Backpropagation

Convolutional Neural Networks

EE5179: Deep Learning for Imaging

Slides based on Pavithra Solai’s article at https://pavisj.medium.com/convolutions-and-backpropagations-46026a8f5d2c,
Jefkine’s article at http://www.jefkine.com/general/2016/09/05/backpropagation-in-convolutional-neural-networks/,
And many slides are from Rukayat Sadiq’s slides on CNN backpropagation
https://deeplearning.cs.cmu.edu/F21/document/recitation/Recitation5/CNN_Backprop_Recitation_5_F21.pdf

Back-propagation algorithm for NN

1. Perform a feedforward pass
   - Computing activations L_1, L_2 and so on ...

2. For each output unit i in layer L_4 (output layer), set

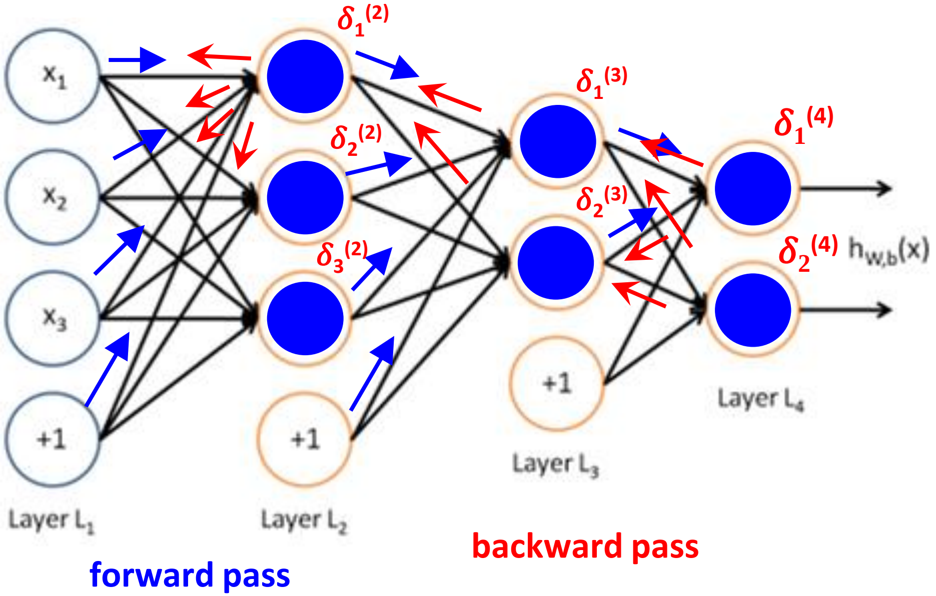
$$\delta^{(n_l)} = -(y - a^{(n_l)}) \bullet f'(z^{(n)})$$

3. Starting from last but one layer to 2nd layer;
   $l = n_l\text{-}1, n_l\text{ -}2, \; ....., \; 2$

$$\delta^{(l)} = \left((W^{(l)})^T \delta^{(l+1)}\right) \bullet f'(z^{(l)})$$

4. Compute the desired partial derivatives, as:

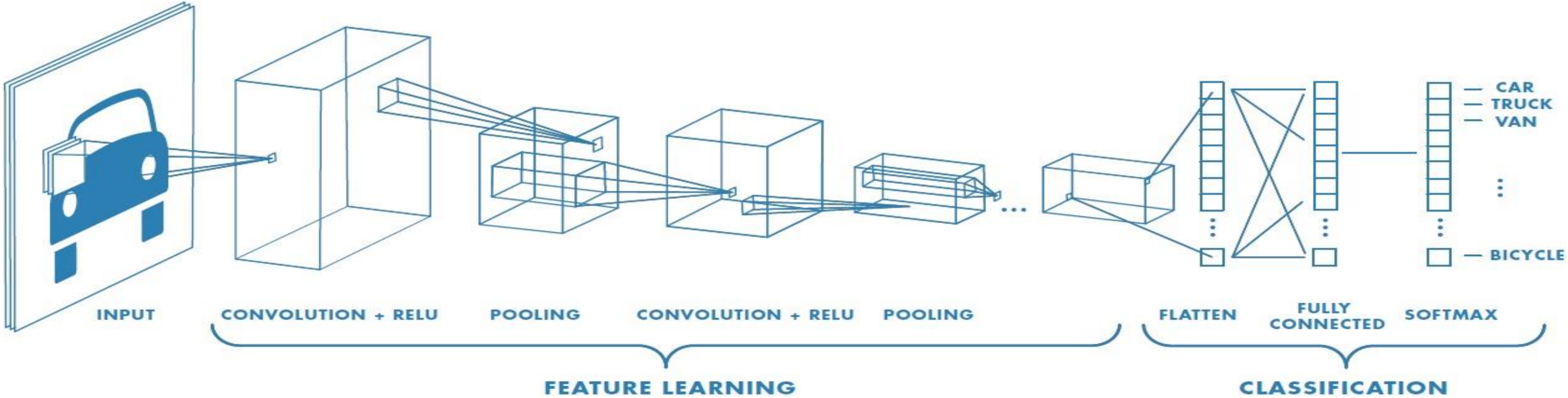
$$\begin{aligned} \nabla_{W^{(l)}} J(W,b;x,y) &= \delta^{(l+1)} (a^{(l)})^T, \\ \nabla_{b^{(l)}} J(W,b;x,y) &= \delta^{(l+1)}. \end{aligned}$$

