8 Tests of Hypotheses

Every statistical investigation aims at collecting information about some aggregate or collection of individuals or of their attributes, rather than the individuals themselves. In statistical language, such a collection is called a *population* or *universe*. For example, we have the population of products turned out by a machine, of lives of electric bulbs manufactured by a company etc. A population is finite or infinite, according as the number of elements is finite or infinite. In most situations, the population may be considered infinitely large. A finite subset of a population is called *a sample* and the process of selection of such samples is called *sampling*. The basic objective of the theory of sampling is to draw inference about the population, using the information of the sample.

Parameters and Statistics

Generally in statistical investigations, our ultimate interest will lie in one or more characteristics possessed by the members of the population. If there is only one characteristic of importance, it can be assumed to be a variable x. If x_i be the value of x for the ith member of the sample, then $(x_1, x_2, \ldots, x_n)$ are referred to as sample observations. Our primary interest will be to know the values of different statistical measures such as mean and variance of the population distribution of x. Statistical measures calculated on the basis of population values of x are called *parameters*. Corresponding measures computed on the basis of sample observations are called *statistics*.

Sampling Distribution

If a number of samples, each of size n (i.e. each containing n elements) are drawn from the same population and if for each sample the value of some statistic, say, mean is calculated, a set of values of the statistic will be obtained.

Note: *The values of the statistic will normally vary from one sample to another, as the values of the population members included in different samples, though drawn from the same population, may be different. These differences in the values of a statistic are said to be due to sampling fluctuations.*

If the number of samples is large, the values of the statistic may be classified in the form of a frequency table. The probability distribution of the statistic that would be obtained if the number of samples, each of same size, were infinitely large is called the *sampling distribution* of the statistic. If we adopt random sampling technique that is, the most popular and frequently used method of sampling (the discussion of which is beyond the scope of this book), the nature of the sampling distribution of a statistic can be obtained theoretically, using the theory of probability, provided the nature of the population distribution is known.

Like any other distribution, a sampling distribution will have its mean, standard deviation and moments of higher order. The standard deviation of the sampling distribution of a statistic is of particular importance in tests of hypotheses and is called *the standard error* of the statistic.

Estimation and Testing of Hypotheses

In sampling theory, we are primarily concerned with two types of problems:

(i) Some characteristic or feature of the population in which we are interested may be completely unknown to us and we may like to make a guess about this characteristic entirely on the basis of a random sample drawn from the population. This type of problem is known as the problem of *estimation*.

(ii) Some information regarding the characteristic or feature of the population may be available to us and we may like to know whether the information is tenable (or can be accepted) in the light of the random sample drawn from the population and if it can be accepted, with what degree of confidence it can be accepted. This type of problem is known as the problem of *testing of hypotheses*.

Tests of Hypotheses and Tests of Significance

When we attempt to make decisions about the population on the basis of sample information, we have to make assumptions or guesses about the nature of the population involved or about the value of some parameter of the population. Such assumptions, which may or may not be true, are called *statistical hypotheses*. Very often, we set up a hypothesis which assumes that there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics. Such a hypothesis of no difference is called a *null hypothesis* and is denoted by H_0. A hypothesis that is different from (or complementary to) the null hypothesis is called an *alternative hypothesis* and is denoted by H_1. A procedure for deciding whether to accept or to reject a null hypothesis (and hence to reject or to accept the alternative hypothesis respectively) is called the *test of hypothesis*.

If θ_0 is a parameter of the population and θ is the corresponding sample statistic, usually there will be some difference between θ_0 and θ since θ is based on sample observations and is different for different samples. Such a difference which is caused due to sampling fluctuations is called *insignificant difference*. The difference that arises either because the sampling procedure is not purely random or that the sample has not been drawn from the given population is known as *significant difference*. This procedure of testing whether the difference between θ_0 and θ is significant or not is called the *test of significance*.

Critical Region and Level of Significance

If we are prepared to accept that the difference between a sample statistic and the corresponding parameter is significant when the sample statistic lies in a certain region or interval, then that region is called the *critical region* or *region of rejection*. The region complementary to the critical region is called the *region of acceptance*.

In the case of large samples, the sampling distributions of many statistics tend to become normal distributions. If t is a statistic in large samples, then t follows a normal distribution with mean E (t), which is the corresponding population parameter, and SD equal to SE (t). Hence $Z = \dfrac{t - E(t)}{\mathrm{SE}(t)}$ is a standard normal variate i.e., Z (called the *test statistic*) follows a normal distribution with mean zero and SD unity.

It is known from the study of normal distribution, that the area under the standard normal curve between $t = -1.96$ and $t = +1.96$ is 0.95. Equivalently the area under the general normal curve of t between $[E(t) - 1.96\ \mathrm{SE}(t)]$ and $[E(t) + 1.96\ \mathrm{SE}(t)]$ is 0.95. In other words, 95% of the values of t will lie between $[E(t) \mp 1.96\ \mathrm{SE}(t)]$ or only 5% of values of t will lie outside this interval.

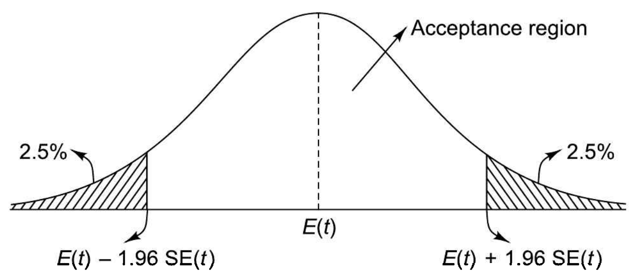

Fig. 8.1

If we are prepared to accept that the difference between t and E (t) is significant when t lies in either of the regions $[-\infty, E(t) - 1.96\mathrm{SE}(t)]$ and $[E(t) + 1.96\mathrm{SE}(t), \infty]$ then these two regions constitute the critical region of t.

The probability that a random value (α) of the statistic lies in the critical region is called the *level of significance* (LOS) and is usually expressed as a percentage. In the other words, the total area of the critical region expressed as α% is the LOS.

From the study of normal distributions, it is known that

$$P\ \{E(t) - 1.96\ \mathrm{SE}(t) < t < E(t) + 1.96\ \mathrm{SE}(t)\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\text{SE}(t)}\right| < 1.96\right\} = 0.95$$

i.e. $$P\{|Z| > 1.96\} = 0.05 \quad \text{or} \quad 5\%$$

Thus when t lies in either of the two regions constituting the critical region given above, the LOS is 5%.

Note *The level of significance can also be defined as the maximum probability with which we are prepared to reject H_0 when it is true. In other words, the total area of the region of rejection expressed as a percentage is called the level of significance.*
(The specification of critical region and the choice of LOS will depend upon the nature of the problem and is a matter of judgement for those who carry out the investigation. Usually the LOS are taken as 5, 2 or 1%.)

Errors in Testing of Hypotheses

The LOS is fixed by the investigator and as such it may be fixed at a higher level by his wrong judgement. Because of this, the region of rejection becomes larger and the probability of rejecting a null hypothesis, when it is true, becomes greater. The error committed in rejecting H_0, when it is really true, is called *Type I error*. This is similar to a good product being rejected by the consumer and hence Type I error is also known as *producer's risk*. The error committed in accepting H_0, when it is false, is called *Type II error*. As this error is similar to that of accepting a product of inferior quality, it is also known as *consumer's risk*.

The probabilities of committing Type I and II errors are denoted by α and β respectively. It is to be noted that the probability α of committing Type I error is the LOS.

One-Tailed and Two-Tailed Tests

If θ_0 is a population parameter and θ is the corresponding sample statistic and if we set up the null hypothesis $H_0 : \theta = \theta_0$, then the alternative hypothesis which is complementary to H_0 can be any one of the following:

(i) $H_1 : \theta \neq \theta_0$, i.e. $\theta > \theta_0$ or $\theta < \theta_0$
(ii) $H_1 : \theta > \theta_0$
(iii) $H_1 : \theta < \theta_0$

H_1 given in (i) is called a two-tailed alternative hypothesis, whereas H_1 given in (ii) is called a right-tailed alternative hypothesis and H_1 given in (iii) is called a left-tailed alternative hypothesis.

When H_0 is tested while H_1 is a one-tailed alternative (right or left), the test of hypothesis is called *a one-tailed test*.

When H_0 is tested while H_1 is two-tailed alternative, the test of hypothesis is called a *two-tailed test*.

The application of one-tailed or two-tailed test depends upon the nature of the alternative hypothesis. The choice of the appropriate alternative hypothesis depends on the situation and the nature of the problem concerned.

Critical Values or Significant Values

The value of the test statistic z for which the critical region and acceptance region are separated is called the *critical value* or the *significant value* of z and denoted by z_α, when α is the LOS. It is clear that the value of z_α depends not only on α but also on the nature of alternative hypothesis.

When $z = \dfrac{t - E(t)}{\text{SE}(t)}$, we have seen that

$P(|z| < 1.96) = 95\%$ and $P(|z| > 1.96) = 5\%$.

Thus $z = \pm 1.96$ separates the critical region and the acceptance region at 5% LOS for a two-tailed test. That is the critical values of z in this case are ± 1.96.

In general, the critical value z_α for the LOS α is a given by the equation $P(|z| > z_\alpha) = \alpha$ for a two-tailed test, by the equation $P(z > z_\alpha) = \alpha$ for the right-tailed test and by the equation.

$P(z < -z_\alpha) = \alpha$ for the left-tailed test. These equations are solved by using the normal tables.

Note *If z_α is the critical value of z corresponding to the LOS α in the right-tailed test, then $P(z > z_\alpha) = \alpha$.*

By symmetry of the standard normal distribution followed by z, $P(z < -z_\alpha) = \alpha$.

$$\therefore \quad P(|z| > z_\alpha) = P\{(z > z_\alpha) + (z < -z_\alpha)\}$$
$$= P\{z > z_\alpha) + P(z < -z_\alpha)$$
$$= 2\alpha$$

That is z_α is the critical value of z corresponding to the LOS 2α.

Thus the critical value of z for a single-tailed test (right or left) at LOS α is the same as that for a two-tailed test of LOS 2α.

The critical values for some standard LOSs are given in the following table both for two-tailed and one-tailed tests.

Nature of test	*LOS* 1% (0.01)	2% (0.02)	5% (0.05)	10% (0.1)
Two-tailed	$\|z_\alpha\| = 2.58$	$\|z_\alpha\| = 2.33$	$\|z_\alpha\| = 1.96$	$\|z_\alpha\| = 1.645$
Right-tailed	$z_\alpha = 2.33$	$z_\alpha = 2.055$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left-tailed	$z_\alpha = -2.33$	$z_\alpha = -2.055$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Procedure for Testing of Hypothesis

1. Null hypothesis H_0 is defined.
2. Alternative hypothesis H_1 is also defined after a careful study of the problem and also the nature of the test (whether one-tailed or two-tailed) is decided.

3. LOS α is fixed or taken from the problem if specified and z_α is noted.
4. The test-statistic $z = \dfrac{t - E(t)}{\mathrm{SE}(t)}$ is computed.
5. Comparison is made between $|z|$ and z_α. If $|z| < z_\alpha$, H_0 is accepted or H_1 is rejected, i.e. it is concluded that the difference between t and $E(t)$ is not significant at α% LOS.

On the other hand, if $|z| > z_\alpha$, H_0 is rejected or H_1 is accepted, i.e. it is concluded that the difference between t and $E(t)$ is significant at α % LOS.

Interval Estimation of Population Parameters

It was pointed out that the objective of the theory of sampling is to estimate population parameters with the help of the corresponding sample statistics. Estimation of a parameter by single value is referred to as *point estimation*, the study of which is beyond the scope of this book. However, an alternative procedure is to give an interval within which the parameter may be supposed to lie. This is called *interval estimation*. The interval within which the parameter is expected to lie is called the *confidence interval* for that parameter. The end points of the confidence interval are called *confidence limits* or *fiducial limits*.

We have already seen that

$$P\{|z| \le 1.96\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\mathrm{SE}(t)}\right| \le 1.96\right\} = 0.95$$

i.e. $P\{t - 1.96\mathrm{SE}(t) \le E(t) \le t + 1.96\mathrm{SE}(t)\} = 0.95$

This means that we can assert, with 95% confidence, that the parameter $E(t)$ will lie between $t - 1.96$ SE(t) and $t + 1.96$SE(t). Thus $\{t - 1.96\mathrm{SE}(t), t + 1.96\ \mathrm{SE}(t)$ are the 95% confidence limits for $E(t)$.

Similarly $\{t - 2.58\ \mathrm{SE}(t), t + 2.58\ \mathrm{SE}(t)\}$ is the 99% confidence interval for $E(t)$.

Tests of Significance for Large Samples

It is generally agreed that, if the size of the sample exceeds 30, it should be regarded as a large sample. The tests of significance used for large samples are different from the ones used for small samples for the reason that the following assumptions made for large samples do not hold for small samples:

1. The sampling distribution of a statistic is approximately normal, irrespective of whether the distribution of the population is normal or not.
2. Sample statistics are sufficiently close to the corresponding population parameters and hence may be used to calculate the standard error of the sampling distribution.

Test 1

Test of significance of the difference between sample proportion and population proportion.

Let X be the number of successes in n independent Bernoulli trials in which the probability of success for each trial is a constant, (say) = P. Then it is known that X follows a binomial distribution with mean $E(X) = nP$ and variance $V(X) = nPQ$.

When n is large, X follows $N(nP, \sqrt{nPQ})$, i.e. a normal distribution with mean nP and SD $\sqrt{nPQ}$, where $Q = 1 - P$.

$$\therefore \quad \frac{X}{n} \text{ follows } N\left\{\frac{nP}{n}, \sqrt{\frac{nPQ}{n^2}}\right\}$$

Now $\frac{X}{n}$ is the proportion of successes in the sample consisting of n trials, which is denoted by p. Thus the sample proportion p follows $N\left(P, \sqrt{\frac{PQ}{n}}\right)$.

Therefore, the test statistic z is given by $z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}}$.

If $|z| \le z_\alpha$, the difference between the sample proportion p and the population proportion P is not significant at α % LOS.

Note 1. *If P is not known, we assume that p is nearly equal to P, and hence, $SE(p)$ is taken as $\sqrt{\frac{pq}{n}}$. Thus*

$$z = \frac{p - P}{\sqrt{\dfrac{pq}{n}}}$$

2. *95% confidence limits for P are then given by*

$$\frac{|P - p|}{\sqrt{\dfrac{pq}{n}}} \le 1.96,$$

i.e. they are $\left(p - 1.96\sqrt{\frac{pq}{n}}, p + 1.96\sqrt{\frac{pq}{n}}\right)$.

Test 2

Test of significance of the difference between two sample proportions.

Let p_1 and p_2 be the proportions of successes in two large samples of size n_1 and n_2, respectively, drawn from the same population or from two populations with the same proportion P.

Then p_1 follows $N\left(P, \sqrt{\frac{PQ}{n_1}}\right)$ and p_2 follows $N\left(P, \sqrt{\frac{PQ}{n_2}}\right)$.

Therefore $p_1 - p_2$, which is a linear combination of two normal variables, also follows a normal distribution.

Now $\quad E(p_1 - p_2) = E(p_1) - E(p_2) = P - P = 0$

$$V(p_1 - p_2) = V(p_1) + V(p_2) \quad (\because \text{ the two samples are independent})$$

$$= PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Therefore, $(p_1 - p_2)$ follows $N\left\{0, \sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right\}$.

Therefore, The test statistic z is given by

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}.$$

If P is not known, an unbiased estimate of P based on both samples, given by

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2},$$

is used in the place of P.

As before, if $|z| \le z_\alpha$, the difference between the two sample proportions p_1 and p_2 is not significant at α % LOS.

Note *A sample statistic θ is said to be an unbiased estimate of the parameter θ_0, if $E(\theta) = \theta_0$. In the present case,*

$$E\left\{\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right\} = \frac{1}{n_1 + n_2}\{n_1 E(p_1) + n_2 E(p_2)\}$$

$$= \frac{1}{n_1 + n_2}(n_1 P + n_2 P) = P$$

Therefore, An unbiased estimate of P is $\left(\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right)$.

Test 3

Test of significance of the difference between sample mean and population mean.

Let $X_1, X_2, \ldots, X_n$ be the sample observations in a sample of size n, drawn from a population that is $N(\mu, \sigma)$.

Then each X_i follows $N(\mu, \sigma)$.

It is known that if X_i $(i = 1, 2, \ldots, n)$ are independent normal variates with mean μ_i and variance σ_i^2, then $\Sigma\, c_i X_i$ is a normal variate with mean $\mu = \Sigma\, c_i\, \mu_i$ and variance $\sigma^2 = \Sigma\, c_i^2\, \sigma_i^2$.

Now putting $c_i = \frac{1}{n}$, $\mu_i = \mu$ and $\sigma_i = \sigma$, we get

$$\Sigma\, c_i X_i = \frac{1}{n}\Sigma X_i = \bar{X} \quad \Sigma\, c_i\, \mu_i = \frac{1}{n}\mu + \frac{1}{n}\mu + \ldots + \frac{1}{n}\mu \ (n \text{ terms}) = \mu$$

and $$\Sigma\, c_i^2\, \sigma_i^2 = \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \ldots + \frac{1}{n^2}\sigma^2 \ (n \text{ terms})$$

$$= \frac{\sigma^2}{n}.$$

Thus, if X_i are n independent normal variates with the same mean μ and same variance σ^2, then their mean $\bar{X}$ follows a $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$. Even if the population, from which the sample is drawn, is non-normal, it is known (from central limit theorem) that the above result holds good, provided n is large.

Therefore, the test statistic z is given by

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}.$$

As usual, if $|z| \le z_\alpha$, the difference between the sample mean $\bar{X}$ and the population mean μ is not significant at α% LOS.

Note 1. *If σ is not known, the sample SD s can be used in its place, as s is nearly equal to σ when n is large.*

2. *95% confidence limits for μ are given by* $\frac{|\mu - \bar{X}|}{\sigma / \sqrt{n}} \le 1.96$, *i.e.* $\left(\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right)$, *if σ is known. If σ is not known, then the 95% confidence interval is* $\left(\bar{X} - \frac{1.96\, s}{\sqrt{n}}, \bar{X} + \frac{1.96\, s}{\sqrt{n}}\right)$.

Test 4

Test of significance of the difference between the means of two samples.

Let $\overline{X}_1$ and $\overline{X}_2$ be the means of two large samples of sizes n_1 and n_2 drawn from two populations (normal or non-normal) with the same mean μ and variances σ_1^2 and σ_2^2 respectively.

Then $\overline{X}_1$ follows $N\left(\mu, \dfrac{\sigma_1}{\sqrt{n_1}}\right)$ and $\overline{X}_2$ follows $N\left(\mu, \dfrac{\sigma_2}{\sqrt{n_1}}\right)$ either exactly or approximately.

Therefore, $\overline{X}_1 - \overline{X}_2$ also follows a normal distribution.

$$E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu - \mu = 0.$$

$$V(\overline{X}_1 - \overline{X}_2) = V(\overline{X}_1) + V(\overline{X}_2)$$

($\because$ $\overline{X}_1$ and $\overline{X}_2$ are independent, as the samples are independent)

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Thus $(\overline{X}_1 - \overline{X}_2)$ follows $N\left\{0, \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}\right\}$.

Therefore, The test statistic z is given by

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \tag{1}$$

If $|z| \le z_\alpha$, the difference between $(\overline{X}_1 - \overline{X}_2)$ and 0 or the difference between $\overline{X}_1$ and $\overline{X}_2$ is not significant at α % LOS.

Note 1. *If the samples are drawn from the same population, i.e. if $\sigma_1 = \sigma_2 = \sigma$, then*

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \tag{2}$$

2. *If σ_1 and σ_2 are not known and $\sigma_1 \ne \sigma_2$, σ_1 and σ_2 can be approximated by the sample SDs s_1 and s_2. Hence, in such a situation [from (1)],*

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} \tag{3}$$

3. *If σ_1 and σ_2 are equal and not known, then $\sigma_1 = \sigma_2 = \sigma$ is approximated by $\sigma^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$. Hence, in such a situation, [from (2)],*

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}},$$

i.e.
$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}} \qquad (4)$$

4. *The difference in the denominators of the values of z given in (3) and (4) may be noted.*

Test 5

Test of significance of the difference between sample SD and population SD.

Let s be the SD of a large sample of size n drawn from a normal population with SD σ. Then it is known that s follows a $N\left(\sigma, \dfrac{\sigma}{\sqrt{2n}}\right)$ approximately.

Therefore, The test statistic z is given by

$$z = \frac{s - \sigma}{\sigma / \sqrt{2n}}$$

As before, the significance of the difference between s and σ is tested.

Test 6

Test of significance of the difference between SDs of two large samples.

Let s_1 and s_2 be the SDs of two large samples of sizes n_1 and n_2 drawn from a normal population with SD σ.

s_1 follows $N\left(\sigma, \dfrac{\sigma}{\sqrt{2n_1}}\right)$ and s_2 follows $N\left(\sigma, \dfrac{\sigma}{\sqrt{2n_2}}\right)$.

Therefore, $(s_1 - s_2)$ follows $N\left\{0, \sigma\sqrt{\dfrac{1}{2n_1} + \dfrac{1}{2n_2}}\right\}$.

Therefore, the test statistic z is given by

$$z = \frac{s_1 - s_2}{\sigma\sqrt{\dfrac{1}{2n_1} + \dfrac{1}{2n_2}}}.$$

As usual, the significance of the difference between s_1 and s_2 is tested.

Note: If σ is not known, it is approximated by $\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)}$, when n_1 and n_2 are large. In this situation,

$$z = \frac{s_1 - s_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\dfrac{1}{2n_1} + \dfrac{1}{2n_2}\right)}}$$

i.e.
$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_2} + \dfrac{s_2^2}{2n_1}}}$$

Worked Examples 8(A)

Example 1 *Experience has shown that 20% of a manufactured product is of top quality. In one day's production of 400 articles, only 50 are of top quality. Show that either the production of the day chosen was not a representative sample or the hypothesis of 20% was wrong. Based on the particular day's production, find also the 95% confidence limits for the percentage of top quality product.*

Solution $H_0 : P = \dfrac{1}{5}$, i.e. 20% of the products manufactured is of top quality.

$H_1 : P \neq \dfrac{1}{5}$.

p = Proportion of top quality products in the sample

$$= \frac{50}{400} = \frac{1}{8}$$

From the alternative hypothesis H_1, we note that two-tailed test is to be used. Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{p - P}{\sqrt{\dfrac{PQ}{n}}} = \frac{\dfrac{1}{8} - \dfrac{1}{5}}{\sqrt{\dfrac{1}{5} \times \dfrac{4}{5} \times \dfrac{1}{400}}}$$

since the size of the sample is equal to 400.

i.e $$z = -\frac{3}{40} \times 50 = -3.75$$

Now $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5% level.

Also H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95% confidence limits for P are given by

$$\frac{|p-P|}{\sqrt{\frac{pq}{n}}} \le 1.96$$

Note *We have taken* $\sqrt{\frac{pq}{n}}$ *in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.*

i.e. $$p - \sqrt{\frac{pq}{n}} \times 1.96 \le P \le p + \sqrt{\frac{pq}{n}} \times 1.96$$

i.e. $$0.125 - \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96 \le P \le 0.125 + \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96$$

i.e. $0.093 \le P \le 0.157$

Therefore, 95% confidence limits for the percentage of top quality product are 9.3 and 15.7.

Example 2 *The fatality rate of typhoid patients is believed to be 17.26% . In a certain year 640 patients suffering from typhoid were treated in a metropolitan hospital and only 63 patients died. Can you consider the hospital efficient?*

Solution $H_0 : p = P$, i.e. the hospital is not efficient.

$H_1 : p < P$.

One-tailed (left-tailed) test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = -2.33$.

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}, \text{ where } p = \frac{63}{640} = 0.0984$$

$$P = 0.1726, \qquad Q = 0.8274$$

$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726 \times 0.8274}{640}}} = -4.96$$

$\therefore$ $|z| > |z_\alpha|$

Therefore, difference between p and P is significant. i.e., H_0 is rejected and H_1 is accepted.

That is, the hospital is efficient in bringing down the fatality rate of typhoid patients.

Example 3 *A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leave without making a purchase. A random sample of 50 shoppers showed that 35 of them left without making a purchase. Are these sample results consistent with the claim of the salesman? Use an LOS of 0.05.*

Solution Let P and p denote the population and sample proportions of shoppers not making a purchase.

$$H_0 : p = P$$

$$H_1 : p > P, \text{ since } p = 0.7 \quad \text{and} \quad P = 0.6$$

One-tailed (right-tailed) test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.645$.

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.7-0.6}{\sqrt{\frac{0.6\times 0.4}{50}}} = 1.443$$

$\therefore \qquad |z| < z_\alpha$

Therefore, the difference between p and P is not significant at 5% level, i.e., H_0 is accepted and H_1 is rejected.

That is, the sample results are consistent with the claim of the salesman.

Example 4 *Show that for a random sample of size 100, drawn with replacement, the standard error of sample proportion cannot exceed 0.05.*

Solution The items of the sample are drawn one after another with replacement.

Therefore, the proportion (probability) of success in the population, i.e. P-remains a constant.

We know that the sample proportion p follows $N\left(P, \sqrt{\frac{PQ}{n}}\right)$,

$$\text{i.e. SE of } p = \sqrt{\frac{PQ}{n}} = \frac{1}{10}\sqrt{PQ} \quad (\because n = 100) \tag{1}$$

$$\text{Now} \qquad \left(\sqrt{P}-\sqrt{Q}\right)^2 \geq 0$$

$$\text{i.e.} \qquad P + Q - 2\sqrt{PQ} \geq 0$$

$$\text{i.e.} \qquad 1 - 2\sqrt{PQ} \geq 0 \quad \text{or} \quad \sqrt{PQ} \leq \frac{1}{2} \tag{2}$$

Using (2) in (1), we get,

SE of $p \le \dfrac{1}{20}$,

i.e. SE of p cannot exceed 0.05.

Example 5 *A cubical die is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Show that the die cannot be regarded as an unbiased one, and find the extreme limits between which the probability of a throw of 3 or 4 lies.*

Solution H_0 : the die is unbiased, i.e. $P = \dfrac{1}{3}$ (= the probability of getting 3 or 4).

$H_1 : P \neq \dfrac{1}{3}$.

Two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

Although we may test the significance of the difference between the sample and population proportions, we shall test the significance of the difference between the number of successes X in the sample and that in the population.

When n is large, X follows $np, \sqrt{nPQ}$ [refer to Test 1].

$$\therefore \qquad z = \frac{X - nP}{\sqrt{nPQ}} = \frac{3240 - \left(9000 \times \frac{1}{3}\right)}{\sqrt{9000 \times \frac{1}{3} \times \frac{2}{3}}} = 5.37$$

$$\therefore \qquad |z| > z_\alpha$$

Therefoe, the difference between X and nP is significant, i.e., H_0 is rejected.

That is the dice cannot be regarded as unbiased.

If X follows a $N(\mu, \sigma)$, then the reader can easily verify that $P(\mu - 3\sigma \le X \le \mu + 3\sigma) = 0.9974$.

The limits $\mu \pm 3\sigma$ are considered as the extreme (confidence) limits within which X lies.

Accordingly, the extreme limits for P are given by

$$\frac{|P - p|}{\sqrt{\frac{pq}{n}}} \le 3 \qquad \text{[refer to Example 1]}$$

$$\text{i.e.} \qquad p - 3\sqrt{\frac{pq}{n}} \le P \le p + 3\sqrt{\frac{pq}{n}}$$

$$\text{i.e.} \qquad 0.36 - 3\sqrt{\frac{0.36 \times 0.64}{9000}} \le P \le 0.36 + 3\sqrt{\frac{0.36 \times 0.64}{9000}}$$

$$\text{i.e.} \qquad 0.345 \le P \le 0.375$$

Example 6 *In a large city A, 20% of a random sample of 900 school boys had a slight physical defect. In another large city B, 18.5 percent of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant?*

Solution $p_1 = 0.2, p_2 = 0.185, n_1 = 900$ and $n_2 = 1600$.

$H_0: p_1 = p_2.$

$H_1: p_1 \neq p_2.$

Two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \tag{1}$$

Since P, the population proportion, is not given, we estimate it as

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.1904.$$

Using in (1), we have

$$z = \frac{0.2 - 0.185}{\sqrt{0.1904 \times 0.8096 \times \left(\frac{1}{900} + \frac{1}{1600}\right)}} = 0.92$$

$\therefore \qquad |z| \leq z_\alpha.$

Therefore The difference between p_1 and p_2 is not significant at 5% level.

Example 7 *Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.*

Solution Let p_1 and p_2 be the proportions of the consumers before and after the increase in duty respectively.

Then $p_1 = \frac{800}{1000} = \frac{4}{5}$ and $p_2 = \frac{800}{1200} = \frac{2}{3}$.

$H_0: p_1 = p_2.$

$H_1: p_1 > p_2.$

One-tailed (right-tailed) test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.33$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where $P \simeq \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \dfrac{800 + 800}{2200} = 0.7273,$

$$= \frac{0.8 - 0.67}{\sqrt{0.7273 \times 0.2727 \times \left(\frac{1}{1000} + \frac{1}{1200}\right)}}$$

$$= \frac{0.13 \times \sqrt{1000 \times 1200}}{\sqrt{0.7273 \times 0.2727 \times 2200}} = 6.82$$

$\therefore$ $|z| > z_\alpha$

Therefore the difference between p_1 and p_2 is significant at 1% level, i.e., H_0 is rejected and H_1 is accepted.

That is, there is significant decrease in the consumption of tea after the increase in duty.

Example 8 *15.5% of a random sample of 1600 undergraduates were smokers, whereas 20% of a random sample of 900 postgraduates were smokers in a state. Can we conclude that less number of undergraduates are smokers than the postgraduates?*

Solution Here $p_1 = 0.155$, $p_2 = 0.2$, $n_1 = 1600$ and $n_2 = 900$.

H_0: $p_1 = p_2$.

H_1: $p_1 < p_2$.

One-tailed (left-tailed) test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = -1.645$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where $P \simeq \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = 0.1712,$

$$= \frac{0.155 - 0.2}{\sqrt{0.1712 \times 0.8288 \times \left(\frac{1}{1600} + \frac{1}{900}\right)}}$$

$$= \frac{-0.045 \times 1200}{\sqrt{0.1712 \times 0.8288 \times 2500}},$$

$$= -2.87$$

$\therefore$ $|z| > |z_\alpha|$

Therefore the difference between p_1 and p_2 is significant, i.e. H_0 is rejected and H_1 is accepted.

That is, the habit of smoking is less among the undergraduates than among the postgraduates.

Example 9 *A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm, and the SD is 10 cm?*

Solution Here $\bar{x} = 160$, $n = 100$, $\mu = 165$ and $\sigma = 10$.

H_0: $\bar{x} = \mu$ (i.e. the difference between $\bar{x}$ and μ is not significant).

H_1: $\bar{x} \neq \mu$.

Two-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.58$.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{160 - 165}{10 / \sqrt{100}} = -5$$

$\therefore \quad |z| > z_\alpha$.

Therefore, the difference between $\bar{x}$ and μ is significant at 1% level, i.e., H_0 is rejected.

That is, it is not statistically correct to assume that $\mu = 165$.

Example 10 *The mean breaking strength of the cables supplied by a manufacturer is 1800 with a SD of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. To test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% LOS?*

Solution $\bar{x} = 1850$, $n = 50$, $\mu = 1800$ and $\sigma = 100$.

H_0: $\bar{x} = \mu$.

H_1: $\bar{x} > \mu$.

One-tailed (right-tailed) test is to be used.

Let LOS = 1%. Therefore, $z_\alpha = 2.33$.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{1850 - 1800}{100 / \sqrt{50}} = 3.54$$

$\therefore \quad |z| > z_\alpha$

Therefore the difference between $\bar{x}$ and μ is significant at 1% level, i.e., H_0 is rejected and H_1 is accepted.

That is, based on the sample data, we may support the claim of increase in breaking strength.

Example 11 *The mean value of a random sample of 60 items was found to be 145, with a SD of 40. Find the 95% confidence limits for the population mean. What size of the sample is required to estimate the population mean within 5 of its actual value with 95% or more confidence, using the sample mean?*

Solution Since the population SD σ too is not given, we can approximate it by the sample SDs. Therefore 95% confidence limits for μ are given by

$$\frac{|\mu - \bar{x}|}{s / \sqrt{n}} \leq 1.96,$$

i.e. $$\bar{x} - 1.96\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{s}{\sqrt{n}}.$$

i.e. $$145 - \frac{1.96 \times 40}{\sqrt{60}} \le \mu \le 145 + \frac{1.96 \times 40}{\sqrt{60}}$$

i.e. $$134.9 \le \mu \le 155.1$$

We have to find the value of n such that

$$P\{\bar{x} - 5 \le \mu \le \bar{x} + 5\} \ge 0.95$$

i.e. $$P\{-5 \le \mu - \bar{x} \le 5\} \ge 0.95$$

i.e. $$P\{|\mu - \bar{x}| \le 5\} \ge 0.95 \quad \text{or}$$

$$P\{|\bar{x} - \mu| \le 5\} \ge 0.95$$

$$\therefore\ P\left\{\frac{|\bar{x} - \mu|}{\sigma/\sqrt{n}} \le \frac{5}{\sigma/\sqrt{n}}\right\} \ge 0{\cdot}95$$

i.e. $$P\left\{|z| \le \frac{5\sqrt{n}}{\sigma}\right\} \ge 0{\cdot}95,$$

where z is the standard normal variate. (1)

We know that $P\{|z| \le 1.96\} = 0{\cdot}95$. Therefore, the least value of $n = n_L$ that will satisfy (1) is given by

$$\frac{5\sqrt{n_L}}{\sigma} = 1.96$$

i.e. $$\sqrt{n_L} = \frac{1.96\,s}{5} \quad (\because\ \sigma \simeq s)$$

i.e. $$n_L = \left(\frac{1.96 \times 40}{5}\right)^2$$

i.e. $$n_L = 245.86$$

Therefore, least size of the sample is 246.

Example 12 *A normal population has a mean of 0.1 and SD of 2.1. Find the probability that the mean of a sample of size 900 drawn from this population will be negative.*

Solution Since $\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$, $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ is the standard normal variate.

Now $$P(\bar{x} < 0) = P\{\bar{x} - 0.1 < -0.1\}$$

$$= P\left\{\frac{\bar{x} - 0.1}{(2.1)/\sqrt{900}} < \frac{-0.1}{(2.1)/\sqrt{900}}\right\}$$

$$= P\{z < -1.43\}$$
$$= P\{z > 1.43\},$$
(by symmetry of the standard normal distribution)
$$= 0.5 - P\{0 < z < 1.43\}$$
$$= 0.5 - 0.4236 \text{ (from the normal tables)}$$
$$= 0.0764$$

Example 13 *In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, the mean is 15. Could the samples have been drawn from the same population with SD 4?*

Solution Here $\bar{x}_1 = 20$, $n_1 = 500$, $\bar{x}_2 = 15$, $n_2 = 400$; $\sigma = 4$

H_0: $\bar{x}_1 = \bar{x}_2$, i.e. the samples have been drawn from the same population.

H_1: $\bar{x}_1 \neq \bar{x}_2$.

Two-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.58$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \qquad \text{(refer to Note 1 under Test 4)}$$

$$= \frac{20 - 15}{4\sqrt{\dfrac{1}{500} + \dfrac{1}{400}}} = 18.6$$

$\therefore \quad |z| > z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at 1% level, i.e., H_0 is rejected

That is, the samples could not have been drawn from the same population.

Example 14 *A simple sample of heights of 6400 English men has a mean of 170 cm and a SD of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a SD of 6.3 cm. Do the data indicate that Americans are, on the average, taller than the Englishmen?*

Solution Here $n_1 = 6400$, $\bar{x}_1 = 170$ and

$s_1 = 6.4$; $n_2 = 1600$, $\bar{x}_2 = 172$ and $s_2 = 6.3$.

H_0: $\mu_1 = \mu_2$ or $\bar{x}_1 = \bar{x}_2$,

i.e. the samples have been drawn from two different populations with the same mean.

H_1: $\bar{x}_1 < \bar{x}_2$ or $\mu_1 < \mu_2$.

Left-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = -2.33$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

[$\because$ $\sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$. Refer to Note 2 under Test 4]

$$= \frac{170 - 172}{\sqrt{\dfrac{(6.4)^2}{6400} + \dfrac{(6.3)^2}{1600}}} = -11.32$$

$\therefore$ $|z| > |z_\alpha|$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 1% level, i.e., H_0 is rejected and H_1 is accepted.

That is, the Americans are, on the average, taller than the Englishmen.

Example 15 *Test the significance of the difference between the means of the samples, drawn from two normal populations with the same SD using the following data:*

	Stize	*Mean*	*SD*
Sample 1	100	61	4
Sample 2	200	63	6

Solution H_0: $\bar{x}_1 = \bar{x}_2$ or $\mu_1 = \mu_2$.

H_1: $\bar{x}_1 \neq \bar{x}_2$ or $\mu_1 \neq \mu_2$.

Two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}}$$

(refer to Note 3 under Test 4; the populations have the same SD)

$$= \frac{61 - 63}{\sqrt{\dfrac{4^2}{200} + \dfrac{6^2}{100}}} = -3.02$$

$\therefore$ $|z| > z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 5% level, i.e., H_0 is rejected and H_1 is accepted.

That is, the two normal populations, from which the samples are drawn, may not have the same mean, though they may have the same SD.

Example 16 *The average marks scored by 32 boys is 72 with a SD of 8, while that for 36 girls is 70 with a SD of 6. Test at 1% LOS whether the boys perform better than girls.*

Solution H_0: $\bar{x}_1 = \bar{x}_2$ (or $\mu_1 = \mu_2$).

H_1: $\bar{x}_1 > \bar{x}_2$.

Right-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.33$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

(The two populations are assumed to have SD s $\sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$)

$$= \frac{72 - 70}{\sqrt{\frac{8^2}{32} + \frac{6^2}{36}}} = 1.15$$

$\therefore$ $|z| < z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (μ_1 and μ_2) is not significant at 1% level, i.e. H_0 is accepted and H_1 is rejected.

That is, statistically we cannot concluede that boys perform better than girls.

Example 17 *The heights of men in a city are normally distributed with mean 171 cm and SD 7 cm., while the corresponding values for women in the same city are 165 cm and 6 cm respectively. If a man and a woman are chosen at random from this city, find the probability that the woman is taller than the man.*

Solution Let $\bar{x}_1$ and $\bar{x}_2$ denote the mean heights of men and women respectively.

Then $\bar{x}_1$ follows a N (171, 7) and $\bar{x}_2$ follows a N (165, 6). Therefore, $\bar{x}_1 - \bar{x}_2$ also follows a normal distribution.

$$E(\bar{x}_1 - \bar{x}_2) = E(x_1) - E(\bar{x}_2) = 171 - 165 = 6$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2) = 49 + 36 = 85 \qquad \text{(refer to Test 4)}$$

Therefore, SD of $(\bar{x}_1 - \bar{x}_2) = \sqrt{85} = 9.22$.

Thus, $\bar{x}_1 - \bar{x}_2$ follows a N (6, 9.22).

$$\therefore \quad P(\bar{x}_2 > \bar{x}_1) = P(\bar{x}_1 - \bar{x}_2 < 0)$$

$$= P\left\{\frac{(\bar{x}_1 - \bar{x}_2) - 6}{9.22} < \frac{-6}{9.22}\right\}$$

$= P\{z < -0.65\}$ (where z is the standard normal variate).

$= P\{z > 0.65\}$ (by symmetry).

$$= 0.5 - P\,(0 < z < 0.65)$$
$$= 0.5 - 0.2422 = 0.2578.$$

Example 18 *Two populations have the same mean, but the SD of one is twice that of the other. Show that in samples, each of size 500, drawn under simple random conditions, the difference of the means will, in all probability, not exceed 0.3 σ, where σ is the smaller SD.*

Solution Let $\bar{x}_1$ and $\bar{x}_2$ be the means of the samples of size 500 each. Let their SD's be σ and 2σ respectively.

Then $\bar{x}_1$ follows a $N\left(\mu, \dfrac{\sigma}{\sqrt{500}}\right)$ and $\bar{x}_2$ follows $N\left(\mu, \dfrac{2\sigma}{\sqrt{500}}\right)$.

Therefore, $\bar{x}_1 - \bar{x}_2$ also follows a normal distribution.

$$E\,(\bar{x}_1 - \bar{x}_2) = E\,(\bar{x}_1) - E\,(\bar{x}_2) = \mu - \mu = 0$$
$$V\,(\bar{x}_1 - \bar{x}_2) = V\,(\bar{x}_1) + V\,(\bar{x}_2)$$
$$= \frac{\sigma^2}{500} + \frac{4\sigma^2}{500} = \frac{\sigma^2}{100}$$

Therefore, SD of $(\bar{x}_1 - \bar{x}_2) = \dfrac{\sigma}{10}$

Thus $(\bar{x}_1 - \bar{x}_2)$ follows $N\left(0, \dfrac{\sigma}{10}\right)$.

$$\therefore \qquad P\,\{|\bar{x}_1 - \bar{x}_2| \le 0.3\,\sigma\}$$
$$= P\left\{\frac{|(\bar{x}_1 - \bar{x}_2) - 0|}{\sigma/10} \le \frac{0.3\,\sigma}{\sigma/10}\right\}$$
$$= P\,\{|z| \le 3\}. \text{ (where } z \text{ is the standard normal variate)}$$
$$= 0.9974 \simeq 1$$

Therefore, $|\bar{x}_1 - \bar{x}_2|$ will not exceed 0.3 σ almost certainly.

Example 19 *A manufacturer of electric bulbs, according to a certain process, finds the SD of the life of lamps to be 100 h. He wants to change the process, if the new process results in a smaller variation in the life of lamps. In adopting a new process, a sample of 150 bulbs gave an SD of 95 h. Is the manufacturer justified in changing the process?*

Solution Here $\sigma = 100$, $n = 150$ and $s = 95$

H_0: $s = \sigma$

H_1: $s < \sigma$

Left-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = -1.645$

$$z = \frac{s - \sigma}{\sigma / \sqrt{2n}} = \frac{95 - 100}{100 / \sqrt{300}} = -0.866$$

$\therefore \qquad |z| < |z_\alpha|$

Therefore, the difference between s and σ is not significant at 5% level, i.e., H_0 is accepted and H_1 is rejected.

That is, the manufacturer is not justified in changing the process.

Example 20 *The SD of a random sample of 1000 is found to be 2.6 and the SD of another random sample of 500 is 2.7. Assuming the samples to be independent, find whether the two samples could have come from populations with the same SD.*

Solution Here $n_1 = 1000$, $s_1 = 2.6$; $n_2 = 500$ and $s_2 = 2.7$.

H_0: $s_1 = s_2$ (or $\sigma_1 = \sigma_2$).

H_1: $s_1 \neq s_2$.

Two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_2} + \dfrac{s_2^2}{2n_1}}} \quad \text{(since } \sigma \text{ is not known)}$$

$$= \frac{2.6 - 2.7}{\sqrt{\dfrac{(2.6)^2}{1000} + \dfrac{(2.7)^2}{2000}}} = -0.98$$

$\therefore \qquad |z| < z_\alpha$

Therefore, the difference between s_1 and s_2 (and hence between σ_1 and σ_2) is not significant at 5% level.

i.e., H_0 is accepted.

That is, the two samples could have come from populations with the same SD.

Exercise 8(A)

Part A (Short answer questions)

1. What is the difference between population and sample?
2. Distinguish between parameter and statistic.
3. What do you mean by sampling distribution?
4. What is meant by standard error?

5. What do you mean by estimation?
6. What is meant by testing of hypothesis?
7. Define null hypothesis and alternative hypothesis.
8. What is meant by test of significance?
9. What do you mean by critical region and acceptance region?
10. Define level of significance.
11. Give the general form of a test statistic.
12. Define type I and type II errors.
13. Define producer's risk and consumer's risk.
14. What is the relation between type I error and LOS?
15. Define one-tailed and two-tailed tests.
16. Define critical value of a test statistic.
17. What is the relation between the critical value and LOS?
18. What is the relation between the critical values for a single-tailed test and a two-tailed test?
19. Write down the 1% and 5% critical values for right-tailed and two-tailed tests for large samples.
20. What do you mean by interval estimation and confidence limits?
21. Write down the general form of 95% confidence limits of a population parameter in terms of the corresponding sample statistic.
22. What is the standard error of the sample proportion when the population proportion is (i) known and (ii) not known?
23. What is the standard error of the difference between two sample proportions when the population proportion is (i) known and (ii) not known?
24. What do you mean by unbiased estimate? Give an example.
25. Write down the form of the 98% confidence interval for the population mean in terms of (i) population SD; and (ii) Sample SD.
26. What is the standard error of the difference between the means of two large samples drawn from different populations with (i) known SDs and; (ii) unknown SDs?
27. What is the standard error of the difference between the means of two large samples drawn from the same population with (i) known SD and (ii) unknown SD?
28. What is the standard error of the difference between the SD's of two large samples drawn from the same population with (i) known SD and (ii) unknown SD?

Part B

29. Out of 200 individuals, 40% show a certain trait and the number expected on a certain theory is 50%. Find whether the number observed differs significantly from expectation.
30. A coin is thrown 400 times and is found to result in 'Head' 245 times. Test whether the coin is a fair one.

31. A manufacturer of light bulbs claims that on the average 2 % of the bulbs manufactured by his firm are defective. A random sample of 400 bulbs contained 13 defective bulbs. On the basis of the this sample, can you support the manufacturer's claim at 5% LOS?
32. 100 people were affected by cholera and out of them only 90 survived. Would you reject the hypothesis that the survival rate, if affected by cholera, is 85% in favour of the hypothesis that it is more at 5% LOS?
33. A random sample of 400 mangoes was taken from a big consignment and 40 were found to be bad. Prove that the percentage of bad mangoes in the consignment will, in all probability, lie between 5.5 and 14.5.
34. A random sample of 64 articles produced by a machine contained 14 defectives. Is it reasonable to assume that only 10% of the articles produced by the machine are defective? If not, find the 99% confidence limits for the percentage of defective articles produced by the machine.
35. Certain crosses of the pea gave 5321 yellow and 1804 green seeds. The expectation is 25% of green seeds based on a certain theory. Is this divergence significant or due to sampling fluctuations?
36. During a countrywide investigation, the incidence of TB was found to be 1% . In a college with 400 students, 5 are reported to be affected whereas in another college of 1200 students, 10 are found to be affected. Does this indicate any significant difference?
37. A random sample of 600 men chosen from a certain city contained 400 smokers. In another sample of 900 men chosen from another city, there were 450 smokers. Do the data indicate that (i) the cities are significantly different with respect to smoking habit among men? and (ii) the first city contains more smokers than the second?
38. A sample of 300 spare parts produced by a machine contained 48 defectives. Another sample of 100 spare parts produced by another machine contained 24 defectives. Can you conclude that the first machine is better than the second?
39. In two large populations, there are 30 and 25%, respectively, of fair-haired people. Is this difference likely to be hidden in samples of sizes 1200 and 900, respectively, drawn from the two populations?

$$\left[\textbf{Hint: } H_0 : P_1 - P_2 = 0, \text{ or } P_1 = P_2; \quad H_1 : P_1 \neq P_2 \quad z = \frac{p_1 - p_2}{\sqrt{\dfrac{P_1 Q_1}{n_1} + \dfrac{P_2 Q_2}{n_2}}}\right]$$

40. A machine produces 16 defective bolts in a batch of 500 bolts. After the machine is overhauled, it produces 3 defective bolts in a batch of 100 bolts. Has the machine improved?
41. There were 956 births in a year in town A, of which 52.5% were males, while in towns A and B combined together this proportion in a total of

1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?

42. A cigarette manufacturing company claims that its brand A cigarettes outsells its brand B by 8%. It is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B. Test at 5% LOS. whether the 8% difference is a valid claim.

[**Hint**: $H_0 : P_1 - P_2 = .08$; $H_1 : P_1 - P_2 \neq .08$ and

$$z = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$]

43. A sample of 900 items is found to have a mean of 3.47 cm. Can it be reasonably regarded as a simple sample from a population with mean 3.23 cm and SD 2.31 cm?
44. A sample of 400 observations has mean 95 and SD 12. Could it be a random sample from a population with mean 98? What should be the maximum value of the population mean so that the sample can be regarded as one drawn from it almost certainly?
45. A manufacturer claims that, the mean breaking strength of safety belts for air passengers produced in his factory is 1275 kgs. A sample of 100 belts was tested and the mean breaking strength and SD were found to be 1258 and 90 kg respectively. Test the manufacturer's claim at 5% LOS.
46. An IQ test was given to a large group of boys in the age group of 18–20 years, who scored an average of 62.5 marks. The same test was given to a fresh group of 100 boys of the same age group. They scored an average of 64.5 marks with a SD 12.5 marks. Can we conclude that the fresh group of boys have better IQ?
47. The guaranteed average life of a certain brand of electric bulb is 1000 h, with a SD of 125 h. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. What should be the minimum sample size?
48. A random sample of 100 students gave a mean weight of 58 kg with a SD of 4 kg. Find the 95% and 99% confidence limits of the mean of the population.
49. The means of two simple samples of 1000 and 2000 items are 170 cm and 169 cm respectively. Can the samples be regarded as drawn from the same population with SD 10 at 5% LOS?

50. The mean and SD of a sample of size 400 are 250 and 40 respectively. Those of another sample of size 400 are 220 and 55. Test at 1% LOS whether the means of the two populations from which the samples have been drawn are equal.
51. Intelligence tests were given to two groups of boys and girls of the same age group chosen from the same college and the following results were got:

	Size	*Mean*	*SD*
Boys	100	73	10
Girls	60	75	8

Examine whether the difference between the means is significant or not.
52. A sample of 100 bulbs of brand *A* gave a mean lifetime of 1200 h, with an SD of 70 h, while another sample of 120 bulbs of brand *B* gave a mean lifetime of 1150 h, with a SD of 85 h. Can we conclude that brand *A* bulbs are superior to brand *B* bulbs?
53. In a college, 60 junior students are found to have a mean height of 171.5 cm and 50 senior students are found to have a mean height of 173.8 cm. Can we conclude, based on these data, that the juniors are shorter than seniors at (i) 5% LOS and (ii) 1% LOS, assuming that the SD of students of that college is 6.2 cm?
54. Two samples drawn from two different populations gave the following results:

	Size	*Mean*	*SD*
Sample I	400	124	14
Sample II	250	120	12

Find the 95% confidence limits for the difference of the population means.

$$\left[\textbf{Hint: } (\bar{x}_1 - \bar{x}_2) - 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right]$$

55. Two samples drawn from two different populations gave the following results:

	Size	*Mean*	*SD*
Sample I	100	582	24
Sample II	100	540	28

Test the hypothesis, at 5% LOS, that the difference of the means of the populations is 35.

$$\left[\textbf{Hint: } z = \frac{(\bar{x}_1 - \bar{x}_2) - 35}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \right]$$

56. Two populations have their means equal, but the SD of one is twice the other. Show that in samples, each of size 2000, drawn one from each, the difference of the means will, in all probability, not exceed 0.15σ, where σ is the smaller SD.
57. In a certain random sample of 72 items, the SD is found to be 8. Is it reasonable to suppose that it has been drawn from a population with SD 7?
58. In a random sample of 200 items, drawn from a population with SD 0.8, the sample SD is 0.7. Can we conclude that the sample SD is less than the population SD at 1% LOS?
59. The SD of a random sample of 900 members is 4.6 and that of another independent sample of 1600 members is 4.8. Examine if the two samples could have been drawn from a population with SD 4.0?
60. Examine whether the two samples for which the data are given in the following table could have been drawn from populations with the same SD:

	Size	*SD*
Sample I	100	5
Sample II	200	7

Tests of Significance for Small Samples

The tests of significance discussed in the previous section hold good only for large samples, i.e. only when the size $n \geq 30$. When the sample is small, i.e. $n < 30$, the sampling distributions of many statistics are not normal, even though the parent populations may be normal. Moreover the assumption of near equality of population parameters and the corresponding sample statistics will not be justified for small samples. Consequently we have to develop entirely different tests of significance that are applicable to small samples.

Student's *t*-Distribution

A random variable T is said to follow student's t-distribution or simply t-distribution, if its probability density function is given by

$$f(t) = \frac{1}{\sqrt{v}\,\beta\left(\frac{v}{2}, \frac{1}{2}\right)} \cdot \left(1 + \frac{t^2}{v}\right)^{-(v+1)/2}, \; -\infty < t < \infty,$$

where v denoteds the number of degrees of freedom of the t-distribution.

Note *t-distribution was defined by the mathematician W.S.D Gosset, whose pen name is Student.)*

Properties of t-Distribution

1. The probability curve of the t-distribution is similar to the standard normal curve, and is symmetric about $t = 0$, bell-shaped and asymptotic to the t-axis, as shown in the fig. 8.2.
2. For sufficiently large value of v, the t-distribution tends to the standard normal distribution.
3. The mean of the t-distribution is zero.
4. The variance of the t-distribution is $\dfrac{v}{v-2}$, if $v > 2$ and is greater than 1, but it tends to 1 as $v \to \infty$.

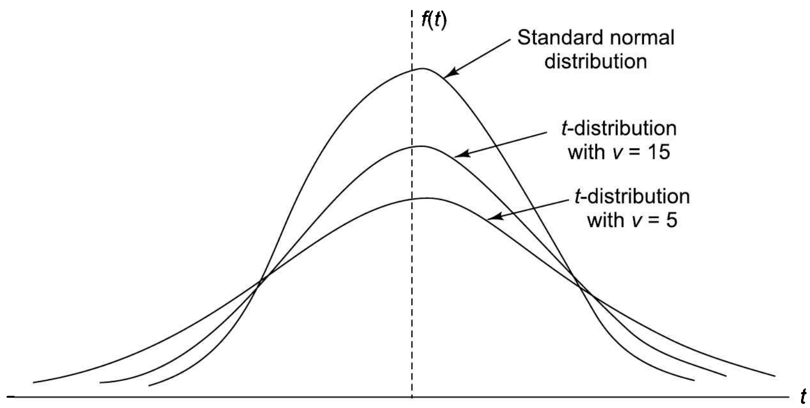

Fig. 8.2

Uses of t-Distribution

The t-distribution is used to test the significance of the difference between

1. The mean of a small sample and the mean of the population,
2. The means of two small samples and
3. The coefficient of correlation in the small sample and that in the population, assumed zero.

Note on Degree of Freedom

The number of degrees of freedom, usually denoted by the Greek alphabet v, can be interpreted as the number of useful bits of information generated by a sample of given size for estimating a population parameter. Suppose we wish to find the mean of a sample with observations $x_1, x_2, ..., x_n$. We have to use all the n values taken by the variable with full freedom (i.e. without putting any constraint or restriction on them) for computing $\bar{x}$. Hence $\bar{x}$ is said to have n degrees of freedom.

Suppose we wish to further compute the SDs of this sample, using the formula $s^2 = \frac{1}{n}\sum\left(x_i - \bar{x}\right)^2$. Though we use the n values $x_1 - \bar{x}$, $x_2 - \bar{x}$, ..., $x_n - \bar{x}$ for this computation, they do not have n degrees of freedom, as they depend on $\bar{x}$, which has already been calculated and fixed. Since there is one restriction regarding the value of $\bar{x}$, s is said to have $(n - 1)$ degrees of freedom.

If we compute another statistic of the sample based on $\bar{x}$ and s, then that statistic will be assumed to have $(n - 2)$ degrees of freedom and so on.

Thus the number of independent variates used to compute the test statistic is known as the number of degrees of freedom of that statistic. In general, the number of degrees of freedom is given by $\nu = n - k$, where n is the number of observations in the sample and k is the number of constraints imposed on them or k is the number of values that have been found out and specified by prior calculations.

Critical Values of t and the t-Table

The critical value of t at level of significance (LOS) α and degrees of freedom ν is given by $P\{|t| > t_\nu(\alpha)\} = \alpha$ for two-tailed test, as in the case of normal distribution and large samples, and by $P\{t > t_\nu(\alpha)\} = \alpha$ for the right-tailed test, also as in the case of normal distribution. The critical value of t for a single (right-or-left) tailed test at LOS α corresponding to ν degrees of freedom is the same as that for a two-tailed test at LOS 2α corresponding to the same degrees of freedom.

Critical values $t_\nu(\alpha)$ of the t-distribution for two-tailed tests, corresponding to a few important LOS and a range of values of ν, have been published by Prof. R.A. Fisher in the form of a table, called the t-table, which is given in the end of this chapter.

Test 1

Test of significance of the difference between sample mean and population mean.

If $\bar{x}$ is the mean of a sample of size n, drawn from a population $N(\mu, \sigma)$, we have seen that $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ follows $N(0, 1)$.

If σ, the SD of the population is not known, we have to estimate it using the sample SDs. From the theory of estimation, it is known that $s\sqrt{\dfrac{n}{n-1}}$ is an unbiased estimate of σ with $(n - 1)$ degrees of freedom. When n is large, $\dfrac{n}{n-1} \simeq 1$ and hence s was taken as a satisfactory estimate of σ and hence $z = \dfrac{\bar{x} - \mu}{s / \sqrt{n}}$ was assumed to follow a $N(0, 1)$. But when n is small, we cannot use s as an estimate of σ, since

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-\mu}{s\sqrt{\frac{n}{n-1}}\cdot\frac{1}{\sqrt{n}}} = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$$

Now $\frac{\bar{x}-\mu}{s/\sqrt{n-1}}$ does not follow a normal distribution, but follows a t-distribution with number of degrees of freedom $\nu = n - 1$. Hence $\frac{\bar{x}-\mu}{s/\sqrt{n-1}}$ is denoted by t and is taken as the test statistic.

Sometimes $t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$ is also taken as $t = \frac{\bar{x}-\mu}{S/\sqrt{n}}$,

where $S^2 = \frac{1}{n-1}\sum_{r=1}^{n}\left(x_r - \bar{x}\right)^2$ and is called student's t. We shall use only

$t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$, where s is the sample SD.

We get the value of $t_\nu(\alpha)$ for the LOS α and $\nu = n - 1$ from the t-table.

If the calculated value of t satisfies $|t| < t_\nu(\alpha)$, the null hypothesis H_0 is accepted at LOS α otherwise H_0 is rejected at LOS α.

Note *95% confidence interval of μ is given by*

$$\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}, \text{ since } P\left\{\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}\right\} = 0.95$$

i.e. by $\bar{x} - t_{0.05}\frac{s}{\sqrt{n-1}} \le \mu \le \bar{x} + t_{0.05} \times \frac{s}{\sqrt{n-1}}$, *where* $t_{0.05}$ *is the 5% critical value of t for* $\nu\ (= n - 1)$ *degrees of freedom for a two-tailed test.*

Test 2

Test of significance of the difference between means of two small samples drawn from the same normal population.

In Test 4 for large samples, the test statistic used to test the significance of the difference between the means of two samples from the same normal population was taken as

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}},$$

which follows a N (0, 1) (1)

If σ is not known, we may assume that $\sigma \simeq \sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}}$, when n_1 and n_2 are large, where s_1 and s_2 are the sample SDs. This assumption no longer holds good when n_1 and n_2 are small.

In fact, it is known from the theory of estimation, that an estimate of σ is $\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}$ with $(n_1 + n_2 - 2)$ degrees of freedom, when n_1 and n_2 are small.

Using this value of σ in (1), the test statistic becomes

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}},$$

which does not follow a N (0, 1), but follows a t-distribution, with $\nu = (n_1 + n_2 - 2)$ degrees of freedom. Hence t-test is applied in this case.

Note 1. *If f $n_1 = n_2 = n$ and if the samples are independent i.e., the observations in the two samples are not at all related, then the test statistic is given by*

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}} \quad \text{with} \quad \nu = 2n - 2 \tag{2}$$

2. *If $n_1 = n_2 = n$ and if the pairs of values of x_1 and x_2 are associated in some way (or correlated), the formula (2) for t in Note (1) should not be used. In this case, we shall assume that $H_0 : \bar{d}\ (= \bar{x} - \bar{y}) = 0$ and test the significance of the difference between $\bar{d}$ and 0, using the test statistic $t = \dfrac{\bar{d}}{s/\sqrt{n-1}}$ with $\nu = n - 1$, where $d_i = x_i - y_i$*

$(i = 1, 2, \ldots\ldots, n)$, $\bar{d} = \bar{x} - \bar{y}$; and $s = SD$ of $d = \dfrac{1}{n}\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2$.

Snedecor's *F*-Distribution

A random variable F is said to follow snedecor's F-distribution or simply F-distribution if its probability density function is given by

$$f(F) = \frac{(v_1/v_2)^{v_1/2}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \cdot \frac{F^{v_1/2-1}}{\left(1 + \frac{v_1 F}{v_2}\right)^{(v_1+v_2)/2}}, \quad F > 0.$$

Note *The mathematical variable corresponding to the random variable F is also taken as F. v_1 and v_2 used in f(F) are the degrees of freedom associated with the F-distribution.*

Properties of the F-Distribution

1. The probability curve of the F-distribution is roughly sketched in Fig. 8.3.

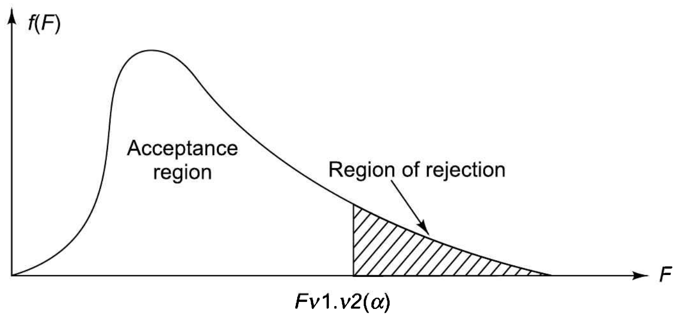

Fig. 8.3

2. The square of the t-variate with n degrees of freedom follows a F-distribution with 1 and n degrees of freedom.
3. The mean of the F-distribution is $\dfrac{v_2}{v_2 - 2}$ $(v_2 > 2)$.
4. The variance of the F-distribution is

$$\frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)} \quad (v_2 > 4)$$

Use of F-Distribution

F-distribution is used to test the equality of the variance of the populations from which two small samples have been drawn.

F-test

F-test of significance of the difference between population variances and F-table.

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_2^2 and then test their equality. It is known that $\hat{\sigma}_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}$ with the number of degrees of freedom $\nu_1 = n_1 - 1$ and $\hat{\sigma}_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}$ with the number of degrees of freedom $\nu_2 = n_2 - 1$.

It is also known that $F = \dfrac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$ follows a F-distribution with ν_1 and ν_2 degrees of freedom. If $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, then $F = 1$. Hence our aim is to find how far any observed value of F can differ from unity as a result of fluctuations of sampling.

Snedecor has prepared tables that give, for different values of ν_1 and ν_2, the 5% and 1% critical values of F. An extract from these tables is given at the end of this chapter. If F denotes the observed (calculated) value and $F_{\nu_1,\nu_2}(\alpha)$ denotes the critical (tabulated) value of F at LOS α, then $P\{F > F_{\nu_1,\nu_2}(\alpha)\} = \alpha$.

(**Note:** F-test is not a two-tailed test and is always a right-tailed test, since F cannot be negative. Thus if $F > F_{\nu_1,\nu_2}(\alpha)$, then the difference between F and 1, i.e. the difference between $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ is significant at LOS α. In other words, the samples may not be regarded as drawn from the same population or from populations with the same variance. If $F < F_{\nu_1,\nu_2}(\alpha)$, the difference is not significant at *LOS* α).

Note

1. *We should always make $F > 1$. This is done by taking the larger of the two estimates of σ^2 as σ_1^2 and by assuming that the corresponding degree of freedom as ν_1.*
2. *To test if two small samples have been drawn from the same normal population, it is not enough to test if their means differ significantly or not, because in this test we assumed that the two samples came from the same population or from populations with equal variance. So, before applying the t-test for the significance of the difference of two sample means, we should satisfy ourselves about the equality of the population variances by F-test.*

Worked Examples 8(B)

Example 1 *Tests made on the breaking strength of 10 pieces of a metal gave the following results: 578, 572, 570, 568, 572, 570, 570, 572, 596 and 584 kg. Test if the mean breaking strength of the wire can be assumed as 577 kg.*

Solution Let us first compute sample mean $\bar{x}$ and sample SDs and then test if $\bar{x}$ differs significantly from the population mean $\mu = 577$.

We take the assumed mean $A = \dfrac{568 + 596}{2} = 582$

$$d_i = x_i - A$$

$\therefore$ $$x_i = d_i + A$$

$\therefore$ $$\bar{x} = \frac{1}{n}\sum x_i = \frac{1}{n}\sum d_i + A$$

$$= \frac{1}{10} \times (-68) + 582 = 575.2 \text{ (see the table below)}$$

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2 \text{ (see the table below)}$$

$$= \frac{1}{10} \times 1144 - \left(\frac{1}{10} \times (-68)\right)^2 = 68.16$$

$\therefore$ $$s = 8.26$$

Now $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{575{\cdot}2 - 577}{8{\cdot}26/\sqrt{9}}$$

$$= -0.65$$

and $$\nu = n - 1 = 9.$$

x_i	$d_i = x_i - A$	d_i^2
578	– 4	16
572	– 10	100
570	– 12	144
568	– 14	196
572	– 10	100
570	– 12	144
570	– 12	144
572	– 10	100
596	14	196
584	2	4
Total	– 68	1144

$$H_0: \bar{x} = \mu \quad \text{and} \quad H_1: \bar{x} \neq \mu.$$

Two tailed test is to be used. Let LOS be 5%.

From the t-table, for $\nu = 9$, $t_{0.05} = 2.26$. Since $|t| < t_{0.05}$, the difference between $\bar{x}$ and μ is not significant or H_0 is accepted. Therefore, the mean breaking strength of the wire can be assumed as 577 kg at 5% LOS.

Example 2 *A machinist is expected to make engine parts with axle diameter of 1.75 cm. A random sample of 10 parts shows a mean diameter 1.85 cm with an SD of 0.1 cm. On the basis of this sample, would you say that the work of the machinist is inferior?*

Solution Here $\bar{x} = 1.85$, $s = 0.1$, $n = 10$ and $\mu = 1.75$.

$$H_0: \bar{x} = \mu; \quad H_1: \bar{x} \neq \mu.$$

Two-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{0\cdot10}{0\cdot1/\sqrt{9}} = 3 \quad \text{and} \quad \nu = n - 1 = 9$$

From the t-table, for $\nu = 9$, $t_{0.05} = 2.26$ and $t_{0.01} = 3.25$.

$\therefore$ $|t| > t_{0.05}$ and $|t| < t_{0.01}$

Therefore, H_0 is rejected and H_1 is accepted at 5% level, but H_0 is accepted and H_1 is rejected at 1% level. That is at 5% LOS the work of the machinist can be assumed to be inferior, but at 1% LOS the work cannot be assumed to be inferior.

Example 3 *A certain injection administered to each of 12 patients resulted in the following increases of blood pressure:*

5, 2, 8, – 1, 3, 0, 6, – 2, 1, 5, 0, 4.

Can it be concluded that the injection will be, in general, accompanied by an increase in BP?

Solution The mean of the sample is given by

$$\bar{x} = \frac{1}{n}\sum x = \frac{31}{12} = 2.58$$

The SDs of the sample is given by

$$s^2 = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2 = \frac{1}{12} \times 185 - (2.58)^2 = 8.76$$

$\therefore$ $s = 2.96$

H_0: $\bar{x} = \mu$,

where $\mu = 0$, i.e. the injection will not result in increase in BP.

H_1: $\bar{x} > \mu$.

Right-tailed test is to be used. Let LOS be 5%. Now $t_{0.05\%}$ for one-tailed test for ($\nu = 11$) = $t_{10\%}$ for two-tailed test for ($\nu = 11$) = 1.80 (from t-table)

Now $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{2\cdot58 - 0}{2\cdot96/\sqrt{11}} = 2.89$$

We see that $|t| > t_{10\%}$ ($\nu = 11$).

Therefore, H_0 is rejected and H_1 is accepted. That is, we may conclude that the injection is accompanied by an increase in BP.

Example 4 *The mean lifetime of a sample of 25 bulbs is found as 1550 hours with a SD of 120 h. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 h. Is the claim acceptable at 5% LOS?*

Solution Here $\bar{x} = 1550$, $s = 120$, $n = 25$ and $\mu = 1600$.

H_0: $\bar{x} = \mu$ and H_1: $\bar{x} < \mu$.

Left-tailed test is to be used. LOS be 5%.

Now $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{-50\sqrt{24}}{120} = -2.04 \quad \text{and} \quad \nu = 24$$

$t_{0.05}$ for one-tailed test = $t_{0.1}$ for two-tailed test (for $\nu = 24$) = 1.71.

We see that $|t| > |t_{0.1}|$.

Therefore, H_0 is rejected and H_1 is accepted at 5% LOS. That is, the claim of the company cannot be accepted at 5% LOS.

Example 5 *The heights of 10 males of a given locality are found to be 175, 168, 155, 170, 152, 170, 175, 160, 160 and 165 cms. Based on this sample, find the 95% confidence limits for the height of males in that locality.*

Solution We shall first find the mean $\bar{x}$ and SDs of the sample, by taking the assumed mean $A = 165$ (see the following table).

$$d_i = x_i - A$$

$$\therefore \quad \bar{x} = A + \bar{d}$$

$$= 165 + \frac{1}{10} \times 0 = 165$$

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 578 = 57.8$$

$$\therefore \quad s = 7.6$$

From the t-table,

$$t_{0.05}\,(\nu = 9) = 2.26.$$

The 95% confidence limits for μ are

$$\left(\bar{x} - 2{\cdot}26\frac{s}{\sqrt{n-1}},\ \bar{x} + 2{\cdot}26\frac{s}{\sqrt{n-1}}\right)$$

i.e.

$$\left(165 - \frac{2{\cdot}26 \times 7{\cdot}6}{\sqrt{9}},\ 165 + \frac{2{\cdot}26 \times 7{\cdot}6}{\sqrt{9}}\right)$$

i.e. (159.3, 170.7)

i.e., the heights of males in the locality are likely to lie within 159.3 cm and 170.7cm.

x_i	$d_i = x_i - A$	d_i^2
175	10	100
168	3	9
155	– 10	100
170	5	25
152	– 13	169
170	5	25
175	10	100
160	– 5	25
160	– 5	25
165	0	0
Total	0	578

Example 6 *Two independent samples of sizes 8 and 7 contained the following values:*

Sample 1 :	*19*	*17*	*15*	*21*	*16*	*18*	*16*	*14*
Sample 2 :	*15*	*14*	*15*	*19*	*15*	*18*	*16*	

Is the difference between the sample means significant?

Solution

	Sample 1			*Sample 2*	
x_1	$d_1 = x_1 - 18$	d_1^2	x_2	$d_2 = x_2 - 16$	d_2^2
19	1	1	15	– 1	1
17	–1	1	14	– 2	4
15	– 3	9	15	– 1	1
21	3	9	19	3	9
16	– 2	4	15	– 1	1
18	0	0	18	2	4
16	– 2	4	16	0	0
14	– 4	16			
Total	– 8	44	Total	0	20

For sample 1,

$$\bar{x}_1 = 18 + \bar{d}_1 = 18 + \frac{1}{8}\sum d_1$$

$$= 18 + \frac{1}{8} \times (-8) = 17.$$

$$s_1^2 = \frac{1}{n_1}\sum d_1^2 - \left(\frac{1}{n_1}\sum d_1\right)^2$$

$$= \frac{1}{8} \times 44 - \left[\frac{1}{8} \times (-8)\right]^2 = 4.5$$

$\therefore$ $s_1 = 2.12.$

For sample 2,

$$\bar{x}_2 = 16 + \bar{d}_2 = 16 + \frac{1}{7}\sum d_2 = 16$$

$$s_2^2 = \frac{1}{n_2}\sum d_2^2 - \left(\frac{1}{n_2}\sum d_2\right)^2$$

$$= \frac{1}{7} \times 20 - \left(\frac{1}{7} \times 0\right)^2 = 2.857$$

$\therefore$ $s_2 = 1.69$

$$H_0: \bar{x}_1 = \bar{x}_2 \quad \text{and} \quad H_1: \bar{x}_1 \neq \bar{x}_2$$

Two-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{17 - 16}{\sqrt{\left(\frac{8 \times 4.5 + 7 \times 2.857}{13}\right)\left(\frac{1}{8} + \frac{1}{7}\right)}}$$

$$= 0.93$$

Also $\quad v = n_1 + n_2 - 2 = 13.$

From the t-table, $t_{0.05}$ $(v = 13) = 2.16$

Since $|t| < t_{0.05}$, H_0 is accepted and H_1 is rejected.

That is, the two sample means do not differ significantly at 5% LOS.

Example 7 *The following data represent the biological values of protein from cow's milk and buffalo's milk at a certain level.*

Cow's milk	:	1.82	2.02	1.88	1.61	1.81	1.54
Buffalo's milk	:	2.00	1.83	1.86	2.03	2.19	1.88

Examine if the average values of protein in the two samples significantly differ.

Solution $n = 6$

$$\bar{x}_1 = \frac{1}{6} \times 10.68 = 1.78$$

$$s_1^2 = \frac{1}{6} \times \Sigma x_1^2 - (\bar{x}_1)^2 = \frac{1}{6} \times 19.167 - (1.78)^2 = 0.0261$$

$$\bar{x}_2 = \frac{1}{6} \times 11.79 = 1.965$$

$$s_2^2 = \frac{1}{6} \times \Sigma x_2^2 - (\bar{x}_2)^2 = \frac{1}{6} \times 23.2599 - (1.965)^2 = 0.0154$$

As the two samples are independent, the test statistic is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2 + s_2^2}{n - 1}}}$$

with $v = 2n - 2$ (refer to Note 2 under Test 2)

$$t = \frac{1.78 - 1.965}{\sqrt{\frac{0.0261 + 0.0154}{5}}} = \frac{-0.185}{\sqrt{.0083}} = -2.03 \text{ and } \quad v = 10$$

$$H_0: \bar{x}_1 = \bar{x}_2 \quad \text{and} \quad H_1: \bar{x}_1 \neq \bar{x}_2.$$

Two-tailed test is to be used. Let LOS be 5%.

From t-table, $t_{0.05}$ $(v = 10) = 2.23$.

Since $|t| < t_{0.05}$ (ν = 10). H_0 is accepted. That is, the difference between the mean protein values of the two varieties of milk is not significant at 5% level.

Example 8 *Samples of two types of electric bulbs were tested for length of life and the following data were obtained.*

	Size	*Mean*	*SD*
Sample 1	*8*	*1234 h*	*36 h*
Sample 2	*7*	*1036 h*	*40 h*

Is the difference in the means sufficient to warrant that type 1 bulbs are superior to type 2 bulbs?

Solution *Here* $\bar{x}_1 = 1234$, $s_1 = 36$, $n_1 = 8$; $\bar{x}_2 = 1036$, $s_2 = 40$, $n_2 = 7$

H_0: $\bar{x}_1 = \bar{x}_2$; H_1: $\bar{x}_1 > \bar{x}_2$.

Right-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{198}{\sqrt{\left(\frac{21568}{13}\right)\left(\frac{1}{8} + \frac{1}{7}\right)}} = \frac{198}{21{\cdot}0807}$$

$$= 9.39$$

$$\nu = n_1 + n_2 - 2 = 13$$

$t_{0.05}$ (ν = 13) for one-tailed test = $t_{10\%}$ (ν = 13) for two-tailed test = 1.77 (from t-table).

Now $t > t_{0.1}$ (ν = 13). Therefore, H_0 is rejected and H_1 is accepted. That is, type 1 bulbs may be regarded superior to type 2 bulbs at 5% LOS.

Example 9 *The mean height and the SD height of 8 randomly chosen soliders are 166.9 and 8.29 cm respectively. The corresponding values of 6 randomly chosen sailors are 170.3 and 8.50 cm respectively. Based on this data, can we conclude that soldiers are, in general, shorter than sailors?*

Solution Here $\bar{x}_1 = 166.9$, $s_1 = 8.29$, $n_1 = 8$; $\bar{x}_2 = 170.3$, $s_2 = 8.50$, $n_2 = 6$.

H_0: $\bar{x}_1 = \bar{x}_2$; H_1: $\bar{x}_1 < \bar{x}_2$.

Left-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{-3{\cdot}4}{\sqrt{\left(\frac{983{\cdot}29}{12}\right)\left(\frac{1}{8} + \frac{1}{6}\right)}}$$

$$= -0.695$$

$$\nu = n_1 + n_2 - 2 = 12$$

$t_{0.05}$ (ν = 12) for one-tailed test = $t_{0.1}$ (ν = 12) for two-tailed test = 1.78 (from t-table)

Now $|t| < t_{0.1}$ ($\nu = 12$). Therefore, H_0 is accepted and H_1 is rejected. That is, based on the given data, we cannot conclude that soldiers are, in general, shorter than sailors.

Example 10 *The following data relate to the marks obtained by 11 students in 2 tests, one held at the beginning of a year and the other at the end of the year after intensive coaching.*

Test 1:	*19*	*23*	*16*	*24*	*17*	*18*	*20*	*18*	*21*	*19*	*20*
Test 2:	*17*	*24*	*20*	*24*	*20*	*22*	*20*	*20*	*18*	*22*	*19*

Do the data indicate that the students have benefited by coaching?

Solution The given data relate to the marks obtained in 2 tests by the same set of students. Hence the marks in the 2 tests can be regarded as correlated and so the t-test for paired values should be used.

Let $d = x_1 - x_2$, where x_1, x_2 denote the marks in the 2 tests.

Thus the values of d are 2, – 1 – 4, 0, – 3, – 4, 0, – 2, 3, – 3, 1.

$$\Sigma d = -11 \quad \text{and} \quad \Sigma d^2 = 69$$

$$\therefore \qquad \bar{d} = \frac{1}{n}\Sigma d = \frac{1}{11} \times (-11) = -1$$

$$s^2 = s^2{}_d = \frac{1}{n}\Sigma d^2 - (\bar{d})^2 = \frac{1}{11} \times 69 - (-1)^2 = 5.27$$

$$\therefore \qquad s = 2.296$$

H_0: $\bar{d} = 0$, i.e. the students have not benefitted by coaching; H_1: $\bar{d} < 0$ (i.e. $\bar{x}_1 < \bar{x}_2$).

One-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-1}{2{\cdot}296/\sqrt{10}} = -1.38 \quad \text{and} \quad \nu = 10$$

$t_{0.05}$ (ν = 10) for one-tailed test = $t_{0.1}$ (ν = 10) for two-tailed test = 1.81 (from t-table).

Now $|t| < t_{10}$ ($\nu = 10$). Therefore, H_0 is accepted and H_1 is rejected, i.e., there is no significant difference between the two sets of marks.

That is, the students have not benefitted by coaching.

Example 11 *A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variance?*

Solution Here, $n_1 = 13$, $\hat{\sigma}_1^2 = 3.0$ and $\nu_1 = 12$, $n_2 = 15$, $\hat{\sigma}_2^2 = 2.5$ and $\nu_2 = 14$.

H_0: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, i.e. the two samples have been drawn from populations with the same variance. $H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

Let LOS. be 5%.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{3 \cdot 0}{2 \cdot 5} = 1.2$$

$$\nu_1 = 12 \quad \text{and} \quad \nu_2 = 14.$$

$F_{0.05\%}$ ($\nu_1 = 12$, $\nu_2 = 14$) = 2.53, from the F-table. Since $F < F_{0.05}$, H_0 is accepted. That is the two samples could have come from two normal populations with the same variance.

Example 12 *Two samples of sizes 9 and 8 gave the sums of squares of deviations from their respective means equal to 160 and 91 respectively. Can they be regarded as drawn from the same normal population?*

Solution $n_1 = 9$, $\Sigma(x_i - \bar{x})^2 = 160$, i.e. $n_1 s_1^2 = 160$

$n_2 = 8$, $\Sigma(y_i - \bar{y})^2 = 91$, i.e. $n_2 s_2^2 = 91$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{1}{8} \times 160 = 20\,; \quad \hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{1}{7} \times 91 = 13$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = n_1 - 1 = 8$ and $\nu_2 = n_2 - 1 = 7$

$$H_0: \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1: \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2.$$

Let the LOS be 5%.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{20}{13} = 1.54$$

$F_{0.05}$ ($\nu_1 = 8$, $\nu_2 = 7$) = 3.73, from the F-table. Since $F < F_{0.05\%}$, H_0 is accepted.

That is, the two samples could have come from two normal populations with the same variance.

We cannot say that the samples have come from the same population, as we are unable to test whether the means of the samples differ significantly or not.

Example 13 *Two independent samples of 8 and 7 items respectively had the following values of the variable.*

Sample 1 :	*9*	*11*	*13*	*11*	*15*	*9*	*12*	*14*
Sample 2 :	*10*	*12*	*10*	*14*	*9*	*8*	*10*	

Do the two estimates of population variance differ significantly at 5% LOS?

Solution For the first sample, $\Sigma x_1 = 94$ and $\Sigma x_1^2 = 1138$.

$$\therefore \qquad s_1^2 = \frac{1}{n_1} \sum x_1^2 - \left(\frac{1}{n_1} \sum x_1 \right)^2$$

$$= \frac{1}{8} \times 1138 - \left(\frac{1}{8} \times 94\right)^2 = 4.19$$

For the second sample, $\Sigma x_2 = 73$ and $\Sigma x_2^2 = 785$.

$$\therefore \qquad s_2^2 = \frac{1}{n_2} \sum x_2^2 - \left(\frac{1}{n_2} \sum x_2\right)^2$$

$$= \frac{1}{7} \times 785 - \left(\frac{1}{7} \times 73\right)^2 = 3.39$$

$$\hat{\sigma}_1^2 = \frac{n_1}{n_1 - 1} s_1^2 = 4.79 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{n_2}{n_2 - 1} s_2^2 = 3.96$$

since $\qquad \hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = 7$ and $\nu_2 = 6$

$$H_0: \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1: \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{4.79}{3.96} = 1.21$$

$F_{0.05}$ ($\nu_1 = 7$, $\nu_2 = 6$)= 4.21, from the F-table. Since $F < F_{0.05}$, H_0 is accepted. That is, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ do not differ significantly at 5% LOS.

Example 14 *Two random samples gave the following data:*

Sample no.	*Size*	*Mean*	*Variance*
1	*8*	*9.6*	*1.2*
2	*11*	*16.5*	*2.5*

Can we conclude that the two samples have been drawn from the same normal population?

Solution Refer to Note 2 under F-test. To conclude that the two samples have been drawn from the same population, we have to check first that the variances of the populations do not differ significantly and then check that the sample means (and hence the population means) do not differ significantly.

$$\hat{\sigma}_1^2 = \frac{8 \times 1.2}{7} = 1.37 \; ; \quad \hat{\sigma}_2^2 = \frac{11 \times 2.5}{10} = 2.75$$

$$F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{2.75}{1.37} = 2.007 \text{ with } \nu_1 = 10 \text{ and } \nu_2 = 7.$$

From the F-table, $F_{0.05}$ (10, 7) = 3.64.

If H_0: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$ and H_1: $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$, then, H_0 is accepted, since $F < F_{0.05}$.

That is, the variances of the populations from which samples are drawn may be regarded as equal.