

Class-15

$$GF(p^n)$$

$p \rightarrow$ prime Number

$$x \in GF(p^n)$$

$$x^{p^n-1} \equiv 0 \pmod{p}$$

Class-16

* Method of symmetrically repeated difference

→ Pure and Mixed differences :

Consider a module M containing n -elements. To each element of the module, let there correspond m -treatments the treatments corresponding to the element a being denoted by a_1, a_2, \dots, a_m . The treatment a_i is said to belong to the i th class. Thus we have mn treatments n belonging to each of the m -classes. With any ordered pair of distinct treatments a_i and b_j , we associate the difference $a-b$ of the type $[i, j]$. Each difference is an element of the module and is of a certain type. If $i=j$, the difference is said to be a pure difference. Obviously in this case $a \neq b$ as the treatments are distinct. If $i \neq j$, the difference is said to be a mixed difference.

Example

$M =$ residue class mod 5

$$M = \{0, 1, 2, 3, 4\}, n = 5$$

To each element a of M , let there corresponds two treatments, a_1 and a_2 ($m=2$)

Treatments	ordered pair	Difference	Difference type
$0_1, 0_2$	$(0_1, 1_1)$	4	[1, 1] Pure
$1_1, 1_2$	$(1_1, 0_1)$	1	
$2_1, 2_2$	$(0_1, 2_1)$	3	[1, 1] Pure
$3_1, 3_2$	$(2_1, 0_1)$	2	
$4_1, 4_2$	$(3_2, 1_2)$	2	[1, 2] Mixed
	$(1_2, 3_1)$	3	[2, 1] Mixed
	$(3_2, 4_1)$	4	[2, 1] mixed
	$(3_1, 4_2)$	4	[1, 2] Mixed

Now, suppose a block B containing k distinct treatments. From this block we can get $\{k \times (k-1)\}$ ordered pairs of treatments, giving rise to $\{k \times (k-1)\}$ differences. These differences are called differences arising out of the block B.

→ Refer to the previous example :

Suppose $B = (2_1, 4_2, 0_2)$

Differences arising out of the block B

ordered pair	Difference	Difference type
$(2_1, 4_2)$	3	[1, 2] Mixed
$(2_1, 0_2)$	2	[1, 2] Mixed
$(4_2, 2_1)$	2	[2, 1] Mixed
$(4_2, 0_2)$	4	[2, 2] Pure
$(0_2, 2_1)$	3	[2, 1] Mix.
$(0_2, 4_2)$	1	[2, 2] Pure

→ Since there are m -classes and the no. of elements in the module is n , there are $m-1$ $(n-1)$ pure differences of the type $[i, i]$, for each $i = 1, 2, \dots, m$.

Total $m(n-1) \rightarrow$ pure differences. Similarly there are n mixed differences of each type $[i, j]$ for $i, j = 1(1)m$
 $i \neq j$

~~m-1~~ Total $n m (m-1) \rightarrow$ Mixed differences.

→ Now, consider a set of t -blocks, B_1, B_2, \dots, B_t .

If among the differences arising from these t -blocks, each possible difference occurs a constant no. of times (say, λ), the differences are said to be symmetrically repeated.

Example

$$M = \{0, 1, 2, 3, 4\}$$

$$m = 3, n = 5, t = 7, \lambda = 1, K = 3$$

To each $a \in M$, there corresponds 3 treatments a_1, a_2, a_3

consider 7 blocks $B_1 = (0_1, 1_1, 0_2)$; $B_2 = (0_2, 1_2, 2_3)$;

$B_3 = (0_3, 1_3, 2_1)$; $B_4 = (0_1, 2_1, 3_2)$; $B_5 = (0_2, 2_2, 0_3)$;

$B_6 = (0_3, 2_3, 0_1)$; $B_7 = (0_1, 2_2, 1_3)$

Blocks	$[1, 1]$	$[2, 2]$	$[3, 3]$	$[1, 2]$	$[1, 3]$	$[2, 3]$	$[2, 1]$	$[3, 1]$	$[3, 2]$
$(0_1, 1_1, 0_2)$	4, 1	—	—	0, 1	—	—	0, 4	—	—
$(0_2, 1_2, 2_3)$	—	4, 1	—	—	—	3, 4	—	—	1, 2
$(0_3, 1_3, 2_1)$	—	—	4, 1	—	2, 1	—	—	3, 4	—
$(0_1, 2_1, 3_2)$	3, 2	—	—	2, 4	—	—	1, 3	—	—
$(0_2, 2_2, 0_3)$	—	3, 2	—	—	—	0, 2	—	—	0, 3
$(0_3, 2_3, 0_1)$	—	—	3, 2	—	0, 3	—	—	0, 2	—
$(0_1, 2_2, 1_3)$	—	—	—	3	4	1	2	1	4

From the table we see that among the non-zero differences arising out of the t blocks, each difference is repeated symmetrically. It must be noted that 0 cannot occur as a pure difference since the block contents are distinct.