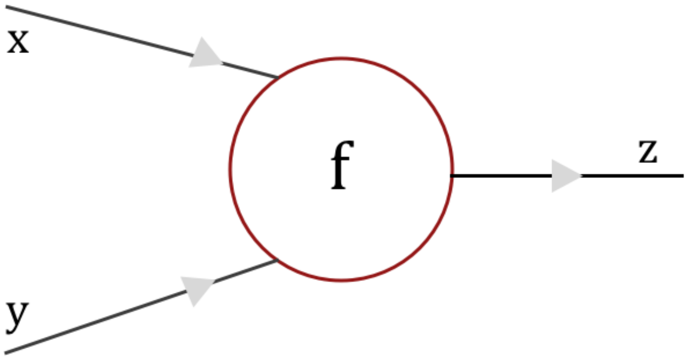


*Slide courtesy, sparse autoencoder by Andrew Ng

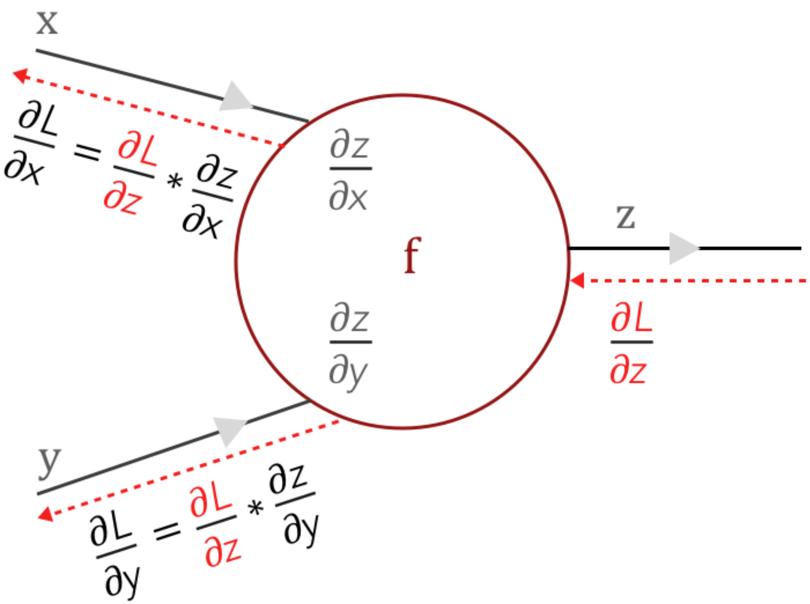
Review of Conv Nets

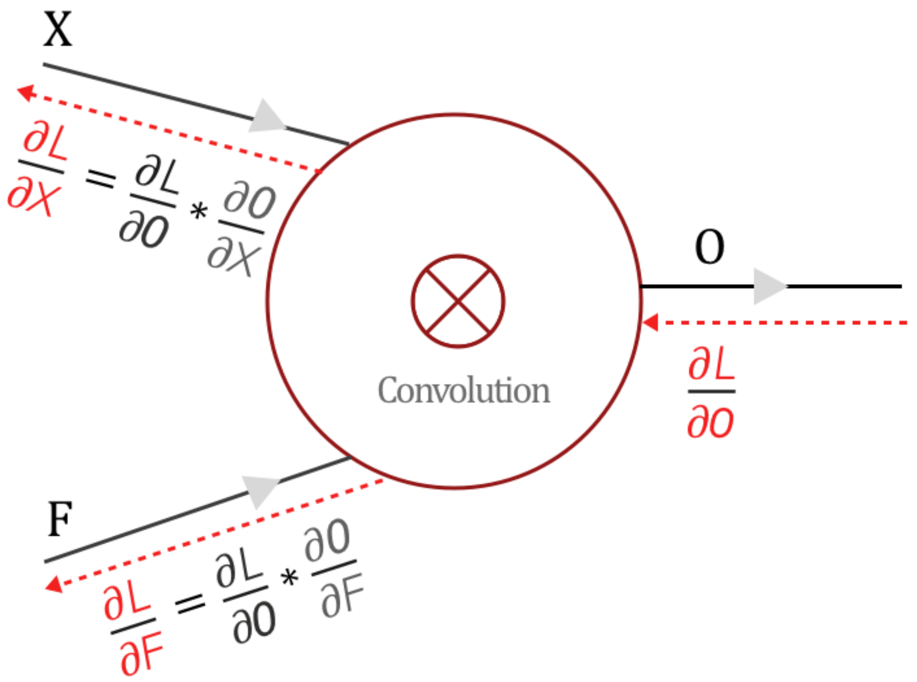
