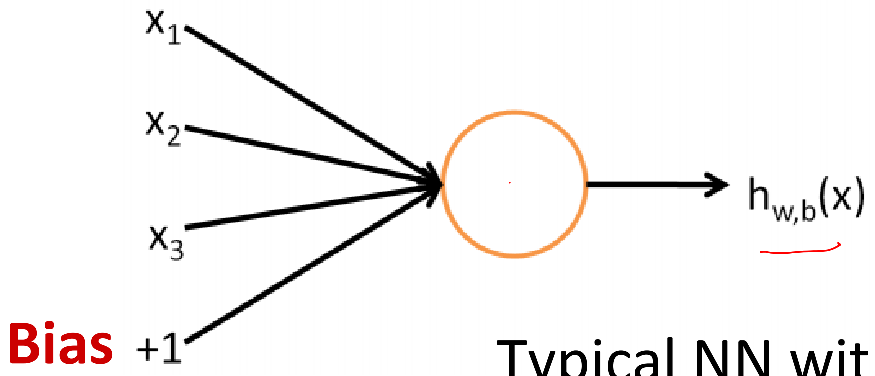
Back Propagation Algorithm for MLP

EE 5179

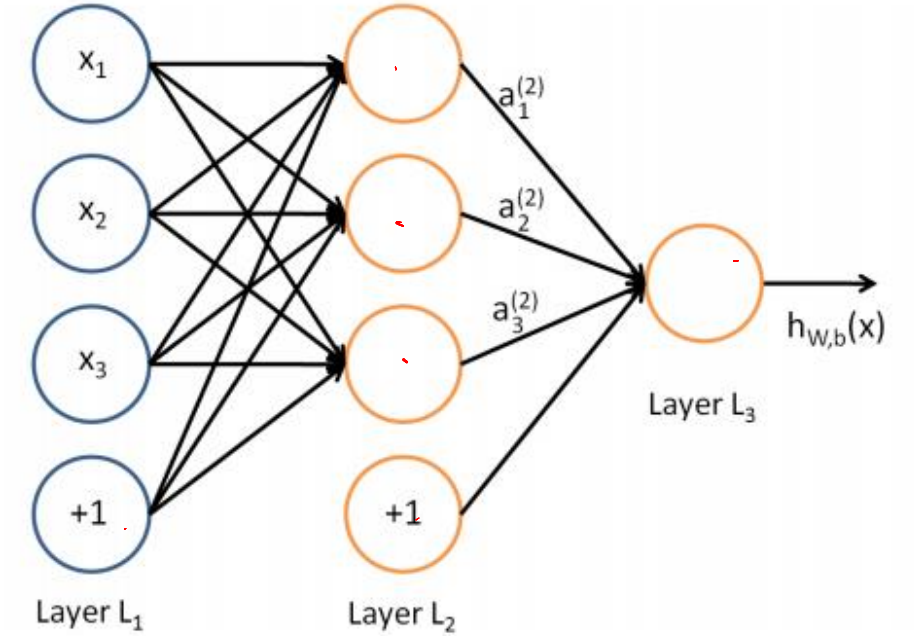
Instructor: Kaushik Mitra

How to learn the parameters?