→ First Fundamental Theorem of method of symmetrically repeated difference:

Let, M be a module containing n -elements, $x^{(0)}, x^{(1)}, \dots, x^{(n-1)}$ and to each element $x^{(u)}$, let there correspond m treatments $x_1^{(u)}, x_2^{(u)}, \dots, x_m^{(u)}$ ($u=0, 1, 2, \dots, n-1$). Thus there are mn treatments.

Let, it is possible to find a set of t blocks, B_1, B_2, \dots, B_t , satisfying the following conditions,

- (i) Each block contains k distinct treatments.
- (ii) Among the kt treatments occurring in the t -blocks, exactly r -treatments belong to each of the classes.
- (iii) The differences arising from the t -blocks are symmetrically repeated, λ times each.

Then the nt blocks obtained by developing the initial blocks B_1, B_2, \dots, B_t provide us with a solution of a BIBD with parameters $v=mn, b=nt, r, k, \lambda$.

→ Given any block B_s containing k -distinct treatments one can obtain n blocks $B_{s,\theta}$ where θ ranges over the elements of M as follows:
corresponding to any treatment $x_i^{(u)}$ of the i th class in B_s , we take the treatment $x_i^{(v)} = x_i^{(u)} + \theta$ of the i th class in $B_{s,\theta}$. The n -blocks $B_{s,\theta}$ are

said to be obtained by developing the block B_s ...

Example: Refer to previous example

$$n = 5$$

$$m = 3$$

$$t = 7$$

$$\lambda = 1$$

$$K = 3$$

$$v = 15$$

$$b = 35$$

$$K = 3$$

$$r = 7$$

$$\lambda = 1$$

$(0_1, 1_1, 0_2)$	$(0_2, 1_2, 2_3)$	$(0_3, 1_3, 2_1)$
$(1_1, 2_1, 1_2)$	$(1_2, 2_2, 3_3)$	$(1_3, 2_3, 3_1)$
$(2_1, 3_1, 2_2)$	$(2_2, 3_2, 4_3)$	$(2_3, 3_3, 4_1)$
$(3_1, 4_1, 3_2)$	$(3_2, 4_2, 0_3)$	$(3_3, 4_3, 0_1)$
$(4_1, 0_1, 4_2)$	$(4_2, 0_2, 1_3)$	$(4_3, 0_3, 1_1)$
$(0_1, 2_1, 3_2)$	$(0_2, 2_2, 0_3)$	$(0_3, 2_3, 0_1)$
$(1_1, 3_1, 4_2)$	$(1_2, 3_2, 1_3)$	$(1_3, 3_3, 1_1)$
$(2_1, 4_1, 0_2)$	$(2_2, 4_2, 2_3)$	$(2_3, 4_3, 2_1)$
$(3_1, 0_1, 1_2)$	$(3_2, 0_2, 3_3)$	$(3_3, 0_3, 3_1)$
$(4_1, 1_1, 2_2)$	$(4_2, 1_2, 4_3)$	$(4_3, 1_3, 4_1)$
$(0_1, 2_2, 1_3)$		
$(1_1, 3_2, 2_3)$		
$(2_1, 4_2, 3_3)$		
$(3_1, 0_2, 4_3)$		
$(4_1, 1_2, 0_3)$		

Example-2

$$M = \{0, 1, 2, 3, 4, 5, 6\}$$

$$n = 7$$

$$m = 1$$

$$v = b = 7$$

$$r = k = 4$$

$$\lambda = 2$$

$t = 1$ (no. of initial block)

Initial block : (3, 5, 6, 7)

4, 6, 7, 1

5, 7, 1, 2

6, 1, 2, 3

7, 2, 3, 4

1, 3, 4, 5

2, 4, 5, 6

Class-17

→ Second fundamental theorem of the method of symmetric difference :

