[1) Outline vol. 1
2) Rohatgi
3) Lehman]

||Testing of Hypothesis||

Hypothesis: Statement

Statistical Hypothesis: Statement or assertion about population parameter.

Test: Rule based on sample obsⁿs.
[By that rule the statement (hypothesis) is verified.]

Test of Hypothesis: To verify the information of popⁿ parameter by a specific rule based on sample observations.

Types of Hypothesis:- [Let us consider the parametric space Θ. The information which is already given about the population parameter is known as null hypothesis information.]
[The part of parametric space which hold that information is called null hypothesis, Θ_0]
[On the contrary, the complement of Θ_0 is called ~~alternative~~ alternative hypothesis and is denoted by Θ_1]

Naturally, $\boxed{\Theta \equiv \Theta_0 \cup \Theta_1}$ $\boxed{\Theta \in \mathbb{R}}$

$\Rightarrow$ [Θ_0 and Θ_1 are mutually exclusive and exhaustive.]

[A specific statement belonging to Θ_0 is denoted by H_0 whereas a specific statement $\in \Theta_1$ is denoted by H_1]

How to decide on correct statement:
By the policy of statistical inference, the decision will be based on the sample. Let us assume that $\mathcal{X}$ is the sample space (space which generates the prob. structure of $\underline{x} = (x_1, x_2 \dots x_n)$ $\underline{x} \in \mathcal{X}$) On the basis of $\underline{x}$, Θ_0 can be considered

as true (accept) or Θ_0 can not be (reject). The $\underset{\sim}{x}$'s which lead to reject Θ_0 is considered to form a subset $\omega \in \mathcal{X}$, while the $\underset{\sim}{x}$'s which lead to accept Θ_0 is complementary set of W (let us name it as W^c)

$$\therefore \boxed{\mathcal{X} = W \cup W^c}$$

$\mathcal{X}$ \ Θ	Θ_0 true	Θ_0 false [Θ_1 true]
Reject Θ_0	mismatch	match
Accept Θ_0	match	mismatch

P(Type I) = Prob. of rejecting a true Θ_0 $= \boxed{P(\underset{\sim}{x} \in W / \Theta_0)}$

P(Type II) = Prob. of accepting a false Θ_0 $= \boxed{P(\underset{\sim}{x} \in W^c / \Theta_1)}$ $\boxed{= \beta}$

- Prob. type I error is called size of a test.
- Upper limit of prob. of type I error is called level of significance (α)

  - For cont. distribution, size $= \alpha$
  - For discrete " , size $\leq \alpha$

Our objective is to minimize both of the errors. But minimization of two errors is not possible simultaneously.

As if α is minimized β will get high.

Thus the rule of testing of hypothesis is to fix α, try to minimize β.

Generally α is fixed at 5%, 1% so on.

We try to maximize $\boxed{1-\beta}$ (power) which leads to the search of most powerful test.

Prob

i) Let, $X \sim$ Poisson(θ)

Let, X and Y be two iid obs^n from $P(\theta)$.

We want to test, $H_0 : \theta = 1$

$H_1 : \theta = 2$

Reject H_0 if $X = 0$ or ($X = 1$ and $X + Y \leq 2$). Find out prob. (type I error)

$\Rightarrow \quad P(X \in W / H_0)$

$= P(X=0 \cup (X=1 \cap X+Y \leq 2)\} / H_0)$

$= P(X=0/\theta=1) + P(X=1 \cap Y=0/\theta=1) + P(X=1 \cap Y=1/\theta=1)$

$= e^{-1} + e^{-2} + e^{-2} = \boxed{e^{-1} + 2e^{-2}}$

2) Let $X_1, X_2, \ldots, X_5$ be r.s from $Bin(1,\theta)$ for testing, $H_0: \theta \leq 0.5$ against $H_1: \theta > 0.5$ consider two tests T_1 and T_2

T_1: Reject H_0 iff $\Sigma X_i = 5$

T_2: Reject H_0 iff $\Sigma X_i \geq 3$

What $\beta_1 + \beta_2 = ?$

$\Rightarrow$

$\Sigma X_i \sim Bin(5, \theta)$

$\beta_1 = P(\Sigma X_i \neq 5 / \theta > 0.5) + P(\Sigma X_i < 3 / \theta > 0.5)$

$= P(\Sigma X_i \neq 5 / \theta = 0.5) + P(\Sigma X_i < 3 / \theta = 0.5)$

3) $X_1, X_2 \sim U(0, \theta)$

$H_0: \theta \in (0, 1] \cup [2, \infty)$

$H_1: \theta \in (1, 2)$

$W = \{(x_1, x_2) \in \mathbb{R} : 5/4 < \max(x_1, x_2) < 7/4\}$

The size of the critical region?

$\Rightarrow$

$f(x_{(n)}) = n\left(\frac{x_{(n)}}{\theta}\right)^{n-1} \frac{1}{\theta}$

$\alpha_1 = Pr\left(5/4 < X_{(n)} < 7/4 / \theta = 1\right)$

$\alpha_2 = Pr\left(5/4 < X_{(n)} < 7/4 / \theta = 2\right)$

$\boxed{\text{Size} = \text{Supremum}(\alpha_1, \alpha_2)}$

What is the problem of critical region concept?

Critical region concept fails if the structures of Θ_0 and Θ_1 are unions (intersection) of separate sets or Θ_0 is a nested family of spaces. Not only that; sometime the optimization technique on finding test with maximum power does not lead to a convex set of feasible solutions.

[To handle these odd situations, instead of critical region concept, we invite the concept of test fn. (critical)]

Critical (Test) function:

A critical fn (test fn) $\phi(\underset{\sim}{x})$ is a function originated from a two point set. $A = \{a_0, a_1\}$ where a_i stands for rejection of Θ_i. $i = 0, 1$.

$$\boxed{\phi(\underset{\sim}{x}) = P\{a_i / \underset{\sim}{x} \in \mathcal{X}\}}$$

Clearly, $0 \leq \phi(\underset{\sim}{x}) \leq 1$. On the other hand, ϕ is a mapping from $W \xrightarrow{\phi} [0,1]$

$\therefore$ [A test is a triplet containing $(\phi, \Theta_0, \Theta_1)$]

Remark

The function $\beta_\theta(\phi) = E_\theta(\phi(\underset{\sim}{x}))$ is called power function of ϕ.

Under, $\theta \in \Theta_0$ $\underset{\theta \in \Theta_0}{\beta(\phi)}$ is called size of the test.

$\underset{\theta \in \Theta_0}{\operatorname{Sup}} \beta(\phi)$ is called level of significance (α)

Remark

A test which is completely specify by the critical (test) fn ϕ where $0 < \phi < 1$ $\forall \underset{\sim}{x}$ is called randomized test. In other words, for a randomized test, prob. of rejecting Θ_0 is a func.

On the contrary, if ϕ behaves as an indicator fn, the test is known as non-randomized test.

$$\left[\phi(\underset{\sim}{x}) = \begin{cases} 1 & \underset{\sim}{x} \in W \\ 0 & x \in W^c \end{cases} \right]$$

Example 1

Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, 1)$ where μ is unknown but $\mu \in \Theta = \{\mu_0, \mu_1\}$

Let $H_0: X_i \sim N(\mu_0, 1)_{\mu=\mu_0} \Theta_0 = \{\mu_0\}$

$H_1: X_i \sim N(\mu_1, 1)_{\mu=\mu_1} \Theta_1 = \{\mu_1\}$

We would accept if the sample mean $\bar{x}$ is close to μ_0 than μ_1, i.e. We reject if $\bar{x} > k$. This constant k is determined from level of significance α.

$$P_{\mu_0}(\bar{x} > k) = P\left\{\frac{\bar{x}-\mu_0}{1/\sqrt{n}} > \frac{k-\mu_0}{1/\sqrt{n}}\right\} = \alpha$$

$$\Rightarrow k = \mu_0 + \frac{\tau_\alpha}{\sqrt{n}}, \quad \tau \sim N(0,1)$$

Reject H_0 if, $\tau > \mu_0 + \tau_\alpha/\sqrt{n}$

The non-randomized test

$$\phi(x) = \begin{cases} 1 & \text{if } \bar{x}-\mu_0/1/\sqrt{n} > \mu_0 + \tau_\alpha/\sqrt{n} \\ 0 & \text{o.w} \end{cases}$$

Remark

Is it necessary always to form randomized test for a discrete random variable?

$\boxed{\text{NO}}$ It depends on α. $\begin{Bmatrix} \text{non randomized} \\ \text{randomized} \end{Bmatrix}$

Example

An urn contains 10 marbles, of which m are white and $(10-m)$ are black. To test $H_0: m=5$ against $H_1: m=6$ one draws 3 marbles without replacement. The null hypothesis is rejected if a sample contains 2 or 3 white marbles. Construct a test and its power.

$\underline{\text{Sol}^n}$

X : No. of white marbles,

$$P(X=x) = \frac{\binom{m}{x}\binom{10-m}{3-x}}{\binom{10}{3}}, \quad x = 0,1,2,3$$

$$\phi(x) = \begin{cases} 1 & \text{if } x=2 \text{ or } x=3 \\ 0 & \text{o.w} \end{cases}$$

$$\text{size} = \alpha = E_{H_0}(\phi(x)) = \boxed{\sum_{x=0}^{3} x \binom{5}{x}\binom{10-5}{}}$$

$$= P_{H_0}(X=2) + P_{H_0}(X=3)$$

$$= \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} + \frac{\binom{5}{3}\binom{5}{0}}{\binom{10}{3}}$$

$$= 0.5$$

If α is fixed at 0.5 before the experiment then ϕ will be a non-randomized test.

If $\alpha = 0.6$,
we need to look for randomized pts. where prob. of rejecting null hypothesis is a random event (to adjust the excess prob. of $0.6 - 0.5 = 0.1$)

$$0.6 = P_{H_0}[X=2] + P_{H_0}[X=3] + \gamma . P_{H_0}[X=0]$$

γ = prob. of rejecting H_0 when $X=0$

$$\Rightarrow 0.6 = 0.5 + \gamma \times 0.8$$

$$\Rightarrow \gamma = 1/8 = 0.12$$

$$\phi(x) = \begin{cases} 1 & \text{for } X = 2,3 \\ 0.12 & \text{for } X = 0 \longrightarrow \text{randomized pt.} \\ 0 & \text{for } X = 1 \end{cases}$$

$$E_{H_1}(\phi(x)) = P_{H_1}(X=2) + P_{H_1}(X=3) + 0.12\, P_{H_1}(X=0)$$

$$= \frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} + \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} + 0.12 \times \frac{\binom{6}{0}\binom{4}{3}}{\binom{10}{3}}$$

$$= 0.61$$

$$\phi(x) = \begin{cases} 1 & \text{for } x = 2, 3 \\ \gamma_1 & x = 1 \\ 0 & x = 0 \end{cases}$$

$$0.6 = E_{H_0}(\phi(x)) = P_{H_0}[X=2] + P_{H_0}[X=3] + \gamma_1 \cdot P_{H_0}[X=1]$$

$$= 0.5 + \gamma_1 \frac{\binom{5}{1}\binom{5}{2}}{\binom{10}{3}} \Rightarrow 0.1 = \gamma_1 \times 0.42$$

$$E_{H_1}(\phi(x)) = 0.34$$

2nd checking point is the comparison of power. Our objective is to $\downarrow P$(type-II error) i.e. Maximize the power.

- Remember:
  1) $\gamma \not> 1$, $\gamma \not< 0$
  2) Checking of maximum power as it ensures minimum of β.

Properties of ϕ

1) ϕ is a Borel measurable mapping of W on [0,1]
2) Some simple examples of ϕ.
   i) $\phi(\underset{\sim}{x}) = 1 \quad \forall \underset{\sim}{x}$
   ii) $\phi(\underset{\sim}{x}) = 0$
   iii) $\phi(\underset{\sim}{x}) = \alpha \quad \forall \underset{\sim}{x}$
   iv) $\phi(\underset{\sim}{x}) = \begin{cases} 1 ; & \underset{\sim}{x} \in W \\ 0 ; & \underset{\sim}{x} \in W^c \end{cases}$
   v) $\phi(\underset{\sim}{x}) = \begin{cases} 1 & x \in W \\ \gamma & x = c_i ; i = 1, 2, \ldots, n \\ 0 & \text{o.w} \end{cases}$

Q. How to construct a test which always gives the largest power?

Fundamental Neyman-Pearson Lemma:

Statement:- Let f_0 and f_1 be densities (mass fn.) w.r.t some prob. measure μ then consider $\left[H_0: \text{An obs}^n\ X \in f_0\right]$ vs $\left[H_1: \text{That obs}^n\ \underset{\sim}{x} \in f_1\right]$

Fix an α which lies b/w 0 and 1, i.e $\alpha \in (0,1)$

Denote, $R(\underset{\sim}{x}) = \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})}$,

Further choose $K_\alpha \rightarrow P_{H_0}\{R(\underset{\sim}{x}) > K_\alpha\} \leq \alpha$

and, $P_{H_0}\{R(\underset{\sim}{x}) \geq K_\alpha\} > \alpha$

Under the above setup there always exist a test fn. ϕ such that,

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } R(\underset{\sim}{x}) > K_\alpha \\ \gamma & \text{if } R(\underset{\sim}{x}) = K_\alpha \\ 0 & R(\underset{\sim}{x}) < K_\alpha \end{cases}$$

Where, $$\gamma = \frac{\alpha - P_{H_0}\{R(\underset{\sim}{x}) > K_\alpha\}}{P_{H_0}\{R(\underset{\sim}{x}) = K_\alpha\}}$$

For this ϕ, i) $E_{H_0}[\phi(x)] = \alpha \rightarrow$ size cond^n.

ii) $E_{H_1}[\phi(x)] \geq E_{H_1}[\psi(\underset{\sim}{x})] \rightarrow$ power cond^n, where ψ is another test fn having $E_{H_0}(\psi(x)) \leq \alpha$.

Remark N-P lemma provides a recipe for constructing the most powerful test. <hint (ii)>

Proof) proof of <i>

$$E_{H_0}[\phi(\underset{\sim}{x})]$$
$$= 1 \cdot P_{H_0}\{R(\underset{\sim}{x}) > K_\alpha\} + \gamma \cdot P\{R(\underset{\sim}{x}) = K_\alpha\} + 0 \cdot P\{R(\underset{\sim}{x}) < K_\alpha\}$$

$$= P_{H_0}\{R(\underset{\sim}{x}) > K_\alpha\} + \frac{\alpha - P_{H_0}\{R(\underset{\sim}{x}) > K_\alpha\}}{P_{H_0}\{R(\underset{\sim}{x}) = K_\alpha\}} \cdot P_{H_0}\{R(\underset{\sim}{x}) = K_\alpha\} + 0$$

$$= \alpha$$

<u>proof of (ii)</u>

Suppose ψ is another test fn and,

$E_{H_0}[\psi(\underset{\sim}{x})] \leq \alpha$.

$E_{H_1}[\phi(\underset{\sim}{x})] - E_{H_1}[\psi(\underset{\sim}{x})]$

$= \int_{\mathcal{X}} (\phi - \psi) f_1(\underset{\sim}{x})\, d\underset{\sim}{x}$ [One may consider summation instead of $\int$]

$= \int_{\mathcal{X}} (\phi - \psi)(f_1 - K_\alpha f_0)\, d\underset{\sim}{x} + K_\alpha \int_{\mathcal{X}} (\phi - \psi) f_0\, d\underset{\sim}{x}$

$\therefore$ The last term is greater than or equal to zero, as $\int_{\mathcal{X}} \phi f_0\, d\underset{\sim}{x} = \alpha$ but $\int_{\mathcal{X}} \psi f_0\, d\underset{\sim}{x} \leq \alpha$

Now, the first term $= \int_{R(\underset{\sim}{x}) > K_\alpha} (\phi - \psi)(f_1 - K_\alpha f_0)\, d\underset{\sim}{x}$

$+ \int_{R(\underset{\sim}{x}) = K_\alpha} (\phi - \psi)(f_1 - K_\alpha f_0)\, d\underset{\sim}{x} + \int_{R(\underset{\sim}{x}) < K_\alpha} (\phi - \psi)(f_1 - K_\alpha f_0)\, d\underset{\sim}{x}$

When, $R(\underset{\sim}{x}) > K_\alpha \Rightarrow f_1 > f_0 K_\alpha$

$(f_1 - K_\alpha f_0) > 0$ and $\phi = 1$ but ψ being another fn ≤ 1

$\therefore$ in $\{R(\underset{\sim}{x}) > K_\alpha\}$ $\therefore (\phi - \psi) \geq 0$

$\Rightarrow \int (\phi - \psi)(f_1 - K_\alpha f_0)\, d\underset{\sim}{x} \geq 0$

The second term when, $R(x) = K_\alpha \Rightarrow f_1 = K_\alpha f_0$ is always zero.

When, $R(x) < K_\alpha$, i.e $f_1 < K_\alpha f_0$, $(f_1 - K_\alpha f_0) < 0$ and ϕ being 0 yields $\phi - \psi < 0$

$$\Rightarrow \int (\phi - \psi)(f_1 - K_\alpha f_0)\, dx > 0$$

$\therefore$ In essence,

$$\int_{\mathcal{X}} (\phi - \psi)(f_1 - K_\alpha f_0)\, dx \geq 0$$

Finally, we conclude,

$$E_{H_1}[\phi(x)] - E_{H_1}[\psi(x)] \geq 0$$

$$\Rightarrow E_{H_1}[\phi(x)] > E_{H_1}[\psi(x)]$$

It assures $\phi(x)$ is the most powerful test.

