Opinion Formation on Networks

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Abstract

Many people rely primarily on social media for sources of news and information, and the content that is shared on social media can greatly impact people's opinions. Bounded-confidence models of opinion dynamics allow one to model the dynamics of how opinions spread online as a function of both media ideology and non-media account opinions. In a subset of these models, agents start with some opinion which takes a continuous value from $[-1, 1]$ and evolves when agents interact with other agents whose opinion is sufficiently similar to their own. We build on recent work by adding heterogeneous media influence to such bounded-confidence models. In addition, as the number of the agents in such models increases, the model becomes very computationally costly with respect to processing power and time. Therefore, we derive a macroscopic equation for a bounded-confidence model that more efficiently finds how the distribution of opinions on a social-media network evolves over time. In ongoing research, we are examining how closely this macroscopic equation approximates our original bounded-confidence model, investigating the possible stationary states of the macroscopic equation, and exploring other properties of the macroscopic model.

1 Background and Motivation

Social media, such as Instagram, Twitter, Facebook and YouTube, have become influential sources of information for many people. For example, more than two thirds of American adults said, in a 2018 survey, that social media is their primary source of news [1]. Modeling the way in which opinions spread on these online networks may yield insight into important issues, such as consumer behavior, political polarization and the spread of misinformation online [1, 2].

The scientific literature includes many different approaches for studying the spread of opinions on social media: researchers in [3, 4, 5] mathematically modelled the spread of opinions using ordinary differential equations. Some of these equations describe what proportion of a population holds a specific opinion over time. A primary issue with these models is that they do not take into account the effects of network structure or heterogeneity in an account's followers, as well as other potentially influential account characteristics. However, such models can be analytically tractable, as well as less computationally demanding when compared to the models discussed in the next paragraph.

Recent work uses a variation on the DeGroot model, first developed by Deffuant et al. in [6], to capture the effect of a social media's network structure on opinion dynamics. We describe this model in Section 2. Meng et al. in [2] investigated the effect of network structure, among other things, on the evolution of opinions predicted by the Deffuant model. More recently, Brooks and Porter in [1] introduced media accounts to the Hegselmann—Krause bounded-confidence model;

these are influencer accounts that spread a specific opinion without changing their own. They simulated the model for different parameter values and network structures to quantify the impact of the media accounts on non-media opinions. One important issue with these models is that when they simulate the opinion dynamics of each individual account over time, it is costly with respect to time and processing power. Therefore it can be impractical to find solutions for the steady-state behavior of a sufficiently large network. We base our approach on [8], where a continuous limit of the model in [1] was derived that describes how the distribution of opinions evolve over time when media opinion is homogeneous. We aim to extend this work by deriving a similar macroscopic equation when media opinion is heterogeneous and compare it to the discrete bounded-confidence model, described in Section 2.

2 Microscopic Model Framework

We start from the model that was recently introduced in [1]. The structure of an online network such as Twitter can be modelled as a directed graph $G = (V, E)$, where V is a set of nodes (where each node represents an account) and E is a set of edges (where each edge reflects whether an account follows another account). That is, the set E has a directed edge from account i to account j if account j follows account i. If account j follows account i, then account j receives information from account i. We can then build an adjacency matrix $\boldsymbol{A}$, where $\boldsymbol{A}_{ij} = 0$ if i does not follow j and $\boldsymbol{A}_{ij} = 1$ if i follows j. We take $\boldsymbol{A}_{ii} = 1$ for all i, as we assume each account has access to its own posts.

The number of individual accounts (N) and the number of media accounts (M) in total make up the network. Both types of accounts output opinions, but media accounts do not change their own opinions, while individual accounts change their opinions in response to nodes that they follow.

Each node i has opinion x_i^k at discrete time step k, where $x \in [-1, 1]^d$, where d is the dimensionality in opinion space of each node. To use political opinions as an analogy and setting $d = 1$, an opinion $x = -1$ perhaps represents a strongly liberal opinion, $x = 0$ perhaps represents a moderate opinion and $x = +1$ perhaps represents a strongly conservative opinion.