$$h_{W,b}(x) = f(W^T x) = f(\sum_{i=1}^{3} W_i x_i + b)$$

Typical NN with many such units

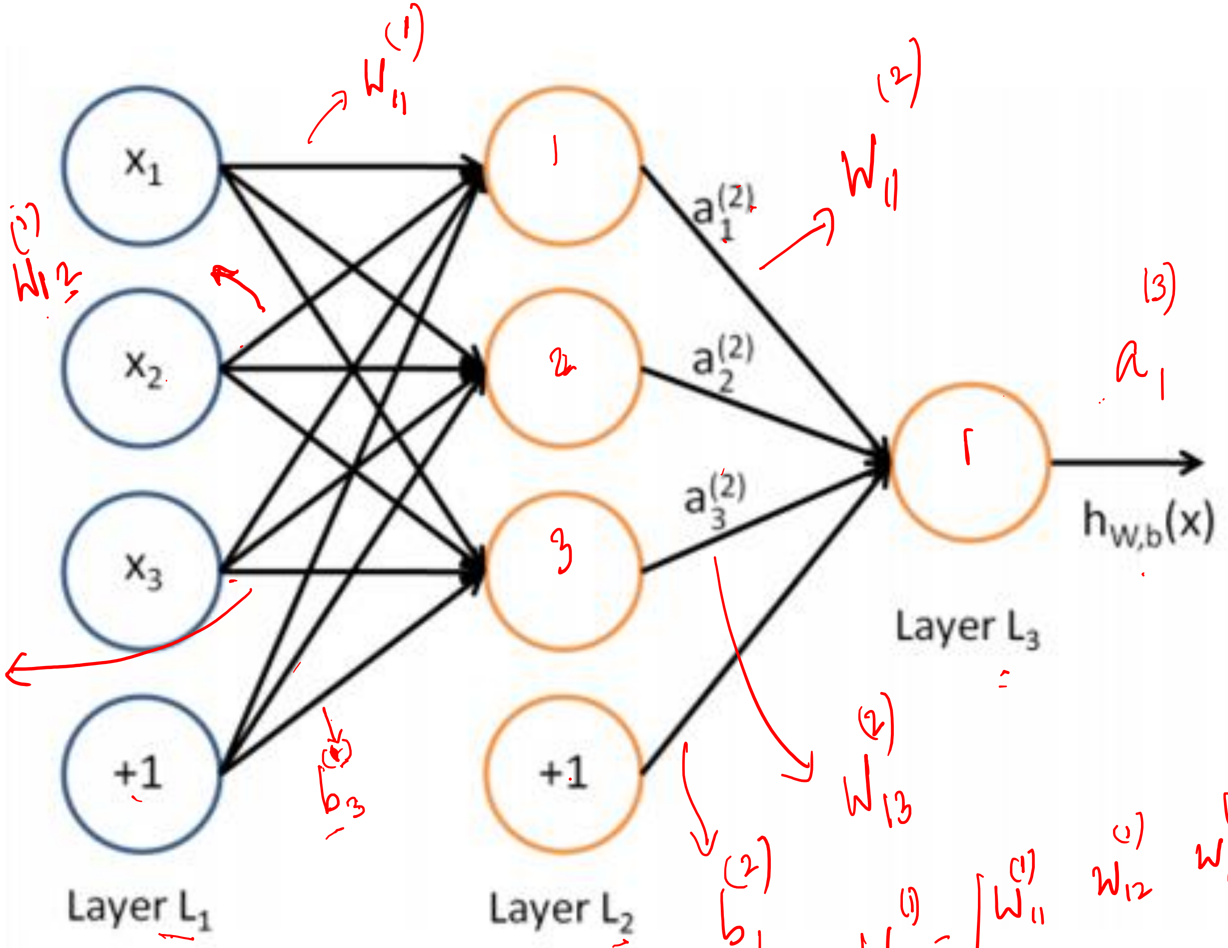


- One hidden layer
    - 3 neuron units
- One output

*Slide courtesy, sparse autoencoder by Andrew Ng

How to learn the parameters?

$W_{i \leftarrow j}^{(l)}$



$W^{(1)}_{3\times 3} = \begin{pmatrix} W^{(1)}_{11} & W^{(1)}_{12} & W^{(1)}_{13} \\ W^{(1)}_{31} & - & - \end{pmatrix}$

L_l – Layer l

$a_i^{(l)}$ – activation of unit i in layer l

$W_{ij}^{(l)}$ – Weight from j^{th} unit in l to i^{th} unit in $l+1$

$b_i^{(l)}$ – bias to unit i in layer $l+1$

Parameters:

3×1 $\begin{pmatrix} b^{(1)}_1 \\ b^{(1)}_2 \\ b^{(1)}_3 \end{pmatrix}$ $\begin{pmatrix} W^{(2)}_{11} & W^{(2)}_{12} & W^{(2)}_{13} \end{pmatrix}$ $\begin{pmatrix} b^{(2)}_1 \end{pmatrix}$

$$(W^{(1)}, b^{(1)}, W^{(2)}, b^{(2)})$$

$$W^{(1)} \in \mathbb{R}^{3\times 3},\ W^{(2)} \in \mathbb{R}^{1\times 3}$$

*Slide courtesy, sparse autoencoder by Andrew Ng

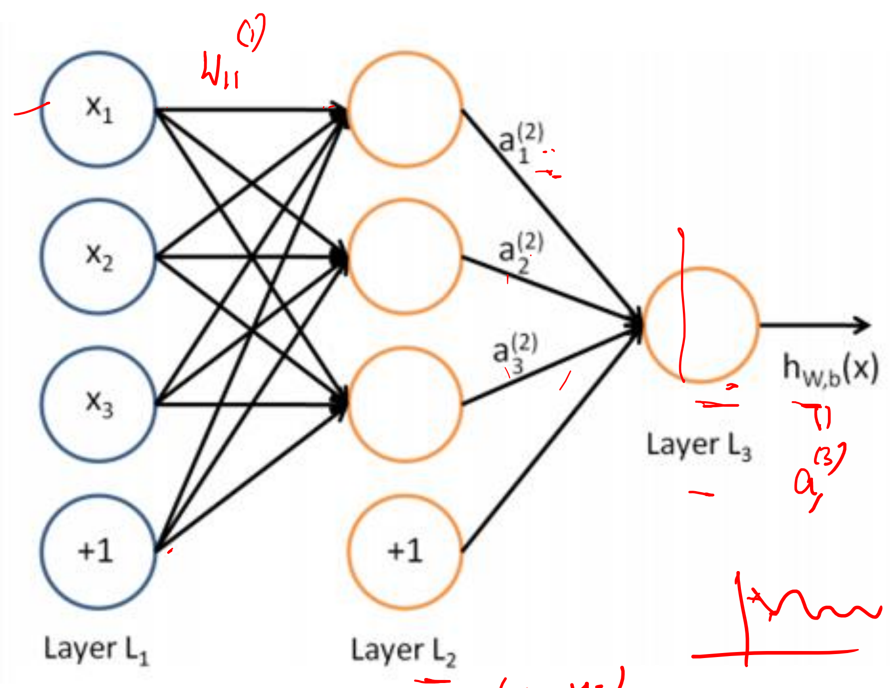
How to learn the parameters?

