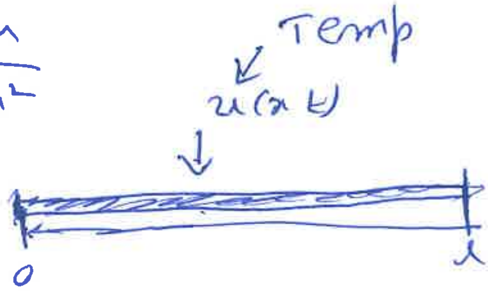
Partial Differential Equations

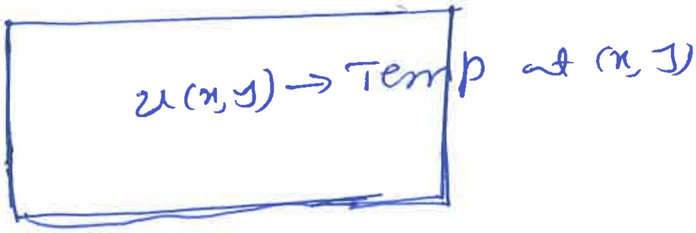
- → linear
- → 2^{nd} order

→ Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

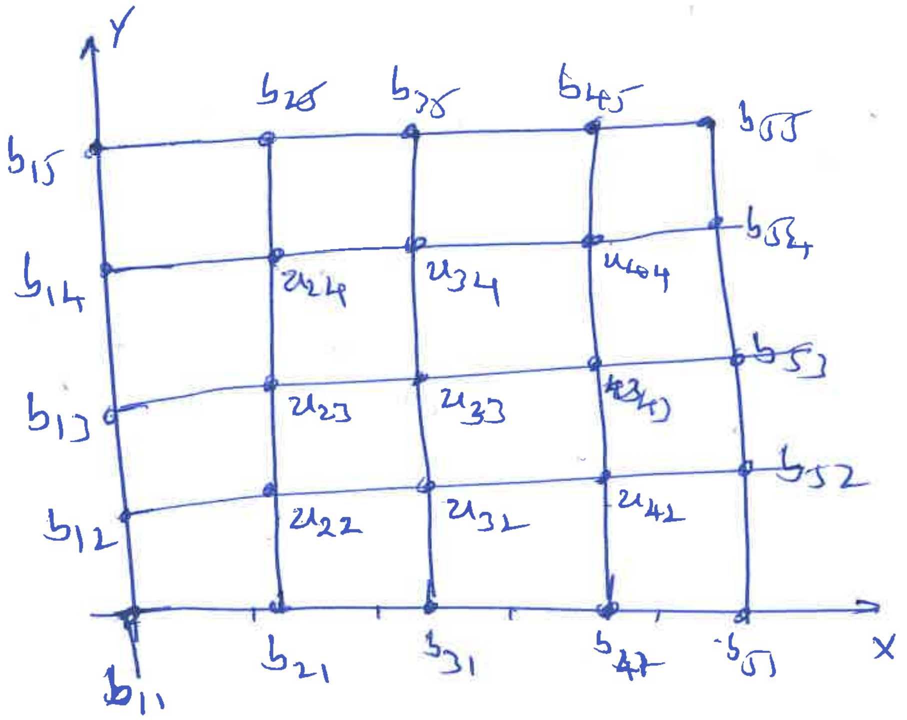
→ Heat equation: $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

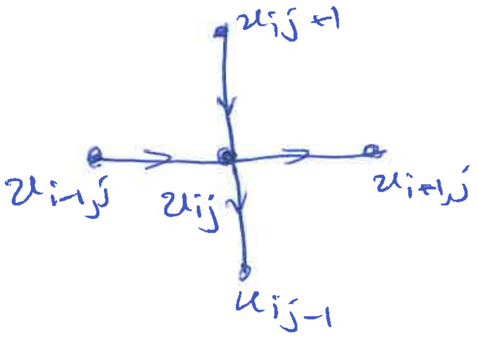


→ Laplace equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$



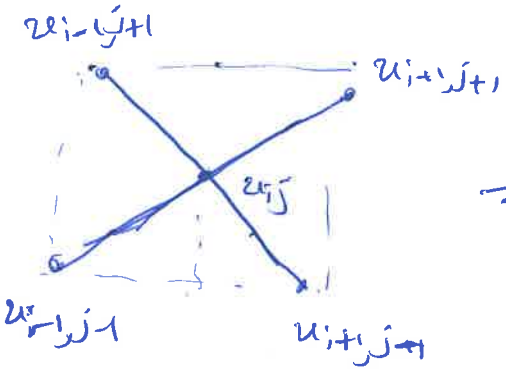
(I) Numerical solution of Laplace Equation





std five point formula

$$\Rightarrow u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{ij+1} + u_{ij-1}\right)$$



Diagonal five point formula

$$\Rightarrow u_{ij} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right)$$

Using above formulae find u_{33}, u_{24}, u_{44}
u_{34} u_{22}, u_{42}
u_{23} u_{43}

Improved values of u_{ij} are found by repeated application of std 5-point formula

$$u_{ij} = \frac{1}{4}\left[u_{i-1,j} + \overbrace{u_{i+1,j} + u_{i,j+1} + u_{i,j-1}}^{\text{latest available value}}\right]$$

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Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in Fig. 33.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or
$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad \text{...(2)}$$

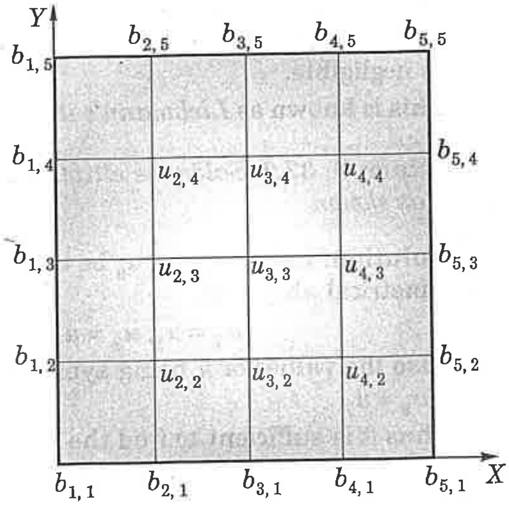


Fig. 33.3

This shows that the value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Fig. 33.4.

