9 Queueing Theory

There are many situations in daily life when a queue is formed. For example, machines waiting to be repaired, patients waiting in a Doctor's room, cars waiting at a traffic signal and passengers waiting to buy tickets in counters form queues. Queue is formed if the service required by the customer (machine, patient, car, etc.) is not immediately available, that is, if the current demand for a particular service exceeds the capacity to provide the service.

Queues may be decreased in size or prevented from forming by providing additional service facilities which results in a drop in the profit. On the other hand, excessively long queues may result in lost sales and lost customers. Hence the problem of interest is how to achieve a balance between the cost associated with long waiting (queues) and the cost associated with the prevention of waiting in order to maximise the profits. As queueing theory provides an answer to this problem, it has become a topic of interest. Before we consider the solutions of queueing problems, we shall consider the general framework of a queueing system.

Although there are many types of queueing systems, all of them can be classified and described according to the following characteristics:

1. The input (or arrival) pattern

The input describes the manner in which the customers arrive and join the queueing system. It is not possible to observe and control the actual moment of arrival of a customer for service. Hence the number of arrivals in one time period or the interval between successive arrivals is not treated as a constant, but a random variable. So the mode of arrival of customers is expressed by means of the probability distribution of the number of arrivals per unit of time or of the inter-arrival time.

We shall deal with only those queueing systems in which the number of arrivals per unit of time has a poisson distribution with mean λ. In this case, the time between consecutive arrivals has an exponential distribution with mean $\frac{1}{\lambda}$ [Refer to Property 4 of the poisson process discussed in the Chapter 6].

Further the input process should specify the number of queues that are permitted to form, the maximum queue length and the maximum number of customers requiring service, viz., the nature of the source (finite or infinite) from which the customers emanate.

2. The service mechanism (or pattern)

The mode of service is represented by means of the probability distribution of the number of customers serviced per unit of time or of the inter-service time. We shall deal with only those queueing systems in which the number of customers serviced per unit of time has a Poisson distribution with mean μ or equivalently the inter-service time (viz. the time to complete the service for a customer) has an exponential distribution with mean $\dfrac{1}{\mu}$.

Further the service process should specify the number of servers and the arrangement of servers (in parallel, in series, etc.), as the behaviour of the queueing system depends on them also. The following figures represent the framework of queueing systems in which only one queue is permitted to form:

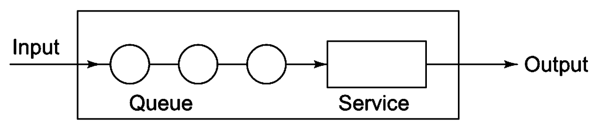

Fig. 9.1 Single server queueing system

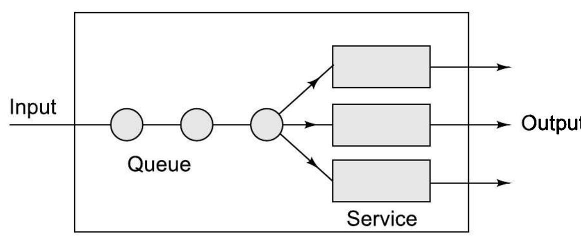

Fig. 9.2 Multiple servers (in parallel) queueing system

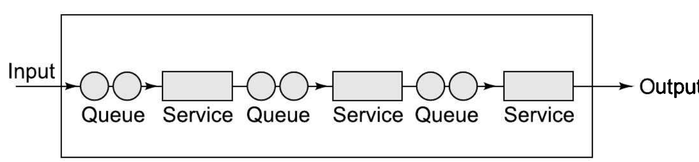

Fig. 9.3 Multiple servers (in series) queueing system

3. The queue discipline

The queue discipline specifies the manner in which the customers form the queue or equivalently the manner in which they are selected for service, when a queue has been formed. The most common discipline is the *FCFS* (First Come First Served) or *FIFO* (First in First out) as per which the customers are served in the strict order of their arrival. If the last arrival in the system is served first, we have the *LCFS* or *LIFO* (last in First Out) discipline. If the service is given in random order, we have the *SIRO* discipline. In the queueing systems which we deal with, we shall assume that service is provided on the *FCFS* (First come First served) basis.

Symbolic Representation of a Queueing Model

Usually a queueing model is specified and represented symbolically in the form $(a/b/c)$:(d/e), where a denotes the type of distribution of the number of arrivals per unit time, b the type of distribution of the service time, c the number of servers, d the capacity of the system, viz., the maximum queue size and e the queue discipline.

Accordingly, the first four models which we will deal with will be denoted by the symbols $(M/M/1)$:$(\infty/FIFO)$, $(M/M/s)$: $(\infty/FIFO)$, $(M/M/1)$: $(k/FIFO)$ and $(M/M/s)$: $(k/FIFO)$.

In the above symbols the letter 'M' stands for Markov' indicating that the number of arrivals in time t and the number of completed services in time t follow Poisson process which is a continuous time Markov chain.

Difference Equations Related to Poisson Queue Systems

If the characteristics of a queueing system (such as the input and output parameters) are independent of time or equivalently if the behaviour of the system is independent of time, the system is said to be in *steady-state*. Otherwise it is said to be in *transient-state*.

Let $P_n(t)$ be the probability that there are n customers in the system at time t $(n > 0)$. Let us first derive the differential equation satisfied by $P_n(t)$ and then deduce the difference equation satisfied by P_n(probability of n customers at any time) in the steady-state.

Let λ_n be the average arrival rate when there are n customers in the system (both waiting in the queue and being served) and let μ_n be the average service rate when there are n customers in the system.

Note *The system being in steady-state does not mean that the arrival rate and service rate are independent of the number of customers in the system.*

Now $P_n(t + \Delta t)$ is the probability of n customers at time $t + \Delta t$.

The presence of n customers in the system at time $t + \Delta t$ can happen in any one of the following four mutually exclusive ways:

(i) Presence of n customers at t and no arrival or departure during Δt time.

(ii) Presence of $(n - 1)$ customers at t and one arrival and no departure during Δt time.

(iii) Presence of $(n + 1)$ customers at t and no arrival and one departure during Δt time.

(iv) Presence of n customers at t and one arrival and one departure during Δt time (since more than one arrival/departure during Δt is ruled out)

$$\therefore \quad P_n(t + \Delta t) = P_n(t)\,(1 - \lambda_n \Delta t)\,(1 - \mu_n \Delta t) + P_{n-1}(t)\,\lambda_{n-1}\Delta t\,(1 - \mu_{n-1}\Delta t) + P_{n+1}(t)\,(1 - \lambda_{n+1}\Delta t)\,\mu_{n+1}\Delta t + P_n(t)\cdot\lambda_n\Delta t\cdot\mu_n\Delta t$$

[since P(an arrival occurs during Δt time) $= \lambda\,\Delta t$ etc.]

i.e., $P_n(t + \Delta t) = P_n(t) - (\lambda_n + \mu_n) P_n(t) \Delta t + \lambda_{n-1} P_{n-1}(t) \Delta t + \mu_{n+1} P_{n+1}(t)$, on omitting terms containing $(\Delta t)^2$ which is negligibly small.

$$\therefore \quad \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) \quad (1)$$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have

$$P_n'(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t) \quad (2)$$

Equation (2) does not hold good for $n = 0$, as $P_{n-1}(t)$ does not exist. Hence we derive the differential equation satisfied by $P_0(t)$ independently. Proceeding as before,

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda_0 \Delta t) + P_1(t)(1 - \lambda_1 \Delta t) \mu_1 \Delta t,$$

[by the possibilities (i) and (iii) given above and as no departure is possible when $n = 0$]

$$\therefore \quad \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (3)$$

Taking limits on both sides of (3) as $\Delta t \to 0$, we have

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (4)$$

Now in the steady-state, $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t)$ and $P_0'(t)$ become zero. Hence the differential equations (2) and (4) reduce to the difference equations

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \quad (5)$$

and

$$-\lambda_0 P_0 + \mu_1 P_1 = 0 \quad (6)$$

Values of P_0 and P_n for Poisson Queue Systems

From Equation (6) derived above, we have

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad (7)$$

Putting $n = 1$ in (5) and using (7), we have

$$\mu_2 P_2 = (\lambda_1 + \mu_1) P_1 - \lambda_0 P_0$$

$$= (\lambda_1 + \mu_1) \frac{\lambda_0}{\mu_1} P_0 - \lambda_0 P_0 = \frac{\lambda_0 \lambda_1}{\mu_1} P_0$$

$$\therefore \quad P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 \quad (8)$$

Successively putting $n = 2, 3, \ldots$ in (5) and proceeding similarly, we can get

$P_3 = \dfrac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$ etc.

Finally

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \cdots \mu_n} . P_0, \text{ for } n = 1, 2, \ldots \quad (9)$$

Since the number of customers in the system can be 0 or 1 or 2 or 3 etc., which events are mutually exclusive and exhaustive, we have $\sum_{n=0}^{\infty} P_n = 1$.

i.e., $P_0 + \sum_{n=1}^{\infty} \left(\dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right) P_0 = 1$

$$\therefore \quad P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right)} \tag{10}$$

Equations (9) and (10) will be used to derive the important characteristics of the four queueing models.

Characteristics of Infinite Capacity, Single Server Poisson Queue Model I [M/M/1): (∞/FIFO) model], when $\lambda_n = \lambda$ and $\mu_n = \mu$ $(\lambda < \mu)$

1. *Average number L_s of customers in the system:* Let N denote the number of customers in the queueing system (i.e., those in the queue and the one who is being served).
   N is a discrete random variable, which can take the values 0, 1, 2, …, ∞ such that $P(N = n) = P_n = \left(\dfrac{\lambda}{\mu}\right)^n P_0$, from Equation (9) of the previous discussion.
   From Equation (10), we have

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\dfrac{\lambda}{\mu}\right)^n} = \frac{1}{\sum_{n=0}^{\infty} \left(\dfrac{\lambda}{\mu}\right)^n} = 1 - \frac{\lambda}{\mu}$$

$$\therefore \quad P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Now

$$L_s = E(N) = \sum_{n=0}^{\infty} n \times P_n$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{-2}, \text{ by binomial summation}$$

$$= \frac{\dfrac{\lambda}{\mu}}{1 - \dfrac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda} \tag{1}$$

2. *Average number L_q of customers in the queue or Average length of the queue:*

If N is the number of customers in the system, then the number of customers in the queue is $(N - 1)$

$$\therefore \quad L_q = E(N-1) = \sum_{n=1}^{\infty} (n-1)P_n$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} (n-1) \left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=2}^{\infty} (n-1) \left(\frac{\lambda}{\mu}\right)^{n-2}$$

$$= \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) \left(1 - \frac{\lambda}{\mu}\right)^{-2}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^2}{1 - \frac{\lambda}{\mu}} = \frac{\lambda^2}{\mu(\mu - \lambda)} \tag{2}$$

3. *Average number L_w of customers in nonempty queues*

$L_w = E\{(N-1)/(N-1) > 0\}$, since the queue is non-empty

$$= \frac{E(N-1)}{P(N-1>0)} = \frac{\lambda^2}{\mu - \lambda} \times \frac{1}{\sum\limits_{n=2}^{\infty} P_n}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} \times \frac{1}{\sum\limits_{n=2}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} \times \frac{1}{\left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) \sum\limits_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

$$= \frac{\mu}{\mu - \lambda} \times \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{-1}} = \frac{\mu}{\mu - \lambda} \tag{3}$$

4. *Probability that the number of customers in the system exceeds k*

$$P(N > k) = \sum_{n=k+1}^{\infty} P_n = \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(\frac{\lambda}{\mu}\right)^{k+1}\left(1-\frac{\lambda}{\mu}\right)\sum_{n=k+1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n-(k+1)}$$

$$= \left(\frac{\lambda}{\mu}\right)^{k+1}\cdot\left(1-\frac{\lambda}{\mu}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n}$$

$$= \left(\frac{\lambda}{\mu}\right)^{k+1}\cdot\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-1} = \left(\frac{\lambda}{\mu}\right)^{k+1} \quad (4)$$

5. *Probability density function of the waiting time in the system.*

Let W_s be the continuous random variable that represents the waiting time of a customer in the system, viz, the time between arrival and completion of service.

Let its pdf be $f(w)$ and let $f(w/n)$ be the density function of W_s subject to the condition that there are n customers in the queueing system when the customer arrives,

Then $$f(w) = \sum_{n=0}^{\infty} f(w/n)\, P_n \quad (5)$$

Now $f(w/n)$ = pdf of sum of $(n + 1)$ service times (one part-service time of the customer being served + n complete service times)

= pdf of sum of $(n + 1)$ independent random variables, each of which is exponentially distributed with parameter μ

$= \dfrac{\mu^{n+1}}{n!}\, e^{-\mu w}\, w^n;\ w > 0$ which is the pdf of Erlang distribution.

[$\because$ The mgf of the exponential distribution (μ) is $\left(1-\dfrac{t}{\mu}\right)^{-1}$ and hence the mgf of the sum of $(n + 1)$ independent exponential (μ) variables is $\left(1-\dfrac{t}{\mu}\right)^{n+1}$, which is the mgf of Erlang distribution with parameters μ and $(n + 1)$] (refer to Erlang distribution in chapter 5)

$$\therefore \quad f(w) = \sum_{n=0}^{\infty}\frac{\mu^{n+1}}{n!}\, e^{-\mu w}\, w^n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right), \text{ by } \quad (5)$$

$$= \mu e^{-\mu w}\left(1-\frac{\lambda}{\mu}\right)\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda w)^n$$

$$= \mu\left(1-\frac{\lambda}{\mu}\right) e^{-\mu w}\, e^{\lambda w}, \text{ by exponential summation}$$

$$= (\mu - \lambda)\, e^{-(\mu-\lambda)w} \quad (6)$$

which is the pdf of an exponential distribution with parameter $(\mu - \lambda)$.

6. *Average waiting time of a customer in the system:*

   W_s follows an exponential distribution with parameter $(\mu - \lambda)$.

$$\therefore \quad E(W_s) = \frac{1}{\mu - \lambda} \tag{7}$$

($\because$ the mean of an exponential distribution is the reciprocal of its parameter).

7. *Probability that the waiting time of a customer in the system exceeds t*

$$P(W_s > t) = \int_t^\infty f(w)\, \mathrm{d}w$$

$$= \int_t^\infty (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, \mathrm{d}w$$

$$= [- e^{-(\mu - \lambda)w}]_t^\infty = e^{-(\mu - \lambda)t} \tag{8}$$

8. *Probability density function of the waiting time W_q in the queue:*

   W_q represents the time between arrival and reach of service point.

Let the pdf of W_q be $g(w)$ and let $g(w/n)$ be the density function of W_q subject to the condition that there are n customers in the system or there are $(n - 1)$ customers in the queue apart from one customer receiving service. Now $g(w/n)$ = pdf of sum of n service times [one residual service time + $(n - 1)$ full service times]

$$= \frac{\mu^n}{(n-1)!}\, e^{-\mu w}\, w^{n-1};\ w > 0$$

$$\therefore \qquad g(w) = \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!}\, e^{-\mu w}\, w^{n-1} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-\mu w} \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, (\lambda w)^{n-1}$$

$$= \frac{\lambda}{\mu}\, (\mu - \lambda)\, e^{-\mu w}\, e^{\lambda w}$$

$$= \frac{\lambda}{\mu}\, (\mu - \lambda)\, e^{-(\mu - \lambda)w};\ w > 0 \tag{9}$$

and $$g(w) = 1 - \frac{\lambda}{\mu},\ \text{when } w = 0$$

Note *1. W_q is a continuous random variable in $w > 0$ and it takes the value 0 with a non-zero probability.*

2. Wq does not follow an exponential distribution.

9. *Average waiting time of a customer in the queue.*

$$E(W_q) = \frac{\lambda}{\mu}(\mu-\lambda)\int_0^\infty w\, e^{-(\mu-\lambda)w}\,dw$$

$$= \frac{\lambda}{\mu}\int_0^\infty x\, e^{-x}\,\frac{dx}{\mu-\lambda}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}\left[x(-e^{-x}) - e^{-x}\right]_0^\infty$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} \tag{10}$$

10. *Average waiting time of a customer in the queue, if he has to wait*

$$E(W_q/W_q>0) = \frac{E(W_q)}{P(W_q>0)}$$

$$= \frac{E(W_q)}{1-P(W_q=0)}$$

$$= \frac{E(W_q)}{1-P\text{ (no customer in the queue)}}$$

$$= \frac{E(W_q)}{1-P_0}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}\times\frac{\mu}{\lambda} \qquad \left(\because P_0 = 1-\frac{\mu}{\lambda}\right)$$

$$= \frac{1}{\mu-\lambda} \tag{11}$$

Relations Among $E(N_s)$, $E(N_q)$, $E(W_s)$ and $E(W_q)$

(i) $E(N_s) = \dfrac{\lambda}{\mu-\lambda} = \lambda E(W_s)$ $\quad [\because E(N_s) = L_s]$

(ii) $E(N_q) = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \lambda E(W_q)$ $\quad [\because E(N_q) = L_q]$

(iii) $E(W_s) = E(W_q) + \dfrac{1}{\mu}$

(iv) $E(N_s) = E(N_q) + \dfrac{\lambda}{\mu}$

Note *1. If any one of the quantities $E(N_s)$, $E(N_q)$, $E(W_s)$ and $E(W_q)$ is known, the other three can be found out using the relations given above.*

2. The above relations, called Little's formulas hold good for the models with infinite capacity, but with a slight modification for the models with finite capacity.

Characteristics of Infinite Capacity, Multiple Server Poisson Queue Model II [M/M/s]: (∞/FIFO) model], When $\lambda_n = \lambda$ for all $n(\lambda < s\mu)$

1. *Values of P_0 and P_n:*

For the Poisson queue system, P_n is given by

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \cdots \mu_n} \times P_0, n \geq 1, \tag{1}$$

where
$$P_0 = \left[1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right)\right]^{-1} \tag{2}$$

If there is a single server, $\mu_n = \mu$ for all n. But there are s servers working independently of each other.

If there be less than s customers, i.e., if $n < s$, only n of the s servers will be busy and the others idle and hence the mean service rate will be $n\mu$

If $n \geq s$, all the s servers will be busy and hence the mean service rate = $s\mu$.

i.e.,
$$\mu_n = \begin{cases} n\mu, & \text{if } 0 \leq n < s \\ s\mu, & \text{if } n \geq s \end{cases} \tag{3}$$

Using (3) in (1) and (2), we have

$$P_n = \frac{\lambda^n}{1\,\mu \cdot 2\,\mu \cdot 3\mu \cdots n\mu}, P_0, \text{ if } 0 \leq n < s$$

$$= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \text{ if } 0 \leq n < s \tag{4}$$

and
$$P_n = \frac{\lambda^n}{\{1\,\mu \cdot 2\,\mu \cdots (s-1)\mu\}\{s\mu \cdot s\mu \cdots (n-s+1) \text{ times}\}} P_0$$

$$= \frac{\lambda^n}{(s-1)!\,\mu^{s-1}(s\mu)^{n-s+1}} \cdot P_0$$

$$= \frac{1}{s!\,s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, \text{ if } n \geq s \tag{5}$$

Now P_0 is given by $\sum_{n=0}^{\infty} P_n = 1$

i.e.
$$\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{\infty} \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n\right] P_0 = 1$$

i.e.,
$$\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!}\sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^n\right] P_0 = 1$$

i.e.,
$$\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{s^s}{s!}\left(\frac{\lambda}{\mu s}\right)^s \frac{1}{1-\dfrac{\lambda}{\mu s}}\right] P_0 = 1$$

i.e.,
$$\left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\dfrac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}\right] P_0 = 1$$

or
$$P_0 = \frac{1}{\left\{\displaystyle\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\dfrac{1}{s!\left(1-\dfrac{\lambda}{\mu s}\right)}\left(\dfrac{\lambda}{\mu}\right)^s\right\}} \tag{6}$$

2. *Average number of customers in the queue or average queue length*

$$L_q = E(N_q) = E(N-s) = \sum_{n=s}^{\infty} (n-s)P_n$$

$$= \sum_{x=0}^{\infty} x\, P_{x+s}$$

$$= \sum_{x=0}^{\infty} x \times \frac{1}{s!\, s^x}\left(\frac{\lambda}{\mu}\right)^{s+x} \cdot P_0$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0 \sum_{x=0}^{\infty} x\left(\frac{\lambda}{\mu s}\right)^x$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda}{\mu s} \cdot P_0 \frac{1}{\left(1-\dfrac{\lambda}{\mu s}\right)^2}$$

$$= \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} P_0 \tag{7}$$

3. *Average number of customers in the system*
   By Little's formula (iv),

$$E(N_s) = E(N_q) + \frac{\lambda}{\mu}$$

$$= \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} P_0 + \frac{\lambda}{\mu} \tag{8}$$

Result (8) can also be directly derived by using the definition $E(N_s) = \sum_{n=0}^{\infty} n\ P_n$.

4. *Average time a customer has to spend in the system*
   By Little's formula (i)

$$E(W_s) = \frac{1}{\lambda} E(N_s)$$

$$= \frac{1}{\mu} + \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_0 \tag{9}$$

5. *Average time a customer has to spend in the queue*
   By Little's formula (ii),

$$E(W_q) = \frac{1}{\lambda} E(N_q)$$

$$= \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_0 \tag{10}$$

6. *Probability that an arrival has to wait*
   Required probability = Probability that there are s or more customers in the system
   i.e., $P\ (W_s > 0) = P(N \geq s)$

$$= \sum_{n=s}^{\infty} P_n = \sum_{n=s}^{\infty} \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot P_0 \sum_{n=s}^{\infty}\left(\frac{\lambda}{\mu s}\right)^{n-s}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)} \tag{11}$$

7. *Probability that an arrival enters the service without waiting*
   Required probability

$$= 1 - P(\text{an arrival has to wait})$$

$$= 1 - \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)} \tag{12}$$

8. *Mean waiting time in the queue for those who actually wait.*

$$E(W_q/W_s > 0) = \frac{E(W_q)}{P(W_s > 0)}$$

$$= \frac{1}{\mu}\cdot\frac{1}{s\cdot s!}\frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 \times \frac{s!\left(1-\frac{\lambda}{\mu s}\right)}{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0} \quad \text{[using (10) and (11)]}$$

$$= \frac{1}{\mu s\left(1-\frac{\lambda}{\mu s}\right)} = \frac{1}{\mu s - \lambda} \tag{13}$$

9. *Probability that there will be someone waiting*
   Required probability = $P(N \geq s + 1)$

$$= \sum_{n=s+1}^{\infty} P_n = \sum_{n=s}^{\infty} P_n - P(N = s)$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)} - \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!} \quad \text{[using (10) and (5)]}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!} \cdot \frac{\left(\frac{\lambda}{\mu s}\right)}{1 - \frac{\lambda}{\mu s}} \tag{14}$$

10. *Average number of customers (in non-empty queues), who have to actually wait.*

$$L_w = E(N_q / N_q \geq 1) = E(N_q)/P(N \geq s)$$

$$= \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1} \cdot P_0}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot \frac{s!\left(1 - \frac{\lambda}{\mu s}\right)}{\left(\frac{\lambda}{\mu}\right)^s P_0}$$

$$= \frac{\left(\frac{\lambda}{\mu s}\right)}{1 - \frac{\lambda}{\mu s}} \tag{15}$$

Characteristics of Finite Capacity, Single Server Poisson Queue Model III [(M/M/1): (k/FIFO) Model]

1. *Values of P_0 and P_n*

For the Poisson queue system, $P_n = P(N = n)$ in the steady-state is given by the difference equations

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0; \; n > 0$$

and $\quad - \lambda_0 P_0 + \mu_1 P_1 = 0; \; n = 0$

This model represents the situation in which the system can accommodate only a finite number k of arrivals. If a customer arrives and the queue is full, the customer leaves without joining the queue.