When we update the system at each discrete time step $k + 1$, each node in the network averages the opinions of all accounts at time k to which it is 'receptive' to and sets that average to be its new opinion value. An account i is receptive to another account j if i follows j and $\text{dist}(x_i - x_j) < c$, where dist reflects the distance between the opinions of nodes i and j. A smaller c implies that accounts must be more similar in opinion to influence each other, while the opposite is true for a larger c. The parameter c encodes the idea that accounts are only influenced by accounts with sufficiently similar opinions [9]; this opinion cutoff is a central part of bounded-confidence models [2]. The value of c defines a confidence interval for any opinion x, which we call the 'receptiveness interval', such that only nodes with opinions in that interval around x can influence x.

To define an update rule for each account i, we denote the set of accounts to which i is receptive to by I_i. We can now define the update rule for each account i:

$$x_i^{k+1} = \frac{1}{|I_i|} \sum_{j=1}^{N+M} \mathbf{A}_{ij} x_j^k \eta(|x_i^k - x_j^k|) \quad \text{for} \quad i \in \{1, ..., N + M\}, \tag{1}$$

where

$$\eta(x) = \begin{cases} 1\,, & \text{if } x < c \\ 0\,, & \text{if } x \geq c\ . \end{cases}$$

Any media account i has $I_i = \{i\}$ because media accounts only follow themselves. This is an important assumption of the model because it treats the opinions of media accounts as static, unlike the opinions of other accounts.

To study how the model (1) predicts how opinions on a network changes over time, we simulate it on various network topologies. We first create a initial network with N nodes, and we then add some number of M media nodes. We then set the initial value of each node's opinion, pulling these values from a distribution over the opinion space. We then update all nodes repeatedly until every node's opinion is less than the the value ε different from the node's previous opinion. This is when we say that the model reaches 'convergence', and we choose to terminate the simulation. We also impose an upper bound S on the number of updates, such that if the model does not converge within this specified number of updates, then the model terminates without convergence being reached. However, we did not extensively explore use of these simulations in this paper.We leave this for future work.

3 Deriving a Macroscopic Equation for the Opinion Distribution

We can describe the microscopic model from Section 2 as an agent-based model (ABM), where the 'agents' are accounts. In this ABM, we track the opinion value of each agent individually and evolve an agent's opinion over time according to the update rule (1). The benefits of such a model are that it provides a granular level of understanding for the opinion held by each agent, as well the rate at which that opinion is changing. The downside of such a model is that as $N \to \infty$ the dimensionality of the model increases, and it becomes very computationally expensive to track the opinion of each agent.

Instead of tracking each agent's opinion over time, we aim to derive a macroscopic equation that describes the evolution of the distribution of agent opinions over time. This significantly reduces the dimensionality of the problem and it also allows us to gain insights into large-scale opinion patterns. We proceed in four steps to obtain such a macroscopic description, following the work of [8] closely for Sections 3.1 − 3.4:

1. We find an equation for $\{x_i^{\tau,k}\}_{i=1,k=0}^{N,T}$ with fixed time step τ by altering model (1).

2. We take the limit as $\tau \to 0$ and find a set of equations for $\{x_i(t)\}_{i=1}^{N}$.

3. We let $N \to \infty$ and derive a corresponding equation for $x(t)$.

4. We use the particle-model equation from [4,12] to derive the corresponding macroscopic equation for the distribution $\rho(t, x)$ of individual opinions.

3.1 Heterogeneous-Media Discrete-Time Model

The adjacency matrix for a directed graph of N non-media nodes and M media nodes can be written as

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{J} & \mathbf{B} \\ \hline \mathbf{0} & \mathbf{I}_M \end{array} \right) ,$$

where $\mathbf{A}_{ij} = 1$ if account i follows account j and $\mathbf{A}_{ij} = 0$ if account i does not follow account j. Every node follows itself. The $N \times N$ submatrix $\mathbf{J}$ in the top left encodes follower-ship between non-media nodes. Throughout the remainder of the paper we work with a complete network where $\mathbf{J}$ is a matrix of 1s. The $N \times M$ submatrix $\mathbf{B}$ in the top right encodes non-media nodes that follow media nodes. The $M \times N$ submatrix $\mathbf{0}$ in the bottom left encodes media nodes that follow non-media nodes. An assumption of our model is that the opinions of media nodes are static over time, so $\mathbf{0}$ consists entirely of 0s as no media nodes follow non-media nodes. The $M \times M$ submatrix $\mathbf{I}_M$ in the bottom right encodes the followership between media accounts. This is an identity matrix, as it is an assumption of our model that media opinions are static, so media nodes only follow themselves.

