2.3 Compute Integration

▶ I think you have enough skill to solve basic integration of higher secondary level. If you don't have prerequisite knowledge then just go through those chapters.

2.3.1 Line Integral

[Do It Yourself] 2.60. *Let* $P = \int_0^1 \frac{dx}{\sqrt{8-x^2-x^3}}$ *which of the following statements is TRUE*
(A) $\sin^{-1}(\frac{1}{2\sqrt{2}}) \le P \le \frac{1}{\sqrt{2}}\sin^{-1}(\frac{1}{2})$ *(B)* $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{1}{2}) \le P \le \sin^{-1}(\frac{1}{2})$
(C) $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{1}{2\sqrt{2}}) \le P \le \sin^{-1}(\frac{1}{2\sqrt{2}})$ *(D)* $\sin^{-1}(\frac{1}{2}) \le P \le \frac{\sqrt{3}}{2}\sin^{-1}(\frac{1}{2})$.
[*Hint* : $\int_0^1 \frac{dx}{\sqrt{8-x^2}} \le \int_0^1 \frac{dx}{\sqrt{8-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{8-2x^2}}$]

[Do It Yourself] 2.61. *Solve:* $\int \sqrt{x^2-a^2}dx$, $\int \sqrt{x^2+a^2}dx$, $\int \frac{1}{\sqrt{x^2-a^2}}dx$, $\int \frac{1}{\sqrt{x^2+a^2}}dx$, $\int \frac{1}{x^2-a^2}dx$, $\int \frac{1}{x^2+a^2}dx$, $\int \ln x dx$, $\int \ln(x+1)dx$, $\int \log x dx$.

[Do It Yourself] 2.62. *Solve:* $\int \frac{1}{(x-1)(x-2)(x-3)}dx$, $\int \frac{2x+3}{x^2+1}dx$, $\int \frac{1}{\sin^2 x}dx$, $\int \sin^3 x dx$, $\int \cos^3 x dx$, $\int \sin^4 x dx$, $\int \cos^4 x dx$.

[Do It Yourself] 2.63. *Solve:* $\int_{-1}^2 [x]dx$, $\int_{-1}^2 [x^2]dx$, $\int_{-1}^2 [x]^2 dx$, $\int_{-1}^2 [x+1]dx$, $\int_{-1}^2 |x|dx$.

Example 2.10. *Evaluate:* $\lim_{n\to\infty}\Big[\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+\frac{3}{n^2+3^2}+\cdots+\frac{1}{2n}\Big]$.

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\Big[\frac{n}{n^2+1^2}+\frac{2n}{n^2+2^2}+\frac{3n}{n^2+3^2}+\cdots+\frac{n.n}{n^2+n^2}\Big]$$

$$= \lim_{n\to\infty}\frac{1}{n}\Big[\frac{\frac{1}{n}}{1+(\frac{1}{n})^2}+\frac{\frac{2}{n}}{1+(\frac{2}{n})^2}+\frac{\frac{3}{n}}{1+(\frac{3}{n})^2}+\cdots+\frac{\frac{n}{n}}{1+(\frac{n}{n})^2}\Big] = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\frac{r}{n}}{1+(\frac{r}{n})^2}$$

$$= \int_0^1 \frac{x}{1+x^2}dx = [\frac{1}{2}\ln(1+x^2)]\Big|_0^1 = \frac{1}{2}\ln 2.$$

[Do It Yourself] 2.64. *Find the following limits:*

1. $\lim_{n\to\infty}\Big[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{1}{2n}\Big]$.
2. $\lim_{n\to\infty}\Big[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\Big]$.

2.3.2 Double Integral

▶ Repeated Integrals: $\displaystyle\int_a^b \Big[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\Big]\, dx$ or, $\displaystyle\int_c^d \Big[\int_{\psi_1(y)}^{\psi_2(y)} g(x,y)dx\Big]\, dy.$

▶ If the inner functions are easily integrable then evaluate using repeated integral else change the order and integrate.