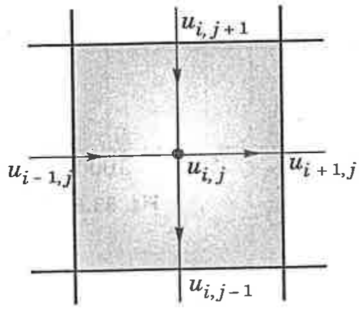


Fig. 33.4

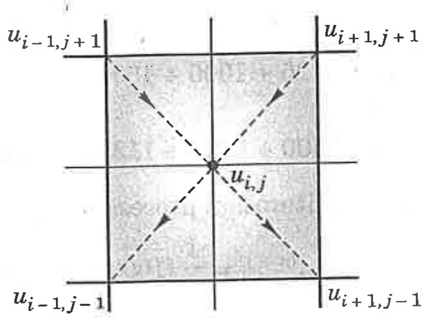


Fig. 33.5

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \quad \text{...(3)}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal 5-point formula** which is represented in Fig. 33.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

Now to find the initial values of u at the interior mesh points, we first use diagonal five point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})\,;\; u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{5,5} + u_{3,3})\,;\; u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.* $u_{2,3}$, $u_{3,4}$, $u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}),\; u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}),\; u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by repeated application of (2) in the form

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

This formula utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. This process is repeated till the difference of values in one round and the next becomes negligible.

This is known as *Liebmann's iteration process.*

Example 33.2. *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mash of Fig. 33.6 with boundary values as shown.* (Anna, 2012 ; Rohtak, 2005 ; V.T.U., 2005)

Solution. Let $u_1, u_2, ..., u_9$ be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB.

$\therefore$ $u_7 = u_1, u_8 = u_2, u_9 = u_3.$

Also the values of u being symmetrical about CD, $u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4 and u_5.

Now we find their initial value in the following order :

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \quad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \quad \text{(Std. formula)}$$

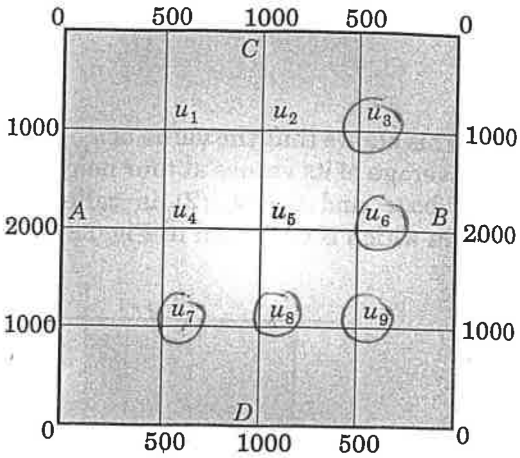


Fig. 33.6

We carry out the iteration process using the formulae :

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1032 + 1000 + 1500) = 1141$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1032) = 1391$$

$$u_5^{(1)} = \frac{1}{4}(1391 + 1391 + 1141 + 1141) = 1266$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + 1141 + 500 + 1391) = 1008$$

$$u_2^{(2)} = \frac{1}{4}(1008 + 1008 + 1000 + 1266) = 1069$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1266 + 1008 + 1008) = 1321$$

$$u_5^{(2)} = \frac{1}{4}(1321 + 1321 + 1069 + 1069) = 1195$$

Third iteration :

$$u_1^{(3)} = \frac{1}{4}(1000 + 1069 + 500 + 1321) = 973$$

$$u_2^{(3)} = \frac{1}{4}(973 + 973 + 1000 + 1195) = 1035$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1195 + 973 + 973) = 1288$$

$$u_5^{(3)} = \frac{1}{4}(1288 + 1288 + 1035 + 1035) = 1162$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4}(1000 + 1135 + 500 + 1288) = 956$$

$$u_2^{(4)} = \frac{1}{4}(956 + 956 + 1000 + 1162) = 1019$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1162 + 956 + 956) = 1269$$

$$u_5^{(4)} = \frac{1}{4}(1269 + 1269 + 1019 + 1019) = 1144$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4}(1000 + 1019 + 500 + 1269) = 947$$

$$u_2^{(5)} = \frac{1}{4}(947 + 947 + 1000 + 1144) \approx 1010$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1144 + 947 + 947) \approx 1260$$

$$u_5^{(5)} = \frac{1}{4}(1260 + 1260 + 1010 + 1010) = 1135$$

Similarly, $u_1^{(6)} = 942,\ u_2^{(6)} = 1005,\ u_4^{(6)} = 1255,\ u_5^{(6)} = 1130$

$u_1^{(7)} = 940,\ u_2^{(7)} = 1003,\ u_4^{(7)} = 1253,\ u_5^{(7)} = 1128$

$u_1^{(8)} = 939,\ u_2^{(8)} = 1002,\ u_4^{(8)} = 1252,\ u_5^{(8)} = 1127$

$u_1^{(9)} = 939,\ u_2^{(6)} = 1001,\ u_4^{(9)} = 1251,\ u_5^{(9)} = 1126$

Thus there is negligible difference between the values obtained in the eighth and ninth iterations.
Hence $u_1 = 939$, $u_2 = 1001$, $u_4 = 1251$ and $u_5 = 1126$.

