Real Analysis in a Glimpse

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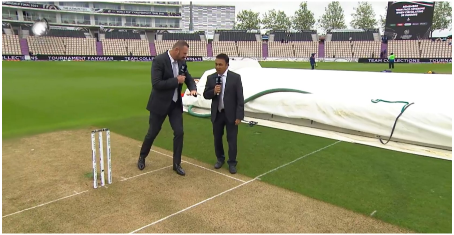
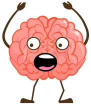
Outline

What is Real Analysis ?

Analogous to Real Analysis



Short Recap in 7 Slides



⚠ **Warning:** *Completing in 7 slides is injurious to health!!*

A Fact!!

- The elements in $\mathbb{R}$ that are not in $\mathbb{Q}$ is **Irrational Numbers**. This was given by the Greek Mathematician Pythagoras when he couldn't get the exact no whose square will be **2**

Important Definitions

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

This is called ϵ - **neighbourhood** set of 'a'.

Eg:

- S={-2,-1,0,1,2,3,4 } here **Supremum**= 4 & **Infimum**= -2, This a **Bound** set.
- S={.......,-2,-1,0,1,2,3,4 } here **Supremum**= 4 & **Infimum**= d.n.e, This set is **unbounded** even though it's bounded above.
- S={.......,-2,-1,0,1,2,3,4,........ } here **Supremum**= d.n.e & **Infimum**= d.n.e, This set is **unbounded**.

Sequence of a Real Numbers

Definition

A sequence of real numbers is a real-valued function defined on the set of natural numbers, i.e. a function f:$\mathbb{N} \longrightarrow \mathbb{R}$ if a$_n = f(n) for \ n \ \epsilon$ N, then we write the sequence f as (a$_n$)$or(a_1, a_2, ...)$.

- A sequence of real numbers is also called a **real sequence**.
- A sequence of Real numbers is represented usually as $\{x_n\}, \{u_n\}$, $(u_n), (x_n)$ or as $\langle x_n \rangle$, $\langle u_n \rangle$.
- A sequence of a real number can be :
  1. Convergent
  2. Divergent
  3. Oscillatory

Convergence

- A sequence (a_n) in $\mathbb{R}$ is said to converge to a real number **A** if for every $\epsilon > 0$, there exists positive integer N (in general depending on ϵ) such that

$$|a_n - a| < \epsilon \quad \forall n \geq N$$

  and in that case, the number a is called a **limit of the sequence** (a_n), and (a_n) is called a **convergent sequence**
- Convergence can be of two types:
  1. Pointwise Convergence
  2. Uniform Convergence
- **[SPOILER]** Every Uniformly convergent sequence and series is point wise convergent.

Series of Real Numbers

Definition

For any sequence of real numbers $< a_n >_{n=1}^{\infty}$,the associated series is defined as the ordered formal sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

The Sequence of partial sums associated with the series is denoted by $\{S_n\}_{n=1}^{\infty}$

$$S_n = a_1 + a_2 + a_3 + ... + a_n$$

Zeno Paradox

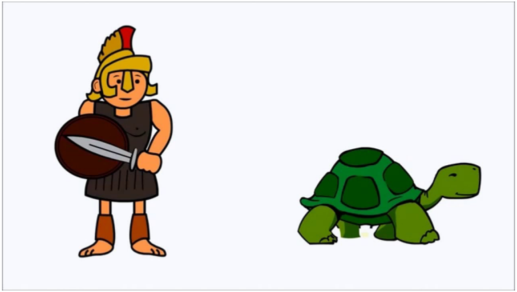


Figure: Achilles (left), Tortoise (right)

Convergence

Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence of partial sums converge.

$$\lim_{n \to \infty} S_n = S$$

Some Conclusions

Theorems

- Limit of a convergent sequence is unique.
- A convergent sequence of real numbers is bounded.

Sequence of a Function

Definition

Let f_n be a real valued function on A $\subseteq \mathbb{R}$ for each n ϵ $\mathbb{N}$. Then the sequence $\{f_1, f_2, f_3, ..., f_n\ \}$ is called sequence of real valued function on A.

- **Notation:** $\{f_n : A \longrightarrow \mathbb{R}, n\ \epsilon\ \mathbb{N}\ \}$ or $\{f_n\}$ or $\langle \mathrm{f}_n \rangle$ *or* (f_n)
- **Example:** f_n is a real valued function defined by f_n(x)= x^n then $\{f_1(x), f_2(x), ..., f_n\}$=$\{x, x^2, x^3, ..., x^n\}$

Series of a Function

Definition

Let $\{f_n\}$ be a sequence on A $\subseteq \mathbb{R}$ for each n ϵ $\mathbb{N}$.Then the expression $\{f_1 + f_2 + f_3 + ... + f_n\} = \sum_{n=1}^{\infty} f_n$ is called series of real valued function on A.