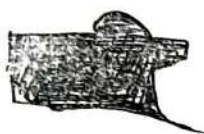
Let, M be a module containing n -elements $x^{(0)}, x^{(1)}, \dots, x^{(n-1)}$, and to each element $x^{(u)}$, let there correspond m -treatments $x_1^{(u)}, x_2^{(u)}, \dots, x_m^{(u)}$ [$u = 0(1)n-1$]. To these mn treatments we adjoin a new treatment α (called the invariant treatment) so that we have $v = mn + 1$ treatments. Consider these $(mn + 1)$ treatments. Let, it be possible to find a set of $(t + u)$ block $B_1, B_2, \dots, B_t, B'_1, B'_2, \dots, B'_u$ satisfying the following conditions :

- (i) Each of the blocks B_1, B_2, \dots, B_t contains k -distinct treatments $x_i^{(u)}$, whereas each of the blocks B'_1, B'_2, \dots, B'_u contains α and $(k-1)$ distinct treatments $x_i^{(u)}$
- (ii) Among the kt treatments occurring in the blocks B_1, B_2, \dots, B_t , exactly $nu - \lambda$ belong to each of the m -classes, whereas among the $u(k-1)$ treatments occurring in the blocks B'_1, B'_2, \dots, B'_u exactly λ belong to

each of the m -classes.

(iii) The differences arising from the $(t+u)$ blocks $B_1, B_2, \dots, B_t, B_1'', B_2'', \dots, B_u''$ are symmetrically repeated λ times, where the blocks B_j'' are obtained from B_j' by deleting the treatment d , for $j = 1(1)u$.

Then the $n(t+u)$ blocks, obtained by developing the initial blocks (B_1, B_2, \dots, B_t) and B_1', B_2', \dots, B_u' provide a solution of a BIBD with parameters $v = mn + 1$, $b = n(t+u)$, $r = nu$, k, λ .



→ Steiner's Triplet :- BIBD with $k=3, \lambda=1$.

A Steiner's triplet is an arrangement of v objects in triplet such that every pair of objects appears exactly once in each triplet. Obviously if we treat our objects as treatments, Steiner's triplet is a BIBD with $k=3$ and $\lambda=1$. From the parametric relations of BIBD, we have : $bk = vr$, $r(k-1) = \lambda(v-1)$

$$\therefore 3b = vr$$

$$2r = v-1 \Rightarrow \boxed{v = 2r+1}$$

$$\Rightarrow 3b = r(2r+1)$$

$$\Rightarrow b = \frac{r(2r+1)}{3}$$

$\therefore r$ must be of the form $3t+1, 3t$

Accordingly we get following two series of Steiner's Triplet :

$$S_{11} :- v = 6t+3$$

$$b = (3t+1)(2t+1)$$

$$r = 3t+1$$

$$k = 3$$

$$\lambda = 1$$

$$S_{12} :- v = 6t+1$$

$$b = t(6t+1)$$

$$r = 3t$$

$$k = 3$$

$$\lambda = 1$$

Solution of S_{11}

Let, M be the module of residue classes mod $(2t+1)$ and let its elements be $0, 1, 2, \dots, 2t$. To every element $u \in M$, let there correspond 3 treatment u_1, u_2, u_3 so that there are $(6t+3)$ treatments.

Consider the pairs:

$$[1, 2t]$$

$$[2, 2t-1]$$

$$[i, 2t-i+1]$$

$$[t, t+1]$$

The differences arising out from the i th pair are $(2t+1-2i)$ and $2i$. Taking $i=1(1)t$, we get all the non-zero elements of M , as the differences from the pair.

Now consider the following ^{sets} blocks of initial blocks:

$$[1_1, (2t)_1, 0_2]; [2_1, (2t-1)_1, 0_2]; \dots [t_1, (t+1)_1, 0_2]$$

$$[1_2, (2t)_2, 0_3]; [2_2, (2t-1)_2, 0_3]; \dots [t_2, (t+1)_2, 0_3]$$

$$[1_3, (2t)_3, 0_1]; [2_3, (2t-1)_3, 0_1]; \dots [t_3, (t+1)_3, 0_1]$$

$$[0_1, 0_2, 0_3]$$

————— (*)

It is clear that the pure differences of the types $[1, 1]$, $[2, 2]$ and $[3, 3]$ arising from the initial blocks are repeated just once. Again among the pairs, every non-zero element of M occurs once. Hence among the mixed differences of the type arising from the first row, every non-zero element of M occurs just once. The mixed difference 0 of the type $[1, 2]$ arises from the last initial block

No mixed difference of the type $[1, 2]$ can occur from 2nd and 3rd row. Hence every elements of M occurs exactly once among the mixed differences of type $[1, 2]$ arising from the initial blocks. Similarly we can proof the same thing for the mixed difference of other types.

