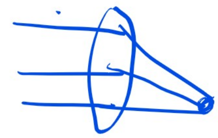
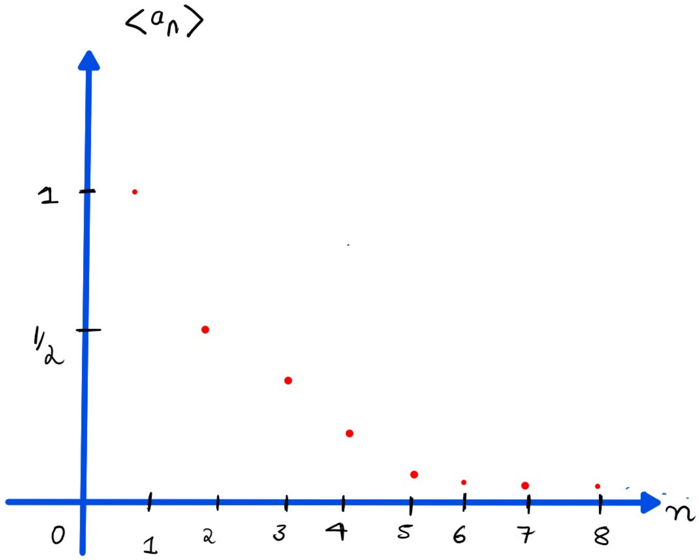
REAL ANALYSIS LECTURE-3

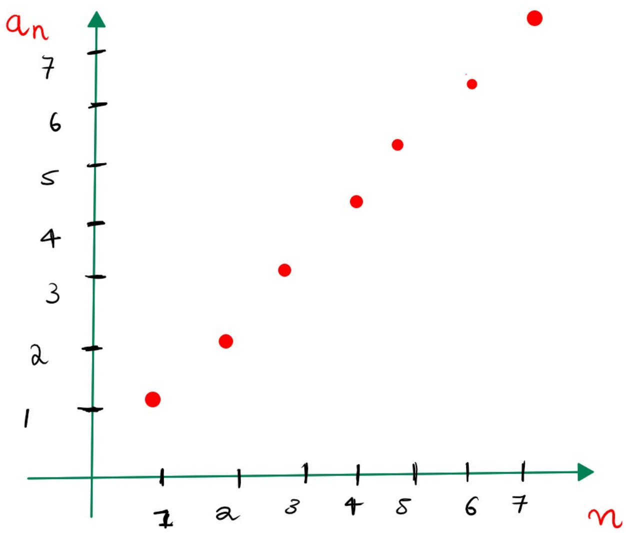
1) $\langle a_n \rangle_{n=1}^{\infty} = \frac{1}{n}$