Layer 2,

$$a_1^{(2)} = f(W_{11}^{(1)}x_1 + W_{12}^{(1)}x_2 + W_{13}^{(1)}x_3 + b_1^{(1)})$$
$$a_2^{(2)} = f(W_{21}^{(1)}x_1 + W_{22}^{(1)}x_2 + W_{23}^{(1)}x_3 + b_2^{(1)})$$
$$a_3^{(2)} = f(W_{31}^{(1)}x_1 + W_{32}^{(1)}x_2 + W_{33}^{(1)}x_3 + b_3^{(1)})$$

Layer 3,

$$h_{W,b}(x) = a_1^{(3)} = f(W_{11}^{(2)}a_1^{(2)} + W_{12}^{(2)}a_2^{(2)} + W_{13}^{(2)}a_3^{(2)} + b_1^{(2)})$$



(x_i, y_i)

$\mathcal{L}(W,b) = \sum_i \frac{1}{2}\|y_i - h_{W,b}(x_i)\|^2$

$W := W - \alpha \frac{\partial L}{\partial W}$

forward recursion

Let, $z_i^{(l)}$ denote weighted sum for the $a_i^{(l)}$ activation

$a^{(l)} \stackrel{?}{=} x$

$$z^{(2)} = W^{(1)}x + b^{(1)}$$
$$a^{(2)} = f(z^{(2)})$$
$$z^{(3)} = W^{(2)}a^{(2)} + b^{(2)}$$
$$h_{W,b}(x) = a^{(3)} = f(z^{(3)})$$

$z_i^{(l)} = \sum_j W_{ij}^{(l-1)} a_j^{(l-1)} + b_i^{(l-1)}$

$a_i^{(l)} = f(z_i^{(l)})$

$$\boxed{\begin{aligned} z^{(l+1)} &= W^{(l)}a^{(l)} + b^{(l)} \\ a^{(l+1)} &= f(z^{(l+1)}) \end{aligned}}$$

*Slide courtesy, sparse autoencoder by Andrew Ng

How to learn the parameters?

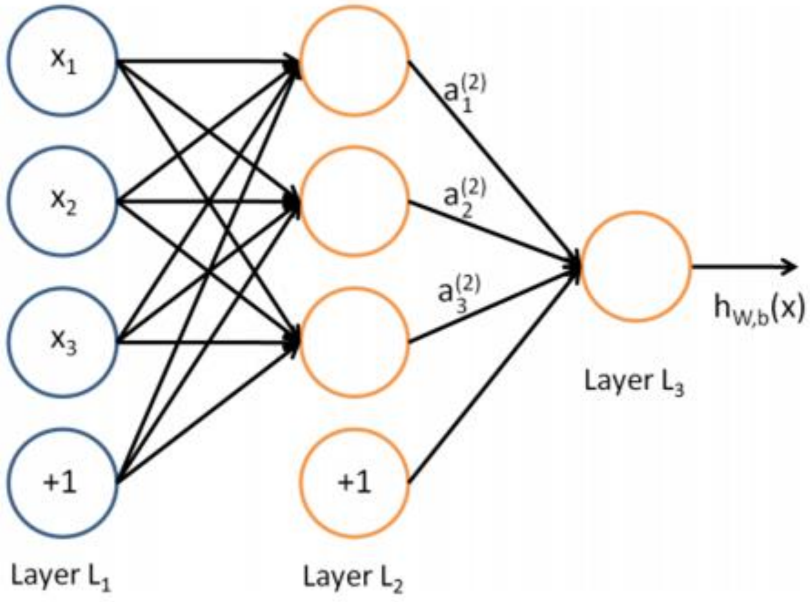
Given *m* training examples

$$\{(x^{(1)}, y^{(1)}), \ldots, (x^{(m)}, y^{(m)})\}$$

Minimize:

$$J(W,b;x,y) = \frac{1}{2}\left\|h_{W,b}(x) - y\right\|^2$$

$$\begin{aligned} J(W,b) &= \left[\frac{1}{m}\sum_{i=1}^{m} J(W,b;x^{(i)},y^{(i)})\right] \\ &= \left[\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{2}\left\|h_{W,b}(x^{(i)}) - y^{(i)}\right\|^2\right)\right] \end{aligned}$$