Example Let X be r.v having p.m.f

x	1	2	3	4	5	6
f_0	·01	0.01	0.01	0.01	0.01	0.95
f_1	0.05	0.04	0.03	0.02	0.01	0.85

Purpose an M.P with $\alpha = 0.03$ and $\alpha = 0.025$,

$\Longrightarrow$

x	1	2	3	4	5	6
$R(x) = \frac{f_1}{f_0}$	5	4	3	2	1	0.89

$\alpha = 0.03$, Arrange $R(x)$ in dec. order,

$$5 > 4 > 3 > 2 > 1 > 0.89$$

$P\{R(x) = 5\} = P_{H_0}\{X = 1\} = 0.01$

$P\{R(x) = 4\} = P_{H_0}\{X = 2\} = 0.01$

$P\{R(x) = 3\} = P_{H_0}\{X = 3\} = \underline{0.01}$

0.03

$$\phi(x) = \begin{cases} 1 & \text{if } x = 1,2,3 \\ 0 & x = 4,5,6 \end{cases}$$

$\alpha = 0.025$,

$P\{R(x) = 5\} = P\{X=1\} = 0.01$

$P\{R(x) = 4\} = P\{X=2\} = 0.01$

$\overline{0.02}$

$\alpha = 0.025$

Adjustment of prob. is, $0.025 - 0.02$
$= 0.005$

$K_\alpha = 3$

$$\left[\gamma = \frac{0.005}{P_{H_0}\{X=3\}} = \frac{0.005}{0.01} = 0.5\right]$$

$$\phi(x) = \begin{cases} 1 & x = 1,2 \\ 0.5 & x = 3 \\ 0 & \text{o.w} \end{cases}$$

__Example__ Let $X \sim f_j$; $j = 0, 1$

X	1	2	3	4	5
f_0	1/5	1/5	1/5	1/5	1/5
f_1	1/6	1/4	1/6	1/4	1/6

Find an MP test of size $\frac{2}{5}$ and $\frac{1}{3}$ (H.W)

$\Rightarrow$

X	1	2	3	4	5
$R(x) = \frac{f_1}{f_0}$	5/6	5/4	5/6	5/4	5/6

$\frac{5}{6} = 5/6 = 5/6 < 5/4 = 5/4$

$P_{H_0}\{R(x) = 5/4\} = P_{H_0}(X=2) = P_{H_0}(X=4) = 1/5$

$$\phi(x) = \begin{cases} 1 & x = 2,4 \\ 0 & x = 1,3,5 \end{cases}$$

H.W $X_1, X_2, \ldots, X_n \sim iid\ N(\theta, 1)$,

$H_0: \theta = \theta'$

vs, $H_1: \theta = \theta''$. $[\theta'' > \theta']$

Find an MP test.

Exercise

Find an MP test of size α where,

$$H_0: X \sim f_0 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}; -\infty < x < \infty$$

$$H_1: X \sim f_1 = \frac{1}{2} e^{-|x|}, -\infty < X < \infty$$

Based on a single obs^n.

$\Rightarrow$ We reject H_0 if $\Rightarrow$

For a specific α,

$$\frac{f_1}{f_0} = \frac{\frac{1}{2} e^{-|x|}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} > K_\alpha$$

Where,

$$P_{H_0}\left\{\frac{\frac{1}{2} e^{-|x|}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} > K_\alpha\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{e^{-|x| + x^2/2} > K'_\alpha\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{-|x| + \frac{x^2}{2} > K''_\alpha\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{x^2 - 2|x| > K''_\alpha\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{|x|^2 - 2|x| > K'''_\alpha\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{|x| > K^*\right\} = \alpha$$

$$\Rightarrow P_{H_0}\left\{|x| < K^*\right\} = 1 + \alpha$$

$$\Rightarrow P_{H_0}\left\{-K^* < x < K^*\right\} = 1 - \alpha$$

$\therefore K^* = \tau_{\alpha/2}$

We reject H_0 if $X > \tau_{\alpha/2}$ or $X < -\tau_{\alpha/2}$

$$\phi(x) = \begin{cases} 1 & \text{if } X > \tau_{\alpha/2} \text{ or } X < -\tau_{\alpha/2} \\ 0 & \text{o.w} \end{cases}$$

Ex For the hypergeometric problem, construct the MP test with $\alpha = 1/3$, $H_0: m = 5$ (→ no. of white balls) $\quad N = 10$, $n = 3$

$H_1: m = 6$ (→ no. of black ")

$\Rightarrow$

$$P(X = x) = \frac{\binom{m}{x}\binom{10-m}{3-x}}{\binom{10}{3}} \quad ; \quad x = 0, 1, 2, 3$$

$X = x$	$P_{H_0}(X = x)$	$P_{H_1}(X = x)$	$R(x) = \frac{P_{H_1}}{P_{H_0}}$
0	$\frac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} = 1/12$	$\frac{\binom{6}{0}\binom{4}{3}}{\binom{10}{3}} = \frac{1}{30}$	0.4
1	5/12	3/10	0.72
2	5/12	1/2	1.2
3	1/12	1/6	2

$2 > 1.2 > 0.72 > 0.4$

$P_{H_0}\{X = 3\} = \frac{1}{12}$

$P_{H_0}\{X = 2\} = 5/12$

$P_{H_0}\{X = 0\} = 1/12$,

$$\therefore \gamma = \frac{\alpha - P(R(x) > K_\alpha)}{P(R(x) = 1.2)} \quad (K_\alpha \to 1.2)$$

$$= \frac{1/3 - P_{H_0}(X = 3)}{P_{H_0}(X = 2)}$$

$$= \frac{1/3 - 1/12}{5/12}$$

$$= 0.6$$

$$\phi(x) = \begin{cases} 1 & X = 3 \\ 0.6 & \cancel{X = 0} \; X = 2 \\ 0 & X = 0, 1 \end{cases}$$

H.W $E_{H_1}(\phi(x))$. $\phi(x) = \begin{cases} 1 & X = 3, 0 \\ \gamma & X = 2 \\ 0 & X = 1 \end{cases}$

$$\therefore \gamma = \frac{1/3 - \frac{1}{6}}{5/12}$$

$$= \frac{1}{6} \times \frac{12}{5} = \frac{2}{5}$$

$$p(X=x) = \frac{\binom{m}{x}\binom{10-m}{3-x}}{\binom{10}{3}}$$

For $H_1 : m=6$,

$$P_{H_1}(X=x) = \frac{\binom{6}{x}\binom{4}{3-x}}{\binom{10}{3}}$$

$$P_{H_1}(X=3) = \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} = \frac{\frac{6\times5\times4}{6}}{\frac{10\times9\times8}{6}}$$

$$= \frac{\cancel{6}\times\cancel{5}\times\cancel{4}}{\cancel{12}_3\,\cancel{72}\times\cancel{10}_2} = \frac{1}{6}$$

$$P_{H_1}(X=0) = \frac{\binom{6}{0}\binom{4}{3}}{\binom{10}{3}} = \frac{4}{\frac{10\times9\times8}{6}}$$

$$= \frac{1}{30}$$

$\gamma = 0.4$

$$P_{H_1}(X=2) = \frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} = \frac{\frac{6\times5}{2}\times4}{\frac{10\times9\times8}{6}}$$

$$= \frac{6\cancel{0}\times6}{72\times1\cancel{0}} = \frac{1}{2}$$

$\therefore E_{H_1}[\phi(x)]$

$$= 1.\frac{1}{6} + 1.\frac{1}{30} + 0.4\times\frac{1}{2}$$

$$= \frac{10+2+12}{60} = \frac{24}{60} = 0.4$$

H.W

$X_i \overset{iid}{\sim} N(\theta,1) \quad \forall\, i = 1(1)n$

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2} \quad i=1(1)n \;;\; -\infty < x_i < \infty$$

$\therefore$ Joint prob. density f^n is,

$$\prod_{i=1}^{n} f_{X_i}(x_i) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$\therefore \frac{\prod_{i=1}^{n} f_{X_i/H_1}(x_i)}{\prod_{i=1}^{n} f_{X_i/H_0}(x_i)} \geq k_\alpha \quad , \alpha = \text{fixed size},$$

$$\Rightarrow e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta'')^2 + \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta')^2} \geq k_\alpha$$

$$\Rightarrow -\frac{1}{2}\left[\sum_{i=1}^{n}\left[(x_i-\theta'')^2 - (x_i-\theta')^2\right]\right] \geq \ln k_\alpha$$

$$\Rightarrow \sum_{i=1}^{n}\left[2x_i - \theta'' - \theta'\right]\left[\theta''-\theta'\right] \geq 2\ln k_\alpha$$

$$\Rightarrow 2(\theta''-\theta')\sum_{i=1}^{n} x_i - (\theta''-\theta')(\theta'+\theta'') \geq k_\alpha'$$

$$\Rightarrow \sum_{i=1}^{n} x_i \geq k_\alpha''$$

$$\phi(x) = \begin{cases} 1 & ; \bar{x} \geq k_\alpha'' \\ 0 & \text{o.w} \end{cases}$$

$$\therefore P_{H_0}(\bar{x} > k_\alpha'') = \alpha$$

$$\Rightarrow \alpha = P_{H_0}\left(\frac{\bar{x}-\theta'}{1/\sqrt{n}} > \frac{k_\alpha''-\theta'}{1/\sqrt{n}}\right)$$

$$\Rightarrow k_\alpha'' = \theta' + \frac{1}{\sqrt{n}}\tau_\alpha$$

■ The objective catered by N-P lemma is adding more points in W such that the size of the test would be as close as chosen α, before going for randomization.
The point adding rule in W is to include the points on the basis of largest $R(x)$ to smallest.

Ex:

$H_0: m=5$
$H_1: m=6$

$$P(X=x) = \frac{\binom{m}{x}\binom{10-m}{3-x}}{\binom{10}{3}} ; X=0,\dots,3$$

$X=x$	$P_{H_0}(X=x)$	$P_{H_1}(X=x)$	$R(x) = P_{H_1}/P_{H_0}$
0	$1/12$	$1/30$	0.4
1	$5/12$	$3/10$	0.72
2	$5/12$	$1/2$	1.2
3	$1/12$	$1/6$	2

$$\boxed{\alpha = \frac{1}{3}}$$

$$P_{H_0}(X=3) = \frac{1}{12} < \frac{1}{3}$$

$$P_{H_0}(X=3) + P_{H_0}(X=2) = \frac{1}{12} + \frac{5}{12} = \frac{1}{2} > \frac{1}{3}$$

$$P_{H_0}(X=3) + P_{H_0}(X=1) = 1/2 > 1/3$$

$$P_{H_0}(X=3) + P_{H_0}(X=0) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} < \frac{1}{3} = 0.33$$

Case-I

$W = \{3, 0\}$ $\quad E_{H_0}(\phi(x)) = 0.33 \Rightarrow$ size bu

$$1 \cdot P(x=3) + 1 \cdot P(x=0) + \gamma P(x=1) = 0.33$$

$$\Rightarrow \frac{1}{12} + \frac{1}{12} + \gamma \cdot 5/12 = 1/3$$

$$\Rightarrow \gamma = 0.4$$

Case-II $\quad 1 \cdot P(x=3) + 1 \cdot P(x=0) + \gamma P(x=2) = 1/3$

$$\Rightarrow \gamma = 0.4$$

$$\phi_1(x) = \begin{cases} 1 & ; x=0,3 \\ 0.4 & x=1 \\ 0 & x=2 \end{cases} \qquad \phi_2(x) = \begin{cases} 1 & ; x=0,3 \\ 0.4 & ; x=2 \\ 0 & ; x=1 \end{cases}$$

$E_{H_1}(\phi_1) = 1\cdot\frac{1}{30} + 1\cdot\frac{1}{6} + 0.4\times 3/10$

$= 0.32$

$E_{H_1}(\phi_2) = 1\cdot\frac{1}{30} + 1\cdot\frac{1}{6} + 0.4\times\frac{1}{2}$

$= 0.4$

Ex

x	1	2	3	4	5
f_0	1/5	1/5	1/5	1/5	1/5
f_1	1/6	1/4	1/6	1/4	1/6

$\alpha = 2/5 \to$ Done,

$\alpha = 1/3$, $\phi_1(x) = \begin{cases} 1 & x=4 \\ 2/3 & x=2 \\ 0 & \text{o.w} \end{cases}$ $\phi_2(x) = \begin{cases} 1 & x=2 \\ 2/3 & x=4 \\ 0 & \text{o.w} \end{cases}$

$E_{H_1}(\phi_1(x)) = \frac{1}{4} + 2/3\times\frac{1}{4} = 5/12$, $E_{H_1}[\phi_2(x)] = \frac{1}{4} + 2/3\times 1/4 = 5/12$

- Most powerful test is not unique.
- Uniformly most powerful test is not unique.
- For non-randomized test mp test is unique, so is UMP.

Observations of N-P test

① If $K_\alpha = \infty$, the test $\phi(x) = \begin{cases} 1 & f_0(x)=0 \\ 0 & f_0(x)>0 \end{cases}$

is called most powerful test of size 0.

(prob. of rejecting a true H_0 is zero)

Ideal M-P test construction.

② motivation of considering $R(x)$: (why $R(x)$?)

Let us have one example:-

x	0	1	2
$H_0: p_0$	0.05	0.05	0.90
$H_1: p_1$	0.9	0.08	0.02

$H_0: x \in p_0$ $\alpha = 0.05$

$H_1: x \in p_1$

$R(x) = \frac{0.9}{0.05} = 18$ ~~or~~

$\frac{0.08}{0.05} = 8/5$

$\frac{0.02}{0.9} = \frac{1}{45}$

$X = 1 \in W_1 \qquad X = 0 \in W_2$

$P_{H_0}(X=1) = 0.05 \qquad P_{H_0}(W_2) = 0.05$

$P_{H_1}(X=1) = 0.08 \qquad P_{H_1}(X=0) = 0.9$

Largest $R(x)$ ensures highest power,

$R(x)$ is considered in N-P lemma.

(3) N-P lemma does not require only the sample obsⁿ for testing, it can test between two prob. distⁿs. Not only that for ~~param~~ test on parameters the sample obsⁿs are not required to be iid.

(4) For a N.P. fnⁿ $\phi = 1$ in W

and $\phi(x) = 0$ mean W^c.

Every most-powerful test of size α must attain a power tending to 1. (i.e power $\to 1$).

Hence if size $< \alpha$ and power < 1, one may include more points in critical region.

(5) For an MP test, Power > size (unbiased test)

proof

Suppose ϕ is an MP test with size α.

Denote its power by β_ϕ. Let us consider another test $\psi \equiv \alpha$ such that $E_{H_0}(\psi) = \alpha$, as well as $E_{H_1}(\psi) = \alpha$

In, $\{R(x) > K_\alpha\}$ ϕ takes 1 as per N-P recipe

in, $\{R(x) = K_\alpha\}$ ϕ " γ as per N-P recipe

and, in $\{R(x) < K_\alpha\}$ ϕ " 0 " " "

$\therefore$ In $\{R(x) > k_\alpha\}$, $\phi > \psi$

while, $\{R(x) = k_\alpha\}$, $\phi = \psi = \alpha$

and, $\{R(x) < k_\alpha\}$; $\phi < \psi$

~~Therefore~~ $\therefore$ In rejection region $\boxed{\underset{(\text{power})}{\beta_\phi} > \underset{(\text{size})}{\alpha}}$

$\therefore$ Acceptance region. $\boxed{\text{power} < \text{size}}$

For an, MP test $\boxed{\text{power} > \text{size}}$

(6) Suppose a test $\bar{\phi} = 1 - \phi$ where ϕ is a test fn. for testing $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$ where

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } R(x) > k_\alpha \\ \gamma & \text{if } R(x) = k_\alpha \\ 0 & \text{if } R(x) < k_\alpha \end{cases}, \text{ given } E_{H_0}[\phi(x)] = \alpha$$

- Then, $\bar{\phi}$ is also a MP test.

Q.
- Find the size of $\bar{\phi}$
- Find the power of $\bar{\phi}$

Proof ⇒

$H_0 : X \sim f_0$

$H_1 : X \sim f_1$

$$\phi(x) = \begin{cases} 1 & \text{if } R(x) > k_\alpha \\ \gamma & \text{if } R(x) = k_\alpha \\ 0 & \text{if } R(x) < k_\alpha \end{cases}$$

With, $E_{H_0}[\phi(x)] = \alpha$, $E_{H_1}[\phi(x)] = \beta_\phi$

Now, $\bar{\phi} = 1 - \phi$

$$\bar{\phi}(x) = 1 - \phi(x) = \begin{cases} 0 & \text{if } R(x) > k_\alpha \\ 1-\gamma & \text{if } R(x) = k_\alpha \\ 1 & \text{if } R(x) < k_\alpha \end{cases}$$

Since, N-P lemma states that in rejection region

the test f^n takes value 1, $R(m) < K$ is the rejection region for, $H_0': X \sim f_1$
$H_1': X \sim f_0$

$$\text{Size of } \bar{\Phi} = E_{H_0'}[\bar{\Phi}]$$
$$= E_{H_0'}[1-\Phi]$$
$$= E_{f_1}[1-\Phi]$$
$$= E_{H_1}[1-\Phi]$$
$$= 1 - E_{H_1}[\Phi] = 1-\beta_\Phi$$

$$\text{Power of } \bar{\Phi} = E_{H_1'}[\bar{\Phi}]$$
$$= E_{H_1'}[1-\Phi]$$
$$= E_{f_0}[1-\Phi]$$
$$= E_{H_0}[1-\Phi] = 1 - E_{H_0}[\Phi]$$
$$= 1-\alpha$$

Since, Φ is a MP test.

Power $\geq$ size

$\beta_\Phi \geq \alpha$

$\Rightarrow -\alpha \geq -\beta_\Phi$

$\Rightarrow 1-\alpha \geq 1-\beta_\Phi$

$\Rightarrow$ power of $\bar{\Phi} \geq$ size of $\bar{\Phi}$

$\Rightarrow \bar{\Phi}$ is a MP test.

[Complementation of test]

<u>H.W</u> X a single obs^n.

$H_0: X \sim U(0,1)$

$H_1: X \sim Beta(2,1)$

Construct an MP test.

Find the power.

How to inc. power keeping a single obs^n in the experiment.

