Chapter 1

Differential Calculus

1.1 Recap of Function of One Variable

▶ Various Intervals: Let $a, b \in \mathbb{R}$ with $a < b$. An open interval $\{x : x \in \mathbb{R}, a < x < b\}$ is a subset of $\mathbb{R}$ and denoted by (a, b). A closed interval $\{x : x \in \mathbb{R}, a \leq x \leq b\}$ is a subset of $\mathbb{R}$ and denoted by $[a, b]$. A semi-open or, semi-closed interval is $[a, b), (a, b]$. Some unbounded intervals are $(-\infty, a), (-\infty, a], (a, \infty), [a, \infty), (-\infty, \infty)$ etc.

▶ Neighbourhood: A neighbourhood of a point $\alpha \in \mathbb{R}$ is an open interval (a, b) such that $\alpha \in (a, b)$. It is denoted by $N(\alpha)$. A *δ − neighbourhood* of a point $\alpha \in \mathbb{R}$ is an open interval $(\alpha - \delta, \alpha + \delta)$ for $\delta > 0$ i.e. the set of all points $x \in \mathbb{R}$ such that $|x - \alpha| < \delta$. It is denoted by $N(\alpha, \delta)$.

▶ Interior Point: A point $x \in S$ is said to be an interior point of S if $\exists$ a $N(x)$ such that $N(x) \subset S$. It is denoted by S^o. For ex. $S = (2, 3] \Rightarrow$ only 3 is not an interior point rest are all interior points $\Rightarrow S^o = (2, 3)$.

▶ Deleted Neighbourhood: $N(\alpha) \setminus \{\alpha\}$ is called deleted neighbourhood of α and denoted by $N'(\alpha)$. Again $N(\alpha, \delta) \setminus \{\alpha\}$ is called *deleted δ − neighbourhood* of α and denoted by $N'(\alpha, \delta)$. For ex. $N'(2, 0.1) = (1.9, 2.1) \setminus \{2\} = (1.9, 2) \cup (2, 2.1)$.

▶ Limit Point/ Cluster Point: A point $x \in S$ is said to be a limit point (cluster point) of S if $N'(x, \delta) \cap S \neq \phi$. For ex. $S = (2, 3] \Rightarrow$ each point of S is a limit point of S, 2 is also a limit point of S. Again, $\mathbb{N}$ has no limit points.

▶ Isolated Point: A point $y \in S$ is said to be an isolated point of S if y is not a limit point of S. For ex. each point of $\mathbb{N}$ is an isolated point.

▶ Open Set: A set S is said to be an open set if each point of S is an interior point of S. For ex. $(1, 2), \mathbb{R}, \phi$, etc. Note that, S^o is the largest open set contained in S.

1.1.1 Limit, Continuity and Differentiability

■ A function $f(x)$ is said to tend to a limit l as x tends to c if for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\boxed{|f(x) - l| < \varepsilon, \text{ when } 0 < |x - c| < \delta}$. We write $\lim_{x \to c} f(x) = l$.
■ Then f is said to be continuous at c $(a < c < b)$ if $\lim_{x \to c} f(x) = f(c)$ i.e. for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\boxed{|f(x) - f(c)| < \varepsilon, \text{ when } 0 < |x - c| < \delta}$.
■ Let $f(x)$ be a function defined on an interval $I = [a, b]$. Let c be an interior point of I, the function f is said to be *differentiable at* c if $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists. If the limiting value is l, then l is said to be the derivative of f at c and is denoted by $f'(c)$.
■ We will study these in detail on the course '*Mathematical Analysis*'.

