GATE: BM - 36.2022

EE22BTECH11219 - Rada Sai Sujan

QUESTION

In the complex z-domain, the value of integral $\oint_C \frac{z^3-9}{3z-i}\, dz$ is

(a) $\frac{2\pi}{81} - 6i\pi$

(b) $\frac{2\pi}{81} + 6i\pi$

(c) $-\frac{2\pi}{81} + 6i\pi$

(d) $-\frac{2\pi}{81} - 6i\pi$

(GATE 2022 BM)

Solution:

Simplyfying the Contour Integral to the standard form we get,

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{1}{3}\oint_C \frac{z^3-9}{z-\frac{i}{3}}\, dz \qquad (1)$$

From Cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \sum R_j \qquad (2)$$

We can observe a non-repeated pole at $z = \frac{i}{3}$ and thus a$=\frac{i}{3}$,

$$R = \lim_{z \to a} (z - a)\, f(z) \qquad (3)$$

$$\implies R = \frac{1}{3}\lim_{z \to \frac{i}{3}}\left(z - \frac{i}{3}\right)\frac{z^3-9}{z-\frac{i}{3}} \qquad (4)$$

$$= \frac{-i}{81} - 3 \qquad (5)$$

Therefore, from (2) and (5)

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{2\pi}{81} - 6i\pi \qquad (6)$$