Q) Ex: Sample size 1

$H_0 : X \sim N(0,1)$ $-\infty < x < \infty$

$H_1 : X \sim C(0,1)$

$$R(x) = \frac{f_1}{f_0} = \sqrt{\frac{2}{\pi}} \cdot \frac{e^{x^2/2}}{(1+x^2)}$$

[⊛ Distn vs Distn test is a non-randomized test]

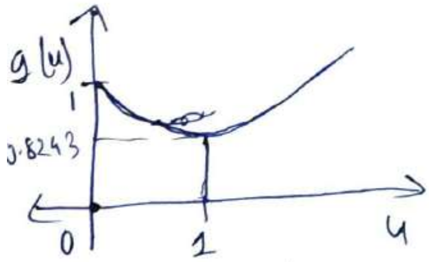
$\Rightarrow$ MP test is, $\phi(x) = \begin{cases} 1 & R(x) = 0.798 \frac{e^{x^2/2}}{1+x^2} > k_\alpha \\ 0 & o.w \end{cases}$ $\frac{e^{x^2/2}}{1+x^2} > \frac{k_\alpha}{0.798}$

Let us consider, $x^2 = u$, $0 < u < \infty$

$$g(u) = \frac{e^{u/2}}{(1+u)}$$

$$\Rightarrow g'(u) = \frac{(1+u)\frac{1}{2}e^{u/2} - e^{u/2} \cdot 1}{(1+u)^2}$$

$$= \frac{e^{u/2}(u-1)}{2(1+u)^2}$$

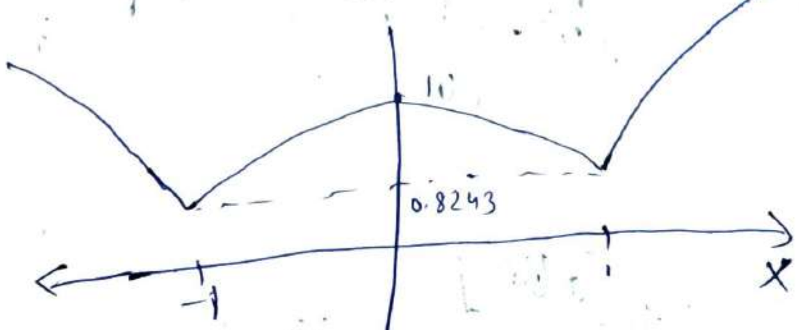


$u = 0$, $g'(0) < 0$ $u=0$, $g(u) = 1$

$u > 1$, $g'(u) > 0$ $u=1$, $g(u) = 0.8243$

$0 < u < 1$, $g'(u) < 0$

$u = x^2$



$R(x)\uparrow$ $x > 1$
$R(x)\uparrow$ $x < -1$
$R(x)\downarrow$

M.P test

$$\phi(x) = \begin{cases} 1 & ; |x| > 1 \\ 0 & o.w \end{cases}$$

$E_{H_0}[\phi(x)]$
$= P_{H_0}[X>1] + P_{H_0}[X<-1]$
$= 1-\Phi(1) + 1-\Phi(1)$
$= 2-2\Phi(1)$

Case-II

If α is given as $\alpha = 0.05$,

MP test, $\therefore \phi(x) = \begin{cases} 1 & ; |x| > k_\alpha \\ 0 & |x| < k_\alpha \end{cases}$

where k_α is determined from
$E_{H_0}[\phi(x)] = 0.05$
$\Rightarrow P(|x| > k_\alpha) = 0.05$; $k_\alpha > 1$

H.W

$R(x) = \frac{f_1}{f_0} = \frac{\frac{1}{B(2,1)} x^{(2-1)}(1-x)^{1-1}}{1}$

$= \frac{x \sqrt{3}}{\sqrt{2}\sqrt{1}}$

$= 2x$

M.P test

$\phi(x) = \begin{cases} 1 & 2 > 2x > k_\alpha \Rightarrow x > k_\alpha/2 \quad \left[0 < \frac{k_\alpha}{2} < 1\right] \\ 0, & \text{o.w} \end{cases}$

Size,
$E_{H_0}[\phi(x)]$
$= 1 \cdot \int_{\frac{k_\alpha}{2}}^{1} 1\, dx$
$= 1 - \frac{k_\alpha}{2}$
Let $1 - \frac{k_\alpha}{2} = \alpha$
$\Rightarrow \frac{k_\alpha}{2} = 1-\alpha$

Power

$E_{H_1}[\phi(x)]$
$= 1 \cdot P\left[\frac{k_\alpha}{2} < x < 1\right]$
$= 1 \cdot \int_{\frac{k_\alpha}{2}}^{1} 2x\, dx = x^2 \Big|_{\frac{k_\alpha}{2}}^{1} = 1 - \frac{k_\alpha^2}{4}$
$= \alpha[2-\alpha]$
$= 2\alpha - \alpha^2$

Three special types of problems

1) Let X be a r.v having $f \equiv \{f_0, f_1\} \in \Theta$, where

$f_0(x) = 1 \quad ; \quad 0 < x < 1$

$f_1(x) = \frac{1}{2} \quad ; \quad 0 < x < 2$

For testing $H_0 : f = f_0$ against $H_1 : f = f_1$ on a single obs^n, find out power of mp test, when $\alpha = 0.05$

Ans

$$0 < x < 2 = \underbrace{(0 < x < 1)}_{\text{common range}} \cup \underbrace{(1 < x < 2)}_{\text{uncommon range}}$$

Randomized kt. may be created

As per N-P principle, an mp test is,

$$\phi(x) = \begin{cases} 1 & \text{when } 1 < x < 2 \\ \gamma & \text{"} \quad 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

$E_{H_0}[\phi(x)] = 0.05$ (Given)

$\Rightarrow 0.05 = 1 . P_{H_0}[1 < x < 2] + \gamma P_{H_0}[0 < x < 1]$

$\Rightarrow 0.05 = 1.0 + \gamma \int_0^1 1 . dx$

$\Rightarrow \gamma = 0.05$

$$\phi(x) = \begin{cases} 1 \\ 0.05 \\ 0 \end{cases} \rightarrow \langle \text{mp test} \rangle$$

Power,

$E_{H_1}[\phi(x)] = 1 . P_{H_1}[1 < x < 2] + 0.05 P_{H_1}[0 < x < 1]$

$= 1 . \int_1^2 \frac{1}{2} dx + 0.05 \int_0^1 \frac{1}{2} dx$

$= \frac{1}{2} + \frac{1}{2} \times 0.05$

$= 0.525$

2) Let X take on values 1, 2, 3, 4 with prob. procedure $P_0, P_1, Q \rightarrow$

X	1	2	3	4
P_0	$2/13$	$4/13$	$3/13$	$4/13$
P_1	$4/13$	$2/13$	$1/13$	$6/13$
Q	$4/13$	$3/13$	$\frac{2}{13}$	$4/13$

$\alpha = \frac{5}{13}$, $\boxed{\alpha = \frac{6}{13}}$

Find an MP test for $H_0: X \in P_0$ or $X \in P_1$ [comp]
$H_1: X \in Q$. [simple]

Ans)

Test I $\begin{cases} H_0: X \in P_0 \\ H_1: X \in Q \end{cases}$ and, $\begin{rcases} H_0: X \in P_1 \\ H_1: X \in Q \end{rcases}$ Test II

X	1	2	3	4
R_1	2	$3/4$	$2/3$	1

$R_1(1) > R_1(4) > R_1(2) > R_1(3)$

X	1	2	3	4
R_2	1	$3/2$	2	$2/3$

$R_2(3) > R_2(2) > R_2(1) > R_2(4)$

$P_{H_0}(X=3) = 1/13$
$P_{H_0}(X=2) = 2/13$
$P_{H_0}(X=1) = 4/13$

$P_{H_0}(X=1) = 2/13$

$P_{H_0}(X=4) = \frac{4}{13}$ × $\left[\because \frac{2}{13} + \frac{4}{13} > \frac{5}{13}\right]$

$P_{H_0}(X=2) = 4/13$ × $\left[\because 2/13 + \frac{4}{13} > \frac{5}{13}\right]$

$P_{H_0}(X=3) = 3/13$ ✓ $\left[\because 2/13 + \frac{3}{13} = \frac{5}{13}\right]$

MP test for Test I, $\phi_1(x) = \begin{cases} 1 & \text{if } x = 1, 3 \\ 0 & \text{if } x = 2, 4 \end{cases}$

MP test for Test II, $\phi_2(x) = \begin{cases} 1 & \text{if } x = 3, 1 \\ 0 & \text{if } x = 2, 4 \end{cases}$

Let, $W_1 = \{1, 3\}$ be the critical region for $H_0: X \in P_0$, $H_1: X \in Q$

$W_2 = \{3, 1\}$ " " " " " " $H_{02}: X \in P_1$, $H_1: X \in Q$

W be the critical region for $X \in P_0$ or $X \in P_1$

$\therefore W = W_1 \cap W_2$

$W_1 : X \in W_1$ we reject H_{01}

$W_2 : X \in W_2$ " " H_{02}

$H_0 : H_{01}$ or H_{02}

$W : X \in W$ we reject H_0

= we reject both H_{01} and H_{02}

$$\therefore W = W_1 \cap W_2 = \{1,3\} \cap \{3,1\} = \{1,3\}$$

For, H_0: MP test is:

$$\phi = \begin{cases} 1 & \text{if } x \in \{1,3\} \\ 0 & \text{o.w} \end{cases}$$

3) Let, P_0, P_1, P_2 be the prob.

X	1	2	3	4	5	6
P_0	0.03	0.02	0.02	0.01	0	0.92
P_1	0.06	0.05	0.08	0.02	0.01	0.78
P_2	0.09	0.05	0.12	0	0.02	0.72

$\alpha = 0.01$, determine whether $\exists$ a UMP

$\boxed{\alpha = 0.03}$

on $H_0 : P = P_0 \; (x \in P_0)$

against $H_1 : P = P_1$ and $P = P_2$, $\alpha = 0.01$

Ans

X	1	2	3	4	5	6
$R_1 = P_1/P_0$	2	2.5	4	2	∞	78/92
$R_2 = \frac{P_2}{P_0}$	3	2.5	6	0	∞	72/92

For the 1st series $\left[P(X=3) = 0.02 > 0.01 \times \right.$, $P(X=4) = 0.01 = 0.01$

For the 2nd series $\left[\begin{array}{l} H_0 : P = P_0 \\ H_1 : P = P_2 \end{array} \right.$ $\left[P(X=4) = 0.01 = 0.01 \right]$.

For all choices of parametric space $\{X=4\}$ will satisfy the size condition for both of the test.

For "and" criterion in H_1 we construct a convex combinaⁿ of P_1 and P_2.

$$P^* = \delta P_1 + (1-\delta) P_2 ; 0 < \delta < 1$$

The test is analogous of testing

$H_0 : X \in P_0$

$H_1 : X \in P^*$

For testing H_0 against H_1 the critical region does not change, as critical region depends on the null hypothesis.

$\{X = 4\}$ is the common critical region.

$\therefore$ For power, $\delta . P_{H_0} [X \in 4 / P_1] + (1-\delta) P [X \in 4 / P_2]$

$= \delta \times 0.02 + (1-\delta) \times 0$

$= 0.02\delta ; 0 < \delta < 1$

Now, if $0 < \delta < \frac{1}{2}$, $\delta = 0 \Rightarrow$ Power $= 0 <$ size

(No UMP exists) $\delta = 1/3 \Rightarrow$ Power $= 0.006 <$ " $(\alpha = 0.01)$

$\delta = 1/2 \Rightarrow$ Power $= 0.01 =$ size

Unbiased condⁿ doesn't satisfy when $0 < \delta < \frac{1}{2}$.

$\frac{1}{2} < \delta < 1$, $\delta = 3/4 \Rightarrow$ Power $= 0.015 > 0.01$

$\delta = 1 \Rightarrow$ Power $= 0.02 > 0.01$

UMP test exists if $\frac{1}{2} < \delta < 1$,

If the mixing proporⁿ lies within $(1/2, 1)$ UMP test exists.

$W = \{4, 5\}$

$\delta . P [X \in 4, 5 / P_1] + (1-\delta) P [X \in 4, 5 / P_2]$

$= \delta [0.02 + 0.01] + (1-\delta) [0 + 0.02]$

$= 0.02 + 0.01\delta > 0.01$

$W = \{4, 5\}$ for any $0 < \delta < 1$ UMP exists.

Relationship b/w sufficient statistic and N-P test fn

Let $T(x)$ be sufficient for θ.

Consider the test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$

An MP test is,

$$\phi(x) = \begin{cases} 1 & \text{if } R(\underset{\sim}{x}) > k_\alpha \\ \gamma & R(\underset{\sim}{x}) = k_\alpha \\ 0 & R(\underset{\sim}{x}) < k_\alpha \end{cases}$$

$$\equiv \phi(x) = \begin{cases} 1 & \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} > k_\alpha \\ \gamma & \quad " \quad = k_\alpha \\ 0 & \quad " \quad < k_\alpha \end{cases}$$

where $f_1(\underset{\sim}{x})$ be the joint distn under H_1 and $f_0(\underset{\sim}{x})$ be the same under H_0.

Think on Fisher Neyman criterion on sufficient statistics,

If $T(x)$ is suff. stat then $f(\underset{\sim}{x})$ is,

$$f(\underset{\sim}{x}) = g_\theta(T) \cdot h(\underset{\sim}{x})$$

$$\equiv \phi(x) = \begin{cases} 1 & \dfrac{g_{\theta_1}(T) \cdot h(x)}{g_{\theta_0}(T) \cdot h(x)} > k_\alpha \\ \gamma & \quad " \quad = k_\alpha \\ 0 & \quad " \quad < k_\alpha \end{cases}$$

$$\equiv \phi(x) = \begin{cases} 1 & R_\theta(T) > k_\alpha \\ \gamma & \quad " \quad = k_\alpha \\ 0 & \quad " \quad < k_\alpha \end{cases}$$

$\Rightarrow \phi$ is an MP test based on sufficient statistic.

Remark ① A N-P, MP test can be constructed on suff. statistic of the parameter under study.

② If ϕ is a test fn, T being the suff. stat. then $E(\phi(x)/T)$ is analogous of $E[\phi(x)]$. Moreover $\phi(x)/T$ is an MP test.

• [Remember - Rao - Blackwell thm]

Ex: $N(\mu, \sigma^2)$; σ^2 is known

$H_0 : \mu = \mu_0$, $H_1 : \mu = \mu_1$ } Let, $x_1, x_2, \ldots, x_n$ be a r.s.

$\Rightarrow$ $\bar{x}$ is the sufficient statistic for μ.

An MP test will be,

$$\phi(x) = \begin{cases} 1 & \text{if } \bar{x} > c \\ 0 & \text{if } \bar{x} < c \end{cases}$$

c can be determined,

$$E_{H_0}[\phi(\bar{x})/\bar{x}] = \alpha$$

$$\Rightarrow P_{H_0}[\bar{x} > c] = \alpha$$

Ex-1 Problems on fundamental N-P lemma

1) Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from $P(\lambda)$. Construct an MP test with size $\alpha = 0.05$, for,

$H_0 : \lambda = 1$

$H_1 : \lambda = 2$

Also find the power of the test.

2) Let $x_1, x_2, \ldots, x_5$ be iid $Ber(p)$. ~~Test~~ Construct an M-P test for $H_0 : p = 0.3$ against $H_1 : p = 0.5$ with $\alpha = 0.05$

Furthermore find the power for $p = 0.6$, $p = 0.7$, $p = 0.9$.

Hint [1] For Poisson, suff. stat for λ is, $Y = \sum_{i=1}^{5} x_i \sim Poi(5\lambda)$

$$\phi(Y) = \begin{cases} 1 & Y > C \\ \gamma & Y = C \\ 0 & Y < C \end{cases}$$

$$E_{H_0}[\phi] = 0.05$$

$\Rightarrow 1 \cdot P(Y > C) + \gamma P[Y = C] = 0.05$

$\Rightarrow 1 - P(Y \leq C) + \gamma \cdot P(Y = C) = 0.05$ Under H_0,

$\Rightarrow 0.95 = P(Y \leq C) + \gamma \cdot P(Y = C)$ $Y \sim Poi(5)$

C	$P(Y \leq C)$	$P(Y = C)$
1		

Hint 2) ~~Suff statistic~~ $Y = \sum_{i=1}^{5} x_i \sim Bin(5, p)$ $\phi(Y) = \begin{cases} 1 & \Sigma x_i > C \\ \gamma & \Sigma x_i = C \\ 0 & \Sigma x_i < C \end{cases}$

Monotone likelihood ratio property: (MLR)

Since N-P lemma suggest on the test construction of simple null vs simple alt. hypothesis, for the test construction of simple null vs composite alt. hypothesis, technically for each $\theta > \theta_0$ (null hyp. figure) the nature of $R(x)$ [Ratio of likelihood fn's] should be checked. Naturally Pointwise checking is impossible.

Therefore if we study a characteristic of the family of dist's for all θ, that will give an extension to N-P most powerful test construction. The characteristic is how $R(\cdot)$ behaves as $\theta \uparrow$. If it shows some monotonicity we can take any $\theta > \theta_0$ and imagine out the construction of MP test on that θ. As for all $\theta > \theta_0$, the test fn remains the same, that test fn is called uniformly most powerful test.

$\boxed{\text{Def}}$ A family $\{f_\theta : \theta \in \Theta\}$ is said to satisfy MLR property in some real valued statistic $T(x)$ for $\theta_0 < \theta \in \Theta_1$ if the ratio $R_\theta(x) = \dfrac{f_\theta(x)}{f_{\theta_0}(x)}$ is monotonically non-decreasing with $T(x)$.

Theorem

If a family has the MLR in $T(x)$. An UMP test for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ exists in the following form of test.

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > k \\ \gamma & \text{if } T(x) = k \\ 0 & \text{o.w} \end{cases}$$

[k and γ are to be determined from,
$E_{H_0}[\phi(x)] = \alpha$].

Trick

[① $\theta > \theta_0$
② $T_1(x) > T(y)$ [suff. stat]
③ $R(x) \geq R(y)$]

Ex/ Construct an UMP test for $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda > \lambda_0$ when a single obs^n comes from $Poi(\lambda)$

Ans Fix $\boxed{\lambda_1 > \lambda_0}$

$$R(x) = \frac{f_{\lambda_1}(x)}{f_{\lambda_0}(x)} = \frac{e^{-\lambda_1}\lambda_1^x}{x!} \times \frac{x!}{e^{-\lambda_0}\lambda_0^x}$$

$$= e^{-(\lambda_1 - \lambda_0)}\left(\frac{\lambda_1}{\lambda_0}\right)^x, \quad x = 0, 1, 2, \ldots$$

Take another sample obs^n

$X > Y$

$$R(Y) = e^{-(\lambda_1 - \lambda_0)}\left(\frac{\lambda_1}{\lambda_0}\right)^y$$

$\therefore R(x)\uparrow$ as $x\uparrow$

$\therefore$ This family has the MLR property in x.

Ex 11 Let x have p.m.f. $P(X=x) = \dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$; $x=0,1,2,\ldots \min(n,m)$

Does the family possess MLR property. $H_0: \theta=m$, $H_1: \theta > m$

$$\left[\begin{array}{l} \text{UMP test is } \phi(x) = \begin{cases} 1 & x>c \\ \gamma & x=c \\ 0 & x<c \end{cases} \\ E_\theta[\phi(x)] = \alpha \end{array}\right]$$

Ans

$H_0: \theta = m$, $H_1: \theta = m+1$ [Other parameters N, n are fixed]

$$R(x) = \frac{P_{m+1}}{P_m} = \frac{\binom{m+1}{x}\binom{N-m-1}{n-x}}{\binom{m}{x}\binom{N-m}{n-x}}$$

$$= \frac{m+1}{m+1-x} \cdot \frac{N-m-n+x}{N-m}$$

Fix, $x=0$ and $x=1$

$$R(0) = \frac{N-m-n}{N-m}$$

$$R(1) = \frac{m+1}{m} \cdot \frac{N-m-n+1}{N-m}$$

$$= \left(1+\frac{1}{m}\right)\left(\underbrace{\frac{N-m-n}{N-m}}_{R_0} + \frac{1}{N-m}\right)$$

$\Rightarrow R(1) = R(0) + \ldots$

$\therefore R(1) > R(0)$

$\therefore R(x)\uparrow$ as $x\uparrow$

$\therefore$ This family possesses MLR in x.

Ex $x_1, x_2, \ldots, x_n \sim U[0, \theta]$
Does the family MLR?