1.2 Function of Several Variables

▶ The neighborhood of (α, β) is a subset S of $\mathbb{R}^2$ such that $(\alpha, \beta) \in S$.
▶ Let $\delta > 0$ then $S = \{(x, y) : |x - \alpha| < \delta, |y - \beta| < \delta\} \subset \mathbb{R}^2$ is called a square $\delta - nbd$ of (α, β). It is denoted by $R_\delta(\alpha, \beta)$.
▶ Let $\delta > 0$ then $S = \{(x, y) : (x - \alpha)^2 + (y - \beta)^2 < \delta^2\} \subset \mathbb{R}^2$ is called a circular $\delta - nbd$ of (α, β). It is denoted by $B_\delta(\alpha, \beta)$.
▶ The deleted neighborhood of (x, y) is $N'_\delta(x, y) = N_\delta(x, y) \setminus \{(x, y)\}$.
▶ Let $S \subset \mathbb{R}^2$ and $(x, y) \in S$. Then (x, y) is called an interior point of S if $\exists\ \delta > 0$ such that $N_\delta(x, y) \subset S$.
▶ S is said to be an open set if every point of S is its interior point.
▶ A point $(x, y) \in \mathbb{R}^2$ is said to be a limit point of a set $S \subset \mathbb{R}^2$ if $\forall \delta > 0$, $N'_\delta(x, y) \cap S \neq \phi$.
▶ Let $S \subset \mathbb{R}^2$ is said to be bounded if $\exists$ a rectangle $R = [a, b] \times [c, d]$ such that $S \subset R$.

1.3 Limit

We already studied the limit in one variable. Now we will study some problems related to one variable.

[Do It Yourself] **1.1.** *Let* $L = \lim_{n \to \infty} n\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{k}{n}\right) - kf(0)\right]$, *where* k *is a positive integer. If* $f(x) = \sin x$, *then* L *is equal to*
(A) $\frac{(k+1)(k+2)}{6}$. *(B)* $\frac{(k+1)(k+2)}{2}$. *(C)* $\frac{k(k+1)}{2}$. *(D)* $k(k+1)$.

[Do It Yourself] **1.2.** $\lim_{n \to \infty} \dfrac{1 - 2 + 3 - 4 + \cdots + (-2n)}{\sqrt{n^2 + 1} + \sqrt{n^2 - 1}}$ *equals.*
(A) ∞. *(B)* $1/2$. *(C)* 0. *(D)* $-1/2$.

[Do It Yourself] 1.3. $\lim_{n\to\infty} \dfrac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(\pi^n+e^n)^{1/n}\ln n}$ *equals*
(A) $\frac{1}{\pi}$. *(B)* $\frac{1}{e}$. *(C)* $\frac{e}{\pi}$. *(D)* $\frac{\pi}{e}$.
$[Hint:\ Use\ Harmonic\ sum\ = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \approx \ln(n)+\frac{1}{2}+\frac{1}{2n}]$

1.3.1 Double Limit (Two Variable)

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ be a function and (a,b) be a limit point of D. Let $(x,y) \in D$ and $(x,y) \to (a,b)$ in any manner. The function $f(x,y)$ is said to tend to limit l if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x,y)-l| < \varepsilon$ for $(x,y) \in N_\delta(a,b)$ [Square or, Circular]. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ is called <u>double limit</u>.

■ $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ means for given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(x,y)-l| < \varepsilon$.
If $0 < |x-a| < \delta,\ 0 < |y-b| < \delta$ $\boxed{\text{or,}}$ $0 < (x-a)^2+(y-b)^2 < \delta^2$.
■ If double limit exists then<u>along any smooth curve or path</u> we can find the limit. This approach is helpful to find the limiting value when it is given that the double limit exists.

Example 1.1. *Show that* $\lim\limits_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2} = 0$.
$\Rightarrow$ *Let* $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$. *Put* $x = r\cos\theta,\ y = r\sin\theta$.
Therefore, $|f(x,y)-0| = |xy\frac{x^2-y^2}{x^2+y^2}| = |r^2\sin\theta\cos\theta\cos 2\theta| = |\frac{r^2}{4}\sin 4\theta| \le \frac{r^2}{4}$.
So $|f(x,y)-0| < \varepsilon$ *if* $\frac{r^2}{4} \le \varepsilon$ *i.e.* $(x-0)^2+(y-0)^2 < \delta^2\ (=4\varepsilon)$.
Therefore, by definition $\lim\limits_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2} = 0$.