Each node x_i (for $i \in \{1, \dots, N+M\}$) follows the update rule

$$x_i^{k+1} = \frac{1}{|I_i|} \sum_{j=1}^{N+M} \mathbf{A}_{ij} x_j^k \eta(|x_i^k - x_j^k|) , \tag{2}$$

where

$$|I_i| = \sum_{j=1}^{N+M} \mathbf{A}_{ij} \eta(|x_i^k - x_j^k|)$$

represents the number of nodes to which an individual i is receptive.

To obtain a continuous-in-time model from our discrete model we first incorporate a fixed time step τ. Let $x_i^{\tau,k}$ represent the opinion of account i at time $k\tau$, where k is the time step number. We define

$$\mathbf{A}^\tau = \left(\begin{array}{c|c} \mathbf{K}^\tau & \tau\mathbf{B} \\ \hline \mathbf{0} & \mathbf{I}_M \end{array} \right), \text{ where } \mathbf{K}^\tau = \tau\mathbf{J} + (1-\tau) \begin{bmatrix} |I_1| & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & |I_N| \end{bmatrix} .$$

We then propose the τ-dependent update rule

$$x_i^{\tau,k+1} = \frac{1}{|I_i^\tau|} \sum_{j=1}^{N+M} \mathbf{A}_{ij}^\tau x_j^{\tau,k} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) = \frac{1}{|I_i^\tau|} \left(\sum_{j=1}^{N} \mathbf{A}_{ij}^\tau x_j^{\tau,k} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \sum_{j=N+1}^{M+N} \mathbf{A}_{ij}^\tau x_j^{\tau,k} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) \right) . \tag{3}$$

Note that $\mathbf{J}$ is a matrix of all 1s, as we work only with complete networks, so

$$\tau \sum_{j=1}^{N} \mathbf{J}_{ij} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) = \tau \sum_{j=1}^{N} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|)$$

still represents the number of non-media nodes that individual i follows and is receptive to. This special case of a complete network allows us to write simplified equations that do not explicitly require an adjacency matrix, which is important for later steps.

In our network, we suppose that there are M media nodes; each media node has a corresponding opinion value. We then define the set of all unique media opinion values as $\mathbf{M} = \{m_1, m_2, m_3 \ldots\}$, where $1 \leq |\mathbf{M}| \leq M$. We define the number of media nodes with opinion value $m_{\mathbf{m}} \in \mathbf{M}$ that an individual i follows as

$$\alpha_{i,\mathbf{m}} = \sum_{j=1}^{M} \mathbf{B}_{ij} \delta_{m_{\mathbf{m}}}(x_j)\,, \tag{4}$$

where

$$\delta_{m_{\mathbf{m}}}(x_j) = \begin{cases} 1\,, & \text{if } x_j = m_{\mathbf{m}} \\ 0\,, & \text{if } x_j \neq m_{\mathbf{m}}\,. \end{cases}$$

Note that

$$\sum_{j=1}^{M} \mathbf{B}_{ij} \eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k}|) = \sum_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}} \eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)$$

is still the number of media nodes that an individual i is receptive to.

Lemma 3.1. *For all $i \in \{1, \ldots, N\}$ and for all k we have*

$$|I_i^{\tau}| = |I_i| = \sum_{j=1}^{N} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}} \eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)\,.$$

Proof.

$$\text{Let } P = \begin{bmatrix} |I_1| & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & |I_N| \end{bmatrix}.$$

For all $i \in \{1, \ldots, N\}$, we have that

$$\begin{aligned} |I_i^{\tau,k}| &= \sum_{j=1}^{N+M} \mathbf{A}_{ij}^{\tau} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) \\ &= \sum_{j=1}^{N} \mathbf{K}_{ij}^{\tau} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau \sum_{j=1}^{M} \mathbf{B}_{ij} \eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k}|) \\ &= \sum_{j=1}^{N} (1-\tau) \mathbf{P}_{ij} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau \sum_{j=1}^{N} \mathbf{J}_{ij} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau \sum_{j=1}^{M} \mathbf{B}_{ij} \eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k} \end{aligned}$$