Theorem: The initial blocks \otimes provide a solution of the series S_{11} of BIBD with parameters $v = 6t + 3$, $b = (3t+1)(2t+1)$, $r = 3t+1$, $k = 3$, $\lambda = 1$.

→ Example-1 $\gg t = 1$, $[v = 9, b = 12, r = 4, k = 3, \lambda = 1]$

Example-2 $\gg t = 2$, $[v = 15, b = 35, r = 7, k = 3, \lambda = 1]$

↓

Initial blocks

$$(1_1, 4_1, 0_2), (2_1, 3_1, 0_2)$$

$$(1_2, 4_2, 0_3), (2_2, 3_2, 0_3)$$

$$(1_3, 4_3, 0_1), (2_3, 3_3, 0_1)$$

$$(0_1, 0_2, 0_3)$$

Solution of S_{12}

$$v = 6t + 1$$

Assume $6t + 1 = p^n$ where p is a prime and $n \geq 1$.

$$\text{Now, } x^{p^n - 1} \equiv 1 \pmod{p}$$

$$\text{or, } x^{6t} - 1 \equiv 0 \pmod{p}.$$

$$\text{or, } (x^{3t} - 1)(x^{3t} + 1) \equiv 0 \pmod{p}$$

since x is the primitive element, $x^{3t} - 1 \neq 0$

$$\therefore x^{3t} + 1 \equiv 0 \pmod{p}$$

$$\text{or, } (x^t + 1)(x^{2t} - x^t + 1) = 0$$

Now, $x^{t+1} \neq 0$

since, if $x^t = -1$

$$\text{or } x^{2t} = 1$$

$$\text{or } x^{2t} - 1 = 0 \pmod{p}$$

$\Rightarrow \Leftarrow$ as x is primitive element.

$$\therefore x^{2t} - x^{t+1} = 0$$

$$\Rightarrow x^{2t} = x^{t+1}$$

Consider the set of t -initial blocks

$$(x^i, x^{2t+i}, x^{4t+i}), \quad i = 0 \leq i \leq t-1$$

Differences arising out of the initial blocks

$$\pm x^i (x^{2t} - 1); \quad x^{2t+i} (x^{2t} - 1); \quad \pm x^i (x^{4t} - 1)$$

$$\text{Let, } x^{2t} - 1 = x^q$$

$$\therefore x^i (x^{2t} - 1) = x^{i+q}$$

$$x^i (x^{4t} - 1) = x^i (x^{2t} - 1)(x^{2t} + 1)$$

$$= x^i \cdot x^q \cdot x^t \quad [\because x^{2t} = x^{t+1}]$$

$$= x^{i+q+t}$$

$$x^{2t+i} (x^{2t} - 1) = x^{q+2t+i}$$

$$- x^i (x^{2t} - 1) = -x^{q+i} = x^{q+3t+i} \quad \left[\begin{array}{l} x^{3t} + 1 = 0 \\ x^{3t} = -1 \end{array} \right]$$

$$- x^i (x^{4t} - 1) = -x^i \cdot x^{q+t} = x^{q+i+4t}$$

$$- x^{2t+i} (x^{2t} - 1) = -x^{2t+i+q} = x^{5t+i+q}$$

Remembering that $x^{6t} = 1$ we see that among the differences arising out of the initial block every non-zero element of $GF(6t+1)$ repeated exactly once.

Theorem

The initial blocks $(x^i, x^{2t+i}, x^{4t+i})$, $i = 0(1)\overline{t-1}$

[x being a primitive element of $GF(6t+1)$]

provide a solution of the series S_{12} of BIBD with parameters $v = 6t+1$, $b = t(6t+1)$, $r = 3t$, $k = 3$, $\lambda = 1$ provided v is a prime or prime power.

