Real Analysis in a Glimpse

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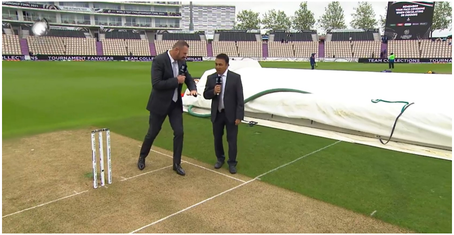
January 31, 2022

Outline

What is Real Analysis ?

Analogous to Real Analysis



Short Recap in 5 Slides

Warning: *Completing in 5 slides is injurious to health !!*

Some Facts!!

- The elements in $\mathbb{R}$ that are not in $\mathbb{Q}$ is **Irrational Numbers**. This was given by the Greek Mathematician Pythagoras when he couldn't get the exact no whose square will be **2**
-

Important Definitions

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

This is called ϵ - **neighbourhood** set of 'a'.
Eg:

- S={-2,-1,0,1,2,3,4 } here **Supremum**= 4 & **Infimum**= -2, This a **Bound** set.
- S={.......,-2,-1,0,1,2,3,4 } here **Supremum**= 4 & **Infimum**= d.n.e, This set is **unbounded** even though it's bounded above.
- S={.......,-2,-1,0,1,2,3,4,........ } here **Supremum**= d.n.e & **Infimum**= d.n.e, This set is **unbounded**.

Sequence of a Real Numbers

Definition

A sequence of real numbers is a real-valued function defined on the set of natural numbers, i.e. a function f:$\mathbb{N} \longrightarrow \mathbb{R}$ if a$_n = f(n) for\ n\ \epsilon$ N, then we write the sequence f as (a$_n$)$or(a_1, a_2, ...)$.

- A sequence of real numbers is also called a **real sequence**.
- A sequence of Real numbers is represented usually as $\{x_n\}, \{u_n\}$, $(u_n), (x_n)$ or as $\langle x_n \rangle$, $\langle u_n \rangle$.
- A sequence of a real number can be :
  1. Convergent
  2. Divergent
  3. Oscillatory

Convergence

- A sequence (a_n) in $\mathbb{R}$ is said to converge to a real number **A** if for every $\epsilon > 0$, there exists positive integer N (in general depending on ϵ) such that

$$|a_n - a| < \epsilon \quad \forall n \geq N$$

  and in that case, the number a is called a **limit of the sequence** (a_n), and (a_n) is called a **convergent sequence**
- Convergence can be of two types:
  1. Pointwise Convergence
  2. Uniform Convergence
- **[SPOILER]** Every Uniformly convergent sequence and series is point wise convergent.

Some Conclusions

Theorems

- Limit of a convergent sequence is unique.
- A convergent sequence of real numbers is bounded.

Sequence of a Function

Definition

Let f_n be a real valued function on A $\subseteq \mathbb{R}$ for each n ϵ $\mathbb{N}$.Then the sequence $\{f_1, f_2, f_3, ..., f_n\ \}$ is called sequence of real valued function on A.

- **Notation:** $\{f_n : A \longrightarrow \mathbb{R}, n \ \epsilon \ \mathbb{N}\ \}$ or $\{f_n\}$ or $\langle \mathrm{f}_n \rangle$ *or* (f_n)
- **Example:** f_n is a real valued function defined by $f_n(\mathrm{x})= x^n$ then $\{f_1(x), f_2(x), ..., f_n\}=\{x, x^2, x^3, ..., x^n\}$

Series of a Function

Definition

Let $\{f_n\}$ be a sequence on A $\subseteq \mathbb{R}$ for each n ϵ $\mathbb{N}$.Then the expression $\{f_1 + f_2 + f_3 + ... + f_n \} = \sum_{n=1}^{\infty} f_n$ is called series of real valued function on A.

- **Example:** $\{f_n\}$ be a sequence of real valued function defined by $f_n(x)=\frac{\cos nx}{n^2}$, x ϵ [0,1] then

$$\sum_{n=1}^{\infty} f_n(x) = \frac{\cos x}{1} + \frac{\cos 2x}{4} + \frac{\cos 3x}{9} + \frac{\cos 4x}{16} + ... + \frac{\cos nx}{n^2}$$

is called series of a real valued function in [0,1]

Pointwise Convergence

- Sequence of function $\{f_n\}$ is said to be pointwise convergent if for each xϵ A sequence $\{f_n\}$ of real numbers converge.

$$\lim_{\mathbf{n}\to\infty} \mathbf{f_n}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \forall \ \mathbf{x} \ \epsilon \ \mathbf{A}$$

- The pointwise convergence means that, given each x ϵ A, $\forall \ \epsilon > 0$, $\exists N \epsilon \ \ \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon \ \forall \ n \geq \ N$

