2.2.6 Arc Length of a Curve

▶ $\boxed{\text{Rule 1}}$: If $f:[a,b]\to\mathbb{R}$ has continuous derivative on $[a,b] \Rightarrow$ Length of the curve is $\displaystyle\int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

▶ $\boxed{\text{Rule 2}}$: The length of the curve $y=f(x)$ within $x=a$ to $x=b$ is $\displaystyle\int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$.

▶ $\boxed{\text{Rule 3}}$: The length of the curve $x=g(y)$ within $y=c$ to $y=d$ is $\displaystyle\int_c^d \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$.

▶ $\boxed{\text{Rule 4}}$: The length of the <u>parametric curve</u> $x=f_1(t),\ y=f_2(t)$ within $t=t_1$ to $t=t_2$ is $\displaystyle\int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt$.

▶ $\boxed{\text{Rule 5}}$: The length of the <u>polar curve</u> $r=f(\theta)$ within $\theta=\theta_1$ to $\theta=\theta_2$ is $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left[\frac{dr}{d\theta}\right]^2}\,d\theta$.

[Do It Yourself] **2.30.** *The length of the curve* $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+7$ *from* $x=1$ *to* $x=8$ *equals (A)* $99/8$*. (B)* $117/8$*. (C)* $99/4$*. (D)* $117/4$*.*
$[Hint:\ Easy\ integration]$

Example 2.2. *Find the perimeter of the astroid* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$*.*
$\Rightarrow$ *The parametric form:* $x=a\sin^3 t,\ y=a\cos^3 t$*.*
Now in first quadrant $x=0,a \Rightarrow t=0,\pi/2$*.*

Length of perimeter in first quadrant is $\int_0^{\pi/2}\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt = 3a\int_0^{\pi/2}\sin t\cos t\,dt=\frac{3a}{2}$.

So the total length is $4.\frac{3a}{2}=6a$*.*

Example 2.3. *Find the length of the loop of the curve* $x=t^2,\ y=t-\frac{t^3}{3}$*.*

$\Rightarrow$ *We first try to draw the curve.* $\left(\begin{array}{ccc|ccc|ccc} t & x & y & t & x & y & t & x & y \\ \hline 0 & 0 & 0 & 2 & 4 & 2/3 & -\sqrt{3} & 3 & 0 \\ 1 & 1 & 2/3 & -2 & 4 & -2/3 & \sqrt{2} & 2 & \sqrt{2}/3 \\ -1 & 1 & -2/3 & \sqrt{3} & 3 & 0 & -\sqrt{2} & 2 & -\sqrt{2}/3 \end{array}\right)$.

We can easily draw the graph and see that loop is on $x-axis$ *from* $x=0$ *to* $x=3$ *i.e.* $t=0$ *to* $t=\sqrt{3}$*. It is both side of* $x-axis$*.*

Length of loop $=2\int_0^{\sqrt{3}}\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt = 2\int_0^{\sqrt{3}}\sqrt{4t^2+(1-t^2)^2}\,dt=4\sqrt{3}$.

Example 2.4. *Show that the length of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is $8a$.*
$\Rightarrow$ *Draw the curve. Now* $y = 0 \Rightarrow \cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi, \cdots$.
So the required length is $\int_0^{2\pi} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = \int_0^{2\pi} \sqrt{\left[a(1-\cos t)\right]^2 + \left[a \sin t\right]^2} dt = 8a$.

Example 2.5. *Find the length of the perimeter of the cardioide $r = a(1 - \cos\theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = 2\pi/3$.*
$\Rightarrow$ *Draw the curve. Now for upper half $\theta = 0$ to $\theta = \pi$.*
So the total length is $2\int_0^{\pi} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = 2\int_0^{\pi} \sqrt{\left[a(1-\cos t)\right]^2 + \left[a \sin t\right]^2} dt = 8a$.

□ *Length of upper half is $4a$. Therefore, we have to show:* $\int_0^{2\pi/3} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = 2a$.

Now $\int_0^{2\pi/3} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = \int_0^{2\pi/3} \sqrt{\left[a(1-\cos t)\right]^2 + \left[a \sin t\right]^2} dt = 2a$.

[Do It Yourself] 2.31. *A tangent is drawn on the curve $y = \frac{1}{3}\sqrt{x^3}, (x > 0)$ at the point $P(1, \frac{1}{3})$ which meets the $x-$ axis at Q. Then find the length of the closed curve $OQPO$, where O is the origin.* $[Hint:\ Curve + two\ lines,\ Easy]$

[Do It Yourself] 2.32. *Consider a differentiable function f on $[0,1]$ with the derivative $f'(x) = 2\sqrt{2x}$. Find the arc length of the curve $y = f(x)$, $0 \le x \le 1$.* $[Hint:\ Easy]$

[Do It Yourself] 2.33. *Find the length of the curve $y = \sqrt{4 - x^2}$ from $x = -\sqrt{2}$ to $x = \sqrt{2}$.* $[Hint:\ Easy]$