Example 33.3. *Given the values of u(x, y) on the boundary of the square in the Fig. 33.7, evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure.*

(V.T.U., 2013 ; Bhopal, 2009 ; Madras, 2003)

Solution. To get the initial values of u_1, u_2, u_3, u_4, we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

We carry out the successive iterations, using the formulae

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[2000 + u_4^{(n)} + u_1^{(n+1)} + 500]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438$$

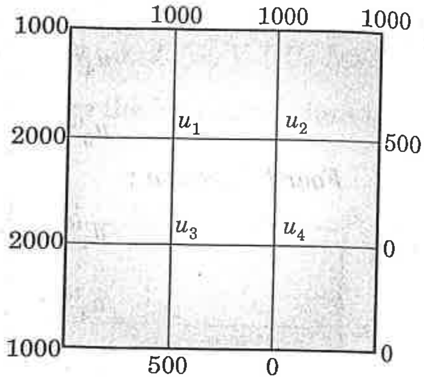


Fig. 33.7

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032$$

$$u_4^{(2)} = \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454$$

Third iteration : (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(2000 + 782 + 1000 + 1032) \approx 1204$$

$$u_2^{(3)} = \frac{1}{4}(1204 + 500 + 1000 + 454) \approx 789$$

$$u_3^{(3)} = \frac{1}{4}(2000 + 454 + 1204 + 500) \approx 1040$$

$$u_4^{(3)} = \frac{1}{4}(1040 + 0 + 789 + 0) \approx 458$$

Similarly, $u_1^{(4)} \approx 1207,\ u_2^{(4)} \approx 791,\ u_3^{(4)} \approx 1041,\ u_4^{(4)} = 458$

and $u_1^{(5)} = 1208,\ u_2^{(5)} = 791.5,\ u_3^{(5)} = 1041.5,\ u_4^{(5)} = 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

Example 33.4. *Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that (Fig. 33.8).* (V.T.U., 2009)

Solution. We first find the initial values in the following order :

$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5$ (Std. formula)

$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4$ (Diag. formula)

$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28$ (Diag. formula)

$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15$ (Diag. formula)

$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65$ (Diag. formula)

$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55$ (Std. formula)

$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52$ (Std. formula)

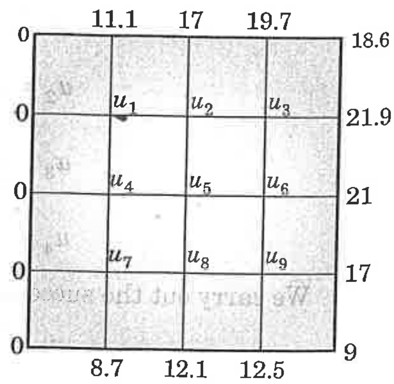


Fig. 33.8

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \quad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \quad \text{(Std. formula)}$$

Now we carry out the iteration process using the Standard formula :

$$u_1^{(n+1)} = \frac{1}{4}\left[(0 + 11.1 + u_4^{(n)} + u_2^{(n)})\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)})\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[(u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9)\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 19.7 + u_7^{(n)} + u_5^{(n)})\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[(u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)})\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[(u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21)\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + (u_4^{(n+1)} + 8.7 + u_8^{(n)})\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[(u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)})\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[(u_8^{(n+1)} + u_6^{(n)} + 12.8 + 17)\right]$$

First iteration (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4}\left(0 + 11.1 + u_4^{(o)} + u_2^{(o)}\right) = \frac{1}{4}\ (0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4}\ (7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4}\ (13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4}\ (0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4}\ (6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4}\ (11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4}\ (0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4}\ (6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4}\ (11.06 + 16.09 + 12.8 + 17) = 12.238$$

Second iteration (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}\ (0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4}\ (7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4}\ (13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4}\ (0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4}(6.58 + 13.64 + 11.06 + 16.09) = 11.84$$
$$u_6^{(2)} = \frac{1}{4}(11.84 + 17.83 + 14.24 + 21) = 16.23$$
$$u_7^{(2)} = \frac{1}{4}(0 + 6.58 + 8.7 + 11.06) = 6.58$$
$$u_8^{(2)} = \frac{1}{4}(6.58 + 11.84 + 12.1 + 14.24) = 11.19$$
$$u_9^{(2)} = \frac{1}{4}(11.19 + 16.23 + 12.8 + 17) = 14.30$$

Third iteration (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(0 + 11.1 + 6.58 + 13.64) = 7.83$$
$$u_2^{(3)} = \frac{1}{4}(7.83 + 17 + 11.84 + 17.83) = 13.637$$
$$u_3^{(4)} = \frac{1}{4}(13.63 + 19.7 + 16.23 + 21.9) = 17.86$$
$$u_4^{(3)} = \frac{1}{4}(0 + 7.83 + 6.58 + 11.84) = 6.56$$
$$u_5^{(3)} = \frac{1}{4}(6.56 + 13.63 + 11.19 + 16.23) = 11.90$$
$$u_6^{(3)} = \frac{1}{4}(11.90 + 17.86 + 14.30 + 21) = 16.27$$
$$u_7^{(3)} = \frac{1}{4}(0 + 6.56 + 8.7 + 11.19) = 6.61$$
$$u_8^{(3)} = \frac{1}{4}(6.61 + 11.90 + 12.1 + 14.30) = 11.23$$
$$u_9^{(3)} = \frac{1}{4}(11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly, $u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94, u_6^{(4)} = 16.28,$
$u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$
$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95, u_6^{(5)} = 16.29,$
$u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$

33.6 SOLUTION OF POISSON'S EQUATION*

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{f(x, y)} \qquad ...(1)$$

This is an *elliptic equation* which can be solved numerically at the interior mesh points of a square network when the boundary values are known. The standard 5-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,\,j+1} + u_{i,\,j-1} - 4u_{i,j} = h^2 f(ih, jh) \qquad ...(2)$$

By applying (2) at each mesh-point, we arrive at linear equations in the pivotal values i, j. These equations can be solved by Gauss-Seidal iteration method (p. 938).