As

$$\mathbf{P_{ij}} = \begin{cases} |I_i|\,, & \text{if } i = j \\ 0\,, & \text{if } i \neq j \end{cases}\,,$$

we then see that

$$(1-\tau)\sum_{j=1}^{N} \mathbf{P}_{ij}\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) = (1-\tau)|I_i|\,,\ \text{since } \eta(0) = 1\,.$$

Consequently,

$$\begin{aligned}
|I_i^{\tau,k}| &= (1-\tau)|I_i| + \tau\sum_{j=1}^{N} \mathbf{J}_{ij}\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau\sum_{j=1}^{M} \mathbf{B}_{ij}\eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k}|) \\
&= (1-\tau)|I_i| + \tau\sum_{j=1}^{N} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau\sum_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k} - m_{\mathbf{m}}|) \\
&= (1-\tau)|I_i| + \tau\sum_{j=1}^{N+M} \mathbf{A}_{ij}\eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k}|) \\
&= (1-\tau)|I_i| + \tau|I_i| \\
&= |I_i|\,.
\end{aligned}$$

□

Proposition 1. *It follows from equation (3) that the following equality is true. This expression denotes the velocity at which an account x's opinion is changing:*

$$\frac{x_i^{\tau,k+1} - x_i^{\tau,k}}{\tau} = -\frac{\frac{1}{N}\sum_{j=1}^{N}(x_i^{\tau,k} - x_j^{\tau,k})\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \frac{1}{N}\sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}(x_i^{\tau,k} - m_{\mathbf{m}})\eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)}{\frac{1}{N}\sum_{j=1}^{N}\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \frac{1}{N}\sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)}\,. \tag{5}$$

Proof. From the τ-dependent update rule we see that

$$\begin{aligned}
x_i^{\tau,k+1} &= \frac{1}{I_i}\left((1-\tau)(x_i^{\tau,k})|I_i| + \tau\sum_{j=1}^{N}\mathbf{J}_{ij}x_j^{\tau,k}\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \tau\sum_{j=1}^{M}\mathbf{B}_{ij}x_{N+j}^{\tau,k}\eta(|x_i^{\tau,k} - x_{N+j}^{\tau,k}|)\right) \\
&= (1-\tau)x_i^{\tau,k} + \frac{\tau}{|I_i|}\left(\sum_{j=1}^{N}x_j^{\tau,k}\eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|}m_{\mathbf{m}}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)\right)\,,
\end{aligned}$$

so

$$\frac{x_i^{\tau,k+1}-x_i^{\tau,k}}{\tau} = -x_i^{\tau,k} + \frac{1}{|I_i|}\left(\sum_{j=1}^{N} x_j^{\tau,k}\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|} m_{\mathbf{m}}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)\right) .$$

By Lemma 3.1,

$$\begin{aligned}
\tfrac{x_i^{\tau,k+1}-x_i^{\tau,k}}{\tau} &= \tfrac{-x_i^{\tau,k}}{|I_i|}\left(\sum_{j=1}^{N}\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)\right) \\
&\quad - \tfrac{1}{|I_i|}\left(-\sum_{j=1}^{N} x_j^{\tau,k}\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) - \sum_{\mathbf{m}=1}^{|\mathbf{M}|} m_{\mathbf{m}}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)\right) \\
&= -\tfrac{1}{|I_i|}\left(\sum_{j=1}^{N}(x_i^{\tau,k}-x_j^{\tau,k})\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}(x_i^{\tau,k}-m_{\mathbf{m}})\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)\right).
\end{aligned}$$

Therefore,

$$\frac{x_i^{\tau,k+1}-x_i^{\tau,k}}{\tau} = -\frac{\sum_{j=1}^{N}(x_i^{\tau,k}-x_j^{\tau,k})\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}(x_i^{\tau,k}-m_{\mathbf{m}})\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)}{\sum_{j=1}^{N}\eta(|x_i^{\tau,k}-x_j^{\tau,k}|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)} .$$

□

3.2 Continuous-Time Particle Model

We express the media and non-media nodes as two separate 'species' because of their different opinion dynamics . (Recall that media node opinions are constant, whereas non-media node opinions are change over time.) This yields a two-species system; similar models can be found in [4,12]. We define the piecewise-constant interpolation function

$$x_i^{\tau}(t) := x_i^{\tau,k} \text{ for } t \in [k\tau,(k+1)\tau]\,, \text{ where } k \in \mathbb{N}\,.$$

