3.5.4 Lin Non-Homogeneous Non-Cons Coeff

▶ Its not easy to get a closed form solution of linear non-homogeneous non-constant coefficients.
▶ If we know a solution then using reduction order method we may find other solutions.
▶ If we know the solution of homogeneous system then using variation of constants method we may find other solutions.
▶ This type of equations may solve through power series solutions.

3.5.5 The Cauchy-Euler's Equation of Order n

▶ Cauchy-Euler differential equation of order n is $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = b(x)$.
▶ We use the transformation $z = \ln(x) \Rightarrow x = e^z$ to solve Cauchy-Euler's Equation.
▶ $z = \ln(x)$, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$, $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{1}{x}\frac{dy}{dz}) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ i.e. $\boxed{x^2 D^2 y = D_1(D_1 - 1)y}$, where $D \equiv \frac{d}{dx}$, $D_1 \equiv \frac{d}{dz}$.
▶ In a similar way, $\boxed{x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)y}$, $\boxed{x^4 D^4 y = D_1(D_1 - 1)(D_1 - 2)(D_1 - 3)y}$ so on.

Example 3.14. *Solve the Ode:* $x^2 y'' - 2xy' + 3y = 0$.
$\Rightarrow$ *It is a Cauchy-Euler's Equation of Order* 2.
Let $z = \ln(x) \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$.
So the given equation transforms to $\frac{d^2y}{dz^2} - \frac{dy}{dz} - 2\frac{dy}{dz} + 3y = 0 \Rightarrow \frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 3y = 0$.
Let $y = e^{mz}$ *be a trial solution of the equation.*
So the auxiliary equation is: $m^2 - 3m + 3 = 0 \Rightarrow m = \frac{3 \pm i\sqrt{3}}{2}$.
Therefore the general solution is $y = e^{3z/2}[c_1 \cos\frac{\sqrt{3}z}{2} + c_2 \sin\frac{\sqrt{3}z}{2}]$ *i.e.* $y = x^{3/2}[c_1 \cos\frac{\sqrt{3}\ln(x)}{2} + c_2 \sin\frac{\sqrt{3}\ln(x)}{2}]$, *where* c_1, c_2 *are arbitrary constants.*

[Do It Yourself] 3.71. *Find the general solution of :* $4x^2y'' - 4xy' + 3y = 0$, $x^3y''' - x^2y'' - 6xy' + 18y = 0$, $x^4y^{(iv)} - 4x^2y'' + 8xy' - 8y = 0$, $x^2y'' - 4xy' + 6y = 4x - 6$, $x^2y'' - 5xy' + 8y = 2x^3$, $x^2y'' + 4xy' + 2y = 4\ln(x)$.

3.6 Series Solution of Linear Ode

Consider the second-order homogeneous linear DE $\boxed{a_0(x)y'' + a_1(x)y' + a_2(x)y = 0}$, and suppose that this equation has no solution that is expressible as a finite linear combination of known elementary functions. Let us assume that it has a solution in the form of an infinite series.

We assume that it has a solution expressible in the form

$$\boxed{c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-x_0)^n}$$

$c_0, c_1, c_2, \cdots$ are constants. The above expression is called a power series in $(x-x_0)$ and the differential equation has a power series solution.
▶ The equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ can be written as $y'' + p(x)y' + q(x)y = 0$.
▶ We will study the conditions under which the differential equation has a series solution. For this we will go through some ideas first.

3.6.1 Ordinary & Singular Point

▶ Analytic at a point: A function $f(x)$ is analytic at $x = x_0$ if its Taylor series $\sum \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ exists and converges to $f(x)$.
▶ Example: e^x, *polynomials*, $\sin(x)$, $\cos(x)$, $\sinh(x)$ are analytic everywhere, rational function are analytic everywhere except the points where denominator is zero i.e. $\frac{1}{(x-1)(x-2)}$ is analytic everywhere except $x = 1, 2$.
▶ Ordinary Point: For an ode: $y'' + p(x)y' + q(x)y = 0$, a point $x = x_0$ is an ordinary point $\Rightarrow p, q$ are analytic at $x = x_0$. If the point is not ordinary then it is a singular point.
▶ Singular point mainly two types : Regular and Irregular.
▶ A point $x = x_0$ is a regular point if $(x-x_0)p(x)$, $(x-x_0)^2q(x)$ are analytic. Otherwise it is called an irregular point.
▶ Regular singular points at infinity: Put $t = 1/x$ and check singularity at $t = 0$. Here $x = 1/t \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$ and so on.

Example 3.15. *Consider the Ode's:* $y'' + (x+1)y' + (x^2-3x+4)y = 0$, $(x-3)y'' + x^2y' + \frac{1}{x}y = 0$, $(x^2-1)y'' + 3xy' + (x+1)y = 0$, *Bessel Equation :* $x^2y'' + xy' + (x^2-n^2)y = 0$. *Discuss the analytic properties of* $p(x), q(x)$.
$\Rightarrow p(x) = x+1$, $q(x) = x^2 - 3x + 4$. *Both of the functions* p, q *are polynomial functions and so they are analytic everywhere. Thus all points are ordinary points of this differential equation.*
☐ $p(x) = \frac{x^2}{x-3}$, $q(x) = \frac{1}{x(x-3)}$. *Here* $x = 0, 3$ *are singular points (regular) of the Ode.*
☐ $p(x) = \frac{3x}{x^2-1}$, $q(x) = \frac{1}{x-1}$. *Here* $x = -1, 1$ *are singular points (regular) of the Ode. Note that:* $\lim_{x\to -1}(x-1)p(x) = finite$, $\lim_{x\to -1}(x-1)^2q(x) = finite$.
☐ $p(x) = \frac{1}{x}$, $q(x) = \frac{x^2-n^2}{x^2}$. *Here* $x = 0$ *is a singular point (regular) of the Ode.*

[Do It Yourself] 3.72. *Discuss the singularities of the Ode:* $x^2(1-x^2)y'' + \frac{2}{x}y' + 3y = 0$.