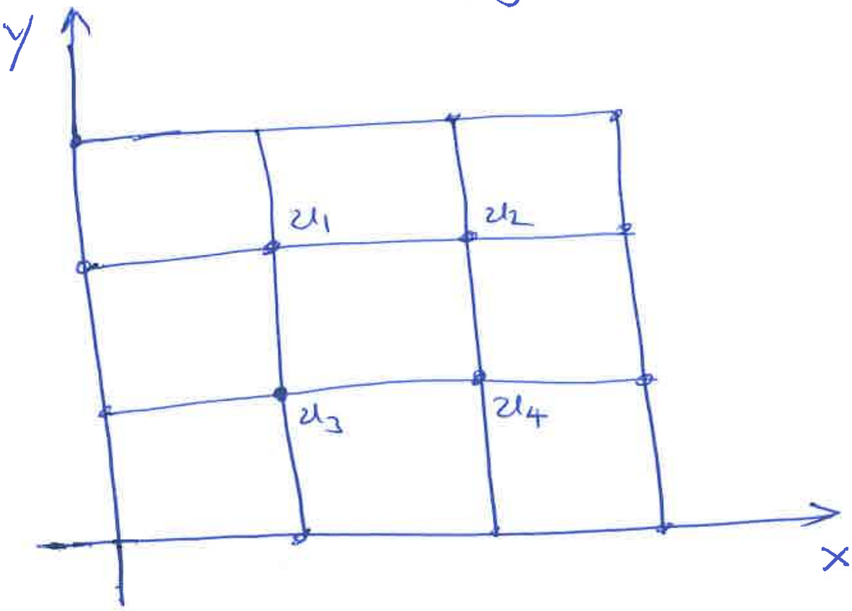
Example 33.5. *Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$.* (Anna, 2005)

Solution. Here $h = 1/3$ $u_{i,j-1}$ $u_{i,j}$ (Fig. 33.9)

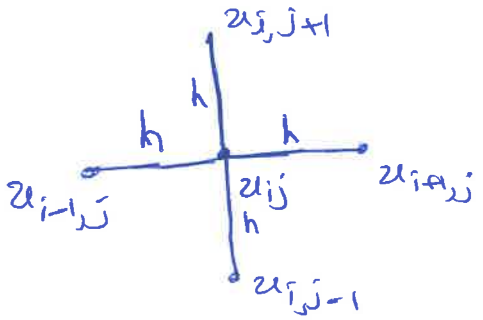
* See p. 882.

Poisson Equation

Equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$



Standard 5-point formula



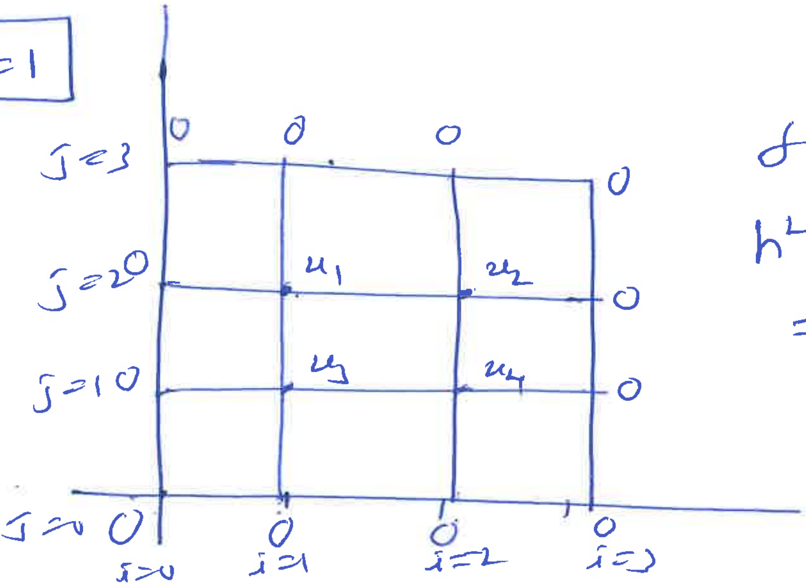
$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2 f(ih, jh)$$

$$u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i,j+1} + u_{i+1,j} + u_{i,j-1}\right) + \frac{h^2}{4} f(ih, jh)$$

$$u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(ih, jh)\right)$$

Ex (2) Solve $\nabla^2 u = -10(x^2+y^2+10)$ over square mesh with sides $x=0=y$, $x=3=y$ with $u=0$ on the boundary & mesh length $=1$.

Sol^n $\boxed{h=1}$



$f(x,y) = -10(x^2+y^2+10)$

$h^2 f(ih, jh)$

$= -10(i^2+j^2+10)$

$u_1 = \frac{1}{4}(0+u_2+0+u_3+150) \Rightarrow u_1 = \frac{1}{4}(u_2+u_3+150)$

$u_2 = \frac{1}{4}(u_1+0+0+u_4+10\times 18) \Rightarrow u_2 = \frac{1}{4}(u_1+u_4+180)$

$u_3 = \frac{1}{4}(0+u_4+u_1+0+10\times 12) \Rightarrow u_3 = \frac{1}{4}(u_1+u_4+120)$

$u_4 = \frac{1}{4}(u_3+0+u_2+0+10\times 15) \Rightarrow u_4 = \frac{1}{4}(u_2+u_3+150)$

