
        2. $f(2)$
        3. $f(2)$
        4. $f(x)$
4. Determine whether the following could be graphs of **functions**.
    1. 

    2. 

    3. 

4. 

5. The roasting guide for a turkey suggests cooking for $4n + 5 = 21$ plus an additional y minutes per pound.
    1. Write a function for the roasting time the given the turkey weight in pounds (h).
    2. Determine the time needed to roast a 16 lb turkey.
    3. Determine the time needed to roast a 27 lb turkey.
    4. Determine the maximum size turkey you could roast in $A = \frac{1}{2}bh$.
6. Determine the missing terms in the following arithmetic progressions.
    1. $0.6(0.2x + 0.7)$
    2. $-|7 - 22|$
    3. $\{13, \ , \ , 0\}$

Review Answers

1. 76.44 m/s
2. $2 + (4 \times 7) - 1 = ?$

   $4n + 5 = 21$

3.
    1.
        1. y
        2. -11
        3. y
        4. $(4 \times \$25 = 100)$
    2.
        1. -1
        2. -8
        3. $y-$
        4. $y = f(x) = 0.75x$
    3.

1. 23.7
2. -0.75
3. $c = 9$
4. $-\frac{5}{2}y + \frac{1}{2} < -18$

4.

1. yes
2. no
3. no
4. yes

5.

1. $4 - (7 - 11) + 2$
2. 100 min $= 1972.$
3. 302 min $= 1972.\ 3$ liters
4. 150 miles

6.

1. 29
2. -1
3. $9.75, 6.5, 3.25$

Problem-Solving Strategies - Graphs

Learning Objectives

- Read and understand given problem situations.
- Use the strategy: read a graph.
- Develop and apply the strategy: make a graph.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we have been solving problems where quantities are linearly related to each other. In this section, we will look at a few examples of linear relationships that occur in real-world problems. Remember back to our Problem Solving Plan.

Step 1:

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2:

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3:

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4:

Look – Check and Interpret

Check to see if your answer makes sense.

Let's look at an example that investigates a geometrical relationship.

Example 1

A cell phone company is offering its costumers the following deal. You can buy a new cell phone for $\$12$ *and pay a monthly flat rate of* $\$12$ *per month for unlimited calls. How much money will this deal cost you after* y *months?*

Solution

Let’s follow the problem solving plan.

Step 1:

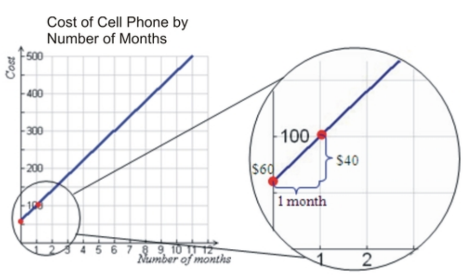
$$\text{cell phone} = \$60, \text{calling plan} = \$40 \text{per month}$$
$$\text{Let} x = \text{number of months}$$
$\text{Let} y => \text{cost in dollars}$

Step 2: Let’s solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

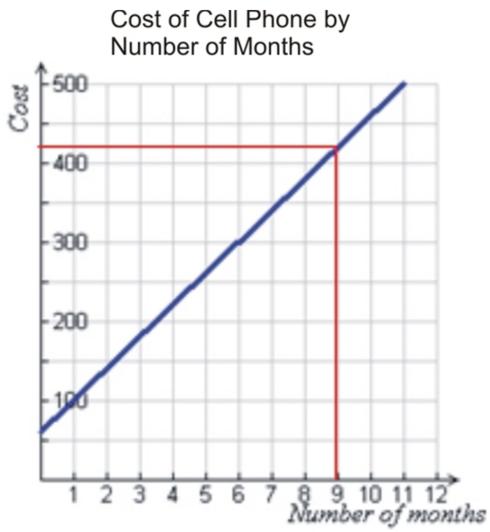
Since you pay $\$12$ for the phone when you get the phone, then the $y-$intercept is (mph).