- **Example:** $\{f_n\}$ be a sequence of real valued function defined by $f_n(x)=\frac{\cos nx}{n^2}$, x ϵ [0,1] then

$$\sum_{n=1}^{\infty} f_n(x) = \frac{\cos x}{1} + \frac{\cos 2x}{4} + \frac{\cos 3x}{9} + \frac{\cos 4x}{16} + ... + \frac{\cos nx}{n^2}$$

is called series of a real valued function in [0,1]

Pointwise Convergence

- Sequence of function $\{f_n\}$ is said to be pointwise convergent if for each xϵ A sequence $\{f_n\}$ of real numbers converge.

$$\lim_{\mathbf{n}\to\infty} \mathbf{f_n(x) = f(x)} \quad \forall\ \mathbf{x} \epsilon\ \mathbf{A}$$

- The pointwise convergence means that, given each x ϵ A, $\forall\ \epsilon > 0$, $\exists N \epsilon\ \ \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon\ \forall\ n \geq\ \ N$

Note

N here depends on both x and ϵ

Examples

Eg 1.1

- Question: $f_n(x) = \frac{x^2+nx}{n}$, x ϵ $\mathbb{R}$

Solution:

for some x ϵ $\mathbb{R}$,

$$f_n(x) = x + \frac{x^2}{n}$$

$$\lim_{n\to\infty}(x + \frac{x^2}{n}) = x$$

therefore $f_n(x) \to$ f(x) $=$ x pointwise on $\mathbb{R}$

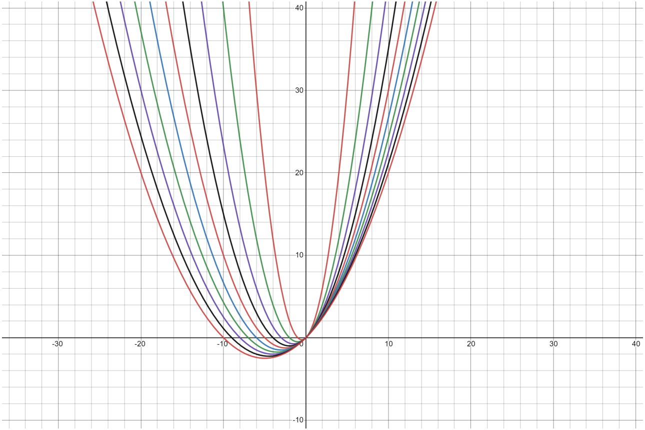


Figure: Plot of $f_n(x) = x + \frac{x^2}{n}$

Examples

Eg 1.2

- Question: g_n(x)= x^n on [0,1]

Solution:

Clearly $g_n(1) = 1$ for all n ϵ $\mathbb{N}$ therefore $g_n(1) \to 1$
and for $0 \leq x < 1$ g$_n(x)$ on [0,1) is 0, therefore,

$$g_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

The pointwise limit function g is not continues at x=1,therefore $g_n(x)$ converges to g on the set [0,1]

Think !!

What about convergence for A ϵ [0,2] ?

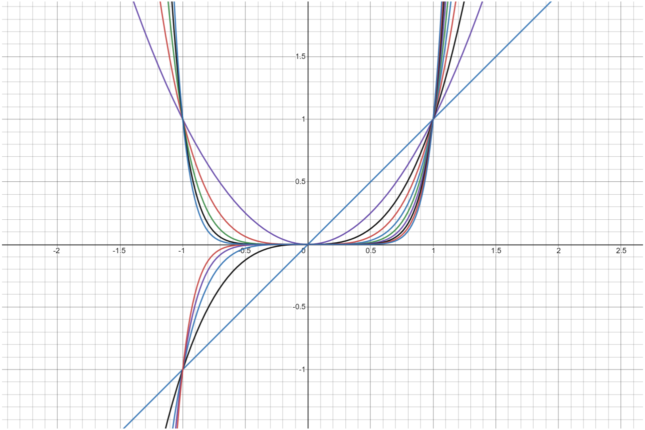


Figure: Plot of $g_n(x) = x^n$

Examples

Eg 1.3

- Question: $f_n(x) = \frac{\sin(nx+n)}{n}$ for x ϵ $\mathbb{R}$

Solution: Let f(x) = 0 for all x ϵ $\mathbb{R}$
Since $\sin(nx + n)$ can have maximum value of 1 or minimum of -1.
Therefore using squeeze theorem of sequences,

$$\frac{-1}{n} \leq \frac{\sin(nx+n)}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 = f_n(x)$$