Ans $f(x) = \begin{cases} \frac{1}{\theta} & ; 0 < x < \theta \\ 0 & o.w \end{cases}$

Let $x_1, x_2, \ldots, x_n$ be a r.s coming from $U(0, \theta)$ and $X_{(n)}$ = largest order statistic is the suff. stat for θ. N-P test construction is possible on the sufficient statistic.

Firs $\theta_1 > \theta_0$ (~~null~~ under null hypothesis)

$$R(\underline{x}) = \frac{f_{\theta_1}}{f_{\theta_0}} = \frac{\left(\frac{1}{\theta_1}\right)^n}{\left(\frac{1}{\theta_0}\right)^n} = \begin{cases} \left(\frac{\theta_0}{\theta_1}\right)^n, & 0 < X_{(n)} < \theta_0 < \theta_1 \\ \infty, & 0 < \theta_0 < X_{(n)} < \theta_1 \\ 0/0, & 0 < \theta_0 < \theta_1 < X_{(n)} \end{cases}$$

$$\left\{\begin{array}{l} f_{\theta_1} = \text{definable} \\ f_{\theta_0} = 0 \end{array}\right\} \rightarrow 0 < \theta_0 < X_{(n)} < \theta_1$$

Further let, another sample, $Y_1, Y_2, \ldots, Y_n$ such that $Y_{(n)} > X_{(n)}$

$$R(\underline{y}_{(n)}) = \begin{cases} \left(\frac{\theta_0}{\theta_1}\right)^n & ; 0 < Y_{(n)} < \theta_0 < \theta_1 \\ \infty & ; 0 < \theta_0 < Y_{(n)} < \theta_1 \\ 0/0 \end{cases}$$

Suppose, 1) $0 < X_{(n)} < Y_{(n)} < \theta_0 < \theta_1$

$$R(x_{(n)}) = R(y_{(n)}) = \left(\frac{\theta_0}{\theta_1}\right)^n$$

2) $0 < X_{(n)} < \theta_0 < Y_{(n)} < \theta_1$

$$R(x_{(n)}) = \left(\frac{\theta_0}{\theta_1}\right)^n < R(y_{(n)}) = \infty$$

3) $0 < \theta_0 < X_{(n)} < Y_{(n)} < \theta_1$

$$R(x_{(n)}) = R(y_{(n)}) = \infty$$

$\therefore$ The family has MLR in $X_{(n)}$.

H.W Let, $X_1, X_2, \ldots, X_n \sim f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0 & o.w \end{cases}$

Check if the family possesses MLR.

Ans

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & x > \theta \\ 0 & o.w \end{cases} \quad \therefore \theta < X_{(1)} < X_{(2)} \cdots < X_{(n)} < \infty$$

$X_{(1)}$ is the sufficient statistic.

$$R(x_{(1)}) = \frac{\prod_{i=1}^{n} f_{\theta_1}(x_i)}{\prod_{i=1}^{n} f_{\theta_0}(x_i)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta_1}}{e^{-\sum_{i=1}^{n} x_i + n\theta_0}}$$

$$= e^{n(\theta_1 - \theta_0)}$$

$Y_1, Y_2, \ldots, Y_n$ be another sample and, $Y_{(1)} > X_{(1)}$

$$\therefore R(x_{(1)}) = \begin{cases} e^{-n(\theta_1 - \theta_0)} & \text{if } \theta_0 < \theta_1 < x_{(1)} \\ 0 & \theta_0 < x_{(1)} < \theta_1 \end{cases}$$

$$R(y_{(1)}) = \begin{cases} e^{-n(\theta_1 - \theta_0)} & ; \theta_0 < \theta_1 < y_{(1)} \\ 0 & \theta_0 < y_{(1)} < \theta_1 \end{cases}$$

if, $\theta_0 < \theta_1 < x_{(1)} < y_{(1)}$

$$R(x_{(1)}) = R(y_{(1)}) = e^{n(\theta_1 - \theta_0)}$$

if, $\theta_0 < x_{(1)} < \theta_1 < y_{(1)}$

$$R(x_{(1)}) < R(y_{(1)})$$

$\therefore R_\theta(x) \uparrow$ as $x_{(1)} \uparrow$

$\Rightarrow$ The family posses MLR

UMP test,

$$\phi(x) = \begin{cases} 1 & x_{(1)} > k_\alpha \\ 0 & \text{o.w} \end{cases}$$

$$\boxed{P_{H_0}(X_{(1)} > k_\alpha) = 0.05}$$

k_α to be determine such that the above condition hold.

<u>MLR in exponential family</u>

The joint distⁿ of $x_1, x_2, \ldots, x_n$ of the exponential family is $f_\theta(\underset{\sim}{x}) = a(\theta)\, h(\underset{\sim}{x})\, e^{c(\theta)T(\underset{\sim}{x})}$ w.r.t some prob. measure μ.

$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$

For the exponential family if $c(\theta)$ is monotonically inc. fuⁿ of θ then the family f possesses MLR in $T(x)$

<u>proof</u> If $c(\theta)$ is inc in θ, for $\theta_1 < \theta_2$.

$c(\theta_1) < c(\theta_2)$. $R(\underset{\sim}{x}) = \dfrac{f_{\theta_2}(\underset{\sim}{x})}{f_{\theta_1}(\underset{\sim}{x})}$ where $H_0: \theta = \theta_1$, $H_1: \theta > \theta_1$, $\theta = \theta_2 > \theta_1$

$$= \frac{a(\theta_2)\, h(\underset{\sim}{x})\, e^{c(\theta_2)T(\underset{\sim}{x})}}{a(\theta_1)\, h(\underset{\sim}{x})\, e^{c(\theta_1)T(\underset{\sim}{x})}}$$

$$= a^*(\theta)_{(\theta_1,\theta_2)} e^{[c(\theta_2) - c(\theta_1)] T(\underset{\sim}{x})}$$

As, $c(\theta) \uparrow_\theta$, $e^{T(\underset{\sim}{x})[c(\theta_2) - c(\theta_1)]}$ is also inc. in $T(\underset{\sim}{x})$

$\therefore R(x) \uparrow_{T(\underset{\sim}{x})}$ provided $c(\theta) \uparrow_\theta$

Hence the proof.

Ex) $x_1, x_2, \dots x_k \sim Bin(n, p)$

Joint pmf,

$$f(\underset{\sim}{x}) = \prod_{i=1}^{k} \binom{n}{x_i} p^{\sum_{i=1}^{k} x_i} (1-p)^{n - \sum_{i=1}^{k} x_i}$$

Let us assume that n is fixed p is unknown parameter.

$$h(x) = \prod_{i=1}^{k} \binom{n}{x_i}$$

$$a(p) = (1-p)^n$$

$$e^{c(p)T(x)} = e^{\Sigma x_i \log\left(\frac{p}{1-p}\right)}$$

$$T(x) = \Sigma x_i$$

$$c(p) = \log\left(\frac{p}{1-p}\right)$$

$$\Rightarrow c'(p) = \frac{d}{dp}\left[\ln \frac{p}{1-p}\right]$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$\therefore c'(p) > 0$

$\Rightarrow c(p)$ is inc. fn. in p

$\therefore Bin(n, p)$ possesses MLR property $\sum_{i=1}^{k} x_i$

Construct a UMP test for $Bin(10, p)$ with,

$$H_0 : p = p_0$$
$$H_1 : p > p_0$$

$$\phi(x) = \begin{cases} 1 & \Sigma x_i > c \\ \gamma & \Sigma x_i = c \\ 0 & \Sigma x_i < c \end{cases}$$

⊛ If family has MLR in $T(x)$ for the parameter test, N-P lemma can be extended to the "UMP test construc^n".

<u>H.W</u> Find the MLR statistic for λ in Poisson(λ)
$x_1, x_2, \ldots, x_n$.

<u>Eg</u> $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ μ is known, σ^2 is unknown | To do: σ^2 known, μ unknown

$$f(\underset{\sim}{x}) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= a(\theta)\, h(x)\, e^{c(\theta) T(\underset{\sim}{x})}$$

$\boxed{a(\theta) =}$ $a(\sigma^2) = \frac{1}{(\sigma^2)^{n/2}}$, $h(x) = 1$

$$c(\sigma^2) = -\frac{1}{2\sigma^2}$$

$$\Rightarrow c'(\sigma^2) = \frac{1}{2\sigma^4} > 0$$

$c(\sigma^2)$ is inc. fu^n

$\therefore T(x) = \sum_{i=1}^{n}(x_i-\mu)^2$ is MLR statistic.

$H_0 : \sigma^2 = \sigma_0^2$
$H_1 : \sigma^2 = \sigma_1^2 \; (< \sigma_0^2)$

$$\phi(x) = \begin{cases} 1 & \sum_{i=1}^{n}(x_i-\mu)^2 < c \\ 0 & \text{o.w} \end{cases}$$

<u>Eg</u> $X \sim G(\alpha, \beta)$, α is known, Let, $x_1, x_2, \ldots, x_n$ be a r.s from the said family. Construct a UMP test for $H_0 : \beta = 4$
$H_1 : \beta > 4$.

p.d.f $f(x) = \dfrac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha}$; $x > 0$

Joint pdf $f(\underset{\sim}{x}) = \dfrac{\beta^{n\alpha} e^{-\beta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\alpha-1}}{(\Gamma\alpha)^n}$

$a(\beta) = \beta^{n\alpha}$

$h(\underset{\sim}{x}) = \prod_{i=1}^{n} x_i^{\alpha-1}$

$c(\beta) = -\beta \quad T(x) = \sum_{i=1}^{n} x_i$

$c(\beta) \downarrow_{\beta\uparrow}$

$\therefore$ Gamma family has MLR in $-\sum_{i=1}^{n} x_i$ for β.

$$\phi(x) = \begin{cases} 1 & \text{if } \sum x_i < c_1 \\ 0 & \text{o.w} \end{cases}$$

c_1 can be determined from the size condn.

<u>β known, α unknown.</u>

$a(\alpha) = \left[\dfrac{\beta^{\alpha}}{\Gamma\alpha}\right]^n$, $h(\underset{\sim}{x}) = e^{-\beta \sum_{i=1}^{n} x_i}$

$e^{c(\alpha) T(\underset{\sim}{x})} = e^{(\alpha-1) \sum_{i=1}^{n} \log x_i}$

$c(\alpha) \uparrow_{\alpha}$

MLR exists for α in $\sum_{i=1}^{n} \log x_i$

$H_0 : \alpha = 0.2$

$H_1 : \alpha > 0.2$

Ex: One obsn. is taken from $f_\theta(x) = \frac{1}{2} e^{-|x-\theta|}$; $-\infty < x < \infty$ (not a member in exponential family). Construct a test for $H_0 : \theta = \theta'$ against $H_1 : \theta > \theta'$.

$$\therefore R(x) = \frac{f_{H_1 : \theta = \theta''(>\theta')}}{f_{H_0 : \theta = \theta'}} = e^{-|x-\theta''| + |x-\theta'|}$$

$$-|x-\theta''| + |x-\theta'|$$

$$= \begin{cases} -(x-\theta'') + (x-\theta') & ; x > \theta'' > \theta' \\ -(\theta''-x) + x-\theta' & ; \theta'' > x > \theta' \\ -(\theta''-x) + (\theta'-x) & ; \theta'' > \theta' > x \end{cases}$$

$$= \begin{cases} \theta'' - \theta' \\ 2x - \theta'' - \theta' \\ -\theta'' + \theta' \end{cases}$$

As, $x \uparrow$, $R(x) \uparrow$ (Remember for $x > \theta'' > \theta'$ and $\theta'' > \theta' > x$, $R(x)$ is constant)

$\therefore$ For, $\theta'' > x > \theta'$, The ~~Laplace~~ Laplace family holds MLR in x.

H.W

$$\text{Joint pmf} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$h(x) = \frac{1}{\prod_{i=1}^{n} x_i!}, \quad a(\lambda) = e^{-n\lambda}, \quad e^{c(\lambda)T(x)} = e^{\sum x_i \log \lambda}$$

$\therefore T(x) = \sum x_i$ $\quad [\because \lambda > 0]$

$c(\lambda) = \log \lambda$

$\Rightarrow c'(\lambda) = \frac{1}{\lambda}$ $\quad \therefore c'(\lambda) > 0$

$\therefore c'(\lambda) \uparrow_\lambda$

Poisson Possesses MLR property $\sum x_i$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

Statement

Not every family of $dist^n$ ~~has~~ posseses MLR ~~problem~~ property.

For those family constructing UMP is not possible, but MP test for a fixed θ under H_1 is possible to construct.

On the basis of construction of MP test for these family, locally MP test are to be constructed.

Ex $X \sim Cauchy(\theta)$

$$f(x) = \frac{1}{\pi\left(1+(x-\theta)^2\right)} ; -\infty < x < \infty$$

$\rightarrow$ $loca^n$ parameter

X be a single obs^n

Fix, $\theta > \theta'$, $H_0 : \theta = \theta'$, $H_1 : \theta > \theta' (\theta'')$

$$R(x) = \frac{f_\theta(x)}{f_{\theta'}(x)} = \frac{1+(x-\theta')^2}{1+(x-\theta)^2}$$

$y > x$, $$R(y) = \frac{1+(y-\theta')^2}{1+(y-\theta)^2}$$

$$x \to \pm\infty, R(x) = \frac{+\infty}{+\infty}$$

Apply L' Hospital Rule,

$$\frac{2(x-\theta')}{2(x-\theta)} = 1$$

$$\therefore \lim_{x \to \pm\infty} R(x) = 1$$

$\therefore R(x)$ is a constant when $x \to \pm\infty$

Cauchy(θ) ~~with~~ does not possess MLR property.

① Propose a test $H_0 : \theta = \theta_0$ ag. $H_1 : \theta > \theta_0$ for $C(\theta)$

$\rightarrow$ Cauchy does not possess MLR.

$\therefore$ It does not have UMP

$\therefore$ we can propose a LMP.

Ex $C(0, \sigma)$ $\longrightarrow$ scale parameter

$H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 > \sigma_0^2$

$$f(x) = \frac{\sigma^2}{\pi(\sigma^2 + x^2)}$$

$$R(x) = \frac{f_{H_1}}{f_{H_0}} = \frac{\sigma_1^2}{\sigma_0^2} \frac{\sigma_0^2 + x^2}{\sigma_1^2 + x^2} \; ; \; \sigma_1^2 > \sigma_0^2$$

$$= \frac{\sigma_1^2}{\sigma_0^2}\left(1 + \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + x^2}\right)$$

Fixing $x < y$

$$R(y) = \frac{\sigma_1^2}{\sigma_0^2}\left(1 + \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + y^2}\right)$$

$\Rightarrow x^2 < y^2$

$\Rightarrow \sigma_1^2 + x^2 < \sigma_1^2 + y^2$

$$\Rightarrow \frac{1}{\sigma_1^2 + x^2} > \frac{1}{\sigma_1^2 + y^2}$$

$$\Rightarrow \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + x^2} < \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + y^2} \quad [\because \sigma_0^2 < \sigma_1^2]$$

$$\Rightarrow 1 + \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + x^2} < 1 + \frac{\sigma_0^2 - \sigma_1^2}{\sigma_1^2 + y^2}$$

$\Rightarrow R(x) < R(y)$

$\therefore$ Cau$(0, \sigma^2)$ has MLR in X.

Ex Show that $U(\theta, \theta+1)$ has MLR. Suppose,

$H_0: \theta = \theta_1$
$H_1: \theta > \theta_1$
$(\theta = \theta_2 > \theta_1)$

$$f(x) = \begin{cases} 1 \; ; & \theta < X < \theta + 1 \\ 0 \; ; & \text{o.w} \end{cases}$$

$$f_{\theta_1} = \begin{cases} 1 \; ; & \theta_1 < X < \theta_1 + 1 \\ 0 & \text{o.w} \end{cases}$$

$$f_{\theta_2} = \begin{cases} 1 \; ; & \theta_2 < X < \theta_2 + 1 \\ 0 & \text{o.w} \end{cases}$$

<u>Case-I</u> $\theta_1+1<\theta_2$

$\theta_1 \quad \theta_1+1 \quad \theta_2 \quad \theta_2+1$

$$R(x)=\begin{cases} \frac{f_{\theta_2}}{f_{\theta_1}}=\frac{0}{1}=0; & \theta_1<x<\theta_1+1 \\ =\frac{0}{0} \text{ undefined}; & \theta_1+1<x<\theta_2 \\ =1/0=\infty, & \theta_2<x<\theta_2+1 \end{cases}$$

<u>Case-II</u> $\theta_2<\theta_1+1$

$\theta_1 \quad \theta_2 \quad \theta_2+1 \quad \theta_2+1$

$$R(x)=\begin{cases} \frac{f_{\theta_2}}{f_{\theta_1}}=\frac{0}{1}=0; & \theta_1<x<\theta_2 \\ =1/1=1; & \theta_2<x<\theta_1+1 \\ =1/0=\infty; & \theta_1+1<x<\theta_2+1 \end{cases}$$

$R(x)\uparrow x$

$U(\theta,\theta+1)$ has MLR in joint sufficient statistics $\{x_{(1)}, x_{(n)}-1\}$

<u>Prob</u> Construct an UMP for,

$H_0: \theta=\theta_1$

$H_1: \theta>\theta_1$

<u>Ex</u> Let, $x_1, x_2, \dots, x_n$ be a r.s from,

$f(x,\lambda)=2\lambda e^{-\lambda x^2}.x, \; x>0$

for testing, $H_0: \lambda=1$ ag. $H_1: \lambda>1$

An ump test will be of the form,

i) $\sum^n x_i < c_n$ when $c_n<c_{n+1} \forall n$

ii) $\sum_{i=1}^n x_i^2 < d_n$ when $d_n<d_{n+1} \forall n$

iii) $\sum_{i=1} x_i < c_n$ " $c_{n+1}<c_n \forall n$

iv) $\sum x_i^2 < d_n$ " $d_n>d_{n+1} \forall n$

Joint pdf

$$f(\underset{\sim}{x}) = (2\lambda)^n \prod_{i=1}^{n} x_i \, e^{-\lambda \sum_{i=1}^{n} x_i^2}$$