[Do It Yourself] 3.73. *Discuss the regular singular points of the Ode:* $x^2y'' + 2xy' + 3y = 0$. $[Ans : 0, \infty]$

[Do It Yourself] 3.74. *Show that infinity is not a regular singular point for the Bessel equation:* $x^2y'' + xy' + (x^2-n^2)y = 0$.

Theorem 3.5. *Suppose x_0 is an ordinary point of the differential equation $y'' + p(x)y' + q(x)y = 0$ then it has two nontrivial linearly independent power series solutions of the form $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ with $|x - x_0| < R$. [We are not going to detail in the convergence of the series].*

■ Note that: $(x - 3)y'' + x^2y' + \frac{1}{x}y = 0$ has singular points $x = 0, 3$. Therefore it has two linearly independent solutions of the form $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ about any point except $x_0 = 0, 3$ i.e. we can't assure that $\sum_{n=0}^{\infty} c_n x^n$ or, $\sum_{n=0}^{\infty} c_n(x - 3)^n$ are solutions of the Ode.

Example 3.16. *Find a power series solution of the IVP:* $(1-x^2)y'' + xy' - y = 0,\ y(0) = 1,\ y'(0) = 1$.

$\Rightarrow$ *We first observe that all points except $x = \pm 1$ are ordinary points for the ode. Thus we could assume solutions of the form $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ for any $x_0 \neq \pm 1$. Here $y(0) = 1,\ y'(0) = 1$, we will choose the solutions in the form $y = \sum_{n=0}^{\infty} c_n(x - 0)^n = \sum_{n=0}^{\infty} c_n x^n$.*

$y = \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=1}^{\infty} nc_n x^{n-1},\ y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$.

So, $(1-x^2)y'' + xy' - y = 0 \Rightarrow (1-x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{} + c_1 x + \sum_{n=2}^{\infty} nc_n x^n - c_0 - c_1 x - \sum_{n=2}^{\infty} c_n x^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n}_{} + c_1 x - c_0 - c_1 x - \underbrace{\sum_{n=2}^{\infty}[n(n-1)c_n - nc_n + c_n]x^n}_{} = 0$$

$$\Rightarrow 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n - c_0 - \underbrace{\sum_{n=2}^{\infty}[n^2 - 2n + 1]c_n x^n}_{} = 0$$

$$\Rightarrow -c_0 + 2c_2 + 6c_3 x + \underbrace{\sum_{n=2}^{\infty}[(n^2 + 2n + 2)c_{n+2} - (n^2 - 2n + 1)c_n]x^n}_{} = 0$$

Equating each term both sides we get,

$-c_0 + 2c_2 = 0,\ 6c_3 = 0,\ (n^2 + 2n + 2)c_{n+2} - (n^2 - 2n + 1)c_n = 0.$

$c_0 = 2c_2,\ c_3 = 0,\ c_{n+2} = \frac{n^2-2n+1}{n^2+3n+2}c_n.$

Now, $c_3 = c_5 = c_7 = \cdots = 0$ *and* $c_4 = \frac{1}{12}c_2, c_6 = \frac{3}{10}c_4,\ c_8 = \frac{25}{56}c_6$ *so on.*

So the solution is: $y = c_0 + c_1 x + \frac{c_0}{2}x^2 + \frac{c_0}{24}x^4 + \frac{3c_0}{240}x^6 + \cdots$

Now given $y(0) = 1 \Rightarrow c_0 = 1$ *and* $y'(0) = 1 \Rightarrow c_1 = 1$.

So the solution is: $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{3}{240}x^6 + \cdots$.

★ *Note*: $c_0 = 2c_2,\ c_3 = 0,\ c_{n+2} = \frac{n^2-2n+1}{n^2+3n+2}c_n.$

Now for two linearly independent solutions (here initial conditions are not given), we can choose the first two terms of the series. The easiest choices are $c_0 = 0, c_1 = 1$ *and* $c_0 = 1, c_1 = 0$. *If any difficulties arise then* $c_0 = 1, c_1 = 1$ *and* $c_0 = 1, c_1 = 0$.

Using the first pair we get the solution: $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{3}{240}x^6 + \cdots$ *Using the second pair we get the solution:* $y = 1 + \frac{1}{24}x^4 + \frac{3}{240}x^6 + \cdots$

[Do It Yourself] 3.76. *Find by power series methods a particular solution of* $y''' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,\ y(1) = 1,\ y'(1) = 0,\ y''(1) = 1$ *and* $y'' + (\sin x)y' + e^x y = 0$.

[*Ans* : $y = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{15} + \cdots$, [*Hint* : *Put* $z = x - 1$ *and solve*], $y = a_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots) + a_1(x - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \cdots)$]