Therefore, for this model,

$$\mu_n = \mu, \; n = 1, 2, 3, \ldots$$

and
$$\lambda_n = \begin{cases} \lambda, \text{ for } n = 0, 1, 2, \cdots (k-1) \\ 0, \text{ for } n = k, k+1, \cdots \end{cases}$$

Using these values in the difference equations given above, we have

$$\mu P_1 = \lambda P_0 \tag{1}$$

$$\mu P_{n+1} = (\lambda + \mu) P_n - \lambda P_{n-1}, \text{ for } 1 \leq n \leq k-1 \tag{2}$$

and $\quad \mu P_k = \lambda P_{k-1}$, for $n = k$ (3) $\quad (\because P_{k+1}$ has no meaning)

From (1), $\quad P_1 = \frac{\lambda}{\mu} P_0$

From (2), $\quad \mu P_2 = (\lambda + \mu) \frac{\lambda}{\mu} \cdot P_0 - \lambda P_0$

$$\therefore \qquad P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \text{ and so on}$$

In general, $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$, true for $0 \le n \le k-1$

From (3) $P_k = \frac{\lambda}{\mu}\cdot\left(\frac{\lambda}{\mu}\right)^{k-1} P_0 = \left(\frac{\lambda}{\mu}\right)^k P_0$

Now $$\sum_{n=0}^{k} P_n = 1$$

i.e., $$P_0 \sum_{n=0}^{k} \left(\frac{\lambda}{\mu}\right)^n = 1$$

i.e., $$P_0 \frac{\left\{1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right\}}{1-\frac{\lambda}{\mu}} = 1,$$

which is valid even for $\lambda > \mu$

$$\therefore \quad P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, \text{ if } & \lambda \ne \mu \qquad (4) \\[2ex] \dfrac{1}{k+1}, \text{ if } & \lambda = \mu, \text{ since } \lim\limits_{\frac{\lambda}{\mu}\to 1} \left\{\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right\} = \dfrac{1}{k+1} \qquad (5) \end{cases}$$

$$\therefore \quad P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \cdot \left[\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right], \text{ if } \lambda \ne \mu & (6) \\[2ex] \dfrac{1}{k+1}, \text{ if } \lambda = \mu & (7) \end{cases}$$

2. *Average number of customers in the system*

$$E(N) = \sum_{n=0}^{k} nP_n = \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \cdot \sum_{n=0}^{k} n\left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\left(1-\frac{\lambda}{\mu}\right)\cdot\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \cdot \sum_{n=0}^{k} \frac{\mathrm{d}}{\mathrm{d}x}(x^n), \text{ where } x = \frac{\lambda}{\mu}$$

$$= \frac{\left(1-\frac{\lambda}{\mu}\right)\cdot\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^{k+1}}{1-x}\right)$$

$$= \frac{(1-x)\,x}{1-x^{k+1}} \cdot \left[\frac{(1-x)\left\{-(k+1)\,x^k\right\} + \left(1-x^{k+1}\right)}{(1-x)^2}\right]$$

$$= \frac{x\left[1-(k+1)\,x^k + k\,x^{k+1}\right]}{(1-x)\left(1-x^{k+1}\right)}$$

$$= \frac{x\left(1-x^{k+1}\right) - (k+1)(1-x)\,x^{k+1}}{(1-x)\left(1-x^{k+1}\right)}$$

$$= \frac{x}{1-x} - \frac{(k+1)\,x^{k+1}}{1-x^{k+1}}$$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, \text{ if } \lambda \neq \mu \tag{8}$$

and
$$E(N) = \sum_{n=0}^{k} \frac{n}{k+1} = \frac{k}{2}, \text{ if } \lambda = \mu \tag{9}$$

3. *Average number of customers in the queue.*

$$E(N_q) = E(N-1) = \sum_{n=1}^{k} (n-1)P_n$$

$$= \sum_{n=0}^{k} n P_n - \sum_{n=1}^{k} P_n$$

$$= E(N) - (1 - P_0) \tag{10}$$

As per Little's formula (iv),

$$E(N_q) = E(N) - \frac{\lambda}{\mu},$$

which is true when the average arrival rate is λ throughout. Now we see that, in step (8), $1 - P_0 \neq \frac{\lambda}{\mu}$, because the average arrival rate is λ as long as there is a vacancy in the queue and it is zero when the system is full.

Hence we define *the overall effective arrival rate*, denoted by λ' or λ_{eff}, by using step (8) and Little's formula as

$$\frac{\lambda'}{\mu} = 1 - P_0 \text{ or } \lambda' = \mu (1 - P_0) \tag{11}$$

Thus, step (8) can be rewritten as

$$E(N_q) = E(N) - \frac{\lambda'}{\mu}, \tag{12}$$

which is the modified Little's formula for this model.

4. *Average waiting times in the system and in the queue:*
   By the modified Little's formulas,

$$E(W_s) = \frac{1}{\lambda'} E(N) \tag{13}$$

and

$$E(W_q) = \frac{1}{\lambda'} E(N_q) \tag{14}$$

where λ' is the effective arrival rate, given by step (9).

Characteristics of Finite Queue, Multiple Server Poisson Queue Model IV [(M/M/s): (k/FIFO) Model]

1. *Values of P_0 and P_n*
   For the Poisson queue system, P_n is given by

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \cdot P_0, n \geq 1, \tag{1}$$

where

$$P_0 = \left\{ 1 + \sum_{n=1}^{k} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) \right\}^{-1} \tag{2}$$

For this $(M/M/s)$: $(k/FIFO)$ model,

$$\lambda_n = \begin{cases} \lambda, & \text{for } n = 0, 1, 2, \cdots, k-1 \\ 0, & \text{for } n = k, k+1, \cdots \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & \text{for} \quad n = 0, 1, 2, \cdots, s-1 \\ s\mu, & \text{for} \quad n = s, s+1, \cdots \end{cases}$$

Using these values of λ_n and μ_n in (2) and noting that $1 < s < k$, we get

$$P_0^{-1} = \left\{1 + \frac{\lambda}{1!\,\mu} + \frac{\lambda^2}{2!\,\mu^2} + \cdots + \frac{\lambda^{s-1}}{(s-1)!\,\mu^{s-1}}\right\} + \left\{\frac{\lambda^s}{(s-1)!\,\mu^{s-1}\cdot\mu s}\right.$$

$$\left. + \frac{\lambda^{s+1}}{(s-1)!\,\mu^{s-1}\cdot(\mu s)^2} + \cdots + \frac{\lambda^k}{(s-1)!\,\mu^{s-1}\cdot(\mu s)^{k-s+1}}\right\}$$

$$= \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\lambda^s}{s!\,\mu^s}\left[1 + \frac{\lambda}{\mu s} + \left(\frac{\lambda}{\mu s}\right)^2 + \cdots + \left(\frac{\lambda}{\mu s}\right)^{k-s}\right]$$

$$= \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu s}\right)^{n-s} \tag{3}$$

$$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0, & \text{for} \quad n \le s \\ \dfrac{1}{s!\,s^{n-s}}\cdot\left(\dfrac{\lambda}{\mu}\right)^n \cdot P_0, & \text{for} \quad s < n \le k \\ 0, & \text{for} \quad n > k \end{cases} \tag{4}$$

2. *Average queue length or average number of customers in the queue*

$$E(N_q) = E(N - s) = \sum_{n=s}^{k} (n-s)P_n$$

$$= \frac{P_0}{s!}\sum_{n=s}^{k} (n-s)\left(\frac{\lambda}{\mu}\right)^n / s^{n-s} \text{ [using (4)]}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!} \sum_{x=0}^{k-s} x\cdot\left(\frac{\lambda}{\mu s}\right)^x$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0\,\rho}{s!} \sum_{x=0}^{k-s} x\,\rho^{x-1} \text{ where } \rho = \frac{\lambda}{\mu s}$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0\,\rho}{s!} \sum_{x=0}^{k-s} \frac{d}{d\rho}\left(\rho^x\right)$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0\,\rho}{s!}\,\frac{d}{d\rho}\left\{\frac{1-\rho^{k-s+1}}{1-\rho}\right\}$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0\,\rho}{s!}\left[\frac{-(1-\rho)(k-s+1)\,\rho^{k-s} + \left(1-\rho^{k-s+1}\right)}{(1-\rho)^2}\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{P_0\,\rho}{s!}\left[\frac{-(k-s)(1-\rho)\rho^{k-s} - (1-\rho)\rho^{k-s} + 1 - \rho^{k-s+1}}{(1-\rho)^2}\right]$$

$$= P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!}\left[\frac{-(k-s)(1-\rho)\,\rho^{k-s} + 1 - \rho^{k-s}(1-\rho+\rho)}{(1-\rho)^2}\right]$$

$$= P_0 \cdot \left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\,[1 - \rho^{k-s} - (k-s)(1-\rho)\,\rho^{k-s}],$$

where $\rho = \dfrac{\lambda}{\mu s}$ (5)

3. *Average number of customers in the system*

$$E(N) = \sum_{n=0}^{k} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{k} nP_n$$

$$= \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{k} (n-s)\,P_n + \sum_{n=s}^{k} sP_n$$

$$= \sum_{n=0}^{s-1} nP_n + E\,(N_q) + s\left\{\sum_{n=0}^{k} P_n - \sum_{n=0}^{s-1} P_n\right\}$$

$$= E(N_q) + s - \sum_{n=0}^{s-1} (s-n)P_n \left(\because \sum_{n=0}^{k} P_n = 1\right) \tag{6}$$

Obviously $\left\{s - \sum_{n=0}^{s-1} (s-n)P_n\right\} \neq \dfrac{\lambda}{\mu}$, so that step (6) represents Little's formula.

In order to make (6) to assume the form of Little's formula, we define the *overall effective arrival rate* λ' or λ_{eff} as follows:

$$\frac{\lambda'}{\mu} = s - \sum_{n=0}^{s-1} (s-n)P_n$$

i.e.,
$$\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)\,P_n\right] \tag{7}$$

With this definition of λ', step (6) becomes

$$E(N) = E(N_q) + \frac{\lambda'}{\mu} \tag{8}$$

which is the modified Little's formula for this model.

4. *Average waiting time in the system and in the queue:*
By the modified Little's formulas,

$$E(W_s) = \frac{1}{\lambda'} E(N) \tag{9}$$

and
$$E(W_q) = \frac{1}{\lambda'} E(N_q) \tag{10}$$

where λ' is the effective arrival rate, given by step (7).

Worked Examples 9

Example 1 *Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.*

(a) Find the average number of persons waiting in the system.
(b) What is the probability that a person arriving at the booth will have to wait in the queue?
(c) What is the probability that it will take him more than 10 min. altogether to wait for the phone and complete his call?
(d) Estimate the fraction of the day when the phone will be in use.
(e) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for at least 3 min. for phone. By how much the flow of arrivals should increase in order to justify a second booth?
(f) What is the average length of the queue that forms from time to time?

Solution Mean inter-arrival time $= \frac{1}{\lambda} = 12$ min.

Therefore mean arrival rate $= \lambda = \frac{1}{12}$ per minute.

Mean service time $= \frac{1}{\mu} = 4$ min.

Therefore, mean service rate $= \mu = \frac{1}{4}$ per minute.

(a) $E(N) = \frac{\lambda}{\mu - \lambda}$, (by formula (1) of model I)

$$= \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = 0.5 \text{ customer}$$

(b) $P(W > 0) = 1 - P(W = 0)$
$= 1 - P$ (no customer in the system)

$$= 1 - P_0$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right) \text{ (by the formula for } P_0 \text{ of model I)}$$

$$= \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \frac{1}{3}$$

(c) $P(W > 10) = e^{-(\mu - \lambda) \times 10}$ [by formula (8) of model I]

$$= e^{-\left(\frac{1}{4} - \frac{1}{12}\right) \times 10}$$

$$= e^{-\frac{5}{3}} = 0.1889$$

(d) $P(\text{the phone will be idle}) = P(N = 0) = P_0$

$$= 1 - \frac{\lambda}{\mu} = \frac{2}{3}$$

$\therefore$ P (the phone will be in use) $= 1 - \frac{2}{3} = \frac{1}{3}$

or the fraction of the day when the phone will be in use $= \frac{1}{3}$.

(e) The second phone will be installed, if $E(W_q) > 3$.

i.e., if $\dfrac{\lambda}{\mu(\mu - \lambda)} > 3$ [by formula (10) of model I]

i.e., if $\dfrac{\lambda_R}{\frac{1}{4}\left(\frac{1}{4} - \lambda_R\right)} > 3,$

where λ_R is the required arrival rate.

i.e., if $\lambda_R > \dfrac{3}{4}\left(\dfrac{1}{4} - \lambda_R\right)$

i.e., if $\lambda_R > \dfrac{3}{28}$

Hence the arrival rate should increase by $\frac{3}{28} - \frac{1}{12} = \frac{1}{42}$ per minute, to justify a second phone.

(f) $E(N_q/\text{the queue is always available})$

$$= E(N_q/N_q > 0)$$

$$= E(N_q/N > 1)$$

$$= \frac{E(N_q)}{P(N > 1)} = \frac{E(N_q)}{1 - P_0 - P_1} = \frac{\lambda^2}{\mu(\mu - \lambda)} \times \frac{1}{1 - \left(1 + \frac{\lambda}{\mu}\right) P_0},$$

[by formula (2) of model 1]

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{1}{1-\left(1+\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{\mu^2}{\lambda^2} = \frac{\mu}{\mu-\lambda} = \frac{1/4}{1/4-1/12} = 1.5 \text{ persons.}$$

Example 2 *Customers arrive at a one-man barber shop according to a Poisson process with a mean interarrival time of 12 min. Customers spend an average of 10 min in the barber's chair.*

(a) *What is the expected number of customers in the barber shop and in the queue?*
(b) *Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.*
(c) *How much time can a customer expect to spend in the barber's shop?*
(d) *Management will provide another chair and hire another barber, when a customer's waiting time in the shop exceeds 1.25 h. How much must the average rate of arrivals increase to warrant a second barber?*
(e) *What is the average time customers spend in the queue?*
(f) *What is the probability that the waiting time in the system is greater than 30 min?*
(g) *Calculate the percentage of customers who have to wait prior to getting into the barber's chair.*
(h) *What is the probability that more than 3 customers are in the system?*

Solution $\frac{1}{\lambda} = 12 \qquad \therefore \lambda = \frac{1}{12}$ *per minute*

$\frac{1}{\mu} = 10 \qquad \therefore \mu = \frac{1}{10}$ *per minute*

(a) $E(N_s) = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{1/10-1/12} = 5$ customers [by formula (1) of model I]

$$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} \qquad \text{[by formula (2) of model I]}$$

$$= \frac{\frac{1}{144}}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)} = 4.17 \text{ customers}$$

(b) P(a customer straight goes to the barber's chair)
= P (No customer in the system)

$$= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{\frac{1}{12}}{\frac{1}{10}} = \frac{1}{6}$$

Therefore, percentage of time an arrival need not wait = 16.7.

(c) $E(W) = \dfrac{1}{\mu - \lambda}$ [by formula (7) of model I]

$$= \frac{1}{\frac{1}{10} - \frac{1}{12}} = 60 \text{ min or } 1 \text{ h}$$

(d) $E(W) > 75$, if $\dfrac{1}{\mu - \lambda_r} > 75$

i.e., if $\lambda_r > \mu - \dfrac{1}{75}$

i.e., if $\lambda_r > \dfrac{1}{10} - \dfrac{1}{75}$

i.e., if $\lambda_r > \dfrac{13}{150}$

Hence, to warrant a second barber, the average arrival rate must increase by $\dfrac{13}{150} - \dfrac{1}{12} = \dfrac{1}{300}$ per minute.

(e) $E(W_q) = \dfrac{\lambda}{\mu(\mu - \lambda)}$

[by formula (10) of model I]

$$= \frac{\frac{1}{12}}{\frac{1}{10}\left(\frac{1}{10} - \frac{1}{12}\right)} = 50 \text{ min}$$

(f) $P(W > t) = e^{-(\mu - \lambda)t}$, [by formula (8) of model I]

$$\therefore \quad P(W > 30) = e^{-\left(\frac{1}{10} - \frac{1}{12}\right) \times 30}$$

$$= e^{-0.5} = 0.6065$$

(g) P(a customer has to wait) = $P(W > 0)$

$$= 1 - P(W = 0) = 1 - P(N = 0) = 1 - P_0$$

$$= \frac{\lambda}{\mu} = \frac{1/12}{1/10} = \frac{5}{6}$$

$\therefore$ Percentage of customers who have to wait

$$= \frac{5}{6} \times 100 = 83.33$$

(h) $P(N > 3) = P_4 + P_5 + P_6 + \ldots$

$$= 1 - \{P_0 + P_1 + P_2 + P_3\}$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)\left\{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right\}$$

[since $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$, for $n \geq 0$, for model I]

$$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{5}{6}\right)^4 = 0.4823$$

Example 3 *At what average rate must a clerk in a supermarket work in order to ensure a probability of 0.90 that the customer will not wait longer than 12 min? It is assumed that there is only one counter at which customers arrive in a Poisson fashion at an average rate of 15 per hour and that the length of the service by the clerk has an exponential distribution.*

Solution $\lambda = 15$/hour; $\mu = \mu_R$/hour

$$P\left(W_q \leq \frac{1}{5}\right) = 0.90$$

i.e., $$P\left(W_q > \frac{1}{5}\right) = 0.10$$

i.e., $$\int_{0.2}^{\infty} g(w)\, \mathrm{d}w = 0.10$$

i.e., $$\int_{0.2}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)\, e^{-(\mu - \lambda)w}\, \mathrm{d}w = 0.10 \quad \text{[by formula (9) of model I]}$$

i.e., $$\left[-\frac{\lambda}{\mu} e^{-(u - \lambda)W}\right]_{0.2}^{\infty} = 0.1$$

i.e., $$\frac{15}{\mu_R} e^{-(\mu_R - 15) \times 0.2} = 0.1$$

i.e., $(15 - \mu_R) \times 0.2 = \log(0.1) - \log 15 + \log \mu_R$

i.e., $0.2\, \mu_R + \log \mu_R = 3 + \log 150 \simeq 8$ (1)

Solving (1), we get $\mu_R = 24$ approximately.

That is, the clerk must serve at the rate of 24 customer per hour.

Example 4 *If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 min before the picture starts and if its takes exactly 1.5 min to reach the correct seat after purchasing the ticket,*

(a) Can he expect to be seated for the start of the picture?

(b) What is the probability that he will be seated for the start of the picture?

(c) *How early must he arrive in order to be 99% sure of being seated for the start of the picture?*

Solution $\lambda = 6/\text{minute}; \mu = 8/\text{minute}$

(a) $$E(W) = \frac{1}{\mu - \lambda} \quad \text{[by formula (7) of model I]}$$

$$= \frac{1}{8-6} = \frac{1}{2} \text{ min}$$

$\therefore$ E(total time required to purchase the ticket and to reach the seat)

$$= \frac{1}{2} + 1\frac{1}{2} = 2 \text{ min}$$

Hence he can just be seated for the start of the picture.

(b) P(total time < 2 min)

$$= P\left(W < \frac{1}{2}\right) = 1 - P\left(W > \frac{1}{2}\right)$$

$$= 1 - e^{-\mu\left(1 - \frac{\lambda}{\mu}\right) \times \frac{1}{2}} \quad \text{[by formula (8) of model I]}$$

$$= 1 - e^{-1} = 0.63$$

(c) $P(W < t) = 0.99$

i.e., $P(W > t) = 0.01$

i.e., $e^{-(\mu - \lambda)t} = 0.1$

i.e., $-2t = \log(0.1) = -2.3$

$\therefore$ $t = 2.3$ min

i.e., P(ticket purchasing time < 2.3) = 0.99

$\therefore$ P[total time to get the ticket and to go to the seat < (2.3 + 1.5)] = 0.99

Therefore the person must arrive at least 3.8 min early so as to be 99% sure of seeing the start of the picture.

Example 5 *A duplicating machine maintained for office use is operated by an office assistant who earns Rs 5 per hour. The time to complete each job varies according to an exponential distribution with mean 6 min. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an 8-h day is used as a base, determine*

(a) *the percentage idle time of the machine,*
(b) *the average time a job is in the system and*
(c) *the average earning per day of the assistant.*

Solution $\lambda = 5/\text{hour}; \mu = \frac{60}{6} = 10/\text{hour}$

(a) P(the machine is idle) $= P(N = 0) = P_0$

$$= 1 - \frac{\lambda}{\mu} \text{ (by the formula for } P_0 \text{ in model I)}$$

$$= 1 - \frac{5}{10} = \frac{1}{2}$$

$\therefore$ Percentage of idle time of the machine = 50

(b) $E(W) = \dfrac{1}{\mu - \lambda}$ [by formula (7) of model I]

$$= \frac{1}{10-5} = \frac{1}{5} \text{ h or 12 min}$$

(c) E(earning per day)

= E(number of jobs done/day) $\times$ earning per job

= E(number of jobs done/day) $\times$ E (time in hour/job) $\times$ earning/hour

$$= (8 \times 5) \times \frac{1}{5} \times 5 = \text{Rs } 40.$$

Example 6 *The mean rate of arrival of planes at an airport during the peak period is 20 per hour, but the actual number of arrivals in any hour follows a Poisson distribution. The airport can land 60 planes per hour on an average in good weather or 30 planes per hour in bad weather, but the actual number landed in any hour follows a Poisson distribution with respective averages. When there is congestion, the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.*

(a) How many planes would be flying over the field in the stack on an average in good weather and in bad weathers?

(b) How long a plane would be in the stack and in the process of landing in good and bad weathers?

(c) How much stack and landing time to allow so that priority to land out of order will have to be requested only 1 in 20 times.

Solution λ = *20 per hour*

$$\mu = \begin{cases} 60 \text{ per hour in good weather} \\ 30 \text{ per hour in bad weather} \end{cases}$$

Note *Landing time is service time; the planes flying over the field in the stack are assumed to form the queue.*

(a) $E(N_q)$ = Average number of planes flying over the field = $\dfrac{\lambda^2}{\mu(\mu - \lambda)}$ [by formula (2) of model I]

$$= \begin{cases} \dfrac{20^2}{60(60-20)}, & \text{in good weather} \\ \dfrac{20^2}{30(30-20)}, & \text{in bad weather} \end{cases}$$

$$= \begin{cases} \dfrac{1}{6}, & \text{in good weather} \\ \dfrac{4}{3}, & \text{in bad weather} \end{cases}$$

(b) $E(W)$ = Average time for flying in the stack and for landing

$$= \frac{1}{\mu - \lambda} \text{ [by formula (7) of model I]}$$

$$= \begin{cases} \dfrac{1}{40}\text{ h} & \text{or 1.5 min in good weather} \\ \dfrac{1}{10}\text{ h} & \text{or 6 min in bad weather} \end{cases}$$

(c) Let t_R be the maximum stack and landing time to be allowed, beyond which priority out of order is to be requested.