- Steps involved
  - Perform convolutions
  - Apply non-linearity
  - Pooling





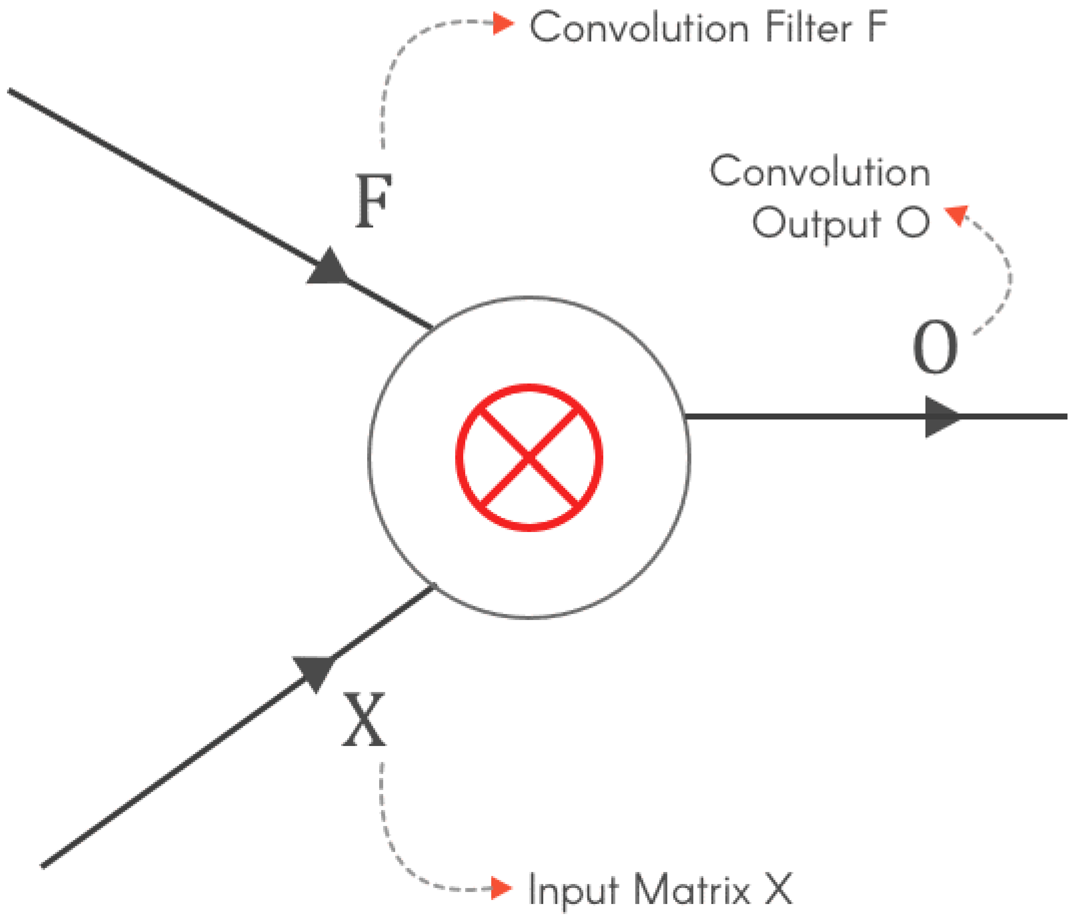
A simple function f which takes x and y as inputs and outputs z





