Binary Relations

Definition

Let A and B be any two sets. A *binary relation* R from A to B, written $R : A \leftrightarrow B$, is a subset of $A \times B$. The notation aRb means $(a, b) \in R$.

- If aRb, we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".

Example

$<: N \leftrightarrow N :\equiv \{(n, m) \mid n < m\}$. $a < b$ means $(a, b) \in <$.

- A binary relation R corresponds to a predicate function $P_R : A \times B \rightarrow \{T, F\}$ defined over the 2 sets A and B.

Examples of Binary Relations

- Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. For instance, we have $0Ra$, $0Rb$, etc..
  - Can we have visualized expressions of relations?
- Let A be the set of all cities, and let B be the set of the 50 states in the USA. Define the relation R by specifying that (a, b) belongs to R if city a is in state b. For instance, (Boulder,Colorado), (Bangor,Maine), (Ann Arbor,Michigan), (Middletown,New Jersey), (Middletown,New York), (Cupertino,California), and (Red Bank,New Jersey) are in R.
- "eats" $:\equiv \{(a, b) \mid \text{organism } a \text{ eats food } b\}$.

Complementary Relations

Definition

Let $R : A \leftrightarrow B$ be any binary relation. Then, $\overline{R} : A \leftrightarrow B$, the *complement* of R, is the binary relation defined by

$$\overline{R} :\equiv \{(a, b) \mid (a, b) \notin R\} = (A \times B) - R.$$

- Note this is just $\overline{R}$ if the universe of discourse is $U = A \times B$; thus the name complement.
- The complement of $\overline{R}$ is R.

Inverse Relations

Definition

Any binary relation $R : A \leftrightarrow B$ has an *inverse relation* $R^{-1} : B \leftrightarrow A$, defined by

$$R^{-1} :\equiv \{(b, a) \mid (a, b) \in R\}.$$

Examples

1. $<^{-1} = \{(b, a) \mid a < b\} = \{(b, a) \mid b > a\} = >.$
2. If $R : People \rightarrow Foods$ is defined by "$aRb \Leftrightarrow a$ eats b", then

$$bR^{-1}a \Leftrightarrow b \text{ is eaten by } a.$$

Examples

Example

Let $A = \{1, 2, 3, 4, 5\}$ and $R : A \leftrightarrow A :\equiv \{(a, b) : a \mid b\}$. What are $\overline{R}$ and R^{-1}?

Solution

- $R = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), \\ (3,3), (4,4), (5,5) \end{array} \right\}$

Examples

Example

Let $A = \{1, 2, 3, 4, 5\}$ and $R : A \leftrightarrow A :\equiv \{(a, b) : a \mid b\}$. What are $\overline{R}$ and R^{-1}?

Solution

- $R = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), \\ (3,3), (4,4), (5,5) \end{array} \right\}$
- $\overline{R} = \left\{ \begin{array}{l} (2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (3,5), \\ (4,1), (4,2), (4,3), (4,5), (5,1), (5,2), (5,3), \\ (5,4) \end{array} \right\}$

Examples

Example

Let $A = \{1,2,3,4,5\}$ and $R : A \leftrightarrow A :\equiv \{(a,b) : a \mid b\}$. What are $\overline{R}$ and R^{-1}?

Solution

- $R = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), \\ (3,3), (4,4), (5,5) \end{array} \right\}$
- $\overline{R} = \left\{ \begin{array}{l} (2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (3,5), \\ (4,1), (4,2), (4,3), (4,5), (5,1), (5,2), (5,3), \\ (5,4) \end{array} \right\}$
- $R^{-1} = \left\{ \begin{array}{l} (1,1), (2,1), (3,1), (4,1), (5,1), (2,2), (4,2), \\ (3,3), (4,4), (5,5) \end{array} \right\}$

Combining Relations

- Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined through set operations.
- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$
\begin{aligned}
R_1 \cup R_2 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}\,, \\
R_1 \cap R_2 &= \{(1, 1)\}\,, \\
R_1 - R_2 &= \{(2, 2), (3, 3)\} \\
R_2 - R_1 &= \{((1, 2), (1, 3), (1, 4)\}\,.
\end{aligned}
$$

- Quiz: What is $R_1 \oplus R_2$?

Composite Relations

- Let $R : A \leftrightarrow B$, and $S : B \leftrightarrow C$. Then the composite $S \circ R$ of R and S is defined as: $S \circ R = \{(a, c) \mid aRb \wedge bSc\}$.

  Example 1 Function composition $f \circ g$ is an example.

  Example 2 $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$.