Example 1.2. *Show that* $\lim\limits_{(x,y)\to(0,0)} \left[x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}\right] = 0$.
$\Rightarrow$ *Let* $f(x,y) = x\sin\frac{1}{y}+y\sin\frac{1}{x}$.
Therefore, $|f(x,y)-0| = |x\sin\frac{1}{y}+y\sin\frac{1}{x}| \le |x|+|y|$.
So $|f(x,y)-0| < \varepsilon$ *if* $|x|+|y| \le \varepsilon$ *i.e.* $|x-0| < \delta\ (=\frac{\varepsilon}{2}),\ |y-0| < \delta\ (=\frac{\varepsilon}{2})$.
Therefore, by definition $\lim\limits_{(x,y)\to(0,0)} \left[x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}\right] = 0$.

[Do It Yourself] 1.4. *Show that i)* $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$, *ii)* $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2} = 0$,
iii) $\lim\limits_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = 1$, *iv)* $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2} = 0$.
$\left[Hint:\ iii)\ |e^{-(x^2+y^2)}-1| \approx |x^2+y^2|,\ in\ nbd\ of\ (0,0),\ iv)\ |\frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}| \approx \frac{1}{2}|\frac{x^2y^2}{x^2+y^2}|\right]$

Example 1.4. *Show that* $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ *does not exist.*

$\Rightarrow$ *Let* $(x,y) \to (0,0)$ *along* $y = mx$.

Then, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$.

The limit is different for different values of m. *So the limit does not exist.*

[**Do It Yourself**] **1.5.** *Show that the limits are* *not exists*

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$, *ii)* $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x-y}$, *iii)* $\lim\limits_{(x,y)\to(0,0)} (x+y)\dfrac{y+(x+y)^2}{y-(x+y)^2}$.

$[Hint: \; x = my^3; \; y = x - mx^3; \; y = x^2 \Rightarrow -1 \; and \; y = 0 \Rightarrow 0]$

1.3.2 Polar Transformation (Two Variable)

■ $x = r\cos\theta, \; y = r\sin\theta$. It works if the denominator contains the term (x^2+y^2) or, some function of (x^2+y^2).

■ For $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ polar transformation gives $\lim\limits_{r\to 0} \dfrac{r^2\sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta \Rightarrow$ Limit does not exist.

■ For $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ polar transformation gives $\lim\limits_{r\to 0} \dfrac{r^2\sin\theta\cos\theta}{r} = 0 \Rightarrow$ Limit exist and equal to 0.

■ For $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x+y}$ polar transformation gives $\lim\limits_{r\to 0} \dfrac{r^2\sin\theta\cos\theta}{r(\sin\theta+\cos\theta)} = \lim\limits_{r\to 0} \dfrac{r\sin\theta\cos\theta}{\sin\theta+\cos\theta}$
$\Rightarrow$ Denominator can be zero $\Rightarrow$ Try not to use polar transformation.

1.3.3 Repeated Limit (Two Variable)

■ $\lim\limits_{x\to a}[\lim\limits_{y\to b} f(x,y)] = \alpha$. Here calculating '$\lim\limits_{y\to b} f(x,y)$' we will take x is fixed.

■ $\lim\limits_{y\to b}[\lim\limits_{x\to a} f(x,y)] = \beta$. Here calculating '$\lim\limits_{x\to a} f(x,y)$' we will take y is fixed.

■ $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l \Rightarrow \lim\limits_{x\to a}[\lim\limits_{y\to b} f(x,y)] = \lim\limits_{y\to b}[\lim\limits_{x\to a} f(x,y)] = l$.

■ If $\lim\limits_{x\to a}[\lim\limits_{y\to b} f(x,y)] \neq \lim\limits_{y\to b}[\lim\limits_{x\to a} f(x,y)] \Rightarrow$ Double limit does not exist.