→ Example

$$t = 2$$

$$v = 13, b = 26, r = 6, k = 3, \lambda = 1.$$

Primitive element of $GF(13)$ is 2

$$i = 0, 1$$

$$\text{Initial Blocks: } (2^0, 2^4, 2^8) \equiv (1, 3, 9)$$

$$(2^1, 2^5, 2^9) \equiv (2, 6, 5)$$

Class-18

→ BIBD with $k=4, \lambda=1$

$$bk = vr$$

$$r(k-1) = \lambda(v-1)$$

$$\therefore 4b = vr$$

$$3r = v-1$$

$$b = \frac{r(3r+1)}{4}$$

$$\Rightarrow r = 4t \text{ or } r = 4t+1$$

$$S_{31} :- v = 12t+1, b = t(12t+1), r = 4t, k = 4, \lambda = 1$$

$$S_{32} :- v = 12t+4, b = (3t+1)(4t+1), r = 4t+1, k = 4, \lambda = 1$$

$$S_{31} :- 12t+1 = p^n \text{ (p is a prime number and } n > 1)$$

$$x^{12t} - 1 = 0 \pmod{p} \Rightarrow x^{6t} + 1 = 0 \left[\because x^{6t} - 1 \neq 0 \text{ as } x \text{ is the primitive element} \right]$$

$$\text{Initial blocks :- } (0, x^{2i}, x^{4t+2i}, x^{8t+2i}) \quad i = 0(1)\overline{t-1}$$

$$\text{Let, } x^{4t} - 1 = x^q$$

e.g : $t=2$

$v=25, b=50, r=8, k=4, \lambda=1$

$v=25=5^2$

Minimal polynomial of $GF(5^2)$ is x^2+2x+3

Initial blocks $(0, x^0, x^8, x^{16})$

$(0, x^2, x^{10}, x^{18})$

$x^2 = -2x-3 = 3x+2 \pmod{5} \rightarrow [ax+b, a, b \in GF(5)]$

$x^3 = 3x^2+2x = 3(-2x-3)+2x = -6x-9+2x = -4x-9 = x+1 \pmod{5}$

$x^4 = x^2+x = -2x-3+x = -x-3 = 4x+2 \pmod{5}$

$x^8 = 16x^2+16x+4 = x^2+x+4 \pmod{5} = -2x-3+x+4 = -x+1 = 4x+1 \pmod{5}$

$x^{10} = (4x+1)(3x+2) = 12x^2+11x+2 = 2x^2+x+2 = 6x+4+x+2 = 7x+6 = 2x+1 \pmod{5}$

$x^{16} = (4x+1)(4x+1) = 16x^2+8x+1 = 48x+32+8x+1 = 56x+33 = x+3 \pmod{5}$

$x^{18} = (x+3)x^2 = (x+3)(3x+2) = 3x^2+2x+9x+6 = 3x^2+11x+6 = 9x+6+11x+6 = 20x+12 = 2 \pmod{5}$

Initial block : $0, 3x+2$

Theorem

A solution of the series S_{32} with parameters

$$v = 12t + 4, b = (3t + 1)(4t + 1)$$

$r = (4t + 1), k = 4, \lambda = 1$ is provided by the initial blocks

$$(x_1^{2i}, x_1^{2t+2i}, x_2^{\beta+2i}, x_2^{\beta+2t+2i}); i = 0(1)\overline{t-1}$$

$$(x_2^{2i}, x_2^{2t+2i}, x_3^{\beta+2i}, x_3^{\beta+2t+2i}), i = 0(1)\overline{t-1}$$

$$(x_3^{2i}, x_3^{2t+2i}, x_1^{\beta+2i}, x_1^{\beta+2t+2i}), i = 0(1)\overline{t-1}$$

$$(d, 0, 0_2, 0_3)$$

Provided that,

- ① $(4t + 1)$ is a prime or prime power.
- ② To every element of $GF(4t + 1)$ there correspond 3 treatments and we adjoin the invariant treatment d to these treatments.
- ③ x be a primitive element of $GF(4t + 1)$.
- ④ It is possible to find an odd integer β such that:
$$\frac{x^{\beta} + 1}{x^{\beta} - 1} = x^q$$
, where q is an odd integer.