$u_1 = u_4$

$\therefore u_2 = \frac{1}{2}(u_1+90)$

$u_3 = \frac{1}{2}(u_1+60)$

$\therefore u_1 = \frac{1}{4}(u_2+u_3+150)$

$4u_1 = \frac{u_1}{2} + 45 + \frac{u_1}{2} + 30 + 150$

$3u_1 = 225 \quad \therefore u_1 = 75$

$u_2 = 82.5$

$u_4 = 75$

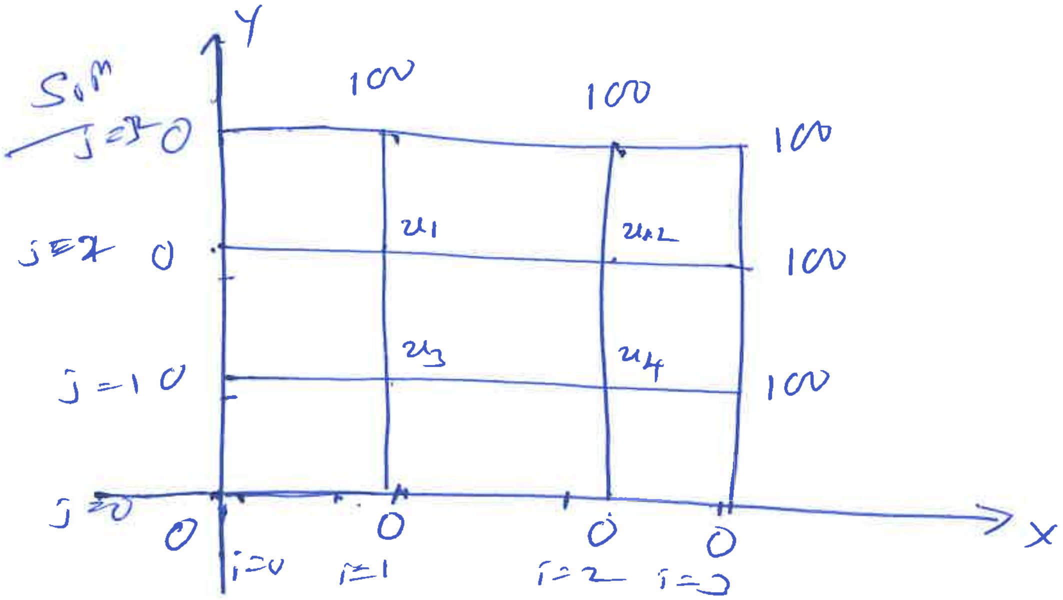
$u_3 = 67.5$

Ex (1) Solve $u_{xx}+u_{yy} = -81xy$, $0<x<1$, $0<y<1$ given

$u(0,y)=0$, $u(x,0)=0$, $u(1,y)=100$, $u(x,1)=100$

& $h = 1/3$

Sol^n



$f(x,y) = -81xy$ $\quad h^2 f(ih, jh) = h^2 \times -81 h^2 ij = -ij$

$u_1 = \frac{1}{4}(0 + u_2 + 100 + u_3 + 1\times 2) = \frac{1}{4}(u_2 + u_3 + 102)$

$u_2 = \frac{1}{4}(u_1 + 100 + 100 + u_4 + 2\times 2) = \frac{1}{4}(u_1 + u_4 + 204)$

$u_3 = \frac{1}{4}(0 + u_4 + u_1 + 0 + 1) = \frac{1}{4}(u_1 + u_4 + 1)$

$u_4 = \frac{1}{4}(u_3 + 100 + u_2 + 0 + 2) = \frac{1}{4}(u_2 + u_3 + 102)$

$\boxed{u_1 = u_4}$

$\therefore u_2 = \frac{1}{2}(u_1 + 102)$

$u_3 = \frac{1}{4}(2u_1 + 1)$

$\therefore 4u_1 = \frac{u_1}{2} + 51 + \frac{u_1}{2} + \frac{1}{4} + 102$

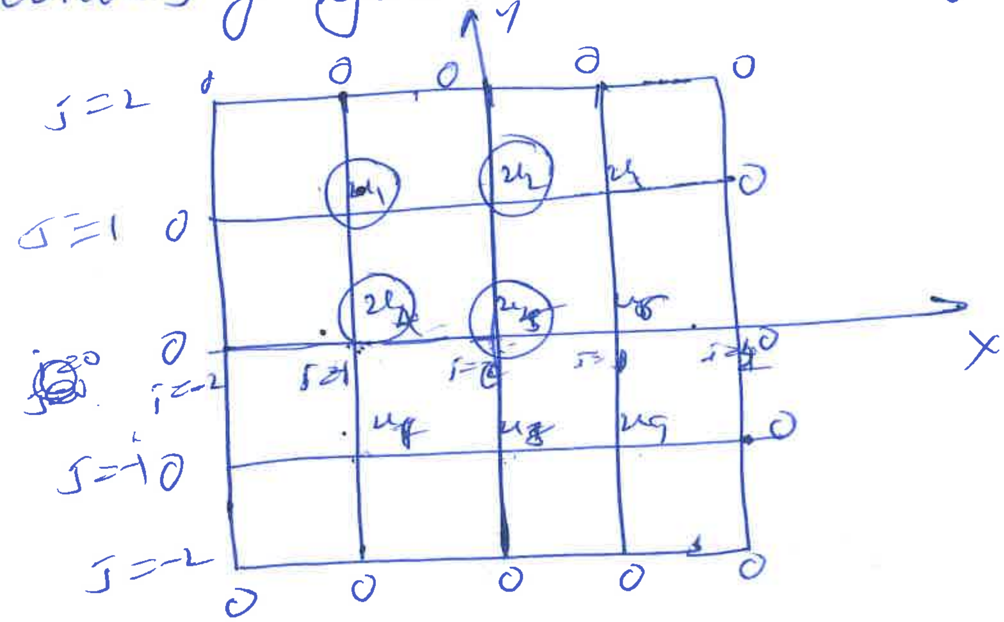
$3u_1 = 153.25$

$\therefore \boxed{u_1 = 51.08}$

$\therefore u_2 = 76.54 \quad u_3 = 25.79$

$u_4 = 51.08$

Ex 3 Solve $\nabla^2 u = 8x^2y^2$ with $u(x,y)=0$ on the boundary of given mesh & mesh length $=1$



Sol^n $h=1$, $f(x,y) = 8x^2y^2$ $\therefore h^2 f(ih, jh) = 8i^2j^2$

$u_1 = u_7 = u_3 = u_9$, $u_2 = u_8$, $u_4 = u_6$

$u_1 = \frac{1}{4}(0 + u_2 + 0 + u_4 - 8\times1\times1)$ $\therefore u_1 = \frac{1}{4}(u_2 + u_4 - 8)$

$u_2 = \frac{1}{4}(u_1 + u_3 + 0 + u_5 - 8\times0)$ $u_2 = \frac{1}{4}(2u_1 + u_5)$

