1.6.2 Maxima and Minima (One Variable)

Let f be a function defined on an interval $I = [a, b]$. Then f is said to have a

- Global Max. on I: If $\exists$ a point $c \in I$ such that $f(c) \geq f(x), \forall x \in I$.
- Global Min. on I: If $\exists$ a point $c \in I$ such that $f(c) \leq f(x), \forall x \in I$.
- Local Max. at $c \in I$: If $\exists$ a nbd $N(c, \delta)$ such that $f(c) \geq f(x), \forall x \in N(c, \delta)$.
- Local Min. at $c \in I$: If $\exists$ a nbd $N(c, \delta)$ such that $f(c) \leq f(x), \forall x \in N(c, \delta)$.

- Extremum $\Rightarrow$ Either maximum or, minimum.
- Global or, Absolute Min/Max $\clubsuit$ Local or, Relative Min/Max.

Theorem 1.5. *If a function $f(x)$ has local extremum value at c. Then if $f'(c)$ exist, $f'(c) = 0$.*

Note that: Since $f'(c) = 0$, the point c is known as stationary point and $f(c)$ is known as stationary value of the function.

[Do It Yourself] 1.29. *Does the converse of the above theorem is true? Discuss.*

Theorem 1.6. *Derivative Test for Extrema: If f is a real valued function defined on $[a, b]$ and c be an interior point of $[a, b]$. If $f'(c) = 0$ then $f(x)$ has a local maximum Local for $f''(c) < 0$ and a local minimum for $f''(c) > 0$.*

■ Now we will study some concepts on local/ global extrema.
▶ Maxima $\Rightarrow$ Left $\uparrow$ and Right $\downarrow$.
▶ Minima $\Rightarrow$ Left $\downarrow$ and Right $\uparrow$.
▷ $f(x) = x$ has no global minima or maxima. It has no local extrema as well.
▷ $f(x) = x, \ x \in \ [-1, 1]$ has global minima -1 and global maxima 1. It has same local extrema.
▶ Every global extremum is a local extremum or, an endpoint extremum.
▶ If $f(x)$ is continuous on $[a, b] \Rightarrow f(x)$ have a global maximum and a global minimum on $[a, b]$. If the interval is not bounded or closed, then there is no guarantee that a continuous function $f(x)$ will have global extrema.

▷ $f(x) = x^2$ has global minima at $x = 0$ and it has no global maxima.
▷ $f(x) = x^2$ has global minima at $x = 1$ and global maxima at $x = 3$ in the interval $x \in [1, 3]$. This is known as endpoint extrema.
▷ $f(x) = x^3$ has no global minima or maxima. It has no local extrema as well.
▷ $f(x) = x^3 - 3x$ has no global minima or maxima. It has two local extrema $-1, 1$. Minima at 1 and maxima at -1. A graph of the function may help.
▷ $f(x) = |x|$ has global minima at $x = 0$ and it can't be found by using derivative. A graph of the function or, increasing-decreasing concept may help.
▷ $f(x) = 1/x$ has no global minima or maxima. It has no local extrema as well.
▷ $f(x) = 1/x,\ x \in\ [-1, 1]$ has global minima -1 and global maxima 1. It has same local extrema.
▷ $f(x) = \sin(x)$ has infinitely global maxima at $x = \pi/2, 5\pi/2, \cdots, -3\pi/2, -7\pi/2, \cdots$ and infinitely global minima at $x = 3\pi/2, 7\pi/2, \cdots, -\pi/2, -5\pi/2, \cdots$.
▶ $f(x) = x + \frac{1}{x}$ has no global minima or maxima. It has local extrema at $-1, 1$. Local maxima is less than local minima. Draw the graph.
▶ $f(x) = x^2(x-2)^2(x-1)$ has no global minima or maxima. It has local extrema at $0, 2$. Local maxima and local minima are same. It also has other local extrema. Draw the graph.
▶ $f(x) = |x - 1| + |x - 2|$ has global minima 1 and no global maxima exists. It has same local minima. Draw the graph.
▷ $f(x) = [x]$ has no global minima or maxima. It has local: minima = maxima = 0; minima = maxima = 1, minima = maxima = -1, minima = maxima = 2, so on.
▷ $f(x) = [x],\ x \in\ [-1, 1]$ has global minima -1 and global maxima 1. It has local: minima = maxima = 0; minima = maxima = 1, minima = maxima = -1.