We also define

$$x_i(t) := \lim_{\tau\to\infty} x_i^{\tau}(t)\,.$$

From Proposition 1, it follows that $(x_i, m_{\mathbf{m}})$ for $i \in \{1,\dots,N\}$ and $\mathbf{m} \in \{\mathbf{1},\dots,|\mathbf{M}|\}$ satisfies the system

$$\begin{cases} \dot{x}_i = -\dfrac{\frac{1}{N}\sum_{j=1}^{N}(x_i^{\tau,k}-x_j^{\tau,k})\eta(|x_i^{\tau,k}-x_j^{\tau,k}|)+\frac{1}{N}\sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}(x_i^{\tau,k}-m_{\mathbf{m}})\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)}{\frac{1}{N}\sum_{j=1}^{N}\eta(|x_i^{\tau,k}-x_j^{\tau,k}|)+\frac{1}{N}\sum_{\mathbf{m}=1}^{|\mathbf{M}|}\alpha_{i,\mathbf{m}}\eta(|x_i^{\tau,k}-m_{\mathbf{m}}|)}\,, \\ \dot{m}_{\mathbf{m}} = 0\,, \end{cases} \tag{6}$$

with initial conditions

$$\begin{cases} x_i(0) & = x_i^0\,, \\ m_{\mathbf{m}}(0) & = m_{\mathbf{m}} \in \mathbf{M}\,, \end{cases}$$

where x_i represents the ideology of non-media node i and $m_j \in \mathbf{M}$ represent the ideology of media node j. Note that all x_i depend on time, while m_j are constant throughout the simulation. We define the function $\mathbf{K} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\mathbf{K}'(x) = \eta(x)$. Consequently,

$$\begin{aligned} \sum_{j=1}^{N+M} \mathbf{A}_{ij}(x_i - x_j)\eta(|x_i - x_j|) &= \sum_{j=1}^{N}(x_i - x_j)\eta(|x_i - x_j|) + \sum_{j=N+1}^{N+M} \mathbf{A}_{ij}(x_i - x_j)\eta(|x_i - x_j|) \\ &= \sum_{j=1}^{N}(x_i - x_j)\eta(|x_i - x_j|) + \sum_{j=1}^{M} \mathbf{B}_{ij}(x_i - x_{N+j})\eta(|x_i - x_{N+j}|) \\ &= \sum_{j=1}^{N}(x_i - x_j)\eta(|x_i - x_j|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}}(x_i - m_{|\mathbf{m}|})\eta(|x_i - m_{|\mathbf{m}|}|) \\ &= \sum_{j=1}^{N} \nabla\mathbf{K}(|x_i - x_j|) + \sum_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}} \nabla\mathbf{K}(|x_i - m_{|\mathbf{m}|}|)\,. \end{aligned}$$

Therefore,

$$\dot{x}_i = -\frac{\frac{1}{N}\sum\limits_{j=1}^{N} \nabla\mathbf{K}(|x_i^{\tau,k} - x_j^{\tau,k}|) + \frac{1}{N}\sum\limits_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}} \nabla\mathbf{K}(|x_i^{\tau,k} - m_{\mathbf{m}}|)}{\frac{1}{N}\sum\limits_{j=1}^{N} \eta(|x_i^{\tau,k} - x_j^{\tau,k}|) + \frac{1}{N}\sum\limits_{\mathbf{m}=1}^{|\mathbf{M}|} \alpha_{i,\mathbf{m}} \eta(|x_i^{\tau,k} - m_{\mathbf{m}}|)}\,. \tag{7}$$

Reformulating equation (6) to produce equation (7) shows the similarity between the equation (7) and the discrete aggregation model in [10]: both equations include summations of the derivative of a function whose input is the distance between two numbers. The notable difference, however, is the presence of the denominator in (7), which significantly complicates the analysis.

