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Module - 1

Eigen Values

Suppose A is any square matrix and X is any column matrix such that $AX = \lambda X$ then λ is called Eigen value and X is called Eigen Vector

Ex: $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

i) $X = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ $\quad AX = \begin{bmatrix} 3 \\ -3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 3X$

↳ Eigen Value

ii) $X = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ $\quad AX = \begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = 2X$

↳ Eigen Value

iii) $X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad AX = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1X$

↳ Eigen Value

Consider:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[(2-\lambda)(3-\lambda)-2\right] - 0 + (-1)\left[2 - 2(2-\lambda)\right] = 0$$

$$1-\lambda\left[6-2\lambda-3\lambda+\lambda^2-2\right] - \left[2-4+2\lambda\right] = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+4) - (2\lambda-2) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+4) - (-2)(1-\lambda) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+4+2) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+6) = 0$$

$\lambda^2 \cancel{-5\lambda+6} =$ $\qquad \lambda^2-5\lambda+6=0$

$1-\lambda=0$ $\qquad \lambda = 3, 2$

$\lambda = 1$

Symmetrix Matrix

A square matrix $A = (a_{ij})_{n\times m}$ is said to be symmetrix if $a_{ij} = a_{ji} \; \forall \; i,j$

Ex: i) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & -6 \\ 5 & -6 & 9 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 5 & -8 & 10 \\ 5 & 3 & -2 & -3 \\ -8 & -2 & 9 & 0 \\ 10 & -3 & 0 & 7 \end{bmatrix}$

Tridiagonal Matrix

A symmetrix matrix of the type $\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$

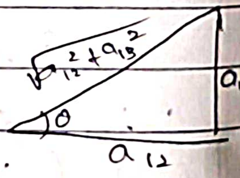
i) $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 5 & 4 & -3 & 0 \\ 0 & -3 & 6 & 1 \\ 0 & 0 & 1 & 8 \end{bmatrix}$

Givens Method

To convert any symmetrix matrix to a tridiagonal matrix

Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$



Step 1: Find $\tan\theta = \dfrac{a_{13}}{a_{12}}$ & obtain $\sin\theta$ & $\cos\theta$

Step 2 Define

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Step 3 : Compute

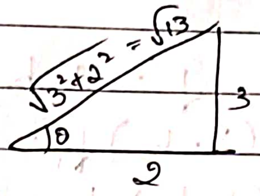
$$B = S^{-1} A S$$

Problem

1. Transform the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ to a tridiagonal matrix using Givens method :

Sol^n Here $a_{12} = 2$, $a_{13} = 3$

$$\tan\theta = \frac{a_{13}}{a_{12}} = \frac{3}{2}$$



① $\sin\theta = \dfrac{3}{\sqrt{13}}$ $\quad \cos\theta = \dfrac{2}{\sqrt{13}}$

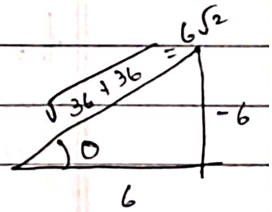
② $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{13} & -3/\sqrt{13} \\ 0 & 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

③ $B = S^{-1} A S$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{13} & 3/\sqrt{13} \\ 0 & -3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{13} & -3/\sqrt{13} \\ 0 & 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3.6055 & 0 \\ 3.6055 & 0.0769 & 0.3846 \\ 0 & 0.3846 & 1.923 \end{bmatrix}$$

2. $A = \begin{bmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{bmatrix}$ $\quad \tan\theta = \dfrac{a_{13}}{a_{12}} = \dfrac{-6}{6}$



$\sin\theta = \dfrac{-6}{6\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$ $\quad \cos\theta = \dfrac{1}{\sqrt{2}}$

$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad \overset{B^{-1}\ A\ B}{B = S^{-1} A S}$

$$= \begin{bmatrix} 12 & 8.45 & 0 \\ 8.45 & 14 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

Givens method to find Eigen values of a Tridiagonal matrix

Consider

$$A = \begin{bmatrix} b_1 & c_1 & 0 \\ c_1 & b_2 & c_2 \\ 0 & c_2 & b_3 \end{bmatrix}$$

Define:

$$f_0 = 1 \qquad f_1 = \lambda - b_1$$
$$f_2 = (\lambda - b_2) f_1 - c_1^2 f_0$$
$$f_3 = (\lambda - b_3) f_2 - c_2^2 f_1$$

$$f_n = (\lambda - b_n) f_{n-1} - c_{n-1}^2 f_{n-2}$$
$$n \geq 2$$

Using givens method find Eigenvalue

1. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Sol: $b_1 = 2 \quad b_2 = 2 \quad b_3 = 2$