  - $R : A \leftrightarrow B$, $R = \{(1, a), (1, b), (2, b), (2, c)\}$.
  - $S : B \leftrightarrow C$, $S = \{(a, x), (a, y), (b, y), (d, z)\}$.
  - $S \circ R = \{(1, x), (1, y), (2, y)\}$.

Relations on a Set

Definition

A (binary) relation from a set A to itself is called a relation on the set A.

- E.g., the "$<$" relation from earlier was defined as a relation on the set $\mathbb{N}$ of natural numbers.
- The *identity relation* I_A on a set A is the set $\{(a, a) \mid a \in A\}$.
- Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?
- How many relations are there on a set with n elements?

Reflexivity

Definition

A relation R on A is *reflexive* if $\forall a \in A, aRa$. A relation is *irreflexive* iff its complementary relation is reflexive.

- E.g., the relation $\geq :\equiv \{(a, b) \mid a \geq b\}$ is reflexive.
- E.g., $<$ is irreflexive.

- "irreflexive"$\neq$"not reflexive"!
- "likes" between people is not reflexive, but not irreflexive either. (Not everyone likes themselves, but not everyone dislikes themselves either.)

Example 7 from Textbook

Example

Consider the following relations on $\{1, 2, 3, 4\}$.

$$\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}, \\
R_2 &= \{(1,1),(1,2),(2,1),(2,2)\}, \\
R_3 &= \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}, \\
R_4 &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}, \\
R_5 &= \left\{ \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3), \\ (3,4),(4,4) \end{array} \right\}, \\
R_6 &= \{(3,4)\}.
\end{aligned}$$

Which of these relations are reflexive, irreflexive, and not reflexive?

Symmetry & Antisymmetry

Definition

- A binary relation R on A is *symmetric* iff $(a, b) \in R \leftrightarrow (b, a) \in R$, i.e. $R = R^{-1}$.
  - E.g., $=$ (equality) is symmetric, and $<$ is not.
  - "is married to" is symmetric, and "likes" is not.
- A binary relation R is *antisymmetric* if $(a, b) \in R \wedge (b, a) \in R \rightarrow a = b$.
  - E.g., $<$ is antisymmetric, and "likes" is not.

- Which relations from Example 7 are symmetric and which are antisymmetric?
- If R_1 is symmetric and R_2 is antisymmetric, is it true that $R_1 \cap R_2 = \varnothing$?

Transitivity

Definition

A relation R is *transitive* iff

$$\forall a, b, c : (a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R.$$

A relation is *intransitive* if it is not transitive.

- E.g., "is an ancestor of" is transitive, and "likes" is intransitive.
- Which of the relations in Example 7 are transitive?
- Is the "divides" relations on the set of positive integers transitive?
- "is within 1 mile of" is . . . ?

The Power of A Relation

Definition

The nth power R^n of a relation R on a set A can be defined recursively by

$$\begin{cases} R^0 :\equiv I_A; \\ R^{n+1} :\equiv R^n \circ R \text{ for all } n \geq 0. \end{cases}$$

The negative powers of R can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.

Whether A Relation Is Transitive Or Not?

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all $n = 1, 2, 3, \cdots$.

- *Think about what $(a, b) \in R^k$ means?*
- *How to prove an "if and only if" statement?*

- Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers R^n for $n = 2, 3, \cdots$.
- Let $R = \{(1,2), (1,3), (2,2), (2,3), (4,3)\}$. Find the powers R^n for $n = 2, 3, \cdots$.

n-ary Relations

Definition

An n-ary relation R on sets $A_1, \cdots, A_n$, written $R : A_1, \cdots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

- The sets A_i are called the *domains* of R.
- The *degree* of R is n.
- R is *functional* in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

- A *relational database* is essentially an n-ary relation R.
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Selection Operators

- Let A be any n-ary domain $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any condition (predicate) on elements (n-tuples) of A.
- Then, the *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.
  - I.e., $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

Selection Operator Example

- Suppose we have a domain $A = StudentName \times Standing \times SocSecNos$.
- Suppose we define a certain condition on A,

$$\begin{aligned} & UpperLevel(name, standing, ssn) \\ : \quad & \equiv [(standing = \text{junior}) \vee (standing = \text{senior})] \end{aligned}$$

- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.
  - That is, where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.
- Then the projection operator on n-tuples

$$P_{\{i_k\}} : A \longrightarrow A_{i_1} \times \cdots \times A_{i_m}$$

is defined by

$$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m}).$$

Projection Example

- Suppose we have a ternary (3-ary) domain $Cars = Model \times Year \times Color$. $(n = 3)$
- Consider the index sequence $\{i_k\} = \{1, 3\}$. $(m = 2)$
- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1, a_2, a_3) = (model, year, color)$ to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color).$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of model/color combinations available.