Then $P(W > t_R) = \dfrac{1}{20}$

i.e., $$e^{-\mu\left(1-\frac{\lambda}{\mu}\right)t_R} = 0.05 \text{ [by formula (8) of model I]}$$

i.e., $$\begin{cases} e^{-40t_R} = 0.05, & \text{for good weather} \\ e^{-10t_R} = 0.05, & \text{for bad weather} \end{cases}$$

i.e., $$\begin{cases} t_R = 0.075 \text{ h or 4.5 min for good weather} \\ \quad = 0.299 \text{ h or 18 min for bad weather} \end{cases}$$

Example 7 *There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour,*

(a) What fraction of the time all the typists will be busy?
(b) What is the average number of letters waiting to be typed?
(c) What is the average time a letter has to spend for waiting and for being typed?
(d) What is the probability that a letter will take longer than 20 min waiting to be typed and being typed?

Solution λ = 15/hour; μ = 6/hour; $s = 3$.

Hence this is a problem in multiple server [$(M/M/s)$: $(\infty/FIFO)$] model, i.e., model II.

(a) P(all the typists are busy) = $P(N \geq 3)$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^3 \cdot P_0}{3!\left(1 - \frac{\lambda}{3\mu}\right)} \qquad \text{[by formula (11) of model II]}$$

$$= \frac{(2.5)^3 P_0}{6 \times \left(1 - \frac{2.5}{3}\right)} \qquad (1)$$

Now
$$P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left(\frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \cdot \left(\frac{\lambda}{\mu}\right)^s\right)}$$

[by formula (6) of model II]

$$= \frac{1}{\left\{1 + 2\cdot 5 + \frac{1}{2} \times (2\cdot 5)^2\right\} + \left\{\frac{1}{6 \times \left(1 - \frac{5}{6}\right)} \times (2\cdot 5)^3\right\}}$$

$$= \frac{1}{22.25} = 0.0449 \qquad (2)$$

Using (2) in (1), we have $P(N \geq 3) = 0.7016$.

Hence the fraction of the time all the typists will be busy = 0.7016.

(b) $$E(N_q) = \frac{1}{s.s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1} \cdot P_0}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \qquad \text{[by formula (7) of model II]}$$

$$= \frac{1}{3 \times 6} \times \frac{(2.5)^4}{\left(1 - \frac{2.5}{3}\right)^2} \times 0.0449 = 3.5078$$

(c) $E(W) = \frac{1}{\lambda} E(N)$ [by Little's formula (i)]

$$= \frac{1}{\lambda}\left\{E\left(N_q\right) + \frac{\lambda}{\mu}\right\}, \text{ [by Little's formula (iv)]}$$

$$= \frac{1}{15}\{3.5078 + 2.5\} = 0.4005 \text{ h}$$

or 24 min, nearly

(d) $$P(W > t) = e^{-\mu t}\left\{1 + \frac{\left(\frac{\lambda}{\mu}\right)^s \left[1 - e^{-\mu t\left(s - 1 - \frac{\lambda}{\mu}\right)}\right] P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)\left(s - 1 - \frac{\lambda}{\mu}\right)}\right\}$$

(This formula has not been derived; it may be assumed.)

$$\therefore \quad P\left(W > \frac{1}{3}\right) = e^{-6 \times \frac{1}{3}} = \left[1 + \frac{(2.5)^3\{1 - e^{(-2 \times -0.5)}\} \times 0.0449}{6\left(1 - \frac{2.5}{3}\right)(-0.5)}\right]$$

$$= e^{-2}\left[1 + \frac{0.7016\,(1 - e)}{(-0.5)}\right]$$

$$= 0.4616$$

Example 8 *Given an average arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rate of 22 customers per hour or at one of two channels in parallel with mean service rate of 11 customers per hour for each of the two channels. Assume both queues to be of Poisson type.*

Solution For the single channel service,

$\lambda = 20$/hour and $\mu = 22$/hour.

$$E(W) = \frac{1}{\mu - \lambda} \qquad \text{[by formula (7) of model I]}$$

$$= \frac{1}{2}\text{ h}$$

For the two channel service,

$\lambda = 20$/hour and $\mu = 11$/hour.

$$E(W) = \frac{1}{\mu} + \frac{1}{\mu}\cdot\frac{1}{s\cdot s!}\cdot\frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \times P_0$$

[by formula (9) of model II]

$$= \frac{1}{11} + \frac{1}{11 \times 2 \times 2} \times \frac{\left(\frac{20}{11}\right)^2}{\left(1 - \frac{20}{22}\right)^2} \times P_0$$

$$= 0.0909 + 9.0909 \times P_0 \qquad (1)$$

Now $$P_0^{-1} = \left\{ \sum_{n=0}^{s-1} \frac{1}{n!} \cdot \left(\frac{\lambda}{\mu} \right)^n \right\} + \left\{ \frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \cdot \left(\frac{\lambda}{\mu} \right)^s \right\}$$

[by formula (6) of model II]

$$= 1 + \frac{20}{11} + \frac{1}{2 \times \frac{1}{11}} \times \left(\frac{20}{11} \right)^2$$

$$= 21$$

$\therefore$ $$P_0 = 0.0476 \qquad (2)$$

Using (2) in (1), we have

$$E(W) = 0.5236 \text{ h}$$

As the average waiting time in single channel service is less than that in two channel service, the customer has to prefer the former.

Example 9 *A telephone company is planning to install telephone booths in a new airport. It has established the policy that a person should not have to wait more than 10% of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 min. How many phone booths should be installed?*

Solution $\lambda = 30$/hour and $\mu = 12$/hour

In order that infinite queue may not build up, *the traffic intensity* $\frac{\lambda}{\mu s} < 1$, for multiserver model.

i.e., $$s > \frac{\lambda}{\mu}$$

i.e., $$s > \frac{30}{12} \; (= 2.5)$$

Therefore, the telephone company must install at least 3 booths.

Now we have to find the number s of telephone booths such that

$P(W > 0) \le 0.10$ or equivalently

$P(N \ge s) \le 0.10$

i.e., we have to find s such that

$$\frac{\left(\frac{\lambda}{\mu} \right)^s \cdot P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \le 0.10$$ [by formula (11) model II]

This inequation is not easily solvable. Hence we proceed by trials and find out the least value of s that satisfies this inequation.

Let $s = 3$:

Then $$P(W > 0) = \frac{(2.5)^3 \cdot P_0}{6\left(1 - \frac{2.5}{3}\right)} = 15.625\, P_0,$$

where $$P_0 = \left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\right]^{-1}$$

[by formula (6) of model II]

i.e., $$P_0 = \left[\left\{1 + 2.5 + \frac{1}{2} \times (2.5)^2\right\} + \frac{(2.5)^3}{6 \times \left(1 - \frac{2.5}{3}\right)}\right]^{-1}$$

$$= (22.25)^{-1} = 0.0449$$

$\therefore$ $P(W > 0) = 15.625 \times 0.0449 = 0.7022 \not< 0.10$

Let $s = 4$:

Then $$P(W > 0) = \frac{(2.5)^4 \cdot P_0}{24\left(1 - \frac{2.5}{4}\right)} = 4.3403\, P_0,$$

where $$P_0 = \left[\left\{1 + 2.5 + \frac{1}{2} \times (2.5)^2 + \frac{1}{6} \times (2.5)^3\right\} + \frac{(2.5)^4}{24\left(1 - \frac{2.5}{4}\right)}\right]^{-1}$$

$$= 0.0737$$

$\therefore$ $P(W > 0) = 4.3403 \times 0.0737 = 0.3199 \not< 0.10$

Similarly, when $s = 5$, $P(W > 0) = 0.1304 \not< 0.10$.

When $s = 6$, $P(W > 0) = 0.047 < 0.10$.

Hence the number of booths to be installed = 6.

Example 10 *A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distributions for both deposits and withdrawals are exponential with mean service time of 3 min per customer. Depositors are found to arrive in a Poisson fashion throughout the day with mean arrival rate of 16 per hour. Withdrawers also arrive in a Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for the customers if each teller could handle both withdrawals and deposits. What would be the effect, if this could only be accomplished by increasing the service time to 3.5 min?*

Solution When there is a separate channel for the depositors, λ_1 = 16/hour, μ = 20/hour

$$\therefore \quad E(W_q \text{ for depositors}) = \frac{\lambda_1}{\mu(\mu - \lambda_1)} \quad \text{[by formula (10) of model I]}$$

$$= \frac{16}{20\,(20-16)} = \frac{1}{5} \text{ h or 12 min}$$

When there is a separate channel for the withdrawers, λ_2 = 14/hour, μ = 20/hour.

$$\therefore \quad E(W_q \text{ for withdrawers}) = \frac{\lambda_2}{\mu(\mu - \lambda_2)}$$

$$= \frac{14}{20(20-14)} = \frac{7}{60} \text{ h or 7 min}$$

If both tellers do both service,

$$s = 2,\ \mu = 20\text{/hour},\ \lambda = \lambda_1 + \lambda_2 = 30\text{/hour}$$

$$\therefore \quad E(W_q \text{ for any customer}) = \frac{1}{\mu}\cdot\frac{1}{s.s!}\cdot\frac{\left(\frac{\lambda}{\mu}\right)^s \times P_0}{\left(1 - \frac{\lambda}{\mu s}\right)^2},$$

[by formula (10) of model II]

$$= \frac{1}{20} \times \frac{1}{2 \times 2} \times \frac{(1.5)^2}{(1-.75)^2} \times P_0 \qquad (1)$$

$$= 0.45 \times P_0$$

Now

$$P_0 = \left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\cdot\left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$$

[by formula (6) of, model II]

$$= \left[1 + 1.5 + \frac{(1.5)^2}{2 \times 0.25}\right]^{-1}$$

$$= \frac{1}{7} \qquad (2)$$

Using (2) in (1)

$$E(W_q \text{ for any customer}) = 0.45 \times \frac{1}{7} \text{ h or 3.86 min}$$

Hence if both tellers do both types of service, the customers get benefited as their waiting time is considerably reduced.

Now if both tellers do both types of service but with increased service time,

$$s = 2,\ \lambda = 30,\ \mu = \frac{60}{3.5} = \frac{120}{7} \text{ per hour.}$$

$E(W_q$ of any customer)

$$= \frac{7}{120} \times \frac{1}{2 \times 2} \times \frac{(1.75)^2}{\left(1 - \frac{7}{8}\right)^2} \times P_0 = 2.86\, P_0, \text{ where}$$

$$P_0 = \left[1 + 1.75 + \frac{(1.75)^2}{2 \times \frac{1}{8}}\right]^{-1} = \frac{1}{15}$$

$$\therefore \quad E(W_q \text{ of any customer}) = \frac{2.86}{15} \text{ h or } 11.44 \text{ min}$$

If this arrangement is adopted, withdrawers stand to lose as their waiting time is increased considerably and depositors get slightly benefited.

Example 11 *A supermarket has two girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 min and if people arrive in Poisson fashion at the rate of 10 per hour,*

(a) what is the probability that a customer has to wait for service?
(b) what is the expected percentage of idle time for each girl?
(c) if the customer has to wait in the queue, what is the expected length of his waiting time?

Solution $s = 2,\ \lambda = \frac{1}{6}$ per minute, $\mu = \frac{1}{4}$ per minute

(a) P(a customer has to wait for service)

$$= P(N \geq 2) = 1 - P_0 - P_1 \tag{1}$$

$$P_0 = \left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\right\}\right]^{-1}$$

[by formula (6) of model II]

$$= \left[1 + \frac{2}{3} + \frac{\left(\frac{2}{3}\right)^2}{2 \times \left(1 - \frac{1}{3}\right)}\right]^{-1} = \frac{1}{2} \tag{2}$$

$$P_1 = \frac{\lambda}{\mu} \cdot P_0, \quad \text{[by formula (4) of model II]}$$

$$= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \quad (3)$$

Using (2) and (3) in (1), we have

$$P(N \geq 2) = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) Fraction of time when the girls are busy $= \dfrac{\lambda}{\mu s} = \dfrac{1}{3}$

$\therefore$ Fraction of time when the girls are idle $= \dfrac{2}{3}$

$\therefore$ Expected percentage of idle time for each girl $= \dfrac{2}{3} \times 100 = 67$

(c) $E(W_q / W_s > 0) = \dfrac{1}{\mu s - \lambda}$ [by formula (13) of model II]

$$= \frac{1}{\frac{1}{4} \times 2 - \frac{1}{6}} = 3 \text{ min}$$

Example 12 *A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.*

(a) What is the probability that an arrival would have to wait in line?

(b) Find the average waiting time, average time spent in the system and the average number of cars in the system.

(c) For what percentage of time would a pump be idle on an average?

Solution $s = 4$, $\lambda = 30$/hour, $\mu = 10$/hour

(a) P(an arrival has to wait) $= P(W > 0)$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \quad \text{[by formula (11) of model II]}$$

$$= \frac{3^4 \times P_0}{24 \times \left(1 - \frac{3}{4}\right)} = 13.5 \times P_0 \quad (1)$$

$$P_0 = \left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\right]^{-1}$$

[by formula (6) of model II]

$$= \left[\left(1 + 3 + \frac{1}{2} \times 9 + \frac{1}{6} \times 27\right) + \frac{3^4}{24 \times \left(1 - \frac{3}{4}\right)}\right]^{-1}$$

$$= 0.0377 \tag{2}$$

Using (2) in (1), $P(W > 0) = 0.5090$

(b) $E(W_q) = \dfrac{1}{\mu} \cdot \dfrac{1}{s \times s!} \times \dfrac{\left(\dfrac{\lambda}{\mu}\right)^s}{\left(1 - \dfrac{\lambda}{\mu s}\right)^2} \times P_0$ [by formula (10) of model II]

$$= \frac{1}{10 \times 4 \times 24} \times \frac{3^4}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 = 0.0509 \text{ h}$$

or 3.05 min

$E(W_s) = \dfrac{1}{\mu} + E(W_q)$ [by formulas (9) and (10) of model II]

$= 6 + 3.05 = 9.05$ min

$$E(N) = \frac{1}{s \times s!} \times \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \times P_0 + \frac{\lambda}{\mu}$$

[by formula (8) of model II]

$$= \frac{1}{4 \times 24} \times \frac{3^5}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 + 3$$

$= 4.53$ cars

(c) The fraction of time when the pumps are busy = traffic intensity

$$= \frac{\lambda}{\mu s} = \frac{3}{4}$$

∴ The fraction of time when the pumps are idle $= \dfrac{1}{4}$

Therefore, required percentage = 25%

Example 13 *In a single server queueing system with Poisson input and exponential service times, if the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 h and the maximum possible number of calling* *units in the system is 2, find $P_n(n \geq 0)$, average number of calling units in the system and in the queue and average waiting time in the system and in the queue.*

Solution The situation in this problem is one of finite capacity, single server Poisson queue models.

$$\lambda = 3,\ \mu = 4 \text{ and } k = 2$$

As $$\lambda \neq \mu,\ P_0 = \frac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \qquad \text{[by formula (4) of model III]}$$

$$= \frac{1 - \dfrac{3}{4}}{1 - \left(\dfrac{3}{4}\right)^{3}} = \frac{16}{37} = 0.4324$$

Since $$\lambda \neq \mu,\ P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}}\right] \qquad \text{[by formula (6) of model III]}$$

$$= (0.4324)\ (0.75)^n$$

$$E(N) = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \qquad \text{[by formula (8) of model IV]}$$

$$= \frac{3}{4-3} - \frac{3 \times \left(\dfrac{3}{4}\right)^3}{1 - \left(\dfrac{3}{4}\right)^3} = 3 - \frac{81}{37} = \frac{30}{37} \simeq 0.8 \text{ calling unit}$$

$$E(N_q) = E(N) - (1 - P_0) \qquad \text{[by formula (10) of model III]}$$

$$= \frac{30}{37} - \left(1 - \frac{16}{37}\right) = \frac{9}{37} = 0.24 \text{ calling unit}$$

$$E(W_s) = \frac{1}{\lambda'}\ E(N) \qquad \text{[by formula (13) of model III]}$$

where $$\lambda' = \mu\,(1 - P_0), \qquad \text{[by formual (11) of model III]}$$

$$= 4\left(1 - \frac{16}{37}\right) = \frac{84}{37}$$

$$\therefore \qquad E(W_s) = \frac{37}{84} \times \frac{30}{37} = \frac{5}{14} \text{ h or 21.4 min}$$

$$E(W_q) = \frac{1}{\lambda'} E(N_q) \text{ [by formula (14) of model III]}$$

$$= \frac{37}{84} \times \frac{9}{37} = \frac{3}{28} \text{ h or 6.4 min}$$

Example 14 *The local one-person barber shop can accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair-cut). Customers arrive according to a Poisson distribution with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (Exponential service time).*

(a) What percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is the expected number of customers waiting for a hair-cut?

(d) How much time can a customer expect to spend in the barber shop?

Solution $\lambda = 5, \mu = 4, k = 5$

(a) P(the barber is idle) $= P(N = 0)$

$$= P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \qquad \text{[by formula (4) of model III]}$$

$$= \frac{1 - \frac{5}{4}}{1 - \left(\frac{5}{4}\right)^6} = 0.0888$$

$\therefore$ Percentage of time when the barber is idle $\simeq 9$.

(b) P (a customer is turned away) $= P(N > 5)$

$$= \left(\frac{\lambda}{\mu}\right)^5 \cdot \left[\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}\right] \text{ [by formula (6) of model III]}$$

$$= \left(\frac{5}{4}\right)^5 \cdot \left[\frac{1 - \frac{5}{4}}{1 - \left(\frac{5}{4}\right)^6}\right]$$

$$= \frac{3125}{11529} = 0.2711$$

Therefore, $0.2711 \times$ potential customers are turned away.

(c) $E(N_q) = E(N) - (1 - P_0)$

$$= \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} - (1 - P_0),$$

[by formulas (6) and (10) of model III]

$$= \left\{ -5 - \frac{6 \times \left(\frac{5}{4}\right)^6}{1 - \left(\frac{5}{4}\right)^6} \right\} - (1 - 0.0888)$$

$$= \frac{6 \times \frac{15625}{4096}}{\frac{11529}{4096}} - 5.9112 \simeq 2.2 \text{ customers}$$

(d) $E(W) = \frac{1}{\lambda'} E(N)$ [by formula (13) of model III]

$$= \frac{1}{\mu(1 - P_0)} \times E(N) = \frac{3.1317}{3.6448} \simeq 0.8592 \text{ h}$$

or 51.5 min

Example 15 *At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals and exponential service distribution, find the probabilities for the numbers of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results get modified?*

Solution

(i) $\lambda = 6$ per hour, $\mu = 6$ per hour, $k = 2 + 1 = 3$

Since $\lambda = \mu$, $P_0 = \frac{1}{k+1}$

$$= \frac{1}{4} \qquad \text{[by formula (5) of model III]}$$

$$P_n = \frac{1}{k+1} = \frac{1}{4} \text{ for } n = 1, 2, 3 \text{ [by formula (7) of model III]}$$

$$E(N) = \frac{k}{2} \qquad \text{[by formula (9) of model III]}$$

$= 1.5$ trains

$$E(W) = \frac{1}{\lambda'} E(N) \qquad \text{[by formula (13) of model III]}$$

$$= \frac{1.5}{\mu(1-P_0)} = \frac{1.5}{6 \times \frac{3}{4}} = \frac{1}{3} \text{ h or 20 min}$$

(ii) $\lambda = 6;\ \mu = 12,\ k = 3$

$$\text{Since } \lambda \neq \mu,\ P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \qquad \text{[by formula (4) of model III]}$$

$$= \frac{1 - \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^4} = \frac{8}{15}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left\{ \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \right\} \qquad \text{[by formula (6) of model III]}$$

$$= \frac{8}{15} \cdot \left(\frac{1}{2}\right)^n, \text{ for } n = 1, 2, 3.$$

$$E(N) = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \qquad \text{[by formula (8) of model III]}$$

$$= 1 - \frac{4 \times \left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{2}\right)^4} = 1 - \frac{4}{15} = \frac{11}{15} \simeq 0.73 \text{ train}$$

$$E(W) = \frac{1}{\lambda'} E(N) \qquad \text{[by formula (13) of model III]}$$

$$= \frac{1}{\mu(1-P_0)} \times E(N)$$

$$= \frac{\frac{11}{15}}{12\left(1 - \frac{8}{15}\right)} = \frac{11}{84} \text{ h or 7.9 min}$$

Example 16 *Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.*

(a) Find the effective arrival rate at the clinic.

(b) What is the probability that an arriving patient will not wait?

(c) What is the expected waiting time until a patient is discharged from the clinic?

Solution

(a) $\lambda = 30$ per hour, $\mu = 20$ per hour, $k = 14 + 1 = 15$

Since $\lambda \neq \mu$, $P_0 = \dfrac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}}$ [by formula (4) of model III]

$$= \frac{1 - \dfrac{3}{2}}{1 - \left(\dfrac{3}{2}\right)^{16}} = 0.00076$$

Effective arrival rate $\lambda' = \mu\,(1 - P_0)$ [by formula (11) of model III]

$= 20 \times (1 - 0.00076)$

$= 19.98$ per hour

(b) P(a patient will not wait)

$= P_0 = 0.00076$

(c) $E(N) = \dfrac{\lambda}{\mu - \lambda} - \dfrac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}}$

$$= -3 - \frac{16 \times \left(\dfrac{3}{2}\right)^{16}}{1 - \left(\dfrac{3}{2}\right)^{16}} = 13 \text{ patients nearly}$$

$$E(W) = \frac{E(N)}{\lambda'} = \frac{13}{19.98} = 0.65 \text{ h or } 39 \text{ min}$$

Example 17 *A 2-person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering barber shop. Customers arrive at the average rate of 4 per hour and spend an average of 12 min in the barber's chair. Compute P_0 P_1, P_7, $E(N_q)$ and $E(W)$.*

Solution The situation in this problem is one of finite capacity, multiserver Poisson queue models.

$$\lambda = 4 \text{ per hour}, \mu = 5 \text{ per hour}, s = 2, k = 2 + 5 = 7$$

(a) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu s}\right)^{n-s}\right]^{-1}$$

[by formula (3) of model IV]

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{4}{5}\right)^n + \frac{1}{2}\cdot\left(\frac{4}{5}\right)^2 \sum_{n=2}^{7}\left(\frac{2}{5}\right)^{n-2}\right]^{-1}$$

$$= \left[1 + \frac{4}{5} + \frac{8}{25}\left\{1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5\right\}\right]^{-1}$$

$$= \left[\frac{9}{5} + \frac{8}{25}\left\{\frac{1-(0.4)^7}{1-0.4}\right\}\right]^{-1} = 0.4287$$

(b) $P_n = \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0$, for $n \le s$ [by formula (4) of model IV]

$$\therefore \qquad P_1 = \left(\frac{4}{5}\right) \times 0.4287 = 0.3430$$

(c) $P_n = \dfrac{1}{s!.s^{n-s}}\left(\dfrac{\lambda}{\mu}\right)^n \cdot P_0$, for $s < n \le k$ [by formula (4) of model IV]

$$\therefore \qquad P_7 = \frac{1}{2 \times 2^{7-2}} \times \left(\frac{4}{5}\right)^7 \times 0.4287$$

$$= 0.0014$$

(d) $E(N_q) = P_0\left(\dfrac{\lambda}{\mu}\right)^s \cdot \dfrac{\rho}{s!(1-\rho)^2}\,[1 - p^{k-s} - (k-s)(1-\rho)\,\rho^{k-s}]$,

where $\rho = \dfrac{\lambda}{\mu s}$ [by formula (5) of model IV]

$$= (0.4287).(0.8)^{2}\cdot \frac{(0.4)}{2 \times (0.6)^2}\,[1 - (0.4)^5 - 5 \times 0.6 \times (0.4)^5]$$

$$= 0.15 \text{ customer}$$

(e) $E(N) = E(N_q) + s - \displaystyle\sum_{n=0}^{s-1} (s-n)\,P_n$ [by formula (6) of model IV]

$$= 0.1462 + 2 - \sum_{n=0}^{1} (2-n)P_n$$

$$= 2.1462 - (2 \times P_0 + 1 \times P_1)$$
$$= 2.1462 - (2 \times 0.4287 + 1 \times 0.3430)$$
$$= 0.95 \text{ customer}$$

$$E(W) = \frac{1}{\lambda'} \cdot E(N) \text{ [by formula (9) of model IV]}$$

where $$\lambda' = \mu \left[s - \sum_{n=0}^{s-1} (s-n) P_n \right] \text{ [by formula (7) of model IV]}$$
$$= 4[2 - (2 \times 0.4287 + 1 \times 0.3430)]$$
$$= 3.1984$$

$$\therefore \quad E(W) = \frac{0{\cdot}9458}{3{\cdot}1984} = 0.2957 \text{ h or } 17.7 \text{ min}$$

Example 18 *At a port there ar 6 unloading berths and 4 unloading crews. When all the berths are full, arriving ships are diverted to an overflow facility 20 kms down the river. Tankers arrive according to a Poisson process with a mean of 1 every 2 h. It takes for an unloading crew, on the average, 10 h to unload a tanker, the unloading time following an exponential distribution. Find*

(a) how many tankers are at the port on the average?
(b) how long does a tanker spend at the port on the average?
(c) what is the average arrival rate at the overflow facility?