3.3 Mean-field Limit

The evolution of opinions in the continuous-time model in equation (6) does not depend on the entries of the media-followership matrix $\mathbf{B}$. It depends only on the number of media with a unique opinion that is followed by each individual; that is, it depends only on $\alpha_{i,1} \dots \alpha_{i,|\mathbf{M}|}$. We thus introduce the set $B = \{b_{N,1}, \dots, b_{N,|\mathbf{M}|}\}$ of functions to formulate (6) in such a way that allows us to take the number of non-media nodes to infinity. Let $b_{N,\mathbf{m}} : \Omega \to \mathbb{R}$ be a function, where Ω represents the opinion space, that satisfies

$$b_{N,\mathbf{m}}(x_i) = \alpha_{i,\mathbf{m}}\,, \text{ for } i \in \{1, \dots, N\}\,, \quad \mathbf{m} \in \{1, \dots, |\mathbf{M}|\}$$

to represent the number of media accounts with opinion value $m_{\mathbf{m}}$ that non-media node i follows. We use $b_{N,\mathbf{m}}$ to encode media followership in our continuum model (8). Assuming that the matrix $\mathbf{B}$ is such that the following limit exists, we then define

$$b_{\mathbf{m}}(x) := \lim_{N \to \infty} \frac{b_{N,\mathbf{m}}(x)}{N}$$

,

so $\frac{\alpha_{i,\mathbf{m}}}{N} \to b_{\mathbf{m}}(x_i)$ as $N \to \infty$. If we let $\rho(t,x)$ denote the distribution of non-media opinions at time t and take the Dirac distribution $\nu(x)$ (where $\nu(x,m) = \sum_{j=1}^{\mathbf{M}} b_j(x)\delta_{m_j}(m)$) as the distribution of media opinions, then we write the mean-field approximation of equation (6) as

$$\dot{x} = -\frac{\int_{\Omega}(x-y)\eta(|x-y|)\rho(y)dy + \sum_{\mathbf{m=1}}^{|\mathbf{M}|}(x-m_{\mathbf{m}})\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}{\int_{\Omega}\eta(x-y)\rho(y)dy + \sum_{\mathbf{m=1}}^{|\mathbf{M}|}\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}\,, \tag{8}$$

where $x(t) \in \Omega \subseteq \mathbb{R}^d$. Equation (8) expresses the averaged particle dynamics and corresponds to the mean-field limit in which we take the number of non-media nodes in the network to infinity, while fixing the number of media nodes.

3.4 Macroscopic Dynamics

From previous work [10], if we know the velocity $\mathbf{v}$ of particles that move as part of a group, the distribution ρ of particles satisfies the continuity equation

$$\partial_t \rho + \operatorname{div}(\mathbf{v}\rho) = 0\,.$$

With the velocity of each particle given by equation (8), the evolution of the macroscopic density ρ is given by

$$\partial_t \rho = \operatorname{div}\left(\rho\left(\frac{\int_{\Omega}(x-y)\eta(|x-y|)\rho(y)dy + \sum_{\mathbf{m=1}}^{|\mathbf{M}|}(x-m_{\mathbf{m}})\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}{\int_{\Omega}\eta(x-y)\rho(y)dy + \sum_{\mathbf{m=1}}^{|\mathbf{M}|}\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}\right)\right)\,, \tag{9}$$

where $b_{\mathbf{m}}(x)$ represents strength of the influence that a media node with ideology $m_{\mathbf{m}}$ exerts on non-media accounts that have opinion x.

4 Stationary States

In section 3 we derived a PDE that describes the change in the distribution of non-media opinions ρ with respect to time:

$$\partial_t \rho = \operatorname{div}\left(\rho\left(\frac{\int_\Omega (x-y)\eta(|x-y|)\rho(y)dy - \sum_{\mathbf{m}=\mathbf{1}}^{|\mathbf{M}|}(x-m_{\mathbf{m}})\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}{\int_\Omega \eta(x-y)\rho(y)dy + \sum_{\mathbf{m}=\mathbf{1}}^{|\mathbf{M}|}\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x)}\right)\right).$$

When analyzing this PDE, we seek to determine the relationship between all initial conditions and all long-term solutions of the PDE. We thus seek to identify potential steady states of this mathematical model (9).