Join Operator

- Puts two relations together to form a sort of combined relation.
- If the tuple (A, B) appears in R_1, and the tuple (B, C) appears in R_2, then the tuple (A, B, C) appears in the join $J(R_1, R_2)$.
- A, B, C can also be sequences of elements rather than single elements.

Join Example

- Suppose R_1 is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.
- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Representing Relations

- Some ways to represent n-ary relations:
  - With an explicit list or table of its tuples.
  - With a function from the domain to $\{T, F\}$.
- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

Using Zero-One Matrices

- To represent a relation R by a matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ if $(a_i, b_j) \in R$, else 0.
- E.g., Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally. The $0-1$ matrix representation of that "Likes" relation:

$$\begin{array}{rc} & \begin{array}{ccc}\text{Susan} & \text{Mary} & \text{Sally}\end{array} \\ \begin{array}{r}\text{Joe} \\ \text{Fred} \\ \text{Mark}\end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Examples

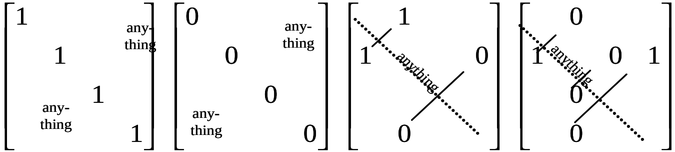
Example

Let $S = \{\text{Spring,Summer,Fall,Winter}\}$ and $F = \{\text{Apple,Berry,Cherry,Durian}\}$. Which ordered pairs are in the relation R represented by the matrix?

	Apple	Berry	Cherry	Durian
Spring	1	0	1	0
Summer	0	0	1	1
Fall	0	1	0	0
Winter	1	0	0	0

Zero-One Reflexive, Symmetric

- Terms: reflexive, non-reflexive[1], irreflexive, symmetric, asymmetric[2], and antisymmetric.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal

Irreflexive: all 0's on diagonal

Symmetric: all identical across diagonal

Antisymmetric: all 1's are across from 0's

---

[1] A relation R on A is non-reflexive if it is not reflexive.

[2] A relation R on A is asymmetric if $\forall a, b \in A : aRb \rightarrow b\overline{R}a$.

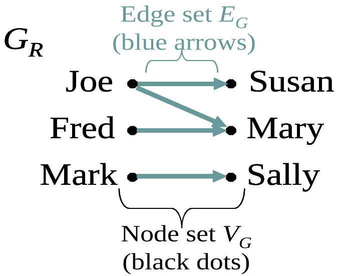
Matrix Operation v.s. Relation Operations

- $\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$; $\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$.
  - $\vee$ and $\wedge$ are element-wise Boolean operators.
- $\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$; $\mathbf{M}_{R^n} = (\mathbf{M}_R)^n$.
  - $\odot$ denotes Boolean matrix multiplications.
- $\mathbf{M}_{R^{-1}} = (\mathbf{M}_R)^T$.
- Quiz: If R is a symmetric relation, $\mathbf{M}_R$ is a symmetric matrix.

Using Directed Graphs

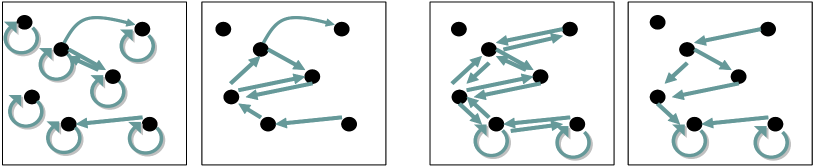
- A directed graph or digraph $G = (V_G, E_G)$ is a set V_G of vertices (nodes) with a set $E_G \subseteq V_G \times V_G$ of edges (*arcs*, *links*). Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R : A \leftrightarrow B$ can be represented as a graph $G_R = (V_G = A \cup B, E_G = R)$.

$$
\begin{array}{c c}
\mathbf{M}_R & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\
\begin{array}{c} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$



Digraph Reflexive, Symmetric

- It is extremely easy to recognize the reflexive/irreflexive/symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop

Irreflexive: No node links to itself

Symmetric: Every link is bidirectional

Antisymmetric: No link is bidirectional

Asymmetric, non-antisymmetric

Non-reflexive, non-irreflexive

Closures of Relations

- For any property X, the "X closure" of a set (or relation) R is defined as the "smallest" superset of R that has the given property.
- The *reflexive closure* of a relation R on A is obtained by adding (a, a) to R for each $a \in A$. I.e., it is $R \cup I_A$.
- The *symmetric closure* of R is obtained by adding (b, a) to R for each (a, b) in R. I.e., it is $R \cup R^{-1}$.
- The *transitive closure* or connectivity relation of R is obtained by repeatedly adding (a, c) to R for each (a, b) and (b, c) in R. I.e., it is
$$R^* = \bigcup_{n \in Z^+} R^n.$$

