3.2 1^{st} Order First Degree Equations

Here we will study some techniques using which we can obtain exact solution of the first-order equations.

▶ The equation is of the form: $\frac{dy}{dx} = f(x,y)$ or, equivalently $\boxed{M(x,y)dx + N(x,y)dy = 0}$.

▶ Total differential 'df' of the function $f(x,y)$ is $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$.

▶ $\boxed{\text{Rule 1}}$: $M(x,y)dx + N(x,y)dy = 0$ is exact iff $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},\ \forall (x,y) \in D}$. Here M, N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

▶ $\boxed{\text{Solve Using Rule 1}}$: If exact then compute: $\int M\ \partial x = f(x,y) + g(x,y)$ and $\int N\ \partial y = f(x,y) + h(x,y)$. Then $f(x,y) + g(x,y) + h(x,y) = c$ is the solution.

[Do It Yourself] 3.16. *If $f(x,y) = x^2y + 2xy^3,\ (x,y) \in \mathbb{R}$ then find 'df'.*

[Do It Yourself] 3.17. *Check whether the differential equations are exact or, not: i) $y^2dx + 2xydy = 0$, ii) $ydx + 2xdy = 0$, iii) $[2x\sin(y) + y^3e^x]dx + [x^2\cos(y) + 3y^2e^x]dy = 0$, iv) $(2xy+1)dx + (x^2+4y)dy = 0$, v) $(3x^2y+2)dx - (x^3+y)dy = 0$, vi) $(\theta^2+1)\cos(r)dr + 2\theta\sin(r)d\theta = 0$, vii) $[y\sec^2(x) + \sec(x)\tan(x)]dx + [\tan(x) + 2y]dy = 0$, viii) $(x/y^2 + x)dx + (x^2/y^3 + y)dy = 0$.*

Example 3.3. *Solve the equation $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.*
$\Rightarrow$ The given equation can be written as $M(x,y)dx + N(x,y)dy = 0$ where $M(x,y) = 3x^2 + 4xy \Rightarrow \frac{\partial M}{\partial y} = 4x$ and $N(x,y) = 2x^2 + 2y \Rightarrow \frac{\partial N}{\partial x} = 4x$.
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so the given equation is exact.
Now $\int M\ \partial x = \int (3x^2 + 4xy)\ \partial x = x^3 + 2x^2y$ and $\int N\ \partial y = \int (2x^2 + 2y)\ \partial y = y^2 + 2x^2y$.
So the solution is $2x^2y + x^3 + y^2 = c$, where c is an arbitrary constant.

[Do It Yourself] 3.18. *Solve $[2x\cos(y) + 3x^2y]dx + [x^3 - x^2\sin(y) - y]dy = 0,\ y(0) = 2$.*
$[Ans: x^2\cos(y) + x^3y - y^2/2 + 2 = 0.]$

[Do It Yourself] 3.19. *Solve the IVP's: i) $(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0,\ y(-2) = 1$, ii) $[2y\sin(x)\cos(x) + y^2\sin(x)]dx + [\sin^2(x) - 2y\cos(x)]dy = 0,\ y(0) = 3$, iii) $(ye^x + 2e^x + y^2)dx + (e^x + 2xy)dy = 0,\ y(0) = 6$, iv) $(\frac{3-y}{x^2})dx + (\frac{y^2-2x}{xy^2})dy = 0,\ y(-1) = 2$.*

3.2.1 Integrating Factor (I.F.)

An integrating factor is a function by which an ordinary differential equation can be multiplied in order to make it integrable. Suppose the equation $M(x,y)dx + N(x,y)dy = 0$ is not exact. Now $\alpha(x,y)$ is said to be an I.F. if $\alpha M dx + \alpha N dy = 0$ is exact.

▶ $\boxed{\text{Rule 2}}$: $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$ is a separable equation and can be solved by $\frac{f_1}{f_2}dx + \frac{g_2}{g_1}dy = 0$.
▶ In the separation process we assumed that $f_2, g_1 \neq 0$. It implies we lost some solutions due to this process.

▶ Homogeneous Function: A function $f(x,y)$ is said to be a homogeneous function of degree n if $f(tx,ty)=t^n f(x,y)$. For ex. $f(x,y)=3xy-y^2 \Rightarrow f(tx,ty)=t^2 f(x,y) \Rightarrow$ *Homogeneous of degree* 2.

▶ Homogeneous Function: A function $f(x,y,z)$ is said to be a homogeneous function of degree n if $f(tx,ty,tz)=t^n f(x,y,z)$.

