Queuing Models

Waiting lines or queues are omnipresent. Businesses of all types, industries, schools, hospitals, cafeterias, book stores, libraries, banks, post offices, petrol pumps, theatres - all have queuing problems. Queues are also found in industry - in shops where machines wait to be repaired, in tool cribs where mechanics wait to receive tools and in telephone exchanges where incoming calls wait to be handled by the operators. Further examples of queues, though less apparent are: Waiting for a telephone operator to answer, a traffic light to change, the morning mail to be delivered and the like.

Waiting line problems arise either because

i) there is too much demand on the facilities so that we say that there is an excess of waiting time or inadequate number of service facilities.

ii) there is too less demand, in which case there is too much idle facility time or too many facilities.

In either case, the problem is to either schedule arrivals or provide proper number of facilities or both so as to obtain an optimum balance between the costs associated with waiting time and idle time.

Queuing theory has also been applied for the solution of problems such as

1. Scheduling of mechanical transport fleets.
2. Scheduling distribution of scarce war material.
3. Scheduling of jobs in Production control.
4. Minimization of congestion due to traffic delay at tool booths.
5. Solution of inventory control problems

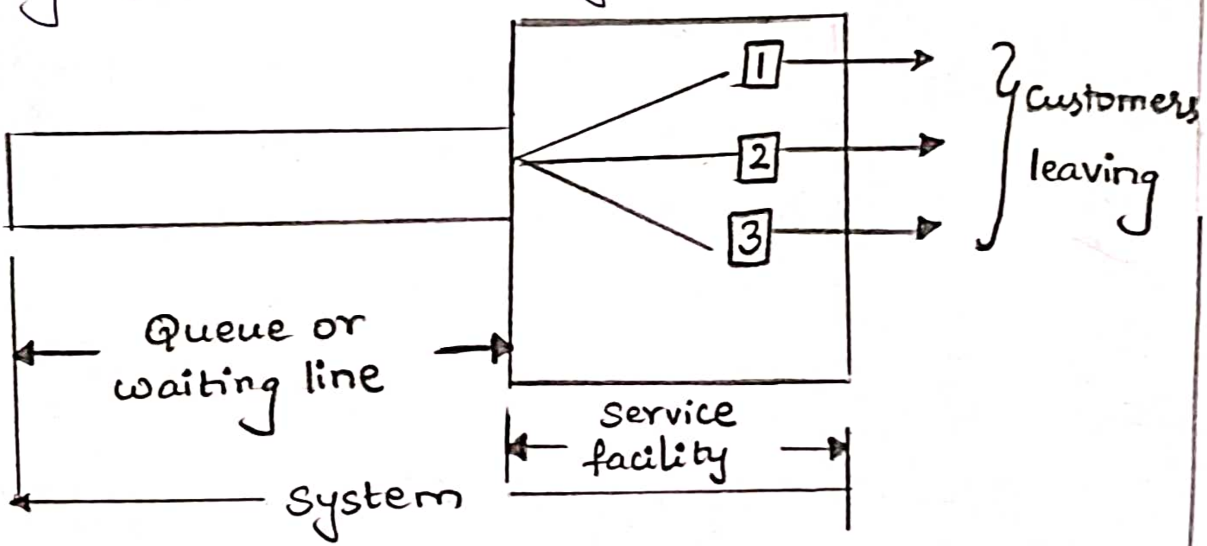
Introduction:

Waiting lines or queues are familiar phenomena, which we observe quite frequently in our daily life. The basic characteristics of a queuing Phenomenon are

1. Units arrive, at regular or irregular intervals of time, at a given point called the service centre. For example, trucks arriving a loading station, customers entering a department store, persons arriving a cinema hall, ships arriving a port, letters arriving a typist's desk, etc. All these units are called entries or arrivals of customers.