$\frac{\partial O}{\partial X}$ & $\frac{\partial O}{\partial F}$ *are local gradients*

$\frac{\partial L}{\partial z}$ *is the loss from the previous layer which has to be backpropagated to other layers*


Convolution Filter F
F
Convolution
Output O
O
X
Input Matrix X

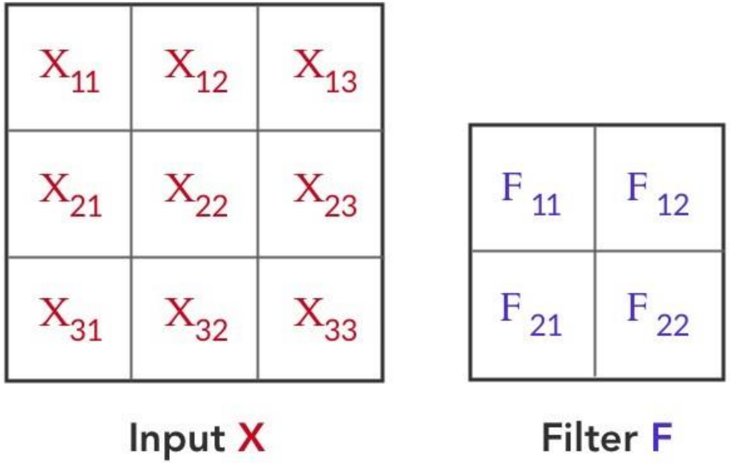
Backpropagation in CNNs

- In the backward pass, we get the loss gradient with respect to the next layer
- In CNNs the loss gradient is computed w.r.t the input and also w.r.t the filter.

Convolution Backprop with single Stride

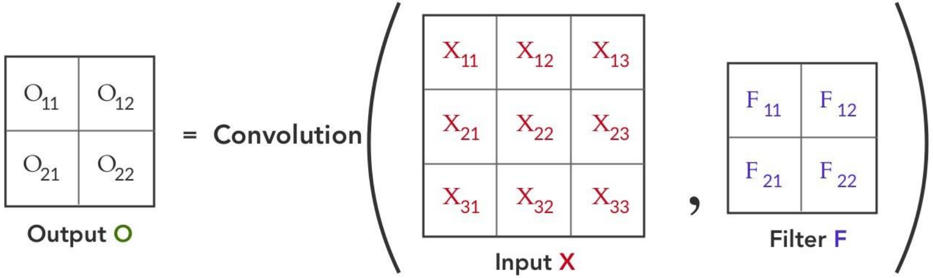
- To understand the computation of loss gradient w.r.t input, let us use the following example:

- Horizontal and vertical stride = 1



Convolution Forward Pass

- Convolution between Input X and Filter F, gives us an output O. This can be represented as:



Convolution Forward Pass

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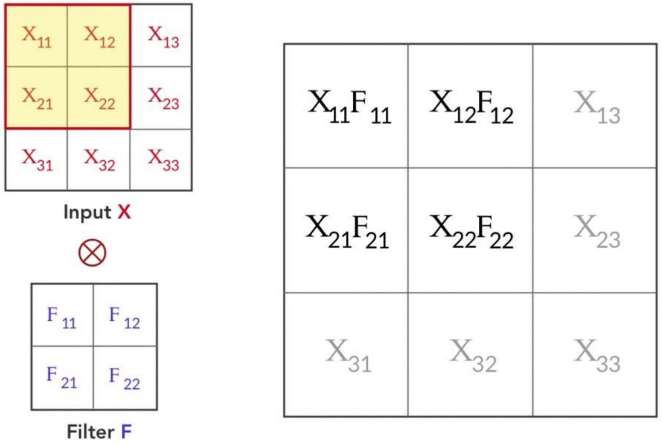
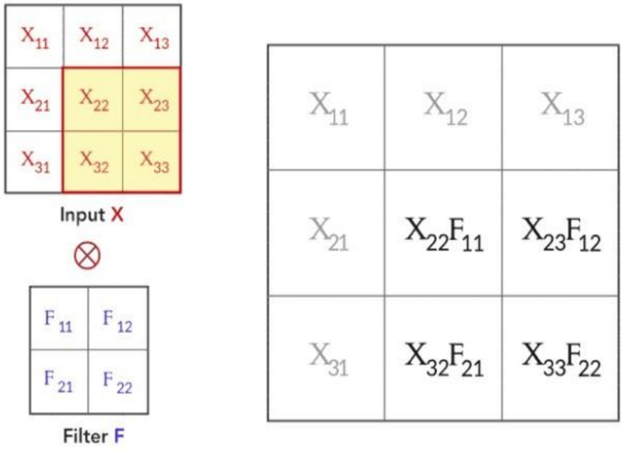