▶ Homogeneous Function-II: A function $f(x,y)$ is said to be a homogeneous function of degree n if $f(x,y)=x^n\phi(\frac{y}{x})$. For ex. $f(x,y)=3xy-y^2=x^2(3\frac{y}{x}-\frac{y^2}{x^2})=x^2\phi(\frac{y}{x})$.

▶ Homogeneous Function-II: A function $f(x,y,z)$ is said to be a homogeneous function of degree n if $f(x,y,z)=x^n\phi(\frac{y}{x},\frac{z}{x})$. If $f(x,y,z)=y^n\psi(\frac{x}{y},\frac{z}{y})$ then also homogeneous.

▶ Homogeneous Equation: $M(x,y)dx+N(x,y)dy=0$ is said to be homogeneous if M,N are homogeneous of same order. Equivalently $\frac{dy}{dx}=f(x,y)$ is homogeneous if $f(x,y)$ is homogeneous.

▶ $\boxed{\text{Rule 3}}$: $Mdx+Ndy=0$ is a homogeneous equation then $\boxed{I.F.=\frac{1}{Mx+Ny}}$.

▶ $\boxed{\text{Rule 4}}$: $Mdx+Ndy=0$ is of the form: $f_1(xy)y\,dx+f_2(xy)x\,dy=0$ then $\boxed{I.F.=\frac{1}{Mx-Ny}}$, provided $Mx-Ny\neq 0$.

[Do It Yourself] 3.20. *Solve i)* $(x-4)y^4\ dx-x^3(y^2-3)\ dy=0$, *ii)* $x\sin(y)\ dx+(x^2+1)\cos(y)\ dy=0,\ y(1)=\pi/2$.
$[Hint: Separable,\ (x^2+1)\sin^2(y)=2.]$

Example 3.4. *Solve the differential equation* $(x^2-3y^2)\ dx+2xy\ dy=0$.
$\Rightarrow$ *Here* $(x^2-3y^2)\ dx+2xy\ dy=0$ *is a homogeneous equation.*
Therefore, $I.F.=\frac{1}{(x^2-3y^2)x+(2xy)y}=\frac{1}{x^3-xy^2}$. *It implies* $\frac{x^2-3y^2}{x^3-xy^2}\ dx+\frac{2xy}{x^3-xy^2}\ dy=0$ *is exact.*
Now $\int\frac{x^2-3y^2}{x^3-xy^2}\ \partial x=\int\frac{x}{x^2-y^2}\ \partial x-3y^2\int\frac{1}{x(x^2-y^2)}\ \partial x=\int\frac{x}{x^2-y^2}\ \partial x-3\int[\frac{x}{x^2-y^2}-\frac{1}{x}]\ \partial x=-2\int\frac{x}{x^2-y^2}\ \partial x+3\int\frac{1}{x}\ \partial x=-\ln(x^2-y^2)+3\ln(x)$.
Again $\int\frac{2xy}{x^3-xy^2}\ \partial y=-\int\frac{-2y}{x^2-y^2}\ \partial y=-\ln(x^2-y^2)$.
So the solution is $-\ln(x^2-y^2)+3\ln(x)=c_1\Rightarrow\frac{x^3}{x^2-y^2}=e^{c_1}=c\Rightarrow x^3=c(x^2-y^2)$ *where* c *is an arbitrary constant.*

[Do It Yourself] 3.21. *Solve the IVP:* $(y+\sqrt{x^2+y^2})dx-xdy=0,\ y(1)=0$.
$[Ans: 2y=x^2-1.]$

[Do It Yourself] 3.22. *Solve the Ode's: i)* $(e^v+1)\cos(u)\ du+e^v(\sin(u)+1)\ dv=0$, *ii)* $(x+y)\ dx-x\ dy=0$, *iii)* $(2xy+3y^2)\ dx-(2xy+x^2)\ dy=0$, *iv)* $v^3\ du+(u^3-uv^2)dv=0$, *v)* $(x\tan\frac{y}{x}+y)\ dx-x\ dy=0$, *vi)* $(\sqrt{x+y}+\sqrt{x-y})\ dx+(\sqrt{x-y}-\sqrt{x+y})\ dy=0$,

3.2.2 Special Integrating Factors (I.F.)

- Suppose $M(x,y)dx + N(x,y)dy = 0$ is not exact.
- $\boxed{\text{Rule 5}}$: If $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = f(x)$ is a function of x. Then $\boxed{I.F. = e^{\int f(x)\ dx}}$.
- $\boxed{\text{Rule 6}}$: If $\frac{1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = g(y)$ is a function of y. Then $\boxed{I.F. = e^{-\int g(y)\ dy}}$.
- $\boxed{\text{Rule 7}}$: Multiply $M(x,y)dx + N(x,y)dy = 0$ by $\boxed{x^\alpha y^\beta}$. The new Ode: $M_1(x,y)dx + N_1(x,y)dy = 0$ and find α, β using the exactness property i.e. $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$.