e.g : $t = 1$

$$v = 16, b = 20, r = 5, k = 4, \lambda = 1$$

$$4t + 1 = 5 \rightarrow \text{Prime or prime power}$$

$$GF(5) \rightarrow \text{Primitive element } 2$$

$$\beta = 1 \Rightarrow \frac{2^1 + 1}{2^1 - 1} = 3 \equiv 8 \pmod{5} \\ = 2^3 \pmod{5}.$$

$$\exists q = 3$$

$$\beta = 1$$

Initial blocks:

$$(2_1^0, 2_1^{2+0}, 2_2^{1+0}, 2_2^{1+2+0})$$

$$(2_2^0, 2_2^{2+0}, 2_3^{1+0}, 2_3^{1+2+0})$$

$$(2_3^0, 2_3^{2+0}, 2_1^{1+0}, 2_1^{1+2+0})$$

$$(\alpha, 0_1, 0_2, 0_3)$$

⇒ Initial blocks: $(1_1, 4_1, 2_2, 3_2)$

$$(1_2, 4_2, 2_3, 3_3)$$

$$(1_3, 4_3, 2_1, 3_1)$$

$$(\alpha, 0_1, 0_2, 0_3)$$

→ BIBD with parameters $b=v=4t-1, r=k=2t-1, \lambda=t-1$

Case-I >> when $4t-1 = p^n$ (Prime or Prime power)

Then the initial block is $(x^0, x^2, x^4, \dots, x^{4t-4})$

where x is a primitive element of $GF(p^n)$

Case-II >> when $4t-1$ is not a prime or prime power

Defn: A square matrix H of order n with entries -1 and $+1$ is said to be Hadamard matrix if

$$HH' = nI. \text{ It can be seen that } H'H = HH' = nI.$$

Also, it follows that if any row or column of a Hadamard matrix is multiplied by (-1) , the matrix remains a Hadamard matrix. A necessary condition for the existence of a Hadamard matrix of order n is that $n \equiv 0 \pmod{4}$, $n=2$ is the trivial case. since, A Hadamard matrix remains Hadamard when any of its rows or columns is multiplied by (-1) , it is always possible to write a Hadama

matrix with its first row and first column with +1 only. This form is called the normal form.

Let, H be a Hadamard matrix of order $n = 4t$ in its normal form and B be a matrix obtained from H by deleting its first row and first column. obviously B is square matrix of order $(4t-1)$.

$$\text{Let, } N = \frac{(B + \underline{1}\underline{1}')}{2}$$

$$\text{where } \underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{(4t-1) \times 1}$$

Then it can be easily proved that N is the incidence matrix of a BIBD with the above mentioned parameters. Conversely if N is the incidence matrix of a BIBD with above parameters, then changing the zero's by -1 in N and bordering the resulting matrix by a row and a column of all +1 we get the Hadamard matrix of order $4t$.

Result

A Hadamard matrix of order $4t$ coexists with a BIBD with parameters $b = v = 4t-1, r = k = 2t-1, \lambda = t-1$.

Illustration $t = 4$

$v = 15$ is not a prime or prime power so, method of difference will not work. we construct H_{16} .

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

; $\otimes \rightarrow$ Kronecker Product

$$H_4 = H_2 \otimes H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

BIBD

2	4	6	8	10	12	14
1	4	5	8	9	12	13
3	4	7	8	11	12	15
1	2	3	8	9	10	11
2	5	7	8	10	13	15
1	6	7	8	9	14	15
3	5	6	8	11	13	14
1	2	3	4	5	6	7
2	4	6	9	11	13	15
1	4	5	10	11	14	15
3	4	7	9	10	13	14
1	2	3	12	13	14	15
2	5	7	9	11	12	14
1	6	7	10	11	12	13
3	5	6	9	10	12	15

Class-19

* 3^n Factorial Experiment :-

Start with 3^2

Two factors A and B, each at 3 levels 0 (Low), 1 (intermediate), 2 (High)

Total $3^2 = 9$ treatment combinations

$a_0 b_0$	$a_0 b_1$	$a_0 b_2$
$a_1 b_0$	$a_1 b_1$	$a_1 b_2$
$a_2 b_0$	$a_2 b_1$	$a_2 b_2$

or

00	01	02
10	11	12
20	21	22

or

(1)	(b)	(b^2)
(a)	(ab)	(ab^2)
(a^2)	($a^2 b$)	($a^2 b^2$)