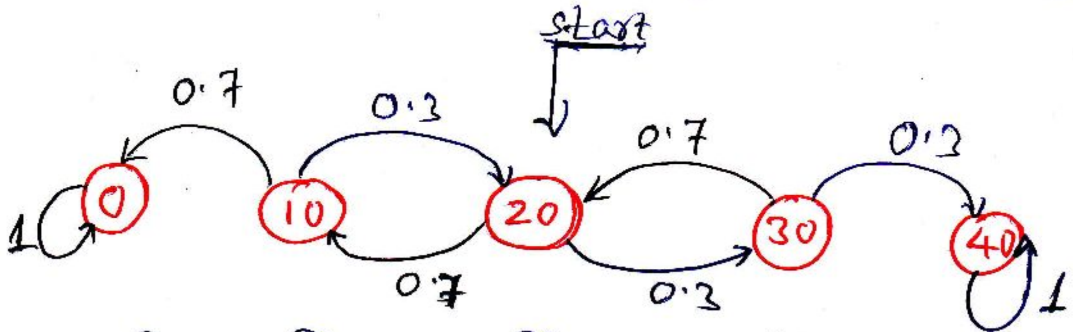
18MAT753 **Stochastic Process** Module-5 (Topic-1) ①

Ex.① Suppose Mr. X has ₹ 20 initially and he bets ₹ 10 each time on a gamble. Probability that Mr. X wins is 0.3 & lose with probability 0.7. If Mr. X has ₹ 0 or ₹ 40 the game is over.

Let X_t is the money Mr. X has after each gamble (each instant of time) & $X_0 = ₹\ 20$



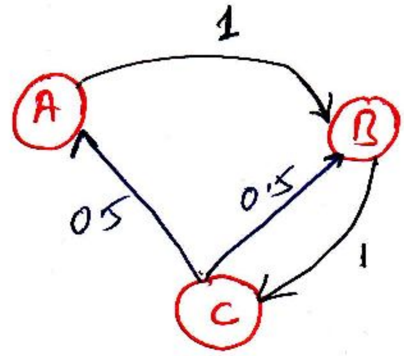
States: ① ② ③ ④ ⑤

sequence of RVs $= \{X_0, X_1, X_2, X_3 \ldots\ldots\}$ → stochastic process

$$P = \begin{array}{c|ccccc} & ① & ② & ③ & ④ & ⑤ \\ \hline ① & 1 & 0 & 0 & 0 & 0 \\ ② & 0.7 & 0 & 0.3 & 0 & 0 \\ ③ & 0 & 0.7 & 0 & 0.3 & 0 \\ ④ & 0 & 0 & 0.7 & 0 & 0.3 \\ ⑤ & 0 & 0 & 0 & 0 & 1 \end{array}$$

→ Stochastic matrix or Transition probability matrix (TPM)

Ex (2) Three boys A, B & C are throwing ball to each other. 'A' always throws the ball to B and B always throws the ball to C, but C is just likely to throw the ball to B as to A. C begins the game.
Let $X_t : (a, b, c)$, where a, b & c takes 1 or 0 depending on a, b or c has the ball or not



States : A, B, C

Sequence of RVs = $\{X_0, X_1, X_2, \dots\}$ ↳ stochastic process

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

→ stochastic matrix (TPM)

Stochastic Process: A set or family of random variables $\{X_t : t \in T \subset R\}$ defined on a sample space S with parameter 't' is called stochastic process & set T is called index set.

The values taken by random variable X_t are called states & set of all possible values is called state space.

Note:

Index set T (Parameter) → Discrete / continuous

state space → Discrete / continuous

A discrete space discrete parameter stochastic process $\langle X_t \rangle = \langle X_0, X_1, X_2, \ldots \rangle$ is said to be a Markov chain if the probability of the state at time (parameter) $t+1$ depends only on the state at time (parameter) t and does not depend on the states before time t.

ie $P[X_{t+1} = i_{t+1} \mid X_t = i_t, X_{t-1} = i_{t-1}, \ldots X_1 = i_1, X_0 = i_0]$

$= P[X_{t+1} = i_{t+1} \mid X_t = i_t]$

Stochastic matrix: A square matrix $P = (P_{ij})$ in which rows & columns represent states of the process is said to be stochastic matrix OR Transition Probability Matrix (TPM) if

(i) $P_{ij} \geq 0 \ \forall i, j$ (ii) Row sum: $\sum_j P_{ij} = 1 \ \forall i$

Here P_{ij} represents probability of process moving from i^{th} state to j^{th} state in single instant of time.