Solution $\lambda = \frac{1}{2}$ per hour, $\mu = \frac{1}{10}$ per hour, $s = 4$, $k = 6$

(a) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s} \right]^{-1}$$

[by formula (3) of model IV]

$$= \left[\left(1 + 5 + \frac{1}{2} \times 5^2 + \frac{1}{6} \times 5^3\right) + \frac{1}{24} \times 5^4 \times \left\{ \left(\frac{5}{4}\right)^0 + \left(\frac{5}{4}\right)^1 + \left(\frac{5}{4}\right)^2 \right\} \right]^{-1}$$
$$= 0.0072$$

$$E(N_q) = P_0 \cdot \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\rho}{s!(1-\rho)^2} [1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}],$$

where $$\rho = \frac{\lambda}{\mu s} \quad \text{[by formula (5) of model IV]}$$

$$= 0.0072 \times 5^4 \times \frac{1.25}{24 \times (.25)^2} [1 - (1.25)^2 - 2 \times (-.25)(1.25)^2]$$
$$= 0.8203 \text{ tanker}$$

$$E(N) = E(N_q) + s - \sum_{n=0}^{s-1} (s-n)\, P_n$$

[by formula (6) of model IV]

$$= 4.8203 - (4P_0 + 3P_1 + 2P_2 + P_3)$$
$$= 4.8203 - \{4 \times .0072 + 3 \times 0.0360 + 2 \times 0.09 + 0.15\}$$
$$= 4.3535 \text{ tankers}$$

(b) $E(W) = \dfrac{1}{\lambda'} E(N)$ [by formula (9) of model IV]

where $\lambda' = \mu \left[s - \sum_{n=0}^{s-1} (s-n)\, P_n \right]$ [by formula (7) of model IV]

$$= \frac{1}{10} [4 - \{4P_0 + 3P_1 + 2P_2 + P_3\}]$$
$$= \frac{1}{10} [4 - 0.4668] = 0.3533$$

$\therefore$ $E(W) = \dfrac{4.3535}{0.3533} = 12.32$ h

(c) When $N = 6$, i.e., when the number of tankers in the port is 6, overflow occurs.

$$P(N = 6) = \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu} \right)^n P_0, \text{ for } n = k$$

[by formula (4) of model IV]

$$= \frac{1}{24 \times 4^2} \times 5^6 \times 0.0072$$
$$= 0.2930$$

Average arrival rate at the overflow facility = (Average arrival rate at the port) $\times$ (Probability that overflow occurs)

$$= \frac{1}{2} \times 0.2930 = 0.586 \text{ per hour}$$

Example 19 *A car servicing station has 2 bays where service can be offered simultaneously. Because of space limitation, only 4 cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time in both the bays is exponentially distributed with $\mu = 8$ cars per day per bay. Find the average number of cars in the service station, the average number of cars waiting for service and the average time a car spends in the system.*

Solution $\lambda = 12$ *per day,* $\mu = 8$ *per day,* $s = 2$, $k = 4$

(a) $P_0 = \left[\sum_{n=0}^{s-1} \dfrac{1}{n!} \left(\dfrac{\lambda}{\mu} \right)^n + \dfrac{1}{s!} \left(\dfrac{\lambda}{\mu} \right)^s \sum_{n=s}^{k} \left(\dfrac{\lambda}{\mu s} \right)^{n-s} \right]^{-1}$

[by formula (3) of model IV]

$$= \left[1 + \frac{1.5}{1} + \frac{1}{2} \times (1.5)^2 \left\{1 + (.75) + (.75)^2\right\}\right]^{-1}$$
$$= 0.1960$$

$$E(N_q) = P_0 \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\rho}{s!(1-\rho)^2} \left[1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}\right],$$

where $\rho = \dfrac{\lambda}{\mu s}$ [by formula (5) of model IV]

i.e., $$E(N_q) = 0.1960 \times (1.5)^2 \times \frac{0 \cdot 75}{2 \times (0 \cdot 25)^2} \times \left[1 - (0.75)^2 - 2 \times 0.25 \times (0.75)^2\right]$$
$$= 0.4134 \text{ car}$$

(b) $E(N)$ = Average number of cars in the service station

$$= E(N_q) + s - \sum_{n=0}^{s-1} (s-n) P_n$$

[by formula (6) of model IV]

$$= 0.4134 + 2 - \sum_{n=0}^{1} (2-n) P_n$$
$$= 2.4134 - (2P_0 + P_1)$$
$$= 2.4134 - (2 \times 0.1960 + 1.5 \times 0.1960)$$
$$= 1.73 \text{ cars}$$

(c) $E(W) = \dfrac{1}{\lambda'} E(N)$ [by formula (9) of model IV]

where $$\lambda' = \mu \left[s - \sum_{n=0}^{s-1} (s-n) P_n\right]$$ [by formula (7) of model IV]
$$= 8[2 - (2P_0 + P_1)]$$
$$= 10.512$$

$\therefore$ $$E(W) = \frac{1.73}{10.512} = 0.1646 \text{ day}$$

Example 20 *A group of engineers has 2 terminals available to aid in their calculations. The average computing job requires 20 min of terminal time and each engineer requires some computation about once every half an hour. Assume that these are distributed according to an exponential distribution. If there are 6 engineers in the group, find*

(a) the expected number of engineers waiting to use one of the terminals and in the computing centre and

(*b*) *the total time lost per day.*

$\lambda = 2$ *per hour,* $\mu = 3$ *per hour,* $s = 2$, $k = 6$

Solution

(a) $$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu s}\right)^{n-s}\right]^{-1}$$

[by formula (3) of model IV]

$$= \left[1 + \frac{2}{3} + \frac{1}{2}\times\left(\frac{2}{3}\right)^2\left\{1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4\right\}\right]^{-1}$$

$= 0.5003$

$$E(N_q) = P_0 \times \left(\frac{\lambda}{\mu}\right)^s \times \frac{\rho}{s!(1-\rho)^2}\,[1 - \rho^{k-s} - (k-s)(1-\rho)\,\rho^{k-s}],$$

where $\rho = \dfrac{\lambda}{\mu s}$ [by formula (5) of model IV]

i.e., $$E(N_q) = 0.5003 \times \left(\frac{2}{3}\right)^2 \times \frac{\left(\frac{1}{3}\right)}{2\times\left(\frac{2}{3}\right)^2}\left[1 - \left(\frac{1}{3}\right)^4 - 4\times\frac{2}{3}\times\left(\frac{1}{3}\right)^4\right]$$

$= 0.0796$

$$E(N) = E(N_q) + s - \sum_{n=0}^{s-1}(s-n)P_n \text{ [by formula (6) of model IV]}$$

$$= 0.0796 + 2 - \sum_{n=0}^{1}(2-n)P_n$$

$= 2.0796 - (2P_0 + P_1)$

$$= 2.0796 - \left(2\times 0.5003 + \frac{2}{3}\times 0.5003\right)$$

$= 0.75$

(b) $E(W_q) = \dfrac{1}{\lambda'}\,E(N_q)$ [by formula (10) of model IV]

where $$\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)\,P_n\right] \text{ [by formula (7) of model IV]}$$

$$= 3\left[2 - \sum_{n=0}^{1}(2-n)\,P_n\right]$$

$= 3\,[2 - (2P_0 + P_1)]$

$$= 3\left[2 - \left(2 \times 0.5003 + \frac{2}{3} \times 0.5003\right)\right]$$

$$= 1.9976$$

$$\therefore \quad E(W_q) = \frac{0.0796}{1.9976} = 0.0398 \text{ h}$$

Every time an engineer approaches the computer centre, he has to lose 0.0398 h by way of waiting.

If the day consists of 8 working hours, he has to approach the centre 16 times.

$\therefore$ Time lost in waiting in a day per engineer

$$= 16 \times 0.0398 = 0.6368 \text{ h}$$

$\therefore$ Total time lost in waiting in a day by all the 6 engineers

$$= 6 \times 0.6368 = 3.82 \text{ h.}$$

Exercise 9

Part A (Short answer questions)

1. What are the characteristics of a queueing system?
2. What do the letters in the symbolic representation $(a/b/c)$: (d/e) of a queueing model represent?
3. What do you mean by transient state and steady-state queueing systems?
4. Write down the difference equations that give the probability that there are n customers ($n \geq 0$) in a Poisson queueing system in steady-state.
5. Write down the formulas for P_0 and P_n in a Poisson queue system in the steady-state.
6. Give the formulas for the average number of customers (i) in the system, (ii) in the queue and (iii) in the non-empty queues for the $(M/M/1)$: $(\infty/FIFO)$ model.
7. Obtain the variance of queue length for the $(M/M/1)$: $(\infty/FIFO)$ model.
8. In the usual notation of a $(M/M/1)$: $(\infty/FIFO)$ queue system, find $P(N > 2)$, if $\lambda = 12$ per hour and $\mu = 30$ per hour.
9. In the usual notation of a $(M/M/1)$: $(\infty/FIFO)$ queue system if $\lambda = 12$ per hour and $\mu = 24$ per hour, find the average number of customers in the system and in the queue.
10. Give the formulas for the waiting time of a customer in the queue and in the system for the $(M/M/1)$: $(\infty/FIFO)$ model.
11. If a customer has to wait in a $(M/M/1)$: $(\infty/FIFO)$ queue system, what is his average waiting time in the queue, if $\lambda = 8$ per hour and $\mu = 12$ per hour?
12. What is the probability that a customer has to wait more than 15 min to get his service completed in a $(M/M/1)$: $(\infty/FIFO)$ queue system, if $\lambda = 6$ per hour and $\mu = 10$ per hour?
13. Write down the probability density function of the waiting time of a customer in the $(M/M/1)$: $(\infty/FIFO)$ queue system.

14. Write down the Little's formulas that hold good for all the Poisson queue models.
15. Write down the Little's formulas that hold good for the infinite capacity Poisson queue models.
16. What is the probability that there are no customers in the $(M/M/s)$: $(\infty/FIFO)$ queueing system?
17. Write down the formula for P_n in terms of P_0 for the $(M/M/s)$: $(\infty/FIFO)$ queueing system.
18. Give the formulas for the average number of customers in the system and in the queue for the $(M/M/s)$: $(\infty/FIFO)$ queueing model.
19. If there are 2 servers in an infinite capacity Poisson queue system with $\lambda = 10$ per hour and $\mu = 15$ per hour, what is the percentage of idle time for each server?
20. In a 3 server infinite capacity Poisson queue model if $\lambda/s\mu = \dfrac{2}{3}$, find P_0.
21. In a 3 server infinite capacity Poisson queue model if $\lambda/s\mu = 2/3$ and P_0 1/9 find the average number of customers in the queue and in the system.
22. If $\lambda/s\mu = \dfrac{2}{3}$ in a $(M/M/s)$: $(\infty/FIFO)$ queue system find the average number of customers in the nonempty queue.
23. What is the probability that an arrival to an infinite capacity 3 server Poisson queue system with $\lambda/s\mu = 2/3$ and $P_0 = 1/9$ will have to wait?
24. What is the probability that an arrival to an infinite capacity 3 server Poisson queue system with $\lambda/s\mu = 2/3$ and $P_0 = 1/9$ enters the service without waiting?
25. Give the formulas for the average waiting time of a customer in the system and in the queue for the $(M/M/s)$:$(\infty/FIFO)$ queueing model.
26. What is the average waiting time of a customer in the 3 server infinite capacity Poisson queue if he happens to wait, given that $\lambda = 6$ per hour and $\mu = 4$ per hour.
27. Give the probability that there is no customer in an $(M/M/1)$:$(k/FIFO)$ queueing system.
28. Write down the probability that there are n customers in an $(M/M/1)$: $(k/FIFO)$ queueing system.
29. If $\lambda = 4$ per hour and $\mu = 12$ per hour in an $(M/M/1)$:$(4/FIFO)$ queueing system, find the probability that there is no customer in the system. If $\lambda = \mu$, what is the value of this probability?
30. Write the formulas for the average number of customers in the $(M/M/1)$: $(k/FIFO)$ queueing system and also in the queue.
31. Define effective arrival rate with respect to an $(M/M/1)$:$(k/FIFO)$ queueing model.
32. How are N_s and N_q related in an $(M/M/1)$:$(k/FIFO)$ queueing model?

33. Write down the Little's formulas for the average waiting time in the system and in the queue for an $(M/M/1)$:$(k/FIFO)$ queueing model.
34. If $\lambda = 3$ per hour, $\mu = 4$ per hour and maximum capacity $k = 7$ in a $(M/M/1)$:$(k/FIFO)$ system, find the average number of customers in the system.
35. Write the formula for the probability that there is no customer in an $(M/M/s)$:$(k/FIFO)$ queue system.
36. Write the formula for the probability that there are n customers in an $(M/M/s)$:$(k/FIFO)$ queueing system.
37. Write down the formula for the average queue length in an $(M/M/s)$:$(k/FIFO)$ queueing model.
38. Define effective arrival rate with respect to an $(M/M/s)$:$(k/FIFO)$ queueing model.
39. How are N_s and N_q related in an $(M/M/s)$:$(k/FIFO)$ queueing model?
40. Write down Little's formulas for the average waiting time in the system and in the queue for an $(M/M/s)$:$(k/FIFO)$ queueing model.

Part B

41. Arrivals at a telephone booth are considered to be Poisson with an average time of 10 min between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 min.
    (a) Find the average number of persons waiting in the system.
    (b) What is the probability that a person arriving at the booth will have to wait in the queue?
    (c) What is the probability that it will take him more than 10 min altogether to wait for phone and complete his call?
    (d) Estimate the fraction of the day when the phone will be in use.
    (e) The telephone department will install a second booth when convinced that an arrival has to wait on the average for at least 3 min for phone. By how much the flow of arrivals should increase in order to justify a second booth?
42. Customers arrive at a one man barber shop according to a Poisson process with a mean interarrival time of 20 min. Customers spend an average of 15 min in the barber's chair.
    (a) What is the expected number of customers in the barber shop in the queue?
    (b) What is the probability that a customer will not have to wait for a hair cut?
    (c) How much can a customer expect to spend in the barber shop?
    (d) Management will put another chair and hire another barber when a customer's average waiting time in the shop exceeds 1.25 h. How much must the average rate of arrivals increase to warrant a second barber?

(e) What is average time customers spend in the queue?

(f) What is the probability that the waiting time in the system is greater than 30 min?

(g) What is the probability that there are more than 3 customers in the system?

43. If customers arrive for service according to a Poisson distribution at the average rate of 5 per day, how fast must they be serviced on the average (assume exponential service time) in order to keep the average number of customers in the system less than 4?

44. Patients arrive at an hospital for emergency service at the rate of one every hour. Currently only one emergency can be handled at a time. Patients spend an average of 20 min for receiving emergency service. How much the average service time need to be decreased to keep the average time to wait and receive the service less then 25 min?

45. A departmental secretary receives an average of 8 jobs per hour. Many are short jobs, while others are quite long. Assume, however, that the time to perform a job has an exponential distribution with a mean of 6 min.

(a) What is the average elapsed time from the time the secretary receives a job until it is completed?

(b) Calculate $E(N)$, $E(W_q)$, $P(W > 2h)$, $P(N > 5)$ and the fraction of time the secretary is busy.

46. A service station expects a customer every 4 min on the average. Service takes, on the average, 3 min. Assume Poisson input and exponential service.

(a) What is the average number of customers waiting for service?

(b) How long can a customer expect to wait for service?

(c) What is the probability that a customer will spend less than 15 min waiting for and getting service?

(d) What is the probability that a customer will spend longer than 10 min waiting for and getting service?

47. A dress shop has 3 sales persons. Assume that arrivals follow Poisson pattern with an average of 10 min between arrivals. Also assume that any salesperson can provide the desired service for any customer. If the time to provide service for a customer is exponentially distributed with a mean of 20 min per customer, calculate $E(N)$, $E(N_q)$ $E(W)$, $E(W_q)$ and P_n for $n = 0$, 1 and 2.

48. If the mean arrival rate is 24 per hour, find from the customer's point of view of the time spent in the system, whether 3 channels in parallel with a mean service rate of 10 per hour is better or worse than a single channel with mean service rate of 30 per hour.

49. Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose a counter at random. If the arrivals at the frontier is Poisson at the rate of λ and the

service time is exponential with parameter $\dfrac{\lambda}{2}$, what is the steady-state average queue at each counter?

50. An insurance company has 3 claim registers in its branch office. People with claims against the company are found to arrive in a Poisson fashion at an average rate of 20/8-h day. The amount of time that an adjuster spends with a claimant is found to have exponential distribution with mean service time 40 min. Claimants are processed in the order of their appearance.
    (a) How many hours a week can an adjuster expect to spend with claimants?
    (b) How much time, on the average, does a claimant spend in the branch office?
51. A telephone exchange has 2 long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 min.
    (a) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?
    (b) If the subscribers will wait and are serviced in turn, what is the expected waiting time?
52. A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.
    (a) What is the probability that an arrival would have to wait in line?
    (b) Find the average waiting time in the queue, average time spent in the system and the average number of cars in the system.
    (c) For what percentage of time would a pump be idle on an average?
53. A one-person barber shop has 6 chairs to accommodate people waiting for a hair cut. Assume that castomers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the average rate of 3 per hour and spend an average of 15 min in the barber's chair.
    (a) What is the probability that a customer can get directly into the barber's chair upon arrival?
    (b) What is the expected number of customers waiting for a hair cut?
    (c) How much time can a customer expect to spend in the barber shop?
    (d) What fraction of potential customers are turned away?
54. Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 min. If the yard can admit 9 trains at a time, calculate the probability that the yard is empty and the average queue length.

55. A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean of 2 min. How many cars are in the car park on an average and what is the probability of a newly arriving customer finding the car park full and leaving to park his car elsewhere.
56. A stenographer is attached to 5 officers for whom she performs stenographic work. She gets calls from the officers at the rate of 4 per hour and takes on the average 10 min to attend to each call. If arrival rate is Poisson and service time is exponential find (a) the average waiting time for an arriving call (b) the average number of waiting calls and (c) the average time an arriving call spends in the system.
57. A 2-person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all the 5 chairs are full, leave without entering the barber shop. Customers arrive at the average rate of 3.7634 per hour and spend an average of 15 min in the barber's chair. Compute P_0, P_1, P_7, $E(N_q)$ and $E(W)$.
58. A barber shop has 2 barbers and 3 chairs for waiting customers. Assume that customers arrive in Poisson fashion at a rate of 5 per hour and that each barber services customers according to exponential distribution with mean of 15 min. Further, if a customer arrives and there are no empty chairs in the shop, he will leave. Find the steady-state probabilities. What is the probability that the shop is empty? What is the expected number of customers in the shop?
59. An automobile inspection station has 3 inspection stalls. Assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most 4 cars waiting (7 in the station) at one time. The arrival pattern is Poisson with a mean of 1 car every minute during the peak hours. The service time is exponential with mean 6 min. Find the average number of customers in the system during peak hours, the average waiting time and the average number of cars per hour that cannot enter the house because of full capacity.
60. A mechanic repairs 4 machines. The mean time between service requirements is 5 h for each machine and forms an exponential distribution. The mean repair time is 1 h and also follows the same distribution pattern. Machine downtime costs Rs. 25 per hour and the machanic costs Rs 55 per day.
    (a) Find the expected number of machines under service and in waiting.
    (b) Determine the expected downtime cost per 8-h day?
    (c) Would it be economical to engage 2 mechanics, each repairing only 2 machines?

Answers

Exercise 9

11. 5 min	12. 0.3679	19. 50%	20. $\frac{1}{9}$	21. $\frac{8}{9}, \frac{26}{9}$
22. 2	23. $\frac{4}{9}$	24. $\frac{5}{9}$	26. 10 min	29. $\frac{81}{121}, \frac{1}{5}$

34. 2.11
41. (a) 0.43 (b) 0.3 (c) 0.097 (d) 0.3
    (e) 0.16 person per minute
42. (a) 3; 2.25 (b) 0.25 (c) 1 h (d) 3.2 per hour
    (e) $\frac{3}{4}$ h (f) 0.61 (g) 0.32
43. 3.84 h/service
44. 17.65 min per patient
45. (a) 30 min (b) 4; 24 min; 0.0183; 0.2621; 0.8
46. (a) 2.25 (b) 9 min (c) 0.7135 (d) 0.8465
47. 2.1739; 0.1739; 21.739 min; 1.739 min; $P_0 = 0.1304$, $P_1 = 0.2608$; $P_2 = 0.2602$
48. Single channel is better
49. $\frac{4}{23}$
50. 22.2 h; 49 min
51. (a) 0.48 (b) 3.2 min
52. (a) 0.3826 (b) 3.05 min; 9.054 min; 4.53 cars; (c) 24.98%
53. (a) 0.2778 (b) 1.3878 (c) 43.8 min (d) 3.7%
54. 0.28; 1.55
55. 0.49, 0.0027
56. (a) 12.45 min (b) 0.79 customer (c) 22.42 min
57. 0.36133; 0.33996; 0.00368; 0.2457; 19 min
58. $P_n = (0.56)\,(0.625)^n$, for $2 \le n \le 5$; $P_1 = 0.35$; $P_0 = 0.28$; 2.956
59. 6.06 cars; 12.3 min; 30.4 cars
60. (a) 1 (b) Total cost = Rs 255
    (c) Total cost with 2 machines = Rs. 270; Use of 2 machines is not economical.

