

GATE: CE - 29.2022

EE22BTECH11219 - Rada Sai Sujan

QUESTION

Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{P}{x_n} \right), \quad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is _____.
(round off to three decimal places)

Solution:

Applying A.M \geq G.M inequality,

$$\begin{aligned} \frac{x_n + \frac{P}{x_n}}{2} &\geq \sqrt{P} \\ \Rightarrow x_{n+1} &\geq \sqrt{P} \end{aligned} \quad (1) \quad (2)$$

Solving the equation,

$$2x_{n+1}x_n - x_n^2 - P = 0 \quad (3)$$

Applying Z-transform we get,

$$X(z) * X(z) = \frac{PZ^{-1}}{(1 - z^{-1})(2 - z^{-1})} \quad (4)$$

$$= P \left(\frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-1}}{2 - z^{-1}} \right) \quad (5)$$

From the transformation pairs,

$$x_{n-a} \xleftrightarrow{Z} z^{-a} X(z) \quad (6)$$

$$x_{n_1} \times x_{n_2} \xleftrightarrow{Z} X_1(z) * X_2(z) \quad (7)$$

$$\frac{u(n-1)}{a^n} \xleftrightarrow{Z} \frac{z^{-1}}{a - z^{-1}} \quad (8)$$

Now, applying inverse Z-transform,

$$x_n^2 = P \left(u(n-1) - \frac{u(n-1)}{2^n} \right) \quad (9)$$

$$\Rightarrow x_n^2 = P \left(1 - \frac{1}{2^n} \right) \quad [\because n \geq 1] \quad (10)$$

Similarly,

$$x_{n+1}^2 = P \left(1 - \frac{1}{2^{n+1}} \right) \quad (11)$$

$$\Rightarrow x_{n+1}^2 - x_n^2 = \frac{P}{2^{n+1}} \quad (> 0) \quad (12)$$

Hence, the system is convergent as it is increasing and bounded,

Now finding the limit of the sequence,

$$x^2 = \lim_{x \rightarrow \infty} P \left(1 - \frac{1}{2^n} \right) \quad (13)$$

$$\Rightarrow x = \pm \sqrt{P} \quad (14)$$

From (2) and (14),

$$x_{n+1} = \sqrt{P} \quad (15)$$

Therefore, for $P = 10$ the value of x_5 is,

$$x_5 = \sqrt{10} \quad (16)$$

$$\therefore x_5 = 3.162 \quad (17)$$