Theorem 1.7. *Higher Order Derivative Test: If f is a real valued function defined on $[a, b]$ and c be an interior point of $[a, b]$. If $f'(c) = f''(c) = f^{(3)}(c) = \cdots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$, then f has*

1. *No extremum at c if $n =$ odd.*
2. *A local extremum at c if $n =$ even. Local maximum for $f^{(n)}(c) < 0$ and local minimum for $f^{(n)}(c) > 0$.*

Example 1.18. *Examine the extreme values of the function $x^4(x + 1)^2$.*
$\Rightarrow$ *Let $f(x) = x^4(x + 1)^2$. So $f'(x) = 2x^3\,(x + 1)\,(3x + 2)$,*
$f^{(2)}(x) = 30x^4 + 40x^3 + 12x^2$,
$f^{(3)}(x) = 120x^3 + 120x^2 + 24x$,
$f^{(4)}(x) = 360x^2 + 240x + 24$. *Now $f'(x) = 0 \Rightarrow x = -1, -2/3, 0$.*
at $x = -1$, $f^{(2)}(x) > 0 \Rightarrow f(x)$has a local minimum at $x = -1$.
at $x = -2/3$, $f^{(2)}(x) < 0 \Rightarrow f(x)$has a local maximum at $x = -2/3$.
at $x = 0$, $f^{(2)}(x) = f^{(3)}(x) = 0, f^{(4)}(x) > 0 \Rightarrow f(x)$has a local minimum at $x = 0$.

[Do It Yourself] 1.30. *Discuss the local maximum or, minimum values of the following functions*

(A) $f(x) = \sin(x) + \cos(x)$ ♠ *For minima and maxima.*

(B) $f(x) = \sin(x)(1 + \cos(x)), x \in [0, 2\pi]$ ♠ *For minima and maxima.*

(C) $f(x) = (1/x)^x$ ♠ *For maxima.*

(D) $f(x) = |x|$ ♠ *For extremum.*

(E) $f(x) = |x - 1| + |x - 2|, x \in [0, 3]$ ♠ *For extremum.*

(F) $f(x) = x - [x]$ ♠ *For extremum at* $x = 0$.

Example 1.19. *Find the global extrema of the function* $f(x) = xe^{-x}$ *for* $x \in [0.5, 3]$.
$\Rightarrow$ *Let* $f(x) = xe^{-x}$. *So* $f'(x) = e^{-x} - xe^{-x}$, $f''(x) = -e^{-x} - e^{-x} + xe^{-x}$
Now $f'(x) = 0 \Rightarrow x = 1$.
at $x = 1/2$, $f(x) = \frac{1}{2\sqrt{e}}$.
at $x = 3$, $f(x) = \frac{3}{e^3}$.
at $x = 1$, $f(x) = \frac{1}{e}$.
Now for $x \in [0.5, 1)$, $f'(x) > 0$ *and for* $x \in (1, 3]$, $f'(x) < 0$. *So* $f(x)$ *has a global maximum at* $x = 1$.

[Do It Yourself] 1.31. *Find global extremum of* $f(x) = \frac{x}{1+x^2}$.

[Do It Yourself] 1.32. *Let* $f(x) = 3(x - 2)^{2/3} - (x - 2)$, $0 \leq x \leq 20$ *and* x_0, y_0 *are the points at which* $f(x)$ *attains its global maxima and minima respectively. Then find* $f(x_0) + f(y_0)$.
[*Hint* : $f'(x) = \frac{2}{(x-2)^{1/3}} - 1$, $[0, 2) \downarrow$, $(2, 10) \uparrow$, $(10, 20] \downarrow$, *so* $x = 0, 2, 10, 20$ *are extreme points, Here* $x_0 = 0$, $y_0 = 2$]

[Do It Yourself] 1.33. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} x^4(2 + \sin(\frac{1}{x})) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then which of the following statement(s) is (are) true?
(A) f *attains its minimum at* 0. (B) f *is monotone.* (C) f *is differentiable at* 0.
(D) $f(x) > 2x^4 + x^3$, *for all* $x > 0$.
[*Hint* : $-1 \leq \sin(\frac{1}{x}) \leq 1 \Rightarrow f(x) > 0 \Rightarrow f(x) > f(0) \Rightarrow (a)$ *is true,* $(b), (c)$ *easy,* $f(1) > 3 \Rightarrow \sin(1) > 1 \Rightarrow (d)$ *is false*]