$u_4 = \frac{1}{4}(0 + u_5 + u_1 + u_7 - 8\times0)$ $u_4 = \frac{1}{4}(2u_1 + u_5)$ $\Big\}$ $u_2 = u_4$

$u_5 = \frac{1}{4}(u_4 + u_6 + u_2 + u_8 - 8\times0)$ $u_5 = \frac{1}{4}(2u_4 + 2u_2)$

$\therefore \cancel{u_1} =$

$\Rightarrow u_1 = \frac{1}{2}(u_2 - 4)$ $\Rightarrow$

$u_5 = u_2$

$2u_1 = \frac{2}{3}u_1 - 4$

$\frac{4}{3}u_1 = -4$

$u_1 = -3$

$u_2 = -2$

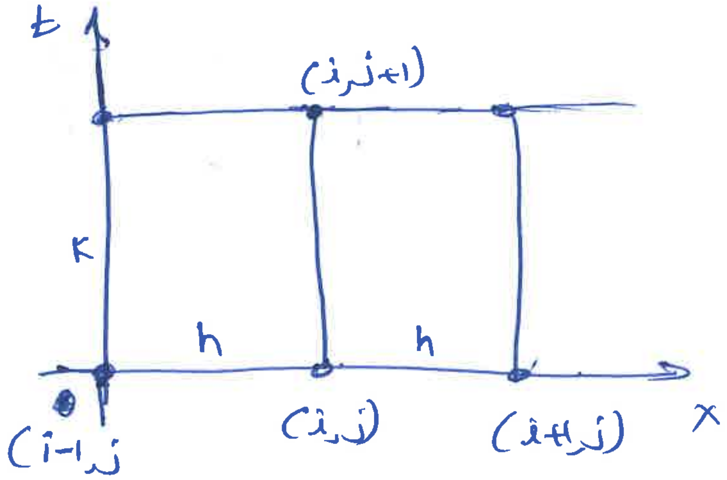
$4u_2 = 2u_1 + u_2$

$3u_2 = 2u_1$

$u_2 = \frac{2}{3}u_1$

Heat Equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{K}{s\rho}$$



(I)

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1-2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}, \quad \alpha = \frac{kc^2}{h^2}$$

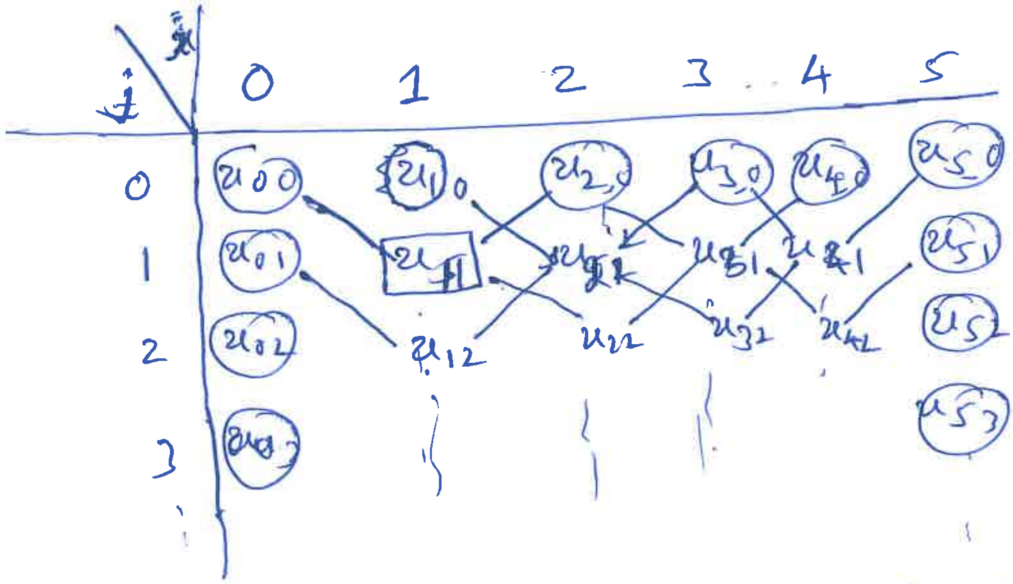
(I) For $\alpha = 1/2$

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

$\alpha \to 0$

$i=1, j=0$

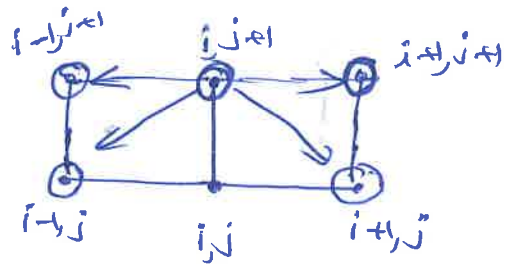
$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right]$