2. One or more service channels or service stations or service facilities (ticket windows, sales girls, typists, docks, etc.) are assembled at the service centre. If the service station is empty (free),

the arriving customer(s) will be served immediately; if not, the arriving customer(s) will wait in line until the service is provided. Once service has been completed, the customer leaves the system. Whenever we have customers coming to a service facility in such a way that either the customers or the facilities have to wait, we have a queuing problem. The figure shows the major constituents of a queuing system (or delay phenomenon). They are:



[fig]

1. Customer: The arriving unit that requires some service to be performed. As already described, the customers may be persons, machines, vehicles, parts etc.

2. Queue (waiting line): The number of customers waiting to be serviced. The queue does not include the customer(s) currently being serviced.

3. Service channel: The process or facility which is performing the services to the customer. This may be single or multi-channel. The number of service channels

is denoted by the symbol c.

A queuing system is specified completely by seven main elements:

1. Input or arrival (inter-arrival) time distribution
2. Output or departure (service) time distribution
3. Service channels
4. Service discipline
5. Maximum number of customers allowed in the system.
6. Calling source or population
7. Customer's behaviour.

Operating characteristics of a queuing system.

Analysis of a queuing system involves a study of its different operating characteristics. Some of them are:

1. Queue length (L_q) – the average number of customers in the queue waiting to get service. This excludes the customer(s) being served.
2. System length (L_s) – the average number of customers in the system including those waiting as well as those being served.
3. Waiting time in the queue (W_q) – the average time for which a customer has to wait in the queue to get service.

4. Total time in the system (W_s) – the average total time spent by a customer in the system from the moment he arrives till he leaves the system. It is taken to be the waiting time plus the service time.

5. Utilization factor (ρ) – It is the proportion of time a server actually spends with the customers. It is also called traffic intensity.

KENDALL'S NOTATION FOR REPRESENTING QUEUING MODELS.

D. G. Kendall (1953) and later A. Lee (1966) introduced useful notation for queuing models. The complete notation can be expressed as:

$$(a/b/c):(d/e/f)$$

where a = arrival (or inter-arrival) time distribution,
b = departure (or service) time distribution,
c = number of parallel service channels in the system,
d = Service discipline,
e = maximum number of customers allowed in the system,
f = Calling source or population.

The following conventional codes are generally used to replace the symbols a, b and d:
Symbols for a and b

M = Markovian (poisson) arrival or departure distribution (or exponential interarrival or service time distribution),

E_k = Erlangian or gamma interarrival or service time distribution with parameter k,

GI = general independent arrival distribution,

G = general departure distribution,

D = deterministic interarrival or service times.

Symbols for d

FCFS = first come, first served,

LCFS = last come, first served,

SIRO = Service in random order,

GD = general service discipline.

The symbols e and f represent a finite (N) or infinite (∞) number of customers in the system and calling source respectively. For instance, $(M/E_k/1)$: $(FCFS/N/\infty)$ represents poisson arrival (exponential interarrival), Erlangian departure, single server, 'first come, first served' discipline, maximum allowable customers N in the system and infinite population model.

Prob: A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming poisson distribution for arrival rate and exponential distribution for service time, find

1) Average number of customers in the system.
2) Average number of customers in the queue or average queue length.
3) Average time a customer spends in the system.
4) Average time a customer waits before being served.

Solution: Arrival rate $\lambda = 9/5 = 1.8$ customers/min,

Service rate $\mu = 10/5 = 2$ customers/minute.

1) Average number of customers in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2 - 1.8} = 9.$$

2) Average number of customers in the queue,

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{(\mu - \lambda)}$$

$$= \frac{1.8}{2} \times \frac{1.8}{2 - 1.8} = 8.1$$

3) Average time a customer spends in the system,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{2 - 1.8} = 5 \text{ minutes.}$$

4) Average time a customer waits in the queue,

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu - \lambda}\right) = \frac{1.8}{2}\left(\frac{1}{2-1.8}\right)$$

$$= 4.5 \text{ minutes.}$$

Prob: (2) A person repairing radios finds that the time spent on the radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Sol: Arrival rate $\lambda = \frac{15}{8 \times 60} = \frac{1}{32}$ units/minute.