[Do It Yourself] 1.34. *Let $f:[0,\pi/2]\to\mathbb{R}$ be defined as $f(x)=\alpha x+\beta\sin(x)$, $\alpha,\beta\in\mathbb{R}$. Let f have a local minimum at $x=\frac{\pi}{4}$ with $f(\frac{\pi}{4})=\frac{\pi-4}{4\sqrt{2}}$. Then find $8\sqrt{2}\alpha+4\beta$. (Ans : 4)*

[Do It Yourself] 1.35. *Let $f:[0,13]\to\mathbb{R}$ be defined by $f(x)=x^{13}-e^{-x}+5x+6$. Then find the minimum value of the function f on $[0,13]$. (Ans : 5)*

[Do It Yourself] 1.36. *Let the function $f:[0,\infty)\to\mathbb{R}$ be given by $f(x)=x^2e^{-x}$. Then the maximum value of f is*
(A) e^{-1}. (B) $4e^{-2}$. (C) $9e^{-3}$. (D) $16e^{-4}$.

[Do It Yourself] 1.37. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and differentiable on $(-\infty,0)\cup(0,\infty)$. Which of the following statements is (are) always TRUE?*
(A) If f is differentiable at 0 and $f'(0)=0$, then f has a local maximum or a local minimum at 0. (B) If f has a local minimum at 0, then f is differentiable at 0 and $f'(0)=0$. (C) If $f'(x)<0$ for all $x<0$ and $f'(x)>0$ for all $x>0$, then f has a global maximum at 0. (D) If $f'(x)>0$ for all $x<0$ and $f'(x)<0$ for all $x>0$, then f has a global maximum at 0.

1.6.3 Maxima and Minima (Two Variables)

► A necessary condition for $f(x,y)$ have an extreme value at (a,b) is that $f_x(a,b)=0,\ f_y(a,b)=0$.
► Converse is not true: For $f(x,y)=|x|+|y|$ the partial derivatives $f_x(0,0),\ f_y(0,0)$ does not exist but f has local minimum at $(0,0)$.
► If $f_x(a,b)=f_y(a,b)=0$ and $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)>0\Rightarrow f(x,y)$ has an extreme value at (a,b). If f_{xx} *or,* $f_{yy}>0\Rightarrow f(x,y)$ has a minimum value and if f_{xx} *or,* $f_{yy}<0\Rightarrow f(x,y)$ has a maximum value.
► If $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)<0\Rightarrow f(x,y)$ has neither a minima nor a maxima i.e. a saddle point at (a,b).
► If $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)=0\Rightarrow$ Further investigation is necessary.
■ Stationary Point: Let $S\subseteq\mathbb{R}^2$ and $f:S\to\mathbb{R}$. An interior point (a,b) of S is said to be a stationary point of f in S if both f_x,f_y exists at (a,b) and $f_x=f_y=0$ at (a,b).

Example 1.20. *Find the maxima and minima of the function* $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.
$\Rightarrow$ *Here* $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. *So* $f_x = 3x^2 - 3$ *and* $f_y = 3y^2 - 12$.
Now $f_x = 0 \Rightarrow x = \pm 1$ *and* $f_y = 0 \Rightarrow y = \pm 2$.
So the function has 4 stationary points: $(1,2), (-1,2), (1,-2), (-1,-2)$.
Now $f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36xy$.
At $(1,2)$, $f_{xx}f_{yy} - f_{xy}^2 > 0$ *and* $f_{xx} > 0 \Rightarrow f(x,y)$ *has minima at* $(1,2)$.
At $(-1,2)$, $f_{xx}f_{yy} - f_{xy}^2 < 0 \Rightarrow f(x,y)$ *has neither maxima nor minima at* $(-1,2)$.
At $(1,-2)$, $f_{xx}f_{yy} - f_{xy}^2 < 0 \Rightarrow f(x,y)$ *has neither maxima nor minima at* $(1,-2)$.
At $(-1,-2)$, $f_{xx}f_{yy} - f_{xy}^2 > 0$ *and* $f_{xx} < 0 \Rightarrow f(x,y)$ *has maxima at* $(-1,-2)$.