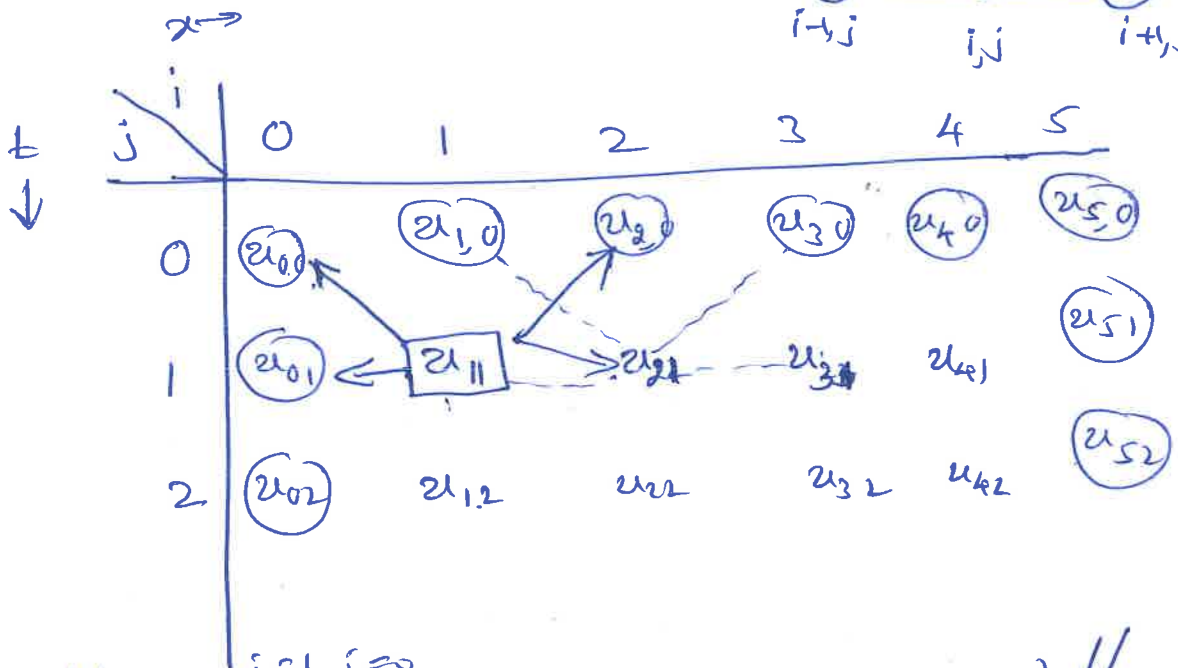
// Bendre Schmidt ~~method~~ formula //

(II) Crank Nicolson Method

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$$



$r = 1$



$i=1, j=0$

$$u_{11} = \frac{1}{4}\left(u_{01} + u_{21} + u_{00} + u_{20}\right)$$

Ex ① Solve $\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$ using Bendre Schmidt method

given $u(x,0) = 4x - x^2$ $u(0,t) = u(4,t) = 0$ at the

points $x = i$, $i = 0,1,2,3,4$, $t = j$, $j = 0,1,2,3,4$

Solⁿ

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1·5	2	1·5	0
3	0	1	1·5	1	0
4	0	0·75	1	0·75	0

Ex (2) Solve $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$ &

$u(x,0) = 5\sin\pi x$, $0 \le x \le 1$, using Bendre Schmidt formula (Take $h = 0.2$)

Solⁿ $c^2 = 1$, $h = 0.2$, $\alpha = \frac{kc^2}{h^2} \Rightarrow k = \frac{h\alpha}{c^2} = \frac{0.2 \times 1/2}{1} = 0.02$

t \ x	0	0.2	0.4	0.6	0.8	1
0	0	0.588	0.951	0.951	0.588	0
0.02	0	0.476	0.770	0.770	0.476	0
0.04	0	0.385	0.623	0.623	0.385	0
0.06	0	0.312	0.504	0.504	0.312	0
0.08	0	0.252	0.408	0.408	0.252	0
0.1	0	0.204	0.33	0.33	0.204	0

Ex (1) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 5$, $t \geq 0$, given $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$. Compute 'u' for the time step with $h = 1$ by Crank Nicholson method

Solⁿ $\boxed{c^2 = 1}$ $\boxed{h = 1}$ & $\boxed{k = 1}$ *

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Formula:

$$u_1 = \frac{1}{4}(0 + 0 + 20 + u_2) \Rightarrow 4u_1 - u_2 = 20 \quad \text{--- (1)}$$

$$u_2 = \frac{1}{4}(20 + 20 + u_1 + u_3) \Rightarrow u_1 - 4u_2 + u_3 = -40 \quad \text{--- (2)}$$

$$u_3 = \frac{1}{4}(20 + 20 + u_2 + u_4) \Rightarrow u_2 - 4u_3 + u_4 = -40 \quad \text{--- (3)}$$

$$u_4 = \frac{1}{4}(20 + 100 + u_3 + 100) \Rightarrow u_3 - 4u_4 = -220 \quad \text{--- (4)}$$

$(1) - 4(2) \Rightarrow 15u_2 - 4u_3 = 180 \Rightarrow u_2 = 20.29$

$4(3) + (4) \Rightarrow 4u_2 - 15u_3 = -380$ $\qquad u_3 = 30.72$

From (1) $u_1 = 10.05$

$u_4 = 62.68$

Ex ④ Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(0, t) = 0$, $u(4, t) = 0$ &

$u(x, 0) = \frac{x}{3}(16 - x^2)$. Find $u(i, j)$; when $i = 0, 1, 2, 3, 4$,

$j = 0, 1, 2$

using Crank Nicholson method

Sol^n $\boxed{c^2 = 1}$

$i = 0$

$t \backslash x$	0	1	2	3	4
0	0	5	8	7	0
1	0	u_1 3.14	u_2 4.57	u_3 3.14	0
2	0	u_4	u_5	u_6	0

we have

$$u_1 = \frac{1}{4}(0 + 8 + 0 + u_2) \Rightarrow 4u_1 - u_2 = 8$$

$$u_2 = \frac{1}{4}(5 + u_1 + 7 + u_3) \Rightarrow u_1 - 4u_2 + u_3 = -12$$

$$u_3 = \frac{1}{4}(8 + u_2 + 0 + 0) \Rightarrow u_2 - 4u_3 = -8$$

$$\Bigg\} \quad u_1 = 3.14, \quad u_2 = 4.57, \quad u_3 = 3.14$$

Next

$$u_4 = \frac{1}{4}(4.57 + u_5) \Rightarrow 4u_4 - u_5 = 4.57$$

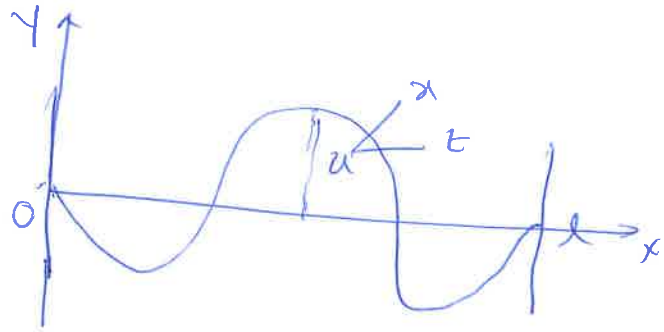
$$u_5 = \frac{1}{4}(3.14 + u_4 + 3.14 + u_6) \Rightarrow u_4 - 4u_5 + u_6 = -6.28$$