Service rate $\mu = \frac{1}{20}$ units / minute.

Number of jobs ahead of the set brought in

= Average number of jobs in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/32}{1/20 - 1/32} = 5/3.$$

Number of hours for which the repairman remains busy in an 8-hour day

$$= 8\frac{\lambda}{\mu} = 8 \times \frac{1/32}{1/20} = 8 \times \frac{20}{32}$$

$$= 5 \text{ hours.}$$

$\therefore$ Time for which repairman remains idle in an 8-hour day

$$= 8 - 5 = 3 \text{ hours.}$$

Prob:(3) A branch of punjab National Bank has only one typist. Since the typing work varies in length (no of pages to be typed), the typing rate is randomly distributed approximating a poisson distribution with mean service rate of 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8-hour work day. If the typewriter is valued at ₹ 1.50 per hour, determine.

1) Equipment utilization
2) The percent time that an arriving letter has to wait.
3) Average system time
4) Average cost due to waiting on the part of typewriter i.e., its remaining idle.

Sol: Arrival rate, $\lambda = 5$ per hour,
Service rate, $\mu = 8$ per hour.

1) Equipment utilization, $\rho = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625$

2) The percent time an arriving letter has to wait

= percent time the typewriter remains busy

= 62.5 %

3) Average system time,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-5} = \frac{1}{3} \text{ hr} = 20 \text{ minutes}$$

4) Average cost due to waiting on the part of the typewriter per day.

$= 8 \times (1 - 5/8) \times ₹1.50 = ₹4.50.$

Prob (4): Workers come to tool store room to receive special tools (required by them) for accomplishing a particular project assigned to them. The average time between two arrivals is 60 seconds and the arrivals are assumed to be in poisson distribution. The average service time (of the tool room attendant) is 40 seconds. Determine:

a) average queue length,
b) average length of non-empty queues,
c) average no of workers in system including the worker being attended,
d) mean waiting time of an arrival,
e) average waiting time of an arrival (worker) who waits, and
f) the type of policy to be established. In other words, determine whether to go in for an additional tool store room attendant which will minimize the combined cost of attendants idle time and the cost of workers waiting time. Assume the charges of a skilled worker ₹ 4 per hour and that of tool store room attendant ₹ 0.75 per hour.

Sol:- Here, $\lambda = 1/60$ per second $= 1$ per minute
$\mu = 1/40$ per second $= 1.5$ per minute

a) average queue length, $L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu - \lambda}$

$$= \frac{1}{1.5} \cdot \frac{1}{1.5-1} = \frac{1}{0.75}$$

$= 4/3$ workers.

b) Average length of non-empty queues,

$L_n = \frac{\mu}{\mu - \lambda} = \frac{1.5}{1.5-1} = 3$ workers.

c) Average no of workers in the System,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1}{1.5 - 1} = 2 \text{ workers.}$$

d) Mean waiting time of an arrival,

$$W_q = \frac{1}{\mu} \cdot \frac{\lambda}{\mu - \lambda}$$

$$= \frac{1}{1.5} \times \frac{1}{1.5 - 1} = 4/3 \text{ minutes.}$$

e) Average waiting time of an arrival who waits,

$$W_n = \frac{1}{\mu - \lambda} = \frac{1}{1.5 - 1} = 2 \text{ minutes}$$

f) Probability that the tool room attendant remains idle,

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{1}{1.5} = \frac{1}{3},$$

∴ idle time cost of the One attendent

$$= 1/3 \times 8 \times ₹\, 0.75 = ₹\, 2/\text{day}$$

waiting time cost of workers

$= W_q \times$ no. of workers arriving / day $\times$ cost of worker.

$$= \left(4/3 \times \frac{1}{60}\right) \times (8 \times 60) \times ₹\, 4$$

$$= ₹\, \frac{128}{3} = ₹\, 42.67/\text{day}.$$

∴ Total cost $= ₹\, (42.67 + 2) = ₹\, 44.67/\text{day}$