$a(\lambda) = (2\lambda)^n$ $\quad c(\lambda) = -\lambda$

$h(x) = \prod_{i=1}^{n} x_i$ $\quad T(x) = \sum_{i=1}^{n} x_i^2$

$$\phi(x) = \begin{cases} 1 & \sum_{i=1}^{n} x_i^2 < d_n \\ 0 & \end{cases}$$

Let, $x_1, x_2, \dots, x_{n+1}$

$$\frac{f_{H_0}(x_1, x_2, \dots, x_{n+1})}{f_{H_0}(x_1, x_2, \dots x_n)} = \frac{(2\lambda)^{n+1} e^{-\lambda \sum_{i=1}^{n+1} x_i^2} \prod_{i=1}^{n+1} x_i}{(2\lambda)^n e^{-\lambda \sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i}$$

$$\cancel{= 2\lambda e^{-\lambda(\sum x_i^2 - \sum x_i)}}$$

$$= 2\lambda e^{-\lambda x_{n+1}^2} x_{n+1}$$

$$= 2e^{-\lambda x_{n+1}^2} x_{n+1}$$

H.W Let, $x_1, x_2, \dots, x_n$ be iid. $f(x;\theta) = \theta x^{\theta-1}; 0 < x < 1$
The set, $\{(x_1, x_2, \dots, x_n) : \sum \log x_i \geq c\}$ (where c is a suitably choosen real numbers) is the MP region for testing.

i) $H_0 : \theta = 1$ against $H_1 : \theta > 1$

ii) $H_0 : \theta = 1$ " $H_1 : \theta > 4$

iii) $H_0 : \theta = 4$ " $H_1 : \theta \leq 1$

iv) $H_0 : \theta = 4$ " $H_1 : \theta \leq 4$

Find c

find $\alpha = 0.05$

$P_{H_0}\left[\sum_{i=1}^{n} \log x_i > c\right]$

$$\Rightarrow f(\underset{\sim}{x}) = \theta^n \prod_{i=1}^{n} x^{\theta-1}$$

$$= \theta^n e^{(\theta-1)\sum_{i=1}^{n} \log x_i}$$

$a(\theta) = \theta^n$

$c(\theta) = (\theta - 1)$

$T(x) = \sum_{i=1}^{n} \log x_i$

MP test,

$$\phi(x) = \begin{cases} 1 & ; \sum_{i=1}^{n} \log x_i > c \\ 0 & ; \text{o.w} \end{cases}$$

This family possesses MLR in $\sum_{i=1}^{n} \log x_i$

Practical

Case-I

$$\phi(x) = \begin{cases} 1 & ; x = 0 \\ 0 & ; x = 1, 2, 3 \end{cases}$$

Power,

$E_{H_1}[\phi(x)]$

$= 1 \cdot P_{H_1}[x = 0] = \theta$

$\alpha = 0.05$

Under H_1; $\theta > 0.05$

Power > size

This is an UMP test

Case - II

$$\phi(x) = \begin{cases} 0.5 & ; x = 1 \\ 0 & ; x = 0, 2, 3 \end{cases}$$

Power,

$E_{H_1}[\phi(x)]$

$= 0.5 \; P_{H_1}[X = 1] = 0.5 \times 2\theta$

$= \theta$

Under H_1 ; $\theta > 0.05 (\alpha)$

Power > size

This is an UMP test

Case - III

$$\phi(x) = \begin{cases} 1 & x = 3 \\ 0 & \text{o.w} \end{cases}$$

Power,

$E_{H_1}[\phi(x)]$

$= 1 \cdot P_{H_1}[X = 3)$

$= 0.1 - \theta$

Under H_1; $\theta > 0.05 (\alpha)$

$0.1 - \theta < 0.05$

Under H_1 Power $\not>$ size

The test is not an UMP test.

Case-2 $\phi(x) = \begin{cases} 1 & x=1 \\ 0 & o.w \end{cases}$

$\alpha = 0.05$

$E_{H_1}[\phi(x)]$

$= P_{H_1}[X=1] = 2\theta$

Under H_1; $\theta > 0.05$

∴ Under H_1, Power > size
∴ This is a UMP test

H.W

$$\frac{f_{H_1}}{f_{H_0}} = \left(\frac{\theta_1}{\theta_0}\right)^n \prod_{i=1}^{n} x_i^{\theta_1-\theta_0} > k$$

$$\Rightarrow (\theta_1-\theta_0)\sum_{i=1}^{n} \log x_i > c$$

For, $H_0: \theta = 1$, $H_1: \theta > 1$

$(\theta_1 - \theta_0) > 0 \Rightarrow \Sigma \log x_i \geq c$

ii) gives MP region

$\Sigma \log x_i \geq c$

iii) and (iv) → does not give the MP region.

Generalized Neyman-Pearson lemma:

Statement: Suppose $f_1, \ldots, f_{m+1}$ be non-negative integral fn of $x_1, x_2, \ldots, x_n$ in some σ-measurable field. Denote a convex set by M →

$$M = \left\{\left(\int_{\mathbb{R}} \phi f_1 d\sigma, \ldots, \int_{\mathbb{R}} \phi f_m d\sigma\right) : \phi \text{ is a test fn}\right\}$$

For fixed constants $c_1, c_2, \ldots, c_m$ let a set be denoted by

$$\mathcal{C}: \left\{\phi : \int_{\mathbb{R}} \phi f_j d\sigma = c_j, 1 \leq j \leq m\right\} \quad \ldots(1)$$

Consider a problem as follows,

$$\max_{\phi} \int \phi f_{m+1} \, d\sigma \text{ subjected to, } \int \phi f_j d\sigma = c_j; \quad 1 \le j \le m$$

The test fn. which solves this maximizing problem is,

$$\phi(x) = \begin{cases} 1 & \text{if } f_{m+1}(\underset{\sim}{x}) > \sum_{j=1}^{m} k_j f_j(\underset{\sim}{x}) \\ \gamma & \text{if } f_{m+1}(\underset{\sim}{x}) = \sum_{j=1}^{m} k_j f_j(\underset{\sim}{x}) \\ 0 & \text{if } f_{m+1}(\underset{\sim}{x}) < \sum_{j=1}^{m} k_j f_j(\underset{\sim}{x}) \end{cases}$$

For constants $k_1, k_2, \dots, k_m$ to be determined by (1).

Such a ϕ also solves the maximization problem when $\mathcal{C}_0$ is replaced by a greater set.

$$\mathcal{C}_0 : \left\{ \phi : \int \phi f_j d\sigma \le c_j ; 1 \le j \le m \right\}$$

Utility

① By generalized N-P lemma, any format of null and alt. hypotheses can be tested.

② Moreover it gives an overall idea about ump test construction (through some linear fn. on distn. fn.s)

③ Moreover one can ~~tested~~ ~~the~~ extend the size space (1) by replacing '=' by '≤'. In essence, we have liberty to extend the size space if required.

Theorem : single parameter

For exponential family $f_\theta(x) = a(\theta)h(x)e^{c(\theta)g(x)}$,

We are to test, $H_0: \theta \leq \theta_1$ or $\theta \geq \theta_2$

ag. $H_1: \theta_1 < \theta < \theta_2$

The UMP test for the above problem is,

$$\phi(x) = \begin{cases} 1 & \text{if } T(\underset{\sim}{x}) \in (c_1, c_2) \\ \gamma_1 & \text{if } T(\underset{\sim}{x}) = c_1 \\ \gamma_2 & \text{if } T(\underset{\sim}{x}) = c_2 \\ 0 & \text{if } T(\underset{\sim}{x}) < c_1 \text{ or } T(\underset{\sim}{x}) > c_2 \end{cases}$$

Where $T(\underset{\sim}{x})$ being the sufficient statistic for θ and $c_1, c_2, \gamma_1, \gamma_2$ are to be determined from,

$$E_{\theta_1}(\phi(\underset{\sim}{x})) = \alpha = E_{\theta_2}(\phi(\underset{\sim}{x}))$$

Proof» PART A

Let $X_1, X_2, \ldots, X_n$ be a r.s from $f_\theta(\underset{\sim}{x}) = a(\theta)h(\underset{\sim}{x})e^{c(\theta)g(x)} d\mu$

~~joint dist~~ joint prob. density fn, $f_\theta(\underset{\sim}{x}) = a(\theta)h(\underset{\sim}{x})e^{c(\theta)g(\underset{\sim}{x})} \prod_{i=1}^{n} d\mu$

[WLG $a(\theta)$, $c(\theta)$ remain the same]

Given any test ϕ based on $\underset{\sim}{X} = (x_1, x_2, \ldots x_n)$ we always construct a test function based on the sufficient statistic $T(\underset{\sim}{x})$ of θ. Therefore for test construction, the above joint p.d.f can be rewritten in terms of suff. statistic.

$$f_\theta(T(\underset{\sim}{x})) = a'(\theta)g'(t)e^{c'(\theta)T(\underset{\sim}{x})} d\sigma$$

where, a', g', c' are the new coeff. w.r.t measure σ.

Moreover, $f_\theta(T(\underset{\sim}{x}))$ may be further remodified as,

$$f_\theta(T(\underset{\sim}{x})) = a'(\theta)e^{c'(\theta)T(\underset{\sim}{x})} d\eta$$

We know that for exponential family MLR exists in $T(\underset{\sim}{x})$ iff $c'(\theta) \uparrow_\theta$

Fix $\theta \in (\theta_1, \theta_2)$

Under this set up, the test $H_1: \theta_1 < \theta < \theta_2$ is a maximization problem as follows,

$$\underset{\phi}{\text{Max}}\ E_\theta[\phi] \text{ subject to constraints}$$

$$E_{\theta_1}[\phi(t)] = E_{\theta_2}[\phi(t)] = \alpha$$

Using the N-P generalized lemma, the ϕ which maximizes the above is of the form,

$$\phi(t) = \begin{cases} 1 & f_\theta(t) > k_1 f_{\theta_1}(t) + k_2 f_{\theta_2}(t) \\ 0 & \text{o.w} \end{cases} \quad \cdots (2)$$

The randomization points γ_1, γ_2 are of no use as exponential family is a cont. one. k_1, k_2 are the constant to be determined from the size cond^n. ①.

$$\equiv \phi(t) = \begin{cases} 1 & \text{if } a'(\theta) e^{c'(\theta) t} > k_1 a'(\theta_1) e^{c'(\theta_1) t} + k_2 a'(\theta_2) e^{c'(\theta_2) t} \\ 0 & < \end{cases} \quad [T(x) = t]$$

$$\Rightarrow \phi(t) = \begin{cases} 1 & \text{if } k_1 \frac{a'(\theta_1)}{a'(\theta)} e^{[c'(\theta_1) - c'(\theta)] t} + \frac{k_2 a'(\theta_2)}{a'(\theta)} e^{[c'(\theta_2) - c'(\theta)] t} < 1 \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow \phi(t) = \begin{cases} 1 & \text{if } a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1 \\ 0 & \text{o.w} \end{cases}$$

where, $a_1 = \frac{k_1 a'(\theta_1)}{a'(\theta)}$, $a_2 = \frac{k_2 a'(\theta_2)}{a'(\theta)}$

$b_1 = c'(\theta_1) - c'(\theta)$, $b_2 = c'(\theta_2) - c'(\theta)$

Due to increasing property of $c'(\theta) \uparrow_\theta$ - $b_1 < 0$, $b_2 > 0$

Next we will show $a_1 > 0$, $a_2 > 0$ by trial and error method.

Case-I Assume, $a_1 < 0$, $a_2 < 0$

$a_1 e^{b_1 t} < 0$, $a_2 e^{b_2 t} < 0$

$\therefore a_1 e^{b_1 t} + a_2 e^{b_2 t} < 0 \quad \forall t$

Always $\Phi(t) = 1 \Rightarrow$ The prob. of accepting a null hypothesis is zero always which indicates acceptance region is null set.

As $E_{\theta_1}[\Phi(t)] = E_{\theta_2}[\Phi(t)] = \alpha < 1$

The above is impossible.

Case-II

$a_1 < 0$, $a_2 > 0$

$e^{b_1 t} \downarrow \quad a_1 e^{b_1 t} \uparrow$

$a_2 > 0$, $b_2 > 0 \quad a_2 e^{b_2 t} \uparrow$

As a whole, $a_1 e^{b_1 t} + a_2 e^{b_2 t} \uparrow$

$\therefore \Phi(x)$ confirms an one sided test

But this is impossible as for $\theta_1 < \theta_2$,

$E_{\theta_1}[\Phi(x)] = E_{\theta_2}[\Phi(x)] = \alpha$

Case II false.

Case III $a_1 > 0$, $a_2 < 0$

Similar as Case II, option is false.

Case IV $a_1 > 0$, $a_2 > 0$, is the right option

$\therefore$ From generalized N-P lemma, for testing, $H_0: \theta < \theta_1$ or $\theta > \theta_2$

ag. $H_1: \theta_1 < \theta < \theta_2$

$$\Phi(t) = \begin{cases} 1 & \text{if } a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1 \\ 0 & \text{o.w} \end{cases}$$

Where both of a_1 and $a_2 > 0$ and

$E_{\theta_1}[\phi(t)] = E_{\theta_2}[\phi(t)]$

$\phi(\cdot)$ is an UMP.

PART B

We'll show within H_0, $\underset{\theta<\theta_1}{E}[\phi(t)] < \alpha$ and $E_{\theta>\theta_2}[\phi(t)] < \alpha$

First consider $\theta < \theta_1$, now the problem is minimize

$$\int \phi f_\theta \, dt ; \cancel{\text{subject}} \ \theta < \theta_1$$

subject to $\int \phi f_{\theta_1} dt = \int \phi f_{\theta_2} dt = \alpha$

By twisting generalized N-P lemma, the soln of the above optimization problem will be a ϕ which is

$$\phi(t) = \begin{cases} 1 & \text{if } f_\theta(t) < k_1 f_{\theta_1}(t) + k_2 f_{\theta_2}(t) \\ 0 & \text{o.w} \end{cases}$$

Let $A = \{t : \phi(t) = 1\}$ be a set of obsn

$$= \left\{ a'(\theta) e^{c'(\theta)t} < k_1 a'(\theta_1) e^{c'(\theta_1)t} + k_2 a'(\theta_2) e^{c'(\theta_2)t} \right\}$$

$$= \left\{ t : a_1^* e^{b_1^* t} < k_1 + a_2^* k_2 e^{b_2^* t} \right\}$$

Where, $a_1^* = \dfrac{a'(\theta)}{a'(\theta_1)}$, $a_2^* = \dfrac{a'(\theta_2)}{a'(\theta_1)}$

$$b_2^* = c'(\theta_2) - c'(\theta_1)$$

$$b_1^* = c'(\theta) - c'(\theta_1)$$

Surely $b_2^* > 0$, $b_1^* < 0$

and, $a_1^* > 0$ and $a_2^* > 0$.

To fix the sign of k_1 and k_2 we use trial and error method.

Suppose $k_1 < 0$ and $k_2 < 0$

Then, $k_1 + a_2^* k_2 e^{b_2^* t} < 0$

where as, $a_1^* e^{b_1^* t} > 0$

$\therefore A$ is a null set.

$\Rightarrow \phi(t) = 0 \quad \forall t$

$\Rightarrow E_{\theta < \theta_1}[\phi(t)] = 0$

It contradicts $E_{\theta_1}[\phi(t)] = \alpha > 0$

$\therefore k_1 \nless 0,\ k_2 \nless 0$

Next, $k_1 > 0$, $k_2 > 0$

$k_1 + a_2^* k_2 e^{b_2^* t} \uparrow^{\infty}$ whereas, $a_1^* e^{b_1^* t} \downarrow$

This means A is an one sided bounded set $A : [c, \infty)$

~~It is a~~ But it is a contradiction.

Cause, $E_{\theta_1}[\phi(t)] = E_{\theta_2}[\phi(t)] = \alpha$ but it should be $E_{\theta_1}[\phi(t)] < E_{\theta_2}[\phi(t)]$

Finally, $k_1 > 0$ and $k_2 < 0$.

Then the desired set has a critical regim of the form,

$$A = \{t : a_1^* e^{b_1^* t} < k_1 + k_2 a_2^* e^{b_2^* t}\}$$

$$= \left\{t : \underbrace{\frac{a_1^*}{k_1}}_{a_1^\Delta} e^{b_1^* t} - \underbrace{\frac{k_2 a_2^*}{k_1}}_{a_2^\Delta} e^{b_2^* t} < 1\right\}$$

$$= \{t : a_1^\Delta e^{b_1^* t} + a_2^\Delta e^{b_2^* t} < 1\}$$

Minimization soln. of $\phi(t)$ is,

$$\Rightarrow \phi(t) = \begin{cases} 1 & \text{if } a_1^{\Delta} e^{b_1^* t} + a_2^{\Delta} e^{b_2^* t} < 1 \\ 0 & \text{o.w} \end{cases}$$

which is just the opposite of UMP critical region in the maximization problem.

Q Construct a region for testing $H_0: \theta < \theta_1$ or $\theta > \theta_2$
$H_1: \theta_1 < \theta < \theta_2$
Where minimum power ~~exists~~ is achived for $H_0: \theta < \theta_1$ or $\theta > \theta_2$

Practical Let $X_1, X_2, \ldots, X_5 \sim Poi(\lambda)$
$H_0: \lambda \leq 1$ against $H_1: \lambda > 1$
Construct an UMP test with $\alpha = 0.1$

Ans For a Poisson distn., the parameter $\lambda > 0$ and $\sum_{i=1}^{5} x_i$ is the sufficient statistic for λ.

The joint p.m.f is $f(\underset{\sim}{x}) = \dfrac{e^{-5\lambda} (\lambda)^{\sum_{i=1}^{5} x_i}}{\prod_{i=1}^{5} x_i!}$

$$= \frac{e^{-5\lambda} e^{(\Sigma x_i) \log \lambda}}{\prod_{i=1}^{5} x_i!}$$

As the above is a member of exponential family with $C(\lambda) = \log \lambda$, UMP test construction can be done on MLR statistic $\sum_{i=1}^{5} x_i = t$. The above test in the question may be reformulated in terms of the ~~limit point~~ common limit pt. of intersection b/w the null hypothetical-closure set and Θ_1-closure set.

$\equiv H_0: \lambda = 1$
against, $H_1: \lambda > 1$

As per N-P lemma, UMP will be,

$$\phi(t) = \begin{cases} 1 & \Sigma x_i > c \\ \gamma & \Sigma x_i = c \\ 0 & \Sigma x_i < c \end{cases}$$

c and γ are to be determined from the size condition.

$$E_{H_0}[\phi(t)] = 0.1$$

$t = \Sigma x_i$, $T = \sum_{i=1}^{5} X_i \sim Poi(5\lambda)$

Under H_0, $T \sim Poi(5)$

$$E(\phi(t)) = 0.1$$

$$\Rightarrow 1 \cdot Pr_{H_0}(T > c) + \gamma P_{H_0}(T = c) = 0.1$$

$$\Rightarrow \gamma P_{H_0}(T = c) = P(T \le c) - 0.9$$

Example
Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, 1)$. ~~Test~~ Test.
$H_0: \mu \le \mu_0$ or $\mu \ge \mu_1$ against $H_1: \mu_0 < \mu < \mu_1$

$\Rightarrow$ Normal popn is a member of exponential family. UMP test construc^n is possible in the suff. statistic concerning the parameter.
As here Generalized N-P lemma, UMP test will be of the form of the suff. stat $T = \sum_{i=1}^{n} X_i$ as follows

$$\phi(t) = \begin{cases} 1 & \text{if } c_1 < \sum_{i=1}^{n} x_i < c_2 \\ \gamma_i & \Sigma x_i = c_i \\ 0 & \text{o.w} \end{cases}$$

Since normal is a cont. dist^n, non-randomized test exists as follows,

$$\phi(t) = \begin{cases} 1 & c_1 < \Sigma x_i < c_2 \\ 0 & \text{o.w} \end{cases}$$

c_1, c_2 have to be determined from the size cond^n.