Example 1.21. *Show that the function* $f(x,y) = y^2 + x^2y + x^4$ *has minima at* $(0,0)$.
$\Rightarrow$ *Here* $f(x,y) = y^2 + x^2y + x^4$. *So* $f_x = 2xy + 4x^3$ *and* $f_y = 2y + x^2$.
Now $f_x = 0$ *and* $f_y = 0$ *at* $(0,0)$.
Now $f_{xx} = 2y + 12x^2,\ f_{yy} = 2,\ f_{xy} = 2x \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 0$ *at* $(0,0)$.
So this is a doubtful case and requires further investigation.
$f(x,y) = (x^2 + \frac{y}{2})^2 + \frac{3y^2}{4}$. *It is a sum of squares of two terms i.e.* $f(x,y) \geq 0 \Rightarrow f(x,y)$ *has minimum value at* $(0,0)$.

[Do It Yourself] 1.39. *Consider the domain* $D = \{(x,y) \in \mathbb{R}^2 : x \leq y\}$ *and the function* $h : D \to \mathbb{R}$ *defined by* $h(x,y) = (x-2)^4 + (y-1)^4,\ (x,y) \in D$. *Then the minimum value of* h *on* D *equals*
(A) 1/2 (B) 1/4 (C) 1/8 (D) 1/16.
[*Hint* : $h(x,y) \geq (x-2)^4 + (x-1)^4,\ h'(x) = 4[(x-2)^3 + (x-1)^3] = 0 \Rightarrow x - 2 = -x + 1 \Rightarrow x = 3/2$. *Also* $h''(x) > 0$. *So* $h(x,y) \geq (3/2-2)^4 + (3/2-1)^4 = 1/8$]

[Do It Yourself] 1.40. *Consider the function* $f(x,y) = x^3 - y^3 - 3x^2 + 3y^2 + 7,\ (x,y) \in \mathbb{R}^2$. *Then the local minimum and maximum of* f *are given by*
(A) 3, 7 (B) 4, 11 (C) 7, 11 (D) 3, 11.

[Do It Yourself] 1.41. *Let* $f(x,y) = x^2 - 400xy^2$ *for all* $(x,y) \in \mathbb{R}^2$. *Then* f *attains its (A) local minimum at* $(0,0)$ *but not at* $(1,1)$ *(B) local minimum at* $(1,1)$ *but not at* $(0,0)$ *(C) local minimum both at* $(0,0)$ *and* $(1,1)$ *(D) local minimum neither at* $(0,0)$ *nor at* $(1,1)$.

[Do It Yourself] 1.42. *The function* $f(x,y) = 3(x^2 + y^2) - 2(x^3 - y^3) + 6xy$ *for all* $(x,y) \in \mathbb{R}^2$ *has*
(A) A point of maxima (B) A point of minima (C) A saddle point (D) No saddle point.

[Do It Yourself] 1.43. *Consider the function* $f(x,y) = x^3 - 3xy^2,\ x,y \in \mathbb{R}$. *Which one of the following statements is TRUE?*
(A) $f(x,y)$ *has a local minimum at* $(0,0)$. *(B)* $f(x,y)$ *has a local maximum at* $(0,0)$ *(C)* $f(x,y)$ *has global maximum at* $(0,0)$ *(D)* $f(x,y)$ *has a saddle point at* $(0,0)$.

1.6.4 Application of Basic Definition (Two Variables)

▶ The function $f(x,y)$ have local maxima at (a,b) if $f(x,y)-f(a,b) \le 0$, $\forall (x,y) \in N(a,b)$.
▶ The function $f(x,y)$ have local minima at (a,b) if $f(x,y)-f(a,b) \ge 0$, $\forall (x,y) \in N(a,b)$.

Example 1.22. *Show that the function* $f(x,y) = x^4 + y^4 - 2x^2$ *has local minima at* $(-1,0),(1,0)$ *and has a saddle at* $(0,0)$.
$\Rightarrow$ *Here* $f(x,y) = x^4 + y^4 - 2x^2$. *So* $f_x = 4x^3 - 4x$ *and* $f_y = 4y^3$.
Now $f_x = 0$ *and* $f_y = 0$ *at* $(-1,0),(1,0),(0,0)$.
Now $f_{xx} = 12x^2 - 4,\ f_{yy} = 12y^2,\ f_{xy} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 0$ *at* $(-1,0),(1,0),(0,0)$.
So this is a doubtful case and requires further investigation.
Now, $f(x,y) - f(-1,0) = x^4 + y^4 - 2x^2 + 1 = (x^2-1)^2 + y^4 \ge 0,\ \forall (x,y) \in N(a,b)$.
Also, $f(x,y) - f(1,0) = x^4 + y^4 - 2x^2 + 1 = (x^2-1)^2 + y^4 \ge 0,\ \forall (x,y) \in N(a,b)$.
Therefore, $f(x,y)$ *has local minima at* $(-1,0),(1,0)$.
Again, $f(x,y) - f(0,0) = x^4 + y^4 - 2x^2 = (x^2-1)^2 + (y^4-1) < 0,\ if\ y = 0,\ |x| < \sqrt{2}$ *and* $f(x,y) - f(0,0) \ge 0,\ if\ x = 0,\ |y| < 1$. *So* $f(x,y) - f(0,0)$ *change sign in any neighborhood of* $(0,0)$ *i.e.* $(0,0)$ *is a saddle point.*