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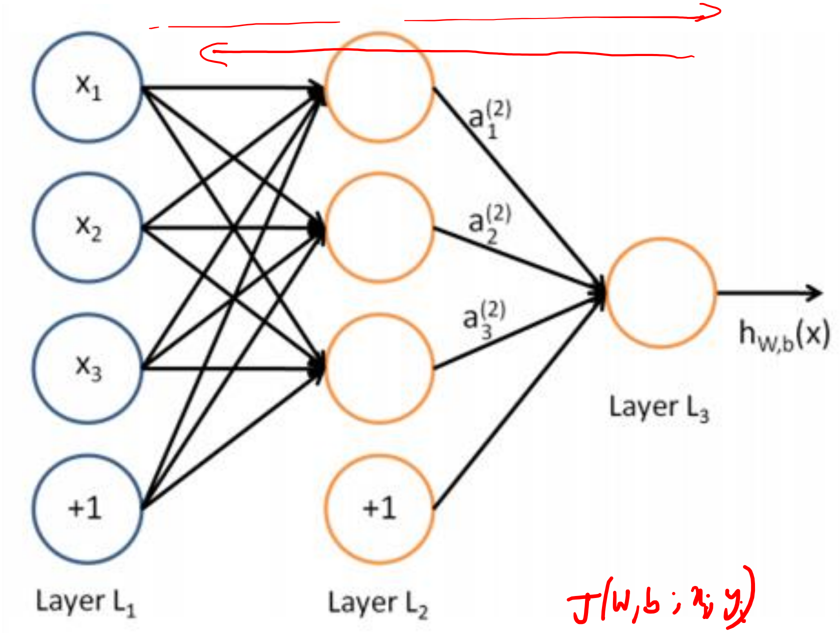
How to learn the parameters?



Minimize:

$$J(W,b;x,y) = \frac{1}{2}\left\| h_{W,b}(x) - y \right\|^2$$

$J(W,b;x_i,y_i)$

Gradient descent:

$$W_{ij}^{(l)} := W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$$

$$b_i^{(l)} := b_i^{(l)} - \alpha \frac{\partial}{\partial b_i^{(l)}} J(W,b)$$

$\frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$

$J(W,b) = \frac{1}{m}\sum_i \frac{1}{2}\| h_{W,b}(x_i) - y_i \|$

$\frac{\partial J}{\partial W_{ij}^{(l)}} = \frac{1}{m}\sum_i \frac{\partial J(W,b;x_i,y_i)}{\partial W_{ij}^{(l)}}$

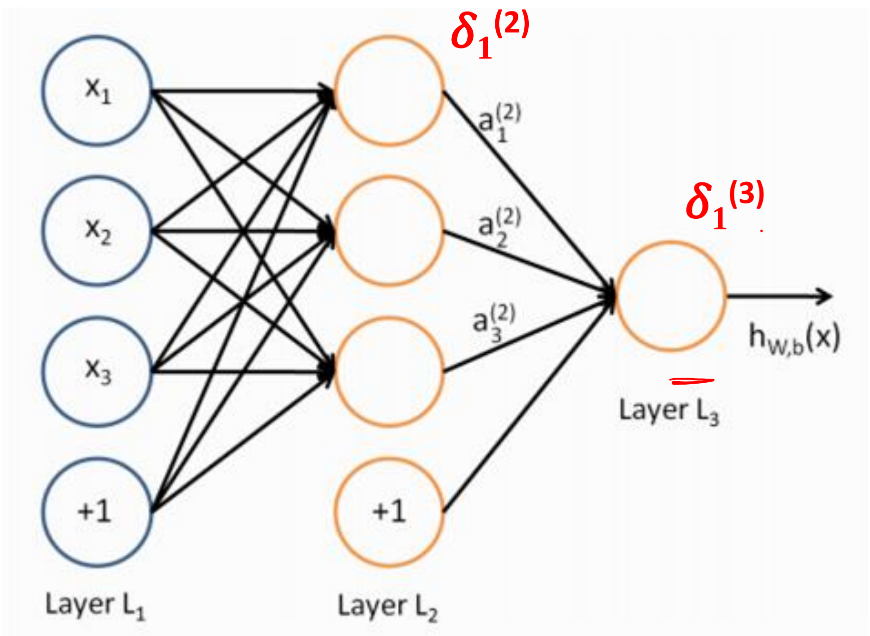
Stochastic

How to evaluate these partial derivatives?

Error back-propagation

*Slide courtesy, [sparse autoencoder by Andrew Ng]

Back-propagation algorithm



Gradient descent:

$$W_{ij}^{(l)} := W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$$

Idea:

First, forward pass the data to calc. all responses

In backward pass, for each unit i in layer l calculate **error** term $\delta_i^{(l)}$ - measures how much unit i is responsible for output error

$$\delta_i^{(l)} = \frac{\partial J}{\partial z_i^{(l)}}$$

$$\delta_1^{(3)} = \frac{\partial}{\partial z_1^{(3)}} \frac{1}{2}\|y - f(z_1^{(3)})\|^2 = -(y - f(z_1^{(3)})) \frac{\partial f(z_1^{(3)})}{\partial z_1^{(3)}}$$

$f'(z_1^{(3)})$

– For output unit in last layer (n_l), this is easy

$\sum_i (y_i - f(z_i))^2$

$$\delta_i^{(n_l)} = \frac{\partial}{\partial z_i^{(n_l)}} \frac{1}{2}\|y - h_{W,b}(x)\|^2 = -(y_i - a_i^{(n_l)}) \cdot f'(z_i^{(n_l)})$$

$a_1^{(3)} = f(z_1^{(3)})$

– How to measure $\delta_i^{(l)}$ for hidden units?