When, $E_{\mu=\mu_0}[\phi(t)] = \alpha$

$$\Rightarrow P_{H_0}[c_1 < \Sigma x_i < c_2] = \alpha$$

$$\Rightarrow P\left(\frac{c_1 - n\mu_0}{\sqrt{n}} < \underbrace{\frac{\Sigma x_i - n\mu_0}{\sqrt{n}}}_{\sim N(0,1)} < \frac{c_2 - n\mu_0}{\sqrt{n}}\right) = \alpha$$

$X_i \sim N(\mu, 1)$
$\sum_{i=1}^{n} X_i \sim N(n\mu, n)$

$$\Rightarrow \alpha = \Phi\left(\frac{c_2 - n\mu_0}{\sqrt{n}}\right) - \Phi\left(\frac{c_1 - n\mu_0}{\sqrt{n}}\right)$$

$$E_{\mu=\mu_1}[\phi(t)] = \alpha$$

$$\Rightarrow \alpha = \Phi\left[\frac{c_2 - n\mu_1}{\sqrt{n}}\right] - \Phi\left(\frac{c_1 - n\mu_1}{\sqrt{n}}\right)$$

Once we have the informa^n on n, μ_1, μ_0 and α we may solve c_1 and c_2.

Practicals

Let $X_1, X_2, \ldots, X_{10}$ be iid $N(\mu, 1)$.

Propose an UMP test for

$H_0 : \mu \le 2$ or, $\mu \ge 3$

ag. $H_1 : 2 < \mu < 3$ with $\alpha = 0.05$

Example

Let, $x_1, x_2, \ldots, x_n$ be a sample of $N(0, \sigma^2)$.

Consider a test $H_0 : \sigma^2 = \sigma_0^2$

ag. $H_1 : \sigma^2 > \sigma_0^2$

$\Rightarrow$ An UMP test $\underset{H_{10}}{}$ $\phi_1(t) = \begin{cases} 1 & T = \Sigma x_i^2 > c_1 \\ 0 & \text{o.w} \end{cases}$

Consider another test, $H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 < \sigma_0^2$.

UMP test; $\underset{H_{20}}{}$ $\phi_2(t) = \begin{cases} 1 & \text{if } \Sigma x_i^2 < c_2 \\ 0 & \text{o.w} \end{cases}$

For, H_{10}, c_1 can be determined from size cond^n, $c_1 = \frac{\sigma_0^2}{n}\chi^2_{\alpha, n}$ $\frac{ns'^2}{\sigma^2} \sim \chi^2_n$ $s'^2 = \Sigma x_i^2$

$$\Rightarrow \frac{n \cdot \Sigma x_i^2}{\sigma_0^2} > \chi^2_{\alpha, n}$$

$$\Rightarrow \Sigma x_i^2 > \frac{\sigma_0^2}{n}\chi^2_{\alpha, n}$$

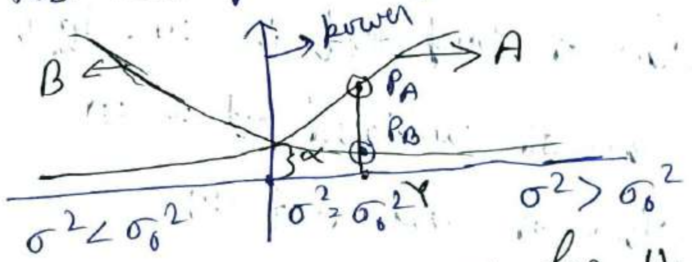
For, H_{20}, $C_2 = \frac{\sigma_0^2}{n}\chi^2_{1-\alpha, n}$

Think about the test, $H_0: \sigma^2 = \sigma_0^2$,
$H_1: \sigma^2 \neq \sigma_0^2$

Clearly, $H_1: H_{10} \cup H_{20}$
$= \{\sigma^2 > \sigma_0^2\} \cup \{\sigma^2 < \sigma_0^2\}$

Intuitively the rejection region of H_1 will be the union of ϕ_1 and $\phi_2 = \{\sum x_i^2 > \frac{\sigma_0^2}{n}\chi^2_{\alpha, n}\} \cup \{\sum x_i^2 < \frac{\sigma_0^2}{n}\chi^2_{1-\alpha, n}\}$

But this is not true as the power curve gives some region where power < size. Every UMP test has power > size.



A: power curve for H_{10}.
B: " " " H_{20}.

When we consider, $H_1: \sigma^2 \neq \sigma_0^2$
What is the power at pt. Y?

Power A and power B

But, $\text{power}_B < \alpha$

We have to impose the condⁿ power > size in this test.

For this test no UMP exist.
We impose unbiasedness condⁿ and then look for UMP test.

Unbiased test (α-similar test / similar test) (A more general optimality criterion to UMP test)

Def^n A size α test ϕ of $H_0: \theta \in \Theta_0$ against $H_1: \theta \in \Theta_1$ is said to be unbiased if $E_\theta[\phi(x)] \geq \alpha, \ \theta \in \Theta_1$

and, $E_\theta[\phi(x)] \leq \alpha, \ \theta \in \Theta_0$

Def^n (UMPU)

Let U_α be the class of all unbiased tests for size α. If $\exists$ a test $\phi \subset U_\alpha$ which has maximum power at $\theta \in \Theta_1$ within that class, ϕ is said to be uniformly most powerful unbiased test (UMPU).

Remark ① $\mathcal{U} \equiv$ Class of ump tests in $\Theta = \Theta_0 \cup \Theta_1$

U_α = Class of unbiased tests in Θ

$\mathcal{X} \equiv$ Class of all test Θ

$$\boxed{\mathcal{U} \subset U_\alpha \subset \mathcal{X}}$$

② Construction of unbiased test.

Unbiased test can be conceptualized from α-similar test. (similar test)

Def^n Let $\Theta = \Theta_0 \cup \Theta_1$

Θ_0 - parametric space under null hyp.

Θ_1 = " " " alt. "

$(\partial H)_0$ = Closure set of Θ_0 (set of all limit pts under H_0)

$(\partial H)_1$ = " " " Θ_1

Consider common boundary or threshold set

$(\partial H)_B = (\partial H)_0 \cap (\partial H)_1$

A test ϕ is called α-similar test with size α if $\beta_\theta(\phi) = E_\theta[\phi(x)] = \alpha$ when $\theta \in \partial\Theta_B$

Theorem Let $\beta_\theta(\phi) = E_\theta[\phi(x)]$ be continuous in θ for any test fun. ϕ. If ϕ is an unbiased size α test of $H_0: \theta \in \Theta_0$ against $H_1: \theta \in \Theta_1$, it is α-similar in $\partial\Theta_B$ and vice-versa.

(α-similar $\Rightarrow$ unbiased $\rightarrow$ omitted (see Lehmann))

proof Suppose a test ϕ is unbiased.

We are to show this test is α-similar.

Let $\theta \in \partial\Theta_B$. Then $\exists$ a sequence $\{\theta_n\}_{n=1}^{\infty}$, $\theta_n \in \Theta_0$ such that $\theta_n \to \theta$, since $\beta_\theta(\phi)$ is ~~continuous~~ continuous.

$$\beta_{\theta_n}(\phi) \to \beta_\theta(\phi)$$

$$\text{Since, } \beta_{\theta_n}(\phi) < \alpha \Rightarrow \beta_\theta(\phi) \leq \alpha \quad \ldots (i)$$

Also, consider a sequence a seq. $\{\theta_n'\} \in \Theta_1$ such that, $\theta_n' \to \theta$. Since $\beta_\theta(\phi)$ is continuous,

$$\beta_{\theta_n'}(\phi) \to \beta_\theta(\phi)$$

Since $\beta_{\theta_n'}(\phi) > \alpha$

$$\Rightarrow \beta_\theta(\phi) \geq \alpha \quad \ldots (ii)$$

Clubbing them together,

$$\beta_\theta(\phi) = \alpha \quad \theta \in \partial\Theta_B$$

<u>Note</u>: If the α-similar test is most powerful. We have α-similar most powerful test or most powerful unbiased test.

Remark

① The continuity of the power curve is not possible always. Then α-similar test does not exist.

② For exponential family power curve is always continuous. Therefore α-similar test always exists for this family.
α-similar may not be unique.

③ Trivial α-similar test.

For every size α test, there always exists an α-similar test, i.e $\phi \equiv \alpha$

Example

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, 1)$.

$H_0: 1 < \mu < 2$

against, $H_1: \mu < 1$ or $\mu > 2$

construct a α-similar test.

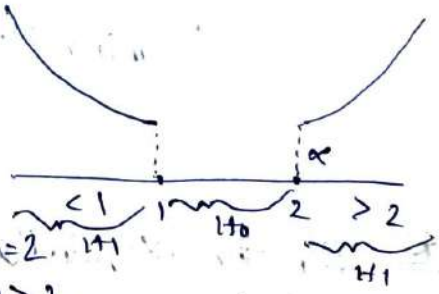
$\partial H_0 = [1, 2]$

$\partial H_1 = (-\infty, 1] \cup [2, \infty)$

$\partial H_0 \cap \partial H_1 = \partial H_B$
$= \{1, 2\}$

A test ϕ is said to be α-similar test if

$$\beta_{\mu=1}(\phi) = \beta_{\mu=2}(\phi) = \alpha$$



$\left[\begin{array}{l} H_0: 1 < \mu < 2 \\ H_1: \mu < 1 \text{ or } \mu > 2 \end{array}\right.$

$\rightarrow \begin{cases} H_{01}: \mu = 1 \\ H_{11}: \mu < 1 \end{cases} \cup \begin{cases} H_{02}: \mu = 2 \\ H_{12}: \mu > 2 \end{cases}$

Result: A UMP test is always α-similar (by default)

Ex

$X_1, X_2, \ldots, X_n \sim N(\mu, 1)$

$H_0: \mu \leq 0$ ag. $H_1: \mu \geq 0$

Since the test is one sided, UMP test exists.

By N-P lemma,

$$\phi(x) = \begin{cases} 1 & \Sigma x_i > k \\ 0 & \text{o.w} \end{cases}$$

Where, $E[\phi(\underline{x})] = \alpha$

Now is it unbiased or α-similar?

$\Theta_0 = (-\infty, 0]$, $\Theta_1 = (0, \infty)$

$\partial\Theta_0 = (-\infty, 0]$, $\partial\Theta_1 = [0, \infty)$

$\partial\Theta_B = \{0\}$

Reformed

$H_0: \mu = 0$

$H_1: \mu > 0$

$$\phi'(x) = \begin{cases} 1 & \text{if } \Sigma x_i > k \\ 0 & \text{o.w} \end{cases} \qquad [\phi \equiv \phi']$$

$E_{H_0}[\phi'(x)] = \alpha$

$\Rightarrow 1 \cdot P_{H_0}(\Sigma x_i > k) = \alpha$

$\Sigma x_i \sim N(n\mu, n)$

Under H_0,

$\Sigma x_i \sim N(0, n)$

$$\Rightarrow P_{H_0}\left(\underbrace{\frac{\Sigma x_i - 0}{\sqrt{n}}}_{Z} > \frac{k}{\sqrt{n}}\right) = \alpha$$

$$\Rightarrow \frac{k}{\sqrt{n}} = \tau_\alpha, \quad \tau \sim N(0,1)$$

$$\Rightarrow k = \sqrt{n}\,\tau_\alpha$$

$$\phi'(x) = \begin{cases} 1 & \text{if } \Sigma x_i > \sqrt{n}\,\tau_\alpha \\ 0 & \text{o.w} \end{cases}$$

$$\boxed{E_{H_1}\left[\phi'(x)\right] > \alpha} \; ? \quad \ldots (1)$$

$$\boxed{E_{H_0}\left(\phi'(x)\right) < \alpha} \; ? \quad \ldots (2)$$

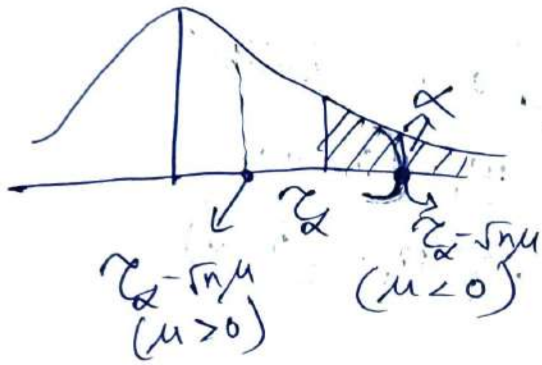
Fix a $\mu < 0$

$$E_{H_0}\left[\phi(x)\right] = 1 \cdot P\left(\Sigma x_i > \sqrt{n}\cdot\tau_\alpha\right)$$

$\Sigma x_i \sim N(n\mu, n)$, $\mu < 0$

$$= P\left(\underbrace{\frac{\Sigma x_i - n\mu}{\sqrt{n}}}_{\tau} > \frac{\sqrt{n}\,\tau_\alpha - n\mu}{\sqrt{n}}\right)$$

$$= P\left(\tau > \tau_\alpha - \sqrt{n}\,\mu\right)$$



$\tau_\alpha - \sqrt{n}\mu$ lies right to τ_α when $\mu < 0$.

$$\therefore E_{H_0}\left[\phi(x)\right] < \alpha$$

Combining (1) and (2) unbiasedness establishes..

$\therefore$ A UMP test is always unbiased.

Theorem:

For exponential family, a UMP test for $H_0: \theta_1 \le \theta \le \theta_2$ against $H_1: \theta < \theta_1$ or $\theta > \theta_2$ does not exist.

A UMP unbiased test can be constructed as below.

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma_1 & \text{if } T(x) = c_1 \\ \gamma_2 & \text{if } T(x) = c_2 \\ 0 & c_1 < T(x) < c_2 \end{cases} \quad (A)$$

<Where $c_1, c_2, \gamma_1, \gamma_2$ are determined from

$E_{\theta_1}\left(\phi(x)\right) = E_{\theta_2}\left(\phi(x)\right) = \alpha$>

Proof ⇒ We consider the above problem as the problem of maximization $E_{\theta'}[\phi(x)]$ for θ outside the interval (θ_1, θ_2), subject to the constraints $E_{\theta_1}[\phi(x)] = E_{\theta_2}[\phi(x)] = \alpha$. These two conditions give the alpha-similarity condition as $\theta_1, \theta_2 \in (\partial H)_B = (\partial H_0) \cap (\partial H_1)$

If the problem is restated in terms of $1-\phi(x)$, it follows that the above theorem is same as A maximizaⁿ problem for.

$$H_0': \theta < \theta_1 \text{ or } \theta > \theta_2$$

$$\text{ag. } H_1': \theta_1 < \theta < \theta_2$$

Borrowing the idea of the previous theorem (already discussed) Solⁿ (A) (structure of ϕ) would give the minimum power.

$\therefore$ For, $H_0: \theta_1 < \theta < \theta_2$

$H_1: \theta < \theta_1$ or $\theta > \theta_2$

ϕ in (A) will give maximum power.

unbiased condition comes from the size condition.

$\therefore$ UMPU exists.

Result:

Consider the exponential family with a single parameter θ. Consider a fixed $\theta_0 \in (H)$ and

$$H_0: \theta = \theta_0 \text{ against } H_1: \theta \neq \theta_0.$$

where $T(x)$ being the suff. stat. for θ.

No UMP test exists. A UMPU exists with form,

$$\phi(T(x)) = \begin{cases} 1 & \text{if } T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma_1 & T(x) = c_1 \\ \gamma_2 & T(x) = c_2 \\ 0 & \text{o.w} \end{cases}$$

where, $c_1, c_2, \gamma_1, \gamma_2$ are to be determined from i) $E_{\theta_0}[\phi(T(x))] = \alpha$

ii) $E_{\theta_0}[T \cdot \phi(T(x))] = \alpha \cdot E_{\theta_0}[T(x)]$

proof» The exponential family distⁿ fuⁿ,

$$f_\theta(x) = a(\theta)\, h(x)\, e^{C(\theta)T(x)} \text{ w.r.t some measure } \mu$$

Renaming all the variables, we reduce the format to $f_\theta(t) = a(\theta) e^{\theta t}$ w.r.t a new measure η.

Next note that, $E_{\theta_0}(\phi(t)) = \alpha$ and since the test is unbiased. $E_\theta[\phi(t)] = \beta_\theta(\phi) \geq \alpha$, $\theta \in \Theta_1$

which implies $\beta_\theta(\phi)$ has a minimum at $\theta = \theta_0$

$\beta_\theta(\phi)$ is cont.

$$\therefore \beta_\theta(\phi) = \int_{\mathbb{R}} \phi(t)\, a(\theta) e^{\theta t}\, d\eta$$

$$\Rightarrow \frac{d}{d\theta}\beta_\theta(\phi) = \frac{d}{d\theta}\int_{\mathbb{R}} \phi(t)\, a(\theta)\, e^{\theta t}\, d\eta$$

$\Rightarrow$ using bounded convergence theorem,

$$\beta_\theta'(\phi) = \int \frac{d}{d\theta}\left[\phi(t)\, a(\theta)\, e^{\theta t}\right] d\eta$$

$$\Rightarrow \beta_\theta'(\phi) = \int \left[\phi(t)\, a'(\theta)\, e^{\theta t} + \phi(t)\, a(\theta) . t . e^{\theta t}\right] d\eta$$

$$\left[a'(\theta) = \frac{d}{d\theta} a(\theta)\right]$$

$$\Rightarrow \frac{\beta_\theta'(\phi)}{a(\theta)} = \frac{a'(\theta)}{a(\theta)} \int \phi(t) e^{\theta t}\, d\eta + \int \phi(t) \cdot t\, e^{\theta(t)}\, d\eta$$

$$\Rightarrow \frac{\beta_\theta'(\phi)}{a(\theta)} = \frac{a'(\theta)}{a(\theta)} E[\phi(t)] + E[t \cdot \phi(t)]$$

Minimizing $\beta_\theta(\phi)$ is analogous of minimizing

$$\frac{\beta_\theta[\phi]}{a(\theta)}.$$

This above is true for any test ϕ.