Each row of a stochastic matrix is called probability vector OR any vector $v = \langle v_1, v_2 \cdots v_n \rangle$ is said to be probability vector if (i) $v_i \geq 0 \ \forall i$ (ii) $\sum_{i=1}^{n} v_i = 1$

Ex: $(0 \ 1)$, $(\frac{1}{2} \ \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{4} \ \frac{1}{2} \ \frac{1}{4})$

A stochastic matrix P is said to be regular if all entries of some power of P are strictly positive.

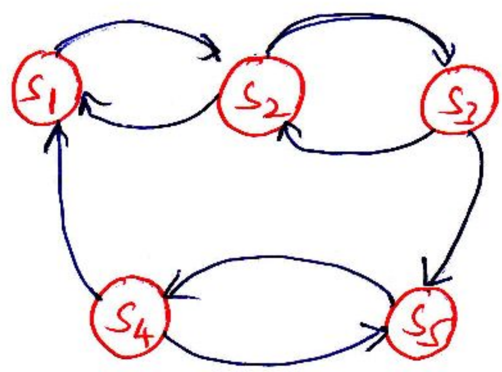
Ex (1) $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is regular because $P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

(2) $P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ is regular as all entries are positive.

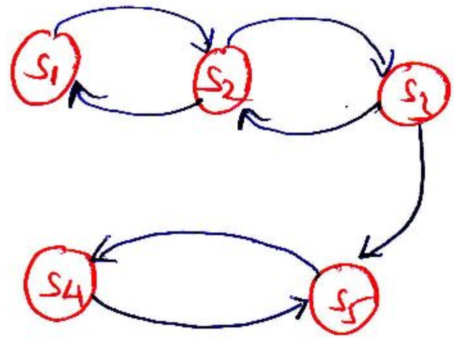
A Markov chain is said to be irreducible ~~regular~~ if corresponding stochastic matrix is regular.

Meaning: If Markov chain is irreducible then regardless the present state one can reach any other state in finite time.

Ex.

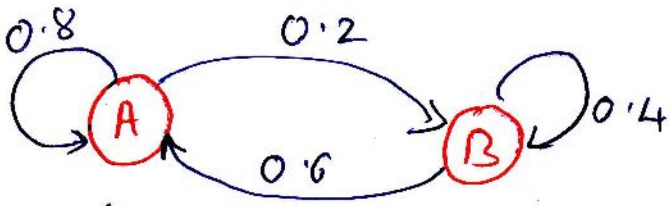


Irreducible



Not irreducible

Example: Throwing ball to each other



TPM: $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}$

A begins the game:

Initial prob. vector: $p^{(0)} = (1 \;\; 0)$

After 1st throw: $p^{(1)} = p^{(0)} P = (1 \;\; 0)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.8 \;\; 0.2)$

After 2nd throw: $p^{(2)} = p^{(1)} P = (0.8 \;\; 0.2)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.72 \;\; 0.28)$

After 3rd throw: $p^{(3)} = p^{(2)} P = (0.688 \;\; 0.312)$

$\vdots$

After 7th throw: $p^{(7)} = p^{(6)} P = (0.667 \;\; 0.333)$

After 8th throw: $p^{(8)} = p^{(7)} P = (0.667 \;\; 0.333)$?

After 9th throw: $p^{(9)} = p^{(8)} P = (0.667 \;\; 0.333)$

Here $(0.667 \;\; 0.333) = (2/3 \;\; 1/3)$ is called fixed probability vector or steady state probability vector.

Meaning: If game continues for a long or on a long run 66% of the time ball is with A & 33% of the time ball is with B.

Property of fixed probability vector

$\rightarrow (2/3 \;\; 1/3)\, P = (2/3 \;\; 1/3)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (2/3 \;\; 1/3)$

In general if v is a fixed probability vector of a process then $vP = v$.