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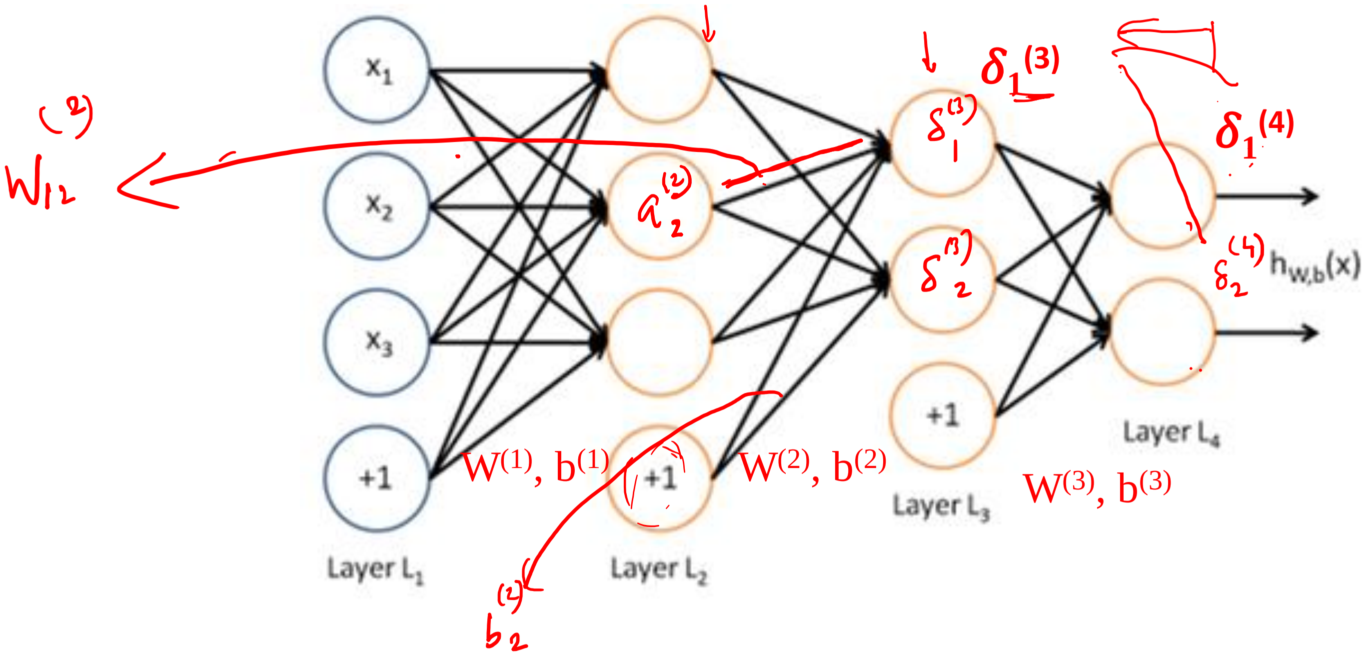
Back-propagation algorithm



$$h_{W,b}(x) = a^{(4)} = f(z^{(4)}); \; z^{(4)} = W^{(3)}a^{(3)} + b^{(3)}$$

Gradient descent:

$$W_{ij}^{(l)} := W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$$

$$J(W,b;x,y) = \frac{1}{2} \left\| h_{W,b}(x) - y \right\|^2$$

For last layer:

$$\frac{\partial J}{\partial W_{ij}^{(3)}} = \boxed{\frac{\partial J}{\partial z_i^4}} \boxed{\frac{\partial z_i^4}{\partial W_{ij}^{(3)}}}$$

$$\boxed{\frac{\partial J}{\partial z_i{}^4} = -(y_i - a_i^{(4)}) \;.f'(z_i^4)}$$

$\delta_i^{(4)}$ **error term**

$$\boxed{\frac{\partial z_i^4}{\partial W_{ij}^3} = a_j^{(3)}}$$

$$\frac{\partial J}{\partial W_{ij}^{(l)}} = \delta_i^{(l+1)} a_j^{(l)} \qquad \frac{\partial J}{\partial b_i^{(l)}} = \delta_i^{(l+1)}$$

$W_{ij}^{(l)}$ $(l+1)$ (l)

$i \leftarrow j$

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Back-propagation algorithm

Gradient descent:

$$W_{ij}^{(l)} := W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$$

$$J(W,b;x,y) = \frac{1}{2} \left\| h_{W,b}(x) - y \right\|^2$$

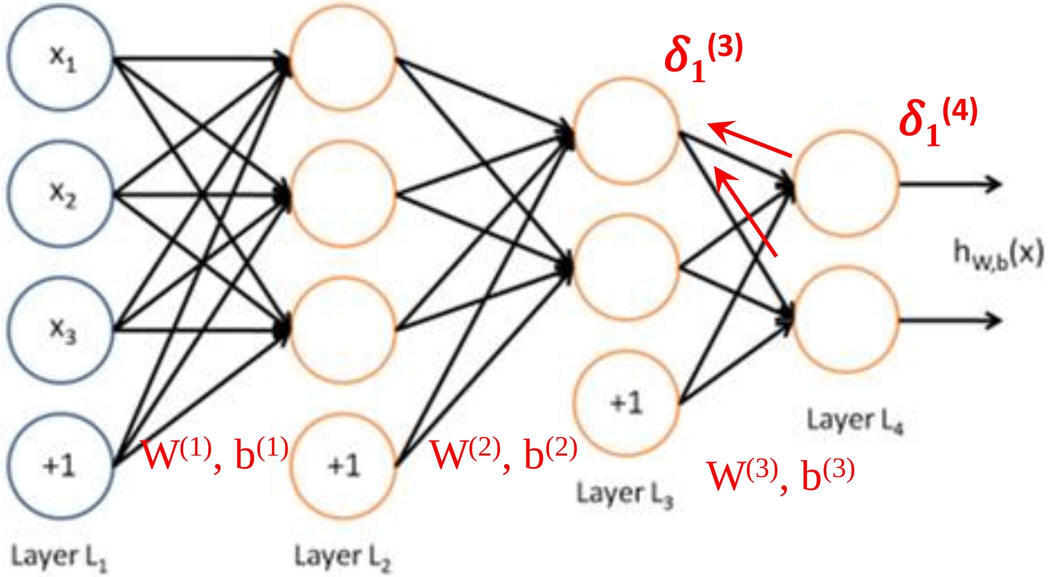
For layers other than last:

$$\frac{\partial J}{\partial W_{ij}^{(2)}} = \boxed{\frac{\partial J}{\partial z_i^{(3)}}} \boxed{\frac{\partial z_i^{(3)}}{\partial W_{ij}^{(2)}}}$$

$a_j^{(2)}$

$$\delta_i^{(l)} = \left(\sum_{j=1}^{s_{l+1}} W_{ji}^{(l)} \delta_j^{(l+1)} \right) f'(z_i^{(l)})$$

$$\frac{\partial J}{\partial W_{ij}^{(l)}} = \delta_i^{(l+1)} a_j^{(l)} \qquad \frac{\partial J}{\partial b_i^{(l)}} = \delta_i^{(l+1)}$$



$$h_{W,b}(x) = a^{(4)} = f(z^{(4)}); \; z^{(4)} = W^{(3)}a^{(3)} + b^{(3)}$$

$$a^{(3)} = f(z^{(3)}); \; z^{(3)} = W^{(2)}a^{(2)} + b^{(2)}$$

$\delta_i^{(3)}$

error term

$$\boxed{\begin{aligned} \frac{\partial J}{\partial z_i^{(3)}} &= \frac{\partial J}{\partial a_i^{(3)}} \frac{\partial a_i^{(3)}}{\partial z_i^{(3)}} \\ &= \left(\sum_j \frac{\partial J}{\partial z_j^{(4)}} \frac{\partial z_j^{(4)}}{\partial a_i^{(3)}} \right) f'(z_i^{(3)}) \end{aligned}}$$

$\delta_j^{(4)}$ Layer - (l+1)

$W_{ji}^{(3)}$

*Slide courtesy, [sparse autoencoder by Andrew Ng]

Back-propagation algorithm

1. Perform a feedforward pass
   - Computing activations L_1, L_2 and so on ...

2. For each output unit i in layer L_4 (output layer), set

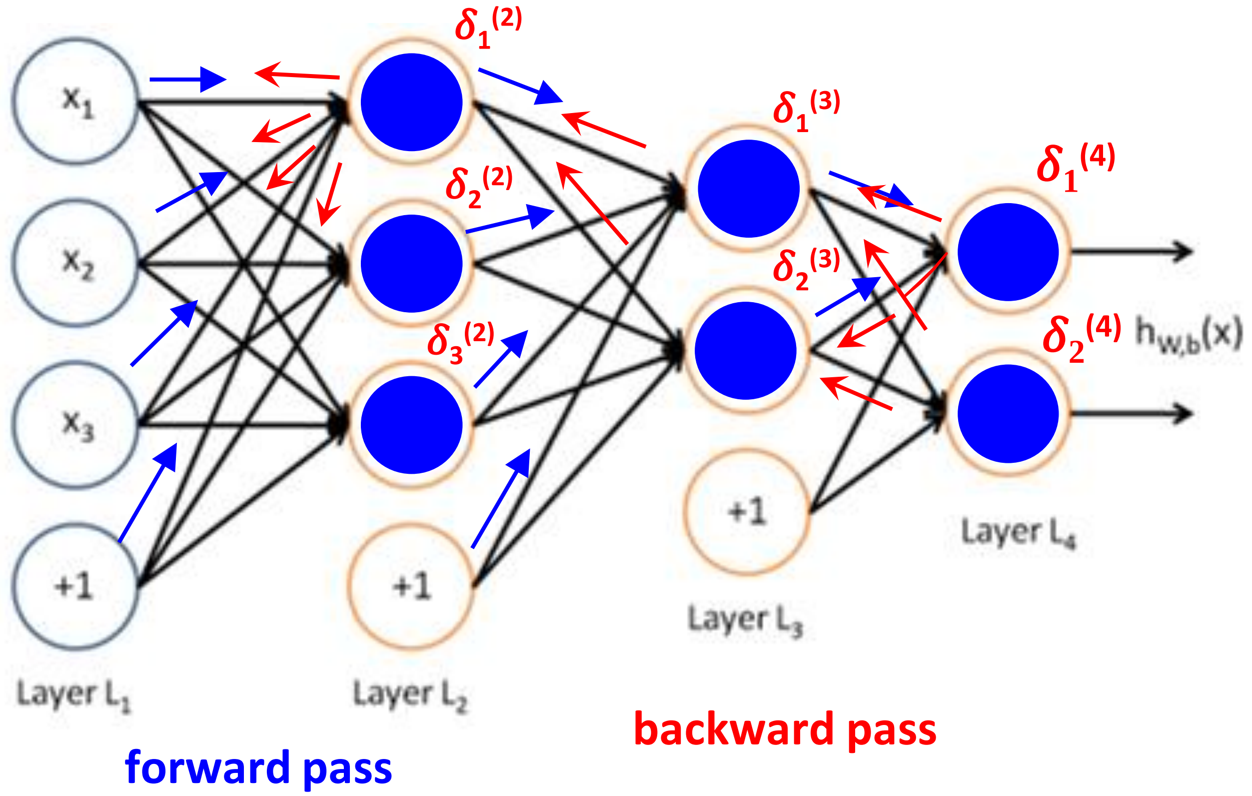
$$\delta_i^{(n_l)} = \frac{\partial}{\partial z_i^{(n_l)}} \frac{1}{2}\left\|y - h_{W,b}(x)\right\|^2 = -(y_i - a_i^{(n_l)}) \cdot f'(z_i^{(n_l)})$$