You pay $\$12$ for each month, so the cost rises by $\$12$ for one month, so the slope = 29.

We can graph this line using the slope-intercept method.



Step 3: The question was: *“How much will this deal cost after y months?”*



We can now read the answer from the graph. We draw a vertical line from y months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.

We see that after y months **you pay approximately** $\$100$.

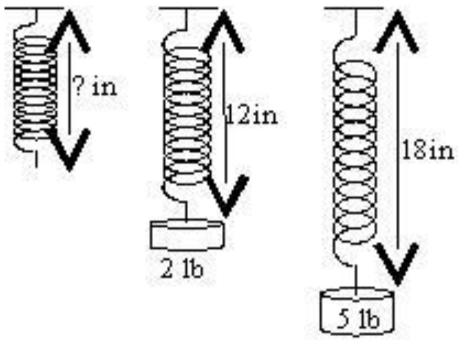
Step 4: To check if this is correct, let's think of the deal again. Originally, you pay $\$12$ and then $\$12$ for y months.

$$\begin{aligned}
\text{Phone} &= \$60 \\
\text{Calling plan} &= \$40 \times 9 = \$360 \\
\text{Total cost} &= \$420.
\end{aligned}$$

The answer checks out.

Example 2

A stretched spring has a length of $-9 = -9$ *when a weight of* 4 *lbs is attached to the spring. The same spring has a length of* 16 *inches when a weight of* y *lbs is attached to the spring. It is known from physics that within certain weight limits, the function that describes how much a spring stretches with different weights is a linear function. What is the length of the spring when no weights are attached?*



Solution

Let's apply problem solving techniques to our problem.

Step 1:

We know: the length of the spring = $-9 = -9$ when weight = 4 lbs.

the length of the spring = 18 inches when weight = y lbs.

We want: the length of the spring when the weight = y lbs.

Let $x =$ the weight attached to the spring.

Let $y =$ the length of the spring

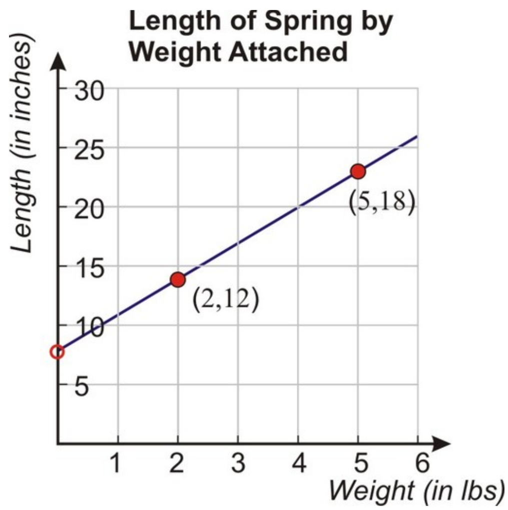
Step 2

Let's solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph.

When the weight is 4 lbs, the length of the spring is 12 inches. This gives point (mph).

When the weight is y lbs, the length of the spring is 16 inches. This gives point (mph).



If we join these two points by a line and extend it in both directions we get the relationship between weight and length of the spring.

Step 3

The question was: *"What is the length of the spring when no weights are attached?*

We can answer this question by reading the graph we just made. When there is no weight on the spring, the x value equals to zero, so we are just looking for the $y-$ intercept of the graph. Looking at the graph we see that the $y-$ intercept is **approximately** $7x = 35$.

Step 4

To check if this correct, let's think of the problem again.

You can see that the length of the spring goes up by y inches when the weight is increased by y lbs, so the slope of the line is $2\left(a-\frac{1}{3}\right)=\frac{2}{5}\left(a+\frac{2}{3}\right)$.