10 Design of Experiments

By 'experiment', we mean collection of data (which usually consist of a series of measurements of some feature of an object) for a scientific investigation, according to certain specified sampling procedures. Statistics provides not only the principles and the basis for the proper planning of the experiments but also the methods for proper interpretation of the results of the experiment.

In the beginning, the study of the design of experiments was associated only with agricultural experimentation. The need to save time and money has led to a study of ways to obtain maximum information with the minimum cost and labour. Such motivations resulted in the subsequent acceptance and wide use of the design of experiments and the related analysis of variance techniques in all fields of scientific experimentation. In this chapter we consider some aspects of experimental design briefly and analysis of data from such experiments using analysis of variance techniques.

Aim of the Design of Experiments

A statistical experiment in any field is performed to verify a particular hypothesis. For example, an agricultural experiment may be performed to verify the claim that a particular manure has got the effect of increasing the yield of paddy. Here the quantity of the manure used and the amount of yield are the two variables involved directly. They are called *experimental variables*. Apart from these two, there are other variables such as the fertility of the soil, the quality of the seed used and the amount of rainfall, which also affect the yield of paddy. Such variables are called *extraneous variables*. The main aim of the design of experiments is to control the extraneous variables and hence to minimise the experimental error so that the results of the experiments could be attributed only to the experimental variables.

Basic Principles of Experimental Design

In order to achieve the objective mentioned above, the following three principles are adopted while designing the experiments—(1) randomisation, (2) replication and (3) local control.

1. Randomisation As it is not possible to eliminate completely the contribution of extraneous variables to the value of the response variable (the amount of yield of paddy), we try to control it by randomisation. The group of experimental units (plots of the same size) in which the manure is used is called the *experimental group* and the other group of plots in which the manure is not used and which will provide a basis for comparison is called the *control group*. If any information regarding the extraneous variables and the nature and magnitude of their effect on the response variable in question is not available, we resort to randomisation. That is, we select the plots for the experimental and control groups in a random manner, which provides the most effective way of eliminating any unknown bias in the experiment.

2. Replication In a comparative experiment, in which the effects of different manures on the yield are studied, each manure is used in more than one plot. In other words, we resort to replication which means repetition. It is essential to carry out more than one test on each manure in order to estimate the amount of the experimental error and hence to get some idea of the precision of the estimates of the manure effects.

3. Local Control To provide adequate control of extraneous variables, another essential principle used in the experimental design is the local control. This includes techniques such as grouping, blocking and balancing of the experimental units used in the experimental design. By *grouping*, we mean combining sets of homogeneous plots into groups, so that different manures may be used in different groups. The number of plots in different groups need not necessarily be the same. By *blocking*, we mean assigning the same number of plots in different blocks. The plots in the same block may be assumed to be relatively homogeneous. We use as many manures as the number of plots in a block in a random manner. By *balancing*, we mean adjusting the procedures of grouping, blocking and assigning the manures in such a manner that a balanced configuration is obtained.

Some Basic Designs of Experiment

1. Completely Randomised Design (C.R.D.)

Let us suppose that we wish to compare 'h' treatments (use of 'h' different manures) and there are n plots available for the experiment.

Let the ith treatment be replicated (repeated) n_i times, so that $n_1 + n_2 + \dots + n_h = n$.

The plots to which the different treatments are to be given are found by the following randomisation principle. The plots are numbered from 1 to n serially. n identical cards are taken, numbered from 1 to n and shuffled thoroughly. The numbers on the first n_1 cards drawn randomly give the numbers of the plots to which the first treatment is to be given. The numbers on the next n_2 cards drawn at random give the numbers of the plots to which the second treatment is to be given and so on. This design is called a completely randomised design, which is used when the plots are homogeneous or the pattern of heterogeneity of the plots is unknown.

Analysis of Variance (ANOVA) The analysis of variance is a widely used technique developed by R.A. Fisher. It enables us to divide the total variation (represented by variance) in a group into parts which are ascribable to different factors and a residual random variation which could not be accounted for by any of these factors. The variation due to any specific factor is compared with the residual variation for significance by applying the F-test, with which the reader is familiar. The details of the procedure will be explained in the sequel.

Analysis of Variance for One Factor of Classification Let a sample of N values of a given random variable X (representing the yield of paddy) be subdivided into 'h' classes according to some factor of classification (different manures).

We wish to test the null hypothesis that the factor of classification has no effect on the variable, viz., there is no difference between various classes, viz., the classes are homogeneous. Let x_{ij} be the value of the j^{th} member of the i^{th} class, which contains n_i members. Let the general mean of all the N values be $\bar{x}$ and the mean of n_i values in the i^{th} class be $\bar{x}_i$.

$$\text{Now} \quad \sum_i \sum_j (x_{ij} - \bar{x})^2 = \sum_i \sum_j \left\{ (x_{ij} - \bar{x}_i) + (x_i - \bar{x}) \right\}^2$$

$$= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i \sum_j (\bar{x}_i - \bar{x})^2$$

$$+ 2 \sum_i \sum_j (x_{ij} - \bar{x}_i)\,(\bar{x}_i - \bar{x})$$

$$= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i \sum_{j=1}^{n_i} (\bar{x}_i - \bar{x})^2$$

$$+ 2 \sum_i (\bar{x}_i - \bar{x}) \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)$$

$$= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i n_i (\bar{x}_i - \bar{x})^2$$

$$\left[\because \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i) = \text{sum of the deviations of } n_i \text{ values of the } x_{ij} \text{ in the } i^{\text{th}} \text{ class from their mean } \bar{x}_i = 0 \right]$$

i.e., $\quad Q = Q_2 + Q_1$, say, where

$Q_1 = \sum_i n_i (\bar{x}_i - \bar{x})^2$ = sum of the squared deviations of class means from the general mean (variation between classes)

$Q_2 = \sum_i \sum_j (x_{ij} - \bar{x}_i)^2$ = sum of the squared deviations of variates from the corresponding class means (variation within classes) and Q = total variation.

Since $Q_2 = Q - Q_1$, viz., the variation Q_2 within classes is got after removing the variation Q_1 between classes from the total variation Q, Q_2 is the residual variation.

If s^2 is the variance of a sample of size n drawn from a population with variance σ^2, then it is known from the theory of estimation that $\left(\frac{ns^2}{n-1}\right)$ is an unbiased estimate of σ^2.

i.e., $$E\left(\frac{ns^2}{n-1}\right) = \sigma^2.$$

Since the items in the i^{th} class with variance $\frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2$ may be considered as a sample of size n_i drawn from a population with variance σ^2,

$$E\left\{\frac{n_i}{n_i - 1} \cdot \frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2\right\} = \sigma^2$$

i.e., $$E\left[\sum_j (x_{ij} - \bar{x}_i)^2\right] = (n_i - 1)\,\sigma^2$$

$\therefore$ $$E\left[\sum_i \sum_j (x_{ij} - \bar{x}_i)^2\right] = \sum_{i=1}^{h}(n_i - 1)\,\sigma^2$$

i.e., $$E(Q_2) = (N - h)\,\sigma^2 \text{ or } E\left\{\frac{(Q)_2}{N - h}\right\} = \sigma^2$$

i.e., $\frac{Q_2}{N-h}$ is an unbiased estimate of σ^2 with $(N - h)$ degrees of freedom.

Now if we consider the entire group of N items with variance $\frac{1}{N}\sum_i \sum_j (x_{ij} - \bar{x})^2$ as a sample of size N drawn from the same population,

$$E\left\{\frac{N}{N - 1} \cdot \frac{1}{N}\sum_i \sum_j (x_{ij} - \bar{x})^2\right\} = \sigma^2$$

i.e., $$E\left(\frac{Q}{N-1}\right) = \sigma^2$$

i.e., $\frac{Q}{N-1}$ is an unbiased estimate of σ^2 with $(N - 1)$ degrees of freedom.

Now $\quad Q_1 = Q - Q_2$

$\therefore \quad E(Q_1) = E(Q) - E(Q_2)$

$$= (N-1)\sigma^2 - (N-h)\sigma^2$$

$$= (h-1)\,\sigma^2 \text{ or } E\left(\frac{Q_1}{h-1}\right) = \sigma^2$$

i.e., $\dfrac{Q_1}{h-1}$ is also an unbiased estimate of σ^2 with $(h-1)$ degrees of freedom.

If we assume that the sampled population is normal, then the estimates $\dfrac{Q_1}{h-1}$ and $\dfrac{Q_2}{N-h}$ are independent and hence the ratio $\dfrac{Q_1/(h-1)}{Q_2/(N-h)}$ follows a F-distribution with $(h-1, N-h)$ degrees of freedom or the ratio $\dfrac{Q_2/(N-h)}{Q_1/(h-1)}$ follows a F-distribution with $(N-h, h-1)$ degrees of freedom. Choosing the ratio which is greater than one, we employ the F-test.

If the calculated value of $F < F_{5\%}$, the null hypothesis is accepted, viz., different treatments do not contribute significantly different yields.

These results are displayed in the form of a table, called the ANOVA table, as given below:

Table 10.1 *ANOVA table for one factor of classification*

Source of variation (S.V.)	*Sum of squares (S.S.)*	*Degrees of freedom (d.f.)*	*Mean square (M.S.)*	*Variance ratio (F)*
Between classes	Q_1	$h-1$	$Q_1/(h-1)$	$\dfrac{Q_1/(h-1)}{Q_2/(N-h)}$
				(OR)
Within classes	Q_2	$N-h$	$Q_2/(N-h)$	$\dfrac{Q_2/(N-h)}{Q_1/(h-1)}$
Total	Q	$N-1$	—	—

Note *For calculating Q, Q_1, Q_2, the following computational formulas may be used:*

$$Q = N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \bar{x}^2\right\}$$

$$= N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \left(\frac{1}{N}\sum\sum x_{ij}\right)^2\right\}$$

$$= \sum\sum x_{ij}^2 - \frac{T^2}{N}, \text{ where } T = \sum\sum x_{ij}$$

Similarly, for the i^{th} class,

$$\sum_j (x_{ij} - \bar{x}_i)^2 = \sum_j x_{ij}^2 - \frac{T_i^2}{n_i}, \text{ where } T_i = \sum_j x_{ij}.$$

$$\therefore \qquad Q_2 = \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 = \sum_i \sum_j x_{ij}^2 - \sum_i \frac{T_i^2}{n_i}$$

Hence

$$Q_1 = Q - Q_2$$

$$= \sum_i \frac{T_i^2}{n_i} - \frac{T^2}{N}$$

2. *Randomised Block Design (R.B.D.)*

Let us consider an agricultural experiment using which we wish to test the effect of 'k' fertilizing treatments on the yield of a crop. We assume that we know some information about the soil fertility of the plots. Then we divide the plots into 'h' blocks, according to the soil fertility, each block containing 'k' plots. Thus the plots in each block will be of homogeneous fertility as far as possible.

Within each block, the 'k' treatments are given to the 'k' plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same 'k' treatments are repeated from block to block. This design is called Randomised Block Design.

Analysis of Variance for two Factors of Classification Let the N variate values $\{x_{ij}\}$ (representing the yield of paddy) be classified according to two factors. Let there be 'h' rows (blocks) representing one factor of classification (soil fertility) and 'k' columns representing the other factor (treatment), so that $N = hk$.

We wish to test the null hypothesis that the rows and columns are homogeneous viz., there is no difference in the yields of paddy between the various rows and between the various columns.

Let x_{ij} be the variate value in the i^{th} row and j^{th} column.

Let $\bar{x}$ be the general mean of all the N values, $\bar{x}_{i*}$ be the mean of the k values in the i^{th} row and $\bar{x}_{*j}$ be the mean of the h values in the j^{th} column.

Now $$x_{ij} - \bar{x} = (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x}) + (\bar{x}_{i*} - \bar{x}) + (\bar{x}_{*j} - \bar{x})$$

$$\begin{aligned} \therefore \sum\sum (x_{ij} - \bar{x})^2 &= \sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})^2 + \sum\sum (\bar{x}_{i*} - \bar{x})^2 \\ &\quad + \sum\sum (\bar{x}_{*j} - \bar{x})^2 + 2\sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})(\bar{x}_{i*} - \bar{x}) \\ &\quad + 2\sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})(\bar{x}_{*j} - \bar{x}) \\ &\quad + 2\sum\sum (\bar{x}_{i*} - \bar{x})(\bar{x}_{*j} - \bar{x}) \end{aligned} \tag{1}$$

Now, the fourth member in the R.H.S. of (1)

$$= 2\sum_i (\bar{x}_{i*} - \bar{x}) \sum_{j=1}^{k} (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})$$

$$= 2\sum_i (\bar{x}_{i*} - \bar{x})(k\,\bar{x}_{i*} - k\,\bar{x}_{i*} - k\,\bar{x} + k\,\bar{x})$$
$$= 0$$

Similarly, the last two members in the R.H.S. of (1) also become zero each.

Also $$\sum_i \sum_j (\bar{x}_{i*} - \bar{x})^2 = k\sum_i (\bar{x}_{i*} - \bar{x})^2 = Q_1, \text{ say}$$
$$\sum_i \sum_j (\bar{x}_{*j} - \bar{x})^2 = h\sum_j (\bar{x}_{*j} - \bar{x})^2 = Q_2, \text{ say}$$

Let $$Q = \sum\sum (x_{ij} - \bar{x})^2 \text{ and}$$
$$Q_3 = \sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})^2$$

Using all these in (1), we get

$Q = Q_1 + Q_2 + Q_3$, where

Q = total variation.

Q_1 = sum of the squares due to the variations in the rows,

Q_2 = that in the columns and

Q_3 = that due to the residual variations.

Proceeding as in one factor of classification, we can prove that $\dfrac{Q_1}{h-1}, \dfrac{Q_2}{k-1}$, $\dfrac{Q_3}{(h-1)(k-1)}$ and $\dfrac{Q}{hk-1}$ are unbiased estimates of the population variance σ^2 with degrees of freedom $h - 1$, $k - 1$, $(h - 1)(k - 1)$ and $(hk - 1)$ respectively. If the sampled population is assumed normal, all these estimates are independent.

$\therefore \dfrac{Q_1/(h-1)}{Q_3'/(h-1)(k-1)}$ follows a F-distribution with $\{h - 1, (h - 1)(k - 1)\}$ degrees of freedom and $\dfrac{Q_2/(k-1)}{Q_3/(h-1)(k-1)}$ follows a F-distribution with $\{k - 1, (h - 1)(k - 1)\}$ degrees of freedom. Then the F-tests are applied as usual and the significance of difference between rows and between columns is analysed.

Table 10.2 *The ANOVA table for the two factors of classifications*

S.V.	*S.S.*	*d.f.*	*M.S.*	*F*
Between rows	Q_1	$h-1$	$Q_1/(h-1)$	$\left[\dfrac{Q_1/(h-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Between columns	Q_2	$k-1$	$Q_2/(k-1)$	$\left[\dfrac{Q_2/(k-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Residual	Q_3	$(h-1)(k-1)$	$Q_3/(h-1)(k-1)$	—
Total	Q	$hk-1$	—	—

Note *The following working formulas that can be easily derived may be used to compute Q, Q_1, Q_2 and Q_3:*

1. $Q = \sum\sum x_{ij}^2 - \frac{T^2}{N}$, *where* $T = \sum\sum x_{ij}$
2. $Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N}$, *where* $T_i = \sum_{j=1}^{k} x_{ij}$
3. $Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N}$, *where* $T_j = \sum_{i=1}^{h} x_{ij}$
4. $Q_3 = Q - Q_1 - Q_2$

It may be verified that $\sum_i T_i = \sum_j T_j = T$.

3. Latin Square Design (L.S.D.)

We consider an agricultural experiment, in which n^2 plots are taken and arranged in the form of an $n \times n$ square, such that the plots in each row will be homogeneous as far as possible with respect to one factor of classification, say, soil fertility and plots in each column will be homogeneous as far as possible with respect to another factor of classification, say, seed quality.

Then n treatments are given to these plots such that each treatment occurs only once in each row and only once in each column. The various possible arrangements obtained in this manner are known as Latin squares of order n. This design of experiment is called the Latin Square Design.

Analysis of Variance for Three Factors of Classifications Let the $N (= n^2)$ variate values $\{x_{ij}\}$, representing the yield of paddy, be classified according to three factors. Let the rows, columns and letters stand for the three factors, say soil fertility, seed quality and treatment respectively.

We wish to test the null hypothesis that the rows, columns and letters are homogeneous, viz., there is no difference in the yield of paddy between the rows (due to soil fertility), between the columns (due to seed quality) and between the letters (due to the treatments).

Let x_{ij} be the variate value corresponding to the i^{th} row, j^{th} column and k^{th} letter.

Let $\bar{x} = \frac{1}{n^2}\sum\sum x_{ij}$, $\bar{x}_{i*} = \frac{1}{n}\sum_j x_{ij}$, $\bar{x}_{*j} = \frac{1}{n}\sum_i x_{ij}$ and $\bar{x}_k$ be the mean of the values of x_{ij} corresponding to the k^{th} treatment.

Now $x_{ij} - \bar{x} = (\bar{x}_{i*} - \bar{x}) + (\bar{x}_{*j} - \bar{x}) + (\bar{x}_k - \bar{x}) + (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} - \bar{x}_k + 2\bar{x})$

$$\therefore \quad \sum\sum (x_{ij} - \bar{x})^2 = n\sum_i (\bar{x}_{i*} - \bar{x})^2 + n\sum_j (\bar{x}_{*j} - \bar{x})^2 + n\sum_k (\bar{x}_k - \bar{x})^2 + \sum_i\sum_j (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} - \bar{x}_k + 2\bar{x})^2$$

($\because$ all the product terms vanish as in two factor classification)

i.e., $\qquad Q = Q_1 + Q_2 + Q_3 + Q_4$

As before we can prove that $\frac{Q_1}{n-1}$, $\frac{Q_2}{n-1}$, $\frac{Q_3}{n-1}$, $\frac{Q_4}{(n-1)(n-2)}$ and $\frac{Q}{n^2-1}$ are unbiased estimates of the population variance σ^2 with degrees of freedom $n-1$, $n-1$, $n-1$, $(n-1)(n-2)$ and (n^2-1) respectively.

If the sampled population is assumed normal, all these estimates are independent.

$\therefore$ Each of $\frac{Q_1/(n-1)}{Q_4/(n-1)(n-2)}$, $\frac{Q_2/(n-1)}{Q_4/(n-1)(n-2)}$ and $\frac{Q_3/(n-1)}{Q_4/(n-1)(n-2)}$ follows a F-distribution with $\{(n-1), (n-1)(n-2)\}$ degrees of freedom.

Then the F-tests are applied as usual and the significance of differences between rows, columns and treatments is analysed.

Table 10.3 *The ANOVA table for three factors of classification*

S.V.	*S.S.*	*d.f.*	*M.S.*	*F*
Between rows	Q_1	$n-1$	$Q_1/(n-1) = M_1$	$\left(\frac{M_1}{M_4}\right)^{\pm 1}$
Between columns	Q_2	$n-1$	$Q_2/(n-1) = M_2$	$\left(\frac{M_2}{M_4}\right)^{\pm 1}$
Between letters	Q_3	$n-1$	$Q_3/(n-1) = M_3$	$\left(\frac{M_3}{M_4}\right)^{\pm 1}$
Residual	Q_4	$(n-1)(n-2)$	$Q_4/(n-1)(n-2) = M_4$	—
Total	Q	n^2-1	—	—

Note *The following working formulas may be used to compute the Q's:*

1. $Q = \sum\sum x_{ij}^2 - \frac{T^2}{n^2}$, *where* $T = \sum\sum x_{ij}$
2. $Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{n^2}$, *where* $T_i = \sum_{j=1}^{n} x_{ij}$
3. $Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{n^2}$, *where* $T_j = \sum_{i=1}^{n} x_{ij}$
4. $Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{n^2}$, *where* T_k *is the sum of all* x_{ij}*'s receiving the* k^{th} *treatment.*
5. $Q_4 = Q - Q_1 - Q_2 - Q_3$.

Also $T = \sum_i T_i = \sum_j T_j = \sum_k T_k$

Comparison of RBD and LSD

1. The number of replications of each treatment is equal to the number of treatments in LSD, whereas there is no such restrictions on treatments and replication in RBD.
2. LSD can be performed on a square field, while RBD can be performed either on a square field or a rectangular field.
3. LSD is known to be suitable for the case when the number of treatments is between 5 and 12, whereas RBD can be used for any number of treatments.
4. The main advantage of LSD is that it controls the effect of two extraneous variables, whereas RBD controls the effect of only one extraneous variable. Hence the experimental error is reduced to a larger extent in LSD than in RBD.

Note on Simplification of Computational Work

The variance of a set of values is independent of the origin and so a shift of origin does not affect the variance calculations. Hence in analysis of variance problems, we can subtract a convenient number from the original values and work out the problems with the new values obtained. Also since we are concerned with the variance ratios, change of scale also may be introduced without affecting the values of F.

Worked Examples 10

Example 1 *A completely randomised design experiment with 10 plots and 3 treatments gave the following results:*

Plot No.	:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Treatment	:	*A*	*B*	*C*	*A*	*C*	*C*	*A*	*B*	*A*	*B*
Yield	:	*5*	*4*	*3*	*7*	*5*	*1*	*3*	*4*	*1*	*7*

Analyse the results for treatment effects.

Solution Rearranging the data according to the treatments, we have the following table:

Treatment	*Yield from plots* (x_{ij})				T_i	T_i^2	n_i	$\frac{T_i^2}{n_i}$
A	5	7	3	1	16	256	4	64
B	4	4	7	–	15	225	3	75
C	3	5	1	–	9	81	3	27
	Total				$T = 40$	–	$N = 10$	166

$$\sum\sum x_{ij}^2 = (25 + 49 + 9 + 1) + (16 + 16 + 49) + (9 + 25 + 1)$$
$$= 84 + 81 + 35 = 200$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 200 - \frac{40^2}{10} = 200 - 160 = 40$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 166 - 160 = 6$$

$\therefore$ $$Q_2 = Q - Q_1 = 40 - 6 = 34$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between classes (treatments)	$Q_1 = 6$	$h - 1 = 2$	3.0	$\frac{4.86}{3.0}$
Within classes	$Q_2 = 34$	$N - h = 7$	4.86	= 1.62
Total	$Q = 40$	$N - 1 = 9$	–	–

From the F-table, $F_{5\%}$ $(v_1 = v_2 = 2) = 19.35$

We note that $F_0 < F_{5\%}$

Let H_0 : The treatments do not differ significantly.

$\therefore$ The null hypothesis is accepted.

i.e., the treatments are not significantly different.

Example 2 *The following table shows the lives in hours of four brands of electric lamps:*

Brand

A : 1610, 1610, 1650, 1680, 1700, 1720, 1800

B : 1580, 1640, 1640, 1700, 1750

C : 1460, 1550, 1600, 1620, 1640, 1660, 1740, 1820

D : 1510, 1520, 1530, 1570, 1600, 1680

Perform an analysis of variance and test the homogeneity of the mean lives of the four brands of lamps.