$c_1 = -1 \quad c_2 = -1$

Now, $f_0 = 1 \qquad f_1 = \lambda - b_1 = \lambda - 2$

$$f_2 = (\lambda - b_2) f_1 - c_1^2 f_0$$
$$= (\lambda - 2)(\lambda - 2) - (-1)^2 \, 1$$
$$= (\lambda - 2)^2 - 1$$

$$f_3 = (\lambda - b_3) f_2 - c_2^2 f_1$$
$$= (\lambda - 2)\left[(\lambda - 2)^2 - 1\right] - (-1)^2 (\lambda - 2)$$
$$= (\lambda - 2)^3 - 2(\lambda - 2)$$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
-1	+	-	+	-	3
0	+	-	+	-	3
1	+	-	$\oplus$	+	2
2	+	0	-	0	*
3	+	+	0	-	1
4	+	+	+	+	0

$\lambda - 2^3 = 0$

$\lambda_1 = 2 \quad \lambda_2 \in (0, 1)$

$\lambda_3 \in (3, 4)$

To find eigen value in $(0, 1)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0.5	+	−	+	−	3

$\lambda_2 \in (0.5, 1)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0.75	+	−	+	+	2

$\lambda_2 \in (0.5, 0.75)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0.63	+	−	+	+	2

0.56

$\lambda_2 \in (0.5, 0.63)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0.57	+	−	+	−	3

$\lambda_2 \in (0.57, 0.63)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0.6	+	−	+	+	2

Approximate $\lambda_2 \simeq 0.6$

$\lambda_3 \in (3, 4)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
3.5	+	+	+	+	0

$\lambda_3 \in (3, 3.5)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
3.25	+	+	+	−	1

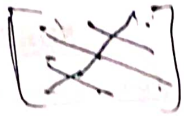
$\lambda_3 \in (3.25, 3.5)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
3.38	+	+	+	−	1

$\lambda_3 \in (3.38, 3.5)$

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
3.44	+	+	+	+	0

$\underline{\lambda_3 \simeq 3.41}$



Householder Method to convert a symmetric matrix into a tridiagonal matrix

$$\text{Suppose } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \text{symmetric matrix}$$

Step 1: Define $v = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

where $x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}(\text{sign of } a_{12})}{S}\right]$

$$x_3 = \frac{a_{13}(\text{sign of } a_{12})}{2x_2 S} \qquad \boxed{x_2^2 + x_3^2 = 1}$$

& $S = \sqrt{a_{12}^2 + a_{13}^2}$

Step 2: Find $P = I - 2vv^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}(0 \; x_2 \; x_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2^2 & x_2x_3 \\ 0 & x_2x_3 & x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2x_2^2 & -2x_2x_3 \\ 0 & -2x_2x_3 & 1-2x_3^2 \end{bmatrix}$$

Step 3: Compute $B = PAP$

where B is tridiagonal matrix

Problems

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ using householder method convert the g matrix to tridiagonal

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sol^n $x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}(\text{sign of } a_{12})}{S}\right]$

$= \frac{1}{2}\left[1 + \frac{2(+1)}{\sqrt{13}}\right]$ $S = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{2^2+3^2}$

$= \sqrt{13}$

$= 0.77735$

$x_3 = \frac{a_{13}(\text{sign of } a_{12})}{2x_2 S} = \frac{3(+1)}{2 \times 0.7735 \times \sqrt{13}} = 0.535183$

2. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

$S = \sqrt{2^2+4^2} = 2\sqrt{5}$

$x_2^2 = \frac{1}{2}\left(1 + \frac{\cancel{2}(+1)}{\cancel{2}\sqrt{5}}\right) = 2x_2^2 = 1 + \frac{1}{\sqrt{5}}$

$= 1 - 2x_2^2 = -1/\sqrt{5}$

$x_3 = \frac{a_{13}(\text{sign of } a_{12})}{2x_2 S} = \frac{4}{2x_2 2\sqrt{5}} \Rightarrow 2x_2x_3 = \frac{2}{\sqrt{5}}$

Also

$1 - 2x_3^2 = 1 - 2(1 - x_2^2)$

$= 1 - 2 + \cancel{2}\frac{1}{\cancel{2}}\left(1 + \frac{1}{\sqrt{5}}\right)$

$= -\cancel{1} + \cancel{1} + \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$

Now, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2x_2^2 & -2x_2x_3 \\ 0 & -2x_2x_3 & 1-2x_3^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{5} & -2/\sqrt{5} \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

finally

$B = PAP$

$$= \begin{bmatrix} 1 & -4.472 & 0 \\ -4.472 & 2.6 & 1.2 \\ 0 & 1.2 & -0.6 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

sol^n $V = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$ $\qquad S = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}(\text{sign of } a_{12})}{S}\right]$$

$$= \frac{1}{2}\left[1 + \frac{2}{\sqrt{5}}\right] \Rightarrow 2x_2^2 = 1 + \frac{2}{\sqrt{5}}$$