Paths in Digraphs/Binary Relations

Definition

A *path* of length n from node a to b in the directed graph G (or the binary relation R) is a sequence $(a, x_1), (x_1, x_2), \cdots, (x_{n-1}, b)$ of n ordered pairs in E_G (or R). A path of length $n \geq 1$ from a to a is called a *circuit* or a *cycle*.

Theorem

There exists a path of length n from a to b in R if and only if $(a, b) \in R^n$.

- An empty sequence of edges is considered a path of length 0 from a to a.
- If any path from a to b exists, then we say that a is connected to b. ("You can get there from here.")

Simple Transitive Closure Algorithm

Lemma

Let A be a set with n element, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

procedure *transClosure*($\mathbf{M}_R$: rank-n 0-1 matrix)
// A procedure computes R^* with 0-1 matrices.
$\mathbf{A} := \mathbf{B} := \mathbf{M}_R$;
for $i := 2$ **to** n **begin**
$\quad\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$; $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$;
end
return B

- This algorithm takes $\Theta(n^4)$ time.

A Faster Transitive Closure Algorithm

procedure *transClosure*($\mathbf{M}_R$: rank-n 0-1 matrix)
$\mathbf{A} := \mathbf{B} := \mathbf{M}_R$;
for $i := 2$ **to** $\lceil \log_2 n \rceil$ **begin**
  $\mathbf{A} := \mathbf{A} \odot \mathbf{A}$; // $\mathbf{A}$ represents R^{2^i}.
  $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$; // "add" into $\mathbf{B}$.
end
return B

- This algorihtm takes only $\Theta(n^3 logn)$ time, BUT NOT CORRECT.

Roy-Warshall Algorithm

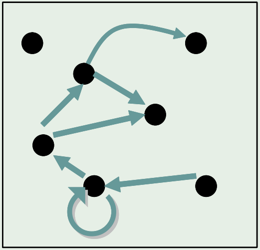
procedure *Warshall*($\mathbf{M}_R$: rank-n 0-1 matrix)
$\mathbf{W} := \mathbf{M}_R$;
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W {This represents R^*.}

- Uses only $\Theta(n^3)$ operations!
- $w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

Examples

Example

Find the symmetric closure, reflexive closure, and transitive closure of the following relation.



Equivalence Relations

Definition

An *equivalence relation* (e.r.) on a set A is simply any binary relation on A that is reflexive, symmetric, and transitive.

- E.g., "$=$" itself is an equivalence relation.
- For any function $f : A \to B$, the relation "have the same f value", or $=_f :\equiv \{(a_1, a_2) \mid f(a_1) = f(a_2)\}$ is an equivalence relation.
  - E.g., let $m =$ "mother of", then $=_m :\equiv$ "have the same mother" is an e.r..

Examples of E.R.'s

Examples

- "Strings *a* and *b* are the same length."

Examples of E.R.'s

Examples

- "Strings a and b are the same length."
- "Integers a and b have the same absolute value."

Examples of E.R.'s

Examples

- "Strings a and b are the same length."
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."

Examples of E.R.'s

Examples

- "Strings a and b are the same length."
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."
- "Integers a and b have the same residue modulo m." (for a given $m > 1$)

Equivalence Classes

Definition

Let R be any equivalence relation on a set A. The *equivalence class* of a is

$$[a]_R :\equiv \{b \mid aRb\}. \text{ (optional subscript } R)$$

- It is the set of all elements of A that are "equivalent" to a according to the E.R. R.
- Each such b (including a itself) is called a representative of $[a]_R$.

Equivalence Class Examples

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."
- "Integers a and b have the same residue modulo m." (for a given $m > 1$)

Equivalence Class Examples

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$.
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."
- "Integers a and b have the same residue modulo m." (for a given $m > 1$)

Equivalence Class Examples

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$.
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."
  - $[a]$ = the set $\{\cdots, a-2, a-1, a, a+1, a+2, \cdots\}$.
- "Integers a and b have the same residue modulo m." (for a given $m > 1$)

Equivalence Class Examples

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$.
- "Real numbers a and b have the same fractional part (i.e., $a - b \in Z$)."
  - $[a]$ = the set $\{\cdots, a-2, a-1, a, a+1, a+2, \cdots\}$.
- "Integers a and b have the same residue modulo m." (for a given $m > 1$)
  - $[a]$ = the set $\{\cdots, a-2m, a-m, a, a+m, a+2m, \cdots\}$.

