L_2 - norm regularization

$$\tilde{J}(w) = J(w) + \frac{\alpha}{2}\|w\|^2$$

$J(w) \rightarrow$ Unregularized loss

Unregularized solution $\leftarrow$ $w^* = \arg\min_w J(w)$

regularized soln. $\leftarrow$ $\tilde{w} = \arg\min_w \tilde{J}(w)$

$H(w^*) \equiv$ Hessian

$$J(w) \simeq J(w^*) + \cancel{(w-w^*)^T \nabla J(w^*)}^{\,0} + \frac{1}{2}(w-w^*)^T H (w-w^*)$$

$$J(w) \simeq J(w^*) + \frac{1}{2}\underbrace{(w-w^*)^T H (w-w^*)}_{\geq 0}$$

H is symmetric, real, positive semi-definite

$\lambda_i \rightarrow$ eigen-values of H

$\lambda_i \geq 0$

$$\rightarrow \boxed{\tilde{w}_i = \left(\frac{\lambda_i}{\lambda_i + \alpha}\right) w_i^*}$$

Weight decay

$$w^* = \begin{pmatrix} w_1^* \\ w_2^* \\ \vdots \end{pmatrix} \qquad \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \vdots \end{pmatrix}$$

$\lambda_i \rightarrow$ eigenvalues of H

$$f(x) = f(x^*) + (x-x^*) f'(x^*) + \frac{(x-x^*)^2}{2!} f''(x^*) + \frac{(x-x^*)^3}{3!} f'''(x^*) + \dots$$

$f(x_1, x_2)$

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

$x^T H x \geq 0$

$\forall x$

$$J(w) \simeq J(w^*) + \frac{1}{2}(w-w^*)^T H (w-w^*)$$

$$\tilde{J}(w) = J(w) + \alpha w^T w$$

$$\tilde{J}(w) \approx J(w^*) + \frac{1}{2}(w-w^*)^T H (w-w^*) + \frac{\alpha w^T w}{2}$$

$$\tilde{w} = \arg\min_{w} \tilde{J}(w)$$

$$\nabla \tilde{J}(\tilde{w}) = 0$$

$$\nabla \tilde{J}(w) = H(w-w^*) + \alpha w$$

$$\nabla \tilde{J}(\tilde{w}) = 0 \implies H(\tilde{w} - w^*) + \alpha \tilde{w} = 0$$

$$(H + \alpha I)\tilde{w} = H w^*$$

$$\boxed{\tilde{w} = (H + \alpha I)^{-1} H w^*}$$

$$H = Q \Lambda Q^T$$

H is p.s.d $\equiv x^T H x \geq 0$

$\lambda_i \rightarrow$ eigen value, real

$v_i \rightarrow$ eigen vector

$$H v_i = \lambda_i v_i$$

$$v_i^T H v_i \geq 0$$

$$\lambda_i v_i^T v_i \geq 0$$

$$\lambda_i \|v_i\|^2 \geq 0$$

$$\lambda_i \geq 0$$

$$\frac{1}{2} \nabla_x (x^T H x) = Hx$$

$$Q = [v_1 \; v_2 \cdots v_n] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$Q^T Q = I \qquad v_i^T v_j = 0, \; i \neq j$$
$$= 1, \; i = j$$

$$Q Q^T = I$$

$$Q^{-1} = Q^T$$

$$\tilde{w} = \left(Q\Lambda Q^T + \alpha Q Q^T\right)^{-1} Q\Lambda Q^T w^*$$

$$\tilde{w} = \left[Q\left(\Lambda + \alpha I\right)Q^T\right]^{-1} Q\Lambda Q^T w^*$$

$$\tilde{w} = Q\left(\Lambda + \alpha I\right)^{-1} \overbrace{\cancel{Q^T Q}}^{I}\, \Lambda Q^T w^*$$

$$= Q\left(\Lambda + \alpha I\right)^{-1} \Lambda Q^T w^*$$

If $Q = I$

$$\tilde{w} = \left(\Lambda + \alpha I\right)^{-1} \Lambda w^*$$

If $Q = I$,

$$\boxed{\tilde{w}_i = \frac{\lambda_i}{(\lambda_i + \alpha)} w_i^*} \Rightarrow \text{wt. decay}$$

If $\lambda_i \gg \alpha$ then $\tilde{w}_i \simeq w_i^*$

If $\lambda_i \ll \alpha$ then $\tilde{w}_i \to 0$

$$H v_i = \lambda_i v_i$$

$$H \underbrace{\left[v_1\ v_2 \dots v_n\right]}_{Q} = \left[Hv_1\ Hv_2 \dots Hv_n\right]$$

$$= \left[\lambda_1 v_1\ \lambda_2 v_2 \dots \lambda_n v_n\right]$$

$$HQ = \underbrace{\left[v_1\ v_2 \dots v_n\right]}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ \phi & & \ddots \ \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\boxed{H = Q\Lambda Q^T}$$

$$\begin{pmatrix} \lambda_1 + \alpha & & 0 \\ & \lambda_2 + \alpha & \\ 0 & & \ddots \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$