Example 1.23. *For the function* $f(x,y) = 2x^4 - 3x^2y + y^2$, *comments about the point* $(0,0)$.
$\Rightarrow$ *Here* $f(x,y) = 2x^4 - 3x^2y + y^2$. *So* $f_x = 8x^3 - 6xy$ *and* $f_y = -3x^2 + 2y$.
Now $f_x = 0$ *and* $f_y = 0$ *at* $(0,0)$.
Now $f_{xx} = 24x^2 - 6y,\ f_{yy} = 2,\ f_{xy} = -6x \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 0$ *at* $(0,0)$.
So this is a doubtful case and requires further investigation.
Now, $f(x,y) - f(0,0) = 2x^4 - 3x^2y + y^2 = (2x^2 - y)(x^2 - y) \le 0,\ if\ 0 < y/2 < x^2 < y$ *and* $f(x,y) - f(0,0) \ge 0,\ if\ y < 0,\ or, x^2 > y > 0,\ or, 2x^2 < y$. *So* $f(x,y) - f(0,0)$ *change sign in any neighborhood of* $(0,0)$ *i.e.* $(0,0)$ *is a saddle point.*

[Do It Yourself] 1.45. *If* $f(x,y) = x^2 + 2x^2y + 2x^4$, *comments about* $(0,0)$.

[Do It Yourself] 1.46. *Find all the critical points of the function* $f(x,y) = x^3 + y^3 + 3xy$ *and examine those points for local maxima and local minima.*

[Do It Yourself] 1.47. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = x^2 + xy + y^2 - x - 100$. *Find the points of local maximum and local minimum, if any, of* f.
$[Ans:\ (2/3, -1/3)\ Point\ of\ minima]$

1.6.5 Extrema with Three Variables

We will check for the extreme values of $f(x,y,z)$. This method is similar to the previous one

- Step 1: Find (a,b,c) such that $f_x = f_y = f_z = 0$.
- Step 2: $J = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$.

  Define $A = f_{xx}$, $B = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$, $C = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$
- $\boxed{A > 0, B > 0, C > 0 \Rightarrow Minimum}$, $\boxed{A < 0, B > 0, C < 0 \Rightarrow Maximum}$, $\boxed{B < 0 \Rightarrow Saddle}$. Otherwise, we will have to use the definition.

[Do It Yourself] 1.49. *Examine the existence for extrema of* $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2zx + yz$. $[Ans:\ Min\ at\ (0,0,0)]$

1.6.6 Lagrange's Method of Multiplier

▶ $dL = L_x dx + L_y dy$.
▶ $d^2L = d(L_x dx + L_y dy) = (dL_x)dx + (dL_y)dy = (L_{xx}dx + L_{xy}dy)dx + (L_{yx}dx + L_{yy}dy)dy = L_{xx}(dx)^2 + 2L_{xy}(dx)(dy) + L_{yy}(dy)^2$.

Example 1.24. *Using Lagrange's method of multiplier find the extreme values of* $f(x,y) = 7x^2 + 8xy + y^2$ *where* $x^2 + y^2 = 1$.
$\Rightarrow$ *The Lagrangian function is* $L(x,y) = 7x^2 + 8xy + y^2 + \lambda(x^2 + y^2 - 1)$. *Here* λ *is Lagrangian multiplier.*
Now for stationary points $L_x = 0 \Rightarrow 2(7+\lambda)x + 8y = 0$ *and* $L_y = 0 \Rightarrow 8x + 2(1+\lambda)y = 0$.
Again $x^2 + y^2 = 1 \Rightarrow x = y = 0$ *is not possible.*
So for nontrivial solution $\begin{vmatrix} 2(7+\lambda) & 8 \\ 8 & 2(1+\lambda) \end{vmatrix} = 0 \Rightarrow \lambda = 1, -9$.
For $\lambda = 1$, $2x + y = 0 \Rightarrow y = -2x$. *Therefore,* $x^2 + y^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{5}}$ *and* $y = \mp\frac{2}{\sqrt{5}}$.
So $(\pm\frac{1}{\sqrt{5}}, \mp\frac{2}{\sqrt{5}})$ *are stationary points.*
Now $dL = L_x dx + L_y dy$
$d^2L = L_{xx}(dx)^2 + 2L_{xy}(dx)(dy) + L_{yy}(dy)^2 = 16[(dx)^2 + (dx)(dy) + (dy)^2]$
Also $x^2 + y^2 = 1 \Rightarrow xdx + ydy = 0 \Rightarrow dy = -\frac{x}{y}dx \Rightarrow d^2L = 16[1 - \frac{x}{y} + \frac{x^2}{y^2}](dx)^2$