To find possible stationary states, we seek conditions that lead to $\partial_t \rho = 0$. For this to be true, then some potential stationary state ρ_∞ will satisfy

$$\int_\Omega (x-y)\eta(|x-y|)\rho_\infty(y)dy - \sum_{\mathbf{m}=\mathbf{1}}^{|\mathbf{M}|}(x-m_{\mathbf{m}})\eta(|x-m_{\mathbf{m}}|)b_{\mathbf{m}}(x) = 0\,. \tag{10}$$

We work with a simplified model where there are no media nodes present in the network, so $b_m(x) = 0$. We are then interested in p_∞ that satisfy

$$\int_\Omega (x-y)\eta(|x-y|)\rho_\infty(y)dy = 0 \tag{11}$$

for all opinions in the opinion distribution, where each opinion is represented by x.

It was found in [8] that a Dirac distribution of non-media opinions centered at points at least c apart are a class of steady states of equation (9). However, a question raised by [8] is whether Dirac distributions are the only possible class of steady states. We make progress on this question by identifying a class of distributions that cannot be steady states for this model in Theorem 4.1. An example of this class of distributions is displayed in Figure 1.

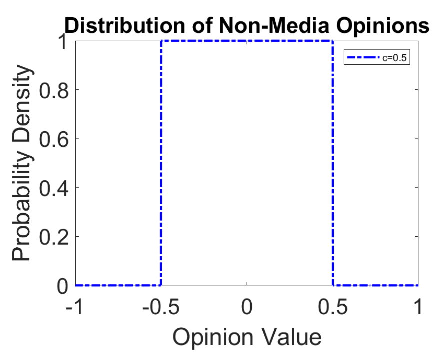

Figure 1: *A hypothetical solution for the steady-state distribution (ρ_∞) of non-media opinions for equation (9).*

Theorem 4.1.

Suppose that the opinion distribution $\rho_\infty(x) = \begin{cases} \bar{\rho}\,, & \text{if } x \le r_2 \;\&\; x \ge r_1 \\ 0\,, & \text{if } x > r_2 \text{ or } x < r_1 \end{cases}$ *for any* $r_1, r_2 \in [-1,1]$*, where* $r_1 < r_2$ *and* $\bar{\rho} > 0$*. It then follows that* ρ_∞ *is not a stable stationary state of equation (9) when no media nodes are present.*

Proof.
The opinion probability distribution ρ_∞ is defined above as a uniform distribution of probability $\bar{\rho} \in [r_1, r_2]$. We want to show that ρ_∞ is not stable over time, and therefore is not a stable stationary state of equation (9). To achieve this, we will do the following:

1. We will assume that ρ_∞ is a stationary state of equation (9).
2. We will then check to see if ρ_∞ satisfies equation (11) for all opinions in the distribution.
3. We will then arrive at a contradiction, where ρ_∞ does not satisfy equation (11).
4. We then conclude that ρ_∞ is not a stationary state of equation (9).

Suppose that ρ_∞ is a stationary state of equation (9). It then follows that ρ_∞ satisfies equation (11). Therefore,

$$\int (x-y)\eta(|x-y|)\rho_\infty(y)dy = 0$$

holds true for all opinions $x \in [r_1, r_2]$. Consider the opinion $\bar{x} = r_1 + \varepsilon$, where ε represents an arbitrarily small and positive value less than c. Equation (11) then holds for nodes with opinion $\bar{x}$.

We start with equation (11) and substitute $\bar{x}$ for x, and implement the bounds of integration,

$$\int_{r_1}^{r_2} (\bar{x}-y)\eta(|\bar{x}-y|)\rho_\infty(y)dy = 0\,.$$

The opinion distribution is piece-wise constant, so we can substitute $\bar{\rho}$ for ρ_∞ to obtain

$$\int_{r_1}^{r_2} (\bar{x}-y)\eta(|\bar{x}-y|)\bar{\rho}dy = 0\,.$$

We then divide both sides by $\bar{\rho}$ to get

$$\int_{r_1}^{r_2} (\bar{x}-y)\eta(|\bar{x}-y|)dy = 0\,.$$

We then substitute $r_1 + \varepsilon$ for $\bar{x}$ and integrate over the bounds of the distribution:

$$\int_{r_1}^{r_2} (r_1+\varepsilon-y)\eta(|r_1+\varepsilon-y|)dy = 0\,.$$

We then alter the bounds of the integral to only include opinions within the confidence interval c of $\bar{x}$. We can then write