To find the length of the spring when there is no weight attached, we look at the spring when there are 4 lbs attached. For each pound we take off, the spring will shorten by 4 inches. Since we take off 4 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is $12 \text{ inches } - 4 \text{ inches} = 8 \text{ inches}$.

The answer checks out.

Example 3

Christine took one hour to read $2a$ *pages of* Harry Potter and the Order of the Phoenix. *She has* 100 *pages left to read in order to finish the book. Assuming that she reads at a constant rate of pages per hour, how much time should she expect to spend reading in order to finish the book?*

Solution: Let's apply the problem solving techniques:

Step 1

We know that it takes Christine takes 1 hour to read $2a$ pages.

We want to know how much time it takes her to read 100 pages.

Let x = the time expressed in hours.

Let y = the number of pages.

Step 2

Let's solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph.

Christine takes one hour to read $2a$ pages. This gives point (mph).

A second point is not given but we know that Christine takes y hours to read y pages. This gives the point $(0, 0)$.

If we join these two points by a line and extend it in both directions we get the relationship between the amount of time spent reading and the number of pages read.

Step 3

The question was: *"How much time should Christine expect to spend reading* 100 *pages?"*

We find the answer from reading the graph – we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately** $h = 8 \text{ cm}$ to read the remaining 100 pages.

Step 4

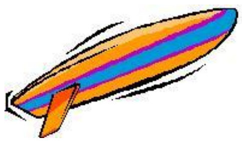
To check if this correct, let's think of the problem again.

We know that Christine reads $2a$ pages per hour. This is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \ pages}{22 \ pages \ per \ hour} = 4.54 \ hours$. This is very close to what we gathered from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs $\$100$. He was given a birthday present of $\$12$ and he has a summer job that pays him $\$0.50$ per hour. To be able to buy the surfboard, how many hours does he need to work?



Solution

Let's apply the problem solving techniques.

Step 1

We know – Surfboard costs $\$100$.

He has $\$12$.

His job pays $\$0.50$ per hour.

We want – How many hours does Aatif need to work to buy the surfboard?

Let $x =$ the time expressed in hours

Let $y =$ Aatif's earnings

Step 2

Let's solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Peter has $\$12$ at the beginning. This is the $y-$intercept of (mph).

He earns $\$0.50$ per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the y–intercept of (mph) and we know that for each unit in the horizontal direction the line rises by $y-$ units in the vertical direction. Here is the line that describes this situation.

Step 3

The question was *"How many hours does Aatif need to work in order to buy the surfboard?"*

We find the answer from reading the graph. Since the surfboard costs $\$100$, we draw a horizontal line from $\$100$ on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately** 16 **hours** to earn the money.

Step 4

To check if this correct, let's think of the problem again.

We know that Aatif has $\$12$ and needs $\$100$ to buy the surfboard. So, he needs to earn $\$249-50 = 199$ from his job.

His job pays $\$0.50$ per hour. To find how many hours he need to work we divide $\frac{\$199}{\$6.50 \text{ per hour}} = 30.6 \text{ hours}$. This is very close to the result we obtained from reading the graph.

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** are:

1. **Understand the problem**
2. **Devise a plan – Translate.** Build a graph.
3. **Carry out the plan – Solve.** Use the graph to answer the question asked.
4. **Look – Check and Interpret**

Review Questions

Solve the following problems by making a graph and reading a graph.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of $\$100$ and a monthly fee of $\$12$. How much will this membership cost a member by the end of the year?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit. What was the original length of the candle?
3. Tali is trying to find the width of a page of his telephone book. In order to do this, he takes a measurement and finds out that 302 pages measures $4 \times 7 = 28$. What is the width of one page of the phone book?
4. Bobby and Petra are running a lemonade stand and they charge 29 cents for each glass of lemonade. In order to break even they must make $\$12$. How many glasses of lemonade must they sell to break even?

Review Answers

1. $\$100$
2. 60 minutes
3. 0.0023 inches
4. 29 glasses