$$1 - 2x_2^2 = -2/\sqrt{5}$$

$$x_3 = \frac{a_{13}(\text{sign of } a_{12})}{2x_2 S}$$

$$2x_2x_3 = \frac{(-1)(+1)}{S} = \frac{-1}{\sqrt{5}}$$

Now

$$1 - 2x_3^2 = 1 - 2(1 - x_2^2)$$

$$= 1 - 2 + 2x_2^2$$

$$= -1 + \cancel{2} \times \frac{1}{\cancel{2}}\left(1 + \frac{2}{\sqrt{5}}\right)$$

$$= -1 + 1 + \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

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$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2x_2^2 & -2x_2x_3 \\ 0 & -2x_2x_3 & 1-2x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Finaly: $B = PAP$

$$= \begin{bmatrix} 1 & -2.236 & 0 \\ -2.236 & -0.6 & -1.2 \\ 0 & -1.2 & 2.6 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

Sol $V = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad s = \sqrt{2^2+3^2} = \sqrt{13}$

$$x_2^2 = \frac{1}{2}\left(1 + \frac{a_{12}(\text{sign } a_{12})}{S}\right)$$

$$= \frac{1}{2}\left(1 + \frac{2}{\sqrt{13}}\right)$$

$$1 - 2x_2^2 = -2/\sqrt{13}$$

$$x_3 = \frac{a_{13}(\pm a_{12})}{2x_2 s} =$$

$$2x_2x_3 = \frac{3(+1)}{\sqrt{13}} = 3/\sqrt{13}$$

$$1-2x_3^2 = 1 - 2(1-x_2^2)$$

$$= 1 - 2 + 2x_2^2$$

$$= -\cancel{1} + \cancel{1} \frac{\cancel{2}}{\cancel{2}}\left(\cancel{1} + \frac{2}{\sqrt{13}}\right)$$

$$= 2/\sqrt{13}$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2/\sqrt{13} & -3/\sqrt{13} \\ 0 & -3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$$

$\therefore B = PAP$

$$= \begin{bmatrix} 1 & -3.605 & 0 \\ -3.605 & 0.0769 & -0.384 \\ 0 & -0.384 & 1.923 \end{bmatrix}$$

Root of a polynomial equation

An equation of the type
$P_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$ is
called a polynomial equation.

Ex:

$P_n(x) \Rightarrow x^3 + 2x^2 - 5x + 3 = 0$

$2x^2 - x + 1 = 0$

$2x^4 - 3x^2 + 2x + 5 = 0$

Birge Vieta Method

Consider, $P_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$
with initial approximation P_0

First iteration

P_0	a_0	a_1	a_2	a_{n-1}	a_n
	$\downarrow$	$+ P_0 b_0$	$+ P_0 b_1$	$+ P_0 b_{n-2}$	$+ P_0 b_{n-1}$
P_0	b_0	b_1	b_2	b_{n-1}	$\boxed{b_n}$
	$\downarrow$	$+ P_0 c_0$	$+ P_0 c_1$	.. $+ P_0 c_{n-2}$	$+ P_0 c_{n-1}$
	c_0	c_1	c_2	$\boxed{c_{n-1}}$	$\cancel{c_n}$

First approximation root

$$P_1 = P_0 - \frac{b_n}{c_{n-1}}$$

Repeat the above method by taking P_1 in place of P_0 to get second approximate late

$$P_2 = P_1 - \frac{b_n}{c_{n-1}}$$

Continue this process till you get the root of desired accuracy

If p is the required root then repeat the above method only first step as shown below

p	a_0	a_1	a_2	a_{n-1}	a_n
	$\cdot$	pb_0	pb_1	pb_{n-2}	pb_{n-1}
	b_0	b_1	b_2	b_{n-1}	$\boxed{b_n \simeq 0}$

then $Q_{n-1}(x) = b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1}$
which is called deflated polynomial.

Problems

1. Find the root of $2x^3 - 5x + 1 = 0$, perform 2 it' with the initial approximate root $P_0 = 0.5$. Find deflated poly'

sol^n

0.5	2	0	-5	1
	↓	1	0.5	-2.25
0.5	2	1	-4.5	$\boxed{-1.25}$
	↓	1	1	-1.75
	2	2	$\boxed{-3.5}$	-3

First approximation root

$$P_1 = P_0 - \frac{b_n}{C_{n-1}} = 0.5 - \frac{(-1.25)}{-3.35} = 0.1428$$

0.1428	2	0	-5	1
	↓	0.2856	0.0407	-0.7081
0.1428	2	0.2856	-4.9592	$\boxed{0.2918}$
	↓	0.2856	0.0815	
	2	0.5712	$\boxed{-4.8776}$	

2nd approximation root

$$P_2 = P_1 - \frac{b_n}{C_{n-1}} = 0.1428 - \frac{0.2918}{-4.8776} = 0.20262$$

$\therefore$ Required root $p = 0.20262$

Ade taxaancel, in genius noble prize (one i'll make it guys) ... how life?