Note

N here depends on both x and ϵ

Examples

Eg 1.1

- Question: $f_n(x) = \frac{x^2+nx}{n}$, x ϵ $\mathbb{R}$

Solution:

for some x ϵ $\mathbb{R}$,

$$f_n(x) = x + \frac{x^2}{n}$$

$$\lim_{n\to\infty}(x + \frac{x^2}{n}) = x$$

therefore $f_n(x) \to$ f(x) = x pointwise on $\mathbb{R}$

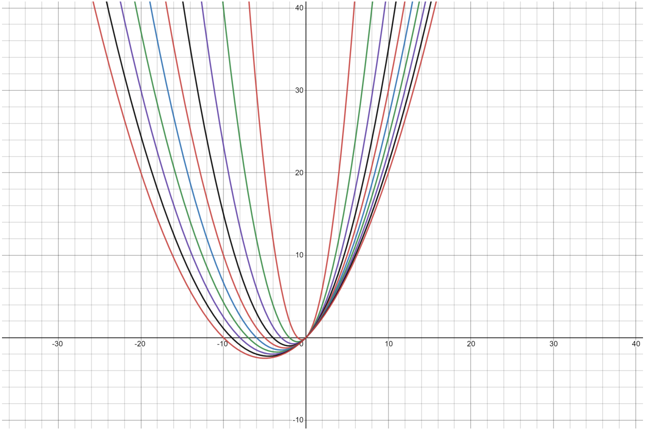


Figure: Plot of $f_n(x) = x + \frac{x^2}{n}$

Examples

Eg 1.2

- Question: g_n(x)= x^n on [0,1]

Solution:

Clearly $g_n(1) = 1$ for all n ϵ $\mathbb{N}$ therefore $g_n(1) \to 1$
and for $0 \leq x < 1$ g$_n(x)$ on [0,1) is 0, therefore,

$$g_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

The pointwise limit function g is not continues at x=1
therefore $g_n(x)$ converges to g on the set [0,1]

Think !!

What about convergence for A ϵ [0,2] ?

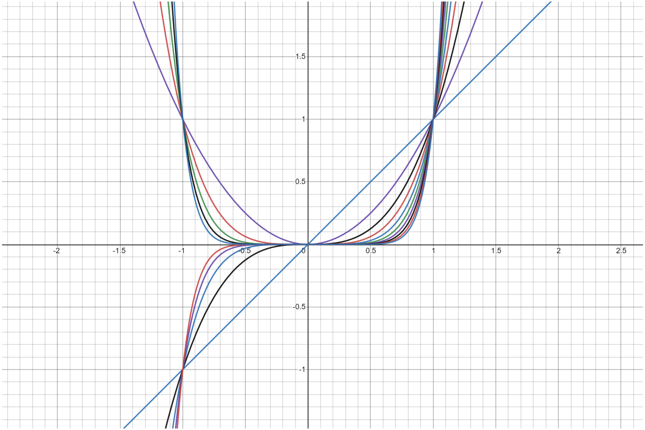


Figure: Plot of $g_n(x) = x^n$

Examples

Eg 1.3

- Question: $f_n(x) = \frac{\sin(nx+n)}{n}$ for x ϵ $\mathbb{R}$

Solution: Let f(x) = 0 for all x ϵ $\mathbb{R}$
Since $\sin(nx + n)$ can have maximum value of 1 or minimum of -1.
Therefore using squeeze theorem of sequences,

$$\frac{-1}{n} \leq \frac{\sin(nx + n)}{n} \leq \frac{1}{n}$$

$$\lim_{n \to \infty} \frac{1}{n} = 0 = f_n(x)$$

Therefore $f_n(x)$ converges to f in $\mathbb{R}$. Also,

$$|f_n(x) - f(x)| = 0 < \epsilon$$

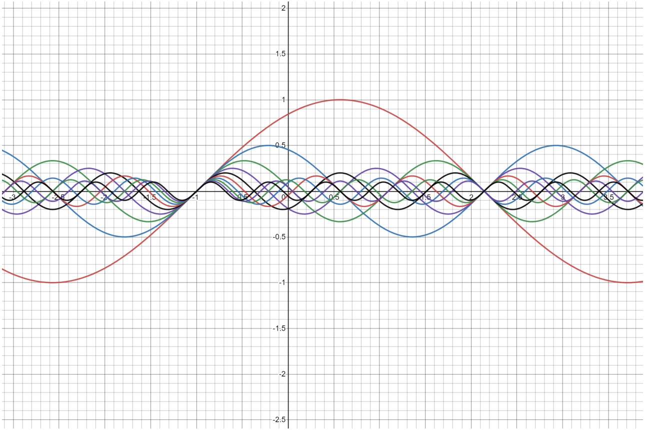


Figure: Plot of $f_n(x) = \frac{\sin(nx+n)}{n}$

Examples

Eg 1.4

- Question: Consider the sequence f_n of functions defined by $f_n(x) = n^2x^n$, $\forall\ 0 \leq x \leq 1$. Determine whether f_n is pointwise convergent.