3. Starting from last but one layer to 2nd layer;
   $l = n_l - 1, n_l - 2, \ldots\ldots, 2$
   - For each node i in layer l, set

$$\delta_i^{(l)} = \left(\sum_{j=1}^{s_{l+1}} W_{ji}^{(l)} \delta_j^{(l+1)}\right) f'(z_i^{(l)})$$

4. Compute the desired partial derivatives, as:

$$\frac{\partial}{\partial W_{ij}^{(l)}} J(W,b;x,y) = a_j^{(l)} \delta_i^{(l+1)} \qquad \frac{\partial}{\partial b_i^{(l)}} J(W,b;x,y) = \delta_i^{(l+1)}.$$



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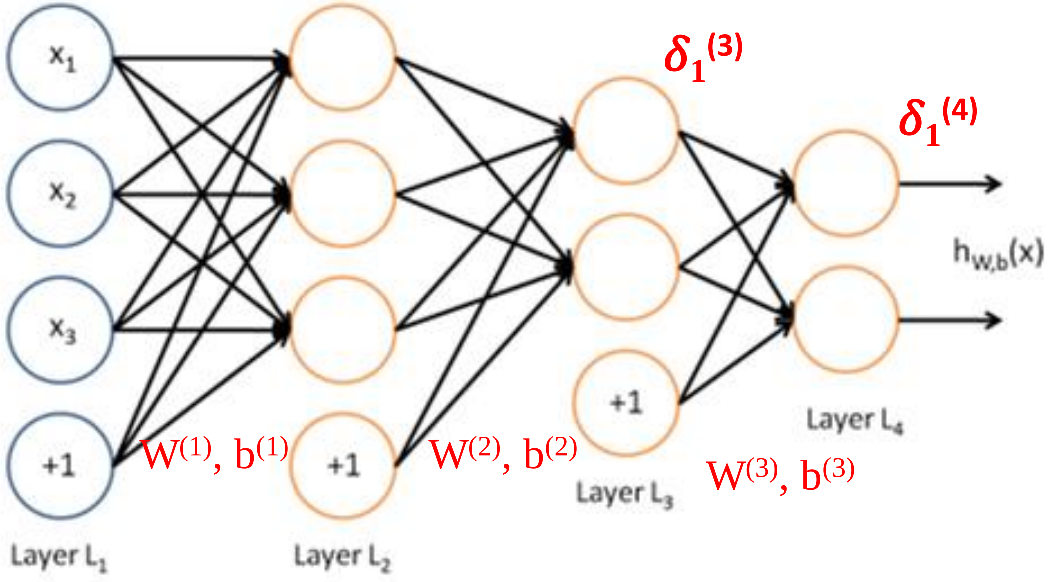
Back-propagation algorithm



$$h_{W,b}(x) = a^{(4)} = f(z^{(4)}); \; z^{(4)} = W^{(3)} a^{(3)} + b^{(3)}$$

Gradient descent:

$$W_{ij}^{(l)} := W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b)$$

$$J(W,b;x,y) = \frac{1}{2} \left\| h_{W,b}(x) - y \right\|^2$$

Partial derivatives:

$$\delta_i^{(l)} = \left(\sum_{j=1}^{s_{l+1}} W_{ji}^{(l)} \delta_j^{(l+1)} \right) f'(z_i^{(l)})$$

$$\frac{\partial J}{\partial W_{ij}^{(l)}} = \delta_i^{(l+1)} a_j^{(l)} \qquad \frac{\partial J}{\partial b_i^{(l)}} = \delta_i^{(l+1)}$$