[Do It Yourself] 1.6. *Show that* $\lim_{(x,y)\to(0,0)}\left[x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}\right]$ *exist but repeated limit does not exist.*
[*Hint* : $\lim_{t\to 0}\sin\dfrac{1}{t}$ *does not exist*].

[Do It Yourself] 1.7. *Let*

$$f(x,y)=\begin{cases}x\sin\frac{1}{y}+\frac{x^2-y^2}{x^2+y^2} & \text{if } y\neq 0\\ 0 & \text{if } y=0\end{cases}$$

, then show that $\lim_{y\to 0}\lim_{x\to 0}f(x,y)$ *exists but* $\lim_{x\to 0}\lim_{y\to 0}f(x,y)$ *and* $\lim_{(x,y)\to(0,0)}f(x,y)$ *does not exist.*

1.4 Continuity (Two Variables)

Let $D\subset\mathbb{R}^2$, $f:D\to\mathbb{R}$ be a function. Suppose $(a,b)\in D$, then we say that $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)}f(x,y)=f(a,b)$ i.e. for each $\varepsilon>0$, $\exists\delta>0$ such that $|f(x,y)-f(a,b)|<\varepsilon$ for $(x,y)\in N_\delta(a,b)$ [Square or, Circular].

■ $f(x,y)$ is continuous at $(a,b)\Rightarrow$ given $\varepsilon>0$, $\exists\ \delta>0$ such that $|f(x,y)-f(a,b)|<\varepsilon$. If $0<|x-a|<\delta,\ 0<|y-b|<\delta$ $\boxed{\text{or,}}$ $0<(x-a)^2+(y-b)^2<\delta^2$.

Example 1.5. *Show that*

$$f(x,y)=\begin{cases}x\sin\frac{1}{y}+y\sin\frac{1}{x} & \text{if } xy\neq 0\\ 0 & \text{if } xy=0\end{cases}$$

is continuous at $(0,0)$.
$\Rightarrow$ *Here* $|f(x,y)-f(0,0)|\leq|x|+|y|<\varepsilon$ *if* $|x-0|<\frac{\varepsilon}{2},\ |y-0|<\frac{\varepsilon}{2}$.
It implies, $|f(x,y)-f(0,0)|<\varepsilon$ *if* $|x-0|<\delta,\ |y-0|<\delta$, *where* $\delta=\frac{\varepsilon}{2}$.
Therefore, $\lim_{(x,y)\to(0,0)}f(x,y)=0=f(0,0)\Rightarrow f(x,y)$ *is continuous at* $(0,0)$.

Example 1.6. *Show that*

$$f(x,y) = \begin{cases} (px+qy)\sin\frac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

is continuous at $(0,0)$.
$\Rightarrow$ *Here* $|f(x,y) - f(0,0)| \leq |p||x| + |q||y| < \varepsilon$ *if* $|x-0| < \frac{\varepsilon}{2|p|}$, $|y-0| < \frac{\varepsilon}{2|q|}$.
It implies, $|f(x,y) - f(0,0)| < \varepsilon$ *if* $|x-0| < \delta$, $|y-0| < \delta$, *where* $\delta = \min\{\frac{\varepsilon}{2|p|}, \frac{\varepsilon}{2|q|}\}$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow f(x,y)$ *is continuous at* $(0,0)$.

[**Do It Yourself**] **1.8.** *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \dfrac{\sin 2(x^2+y^2)}{x^2+y^2} e^{3x\sin\frac{4}{y}} & \text{if } (x,y) \neq (0,0) \\ \alpha & \text{if } (x,y) = (0,0) \end{cases}$$

where α *is a real constants. If* f *is continuous at* $(0,0)$ *then the value of* α *is*
(A) 1. *(B)* 2. *(C)* 3. *(D)* 4.
$[Hint:$ *Since limit exist try to evaluate along* $y = x]$

[**Do It Yourself**] **1.10.** *Let*

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{x+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Check for the continuity of f *at* $(0,0)$.
$[Hint: \ y = mx^2 - x]$