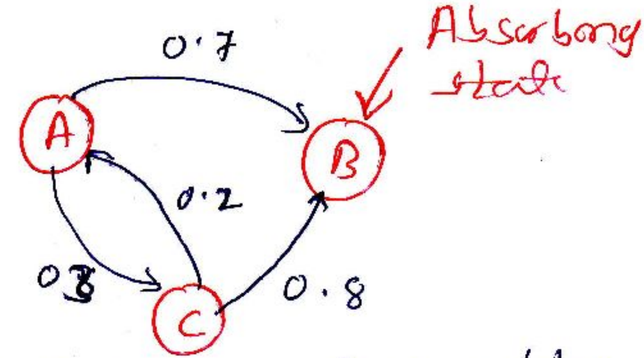
Remember:

(i) For 2 state process: $v = (x \;\; y)$, $x+y=1$

(ii) For 3 state process: $v = (x \;\; y \;\; z)$, $x+y+z=1$

Different states

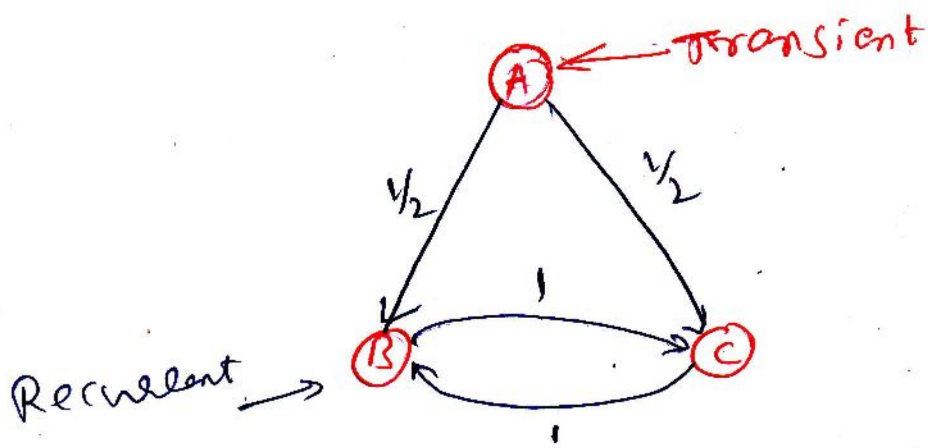
① **Absorbing state**



In a Markov chain, if the process reaches to a certain state after which it continues to remain in the same state is called absorbing state

② **Recurrent state**: A state i is said to be recurrent if starting from state i the chain (process) eventually return to the state i with probability 1

③ **Transient state**: A state i is said to be transient (non-recurrent) if there is positive possibility that the process will not return to that state.



Examples

Ex 1 Find fixed probability vector of a regular stochastic matrix

$$P = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Solⁿ Let $v = (x, y)$ be fixed probability vector

$\Rightarrow x + y = 1 \rightarrow (1)$

we have

$$vP = v$$

$\Rightarrow (x \; y) \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix} = (x \; y)$

$\Rightarrow \frac{x}{4} + \frac{y}{2} = x$ & $\frac{3x}{4} + y/2 = y$

$\Rightarrow y/2 = 3x/4$ & $\frac{3x}{4} = y/2$

$\Rightarrow y = \frac{3x}{2}$ & $y = 3x/2$ From (1) $x + y = 1$

$\Rightarrow x + \frac{3x}{2} = 1 \quad \therefore x = 2/5$ & $y = 3/5$

$\therefore v = (x \; y) = (2/5 \;\; 3/5)$

Ex 2 Find fixed probability vector of a regular stochastic matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

Solⁿ Let $v = (x \; y \; z)$ be fixed probability vector of P & $x + y + z = 1 \rightarrow (1)$

we have

$$\vartheta P = \vartheta \Rightarrow (x\ y\ z)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (x\ y\ z)$$

$$\Rightarrow \frac{y}{6} = x, \quad x + \frac{y}{2} + \frac{2z}{3} = y, \quad y/3 + z/3 = z$$

$$\Rightarrow \boxed{y = 6x} \qquad x + \frac{2z}{3} = y/2 \qquad y/3 = 2z/3$$

$$\Rightarrow \boxed{y = 6x} \quad \boxed{y = 2z}, \quad \boxed{x = z/3}$$

From (1) $x + y + z = 1 \Rightarrow x + 6x + 3x = 1$

$\therefore x = 1/10, \quad y = 6/10, \quad z = 3/10$

$\therefore \vartheta = (1/10\ \ 6/10\ \ 3/10)$

Ex (3) Prove ~~that~~ a Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$ is irreducible and also determine ~~the~~ corresponding stationary probability vector.