$$\int_{r_1}^{r_1+\varepsilon+c} (r_1+\varepsilon-y)dy = 0\,.$$

We evaluate this integral to obtain

$$\int_{r_1}^{r_1+\varepsilon+c} (r_1+\varepsilon)dy = \int_{r_1}^{r_1+\varepsilon+c} ydy\,,$$
$$\implies r_1\varepsilon + r_1c + \varepsilon^2 + \varepsilon c = r_1\varepsilon + r_1 c + \frac{1}{2}\varepsilon^2 + \varepsilon c + \frac{1}{2}c^2\,,$$
$$\implies \frac{1}{2}\varepsilon^2 = \frac{1}{2}c^2\,.$$

We have arrived at a contradiction, as ε was defined to not be equal to c, and therefore ρ_∞ cannot be a stationary state. □

Figure 1 graphically represents a potential steady-state distribution of non-media opinions as described by Theorem 4.1. Any piecewise-constant distribution ρ_∞ satisfies the requirements for Theorem 4.1 and therefore cannot be a steady state solution of equation (9). This eliminates a class of distributions from the search for finding non-Dirac solutions to equation (9).

Moving to a more complicated case, let $|\mathbf{M}|$ unique media nodes (unique with respect to opinion value) be present in the network. As stated before, a potential stationary state ρ_∞ must then satisfy equation (10). To evaluate such an equation, we make use of the properties of the Dirac distribution to convert the summation into an integral. For any $b_m(x_i)$, we write

$$b_m(x_i) = \int b_m(x)\delta_{x_i}(x)dx\,.$$

Therefore, equation (10) can be rewritten as

$$\int (x-y)\eta(|x-y|)\rho_\infty(y)dy - \int (x-m)\eta(|x-m|)\nu(x,m)dm = 0\,, \tag{12}$$

where $\nu(x,m) = \sum_{j=1}^{\mathbf{M}} b_j(x)\delta_{m_j}(m)$.

Although it has been difficult to find non-Dirac steady states, it may be possible to find a distribution $b(x)$ of media accounts that allows a non-Dirac distribution of non-media accounts to exist as a equilibrium solution. We leave this to future work.

5 Conclusions

We began this SURF drawing on the work done by Pan in [8] and by Brooks and Porter in [1]. We took the microscopic model in Section 2 and followed the work of [8] to derive a macroscopic equation for the non-media opinion distributions for heterogeneous media distributions. In doing so, we also wrote down a discrete-in-time and continuous-in-time heterogeneous model, as well as a mean0field limit of the particle model (now continuous in the number of agents). We proved Theorem 4.1, eliminating piece-wise constant distributions as potential class of stable steady-state solutions to equation (9).

6 Future Plans

The primary focus of future research will be on using an existing PDE solver to solve the macroscopic equation (9) numerically. One can then be compared to the microscopic model to determine how accurate the macroscopic equation is. We expect there to be some discrepancies, as the macroscopic equation is an averaged distribution of non-media opinions. Understanding these discrepancies is an interesting facet of future work. We will supply identical initial distributions of opinions to models, and then compare the distributions of the non-media account at convergence. We can calculate the overall error with the relation to the evolution of opinions by comparing how the set of opinions evolve over time in the macroscopic and microscopic model. Additionally, we will analyze the differences in statistics defined in [8], such as χ values, σ^2, the number of opinion groups and R. Some of these statistics will need to be adapted for measuring the influence of heterogeneous media opinions, and entirely new statistics may need to be developed.

We also plan to further investigate the potential stationary states allowed by the macroscopic equation (9) derived, attempting to make further conclusions about what non-Dirac solutions exist, if any. We are interested in using PDE solvers to numerically solve equation (10) and equation (11), examining both homogeneous and heterogeneous media cases. It may also be possible to identify media distributions that lead to steady states in similar models in the literature and test to see if they lead to a steady state in our model.

Finally, we are also working on attempting to apply a change of variables to the macroscopic PDE such that the opinion space becomes $(-1, 1)$ and the η function is changed such that c becomes heterogeneous across every possible opinion. We are looking at finding a suitable function for the change of variables, such as arctan, a sigmoid function and an error function. This is in service of making the PDE easier to solve numerically, as well as potentially making further analysis easier to conduct.

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