Take a trivial test $\phi \equiv \alpha$

$$\beta'_\theta(\alpha) = 0$$

$$\Rightarrow 0 = \frac{a'(\theta)}{a(\theta)} \cdot \alpha + \alpha E(t)$$

$$\Rightarrow \frac{a'(\theta)}{a(\theta)} = -E_\theta(t)$$

$$\therefore \frac{\beta'_\theta(\phi)}{a(\theta)} = -E_\theta(t)\, E_\theta[\phi(t)] + E_\theta[t \cdot \phi(t)]$$

At $\theta = \theta_0$ minimum power occurs.

$$\frac{\beta'_{\theta_0}(\phi)}{a(\theta)} = -E_{\theta_0}(t) \cdot E_{\theta_0}[\phi(t)] + E_{\theta_0}[t \cdot \phi(t)]$$

$$\Rightarrow 0 = -E_{\theta_0}(t)\, \underbrace{E_{\theta_0}[\phi(t)]}_{\alpha} + E_{\theta_0}[t \cdot \phi(t)]$$

$$\Rightarrow \alpha E_{\theta_0}(t) = E_{\theta_0}[t \cdot \phi(t)]$$

The above is the 2nd constraint, $H_0: \theta = \theta_0$ ag. $H_1: \theta \neq \theta_0$

Now, for $\theta \neq \theta_0$ consider maximization problem $\max \int \phi f_\theta(t) dx$ sub to $\int \phi f_{\theta_0}(t) dx = \alpha$ and $\int \phi\, t\, f_{\theta_0}(t) dx = \alpha \int t \cdot f_{\theta_0}(t) dx$

~~Using~~ Using generalized N-P lemma

$$\phi(t) = \begin{cases} 1 & \text{if } a(\theta) e^{\theta t} > k_1 a(\theta_0) e^{\theta_0 t} + k_2 t \cdot a(\theta_0) e^{\theta_0 t} \\ \gamma_1 & \\ \gamma_2 & \\ 0 & < \end{cases}$$

The rejection region (where $\phi = 1$) is of the following form $a_1 + a_2 t < e^{bt}$ [By dividing both sides $a(\theta_0) e^{\theta_0 t}$]

$$\left[\begin{array}{l} \dfrac{a(\theta)\, e^{\theta t}}{a(\theta_0)\, e^{\theta_0 t}} > \dfrac{k_1\, a(\theta_0) e^{\theta_0 t}}{a(\theta_0) e^{\theta_0 t}} + \dfrac{k_2\, t\, a(\theta_0) e^{\theta_0 t}}{a(\theta_0) e^{\theta_0 t}} \\ \Rightarrow e^{bt} > k_1/a + \dfrac{k_2}{a} t \quad \left[\because \dfrac{a(\theta)}{a(\theta_0)} = a\right] \end{array}\right]$$

Now sign of a_1, a_2 and b

Case-I : $b > 0$, then if $a_2 < 0$ the set $A : a_1 + a_2 t < e^{bt}$, will be of the form (x_0, ∞) which indicates $\phi(t)$ is an one sided test, i.e power curve is monotonically increasing:

$$\beta_\theta(\phi) > \alpha, \quad \theta > \theta_0$$

$$\beta_\theta(\phi) < \alpha, \quad \theta < \theta_0$$

Which is a contradiction because of the unbiasedness condition.

$\therefore b > 0, \; a_2 > 0$

Case II : $b < 0$; similar argument establishes $a_2 < 0$

The rejection region $(\phi(t) = 1)$ would be in $(-\infty, x_0') \cup (x_0, \infty)$

Thus ϕ is a UMPU test subject to the condition,

$$E_{\theta_0}[\phi(t)] = \alpha$$

and, $E_{\theta_0}[T \cdot \phi(t)] = \alpha \cdot E_{H_0}(T)$

$$\phi(x) = \begin{cases} 1 & \text{if } T < x_0' \text{ or } T > x_0 \\ \gamma_1 & T = x_0' \\ \gamma_2 & T = x_0 \\ 0 & x_0' < T < x_0 \end{cases}$$

Practical 6

Refer to the problem no. 5.
Do you believe there exist any UMP test, if you shuffle H_0 and H_1.
If so ~~write the~~ propose another alt. test by taking $\alpha = 0.05$.

5) $H_0: 2 \le \mu \le 3$

ag. $H_1: \mu < 2$ or $\mu > 3$

We know that, in that case for exponential family a UMP test does not exist.

A UMP unbiased test can be constructed.

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) < c_1 \text{ or } T(x) > c_2 \\ 0 & \text{o.w} \end{cases}$$

$X_i \overset{iid}{\sim} N(\mu, 1) \ \forall\ i = 1(1)n$

$$\therefore f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2}$$

$\therefore$ Joint pdf will be,

$$\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-n\mu^2/2 + \mu\sum_{i=1}^{n} x_i - \frac{1}{2}\sum_{i=1}^{n} x_i^2}$$

$\therefore$ Here, $T(\underline{x}) = \sum_{i=1}^{n} x_i$

$$T(\underline{x}) = \sum_{i=1}^{n} x_i \sim N(n\mu, n)$$

~~0.5160~~
~~-0.5040~~
~~21.6317~~
~~18.4062~~
~~21.6317~~
~~28.4062~~

Here, $n = 10$ $\quad \therefore E_{\mu=2}[\phi(x)] = 0.05$ $\quad \cancel{E_{\mu=3}[\phi(x)] = 0.05}$

$\therefore T(\underline{x}) \sim N(10\mu, 10)$ $\Rightarrow P[T(\underline{x}) < c_1] + P[T(\underline{x}) > c_2] = 0.05$

$$\Rightarrow P\left[\frac{T(\underline{x}) - 20}{\sqrt{10}} < \frac{c_1 - 20}{\sqrt{10}}\right] + P\left[\frac{T(\underline{x}) - 20}{\sqrt{10}} > \frac{c_2 - 20}{\sqrt{10}}\right] = 0.05$$

$$\Rightarrow \Phi\left(\frac{c_1-20}{\sqrt{10}}\right)+1-\Phi\left(\frac{c_2-20}{\sqrt{10}}\right)=0.05$$

$$\Rightarrow \Phi\left(\frac{c_2-20}{\sqrt{10}}\right)-\Phi\left(\frac{c_1-20}{\sqrt{10}}\right)=0.95$$

Similary,

$$\Phi\left(\frac{c_2-30}{\sqrt{10}}\right)-\Phi\left(\frac{c_1-30}{\sqrt{10}}\right)=0.95$$

Result for symmetric distⁿ.

A simplification of the test $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$, when the prob. distⁿ of sufficient statistic T is symmetric around some pt. a, i.e

$$P_r(T < a-u) = P_r(T > a+u) \quad \forall u$$

Any test ϕ which is constructed on T and satisfies $E_{\theta_0}[\phi] = \alpha$ (size condⁿ) must satisfy the other condⁿ $\langle E_{\theta_0}[T.\phi] = \alpha E_{\theta_0}(T) \rangle$ by default.

(In case of symmetric distⁿ the ~~...~~ 2nd condition is obvious. why?)

$$\Rightarrow \text{since, } E_{\theta_0}(T.\phi) = \underbrace{E_{\theta_0}((T-a)\phi)}_{0} + a E_{\theta_0}[\phi]$$

$$= 0 + a\alpha$$

Therefore, c's and γ's are determined by the property of symmetry around a.

$$\text{Prob}\{T < c_1\} + \gamma_1 \Pr\{T = c_1\} = \alpha/2$$

$$\text{Prob}\{T > c_2\} + \gamma_2 \Pr\{T = c_2\} = \alpha/2$$

Prob $\{a-T$

$$Pr\{T-a < c_1 - a\} + \gamma_1 \cdot Pr\{T-a = c_1 - a\} = \alpha/2$$

and, $Prob\{T - a > c_2 - a\} + \gamma_2\, Pr\{T - a = c_2 - a\} = \alpha/2$

$$\therefore c_1 - a = -(c_2 - a)$$
$$\Rightarrow c_1 - a = -c_2 + a$$
$$\Rightarrow \boxed{c_1 = 2a - c_2}$$
and, $\boxed{\gamma_1 = \gamma_2}$

For symmetric dist^n

$$\phi = \begin{cases} 1 & \text{if } T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma_1 = \gamma_2 & T(x) = c_1 \\ \gamma_2 & T(x) = c_2 \\ 0 & c_1 < T(x) < c_2 \end{cases}$$

($c_1 = 2a - c_2$)

Examples

Let $\underset{\sim}{x}$ be the random sample of size n from $N(\mu, \sigma^2)$. Test, $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 \neq \sigma_0^2$ } μ known

Joint dist^n $f(\underset{\sim}{x}) = \dfrac{1}{(2\pi\,\sigma^2)^{n/2}}\, e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$

μ is known, $x_i' = x_i - \mu$

$$f(\underset{\sim}{x}') = \frac{1}{(2\pi\,\sigma^2)^{n/2}}\, e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i'^2}{\sigma^2}}$$

$T = \sum_{i=1}^{n} x_i'^2$ is suff. stat for σ^2

$$\frac{nT}{\sigma^2} \sim \chi^2_n$$

$f\left(\frac{nT}{\sigma^2}\right) =$ p.d.f of χ^2_n

$$= \frac{e^{-nT/2\sigma^2}}{2^{n/2}\,\Gamma\frac{n}{2}} \cdot \left(\frac{nT}{2\sigma^2}\right)^{n/2-1}$$

$$= c \cdot e^{-nT/2\sigma^2} \left(T/\sigma^2\right)^{n/2 - 1}$$

The UMPU for the above test will be,

$$\phi(T) = \begin{cases} 1 & \frac{nT}{\sigma^2} < c_1 \text{ or } \frac{nT}{\sigma^2} > c_2 \\ 0 & \text{o.w} \end{cases}$$

Where, $E_{\sigma_0^2}[\phi(T)] = \alpha$

and, $E_{\sigma_0^2}[T \cdot \phi(T)] = \alpha \cdot E_{\sigma_0^2}(T)$

We solve the pair of conditions for determining c_1 and c_2

Consider the acceptance region, $c_1 < \frac{nT}{\sigma^2} < c_2$

① $\int_{c_1}^{c_2} c\, e^{-nT/2\sigma_0^2} \left(T/\sigma_0^2\right)^{n/2 - 1} dT = 1 - \alpha$

② $\int_{c_1}^{c_2} \frac{T}{\sigma^2} f\left(\frac{nT}{\sigma^2}\right) dT = (1-\alpha) E\left(\frac{T}{\sigma_0^2}\right)$
$= (1-\alpha)$

$\left[\because E\left(\frac{T}{\sigma_0^2}\right) = 1\right]$

Example

$X \sim Bin(n, p)$

Construct a test for $H_0: p = p_0$ ag. $H_1: p \neq p_0$ on k i.i.d obsns.

Joint p.m.f is $f(\underline{x}, p) = \prod_{i=1}^{k} \binom{n}{x_i} (1-p)^{nk - \sum_{i=1}^{k} x_i} p^{\sum_{i=1}^{k} x_i}$

$= c \cdot \left(\frac{p}{1-p}\right)^{\sum x_i} (1-p)^{nk}$

$= c \cdot e^{\sum x_i \log \frac{p}{1-p}} (1-p)^{nk}$

$\sum_{i=1}^{k} x_i$ is suff. stat for p. As Binomial is a member of exponential family.

UMPU will be constructed on sufficient statistic $\sum_{i=1}^{n} x_i = T \sim Bin(nk, p)$

$$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ \gamma_i & T = c_i \ ; i = 1,2 \\ 0 & \text{o.w} \end{cases}$$

$c_1, c_2, \gamma_1, \gamma_2$ are determined by:

i) $E_{p_0}[\phi(T)] = \alpha$

ii) $E_{p_0}[T.\phi(T)] = \alpha E_{p_0}(T)$

(i) $Pr_{p_0}[T < c_1 \text{ or } T > c_2] + \gamma_1 Pr_{p_0}[T = c_1] + \gamma_2 Pr_{p_0}[T = c_2] = \alpha$

$\Rightarrow Pr[c_1 < T < c_2] + (1-\gamma_1) Pr[T = c_1] + (1-\gamma_2) Pr[T = c_2] = 1-\alpha$

$$\Rightarrow \sum_{t=c_1+1}^{c_2-1} \binom{nk}{t} p_0^t (1-p_0)^{nk-t} + (1-\gamma_1)\binom{nk}{c_1} p_0^{c_1}(1-p_0)^{nk-c_1} + (1-\gamma_2)\binom{nk}{c_2} p_0^{c_2}(1-p_0)^{nk-c_2}$$

(ii) $\Rightarrow E(T(1-\phi(T))) = (1-\alpha) nkp_0$

$$\Rightarrow \sum_{c_1+1}^{c_2-1} t\binom{nk}{t} p_0^t (1-p_0)^{nk-t} + (1-\gamma_1) c_1 \binom{nk}{c_1} p_0^{c_1}(1-p_0)^{nk-c_1} + (1-\gamma_2) c_2 \binom{nk}{c_2} p_0^{c_2}(1-p_0)^{nk-c_2} = (1-\alpha) nkp_0$$

By further simplification and by trial and error method from Binomial ~~table~~ table, one can find $\gamma_1, \gamma_2, c_1, c_2$.

Test on multiparametric exponential family with nuisance parameter

Structure: For a multiparametric exponential family, the dist^n is of, ~~f(x)~~

$$f(x, \theta, \underset{\sim}{\eta}) = c(\theta, \underset{\sim}{\eta})\, e^{\theta u(x) + \eta_1 T_1(x) + \eta_2 T_2(x) + \dots + \eta_k T_k(x)}$$

where, $(\theta, \eta_1, \eta_2, \dots, \eta_k)$ be the vector of parameters.

θ is the parameter of interest and $\underset{\sim}{\eta} = (\eta_1, \eta_2, \dots, \eta_k)$ is the vector of nuisance parameters.

All of the parameters are unknown.

Therefore, it is the optimization problem defined in a $(k+1)$ parametric space ~~where~~ with k dimensional nuisance parametric space.

Neyman's Structure

$X \sim P_\theta \rightarrow$ prob. dist^n

and T be the suff. stat for θ.

A test ~~fn~~ fu^n ϕ is said to have Neyman structure w.r.t suff. stat T.

if, $E_{\Theta_0}[\phi(x) / T = t] = \underset{\substack{\uparrow \\ \text{size}}}{\alpha}$ a.e t.

Remark 1

Neyman structure gives stronger cond^n than the usual size cond^n.

Remark 2 If a test fu^n ϕ is Neyman structured ϕ will be α-similar.

$$\theta \in \partial \Theta_B,\ E[\phi(x)] = E_T\left[E_{X/T}[\phi(x)/T=t]\right]$$
$$= E_T[\alpha] = \alpha$$

Remark 3
Let X be a r.v with $\mathcal{P}_\theta$ and T be the suff. stat. Then all ~~Neyman Structured~~ α-similar test will be Neyman-structured.

Remark 4.
For multiparametric exponential family, Neyman structure will enable to construct UMPU test.

Construction of Hypothesis:

(1) $H_{01}: \theta \leq \theta_0$ ag. $H_{11}: \theta > \theta_0$

(2) $H_{02}: \theta \leq \theta_1$ or $\theta \geq \theta_2$ ag. $H_{12}: \theta_1 < \theta < \theta_2$

(3) $H_{03}: \theta_1 < \theta < \theta_2$ ag. $H_{13}: \theta \geq \theta_2$ or $\theta \leq \theta_1$

(4) $H_{04}: \theta = \theta_0$ ag. $H_{14}: \theta \neq \theta_0$

Assumption:

(1) The parametric space is convex, for assuming $T_1, T_2, \ldots, T_k$ are the suff. statistics for $\eta_1, \eta_2, \ldots, \eta_k$.

Let, $(U, T_i;\ i=1(1)k)$ be the suff. stat for $(\theta, \eta_1, \eta_2, \ldots, \eta_k)$.

(2) Exponential family is a complete family.

Discussion

We consider the joint density of (U, T_i) as,

$$f_{\theta,\underset{\sim}{\eta}}(u, t) = c^*(\theta, \underset{\sim}{\eta})\, e^{\theta u + \sum_{i=1}^{k} \eta_i t_i} \quad \text{w.r.t measure } \mu.$$

The conditional distn of U given $\underset{\sim}{T} = \underset{\sim}{t}$ is also exponential,

$$f(u/t) = c_\theta(t)\, e^{\theta u} \quad [\text{Here } \underset{\sim}{T} = \underset{\sim}{t} \text{ is fixed}]$$

The above is the expression of single parametric exponential family with parameter θ and suff. stat U.

In this conditional situation, $\exists$ a UMP test, based on fixed point of $T=t$.

① $\Rightarrow$

$$\phi(u) = \begin{cases} 1 & u > c(t) \\ \gamma(t) & u = c(t) \\ 0 & u < c(t) \end{cases}$$

~~Where~~ Where $c(t)$ is the critical point to be determined from Neyman structure; i.e

$$E\left[\phi(u)/\underset{\sim}{T} = \underset{\sim}{t}\right] = \alpha \quad \left[\gamma(t) \text{ from size cond}^n\right]$$

② $\Rightarrow$

$$\phi(u) = \begin{cases} 1 & c_1(t) < u < c_2(t) \\ \gamma_i(t) \quad i=1,2 & u = c_i(t) \\ 0 & \text{o.w} \end{cases}$$

$$E_{\theta_1}\left[\phi(u)/\underset{\sim}{T} = \underset{\sim}{t}\right] = E_{\theta_2}\left[\phi(u)/\underset{\sim}{T} = \underset{\sim}{t}\right] = \alpha$$

④ $\Rightarrow$

$$\left.\phi(u) = \begin{cases} 1 & u < c_1(t) \text{ or } u > c_2(t) \\ \gamma_i(t) & u = c_i(t) \\ 0 & \end{cases}\right\} \text{UMPU test}$$

$$E_{\theta_0}\left[\phi(u)/\underset{\sim}{T} = \underset{\sim}{t}\right] = \alpha$$

$$E_{\theta_0}\left[u.\phi(u)/\underset{\sim}{T} = \underset{\sim}{t}\right] = \alpha.\, E_{\theta_0}\left[u/\underset{\sim}{T} = \underset{\sim}{t}\right]$$

Revisiting normal dist^n test in presence of nuisance parameter:

① Consider ~~the test~~ $H_0: \sigma^2 < \sigma_0^2$, $H_1: \sigma^2 \geq \sigma_0^2$; μ is unknown (nuisance parameter).