Therefore $f_n(x)$ converges to f in $\mathbb{R}$. Also,

$$|f_n(x) - f(x)| = 0 < \epsilon$$

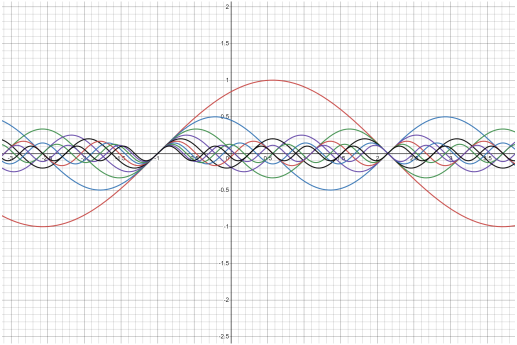


Figure: Plot of $f_n(x) = \frac{\sin(nx+n)}{n}$

Examples

Eg 1.4

- Question: Consider the sequence f_n of functions defined by $f_n(x) = n^2x^n$, $\forall\ 0 \leq x \leq 1$. Determine whether f_n is pointwise convergent.

Solution: Clearly $f_n(0) = 0$ for every n $\epsilon\ \mathbb{N}$
So the sequence f_n is a constant and converges to **0**.
Now for $0 < x < 1$,

$$n^2x^n = n^2e^{n\log_e x}$$

but,

$$\log_e x < 0 \quad when \quad 0 < x < 1$$

$$\lim_{n\to\infty} f_n(x) = 0 \ when \ 0 \leq x < 1$$

Examples

Finally,

$$f_n(1) = n^2$$

$$\lim_{n \to \infty} n^2 = \infty$$

Therefore f_n is not pointwise convergent in [0,1]

Think !!

Will the same sequence $f_n(x)$ be pointwise convergent in [0,1) ?

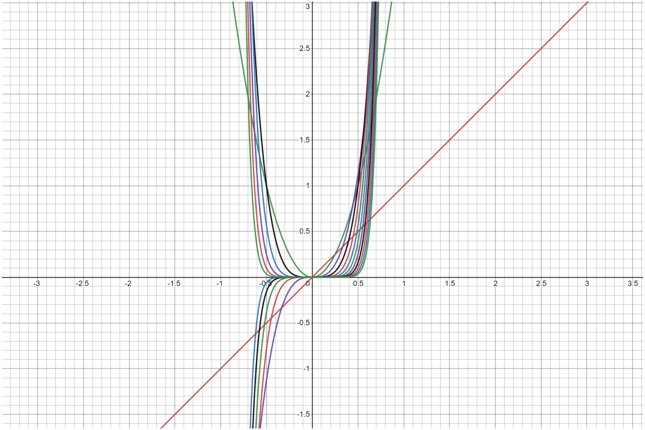


Figure: Plot of $f_n(x) = n^2 x^n$

Uniform Convergence

Definition

Let f_n : A $\rightarrow \mathbb{R}$ and f : A $\rightarrow \mathbb{R}$ be given functions. We say that the sequence $\{f_n\}$ converges uniformly on A to function f if, for every $\epsilon > 0$, there exists an N ϵ $\mathbb{N}$ such that whenever x ϵ A and n $\geq$ N it follows that :

$$|f_n(x) - f(x)| < \epsilon$$

- **Notation:** $f_n(x) \Rightarrow f(x) \ \forall \ x \epsilon$ A or $f_n \Rightarrow f \ \forall \ x \epsilon$ A

Uniform Convergence

Note

- For pointwise convergence, given $\epsilon > 0$, the number N is to be found after x ϵ A is given (so N depends on x), while for the uniform convergence, the number N is to be found that works for every x ϵ A (so N is independent of x).
- Every sequence which is **Uniformly Convergent** is **Pointwise Convergent** and the converse isn't always true.

Examples

Eg 1.5

- Consider $f_n(x) = \frac{x^2+nx}{n}$ and f(x) = x on $\mathbb{R}$. Does f_n converge uniformly on $\mathbb{R}$?

Solution: Since we already know that $f_n(x) \rightarrow$ f(x) on $\mathbb{R}$

$$|f_n(x) - f(x)| = \frac{x^2}{n} < \epsilon \quad \forall\ n \geq N, does\ x\epsilon\mathbb{R}?$$

If such an N existed, we would take x $=\sqrt{N}$ and n = N to obtain 1 $< \epsilon$, a contradiction if our ϵ is chosen < 1. Therefore, the sequence (f_n) **does not converge uniformly** to f on R.