Solⁿ $P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$

$$P^2 = P.P = \frac{1}{36}\begin{pmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$$

$$= \frac{1}{36}\begin{pmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{pmatrix}$$

All entries of P^2 are strictly positive, hence P is regular & Markov chain is irreducible

Let $v = (x\ y\ z)$ be fixed probability vector

& $x+y+z=1$ — (1)

we have

$$vP = v \Rightarrow (x\ y\ z)\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (x\ y\ z)$$

$$\Rightarrow \frac{y}{2}+\frac{z}{2}=x;\quad \frac{2x}{3}+z/2=y;\quad \frac{x}{3}+\frac{y}{2}=z$$

$$\Rightarrow y+z=2x \text{ — (2)} \quad 4x+3z=6y \text{ — (3)} \quad 2x+3y=6z \text{ — (4)}$$

From (1) & (2) $\quad x+2x=1 \quad \therefore x=1/3$

From (3) & (4)

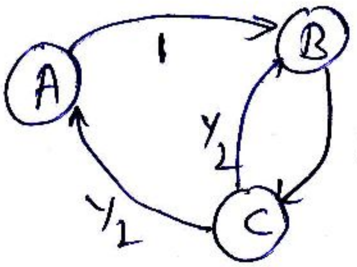
$$\frac{4}{3}+3z=6y \qquad\Rightarrow\qquad 4+9z=18y$$
$$2/3+3y=6z \qquad\qquad 2+9y=18z$$

$$\Rightarrow \begin{matrix} 4+18y+9z=0 \\ 4+18y-36z=0 \end{matrix} \Rightarrow \begin{matrix} 8-27z=0 \quad \therefore z=8/27 \\ \& \ y=10/27 \end{matrix}$$

$$\therefore v = \left(1/3 \quad 10/27 \quad 8/27\right)$$

Ex (4) Three boys A, B & C are throwing ball to each other. A always throws the ball to B & B always throws the ball to C, but C is just likely to throw the ball to B as to A. If C was the person to throw the ball, find the probabilities that (i) A has the ball (ii) B has the ball (iii) C has the ball, after 3 throws.

Sol^n



$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

Initial C has the ball $\therefore p^{(0)} = (0\ 0\ 1)$

After 1st throw: $p^{(1)} = p^{(0)} P$

$$\Rightarrow p^{(1)} = (0\ 0\ 1) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (1/2\ \ 1/2\ \ 0)$$

After 2nd throw: $p^{(2)} = p^{(1)} P$

$$\Rightarrow p^{(2)} = (1/2\ \ 1/2\ \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (0\ \ 1/2\ \ 1/2)$$

After 3rd throw: $p^{(3)} = p^{(2)} P$

$$\Rightarrow p^{(3)} = (0\ \ 1/2\ \ 1/2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (1/4\ \ 1/4\ \ 1/2)$$

After 3 throws, probability that

(i) A has ball $= 1/4 = 0.25$

(ii) B has ball $= 1/4 = 0.25$

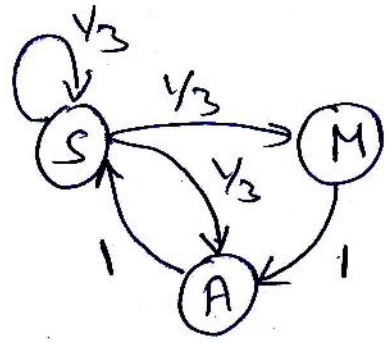
(iii) C has ball $= 1/2 = 0.5$

Ex (5) Every year a man trades his car for a new car. If he has Maruti, he trades it for an Ambassador, if he has an ambass--adar he trades it for santro, however if he has santro he his just likely to trade it for a new Santro as to trade for a Maruti or an Ambassador. In 2016 he bought his

(ii)

first car which was santro, find the probability ~~that~~ he has (i) 2018 santro (ii) 2018 Maruti.