[Do It Yourself] **3.23.** *Solve the Ode's: i)* $(x+2y+3)dx + (2x+4y-1)dy = 0$, *ii)* $(x-2y+1)dx+(4x-3y-6)dy = 0$, *iii)* $(2x+3y+1)dx+(4x+6y+1)dy = 0,\ y(-2) = 2$, *iv)* $(3x-y-6)dx + (x+y+2)dy = 0,\ y(2) = -2$, *v)* $(x^2+y^2+1)dx - 2xydy = 0$, *vi)* $2xydx + (y^2 - x^2)dy = 0$, *vii)* $(xy^2 - e^{1/x^3})dx - x^2ydy = 0$, *viii)* $x^3y^3(2ydx + xdy) - (5ydx + 7xdy) = 0$, *ix)* $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$, *x)* $(x^3 + xy^4)dx + 2y^3dy = 0$, *xi)* $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$.
[*Hint : i)Easy substitution, ii) Transform to origin, ix) Rule 7, x) Rule 5, xi) Rule 6*]

[Do It Yourself] **3.25.** *The solution of the differential equation:* $\frac{dy}{dx} = \frac{y^2 \cos x + \cos y}{x \sin y - 2y \sin x}$, $y(\pi/2) = 0$ *is*
(A) $y^2 \cos x + x \sin y = 0$. *(B)* $y^2 \sin x + x \cos y = \pi/2$.
(C) $y^2 \sin x + x \sin y = 0$. *(D)* $y^2 \cos x + x \cos y = \pi/2$.

3.2.3 First Order Linear ODE

- First Order Linear ODE on y is: $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$.
- $\frac{dy}{dx} + P(x)y = Q(x)$ has $\boxed{I.F. = e^{\int P\ dx}}$. [Prove]

Example 3.5. *Solve the differential equation* $\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x}$.
$\Rightarrow$ *The given equation is a first order linear ODE on y is of the form* $\frac{dy}{dx} + P(x)y = Q(x)$.
Here $P(x) = \frac{2x+1}{x}, Q(x) = e^{-2x}$.
Therefore, $I.F. = e^{\int P\ dx} = \exp[\int(2 + \frac{1}{x})\ dx] = \exp[2x + \ln(x)] = xe^{2x}$.
It implies $xe^{2x}\frac{dy}{dx} + xe^{2x}\frac{2x+1}{x}y = x$ *i.e.* $xe^{2x}\frac{dy}{dx} + (2x+1)e^{2x}y = x$ *is exact.*
Therefore, $\frac{d}{dx}(yxe^{2x}) = x$. *Integrating we get,* $yxe^{2x} = \int xdx + c \Rightarrow xye^{2x} = x^2/2 + c$
So the solution is $xye^{2x} = x^2/2 + c$ *where c is an arbitrary constant.*

[Do It Yourself] **3.29.** *Solve the Ode's: i)* $\frac{dy}{dx} + 3y = 3x^2e{-}3x$, *ii)* $\frac{dy}{dx} + 4xy = 8x$, *iii)* $\frac{dr}{d\theta} + r\tan(\theta) = \cos(\theta)$, *iv)* $xdy + (xy + y - 1)dx = 0$, *v)* $ydx + (xy^2 + x - y)dy = 0$.

3.2.4 Bernoulli's Equations

▶ Bernoulli's Equations form: $\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$. Now $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

▶ If we use the transformation $\boxed{z = y^{1-n}}$ then Bernoulli's Equations reduced to linear equation. [**Prove**]

[Do It Yourself] 3.30. *Solve the Ode's: i)* $\frac{dy}{dx} + y = xy^3$, *ii)* $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$, *iii)* $x\frac{dy}{dx} + y = -2x^6y^4$, *iv)* $dy + (4y - 8y^{-3})xdx = 0$.

