Counting Linearly Independent Eigenvectors

Gooty Suhas

Question

Find the number of linearly independent eigenvectors of the matrix:

$$\vec{A} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Step 1: Block Diagonal Structure

Matrix $\vec{A}$ is block diagonal:

$$\vec{A} = \begin{bmatrix} \vec{B} & 0 \\ 0 & \vec{C} \end{bmatrix} \quad \text{where} \quad \vec{B} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \quad \vec{C} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

Eigenvalues of $\vec{A}$ are the union of eigenvalues of $\vec{B}$ and $\vec{C}$.

Step 2: Discriminant Test for $\vec{B}$

For any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the characteristic polynomial is:

$$\lambda^2 - \mathrm{Tr}(\vec{B})\lambda + \det(\vec{B})$$

Discriminant:

$$\Delta = \mathrm{Tr}(\vec{B})^2 - 4\det(\vec{B})$$

Apply to $\vec{B} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$:

$$\mathrm{Tr}(\vec{B}) = 2 + 1 = 3, \quad \det(\vec{B}) = 2 \cdot 1 - 2 \cdot 2 = -2$$

$$\Delta = 3^2 - 4(-2) = 9 + 8 = 17 > 0$$

So $\vec{B}$ has two distinct real eigenvalues.

Step 3: Eigenvalues of $\vec{C}$

Since $\vec{C}$ is diagonal:

$$\text{Eigenvalues of } \vec{C} = 3, \quad 4$$

Each contributes one linearly independent eigenvector.

Step 4: Final Count

- $\vec{B}$: 2 distinct eigenvalues $\rightarrow$ 2 independent eigenvectors
- $\vec{C}$: 2 distinct eigenvalues $\rightarrow$ 2 independent eigenvectors

$$\text{Total linearly independent eigenvectors} = 2 + 2 = \boxed{4}$$

Conclusion

Therefore, the matrix

$$\vec{A} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

has **4 distinct and non-linearly dependent eigenvectors**.

Additional Proof: Block Diagonal Eigenvalues

Let

$$\vec{A} = \begin{bmatrix} \vec{A}_1 & 0 \\ 0 & \vec{A}_2 \end{bmatrix} \quad \text{where } \vec{A}_1 \in \mathbb{R}^{m\times m},\ \vec{A}_2 \in \mathbb{R}^{(n-m)\times(n-m)}$$

Suppose $\vec{x}_1 \in \mathbb{R}^m$ is an eigenvector of $\vec{A}_1$ with eigenvalue λ. Define
$\vec{x} = \begin{bmatrix} \vec{x}_1 \\ \vec{0} \end{bmatrix} \in \mathbb{R}^n$
Then:

$$\vec{A}\vec{x} = \begin{bmatrix} \vec{A}_1\vec{x}_1 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \lambda\vec{x}_1 \\ \vec{0} \end{bmatrix} = \lambda\vec{x}$$

So $\vec{x}$ is an eigenvector of $\vec{A}$ with eigenvalue λ. The same holds for eigenvectors of $\vec{A}_2$.
Conclusion: Eigenvalues of $\vec{A}$ are the union of eigenvalues of its diagonal blocks.