Solution We subtract 1640 (= the average of the extreme values) from the given values and work out with the new values of x_{ij}

Brand	Lives of lamps (x_{ij})								T_i	n_i	$\frac{T_i^2}{n_i}$
A	– 30	– 30	10	40	60	80	160	–	290	7	12014
B	– 60	0	0	60	110	–	–	–	110	5	2420
C	– 180	– 90	– 40	– 20	0	20	100	180	– 30	8	113
D	– 130	– 120	– 110	– 70	– 40	40	–	–	– 430	6	30817
								Total	– 60	26	45364

$$\sum\sum x_{ij}^2 = (900 + 900 + 100 + 1600 + 3600 + 6400 + 25600)$$
$$+ (3600 + 0 + 0 + 3600 + 12100)$$
$$+ (32400 + 8100 + 1600 + 400 + 0 + 400 + 10000 + 32400)$$
$$+ (16900 + 14400 + 12100 + 4900 + 1600 + 1600)$$
$$= 39100 + 19300 + 85300 + 51500 = 195200$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 1{,}95{,}200 - 138 = 1{,}95{,}062$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 45{,}364 - 138 = 45{,}226$$

$$Q_2 = Q - Q_1 = 1{,}95{,}062 - 45{,}226 = 1{,}49{,}836$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between brands	$Q_1 = 45{,}226$	$h - 1 = 3$	15,075	$\frac{15{,}075}{6{,}811}$
Within brands	$Q_2 = 1{,}49{,}836$	$N - h = 22$	6,811	= 2.21
Total	$Q = 1{,}95{,}062$	$N - 1 = 25$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 3,\ v_2 = 22) = 3.06$

$$F_0 < F_{5\%}$$

Hence the null hypothesis H_0, namely, the means of the lives of the four brands are homogeneous, is accepted viz., the lives of the four brands of lamps do not differ significantly.

Note *We could have used a change of scale also. viz., we could have made the change*

$$\text{New } x_{ij} = \frac{\text{old } x_{ij} - 1640}{10}$$ *and simplified the numerical work still further.*

Example 3 *A car rental agency, which uses 5 different brands of tyres in the process of deciding the brand of tyre to purchase as standard equipment for its fleet, finds that each of 5 tyres of each brand last the following number of kilometres (in thousands):*

Tyre brands

A	*B*	*C*	*D*	*E*
36	46	35	45	41
37	39	42	36	39
42	35	37	39	37
38	37	43	35	35
47	43	38	32	38

Test the hypothesis that the five tyre brands have almost the same average life.

Solution We shift the origin to 40 and work out with the new values of x_{ij}.

Tyre brand	x_{ij}					T_i	n_i	$\frac{T_i^2}{n_i}$	$\sum_{j=1}^{5} x_{ij}^2$
A	– 4	– 3	2	– 2	7	0	5	0	82
B	6	– 1	– 5	– 3	3	0	5	0	80
C	– 5	2	– 3	3	– 2	– 5	5	5	51
D	5	– 4	– 1	– 5	– 8	– 13	5	33.8	131
E	1	– 1	– 3	– 5	– 2	– 10	5	20	40
	Total					– 28	25	58.8	384

$$T = \sum_i T_i = -28; \sum\sum x_{ij}^2 = \sum_i \left(\sum_j x_{ij}^2 \right) = 384$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 384 - \frac{(-28)^2}{25} = 352.64$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 58.8 - 31.36 = 27.44$$

$$Q_2 = Q - Q_1 = 352.64 - 27.44 = 325.20$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between tyre brands	$Q_1 = 27.44$	$h - 1 = 4$	6.86	$\frac{16.26}{6.86}$
Within tyre brands	$Q_2 = 325.20$	$N - h = 20$	16.26	= 2.37
Total	$Q = 352.64$	$N - 1 = 24$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 20, v_2 = 4) = 5.80$

$$F_0 < F_{5\%}$$

Hence 'H_0 : the five tyre brands have almost the same average life) is accepted viz., the five tyre brands do not differ significantly in their lives.

Example 4 *In order to determine whether there is significant difference in the durability of 3 makes of computers, samples of size 5 are selected from each make and the frequency of repair during the first year of purchase is observed. The results are as follows :*

Makes

A	*B*	*C*
5	*8*	*7*
6	*10*	*3*
8	*11*	*5*
9	*12*	*4*
7	*4*	*1*

In view of the above data, what conclusion can you draw?

Solution

Make	x_{ij}					T_i	n_i	$\frac{T_i^2}{n_i}$	$\sum_{j=1} x_{ij}^2$
A	5	6	8	9	7	35	5	245	255
B	8	10	11	12	4	45	5	405	445
C	7	3	5	4	1	20	5	80	100
					Total	100	15	730	800

$$T = \sum T_i = 100\ ;\ \sum\sum x_{ij}^2 = 800;\ N = \sum n_i = 15$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 800 - \frac{100^2}{15} = 133.33$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 730 - 666.67 = 63.33$$

$$Q_2 = Q - Q_1 = 70$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between makes	$Q_1 = 63.33$	$h - 1 = 2$	31.67	$\frac{31.67}{5.83}$
Within makes	$Q_2 = 70$	$N - h = 12$	5.83	5.43
Total	$Q = 133.33$	$N - 1 = 14$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 2, v_2 = 12) = 3.88$

$$F_0 > F_{5\%}$$

Hence the null hypothesis (H_0: the 3 makes of computers do not differ in the durability) is rejected.

viz., there is significant difference in the durability of the 3 makes of computers.

Example 5 *Three varieties of a crop are tested in a randomised block design with four replications, the layout being as given below: The yields are given in kilograms. Analyse for significance*

C48	*A51*	*B52*	*A49*
A47	*B49*	*C52*	*C51*
B49	*C53*	*A49*	*B50*

Solution Rewriting the data such that the rows represent the blocks and the columns represent the varieties of the crop (as assumed in the discussion of analysis of variance for two factors of classification), we have the following table:

Crops

Blocks	*A*	*B*	*C*
1	47	49	48
2	51	49	53
3	49	52	52
4	49	50	51

We shift the origin to 50 and work out with the new values of x_{ij}.

Crops

Blocks	*A*	*B*	*C*	T_i	$T^2{}_i / k$	$\sum_j x_{ij}^2$
1	– 3	– 1	– 2	– 6	36/3 = 12	14
2	1	– 1	3	3	9/3 = 3	11
3	– 1	2	2	3	9/3 = 3	9
4	– 1	0	1	0	0/3 = 0	2
T_j	– 4	0	4	$T = 0$	$\sum \frac{T_j^2}{k} = 18$	36
$T^2{}_j / h$	$\frac{16}{4} = 4$	$\frac{0}{4} = 0$	$\frac{16}{4} = 4$	$\sum \frac{T_i^2}{h} = 8$		
$\sum_i x_{ij}^2$	12	6	18	36		

$$Q = \sum \sum x_{ij}^2 - \frac{T^2}{N} = 36 - \frac{0^2}{12} = 36$$

$$Q_1 = \frac{1}{k} \sum T_i^2 - \frac{T^2}{N} = 18 - 0 = 18$$

$$Q_2 = \frac{1}{h} \sum T_j^2 - \frac{T^2}{N} = 8 - 0 = 8$$

$$Q_3 = Q - Q_1 - Q_2 = 36 - 18 - 8 = 10$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows (blocks)	$Q_1 = 18$	$h - 1 = 3$	6	$\frac{6}{1.67} = 3.6$
Between columns (crops)	$Q_2 = 8$	$k - 1 = 2$	4	$\frac{4}{1.67} = 2.4$
Residual	$Q_3 = 10$	$(h - 1)(k - 1) = 6$	1.67	–
Total	Q = 36	$hk - 1 = 11$	–	–

From F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 6) = 4.76$ and $F_{5\%}$ $(v_1 = 2, v_2 = 6) = 5.14$.

Considering the difference between rows, we see that F_0 (= 3.6) < $F_{5\%}$ (= 4.76).

Hence the difference between the rows is not significant. (H_0 is accepted) viz., the blocks do not differ significantly with respect to the yield.

Considering the difference between columns, we see that F_0 (= 2.4) < $F_{5\%}$ (= 5.14)

Hence the difference between the columns is not significant. (H_0 is accepted) viz., the varieties of crop do not differ significantly with respect to the yield.

Example 6 *Five breeds of cattle B_1, B_2, B_3, B_4, B_5 were fed on four different rations R_1, R_2, R_3, R_4. Gains in weight in kg over a given period were recorded and given below:*

	B_1	B_2	B_3	B_4	B_5
R_1	1.9	2.2	2.6	1.8	2.1
R_2	2.5	1.9	2.3	2.6	2.2
R_3	1.7	1.9	2.2	2.0	2.1
R_4	2.1	1.8	2.5	2.3	2.4

Is there a significant difference between (i) breeds and (ii) rations?

Solution We effect the change of origin and scale using $y_{ij} = \dfrac{x_{ij} - 2}{1/10} = 10(x_{ij} - 2)$ and work out with y_{ij} values.

	B_1	B_2	B_3	B_4	B_5	T_i	$\dfrac{T_i^2}{k}$	$\sum_j y_{ij}^2$
R_1	– 1	2	6	– 2	1	6	7.2	46
R_2	5	– 1	3	6	2	15	45.0	75
R_3	– 3	– 1	2	0	1	–1	0.2	15
R_4	1	– 2	5	3	4	11	24.2	55
T_j	2	– 2	16	7	8	$T = 31$	$\sum \dfrac{T_i^2}{k} = 76.6$	191
T_j^2/h	1	1	64	12.25	16	$\sum T_j^2/h = 94.25$		
$\sum_i y_{ij}^2$	36	10	74	49	22	191		

$$Q = \sum\sum y_{ij}^2 - \frac{T^2}{N} = 191 - \frac{(31)^2}{20} = 142.95$$

$$Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N} = 76.6 - 48.05 = 28.55$$

$$Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N} = 94.25 - 48.05 = 46.20$$

$$Q_3 = Q - Q_1 - Q_2 = 142.95 - (28.55 + 46.20) = 68.20$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (rations)	$Q_1 = 28.55$	$h - 1 = 3$	9.52	9.52/5.68 = 1.68
Between Cols. (breeds)	$Q_2 = 46.20$	$k - 1 = 4$	11.55	11.55/5.68 = 2.03
Residual	$Q_3 = 68.20$	$(h - 1)(k - 1) = 12$	5.68	–
Total	$Q = 142.95$	$hk - 1 = 19$	–	–

From the F-tables, $F_{5\%}$ ($v_1 = 3, v_2 = 12$) = 3.49 and $F_{5\%}$ ($v_1 = 4, v_2 = 12$) = 3.26

With respect to the rows, F_0(= 1.68) < $F_{5\%}$ (= 3.49)

With respect to the columns, F_0 (= 2.03) < $F_{5\%}$ (= 3.26)

Hence the difference between the rations and that between the breeds are not significant.

Example 7 *The following data represent the number of units of production per day turned out by 5 different workers using 4 different types of machines:*

Machine Type

		A	B	C	D
	1	44	38	47	36
	2	46	40	52	43
Workers:	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

(a) Test whether the five men differ with respect to mean productivity.

(b) Test whether the mean productivity is the same for the four different machine types.

Solution We subtract 40 from the given values and work out with new values of x_{ij}.

Machine Type

Worker	A	B	C	D	T_i	T_i^2/k	$\sum_j x_{ij}^2$
1	4	– 2	7	– 4	5	6.25	85
2	6	0	12	3	21	110.25	189
3	– 6	– 4	4	– 8	– 14	49.00	132
4	3	– 2	6	– 7	0	0	98
5	– 2	2	9	– 1	8	16.00	90
T_j	5	– 6	38	– 17	$T = 20$	$\sum \frac{T_i^2}{k} = 181.5$	594
T^2j_i / h	5	7.2	288.8	57.8	$\sum T_j^2 / h = 358.8$		
$\sum_i x_{ij}^2$	101	28	326	139	594		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 594 - \frac{400}{20} = 574$$

$$Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 181.5 - 20 = 161.5$$

$$Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 358.8 - 20 = 338.8$$

$$Q_3 = Q - Q_1 - Q_2 = 574 - (161.5 + 338.8) = 73.7$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows (workers)	$Q_1 = 161.5$	$h - 1 = 4$	40.375	$\frac{40.375}{6.142} = 6.57$
Between Cols. (machines)	$Q_2 = 338.8$	$k - 1 = 3$	112.933	$\frac{112.933}{6.142} = 18.39$
Residual	$Q_3 = 73.7$	$(h - 1)(k - 1) = 12$	6.142	–
Total	$Q = 574$	$hk - 1 = 19$	–	–

From the F-tables, $F_{5\%}(v_1 = 4, v_2 = 12) = 3.26$

and $F_{5\%}(v_1 = 3, v_2 = 12) = 3.49$

With respect to the rows, $F_0(= 6.57) > F_{5\%}\ (= 3.26)$

With respect to the columns, $F_0(= 18.39) > F_{5\%}\ (= 3.49)$

Hence the 5 workers differ significantly and the 4 machine types also differ significantly with respect to mean productivity.

Example 8 *Four doctors each test four treatments for a certain disease and observe the number of days each patient takes to recover. The results are as follows (recovery time in days)*

Treatment

Doctor	*1*	*2*	*3*	*4*
A	*10*	*14*	*19*	*20*
B	*11*	*15*	*17*	*21*
C	*9*	*12*	*16*	*19*
D	*8*	*13*	*17*	*20*

Discuss the difference between (a) doctors and (b) treatments.

We subtract 15 from the given values and work out with the new values of x_{ij}.

Doctor	Treatment 1	2	3	4	T_i	$\frac{T_i^2}{k}$	$\sum_j x_{ij}^2$
A	– 5	– 1	4	5	3	2.25	67
B	– 4	0	2	6	4	4.00	56
C	– 6	– 3	1	4	– 4	4.00	62
D	– 7	– 2	2	5	– 2	1.00	82
T_j	– 22	– 6	9	20	$T = 1$	$\sum \frac{T_i^2}{k} = 11.25$	267
T_j^2/h	121	9	20.25	100	$\sum T_j^2 / h = 250.25$		
$\sum_i x_{ij}^2$	126	14	25	102	267		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 267 - \frac{1}{16} = 266.94$$

$$Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 11.25 - 0.0625 = 11.19$$

$$Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 250.25 - 0.0625 - 250.19$$

$$Q_3 = Q - Q_1 - Q_2 = 266.94 - 261.38 = 5.56$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows (doctors)	$Q_1 = 11.19$	$h - 1 = 3$	3.73	$\frac{3.73}{0.62} = 6.02$
Between cols. (treatments)	$Q_2 = 250.19$	$k - 1 = 3$	83.40	$\frac{83.40}{0.62} = 134.52$
Residual	$Q_3 = 5.56$	$(h - 1)(k - 1) = 9$	0.62	–
Total	$Q = 266.94$	$hk - 1 = 15$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 9) = 3.86$

Since $F_0 > F_{5\%}$ with respect to rows and columns, the difference between the doctors is significant and that between the treatments is highly significant.

Example 9 *The following data resulted from an experiment to compare three burners B_1, B_2 and B_3. A Latin square design was used as the tests were made on 3 engines and were spread over 3 days.*

	Engine 1	*Engine 2*	*Engine 3*
Day 1	$B_1 - 16$	$B_2 - 17$	$B_3 - 20$
Day 2	$B_2 - 16$	$B_3 - 21$	$B_1 - 15$
Day 3	$B_3 - 15$	$B_1 - 12$	$B_2 - 13$

Test the hypothesis that there is no difference between the burners.

Solution We subtract 16 from the given values and work out with new values of x_{ij}.

	E_1	E_2	E_3	T_i	$\frac{T_i^2}{n}$	$\sum_j x_{ij}^2$
D_1	$0(B_1)$	$1(B_2)$	$4(B_3)$	5	8.33	17
D_2	$0(B_2)$	$5(B_3)$	$-1(B_1)$	4	5.33	26
D_3	$-1(B_3)$	$-4(B_1)$	$-3(B_2)$	-8	21.33	26
T_j	-1	2	0	$T = 1$	$\sum T_i^2 / n = 35$	69
T_j^2 / n	0.33	1.33	0	$\sum T_i^2 / n = 1.66$		
$\sum_i x_{ij}^2$	1	42	26	69		

Rearranging the data values according to the burners, we have

Burner		x_k		T_k	T_k^2 / n
B_1	0	-1	-4	-5	8.33
B_2	1	0	-3	-2	1.33
B_3	4	5	-1	8	21.33
	Total		$T = 1$	$\sum \frac{T_k^2}{n} = 31$	

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 69 - \frac{1}{9} = 68.89$$

$$Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{N} = 35 - \frac{1}{9} = 34.89$$

$$Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{N} = 1.67 - \frac{1}{9} = 1.56$$

$$Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{N} = 31 - \frac{1}{9} = 30.89$$

$$Q_4 = Q - Q_1 - Q_2 - Q_3 = 1.55$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows (days)	$Q_1 = 34.89$	$n - 1 = 2$	17.445	$\frac{17.445}{0.775} = 22.51$
Between Cols. (engines)	$Q_2 = 1.56$	$n - 1 = 2$	0.780	$\frac{0.780}{0.775} = 1.01$
Between letters (burners)	$Q_3 = 30.89$	$n - 1 = 2$	15.445	$\frac{15.445}{0.775} = 19.93$
Residual	$Q_4 = 1.55$	$(n - 1)(n - 2) = 2$	0.775	–
Total	$Q = 68.89$	$n^2 - 1 = 8$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 2, v_2 = 2) = 19.00$

Since $F_0(= 19.93) > F_{5\%}$ $(= 19.00)$ for the burners, there is significant difference between the burners.

Incidentally, since $F_0 > F_{5\%}$ for the rows, the difference between the days is significant and since $F_0 < F_{5\%}$ for the columns, the difference between the engine is not significant.

Example 10 *Analyse the variance in the following Latin square of yields (in kgs) of paddy where A, B, C, D denote the different methods of cultivation*

*D*122	*A*121	*C*123	*B*122
*B*124	*C*123	*A*122	*D*125
*A*120	*B*119	*D*120	*C*121
*C*122	*D*123	*B*121	*A*122

Examine whether the different methods of cultivation have given significantly different yields.

Solution We subtract 120 from the given values and work out with the new values of x_{ij}.

i \\ j	1	2	3	4	T_i	T_i^2/n	$\sum_j x_{ij}^2$
1	D2	A1	C3	B2	8	16	18
2	B4	C3	A2	D5	14	49	54
3	A0	B-1	D0	C1	0	0	2
4	C2	D3	B1	A2	8	16	18
T_j	8	6	6	10	$T = 30$	$\sum T_i^2/n = 81$	92
T_j^2/n	16	9	9	25	$\sum T_j^2/n = 59$		
$\sum_i x_{ij}^2$	24	20	14	34	92		

Rearranging the data according to the letters, we have

Letter		x_k			T_k	T_k^2 / n
A	1	2	0	2	5	6.25
B	2	4	- 1	1	6	9.00
C	3	3	1	2	9	20.25
D	2	5	0	3	10	25.00
		Total			30	60.50

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 92 - \frac{30^2}{16} = 35.75$$

$$Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{N} = 81 - 56.25 = 24.75$$

$$Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{N} = 59 - 56.25 = 2.75$$

$$Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{N} = 60.50 - 56.25 = 4.25$$

$$Q_4 = Q - Q_1 - Q_2 - Q_3 = 35.75 - (24.75 + 2.75 + 4.25) = 4.0$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows	$Q_1 = 24.75$	$n - 1 = 3$	8.25	
Between columns	$Q_2 = 2.75$	$n - 1 = 3$	0.92	
Between letters	$Q_3 = 4.25$	$n - 1 = 3$	1.42	$\frac{1.42}{0.67} = 2.12$
Residual	$Q_4 = 4.0$	$(n - 1)(n - 2) = 6$	0.67	–
Total	$Q = 35.75$	$n^2 - 1 = 15$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 6) = 4.76$.

Since $F_0 (= 2.12) < F_{5\%}$ $(= 4.76)$ with respect to the letters, the difference between the methods of cultivation is not significant.

Exercise 10

Part A (Short answer questions)

1. What do you mean by the term 'experiment' in Design of experiments?
2. What motivated the adoption of design of experiments technique in scientific problems?

3. What is the aim of the design of experiments?
4. Distinguish between experimental and extraneous variables.
5. Name the basic principles of experimental design.
6. What do you mean by experimental group and control group?
7. What are the techniques frequently used in the local control of extraneous variables?
8. Name three basic designs of experiment.
9. What do you mean by analysis of variance.
10. Explain completely randomised design briefly.
11. Write down the format of the ANOVA table for one factor of classification.
12. Explain randomised block design briefly.
13. Write down the format of the ANOVA table for two factors of classification.
14. Explain Latin square design briefly.
15. Is a 2×2 Latin square design possible? Why?
    [**Hint** : No, as the degree of freedom for the residual variation is zero]
16. Write down the format of ANOVA table for three factors of classification.
17. Compare RBD and LSD.
18. What is the main advantage of LSD over RBD?
19. What is the total number of all possible Latin squares of order 3?
20. What is the total number of all possible Latin squares of order 4?

Part B

21. The following tables gives the yields of wheat from 16 plots, all of approximately equal fertility, when 4 varieties of wheat were cultivated in a completely randomised fashion. Test the hypothesis that the varieties are not significantly different.

Plot No. :	1	2	3	4	5	6	7	8	9	10
Variety :	A	B	D	C	B	C	A	D	B	D
Yield :	32	34	29	31	33	34	34	26	36	30
Plot No. :	11	12	13	14	15	16				
Variety :	A	C	B	A	B	C				
Yield :	33	35	37	35	35	32				

22. A random sample is selected from each of 3 makes of ropes and their breaking strength (in certain units) are measured with the following results:

    I : 70, 72, 75, 80, 83
    II : 60, 65, 57, 84, 87, 73
    III : 100, 110, 108, 112, 113, 120, 107

    Test whether the breaking strengths of the ropes differ significantly.
23. The weights in gm of a number of copper wires, each of length 1 metre, were obtained. These are shown classified according to the dye from which they come:

D_1 : 1.30, 1.32, 1.36, 1.35, 1.32, 1.37
D_2 : 1.28, 1.35, 1.33, 1.34
D_3 : 1.32, 1.29, 1.31, 1.28, 1.33, 1.30
D_4 : 1.31, 1.29, 1.33, 1.31, 1.32
D_5 : 1.30, 1.32, 1.30, 1.33

Test the hypothesis that there is no difference between the mean weights of wires coming from different dyes.

24. It is suspected that four machines used in a canning operation fills cans to different levels on the average. Random samples of cans produced by each machine were taken and the fill (in ounces) was measured. The results are tabulated below:

Machine

A	*B*	*C*	*D*
10.20	10.22	10.17	10.15
10.18	10.27	10.22	10.27
10.36	10.26	10.34	10.28
10.21	10.25	10.27	10.40
10.25	–	–	10.30

Do the machines appear to be filling the cans at different average levels?

25. Different numbers of leaves were taken from each of 6 trees and their lengths measured. The following are the lengths in millimetres:

Tree	Lengths							
1	82	87	86	90	81	84		
2	85	84	91	92	88			
3	92	90	84	86	88	93	89	90
4	80	86	87	81	82	82		
5	87	86	88	90	85	86	87	
6	90	86	84	85	85	86	87	84	87

Can all these leaves be regarded as having come from the same species of trees?