Examples

Think !!

What about uniform convergence $\forall$ x ϵ [-b,b] such that N $> \frac{b^2}{\epsilon}$?

Examples

Eg 1.6

- Show that sequence of functions $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ converges uniformly to f(x)=0 on $\mathbb{R}$.

Solution:

$$|f_n(x) - f(x)| = \left|\frac{\sin(nx)}{\sqrt{n}}\right|$$
$$\leq \frac{1}{\sqrt{n}}$$

and therefore if N ϵ $\mathbb{N}$ is such that $\frac{1}{\sqrt{N}} < \epsilon$ then if n $\geq$ N then
$|f_n(x) - 0| < \epsilon \forall$ x ϵ $\mathbb{R}$
Hence, f_n **converges uniformly** to f=0 on $\mathbb{R}$

Examples

Eg 1.7

- Let f_n be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

Try it out yourself!!

(Hint: Approximate $1 + n^2x^2 \approx n^2x^2$)

Cauchy Criteria for Uniform Convergence

Definition

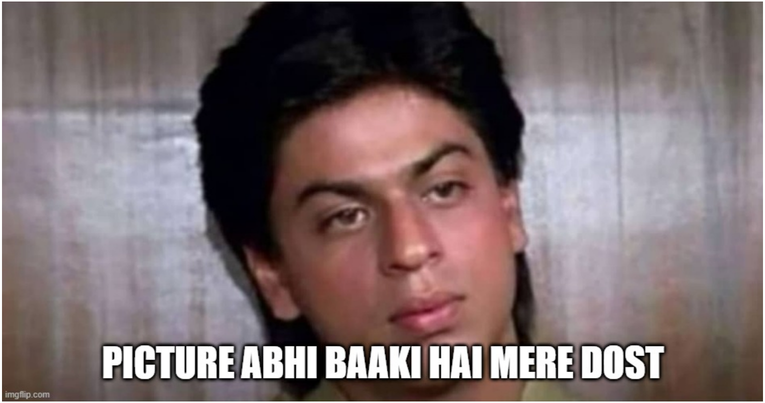
A sequence f_n converges uniformly on A if and only if for a given $\epsilon > 0$, there exists N $>$ 0 such that for all $n \geq m > N$ and for all x ϵ A.

$$|f_m(x) - f_n(x)| < \epsilon$$

Consequences of Uniform Convergence

Consequences

- Let f_n:A$\rightarrow \mathbb{R}$ be a sequence of functions. If f_n converges uniformly to f:A$\rightarrow \mathbb{R}$, and if each f_n is continuous, then f is also continuous.
- If f_n:A$\rightarrow \mathbb{R}$ is bounded on A for every n ϵ $\mathbb{N}$ and f_n $\rightarrow$ f uniformly on A. Then f :A$\rightarrow \mathbb{R}$ is bounded on A.
- If f_n:A$\rightarrow \mathbb{R}$ is a sequence of Riemann integrable functions that converges uniformly to f:A$\rightarrow \mathbb{R}$, then the limit f is also Riemann integrable,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Pointwise Convergence

Defintion

Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$. Let (S_n) be the sequence of partial sums $S_n : A \to \mathbb{R}$, defined by

$$\mathbf{S_n(x)} = \sum_{\mathbf{k=1}}^{\mathbf{n}} f_k(x)$$

Then the series,

$$\mathbf{S(x)} = \sum_{\mathbf{n=1}}^{\infty} \mathbf{f_n(x)}$$

converges pointwise to S : A $\to \mathbb{R}$ on A if $S_n \to$ S as n $\to \infty$ pointwise on A.

Uniform Convergence

Definition

Let (f_n) be a sequence of functions f_n : A$\rightarrow \mathbb{R}$. The series :

$$\sum_{\mathbf{n=1}}^{\infty} \mathbf{f_n}$$

converges uniformly on A if and only if for every $\epsilon > 0$ there exists N ϵ $\mathbb{N}$ such that :

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \epsilon$$

for all x ϵ A and all $n > m > N$.

Uniform Convergence

Proof

Let,

$$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + ... + f_n(x)$$

the sequence $< S_n >$, and therefore the series $\sum f_n$, converges uniformly if and only if for every $\epsilon > 0$ there exists N such that

$$|S_n(x) - S_m(x)| < \epsilon$$

for all x ϵ A and all $n > m > N$. Let's assume $n > m$

Uniform Convergence

Therefore,

$$S_n(x) - S_m(x) = f_{m+1}(x) + f_{m+2}(x) + ... + f_n(x) = \sum_{k=m+1}^{n} f_k(x)$$

Weirstrass M-test

Definition

Let $< f_n >$ be a sequence of functions f_n : A $\to \mathbb{R}$, and suppose that for every n ϵ $\mathbb{N}$ there exists a constant $M_n \geq 0$ such that :

$$|f_n(x)| \leq M_n \quad \forall\ x\ \epsilon\ A$$

and

$$\sum M_n \quad converges$$

Examples

Eg 1.8

- Is the given series uniformly convergent in $[0,2\pi]$?