Sol^n



$$\therefore P = \begin{array}{c c} & \begin{array}{ccc} M & A & S \end{array} \\ \begin{array}{c} M \\ A \\ S \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

Initially (ie on 2016) he has santro

$$\therefore p^{(0)} = (0 \;\; 0 \;\; 1)$$

After 1 year (on 2017) : $p^{(1)} = p^{(0)} P$

$$\Rightarrow p^{(1)} = (0 \;\; 0 \;\; 1) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = (1/3 \;\; 1/3 \;\; 1/3)$$

After 2 years (on 2018) : $p^{(2)} = p^{(1)} P$

$$\Rightarrow p^{(2)} = (1/3 \;\; 1/3 \;\; 1/3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = (1/9 \;\; 4/9 \;\; 4/9)$$

$\therefore$ In 2018 probability of having

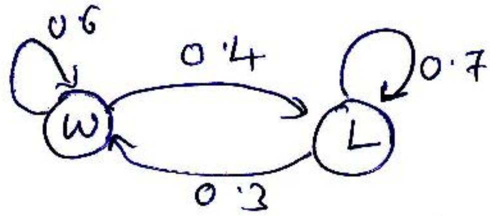
(i) Maruti $= 1/9$

(ii) Ambassador $= 4/9$

(iii) Santro $= 4/9$

Ex ⑥ A man's gambling luck follows a pattern. If he wins a game the probability of winning the next game is 0.6. However, if he loses ~~the~~ a game, the probability of losing the next game is 0.7. There is an even chance that he wins the first game. (i) Find prob of winning 3rd game (ii) In long run how often he wins

Soln



$$P = \begin{array}{c} \\ W \\ L \end{array}\begin{array}{c} \begin{array}{cc} W & L \end{array} \\ \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

(i) Initially (prob. he wins 1st game): $p^{(0)} = (1/2 \;\; 1/2)$

After 1st game (prob. he wins 2nd game):

$$p^{(1)} = p^{(0)} P = (1/2 \;\; 1/2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (0.45, 0.55)$$

After 2nd game (prob. of winning 3rd game):

$$p^{(2)} = p^{(1)} P = (0.45 \;\; 0.55)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (0.435, 0.565)$$

$\therefore$ Probability of winning 3rd game $= 0.435$

(ii) Let $v = (x \;\; y)$ be fixed probability vector & $x + y = 1$

we have $vP = v \Rightarrow (x \;\; y)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (x \;\; y)$

$\Rightarrow 0.6x + 0.3y = x$ & $0.4x + 0.7y = y$

$\Rightarrow 0.3y = 0.4x$ & $0.4x = 0.3y$

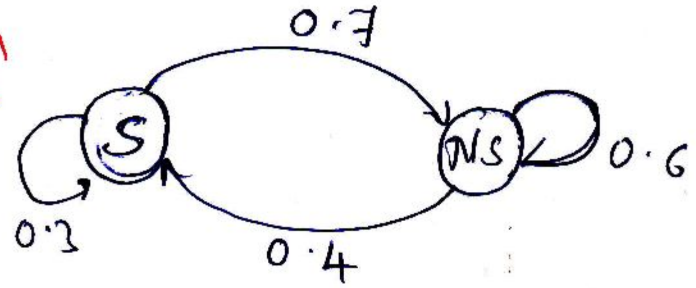
$\Rightarrow y = 4/3\, x$ & $x = 3/4\, y$

From ① $x + y = 1 \Rightarrow x + 4/3\, x = 1 \quad \therefore x = 3/7$ & $y = 4/7$

$\therefore v = (3/7, 4/7) \quad \therefore$ In long run, he wins $3/7$ of times

Ex 7 A Student's study habits are as follows, If he studies one night he is 70% sure not to study next night. On the other hand if he does not study one night he is 60% sure not to study the next night as well. In long run how often does he study?

Soln



$$P = \begin{array}{c|cc} & S & NS \\ \hline S & 0.3 & 0.7 \\ NS & 0.4 & 0.6 \end{array}$$

Let $v = (x \ y)$ be fixed probability vector & $x + y = 1$

we have $vP = v \Rightarrow (x \ y)\begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} = (x \ y)$

$\Rightarrow 0.3x + 0.4y = x$ & $0.7x + 0.6y = y$

$\Rightarrow 0.4y = 0.7x$ & $0.7x = 0.4y$

$\Rightarrow y = 7/4\, x$ & $x = 4/7\, y$

From, $x + y = 1 \Rightarrow x + 7/4\, x = 1 \therefore x = 4/11$ & $y = 7/11$

$\therefore v = (4/11, 7/11)$

In long run, he studies $4/11$ i.e. 36.36% of the time.