$$u_6 = \frac{1}{4}(4.57 + u_5 + 0 + 0) \Rightarrow u_5 - 4u_6 = -4.57$$

$$\Rightarrow u_4 = 1.75 \qquad u_6 = 1.75$$

$$u_5 = 2.45$$

Wave Equation

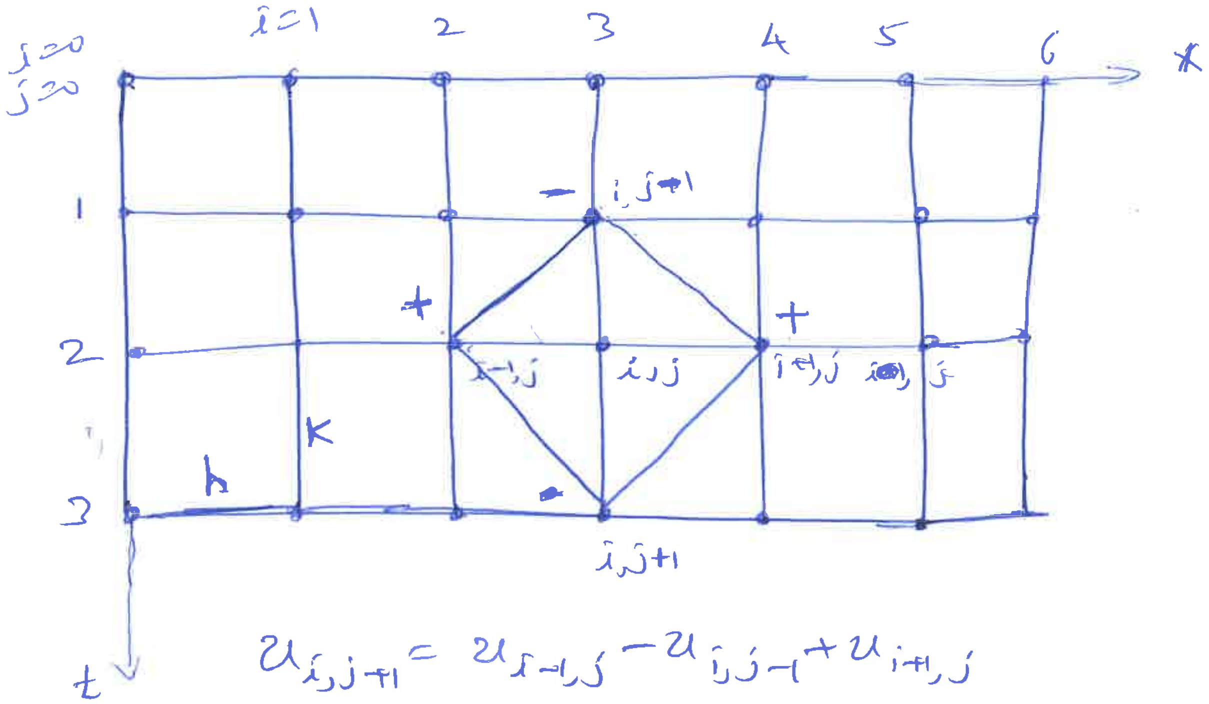


$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Initial conditions:

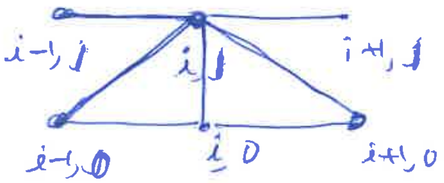
$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le 1$$

$$u(0,t) = \phi(t), \quad u(1,t) = \psi(t)$$



$$u_{i,j+1} = u_{i-1,j} - u_{i,j-1} + u_{i+1,j}$$

$$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right) + g(ih) \quad (*)$$



Ex ① Solve $u_{tt} = 16\, u_{xx}$, ~~taking~~ $h=1$ upto $t = 1.25$, given $u(0,t) = u(5,t) = 0$

$u_t(x,0) = 0$, $u(x,0) = x^2(5-x)$

Sol^n

$k = h/c = \frac{1}{4}$

j	t \ x	0	1	2	3	4	5
	$i \to$	0	1	2	3	4	5
0	0	0	4	12	18	16	0
1	0.25	0	6	11	14	9	0
2	0.5	0	7	8	2	−2	0
3	0.75	0	2	−2	−8	−7	0
4	1	0	−9	−14	−11	−6	0
5	1.25	0	−16	−18	−12	−4	0

$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right)$ | $u_{i,j+1} = u_{i-1,j} - u_{i,j-1} + u_{i+1,j}$

Ex ② solve $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, $u=0$ at $x=0$, $t>0$ &

$u=0$ at $x=4$, $t>0$ & $u=x(4-x)$ & $\frac{\partial u}{\partial t}=0$ at $t=0$ &

$0 \le x \le 4$ ($h=1$, $k=\frac{1}{2}$)
upto $t=2$

Solⁿ

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
½	0	2	3	2	0
1	0	0	0	0	0
3/2	0	-2	-3	-2	0
2	0	-3	-4	-3	0

Ex ③ Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, subject to $u(x,0) = \sin \pi x$,

$0 \le x \le 1$, $u(0,t)=0$, $u(1,t)=0$, $t>0$ (Take $h=0.2$)

Solⁿ $\boxed{c^2=1}$ $h=0.2$, $k=h/c=0.2$

$t \backslash x$	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.2	0	0.4756	0.7695	0.7695	0.4756	0
0.4	0	0.1817	0.294	0.294	0.1817	0
0.6	0	-0.1816	-0.2938	-0.2938	-0.1816	0
0.8	0	-0.4755	-0.7694	-0.7694	-0.4755	0
1.0	0	-0.5878	-0.9511	-0.9511	-5878	0