$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

Convolution Forward Pass

•Convolution between Input X and Filter F, gives us an output O. This can be represented as:



$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

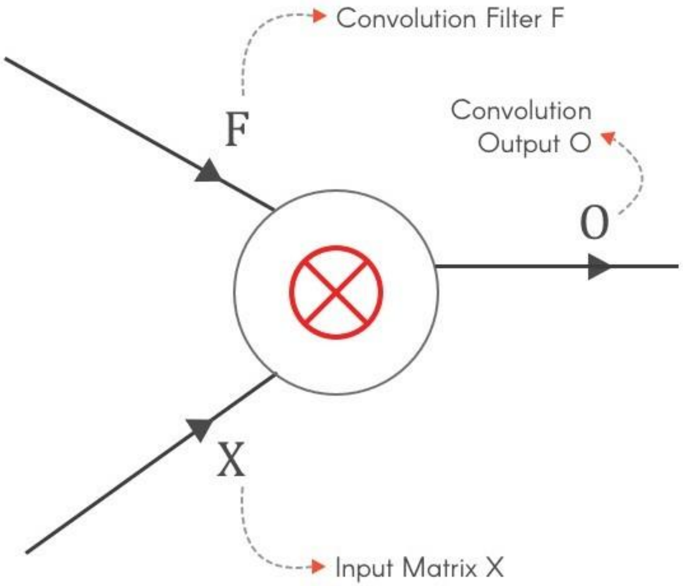
$$O_{12} = X_{12}F_{11} + X_{13}F_{12} + X_{22}F_{21} + X_{23}F_{22}$$

$$O_{21} = X_{21}F_{11} + X_{22}F_{12} + X_{31}F_{21} + X_{32}F_{22}$$

$$O_{22} = X_{22}F_{11} + X_{23}F_{12} + X_{32}F_{21} + X_{33}F_{22}$$

Loss gradient

- We want to calculate the gradients wrt to input ‘X’ and filter ‘F’



Loss gradient w.r.t the filter

We can use the chain rule to obtain the gradient wrt the filter as shown in the equation.

$$\underbrace{\frac{\partial L}{\partial F}}_{\text{Gradient to update Filter F}} = \underbrace{\frac{\partial L}{\partial O}}_{\text{Loss Gradient from previous layer}} * \underbrace{\frac{\partial O}{\partial F}}_{\text{Local Gradients}}$$

For every element of F

$$\frac{\partial L}{\partial F_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial F_i}$$

Loss gradient w.r.t the filter

We can expand the chain rule summation as:

For every element of F

$$\frac{\partial L}{\partial F_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial F_i}$$

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{11}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{11}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{11}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{11}}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{12}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{12}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{12}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{12}}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{21}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{21}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{21}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{21}}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{22}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{22}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{22}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{22}}$$

Loss gradient w.r.t the filter

- Replacing the local gradients of the filter i.e, $\frac{60_{i}}{6F_{i}}$, we get this:

$$\begin{bmatrix} \frac{\partial L}{\partial F_{11}} & \frac{\partial L}{\partial F_{12}} \\ \frac{\partial L}{\partial F_{21}} & \frac{\partial L}{\partial F_{22}} \end{bmatrix} = \text{Convolution}\left(\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} , \begin{bmatrix} \frac{\partial L}{\partial O_{11}} & \frac{\partial L}{\partial O_{12}} \\ \frac{\partial L}{\partial O_{21}} & \frac{\partial L}{\partial O_{22}} \end{bmatrix} \right)$$

where

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \text{Input X} \qquad \begin{bmatrix} \frac{\partial L}{\partial O_{11}} & \frac{\partial L}{\partial O_{12}} \\ \frac{\partial L}{\partial O_{21}} & \frac{\partial L}{\partial O_{22}} \end{bmatrix} = \frac{\partial L}{\partial O} \quad \text{Loss gradient from previous layer}$$