Since $d^2L > 0$ *at* $(\pm\frac{1}{\sqrt{5}}, \mp\frac{2}{\sqrt{5}}) \Rightarrow f(x,y)$ *has minimum value and* $f_{min} = -\sqrt{5}$.
For $\lambda = -9$, $8x - 16y = 0 \Rightarrow x = 2y$. *Therefore,* $x^2 + y^2 = 1 \Rightarrow x = \pm\frac{2}{\sqrt{5}}$ *and* $y = \pm\frac{1}{\sqrt{5}}$.
So $(\pm\frac{2}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}})$ *are stationary points.*
Now $dL = L_x dx + L_y dy$
$d^2L = L_{xx}(dx)^2 + 2L_{xy}(dx)(dy) + L_{yy}(dy)^2 = 16[(dx)^2 + (dx)(dy) + (dy)^2]$
Also $x^2 + y^2 = 1 \Rightarrow xdx + ydy = 0 \Rightarrow dy = -\frac{x}{y}dx \Rightarrow d^2L = -4[1 + \frac{4x}{y} + \frac{4x^2}{y^2}](dx)^2$
Since $d^2L < 0$ *at* $(\pm\frac{2}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}) \Rightarrow f(x,y)$ *has maximum value and* $f_{max} = 9$.

Example 1.25. *Show that* $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ *for* $x, y, z \geq 0$.
$\Rightarrow$ *Let* $f(x,y,z) = xyz$ *and* $x + y + z = s$. *The Lagrangian function is* $L(x,y,z) = xyz + \lambda(x + y + z - s)$. *Here* λ *is Lagrangian multiplier.*
Now for stationary points $L_x = 0 \Rightarrow yz + \lambda = 0$, $L_y = 0 \Rightarrow xz + \lambda = 0$ *and* $L_z = 0 \Rightarrow xy + \lambda = 0$. *So* $x = y = z = s/3$
So the stationary point is $(s/3, s/3, s/3)$.
Now $dL = L_x dx + L_y dy + L_z dz$
$d^2L = L_{xx}(dx)^2 + L_{yy}(dy)^2 + L_{zz}(dz)^2 + 2L_{xy}(dx)(dy) + 2L_{xz}(dx)(dz) + 2L_{yz}(dy)(dz) = \frac{2s}{3}[(dx)(dy) + (dx)(dz) + (dy)(dz)]$
Also $x+y+z = s \Rightarrow dx+dy+dz = 0 \Rightarrow (dx)(dy)+(dx)(dz)+(dy)(dz) = -\frac{(dx)^2+(dy)^2+(dz)^2}{2}dx \Rightarrow$
$d^2L = -\frac{s}{3}[(dx)^2 + (dy)^2 + (dz)^2]$
Since $d^2L < 0$ *at* $(s/3, s/3, s/3) \Rightarrow f(x,y,z)$ *has maximum value and* $f_{max} = (s/3)^3$.
So, $xyz \leq (s/3)^3 \Rightarrow \frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

[Do It Yourself] 1.52. *Show that* $f(x,y,z) = x^m y^n z^p$ *subject to* $x + y + z = a$ *has maximum value* $\frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$ *and stationary point* $(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p})$.

[Do It Yourself] 1.53. *Use Lagrange's Multiplier method, find the shortest distance between* $(-1,4)$ *and the straight line* $12x - 5y + 71 = 0$.
[$Hint:\ L = (x+1)^2 + (y-4)^2 + \lambda(12x - 5y + 71),\ SD = 3$]

[Do It Yourself] 1.54. *Find the minimum distance from the point* $(1,2,0)$ *to the cone* $z^2 = x^2 + y^2$.
[*Hint*: *Try to remove* z^2 *term*]