26. There are 3 typists working in an office. The times (in minutes) they spend for tea-break in addition to the allowed lunch tea break are observed and noted below:

A :	25	18	30	32	35	37	19			
B :	24	22	26	28	30	32	28	26		
C :	28	20	27	19	29	35	30	23	27	32

Can the difference in average times that the 3 typists spend for tea break be attributed to chance variation?

27. Four machines *A*, *B*, *C*, *D* are used to produce a certain kind of cotton fabric. 4 Samples with each unit of size 100 square metres are selected from the outputs of the machines at random and the number of flaws in each 100 square metres are counted, with the following results:

A	B	C	D
8	6	14	20
9	8	12	22
11	10	18	25
12	4	9	23

Do you think that there is a significant difference in the performance of the four machines?

28. The following table shows the yield (in certain units) of lima beans on 20 plots of land subject to 4 different treatments, 5 plots per treatment. Set up an analysis of variance table to test the significance of the differences between the yields due to different treatments.

T_1 :	26.3	30.0	54.2	25.7	52.4
T_2 :	18.5	21.1	29.3	17.2	12.4
T_3 :	36.9	21.8	24.0	18.5	10.2
T_4 :	39.8	28.7	21.2	39.4	29.0

29. To test the significance of the variation of the retail prices of a certain commodity in the 4 principal cities Mumbai, Kolkata, Delhi and Chennai, 7 shops were chosen at random in each city and the prices (in Rs.) observed were as follows:

Mumbai :	100,	97,	91,	87,	87,	81,	79
Kolkata :	102,	100,	98,	97,	94,	86,	80
Delhi :	106,	102,	98,	86,	86,	84,	84
Chennai :	97,	95,	94,	92,	90,	86,	82

Do the data indicate that the prices in the 4 cities are significantly different?

30. Steel wire was made by 4 manufacturers A, B, C and D. In order to compare their products, 10 samples were randomly drawn from a batch of wires made by each manufacturer and the strength of each piece of wire was measured. The (coded) values are given below:

A : 55, 50, 80, 60, 70, 75, 40, 45, 80, 70
B : 70, 80, 85, 105, 65, 100, 90, 95, 100, 70
C : 70, 60, 65, 75, 90, 40, 95, 70, 65, 75
D : 90, 115, 80, 70, 95, 100, 105, 90, 100, 60

Carry out an analysis of variance and give your conclusions.

31. A randomised block experiment was laid out (with 4 blocks, each block containing 4 plots) to test 4 varieties of manure A, B, C, D and the yields per acre are given as below. Test for the significance of the difference among the 4 varieties of manure.

Block I	A155	B152	C157	D156
Block II	B152	C150	D156	A154
Block III	C156	D153	A161	B162
Block IV	D153	A154	B156	C155

32. The following table gives the gains in weights of 4 different types of pigs fed on 3 different rations over a period. Test whether
    (i) the difference in the rations significant
    (ii) the 4 types of pigs differ significantly in gaining weight.

	Types of pig			
Ration	*I*	*II*	*III*	*IV*
A	13.8	15.7	16.0	20.2
B	8.7	11.8	9.0	12.9
C	12.0	16.5	13.3	12.5

33. Four experiments determine the moisture content of samples of a powder, each observer taking a sample from each of six consignments. The assessments are given below:

	Consignment					
Observer	1	2	3	4	5	6
1	9	10	9	10	11	11
2	12	11	9	11	10	10
3	11	10	10	12	11	10
4	12	13	11	14	12	10

Perform an analysis of variance on these data and discuss whether there is any significant difference between consignments or between observers.

34. In order to compare three burners B_1, B_2 and B_3, one observation is made on each burner on each of four successive days. The data are tabulated below:

	B_1	B_2	B_3
Day 1	21	23	24
Day 2	18	17	23
Day 3	18	21	20
Day 4	17	20	22

Perform an analysis of variance on these data and find whether the difference between (i) the days and (ii) the burners significant at 5% LOS.

35. A company appoints 4 salesmen A, B, C and D and observes their sales in 3 seasons summer, winter and monsoon. The figures (in lakhs of Rs) are given in the following table :

	Salesmen			
Season	*A*	*B*	*C*	*D*
Summer	36	36	21	35
Winter	28	29	31	32
Monsoon	26	28	29	29

Carry out an analysis of variance.

36. The following data represent the numbers of units of production per day turned out by 4 different workers using 5 different types of machines:

	Machine type				
Worker	*A*	*B*	*C*	*D*	*E*
1	4	5	3	7	6
2	6	8	6	5	4
3	7	6	7	8	8
4	3	5	4	8	2

On the basis of this information, can it be concluded that (i) the mean productivity is the same for different machines (ii) the workers do not differ with regard to productivity?

37. The number of automobiles arriving at 4 toll gates were recorded for a 2 hours time period (10 A.M. to 12 noon) for each of six working days. The data are as follows :

Day	Gate 1	Gate 2	Gate 3	Gate 4
Mon	200	228	212	301
Tues	208	230	215	305
Wed	225	240	228	288
Thur	223	242	224	212
Fri	228	210	235	215
Sat	220	208	245	200

Determine whether the rate of arrival (i) is the same at each toll gate (ii) differs significantly during the six days or not.

38. The following table gives the number of refrigerators sold by 4 salesmen in 3 months:

	Salesman			
Months	*I*	*II*	*III*	*IV*
May	50	40	48	39
June	46	48	50	45
July	39	44	40	39

Determine whether (i) there is any difference in average sales made by the four salesmen (ii) the sales differ with respect to different months.

39. Four different drugs have been developed for a certain disease. These drugs are used in 3 different hospitals and the results, given below, show the number of cases of recovery from the disease per 100 people who have taken the drugs :

	D_1	D_2	D_3	D_4
H_1	19	8	23	8
H_2	10	9	12	6
H_3	11	13	13	10

What conclusions can you draw based on an analysis of variance ?

40. The following table gives the additional hours of sleep due to 3 soporofic drugs A, B, C tried on one patient each from 4 different age groups. Examine whether age has got any significant effect on the gain in sleep. Also examine whether the 3 drugs are similar in their effects or not

	Age group			
Drug	30–40	40–50	50–60	60–70
A	2.0	1.2	1.0	0.3
B	1.1	0.8	0.0	– 0.1
C	1.5	1.3	0.9	0.1

41. The following table gives the results of experiments on 4 varieties of a crop in 5 blocks of plots. Prepare the ANOVA table to test the significance of the difference between the yields of the 4 varieties :

Variety	B_1	B_2	B_3	B_4	B_5
A	32	34	33	35	37
B	34	33	36	37	35
C	31	34	35	32	36
D	29	26	30	28	29

42. In the table given below are the yields of 6 varieties of a crop in a 4 replicate RBD experiment. Analyse the data:

	Varieties					
Replicates	1	2	3	4	5	6
1	18.5	15.7	16.2	14.1	13.0	13.6
2	11.7	14.25	12.9	14.4	14.9	12.5
3	15.4	14.6	15.5	20.3	18.4	21.5
4	16.5	18.6	12.7	15.7	16.5	18.0

43. Analyse the variance in the following Latin square:

A8	C18	B9
C9	B18	A16
B11	A10	C20

44. Analyse the variance in the following Latin square:

20 B	17 C	25 D	34 A
23 A	21 D	15 C	24 B
24 D	26 A	21 B	19 C
26 C	23 B	27 A	22 D

45. A varietal trial was conducted on wheat with 4 varities A, B, C, D in a Latin square design. The plan of the experiment and the per plot yield are given below.

C25	B23	A20	D20
A19	D19	C21	B18
B19	A14	D17	C20
D17	C20	B21	A15

Analyse the data and interpret the result.

46. The following is the Latin square layout of a design when 4 varieties of seeds are tested. Set up the analysis of variance table and state your conclusions.

A105	B95	C125	D115
C115	D125	A105	B105
D115	C95	B105	A115
B95	A135	D95	C115

47. The table given below shows the yield of a certain crop in kgs per plot. The letters *A*, *B*, *C*, *D* refer to 4 different manurial treatments. Carry out an analysis of variance.

A260	B300	C335	D371
B280	A300	D300	C410
D320	C345	B340	A254
C372	D395	A290	B328

48. The following results were obtained in a textile experiment to compare the effects of sizing treatments *A*, *B*, *C*, *D* on the number of warps breaking per hour. Is the difference between the treatments significant?

		Loom		
Period	1	2	3	4
1	A 54	B 31	C 70	D 45
2	B 59	A 23	D 100	C 22
3	C 40	D 41	B 74	A 33
4	D 83	C 29	A 100	B 28

49. An agricultural experiment on the Latin square plan gave the following results for the yield of wheat per acre, letters corresponding to varieties.

A16	B10	C11	D9	E9
E10	C9	A14	B12	D11
B15	D8	E8	C10	A18
D12	E6	B13	A13	C12
C13	A11	D10	E7	B14

Discuss the variation of yield with each of the factors corresponding to the rows and columns.

50. The following is a Latine square design of five treatments:

A13	B9	C21	D7	E6
D9	E8	A15	B7	C16
B11	C17	D8	E10	A17
E8	A15	B7	C10	D7
C11	D9	E8	A11	B15

Analyse the data and interpret the results.

Answers

Exercise 10

19. 12 20. 576
21. $Q_1 = 46.08$, $Q_2 = 73.67$, $F_0 = 2.50$, $F_{5\%} = 3.49$; Difference between varieties not significant.
22. $Q_1 = 5838.4$, $Q_2 = 1126$, $F_0 = 38.89$, $F_{5\%} = 3.68$; Breaking strengths of ropes differ significantly.
23. $Q_1 = 35.98$, $Q_2 = 99.38$, $F_0 = 1.81$, $F_{5\%} = 2.87$; Mean weights of wires do not differ significantly.
24. $Q_1 = 44.44$, $Q_2 = 696$, $F_0 = 2.98$, $F_{5\%} = 3.35$; No, the machines appear to fill at same level.
25. $Q_1 = 151.95$, $Q_2 = 255$, $F_0 = 4.17$, $F_{5\%} = 2.50$; Leaves have not come from the same species.
26. $Q_1 = 2.52$, $Q_2 = 29.27$, $F_0 = 11.62$, $F_{5\%} = 19.45$; Difference may be attributed to chance variation.
27. $Q_1 = 540.65$, $Q_2 = 85.75$, $F_0 = 25.21$, $F_{5\%} = 3.49$; Performances of the machines differ significantly.
28. $Q_1 = 34845.93$, $Q_2 = 10032.78$, $F_0 = 3.47$, $F_{5\%} = 3.24$; Treatments give significantly different yields.
29. $Q_1 = 94.97$, $Q_2 = 1446.03$, $F_0 = 1.9$, $F_{5\%} = 8.64$; Prices do not differ significantly.
30. $Q_1 = 5151$, $Q_2 = 8348$, $F_0 = 7.41$, $F_{5\%} = 8.60$; Strengths of wire do not differ significantly.
31. $Q_1 = 42.75$, $Q_2 = 6.75$, $Q_3 = 96.25$, $F_0 = 4.75$, $F_{5\%} = 8.82$; Difference between manures is not significant.
32. $Q_1 = 3393.59$, $Q_2 = 878.44$, $Q_3 = 344.36$,
    F_0 (rows) = 9.85 and $F_{5\%} = 5.14$,
    F_0 (columns) = 2.55 and $F_{5\%} = 4.76$;
    Difference between rations significant. Difference between pigs is not significant.
33. $Q_1 = 13.13$, $Q_2 = 9.71$, $Q_3 = 13.12$,

F_0 (rows) = 5.03 and $F_{5\%}$ = 3.29,
F_0 (columns) = 2.23 and $F_{5\%}$ = 5.05;
Difference between observers is significant; Difference between consignments is not significant.

34. Q_1 = 22.00, Q_2 = 28.17, Q_3 = 14.50,
F_0 (rows) = 3.03 and $F_{5\%}$ = 4.76,
F_0 (columns) = 5.83 and $F_{5\%}$ = 5.14%;
Difference between days is not significant; Difference between burners is significant.

35. Q_1 = 32, Q_2 = 42, Q_3 = 136,
F_0 (rows) = 1.42 and $F_{5\%}$ = 19.33,
F_0 (columns) = 1.62, and $F_{5\%}$ = 8.94;
Differences between seasons and between salesmen are not significant.

36. Q_1 = 22.0, Q_2 = 12.8, Q_3 = 30.0;
F_0 (rows) = 2.93 and $F_{5\%}$ = 3.49;
F_0 (columns) = 1.28 and $F_{5\%}$ = 3.26;
Differences between the workers and between machine types are not significant.

37. Q_1 = 2279.83, Q_2 = 1470.05, Q_3 = 820.12,
F_0 (rows) = 1.80 and $F_{5\%}$ = 4.64,
F_0 (columns) = 1.79 and $F_{5\%}$ = 3.29;
Differences between the days and between the gates are not significant.

38. Q_1 = 109.5, Q_2 = 42.0, Q_3 = 64.5,
F_0 (rows) = 5.09 and $F_{5\%}$ = 5.14,
F_0 (columns) = 1.30 and $F_{5\%}$ = 4.76;
Differences between the months and between salesmen are not significant.

39. Q_1 = 55.17, Q_2 = 113.0, Q_3 = 89.5,
F_0 (rows) = 1.85 and $F_{5\%}$ = 5.14;
F_0 (columns) = 2.52 and $F_{5\%}$ = 4.76;
Differences between the hospitals and between the drugs are not significant.

40. Q_1 = 98.17, Q_2 = 341.58, Q_3 = 25.17,
F_0 (rows) = 11.69 and $F_{5\%}$ = 5.14,
F_0 (columns) = 27.11 and $F_{5\%}$ = 4.76;
Age has significant effect on the gain in sleep; Drugs differ significantly in their effect.

41. Q_1 = 134.0, Q_2 = 21.7, Q_3 = 29.5,
F_0 (row) = 18.16 and $F_{5\%}$ = 3.49,
F_0 (columns) = 2.21 and $F_{5\%}$ = 3.26;
Difference between the yields of 4 varieties is significant.

42. Q_1 = 56.76, Q_2 = 12.58, Q_3 = 80.18,
F_0 (rows) = 3.30 and $F_{5\%}$ = 3.34,
F_0 (columns) = 2.27 and $F_{5\%}$ = 4.65;
Difference between the varieties is significant.

43. Q_1 = 11.56, Q_2 = 68.23, Q_3 = 29.56, Q_4 = 68.21,

F_0 (rows) = 5.90, F_0 (cols.) = 1, F_0 (letters) = 2.31,
$F_{5\%}$ (for all) = 19.0;
The differences between rows, between columns and between letters are not significant.

44. $Q_1 = 34.19$, $Q_2 = 22.69$, $Q_3 = 141.19$, $Q_4 = 96.87$, F_0 (rows) = 1.42 and $F_{5\%}$ = 8.94; F_0 (columns) = 2.14 and $F_{5\%}$ = 8.94; F_0(letters) = 2.91 and $F_{5\%}$ = 4.76; Differences between rows, between columns and between letters are not significant.
45. $Q_1 = 46.5$, $Q_2 = 7.5$, $Q_3 = 48.5$, $Q_4 = 10.5$,
F_0 (rows) = 8.86, F_0 (columns) = 1.43, F_0 (letters) = 9.24, $F_{5\%}$ = 4.76. Difference between varieties is significant.
46. $Q_1 = 2$, $Q_2 = 4$, $Q_3 = 22$, $Q_4 = 60$, F_0 (rows) = 15, F_0 (columns) = 7.5, F_0 (letters) = 1.36, $F_{5\%}$ = 8.94; Difference between rows is significant, but differences between columns and between letters are not significant.
47. $Q_1 = 2540.5$, $Q_2 = 2853.75$, $Q_3 = 18690$, $Q_4 = 7515.75$, F_0 (rows) = 1.48, and $F_{5\%}$ = 8.94; F_0 (columns) = 1.32 and $F_{5\%}$ = 8.94; F_0 (letters) = 4.97 and $F_{5\%}$ = 4.76; Differences between rows and between columns are not significant, but difference between treatments is significant.
48. $Q_1 = 376$, $Q_2 = 8184$, $Q_3 = 1547.5$, $Q_4 = 284.5$
F_0 (rows) = 2.64, F_0 (columns) = 57.53, F_0 (letters) = 10.88; $F_{5\%}$ (for all) = 4.76; Difference between periods is not significant; Differences between looms and between treatments are significant.
49. $Q_1 = 2.16$, $Q_2 = 66.56$, $Q_3 = 122.56$, $Q_4 = 5.28$, F_0 (rows) = 1.2, F_0 (columns) = 37.8, F_0 (letters) = 69.6, $F_{5\%}$ (for all) = 3.26; Difference between rows is not significant, but differences between columns and between varieties are significant.
50. $Q_1 = 26$, $Q_2 = 34$, $Q_3 = 224.4$, $Q_4 = 103.6$, F_0 (rows) = 1.33 and $F_{5\%}$ = 5.91, F_0 (columns) = 1.02 and $F_{5\%}$ = 5.91, F_0 (letters) = 6.50 and $F_{5\%}$ = 3.26; Differences between rows and between columns are not significant, but difference between treatments is significant.

Appendix: Important Formulae

1. If A (an event) $\subset$ (sample space),

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{No. of outcomes favourable to } A}{\text{Exhaustive no. of outcome in } S}$$

2. $$P(A \cup B) = \begin{cases} P(A) + P(B) - P(A \cap B) & \text{if } A \text{ and } B \text{ are any 2 events} \\ P(A) + P(B) & \text{if } A \text{ and } B \text{ are mutually exclusive} \end{cases}$$

3. $$P(A \cap B) = \begin{cases} P(A) \cdot P(B/A) & \text{if } A \text{ and } B \text{ are any 2 associated events} \\ P(A) \cdot P(B) & \text{if } A \text{ and } B \text{ are independent} \end{cases}$$

4. If $B_1, B_2 \ldots, B_n$ be a set of exhaustive and mutually exclusive events and A is another event,

   (a) $$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

   (b) $$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)} \, ; \; i = 1, 2, \ldots, n$$

5. $P(r$ succeses in n Bernoulli's trials)
   $= P_n(r) = nC_r P^r q^{n-r}; r = 0, 1, 2, \ldots, n$

6. $$\sum_{r=r_1}^{r_2} nC_r p^r q^{n-r} \simeq \int_{Z_1}^{Z_2} \phi(z)\, dz, \; z_1 = \frac{r_1 - np - \frac{1}{2}}{\sqrt{npq}} \text{ and } z_2 = \frac{r_2 - np + \frac{1}{2}}{\sqrt{npq}}$$

7. $$P_n(r_1, r_2, \ldots, r_k) = \frac{n!}{r_1!\, r_2! \cdots r_k!} p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{rk},$$

$$P(A_i) = p_i, \sum_{i=1}^{k} p_i = 1 \text{ and } \sum_{i-1}^{k} r_i = n$$

8. If $\{p_i\}$ is the probability mass function and $F(x)$ is the distribution function of a discrete RV X,

$$F(x) = \sum_{x_i \le x} p_i$$

9. If $f(x)$ is the probability density function and $F(x)$ is the distribution function of a continuous RV X,

(i) $P(a \le X \le b) = \int_a^x f(x)\,\mathrm{d}x = F(b) - F(a)$;

(ii) $F(x) = \int_{-\infty}^{b} f(x)\,\mathrm{d}x$;

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x)$

10. If $f\{x, y)$ is the joint probability density function of (X, Y),

(i) $P\{(X, Y) \in D\} = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$;

(ii) $F\{(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$

11. Marginal probability distribution of X and Y are $\{x_i, p_{i*}\}$, $i = 1, 2, 3, \ldots$ where $p_{i*} = \sum_j P_{ij}$ and $\{y_j, p_{*j}\}$, $j = 1, 2, 3, \ldots$ where $p_{*j} = \sum_i p_{ij}$ respectively.

12. Conditional probability distributions of X given $Y = y_j$ and of Y given $X = x_i$ are respectively $\left\{x_i, \dfrac{p_{ij}}{p_{*j}}\right\}$, $i = 1, 2, 3, \ldots$ and $\left\{y_j, \dfrac{p_{ij}}{p_{i*}}\right\}$, $j = 1, 2, 3, \ldots$

13. Marginal density functions of X and Y are respectively

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y \text{ and } f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x$$

14. Conditional density functions of X given Y and of Y given X are respectively

$$f_{X/Y}(x) = \frac{f(xy)}{f_Y(y)} \text{ and } f_{Y/X}(y) = \frac{f(xy)}{f_X(x)}$$

15. X and Y are independent if (*i*) $p_{ij} = p_{i*}\ p_{*j}$ (discrete case) and (ii) $f(x, y) = f_X(x) \cdot f_Y(y)$ (continuous case)

16. If $f_X(x)$ and $f_Y(y)$ are density functions of X and Y where $Y = g(x)$,

$$f_Y(y) = \begin{cases} f_X(x)\left|\dfrac{dx}{dy}\right| & \text{if } x = g^{-1}(y) \text{ is single valued} \\ \sum\limits_i f_X(x_i)\left|\dfrac{dx_i}{dy}\right| & \text{if } x = g^{-1}(y) \text{ is many valued} \end{cases}$$

17. If $Z = X + Y$, where X and Y are independent,

$$f_Z(z) = \begin{cases} \int\limits_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\, dy \\ \int\limits_{0}^{z} f_X(z-y) \cdot f_Y(y)\, dy \end{cases}$$

$z > 0$, if $f_X(x) = 0$ for $x < 0$ and $f_Y(y) = 0$ for $y < 0$

18. If $Z = XY$, where X and Y are independent,

$$f_Z(z) = \int\limits_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\right) \cdot f_Y(y)\, dy$$

19. If $Z = \dfrac{X}{Y}$, where X and Y are independent,

$$f_Z(z) = \int\limits_{-\infty}^{\infty} |y|\, f_X(yz) \cdot f_Y(y)\, dy$$

20. $f_{ZW}(z, w) = |J|\, f_{XY}(x, y)$, where $J = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix}$

21. $E(X) = \begin{cases} \sum\limits_i x_i p_i & \text{if } X \text{ is discrete} \\ \int\limits_{R_X} x f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$

22. $\text{Var}(X) = E(X^2) - E^2(X)$, where $E(X^2) = \begin{cases} \sum x_i^2 p_i \\ \int\limits_{R_X} x^2 f(x)\, \mathrm{d}x \end{cases}$

23. $\mu_n = \mu'_n - nC_1 \mu'_{n-1}\mathrm{d} + nC_2 \mu'_{n-2}\mathrm{d}^2 - \ldots + (-1)^n \mathrm{d}^n$, where
$\mu_n = E\{X - E(X)\}^n$, $\mu'_n = E\{X - a)^n$ and $d = E(X) - a$.