$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \rangle$

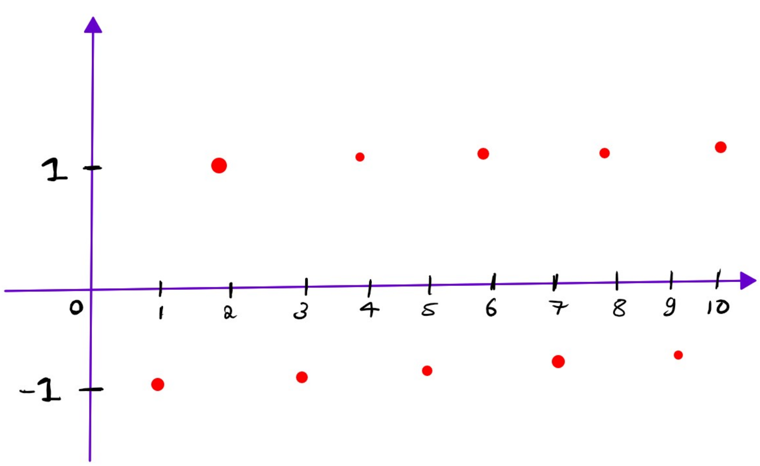




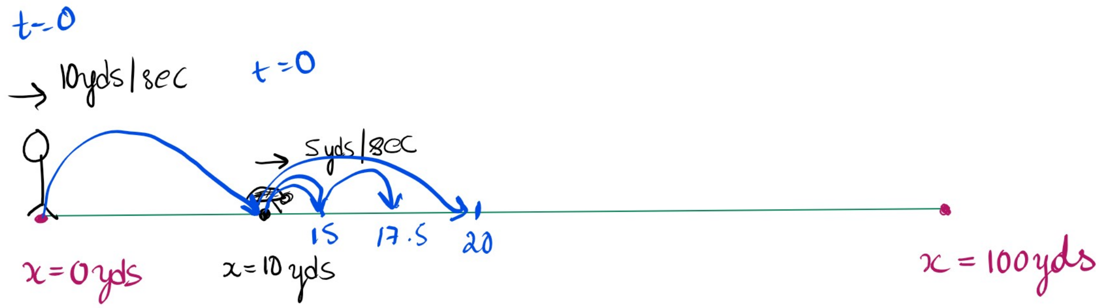
2) $\langle a_n \rangle_{n=1}^{\infty} = n$



3) $\langle a_n \rangle_{n=1}^{\infty} = (-1)^n$



Race between Achiles & Tortoise



Time	Difference
$t = 0$	10 yards
$t = 1$	$5 = 10/2$ yards
$t = 1 + \frac{1}{2}$	$2.5 = 10/4$ yards
$t = 1 + \frac{1}{2} + \frac{1}{4}$	$\cancel{1.}25 = 10/8$ yards
$t = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$0.625 = 10/16$ yards

and so on. In general we have:

Time	Difference
$t = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$	$\frac{10}{2^n}$ yards

'∞'

Time	Difference
$t = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$	$\frac{10}{2^n}$ yards

Now we want to take the limit as n goes to infinity to find out when the distance between Achilles and the tortoise is zero. But that involves adding infinitely many numbers in the above expression for the time, and we (the Greeks and Zeno) don't know how to do that. However, if we define

$$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$$

then, dividing by 2 and subtracting the two expressions:

$$s_n - \frac{1}{2}s_n = 1 - \frac{1}{2^{n+1}}$$

or equivalently, solving for s_n:

$$\lim_{n\to\infty} s_n = 2\left(1 - \underbrace{\frac{1}{2^{n+1}}}_{\to 0}\right). = 2$$

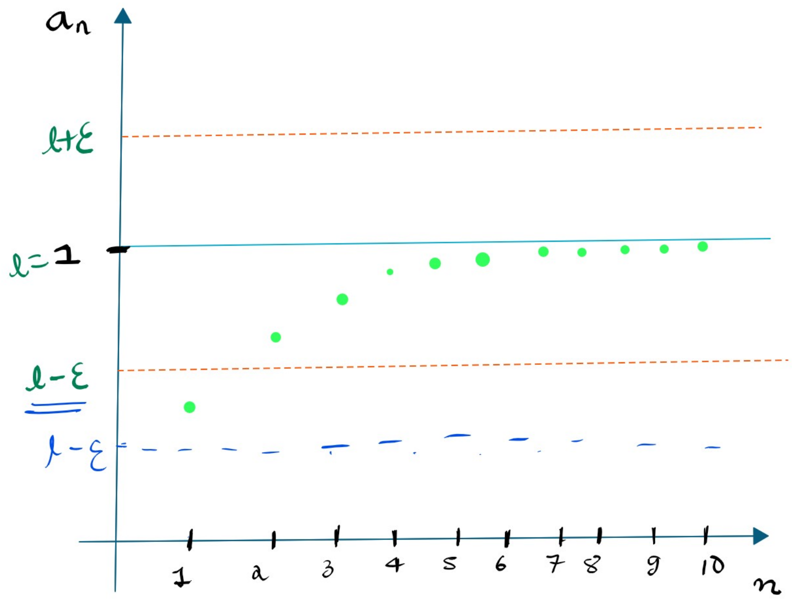
Now s_n is a simple sequence, for which we know how to take limits. In fact, from the last expression it is clear that $\lim s_n = 2$ as n approaches infinity.

Hence, we have - mathematically correctly - computed that Achilles reaches the tortoise after exactly 2 seconds, and then, of course passes it and wins the race. A much simpler calculation not involving infinitely many numbers gives the same result:

- Achilles runs 10 yards per second, so he covers 20 yards in 2 seconds.