Let $x_1, x_2, \ldots, x_n$ be a r.s from $N(\mu, \sigma^2)$

The joint p.d.f,

$$f(\underset{\sim}{x}, \mu, \sigma^2) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= c \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{\mu}{\sigma^2} n\bar{x}}$$

$\theta = -\frac{1}{2\sigma^2}$, $u = \sum_{i=1}^{n} x_i^2$, $\eta = \frac{n\mu}{\sigma^2}$, $T(\underset{\sim}{x}) = \bar{x}$

UMP will be of the form;

$$\phi(u) = \begin{cases} 1 & \Sigma x_i^2 > c(\bar{x}) \\ 0 & \text{o.w} \end{cases} \quad \ldots (A)$$

where, c is chosen in such a way that

$E_{\sigma_0}[\phi(u)/T=t] = \alpha$

$\Rightarrow E_{\sigma_0}[\phi(u)/\bar{x}=t] = \alpha$

Given a value of $\bar{x}$, Σx_i^2 is computationally equivalent to $\sum_{i=1}^{n}(x_i-\bar{x})^2$

But, $\Sigma(x_i-\bar{x})^2$ is ancillary statistic μ while $\bar{x}$ is sufficient stat. for μ.

By Basu's theorem, $\bar{x}$ and $\Sigma(x_i-\bar{x})^2$ are ind.

$$\therefore (A) \equiv \phi(u) = \begin{cases} 1 & \Sigma(x_i-\bar{x})^2 > c(\bar{x}) \\ 0 & \text{o.w} \end{cases}$$

$$\equiv \phi(u) = \begin{cases} 1 & \Sigma(x_i-\bar{x})^2 > c' \\ 0 & \text{o.w} \end{cases}$$

When c' can be determined from,

$E_{\sigma_0^2}[\phi(\Sigma(x_i-\bar{x})^2)] = \alpha$

Remember, $\left\langle \sum_{i=1}^{n}(x_i-\bar{x})^2 \sim \sigma^2 \chi^2_{n-1} \right\rangle$

$\Rightarrow \left\langle P[\sigma_0^2 \chi^2_{n-1} > c'] = \alpha \right\rangle$

② Consider, $H_0 : \mu < \mu_0$ $\quad \sigma^2$ unknown
$H_1 : \mu \geq \mu_0$ $\quad$ nuisance parameter.

$$f(\underset{\sim}{x}) = c \cdot e^{-\frac{1}{2\sigma^2}\Sigma x_i^2 + \frac{n\mu \bar{x}}{\sigma^2}}$$

$$\theta = \frac{n\mu}{\sigma^2}, \quad u = \bar{x}, \quad \eta = -\frac{1}{2\sigma^2}, \quad T = \Sigma x_i^2$$

UMP test exists and will be of the form,

$$\phi(\bar{x}) = \begin{cases} 1 & \bar{x} > c(\Sigma x_i^2) \\ 0 & \text{o.w} \end{cases} \equiv \phi(\bar{x}) = \begin{cases} 1 & \bar{x} > c'(\Sigma(x_i-\bar{x})^2) \\ 0 & \text{o.w} \end{cases}$$

c can be derived by,

$$E\left[\phi(\bar{x}) / \Sigma x_i^2 = t\right] = \alpha$$
$$\equiv E\left[\phi(\bar{x}) / \Sigma(x_i - \bar{x})^2 = t'\right] = \alpha$$

Now, by Basu's theorem, $\bar{x}$ and $\Sigma(x_i - \bar{x})^2$ are independent.

$$\therefore \phi(\bar{x}) = \begin{cases} 1 & \frac{\bar{x}}{\Sigma(x_i-\bar{x})^2} > c'' \\ 0 & \text{o.w} \end{cases}$$

As we know,

$$\left[\frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma^2}}{\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n-1}}} \sim t_{n-1}\right]$$

$$\equiv \phi(\bar{x}) = \begin{cases} 1 & \frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma^2}}{\Sigma(x_i-\bar{x})^2} > c''' \\ 0 & \text{o.w} \end{cases}$$

$$\equiv \phi(\bar{x}) = \begin{cases} 1 & \dfrac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma^2}}{\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n-1}}} > c'''' \\ 0 & \text{o.w} \end{cases}$$

Where, c'''' is determined from,

$$E\left[\phi(\bar{x}) / \Sigma x_i^2\right] = \alpha$$

$$\Rightarrow E\left[\phi(\bar{x}) / \Sigma(x_i-\bar{x})^2\right] = \alpha$$

By Basu's theorem,

$$E\left[\phi(\bar{x})\right] = \alpha$$

$$\Rightarrow 1 \cdot P\left[\dfrac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma^2}}{\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n-1}}} > c''''\right] = \alpha$$

$$\Rightarrow \left\langle P\left[t_{n-1} > c''''\right] = \alpha \right\rangle$$

__Test__ of __parameters from__ two __independent pop^{n}s__ in presence __of other nuisance parameters.__

__Ex 1__ Suppose X and Y are two independent Poisson variable where, $X \sim P(\lambda)$ and $Y \sim P(\mu)$ respectively. Construct a UMP test for $H_0: \mu \le \lambda$ ag. $H_1: \mu > \lambda$.

Ans Re-expressing. $H_0: \mu = \lambda$
ag. $H_1: \mu > \lambda$

We may further re-express H_0 as,

$$H_0: \log \frac{\mu}{\lambda} = 0$$

$$H_1: \log \frac{\mu}{\lambda} > 0$$

Take, $\theta = \log \mu/\lambda$ $\quad H_0: \theta = 0$ ag. $H_1: \theta > 0$

The jt. ~~prob.~~ p.m.f of X and Y is,

$$f_{\mu,\lambda}(x,y) = \frac{e^{-\lambda} \cdot e^{-\mu} \cdot \lambda^x \mu^y}{x!\, y!}$$

$$= \frac{1}{x!\,y!} e^{-(\lambda+\mu)} e^{(x+y)\log\lambda + y\log(\mu/\lambda)}$$

$\theta = \log \mu/\lambda$, $u = y$, $\eta_1 = \log\lambda$ nuisance parameter

$T(x+y) = x+y$

∴ A UMP test is,

$$\phi(u) = \begin{cases} 1 & \text{when } u > c(t) \text{ where } T=t \ (X+Y=t) \\ \gamma & u = c(t) \\ 0 & u < c(t) \end{cases}$$

Here, u and T are not independent.

∴ The cut off pt. will be determined from the conditional distn of $u/T=t$

Now, $Y/\underbrace{X+Y}_{T} = t \sim Bin\left(t, \frac{\mu}{\lambda+\mu}\right)$

∴ From size condn.

$$E_{H_0}(\phi(u)/T=t) = 1 \cdot P_{H_0}[Y > c(t)] + \gamma P_{H_0}[Y = c(t)] = \alpha$$

$c(t)$ and γ can be retrived.

Practical

Let X and Y be ind $P(\lambda)$ and $P(\mu)$

$H_0: \mu \le \lambda$ ag. $H_1: \mu > \lambda$

i) Construct a UMP test

ii) Also find the power of test when, $\lambda = 0.1, \mu = 0.2$ given, $\lambda = 10, \mu = 20$, $\alpha = 0.1$

The test will be constructed on, $U = Y$ where, the critical pt. will be ~~the~~ found from conditional distn $U = Y/X+Y = t$

The prob. table is, followed from, $\frac{Y}{X+Y=t} \sim Bin\left(t, \frac{\mu}{\lambda+\mu}\right)$

Under H_0, $\frac{Y}{X+Y=t} \sim Bin\left(t, \frac{1}{2}\right)$

Y	X+Y=t	P(U=u)
0	1	1/2
1		1/2

Y	$X+Y=t$	$P(U=u)$
0	2	1/4
1		1/2
2		1/4
0	3	1/8
1		3/8
2		3/8
3		1/8

Y	$X+Y=t$	$P(U=u)$
0	4	1/16
1		4/16
2		6/16
3		4/16
4		1/16

$$P_{H_0}[U > C(t)] + \gamma P_{H_0}[U = C(t)] = 0.1$$

$$\Rightarrow P[U < C(t)] + (1-\gamma) P[U = C(t)] = 0.9$$

$C(t)=1$, $P[U<1] + (1-\gamma) P[U=1] = 0.9$

$$\Rightarrow 0.5 + (1-\gamma)\, 0.5 = 0.9$$

$$\Rightarrow \gamma = 1/5$$

When, $T=1$, UMP test,

$$\phi(u) = \begin{cases} 1 & \text{if } U > 1 \\ 1/5 & \text{if } U = 1 \\ 0 & \text{if } U \le 0 \end{cases}$$

But practically it is not feasible,

$\therefore$ We have to scrap it.

$\underline{\underline{T=2}}$ $C(t)=1$,

$$P[U<1] + (1-\gamma) P[U=1] = 0.9$$

$$\Rightarrow 1/4 + (1-\gamma) \times 1/2 = 0.9$$

$$\Rightarrow 1-\gamma = 1.3$$

$$\Rightarrow \gamma = -0.3$$

$\underline{\underline{T=4}}$ $C(t)=1$

$$P[U<1] + (1-\gamma) P[U=1] = 0.9$$

$$\Rightarrow 0.06 + (1-\gamma)\, 0.25 = 0.9$$

$$\Rightarrow 1-\gamma = \frac{0.04}{0.25}$$

$C(t)=2$

$$P[U<2] + [1-\gamma] P[U=2] = 0.9$$

$$\Rightarrow 0.31 + [1-\gamma]\, 0.37 = 0.9$$

$$\Rightarrow [1-\gamma] = \frac{0.59}{0.37}$$

$t=4$

$C(t)=3$

$P[C(t)<3] + (1-\gamma) P[C(t)=3] = 0.9$

$$\Rightarrow 0.68 + (1-\gamma)\, 0.25 = 0.9$$

$$\Rightarrow (1-\gamma) = \frac{0.22}{25} \Rightarrow \gamma = \frac{3}{25}$$

UMP test, For fixed $t=4$

$$\phi(u) = \begin{cases} 1 & U=4 \\ 3/25 & U=3 \\ 0 & U=0,1,2 \end{cases}$$

Under, H_1, $U = \frac{Y}{X+Y} \sim Bin\left(t, \frac{\mu}{\lambda+\mu}\right)$

Power, $= \boxed{\cancel{P}\ \cancel{prob}}\ E_{H_1}[\phi(u)/T=t]$

$= 1.\ P_{H_1}[U=4] + \frac{3}{25} P_{H_1}[U=3]$

$+ 0.\ P_{H_1}[U=0,1,2]$

For this problem, $U \sim Bin(4, 2/3)$

$$= 1.\binom{4}{4}\left(\frac{2}{3}\right)^4 + \binom{4}{3} 3/25 \times \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$$

$$= \frac{16}{81} + \frac{12}{25} \times \frac{8}{81}$$

$$= \frac{400+96}{81\times 25} = \frac{496}{2025}$$

iid

Ex $X \sim Bin(m, p_1)$, $Y \sim Bin(n, p_2)$

$H_0: p_1 = p_2$ ag. $H_1: p_1 \neq p_2$

$\Rightarrow$ Joint pmf, $f(x,y) = \binom{m}{x} p_1^x (1-p_1)^{m-x} \binom{n}{y} p_2^y (1-p_2)^{n-y}$

$$= \binom{m}{x}\binom{n}{y}(1-p_1)^m (1-p_2)^n e^{x \log\frac{p_1}{1-p_1}} \cdot e^{y \cdot \log\frac{p_2}{1-p_2}}$$

$$= \binom{m}{x}\binom{n}{y}(1-p_1)^m (1-p_2)^n e^{x \log\left(\frac{p_1}{1-p_1}\cdot\frac{1-p_2}{p_2}\right) + (x+y)\log\frac{p_2}{1-p_2}}$$

$\theta = \log\frac{p_1(1-p_2)}{p_2(1-p_1)}$, $u = x$, $\eta = \log\frac{p_2}{1-p_2}$, $T = x+y$

Then a UMP can be constructed as,

$$\phi(u) = \begin{cases} 1 & u > c_2(t) \text{ or } u < c_1(t) \\ \gamma_i & u = c_i(t) \\ 0 & u < c(t) \end{cases}$$

Under H_0

Where, $U / T = t = \boxed{\dfrac{X}{X+Y=t}}$

$$= X / X + Y = t \sim \frac{\binom{m}{x}\binom{n}{t-x}}{\binom{m+n}{t}}$$ → Hypergeometric format.

<u>Practical</u>

Construct a test $H_0 : p_1 = p_2$

ag. $H_1 : p_1 > p_2$

Where, $X \sim Bin(m, p_1), Y \sim Bin(n, p_2)$

X and Y are ind.

Construct an UMP test for $\alpha = 0.1$, $m=3$, $n=4$.

<u>Ex</u> Let $X_1, X_2, \ldots X_m$ iid $N(\mu_1, \sigma_1^2)$ > ind.

$Y_1, Y_2, \ldots, Y_n$ iid $N(\mu_2, \sigma_2^2)$

$H_0 : \sigma_2^2 \le k\sigma_1^2$, k fixed

ag. $H_1 : \sigma_2^2 > k\sigma_1^2$

⇒ Joint density of $\underset{\sim}{X}, \underset{\sim}{Y}$

$$f(\underset{\sim}{x}, \underset{\sim}{y}) = c \cdot e^{-\frac{1}{2\sigma_1^2}\sum x_i^2 - \frac{1}{2\sigma_2^2}\sum y_i^2 + \frac{m\mu_1 \bar{x}}{\sigma_1^2} + \frac{n\mu_2 \bar{y}}{\sigma_2^2}}$$

$$= c \cdot e^{-\frac{1}{2\sigma_2^2}\sum y_i^2 + \frac{1}{2k\sigma_1^2}\sum y_i^2 - \frac{1}{2k\sigma_1^2}\sum y_i^2 - \frac{1}{2\sigma_1^2}\sum x_i^2 + \frac{m\mu_1 \bar{x}}{\sigma_1^2} + \frac{n\mu_2 \bar{y}}{\sigma_2^2}}$$

$$= c \cdot e^{\sum y_i^2 \left(\frac{1}{2k\sigma_1^2} - \frac{1}{2\sigma_2^2}\right)} \cdot e^{-\frac{1}{2\sigma_1^2}\left(\frac{\sum y_i^2}{k} + \sum x_i^2\right)} e^{m\mu_1\bar{x}/\sigma_1^2 + n\mu_2\bar{y}/\sigma_2^2}$$

Here, $\theta = \frac{1}{2k\sigma_1^2} - \frac{1}{2\sigma_2^2}$

$u = \Sigma y_i^2$

$\eta_1 = -\frac{1}{2\sigma_1^2}$

$T_1 = \Sigma y_i^2/k + \Sigma x_i^2$

$\eta_2 = \frac{m\mu_1}{\sigma_1^2}$, $T_2 = \bar{x}$

$\eta_3 = n\mu_2/\sigma_2^2$, $T_3 = \bar{y}$

$H_0: \sigma_2^2 \leq k\sigma_1^2$

$H_0: \sigma_2^2 = k\sigma_1^2$

Null hypothesis can be re-expressed as, $H_0: \theta = 0$.

$H_1: \theta > 0$

UMP test,

$$\phi(u) = \begin{cases} 1 & \text{if } u > c(t_1, t_2, t_3) \\ 0 & \text{o.w} \end{cases}$$

Σy_i^2 is ind. of $\bar{x} = T_2$

Σy_i^2 is ind of $\bar{y} = T_3$

(Σy_i^2 ancillary for μ_2, $\bar{y}$ is suff. for μ_2)

Therefore, $\phi(u)$ will be of. Σy_i^2 and $T_1 = \Sigma x_i^2 + \frac{1}{k}\Sigma y_i^2$

$$\phi(u) = \begin{cases} 1 & u > c(t_1) \\ 0 & \end{cases}$$

Distribution of, $\frac{\Sigma y_i^2}{\Sigma x_i^2 + \frac{1}{k}\Sigma y_i^2}$

$\frac{(n-1)s_2^2}{\sigma_2^2} \sim \chi^2_{n-1}$

$\Rightarrow \frac{\Sigma(y_i - \bar{y})^2}{\sigma_2^2} \sim \chi^2_{n-1}$

$\frac{\Sigma(x_i - \bar{x})^2}{\sigma_1^2} \sim \chi^2_{m-1}$

$$\equiv \frac{\Sigma y_i^2}{\Sigma x_i^2} = \frac{\Sigma(y_i - \bar{y})^2/\sigma_2^2 \cdot k}{\Sigma(x_i - \bar{x})^2/\sigma_1^2 \cdot \frac{(n-1)}{(m-1)}}$$

[Under H_0, $\sigma_2^2 = \sigma_1^2 k$]

$$= \frac{\chi^2_{n-1}/n-1}{\chi^2_{m-1}/m-1} \cdot k\left(\frac{n-1}{m-1}\right)$$

$\sim c\, F_{n-1, m-1}$

Nonparametric Inference

1. Gibbons and chakrabarty.

Rank Suppose X_α be the α^{th} obs^n for a set of n i.i.d. obs^{n}s, $\alpha = 1, 2, \dots, n$ from a continuous dist^n: $F_X(x)$.

R_α: rank of X_α or α^{th} smallest obs^n.

: # of obs^{n}s $\leq X_\alpha$

Due to continuity ranks are distinct with prob 1.

Rank is an ordered permutation.

Rank vector $\underset{\sim}{R} = \underset{\sim}{r}$ is a ordered permutation.

$$\underset{\sim}{R} = (R_1 \; R_2 \; R_3 \dots R_n)$$

Rank of 1st obs — Rank of nth obs

$\underset{\sim}{R}$ is the random vector random permutation.

Remmark: $\Pr\{\underset{\sim}{R} = \underset{\sim}{r}\} = \frac{1}{n!}$

$\Pr\{R_\alpha = r_\alpha\} = \frac{1}{n}$; $\alpha = 1, 2, \dots, n$

Marginal dist^n of rank is a discret uniform dist^n.

Remark: $\Pr\{R_\alpha = r_\alpha \cap R_\beta = r_\beta\} = \frac{1}{n(n-1)}$ $\alpha \neq \beta$

Remark:

$$\left[\begin{aligned} E(R_\alpha) &= \frac{n+1}{2} \\ V(R_\alpha) &= \frac{n^2-1}{12} \\ Cov(R_\alpha, R_\beta) &= -\frac{n+1}{12} \end{aligned}\right]$$

<R_α and R_β are not ind>

<Also they are negatively correlated>

$$\left\langle Corr^n(R_\alpha, R_\beta) = -\frac{\frac{n+1}{12}}{\frac{n^2-1}{12}} = -\frac{1}{n-1} \right\rangle ; \; n > 1$$

Linear rank statistic

Let $\underset{\sim}{a} = (a_1, a_2, \dots, a_n)$ and $\underset{\sim}{b} = (b_1, b_2, \dots, b_n)$ be two sets of coeff. (constants) based on n natural number.

Let, $\underset{\sim}{R} = (R_1, R_2, \dots, R_n)$ be the random permutation of $\{1, 2, \dots, n\}$. Then linear rank statistic is,

$$\left\langle T = \sum_{\alpha=1}^{n} a_\alpha b_{R_\alpha} = a_1 b_{R_1} + a_2 b_{R_2} + \dots + a_n b_{R_n} \right\rangle$$