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

Solution:

$$S_n(x) = \frac{\sin nx}{n^2}$$

$$\lim_{n\to\infty} \frac{\sin nx}{n^2} = 0 = S(x)$$

Examples

Continued

$$|S_n(x) - S(x)| = \left|\frac{\sin nx}{n^2}\right| < \epsilon$$

$$\leq \frac{1}{n^2}$$

and therefore if N ϵ $\mathbb{N}$ is such that $\frac{1}{N^2} < \epsilon$ then if n $\geq$ N then
$|S_n(x) - 0| < \epsilon \forall$ x ϵ $\mathbb{R}$
Hence, S_n **converges uniformly** to S=0 on $\mathbb{R}$

Examples

Eg 1.9

- Is the given series uniformly convergent in $[0,2\pi]$?

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

Solution:

Using Weirstrass- M test

$$|f_n(x)| = |\frac{\sin nx}{n^2}| \leq \frac{1}{n^2} = M_n(say) \quad \forall x \in [0, 2\pi] \; n \in \mathbb{N}$$

Examples

Let's now see the second condition

$$\sum M_n = \sum \frac{1}{n^2} \textit{ is convergent}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \textit{ is uniformly convergent on } [0, 2\pi]$$

Examples

Do you remember ??

In p-series,

$$\sum \frac{1}{n^p}$$

- is convergent if p > 1
- is divergent if p ≤ 1

Examples

Eg 1.10

- Is the given series uniformly convergent in $\mathbb{R}$?

$$\sum_{n=1}^{\infty} \frac{1}{(n+x^2)^2}$$

Solution:

Using Weirstrass- M test

$$|f_n(x)| = \left|\frac{1}{(n+x^2)^2}\right| \quad \forall x \in \mathbb{R} \ n \in \mathbb{N}$$

Examples

$$n + x^2 \geq n$$

$$(n + x^2)^2 \geq n^2$$

$$\frac{1}{(n + x^2)^2} \leq \frac{1}{n^2}$$

$$|f_n(x)| = \frac{1}{n^2} = M_n(say) \quad \forall x \in \mathbb{R} \; n\epsilon\mathbb{N}$$

Now $\sum M_n$ should be convergent, clearly its a p-series so it's convergent.

$$\sum_{n=1}^{\infty} \frac{1}{(n + x^2)^2} \; is \; uniformly \; convergent \; on \; \mathbb{R}$$

Power Series

Definition

A power series is an infinite series of power functions:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ...$$

The point c is called the center of the power series and sequence $< a_n >$ is called the sequence of coefficients of the power series.The power series also converges at its center c. In what follows, we always assume c $= 0$.
Example: 1) $(\frac{2^n}{n!})x^n$

Power Series

Let's consider,

$$\sum_{n=0}^{\infty} a_n x^n$$

If there is a +ve Real number 'R' such that the circle $|x - x_0| = R$ consisting all values of x for which $\sum_{n=0}^{\infty} a_n x^n$ is convergent. Then the circle is known as **circle of convergence** and 'R'is known as **Radius of convergence**

Note: If a power series converge only for x=0 $\rightarrow$ *Nowhere convergence*
If a power series converge only for all x $\rightarrow$ *Everywhere convergence*

Power Series

Radius of Convergence

S = $\{x \in R \mid \sum_{n=0}^{\infty} a_n x^n$ converges$\}$ Then,

- S=0. In this case, the power series converges only at center 0.
- S=R.In this case, the power series converges at every $x \epsilon$ R.
- S $\neq$ R & S $\neq$ 0

The power series is :
Convergent if $|x| \leq$ R=sup(S)
Divergent is $|x| >$ R=sup(S)

Power Series

Radius of convergence

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Thanks for your patience!!

References

- Robert G Bartle, Donald R Sherbert, Introduction to Real Analysis, 4^{th} Edition.
- Kenneth A Ross, Elementary Analysis,- The theory of Calculus,2^{nd} Edition.
- Prof. S H Kulkarni, Module-9, Real Analysis, NPTEL.
- Li Jiayi, Joanna, Sequence and Series of Function, University of Hongkong.
- Sequence and Series of Function, Michigan State University.