Eg: ① $\langle a_n \rangle_{n=1}^{\infty} = \dfrac{n}{n+1} \quad \forall n \in \mathbb{N}$

$\langle a_n \rangle = \langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots\ldots \rangle$

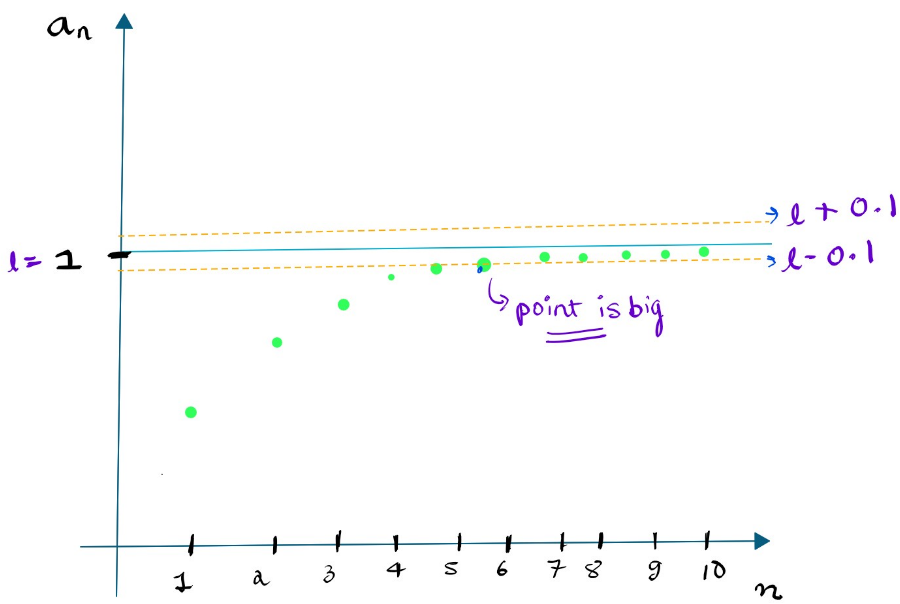


$\boxed{\text{let } \varepsilon > 0}$

& let's just take $\varepsilon = 0.35$

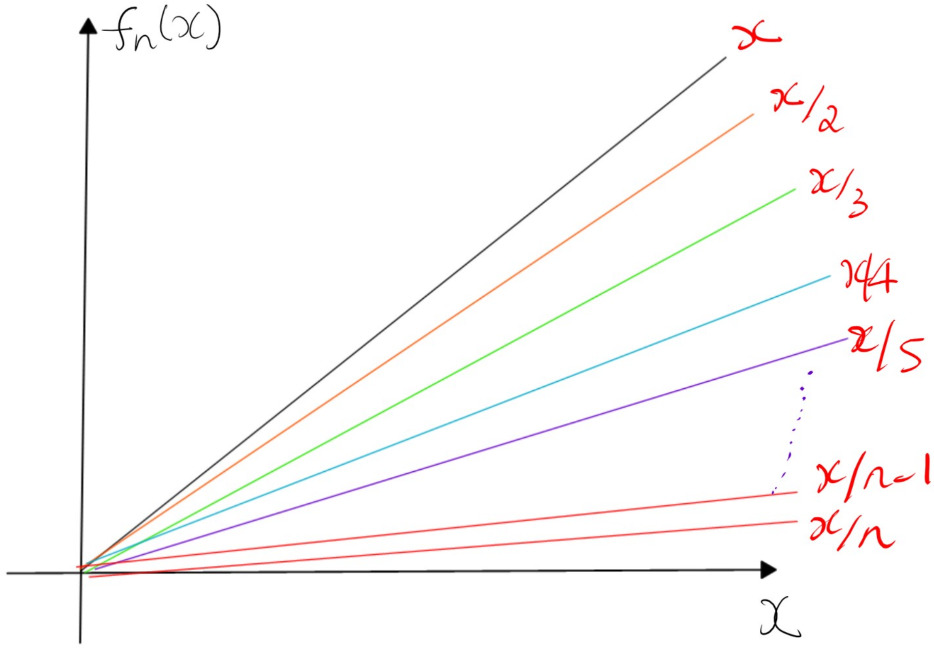
→ so here,

$a_n \in (l - 0.35, l + 0.35) \ \forall n \geq 2$



$\Rightarrow \quad a_n \in (l - 0.1, l + 0.1)$

$\forall n \geq 7$



$$\langle f_n \rangle = \frac{x}{n} \quad \forall n \in \mathbb{N}$$