[Do It Yourself] 2.35. *Let $f : [0, \infty) \to [0, \infty)$ be twice differentiable and increasing function with $f(0) = 0$. Suppose that, for any $t \le 0$, the length of the arc of the curve $y = f(x)$, $x \le 0$ between $x = 0$ and $x = t$ is $\frac{2}{3}[(1+t)^{\frac{3}{2}} - 1]$. Then $f(4) =$ is equal to*
(A) $11/3$. *(B)* $13/3$. *(C)* $14/3$. *(D)* $16/3$.
$[Hint:\ Use\ \int_0^t \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx = \frac{2}{3}[(1+t)^{\frac{3}{2}} - 1],\ and\ find\ f(t)]$

[Do It Yourself] 2.36. *Show that the length of the hypercycloid $(\frac{x}{a})^{2/3} + (\frac{y}{b})^{2/3} = 1$ is* $\frac{4(a^2+ab+b^2)}{a+b}$.

[Do It Yourself] 2.37. *Show that the length of $r = a\cos^3(\frac{\theta}{3})$ is $\frac{3\pi a}{2}$.*

[Do It Yourself] 2.38. *Show that the length of the loop of $x = t^2$, $y = t - \frac{t^3}{3}$ is $4\sqrt{3}$.*

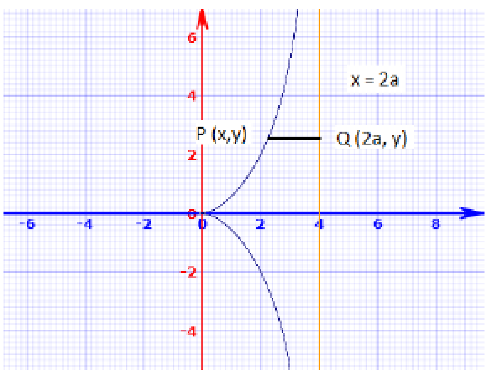
[Do It Yourself] 2.39. *Show that the length of the loop of $9ay^2 = (x - 2a)(x - 5a)^2$ is $4\sqrt{3}a$.*

2.2.7 Volumes of Solids of Revolution

▶ $\boxed{\text{Rotation around } x-axis}$: The volume V generated by revolving the curve $y = f(x)$ about the $x-axis$ and bounded by $x = a$ to $x = b$ is $\boxed{V = \pi \int_a^b y^2 dx}$.

▶ $\boxed{\text{Rotation around } y-axis}$: The volume V generated by revolving the curve $x = g(y)$ about the $y-axis$ and bounded by $y = c$ to $y = d$ is $\boxed{V = \pi \int_c^d x^2 dy}$.

Example 2.6. *Find the volume of the solid obtained by the revolution of the Cissoid $y^2(2a - x) = x^3$ about its asymptote.*
$\Rightarrow$ *The Cissoid $y^2(2a-x) = x^3$ has asymptote $x = 2a$ (In graph we take $a = 2$).*
Now the volume of the solid generated by $y^2(2a - x) = x^3$ around $x = 2a$ has two parts. One is upper side (first quadrant) and another one is lower side (fourth quadrant).



Let, $P(x, y)$ be any point on the curve.
So, $Q(2a, y)$ is the point on $x = 2a$.
It implies $PQ = 2a - x$.
Suppose $x = 2a$ cuts $x - axis$ at R.
So $RQ = \sqrt{\frac{x^3}{2a-x}} \Rightarrow d(RQ) = \frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}} dx$.
So the upper volume is $\pi \int_0^{2a} (PQ)^2 d(RQ)$.
The required volume $V = 2\pi \int_0^{2a} (PQ)^2 d(RQ)$.

Therefore, $V = 2\pi \int_0^{2a} (2a-x)^2 \frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}} dx = 2\pi \int_0^{2a} \sqrt{x(2a-x)}(3a-x)dx$.
Let $x = 2a\sin^2\theta \Rightarrow dx = 2a \sin 2\theta d\theta$. Therefore,
$V = 2\pi \int_0^{\pi/2} \sqrt{x(2a-x)}(3a-x)dx = 4a^3\pi \int_0^{\pi/2} \sin^2 2\theta(2 + \cos 2\theta) = 4a^3\pi(\pi/2) = 2a^3\pi^2$.

[Do It Yourself] 2.42. *Find the volume of the solid generated by revolving the region bounded by the parabola $x = 2y^2 + 4$ and the line $x = 6$ about the line $x = 6$.*
(A) $78\pi/15$. (B) $91\pi/15$. (C) $64\pi/15$. (D) $117\pi/15$.
$[Hint:\ Draw\ curve.\ Here\ V = 2\pi \int_4^6 (6-x)^2 \frac{1}{2\sqrt{2}\sqrt{x-4}} dx]$

[Do It Yourself] 2.44. *Find the volume of the solid formed by revolving the curve $y = x$ between $x = 0$ and $x = 1$ about the $x - axis$. $[Hint:\ Easy]$*

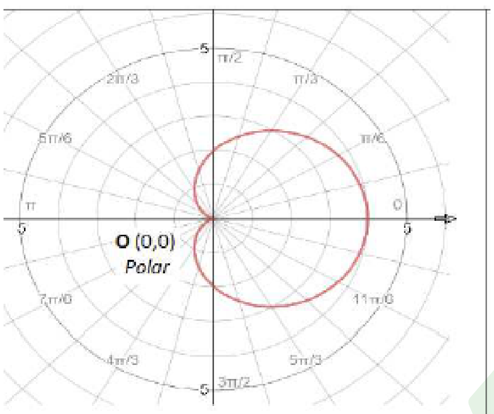
[Do It Yourself] 2.45. *The volume of the solid of revolution generated by revolving the area bounded by the curve $y = \sqrt{x}$ and the straight lines $x = 4$ and $y = 0$ about the $x - axis$, is*
(A) 2π. (B) 4π. (C) 8π. (D) 12π.