[Do It Yourself] 3.33. *Consider the Ode:* $\frac{dy}{dx} + P(x)y = 0$. *i) Show that if f and g are two solutions of this equation and c_1, c_2 are arbitrary constants, then $c_1 f + c_2 g$ is also a solution of this equation. ii) Show that if $f_1, f_2, \cdots, f_n$ are n solutions of this equation and $c_1, c_2, \cdots, c_n$ are n arbitrary constants, then $\sum_{i=1}^{k} c_i f_i$ is also a solution of this equation.*

[Do It Yourself] 3.34. *If f_i be a solution of the Ode:* $\frac{dy}{dx} + P(x)y = Q_i(x)$, $i = 1, 2$. *i) Show that if $f_1 + f_2$ is a solution of* $\frac{dy}{dx} + P(x)y = Q_1(x) + Q_2(x)$. *ii) Use the result solve* $\frac{dy}{dx} + y = \sin(x) + \sin(2x)$.

[Do It Yourself] 3.35. *Solve the Ode:* $(x + y + 2)dy - (y + 2)dx = 0$.

[Do It Yourself] 3.36. *Solve* $(x^2y^3 + xy)dy = dx$.

[Do It Yourself] 3.37. *Find the general solution of the Ode:* $(x^4 - y)dx + (y^4 - x)dy = 0$.

[Do It Yourself] 3.40. *Consider the ordinary differential equation* $x\frac{dy}{dx} + y = x$, $0 < x < 1$. *Which of the following is (are) solution(s) to the above?*
(A) $y = x/2$. *(B)* $y = \frac{x}{2} + \frac{2}{x}$. *(C)* $y = \frac{x}{2} - \frac{2}{x}$. *(D)* $y = 0$.

[Do It Yourself] 3.41. *Let $y(x)$ be the solution to the differential equation* $x^4\frac{dy}{dx} + 4x^3y + \sin x = 0$, $y(\pi) = 1$, $x > 0$. *Then $y(\pi/2)$ is*
(A) $\frac{10(1+\pi^4)}{\pi^4}$. *(B)* $\frac{12(1+\pi^4)}{\pi^4}$. *(C)* $\frac{14(1+\pi^4)}{\pi^4}$. *(D)* $\frac{16(1+\pi^4)}{\pi^4}$.

[Do It Yourself] 3.42. *Solve the Odes: i)* $\frac{dy}{dx} + y = f(x)$, $x \geq 0$, $y(0) = 2$. *Here*
$$f(x) = \begin{cases} 3 & \text{if } 0 \leq x < \pi/2 \\ \cos x & \text{if } x \geq \pi/2 \end{cases}$$

3.2.5 Orthogonal Trajectories

▶ Let $\boxed{F(x, y, c) = 0}$ be a given one-parameter family of curves in the xy-Plane. A Curve that intersects the curves of the above family at right angles is called an orthogonal trajectory of the given family.

▶ So first we transform $F(x, y, c) = 0$ by its ode $f(x, y, \frac{dy}{dx}) = 0$ then replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the ode $f(x, y, -\frac{dx}{dy}) = 0$. We will find the orthogonal trajectory $G(x, y, c_1) = 0$.

▶ In polar co-ordinate: If $F(r, \theta, \frac{dr}{d\theta}) = 0$ is the trajectory $\Rightarrow$ $\boxed{F(r, \theta, -r^2\frac{d\theta}{dr}) = 0}$ is the orthogonal trajectory.

Example 3.6. *Show that the set of orthogonal trajectories of the family of circles $x^2+y^2 = a^2$ is the family of straight lines $y = mx$.*
$\Rightarrow$ *The given family is $x^2 + y^2 = a^2$.*
Differentiate both side w.r.t. x we get, $2x + 2y\frac{dy}{dx} = 0$. It is the differential equation of the given family.
Now the differential equation of the orthogonal family is $2x + 2y(-\frac{dx}{dy}) = 0 \Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$. Integrating we get, $\ln(y) - \ln(x) = c \Rightarrow y = e^c x \Rightarrow y = mx$. Here m is arbitrary constant.

[Do It Yourself] 3.43. *Find the orthogonal trajectories of the following families: $y = cx^2,\ cx^2 + y^2 = 1,\ y = \frac{cx^2}{x+1},\ x^2 - y^2 = cx^3$.*

[Do It Yourself] 3.44. *Find the orthogonal trajectories of the family of ellipses having center at the origin, a focus at the point $(c, 0)$, and semi-major axis of length $2c$.*

[Do It Yourself] 3.45. *A given family of curves is said to be self-orthogonal if its family of orthogonal trajectories is the same as the given family. Show that the family of parabolas $y^2 = 2cx + c^2$ is self-orthogonal.*

[Do It Yourself] 3.46. *Show that the orthogonal trajectories of the family $r^2 = c\sin(2\theta)$ is $r^2 = c\cos(2\theta)$.*