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * X_{11} + \frac{\partial L}{\partial O_{12}} * X_{12} + \frac{\partial L}{\partial O_{21}} * X_{21} + \frac{\partial L}{\partial O_{22}} * X_{22}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * X_{12} + \frac{\partial L}{\partial O_{12}} * X_{13} + \frac{\partial L}{\partial O_{21}} * X_{22} + \frac{\partial L}{\partial O_{22}} * X_{23}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * X_{21} + \frac{\partial L}{\partial O_{12}} * X_{22} + \frac{\partial L}{\partial O_{21}} * X_{31} + \frac{\partial L}{\partial O_{22}} * X_{32}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * X_{22} + \frac{\partial L}{\partial O_{12}} * X_{23} + \frac{\partial L}{\partial O_{21}} * X_{32} + \frac{\partial L}{\partial O_{22}} * X_{33}$$

Loss gradient w.r.t the filter

- If you closely look at it, this represents an operation we are quite familiar with. We can represent it as a **convolution operation between input X** and loss gradient **∂L/∂O as shown below:**

$$\begin{bmatrix} \frac{\partial L}{\partial F_{11}} & \frac{\partial L}{\partial F_{12}} \\ \frac{\partial L}{\partial F_{21}} & \frac{\partial L}{\partial F_{22}} \end{bmatrix} = \text{Convolution}\left(\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \begin{bmatrix} \frac{\partial L}{\partial O_{11}} & \frac{\partial L}{\partial O_{12}} \\ \frac{\partial L}{\partial O_{21}} & \frac{\partial L}{\partial O_{22}} \end{bmatrix} \right)$$

where

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} = \text{Input X} \qquad \begin{bmatrix} \frac{\partial L}{\partial O_{11}} & \frac{\partial L}{\partial O_{12}} \\ \frac{\partial L}{\partial O_{21}} & \frac{\partial L}{\partial O_{22}} \end{bmatrix} = \frac{\partial L}{\partial O} \text{ Loss gradient from previous layer}$$

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * X_{11} + \frac{\partial L}{\partial O_{12}} * X_{12} + \frac{\partial L}{\partial O_{21}} * X_{21} + \frac{\partial L}{\partial O_{22}} * X_{22}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * X_{12} + \frac{\partial L}{\partial O_{12}} * X_{13} + \frac{\partial L}{\partial O_{21}} * X_{22} + \frac{\partial L}{\partial O_{22}} * X_{23}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * X_{21} + \frac{\partial L}{\partial O_{12}} * X_{22} + \frac{\partial L}{\partial O_{21}} * X_{31} + \frac{\partial L}{\partial O_{22}} * X_{32}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * X_{22} + \frac{\partial L}{\partial O_{12}} * X_{23} + \frac{\partial L}{\partial O_{21}} * X_{32} + \frac{\partial L}{\partial O_{22}} * X_{33}$$

Loss gradient w.r.t the input

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For every element of X_i

$$\frac{\partial L}{\partial X_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial X_i}$$

Loss gradient w.r.t the input

- Similarly, we can expand the chain rule summation for the gradient with respect to the input. After substituting the local gradients i.e $\frac{6O_i}{6K_i}$, we have:

$$\frac{\partial L}{\partial X_{11}} = \frac{\partial L}{\partial O_{11}} * F_{11}$$

$$\frac{\partial L}{\partial X_{12}} = \frac{\partial L}{\partial O_{11}} * F_{12} + \frac{\partial L}{\partial O_{12}} * F_{11}$$

$$\frac{\partial L}{\partial X_{13}} = \frac{\partial L}{\partial O_{12}} * F_{12}$$

$$\frac{\partial L}{\partial X_{21}} = \frac{\partial L}{\partial O_{11}} * F_{21} + \frac{\partial L}{\partial O_{21}} * F_{11}$$