Solution: Clearly $f_n(0) = 0$ for every n $\epsilon\ \mathbb{N}$
So the sequence f_n is a constant and converges to $\mathbf{0}$.
Now for $0 < x < 1$,

$$n^2x^n = n^2e^{n\log_e x}$$

but,

$$\log_e x < 0 \quad when \quad 0 < x < 1$$

Therefore,

$$\lim_{n\to\infty} f_n(x) = 0 \ when \ 0 \leq x < 1$$

Examples

Finally,

$$f_n(1) = n^2$$

$$\lim_{n\to\infty} n^2 = \infty$$

Therefore f_n is not pointwise convergent in [0,1]

Think !!

Will the same sequence $f_n(x)$ be pointwise convergent in [0,1) ?

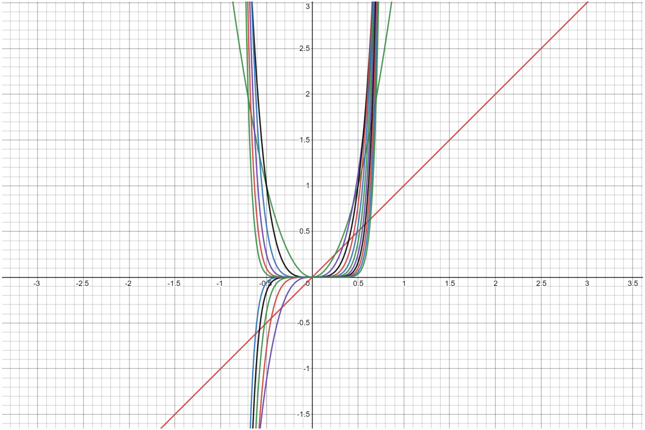


Figure: Plot of $f_n(x) = n^2 x^n$

Uniform Convergence

Definition

Let f_n : A $\rightarrow \mathbb{R}$ and f : A $\rightarrow \mathbb{R}$ be given functions. We say that the sequence $\{f_n\}$ converges uniformly on A to function f if, for every $\epsilon > 0$, there exists an N ϵ $\mathbb{N}$ such that whenever x ϵ A and n $\geq$ N it follows that :

$$|f_n(x) - f(x)| < \epsilon$$

- **Notation:** $f_n(x) \rightrightarrows f(x) \ \forall \ \ x \epsilon$ A or $f_n \rightrightarrows f \ \forall \ \ x \epsilon$ A

Uniform Convergence

Note

- For pointwise convergence, given $\epsilon > 0$, the number N is to be found after x ϵ A is given (so N depends on x), while for the uniform convergence, the number N is to be found that works for every x ϵ A (so N is independent of x).
- Every sequence which is **Uniformly Convergent** is **Pointwise Convergent** and the converse isn't always true.

Examples

Eg 1.5

- Consider $f_n(x) = \frac{x^2+nx}{n}$ and f(x) = x on $\mathbb{R}$. Does f_n converge uniformly on $\mathbb{R}$?

Solution: Since we already know that $f_n(x) \to$ f(x) on $\mathbb{R}$

$$|f_n(x) - f(x)| = \frac{x^2}{n} < \epsilon \quad \forall\ n \geq N, does\ x \epsilon \mathbb{R}?$$

If such an N existed, we would take x $=\sqrt{N}$ and n = N to obtain $1 < \epsilon$, a contradiction if our ϵ is chosen < 1. Therefore, the sequence (f_n) **does not converge uniformly** to f on R.

Think !!

What about uniform convergence $\forall$ x ϵ [-b,b] such that N $> \frac{b^2}{\epsilon}$?

Examples

Eg 1.6

- Show that sequence of functions $f_n(x)=\frac{\sin(nx)}{\sqrt{n}}$ converges uniformly to f(x)=0 on $\mathbb{R}$.

Solution:

$$|f_n(x) - f(x)| = \left|\frac{\sin(nx)}{\sqrt{n}}\right|$$

$$\leq \frac{1}{\sqrt{n}}$$

and therefore if N ϵ $\mathbb{N}$ is such that $\frac{1}{\sqrt{N}} < \epsilon$ then if n $\geq$ N then
$|f_n(x) - 0| < \epsilon \forall$ x ϵ $\mathbb{R}$
Hence, f_n **converges uniformly** to f=0 on $\mathbb{R}$

Examples

Eg 1.7

- Let f_n be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

Solution:

Cauchy Criteria for Uniform Convergence

Definition

A sequence f_n converges uniformly on A if and only if for a given $\epsilon > 0$, there exists N > 0 such that for all $n \geq m > N$ and for all x ϵ A.

Proof:

First let's assume (f_n) converges uniformly on A to a limit function f. Then, for each$\epsilon > 0$, there exists an N ϵ $\mathbb{N}$ such that: