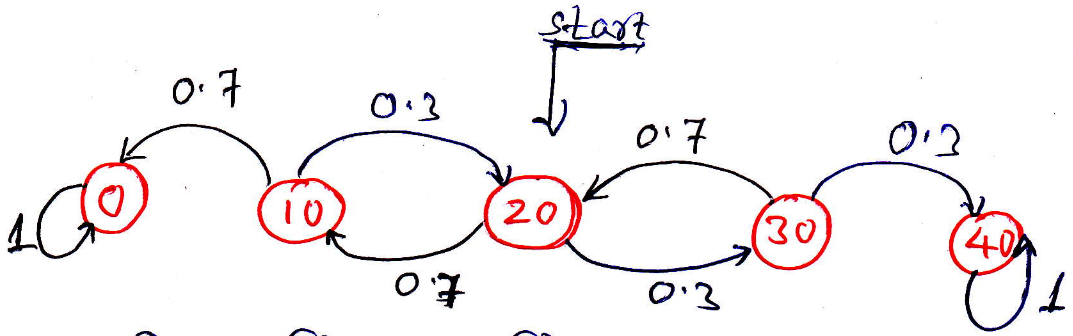
Stochastic Process

Ex. 1 Suppose Mr. X has ₹ 20 initially and he bets ₹ 10 each game on a gamble. Probability that Mr. X wins is 0.3 & lose with probability 0.7. If Mr. X has ₹ 0 or ₹ 40 the game is over.

Let X_t is the money Mr. X has after each gamble (each instant of game) & $X_0 = ₹\,20$



States: 1 2 3 4 5

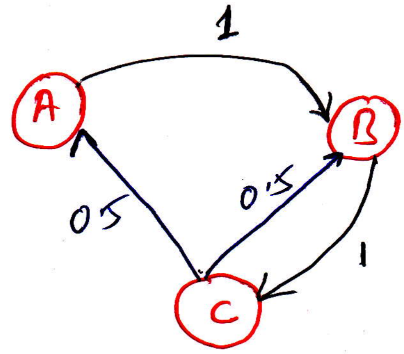
sequence of RVs $= \{X_0, X_1, X_2, X_3, \ldots\ldots\}$ ↳ stochastic process

$$P = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

→ Stochastic matrix or Transition probability matrix (TPM)

Ex 2 Three boys A, B & C are throwing ball to each other. 'A' always throws the ball to B and B always throws the ball to C, but C is just likely to throw the ball to B as to A. C begins the game.

Let $X_t : (a, b, c)$, where a, b & c takes 1 or 0 depending on a, b or c has the ball or not



States : A, B, C

Sequence of Rvs = $\{X_0, X_1, X_2, \ldots\}$ → stochastic process

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 0.5 & 0.5 & 0 \end{array}$$

→ stochastic matrix (TPM)

Stochastic Process: A set or family of random variables $\{X_t : t \in T \subset R\}$ defined on a sample space S with parameter 't' is called stochastic process & set T is called index set.

The values taken by random variable X_t are called states & set of all possible values is called state space.

Note:

Index set T (Parameter) → Discrete / → Continuous

State space → Discrete / → Continuous

A discrete space discrete parameter stochastic process $\langle X_t \rangle = \langle X_0, X_1, X_2, \ldots \rangle$ is said to be a Markov chain if the probability of the state at time (parameter) $t+1$ depends only on the state at time (parameter) t and does not depend on the states before time t.

ie $P[X_{t+1} = i_{t+1} \mid X_t = i_t, X_{t-1} = i_{t-1}, \ldots, X_1 = t_1, X_0 = t_0]$

$= P[X_{t+1} = i_{t+1} \mid X_t = i_t]$

Stochastic matrix: A square matrix $P = (P_{ij})$ in which rows & columns represent states of the process is said to be stochastic matrix OR Transition Probability Matrix (TPM) if

(i) $P_{ij} \geq 0 \ \forall i, j$ (ii) Row sum: $\sum_j P_{ij} = 1 \ \forall i$

Here P_{ij} represents probability of process moving from i^{th} state to j^{th} state in single instant of time

Each row of a stochastic matrix is called probability vector OR any vector $v = \langle v_1, v_2 - v_n \rangle$ is said to be probability vector if (i) $v_i \geq 0 \ \forall i$ (ii) $\sum_{i=1}^{n} v_i = 1$

Ex: $(0, 1)$, $(1/2 \ 1/2)$, $(1/3, 1/3, 1/3)$, $(1/4 \ 1/2 \ 1/4)$.

A stochastic matrix P is said to be **regular** if all entries of some powers of P are strictly positive.

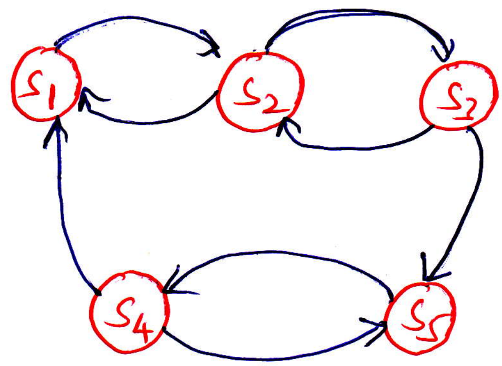
Ex ① $P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ is regular because $P^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$

② $P = \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix}$ is regular as all entries are positive.

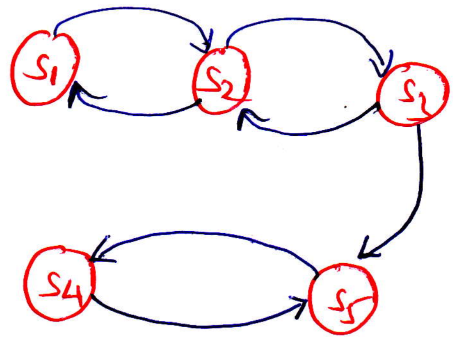
A Markov chain is said to be irreducible ~~regular~~ if corresponding stochastic matrix is regular.

Meaning: If Markov chain is irreducible then regardless the present state one can reach any other state in finite time.

Ex.

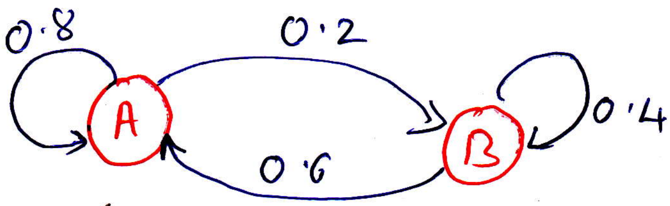


Irreducible



Not irreducible

Example: Throwing ball to each other



TPM: $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}$

A begins the game:

Initial prob. vector: $p^{(0)} = (1 \; 0)$

After 1^{st} throw: $p^{(1)} = p^{(0)} P = (1 \; 0)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.8 \; 0.2)$

After 2^{nd} throw: $p^{(2)} = p^{(1)} P = (0.8 \; 0.2)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.72 \; 0.28)$

After 3^{rd} throw: $p^{(3)} = p^{(2)} P = (0.688 \; 0.312)$

$\vdots$

After 7^{th} throw: $p^{(7)} = p^{(6)} P = (0.667 \; 0.333)$

After 8^{th} throw: $p^{(8)} = p^{(7)} P = (0.667 \; 0.333)$?

After 9^{th} throw: $p^{(9)} = p^{(8)} P = (0.667 \; 0.333)$

Here $(0.667 \ \ 0.333) = (2/3 \ \ 1/3)$ is called fixed probability vector or steady state probability vector.

Meaning: If game continues for a long or on a long run 66% of the time ball is with A & 33% of the time ball is with B.

Property of fixed probability vector

$\rightarrow (2/3 \ \ 1/3) P = (2/3 \ \ 1/3) \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (2/3 \ \ 1/3)$

In general, if v is a fixed probability vector of a process ~~then~~ $vP = v$.

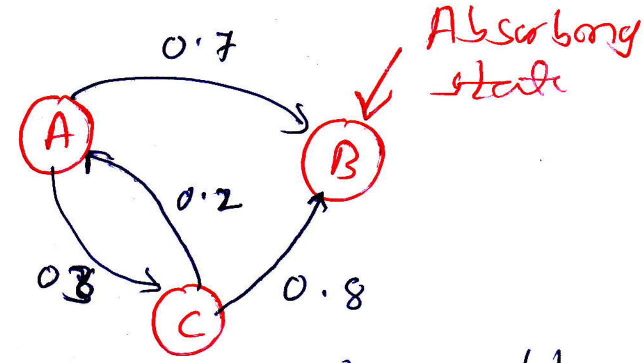
Remember:

(I) For 2 state process: $v = (x \ \ y)$, $x + y = 1$

(II) For 3 state process: $v = (x \ \ y \ \ z)$, $x + y + z = 1$

Different states

① Absorbing state



In a Markov chain, if the process reaches to a certain state after which it continues to remain in the same state is called absorbing state

② Recurrent state:

A state i is said to be recurrent iff starting from state i the chain (process) eventually return to the state i with probability 1

③ Transient state:

A state i is said to be transient (non-recurrent) iff there is positive probability that the process will not return to that state.