24. $E\{g(X, Y)\} = \begin{cases} \sum_i \sum_j g(x_i, y_j)\, p_{ij}, \text{ if } (x, y) \text{ is discrete} \\ \iint\limits_{R_{XY}} g(x, y)\, f(x, y)\, \mathrm{d}x\, dy \; (x, y) \text{ is continuous} \end{cases}$

25. If X and Y are independent, $E(XY) = E(X) \cdot E(Y)$

26. $C_{XY} = E(XY) - E(X) \cdot E(Y)$

27. $\rho_{XY} = C_{XY} / \sqrt{\text{var}(X) \cdot \text{var}(Y)}$

28. $E(X/Y) = \begin{cases} \sum_i x_i\, p_{ij} / p_{*j} & \text{for discrete case} \\ \int\limits_{R_X} x f_{X/Y}(x)\, \mathrm{d}x & \text{for continuous case} \end{cases}$

29. $E(Y/X) = \begin{cases} \sum_j y_j\, p_{ij} / p_{i*} & \text{for discrete case} \\ \int\limits_{R_Y} y f_{Y/X}(y)\, \mathrm{d}y & \text{for continuous case} \end{cases}$

30. Moment Generating Function of X is

$$M(t) = \begin{cases} \sum_i e^{tx_i}\, p_i & \text{for discrete case} \\ \int\limits_{R_X} e^{tx} f(x)\, \mathrm{d}x & \text{for continuous case} \end{cases}$$

31. Characteristic functions of X is

$$\phi(w) = \begin{cases} \sum_r e^{iw_r}\, p_r & \text{for discrete case} \\ \int\limits_{R_X} e^{iwx} f(x)\, \mathrm{d}x & \text{for continuous case} \end{cases}$$

32. $E(X^n) = \begin{cases} \text{coefficient of } \dfrac{t^n}{n!} \text{ in the expansion of } M(t) \\ \text{coefficient of } \dfrac{i^n w^n}{n!} \text{ in the expansion of } \phi(w) \end{cases}$

33. $E(X^n) = \left[\dfrac{d^n}{dt^n} M(t)\right]_{t=0}$ or $\dfrac{1}{i^n}\left[\dfrac{d^n}{dw^n}\phi(w)\right]_{w=0}$

34. $M_{aX+b}(t) = e^{bt} \cdot M_X(at)$

35. $\phi_{aX+b}(w) = e^{ibw}\,\phi_X(w)$

36. Tchebycheff's inequality is

(i) $P\{|X-\mu| \ge c\} \le \dfrac{\sigma^2}{c^2}$ or $P\{|X-\mu| \le c\} \ge 1 - \dfrac{\sigma^2}{c^2}$, $c > 0$

Where $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$

(ii) $P\left\{\left|\dfrac{X-\mu}{k}\right| \ge \sigma\right\} \le \dfrac{1}{k^2}$ or $P\left\{\left|\dfrac{X-\mu}{k}\right| \le \sigma\right\} \ge 1 - \dfrac{1}{k^2}$, $k > 0$

37. If $E(X_i) = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$ = and if $S_n = \sum_{i=1}^{n} X_i$, then S_n follows $N\left\{\sum_{i=1}^{n}\mu_i, \sqrt{\sum_{i=1}^{n}\sigma_i^2}\right\}$, as $n \to \infty$, where $\{X_i\}$ is a sequence of independent RVs.

38. If $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ and $S_n = \sum_{i=1}^{n} X_i$ then S_n follows $N\left\{n\mu, \sigma\sqrt{n}\right\}$ as $n \to \infty$, and $\bar{X} = \dfrac{1}{n} S_n$ follows $N\left\{\mu, \dfrac{\sigma}{\sqrt{n}}\right\}$ as $n \to \infty$, where $\{X_i\}$ is a sequence of independent and identically distributed RVs.

39. For the binomial distribution $B(n, p)$,

(i) $P\{X = r) = nC_r p^r q^{n-r}$; $r = 0, 1, 2, \ldots, n$; $p + q = 1$

(ii) $E\{X) = np$

(iii) $\text{Var}(X) = npq$

(iv) $\mu_{k+1} = pq\left[\dfrac{d\mu_k}{dp} + nk\mu_{k-1}\right]$

(v) $M(t) = (q + pe^t)^n$ and $\phi(w) = (q + pe^{iw})^n$

40. For the Poisson distribution $P(\lambda)$,

(i) $P\{X = r\} = \dfrac{e^{-\lambda} \cdot \lambda^r}{r!}$; $r = 0, 1, 2, \ldots, \infty$

(ii) $E\{X\} = \lambda = \text{Var}(X) = \mu_3$

(iii) $\mu_{k+1} = \lambda\left[\dfrac{d\mu_k}{d\lambda} + k\mu_{k-1}\right]$

(iv) $M(t) = e^{\lambda(e^t - 1)}$ and $\phi(w) = e^{\lambda(e^{iw} - 1)}$

41. For the geometric distribution $G_1(p)$,

(i) $P\{X = r\} = q^r p$; $r = 0, 1, 2, \ldots, \infty$; $p + q = 1$

(ii) $E(X) = \dfrac{q}{p}$

(iii) $\text{Var}(X) = \dfrac{q}{p^2}$

42. For the geometric distribution $G_2(p)$,

(i) $P\{X = r\} = q^{r-1} p$; $r = 1, 2, \ldots, \infty$; $p + q = 1$

(ii) $E(X) = \dfrac{1}{p}$

(iii) $\text{Var}(X) = \dfrac{q}{p^2}$

43. For the hypergemetric distribution $H(N, k, n)$,

(i) $P\{X = r\} = \dfrac{kc_r \cdot (N-k)\,C_{n-r}}{NC_n}$; $r = 0, 1, 2, \ldots, n$

(ii) $E(X) = \dfrac{nk}{N}$ or np; where $p = \dfrac{k}{N}$

(iii) $\text{Var}(X) = \dfrac{nk(N-k)(N-n)}{N^2(N-1)}$ or $npq\left(\dfrac{N-n}{N-1}\right)$,

where $p = \dfrac{k}{N}$ and $q = 1 - p$

44. For the uniform (rectangular) distribution $U(a, b)$,

(i) $f\{x\} = \dfrac{1}{b-a}$; $b > a$; $a < x < b$

(ii) $E(X) = \dfrac{1}{2}(b + a)$

(iii) $\text{Var}(X) = \dfrac{1}{12}(b-a)^2$

(iv) $\mu_k = \begin{cases} 0 & \text{, if } k \text{ is odd} \\ \dfrac{1}{k+1}\left(\dfrac{b-a}{2}\right)^k & \text{, if } k \text{ is even} \end{cases}$

45. For the (negative) exponential distribution $E(\lambda)$,

(i) $f\{x) = \lambda e^{-\lambda x},\ x \geq 0;\ \lambda > 0$

(ii) $E\{X) = \dfrac{1}{\lambda}$

(iii) $\text{Var}(X) = \dfrac{1}{\lambda^2}$

(iv) μ'_k (about the origin) $= E(X^k) = \dfrac{k!}{\lambda^k}$

46. For the Erlang or General Gamma distribution $Er\ (\lambda, k)$,

(i) $f\{x) = \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)},\ x \geq 0;\ \lambda > 0;\ k > 0$

(ii) $E(X) = \dfrac{k}{\lambda}$

(iii) $\text{Var}(X) = \dfrac{k}{\lambda^2}$

(iv) $M(t) = \left(1 + \dfrac{t}{\lambda}\right)^{-k}$ and $\phi(w) = \left(1 + \dfrac{iw}{\lambda}\right)^{-k}$

(v) $\mu' r = E(X^r) = \dfrac{1}{\lambda^r} \cdot \dfrac{\Gamma(k+r)}{\Gamma(k)}$

47. For the Weibull distribution $W(\alpha, \beta)$,

(i) $f\{x) = \alpha\beta\, x^{\beta-1} \cdot e^{-\alpha x\beta};\ x > 0;\ \alpha, \beta > 0$

(ii) $E(X) = \alpha^{-1/\beta} \cdot \Gamma\left(\dfrac{1}{\beta} + 1\right)$

(iii) $\text{Var}(X) = \alpha^{-2/\beta}\left[\Gamma\left(\dfrac{2}{\beta} + 1\right) - \left\{\Gamma\left(\dfrac{1}{\beta} + 1\right)\right\}^2\right]$

(iv) $\mu_r' = E(X^r) = \alpha^{-r/\beta}\,\Gamma\left(\dfrac{r}{\beta}+1\right)$

48. For the normal (Gaussian) distribution $N(\mu, \sigma)$,

(i) $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$

(ii) $E(X) = \mu =$ Median $=$ Mode

(iii) $\mathrm{Var}(X) = \sigma^2$

(iv) $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ and $\phi(w) = e^{i\mu w - \frac{1}{2}\sigma^2 w^2}$

(v) $\mu_{2n+1} = 0$ and $\mu_{2n} = 1.\,3.\,5\,\ldots\,(2n-1)\,\sigma^{2n}$

(vi) Mean deviation $= \sqrt{\dfrac{2}{\pi}}\,\sigma \simeq \dfrac{4}{5}\sigma$

(vii) Quartile deviation $= 0.674\,\sigma \simeq \dfrac{2}{3}\sigma$.

49. For the bivariate normal distribution $N(0, 0; \sigma_x, \sigma_y; r)$

(i) $f(x, y) = \dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$

$$\exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\};\ -\infty < x, y < \infty$$

(ii) $f_X(x) = \dfrac{1}{\sigma_x\sqrt{2\pi}} \exp\,(-x^2/2\sigma_x^2),\ -\infty < x < \infty$

(iii) $f_Y(y) = \dfrac{1}{\sigma_y\sqrt{2\pi}} \exp\,(-y^2/2\sigma_y^2),\ -\infty < y < \infty$

(iv) $f_{X/Y}(x) = \dfrac{1}{\sqrt{2\pi}\left(\sigma_x\sqrt{1-r^2}\right)} \exp\left\{-\dfrac{1}{2\sigma_x^2(1-r^2)}\left(x - \dfrac{r\sigma_x y}{\sigma_y}\right)^2\right\}$,

$-\infty < x < \infty$

(v) $f_{Y/X}(y) = \dfrac{1}{\sqrt{2\pi}\left(\sigma_y\sqrt{1-r^2}\right)} \exp\left\{-\dfrac{1}{2\sigma_y^2(1-r^2)}\left(y - \dfrac{r\sigma_y}{\sigma_x}x\right)^2\right\}$,

$-\infty < y < \infty$.

50. For two-tailed large sample tests,

$|Z_{1\%}| = 2.58, |z_{2\%}| = 2.33, |z_{5\%}| = 1.96$ and $|Z_{10\%}| = 1.645$

51. For one-tailed large sample tests,

$z_{1\%} = \pm\, 2.33,\ Z_{2\%} = \pm\, 2.05,\ z_{5\%} = \pm\, 1.645$ and $z_{10\%} = \pm\, 1.28$

52. In large samples, to test the significance of the difference between

(i) sample proportion p and population propotion P, the test statistic

$$z = \frac{p - P}{\sqrt{pq/n}} \text{ (size of the sample} = n)$$

(ii) two sample proportions p_1 and p_2,

$$z = \frac{P_1 - P_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \text{ where } \hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \text{ and } \hat{Q} = 1 - \hat{P}$$

(iii) sample mean $\bar{X}$ and population mean, μ,

$$z = \begin{cases} \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}, & \text{if } \sigma \text{ is known} \\[2ex] \dfrac{\bar{X} - \mu}{s/\sqrt{n}}, & \text{if } \sigma \text{ is not known (S.D. of sample} = s) \end{cases}$$

(iv) two sample means $\bar{X}$ and $\bar{X}_2$,

$$z = \begin{cases} \left(\bar{X}_1 - \bar{X}_2\right)\Big/\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}} & \text{if } \sigma \neq \sigma_2 \text{ and } \sigma_1 \text{ and } \sigma_2 \text{ are known} \\[2ex] \left(\bar{X}_1 - \bar{X}_2\right)\Big/\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}} & \text{if } \sigma_1 = \sigma_2 \text{ and } \sigma \text{ is known} \\[2ex] \left(\bar{X}_1 - \bar{X}_2\right)\Big/\sigma\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}} & \text{if } \sigma_1 \neq \sigma_2 \text{ and not known} \\[2ex] \left(\bar{X}_1 - \bar{X}_2\right)\Big/\hat{\sigma}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}} & \text{if } \sigma_1 = \sigma_2 \text{ is not known} \\[2ex] \text{(or)}\left(\bar{X}_1 - \bar{X}_2\right)\Big/\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}} & \text{and } \hat{\sigma}^2 = \dfrac{n_1 s_1^2 + n_2 \sigma_2^2}{n_1 + n_2} \end{cases}$$

(v) Sample SDs and population SDσ, $z = \dfrac{s - \sigma}{\sigma / \sqrt{2n}}$

(vi) Two sample SDs s_1 and s_2,

$$z = \frac{s_1 - s_2}{\hat{\sigma}\sqrt{\dfrac{1}{2n_1} + \dfrac{1}{2n_2}}} \text{ or } \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_2} + \dfrac{s_2^2}{1n_1}}}$$

$$\hat{\sigma}^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$$

53. 95% confidence limits for

 (i) the population proportion are $\left(p \mp 1.96\sqrt{\dfrac{pq}{n}}\right)$

 (ii) the population mean are $\left(\bar{x} \mp 1.96\dfrac{\sigma}{\sqrt{n}}\right)$ or $\left(\bar{x} \mp 1.96\dfrac{s}{\sqrt{n}}\right)$ according as σ is known or not known.

54. In small samples, to test the significance of the difference between

 (i) sample mean $\bar{x}$ and population mean μ, the test statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}; \text{ degrees of freedom } \nu = n - 1.$$

 (ii) two sample means $\bar{X}_1$ and $\bar{X}_2$, $t = \dfrac{\bar{X}_1 - \bar{X}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$;

$\nu = n_1 + n_2 - 2$

 (iii) two population variances σ_1^2 and σ_2^2, $F = \dfrac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \dfrac{n_1\ s_1^2/(n_1 - 1)}{n_2\ s_2^2/(n_2 - 1)}$;

$\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$, provided $F > 1$.

55. For χ^2-test of goodness of fit,

$$\chi^2 = \sum_{i=1}^{n} (O_i - E_i)^2/E_i; \ \nu = \text{number of independent frequencies.}$$

56. For χ^2-test of independence of attributes or for the $(m \times n)$ contingency table,

$$\chi^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} (O_{ij} - E_{ij})^2/E_{ij}, \ \nu = (m-1)(n-1), \text{ where } E_{ij} = \frac{O_{i*} \cdot O_{*j}}{N}.$$

57. For a random process $\{X(t)\}$,

 (i) Mean $\mu(t) = E\{X(t)\}$

(ii) Autocorrelation $R_{XX}(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\}$

(iii) Autocovariance $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu(t_1) \cdot \mu(t_2)$

(iv) Correlation coefficient $\rho_{XX}(t_1, t_2) = \dfrac{C(t_1 t_2)}{\sqrt{C(t_1, t_1) \cdot C(t_2, t_2)}}$

58. For two random processes $\{X(t)\}$ and $\{Y(t)\}$,

(i) Cross-correlation $R_{XY}(t_1, t_2) = E\{X(t_1) \cdot Y(t_2)\}$

(ii) Cross-covariance $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1) \cdot \mu_Y(t_2)$

(iii) Cross-correlation coefficient

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1) \cdot C_{YY}(t_2, t_2)}}$$

59. Density function of Wiener process is

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi \alpha t}} e^{-x^2/2\alpha t} \quad -\infty < x < \infty$$

60. Time average of a random process $\{X(t)\}$ over $(-T, T)$ is

$$\bar{X}_T = \frac{1}{2_T} \int_{-T}^{T} X(t)\, \mathrm{d}t$$

61. If $R_{XX}(\tau)$ is the autocorrelation function of $\{X(t)\}$,

(i) $\mu_X^2 = \lim_{\tau \to \infty} \{R_{XX}(\tau)\}$

(ii) $E\{X^2(t)\} = R_{XX}(0)$

62. If $\bar{X}_T$ is the time average of the process $\{X(t)\}$ over $(-T, T)$,

(i) $\mathrm{Var}(\bar{X}_T) = \dfrac{1}{4T^2} \displaystyle\int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2)\, \mathrm{d}t\, \mathrm{d}t$

(ii) $\mathrm{Var}(\bar{X}_T) = \dfrac{1}{2} \displaystyle\int_{0}^{2T} C(\tau) \left\{1 - \frac{|\tau|}{2T}\right\} \mathrm{d}\tau$ if $\{X(T)\}$ is a stationary process.

63. If $R_{XX}(\tau)$ and $S_{XX}(\omega)$ are the autocorrelation function and power spectral density of a stationary process,

(i) $S_{XX}(\omega) = \displaystyle\int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau} \mathrm{d}\tau$

(ii) $R_{XX}(\tau) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} R_{XX}(\omega)\, e^{i\tau\omega} \mathrm{d}\omega$

64. If $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$.

65. The nth order density of a Gaussian process $\{X(t)\}$ is

$$\frac{1}{(2\pi)^{1/2}|\Lambda|^{1/2}} \exp\left\{-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i-\mu_i)(x_i-\mu_j)\right\},$$

where $\mu_i = E\{X(t_i)\}$ and Λ is the nth order square matrix (λ_{ij}),

where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$.

66. If $\{N(t)\}$ is white noise,

(i) $S_{NN}(\omega) = \frac{N_0}{2}$

(ii) $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$

67. If $\{N(t)$ is a band-limited white noise,

(i) $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, \text{ for } |\omega| \le \omega_B \\ 0, \quad \text{elsewhere} \end{cases}$

(ii) $R_{NN}(\tau) = \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right)$

68. Probability law for Poisson process $\{X(t)\}$ is $P\{X(t)=n\} = \frac{e^{-\lambda t}\cdot(\lambda t)^n}{n!}$; $n = 0, 1, 2, \ldots, \infty$

69. If $\{p_{ij}\}$ is the transition probability matrix of a homogeneous Markov chain $\{X_n\}$, $\{p_{ij}^{(n)}\} = (p_{ij})^n$.

For the ($M/M/1$):($\infty/FIFO$) queueing model,

70. $P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right) n \ge 0$

71. $L_s = E$(number of customers in the system) $= \frac{\lambda}{\mu-\lambda}$

72. $L_q = E$(number of customers in the queure) $= \frac{\lambda^2}{\mu(\mu-\lambda)}$

73. $L_w = E$(number of customers in the non-empty queues) $= \frac{\mu}{\mu-\lambda}$

74. $P(N_s > k)$ = Probability that the number of customers in the system exceeds $k = \left(\frac{\lambda}{\mu}\right)^{k+1}$

75. $E(W_s)$ = Average waiting time of a customer in the system = $\frac{1}{\mu - \lambda}$

76. $P(W_s > t) = e^{-(\mu - \lambda)t}$

77. $E(W_q)$ = Average waiting time of a customer in the queue = $\lambda/\mu\,(\mu - \lambda)$.

78. $E(W_q/W_q > 0)$ = Average waiting time of a customer in the queue, if he has to wait = $\frac{1}{\mu - \lambda}$

Little's formulas valid for $(M/M/1)$:$(\infty/F1F0)$ and $(M/M/s)$:$(\infty/F1F0)$ models.

79. $L_S = \lambda \cdot E(W_s)$

80. $L_q = \lambda \cdot E(W_q)$

81. $L_S = \frac{\lambda}{\mu} + L_q$

82. $E(W_s) = \frac{1}{\mu} + E(W_q)$

For the $(M/M/s)$:$(\infty/F1F0)$ queueing model,

83. $$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \cdot P_o & \text{if } 0 \le n < s \\ \frac{1}{s!\,s^{n-s}} \cdot \left(\frac{\lambda}{\mu}\right)^n \cdot P_o & \text{if } n \ge s, \text{ where} \end{cases}$$

$$P_0^{-1} = \sum_{n-0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \left\{\frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}$$

84. $$L_q = E(N_q) = \frac{1}{s.s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_o$$

85. $$L_s = E(N_S) = L_q + \frac{\lambda}{\mu} = \frac{1}{s.s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_o + \frac{\lambda}{\mu}$$

86. $E(W_s) = \dfrac{1}{\lambda} E(N_s) = \dfrac{1}{\mu} + \dfrac{1}{\mu} \cdot \dfrac{1}{s.s!} \cdot \dfrac{\left(\dfrac{\lambda}{\mu}\right)^s}{\left(1 - \dfrac{\lambda}{\mu s}\right)^2} \cdot P_o$

87. $E(W_q) = \dfrac{1}{\lambda} E(N_q) = \dfrac{1}{\mu} \cdot \dfrac{1}{s.s!} \cdot \dfrac{\left(\dfrac{\lambda}{\mu}\right)^s}{\left(1 - \dfrac{\lambda}{\mu s}\right)^2} \cdot P_o$

88. (i) $P(W_s > 0)$ = Probability that an arrival has to wait = $\dfrac{\left(\dfrac{\lambda}{\mu}\right)^s \cdot P_o}{s!\left(1 - \dfrac{\lambda}{\mu s}\right)}$.

(ii) P(an arrival enters the service without waiting) = $1 - P(W_s > 0)$.

89. $E(W_q/W_s > 0)$ = Average waiting time in the queue for the customers who actually wait = $1/\mu s - \lambda$.

90. $E(N_q/N_q \geq 1)$ = Average number of customers in the queue who have to actually wait = $\dfrac{\dfrac{\lambda}{\mu s}}{1 - \dfrac{\lambda}{\mu s}}$.

For the ($M/M/1$):($k/F1F0$) queueing model,

91. $$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n \left\{ \dfrac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \right\} & \text{if} \quad \lambda \neq \mu \quad n \geq 0 \\ \dfrac{1}{k+1} & \text{if} \quad \lambda = \mu \quad n \geq 0 \end{cases}$$

92. $$L_s = E(N_s) = \begin{cases} \dfrac{\lambda}{\mu - \lambda} - \dfrac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} & \text{if } \lambda \neq \mu \\ \dfrac{k}{2} & \text{if } \lambda = \mu \end{cases}$$

93. Overall effective arrival rate $\lambda' = \mu(1 - P_0)$.

94. $L_q = E(N_q) = L_s - \dfrac{\lambda'}{\mu}$ or $L_s - (1 - P_0)$

95. (i) $E(W_s) = \dfrac{1}{\lambda'} L_s$

(ii) $E(W_q) = \dfrac{1}{\lambda'} L_q$

For the (*M/M/s*):(*k/F1F0*) queueing model,

96. $$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n \cdot P_o & \text{for } n \le s \\ \dfrac{1}{s!\,s^{n-s}} \cdot \left(\dfrac{\lambda}{\mu}\right)^n \cdot P_o & \text{for } s \le n \le k \\ 0 & \text{for } n > k, \text{ where} \end{cases}$$

$$P^{-1}{}_0 = \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \cdot \left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s}$$

97. $L_q = E(N_q) = P_o \cdot \left(\dfrac{\lambda}{\mu}\right)^s \cdot \dfrac{p}{s!(1-p)^2} \; \{1 - \rho^{k-s} - (k-s)(1-\rho)\,\rho^{k-s}\}$

where $\rho = \dfrac{\lambda}{\mu s}$

98. Overall effective arrival rate

$$\lambda' = \mu\left[s - \sum_{n=0}^{s-1} (s-n)P_n\right]$$

99. $L_s = E(N_s) = L_q + \dfrac{\lambda'}{\mu}$ or $L_q + s - \displaystyle\sum_{n=0}^{s-1} (s-n)P_n$.

100. (i) $E(W_s) = \dfrac{1}{\lambda'} L_s$

(ii) $E(W_q) = \dfrac{1}{\lambda'} L_{q.}$

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