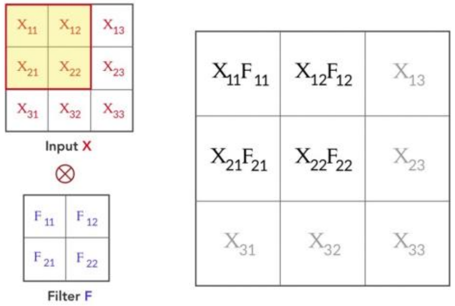
$$\frac{\partial L}{\partial X_{22}} = \frac{\partial L}{\partial O_{11}} * F_{22} + \frac{\partial L}{\partial O_{12}} * F_{21} + \frac{\partial L}{\partial O_{21}} * F_{12} + \frac{\partial L}{\partial O_{22}} * F_{11}$$

$$\frac{\partial L}{\partial X_{23}} = \frac{\partial L}{\partial O_{12}} * F_{22} + \frac{\partial L}{\partial O_{22}} * F_{12}$$

$$\frac{\partial L}{\partial X_{31}} = \frac{\partial L}{\partial O_{21}} * F_{21}$$

$$\frac{\partial L}{\partial X_{32}} = \frac{\partial L}{\partial O_{21}} * F_{22} + \frac{\partial L}{\partial O_{22}} * F_{21}$$

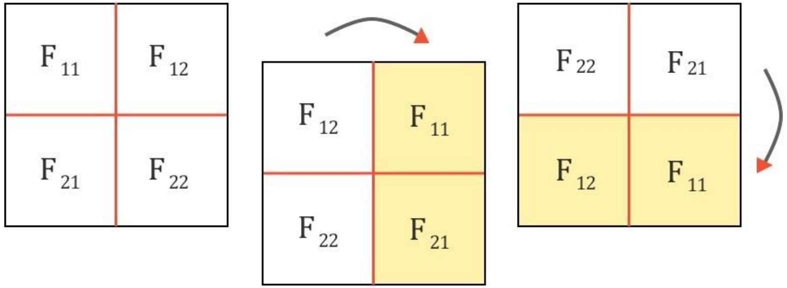
$$\frac{\partial L}{\partial X_{33}} = \frac{\partial L}{\partial O_{22}} * F_{22}$$



$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

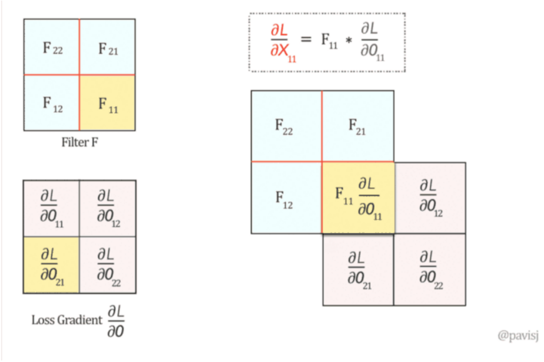
Loss gradient w.r.t the input

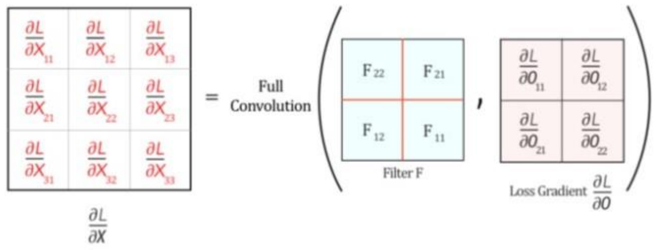
- First, let us rotate the Filter F by 180 degrees. This is done by flipping it first vertically and then horizontally.



Loss gradient w.r.t the input

- We see that the loss gradient wrt the input $\frac{6L}{6K}$ is given as a full convolution between the filter and Loss gradient $\frac{6L}{60}$.





Takeaway

- Both the Forward pass and the Backpropagation of a Convolutional layer are Convolutions

$$\frac{\partial L}{\partial F} = \text{Convolution}\left(\text{Input } X \text{ , Loss gradient } \frac{\partial L}{\partial O}\right)$$

$$\frac{\partial L}{\partial X} = \begin{matrix}\text{Full} \\ \text{Convolution}\end{matrix}\left(\begin{matrix}180^{\circ}\text{rotated} \\ \text{Filter } F\end{matrix} \text{ , } \begin{matrix}\text{Loss} \\ \text{Gradient}\end{matrix} \frac{\partial L}{\partial O}\right)$$

Backprop in Pooling Layer

- Max Pooling
    - the error is just assigned to where it comes from
- Average Pooling
    - The error is multiplied by 1/(NxN) and assigned to the whole pooling block