GATE: CE - 29.2022

EE22BTECH11219 - Rada Sai Sujan

Question

Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*) (GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3}$$

Applying Z-transform we get,

$$X(z) * X(z) = \frac{PZ^{-1}}{(1 - z^{-1})(2 - z^{-1})} \tag{4}$$

$$= P\left(\frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-1}}{2 - z^{-1}}\right) \tag{5}$$

From the transformation pairs,

$$x_{n-a} \xleftrightarrow{Z} z^{-a}X(z) \tag{6}$$

$$x_{n_1} \times x_{n_2} \xleftrightarrow{Z} X_1(z) * X_2(z) \tag{7}$$

$$\frac{u(n-1)}{a^n} \xleftrightarrow{Z} \frac{z^{-1}}{a - z^{-1}} \tag{8}$$

Now, applying inverse Z-tranform,

$$x_n^2 = P\left(u(n-1) - \frac{u(n-1)}{2^n}\right) \tag{9}$$

$$\implies x_n^2 = P\left(1 - \frac{1}{2^n}\right) \quad [\because n \geq 1] \tag{10}$$

Similarly,

$$x_{n+1}^2 = P\left(1 - \frac{1}{2^{n+1}}\right) \tag{11}$$

$$\implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt{\frac{\left(1 - \frac{1}{2^n}\right)}{\left(1 - \frac{1}{2^{n+1}}\right)}} \tag{12}$$

$$= 1 \tag{13}$$

Hence, the system is convergent.
Now finding the limit of the sequence,

$$x^2 = \lim_{x\to\infty} P\left(1 - \frac{1}{2^n}\right) \tag{14}$$

$$\implies x = \pm\sqrt{P} \tag{15}$$

From (2) and (15),

$$x_{n+1} = \sqrt{P} \tag{16}$$

Therefore, for $P = 10$ the value of x_5 is,

$$x_5 = \sqrt{10} \tag{17}$$

$$\therefore x_5 = 3.162 \tag{18}$$