Example 2.7. *A region bounded by $y^2 = x$ and $x^2 = y$ rotates about $x-axis$. Find the volume of the solid of revolution.*

$\Rightarrow$ *We can easily draw the curves and see that the required volume $V = V_1 - V_2$.*
Here V_1 : Volume of the solid by the revolution of $y^2 = x,\ x = 0,\ x = 1$.
V_2 : Volume of the solid by the revolution of $x^2 = y,\ x = 0,\ x = 1$.
Therefore, $V = \pi \int_0^1 x dx - \pi \int_0^1 x^4 dx = \pi(\frac{1}{2} - \frac{1}{5}) = \frac{3\pi}{10}$.

[Do It Yourself] 2.47. *Find the volume of the solid obtained by rotating the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about $x-axis$.*
[$Hint$: Draw the curve and check $V = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3 dx = \frac{32\pi a^3}{105}$.]

Example 2.9. *Show that the volume of the solid obtained by the cardioid $r = a(1 + \cos\theta)$ about about the initial line is $\frac{8\pi a^3}{3}$.*
$\Rightarrow$ *We will transform polar (r, θ) to cartesian (x, y) system. [In plot $a = 2$]*



Now the pole is O $(r = 0, \theta = 0)$.
$\overrightarrow{OX}$ is the initial line.
Rotation: $0 \le r \le 2a$, $0 \le \theta < \pi$.
Now $x = r\cos\theta,\ y = r\sin\theta$.
Curve: $x = a(1 + \cos\theta)\cos\theta$,
$y = a(1 + \cos\theta)\sin\theta$.
$dx = -a\sin\theta(1 + 2\cos\theta)d\theta$.
The required volume $V = \pi \int y^2 dx$.
Now we have to find the range of x.
At $\theta = 0$, $x = 2a$ and $\theta = \pi$, $x = 0$.

The required volume $V = \pi \int_0^{2a} y^2 dx = -\pi \int_\pi^0 a^2(1 + \cos\theta)^2 a \sin^3\theta(1 + 2\cos\theta)d\theta =$ $\pi a^3 \int_0^\pi (1 + \cos\theta)^2 \sin^3\theta(1 + 2\cos\theta)d\theta = \frac{8\pi a^3}{3}$. [take $z = \cos\theta$]

[Do It Yourself] 2.49. *Show that the volume of a sphere with radius a is $\frac{4}{3}\pi a^3$.*

[Do It Yourself] 2.50. *Find the volume generated by the following curve rotating around $x-axis$:*
i) $y = \cos x$ between $x = 0,\ \pi/2$. ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ between $x = 0,\ y = 0$. iii) The area between $9y = 4(9 - x^2)$ and $4x + 3y = 12$. iv) Loop of the curve $x = t^2,\ y = t - \frac{t^3}{3}$.
[Ans : $\frac{\pi^2}{4}$, $\frac{8\pi a^3}{15}$, $\frac{48\pi}{5}$, $\frac{3\pi}{4}$]

[Do It Yourself] 2.51. *Show that the volume of the solid obtained by the cardioid $r = a(1 - \cos\theta)$ about about the initial line is $\frac{8\pi a^3}{3}$.*

2.2.8 Surface of Revolution

▶ $\boxed{\text{Rotation around } x-axis}$: The surface area S generated by revolving the curve $y = f(x)$ about the $x-axis$ and bounded by $x = a$ to $x = b$ is $\boxed{S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$.

▶ $\boxed{\text{Rotation around } y-axis}$: The volume S generated by revolving the curve $x = g(y)$ about the $y-axis$ and bounded by $y = c$ to $y = d$ is $\boxed{S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$.

▶ Finding surface of revolution is similar to finding the **Volumes of Solids of Revolution.**

[Do It Yourself] **2.55.** *Find the area of the solid obtained by rotating the astroid* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ *about* $x - axis$.

[$Hint:\ S = 2\pi \int_{-a}^{a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = 2\pi \int_{-\pi/2}^{\pi/2} a\cos^3 t\sqrt{1+\frac{\sin^2 t}{\cos^2 t}}3a\sin^2 t\cos t dt = \frac{8\pi a^2}{5}$]

[Do It Yourself] **2.56.** *Show that the surface area of the solid obtained by rotating the cycloid* $x = a(t - \sin t),\ y = a(1 - \cos t)$ *about* $x - axis$ *is* $\frac{64\pi a^2}{3}$.