Partitions

Definition

A *partition* of a set A is the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some e.r. on A.

Example

Let $m \in \mathbb{Z}^+$. For any $a, b \in \mathbb{Z}$, we define aRb iff $m \mid a - b$. Then, R is an e.r., and $\{[0], [1], \cdots, [m-1]\}$ is a partition of $\mathbb{Z}$ for R.

- The A_i's are all disjoint and their union is equal to A.
- They "partition" the set into pieces. Within each piece, all members of the set are equivalent to each other.

Partial Orderings

Definition

A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R).

- The "greater than or equal" relation $\geq$ is a partial ordering on the set of integers.
- The divisibility relation $\mid$ is a partial ordering on the set of positive integers.
- The inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Total Orderings

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered set* or *linearly ordered set*, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

- E.g., $(\mathbb{N}, \leq)$.

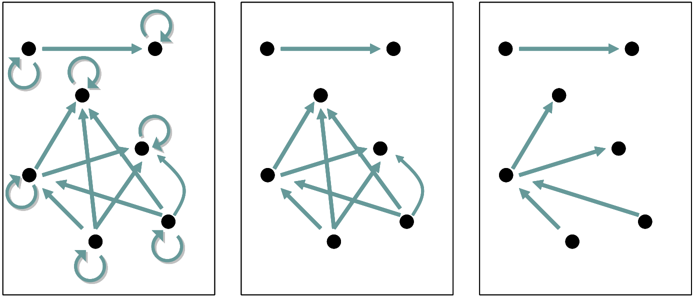
Lexicographic Order

- $(A_1, \preccurlyeq_1)$ and $(A_1, \preccurlyeq_2)$ are posets. For any $(a_1, a_2), (b_1, b_2) \in A_1 \times A_2$, we say $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if and only if $a_1 \preccurlyeq_1 b_1$ or both $a_1 = b_1$ and $a_2 \preccurlyeq_2 b_2$.
- The lexicographic order of the Cartesian product of posets is a partial order.
  - Please prove this by yourself.

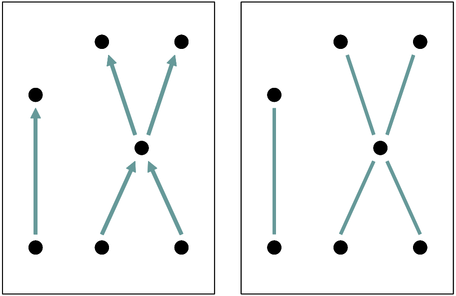
Hasse Diagrams

- Digraphs for finite posets can be simplified by following ideas.
  1. Remove loops at every vertices.
  2. Remove edge that must be present because of the transitivity.
  3. Arrange each edge so that its initial vertex is below its terminal vertex.
  4. Remove all the arrows.
- The simplified diagrams are called *Hasse diagrams*.

Example of Hasse Diagrams



Example of Hasse Diagrams (Cont.)



Maximal and Minimal Elements

Definition

a is a *maximal* (resp., *minimal*) element in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp., $b \prec a$).

Definition

a is the *greatest* (resp., *least*) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp., $a \preccurlyeq b$) for all $b \in S$.

Lemma

Every finite nonempty poset $(S, \preccurlyeq)$ has a minimal element.

Maximal and Minimal Elements (Cont.)

Definition

A is a subset of of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an upper bound (resp., lower bound) of A if $a \preccurlyeq u$ (resp., $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the least upper bound (resp., greatest lower bound) of A if x is an upper bound (resp., lower bound) that is less than every other upper bound (resp., lower bound) of A.

Definition

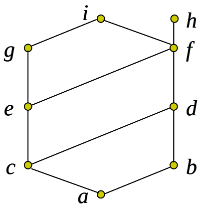
$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

- E.g., $(\mathbb{Z}^+, \leq)$ is *well-ordered* but $(\mathbb{R}, \leq)$ is not.
- There is "well-ordered induction".

Lattices

Definition

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.



Example

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

- Motivation: A project is make up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?
- Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever aRb. $\preccurlyeq$ is said compatible with R.

Topological Sorting for Finite Posets

procedure *topological_sort*(S: finite poset)
$k := 1$
while $S \neq \varnothing$
begin
    $a_k :=$ a minimal element of S
    $S := S - \{a_k\}$
    $k := k + 1$
end $\{a_1, a_2, \cdots, a_n$ is a compatible total ordering of $S\}$