Matrix notation:

$$\delta^{(l)} = \left((W^{(l)})^T \delta^{(l+1)} \right) \bullet f'(z^{(l)})$$

$$\frac{\partial J}{\partial W^{(l)}} = \delta^{(l+1)} (a^{(l)})^T \qquad \frac{\partial J}{\partial b^{(l)}} = \delta^{(l+1)}$$

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Back-propagation algorithm

1. Perform a feedforward pass
   - Computing activations L_1, L_2 and so on ...

2. For each output unit i in layer L_4 (output layer), set

$$\delta^{(n_l)} = -(y - a^{(n_l)}) \bullet f'(z^{(n)})$$

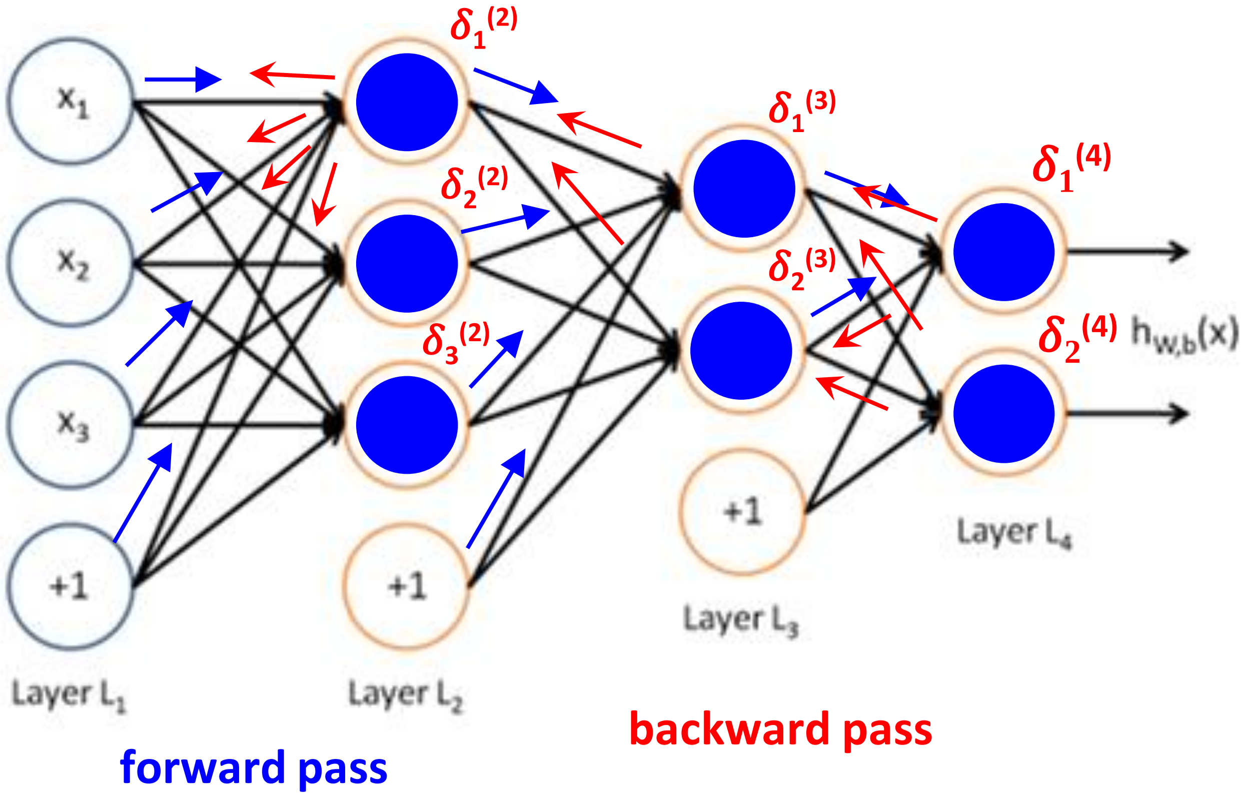
3. Starting from last but one layer to 2nd layer;

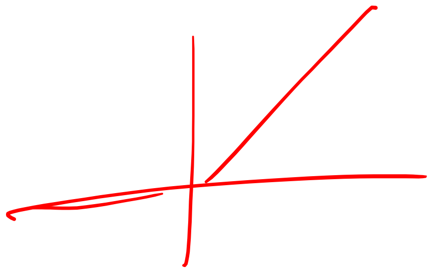
   $l = n_l\text{-}1, n_l\text{ -}2, \;....., 2$

$$\delta^{(l)} = \left((W^{(l)})^T \delta^{(l+1)}\right) \bullet f'(z^{(l)})$$

4. Compute the desired partial derivatives, as:

$$\begin{aligned} \nabla_{W^{(l)}} J(W,b;x,y) &= \delta^{(l+1)} (a^{(l)})^T, \\ \nabla_{b^{(l)}} J(W,b;x,y) &= \delta^{(l+1)}. \end{aligned}$$





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