▶ If the above two methods does not work then we will use the method of transformation.

Example 2.11. *Find the double integral* $\displaystyle\int_0^9 \int_{\sqrt{x}}^3 \frac{1}{1+y^3} dy\ dx.$

$$\Rightarrow \text{Here } I = \int_{x=0}^9 \int_{y=\sqrt{x}}^3 \frac{1}{1+y^3} dy\ dx = \int_{y=0}^3 \int_{x=0}^{y^2} \frac{1}{1+y^3} dx\ dy = \int_{y=0}^3 \frac{y^2}{1+y^3}\ dy = \frac{\ln 28}{3}.$$

Example 2.12. *What is the value of* $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^x e^{\sin y} \sin x dy\ dx?$

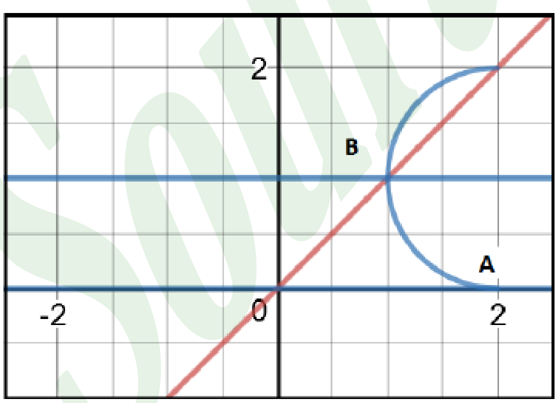
$$\Rightarrow \text{Here } I = \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^x e^{\sin y} \sin x dy\ dx = \int_{y=0}^{\frac{\pi}{2}} \int_{x=y}^{\pi/2} e^{\sin y} \sin x dx\ dy =$$

$$\int_{y=0}^{\pi/2} e^{\sin y}[-\cos\frac{\pi}{2} + \cos y]\ dy = \int_{y=0}^{\pi/2} e^{\sin y} \cos y\ dy = e - 1.$$

Example 2.13. *If* $\displaystyle\int_0^1 \int_y^{2-\sqrt{1-(y-1)^2}} f(x,y) dx dy = \int_0^1 \int_0^{\alpha(x)} f(x,y) dy dx + \int_1^2 \int_0^{\beta(x)} f(x,y) dy dx$
then $\alpha(x)$ *and* $\beta(x)$ *are*
(A) $\alpha(x) = x, \beta(x) = 1 + \sqrt{1-(x-2)^2}$ *(B)* $\alpha(x) = x, \beta(x) = 1 - \sqrt{1-(x-2)^2}$
(C) $\alpha(x) = 1 + \sqrt{1-(x-2)^2}, \beta(x) = x$ *(D)* $\alpha(x) = 1 - \sqrt{1-(x-2)^2}, \beta(x) = x.$

$$\Rightarrow \text{Here } I = \int_{y=0}^1 \int_{x=y}^{2-\sqrt{1-(y-1)^2}} f(x,y) dx\ dy.$$

Now we will draw the area enclosed by the given limits.



Now we draw $y = 0,\ y = 1.$
$x = y,\ x = 2 - \sqrt{1-(y-1)^2}.$
These four curve generates OABO.
Now we will change the order.
Given y *is constant from 0 to 1.*
Changing order we get x *is constant from* $x = 0$ *to* $x = 2.$
First Part: $x = 0,\ x = 1$ *and* $y = 0,\ y = x$
Second Part: $x = 1,\ x = 2$ *and* $y = 0,\ y = 1 - \sqrt{1-(x-2)^2}.$

Therefore, the option B is correct.

[Do It Yourself] 2.65. *The integral* $\int_0^1 \int_{x^2}^{2x} f(x,y)dydx$ *is equal to*

(A) $\int_0^1 \int_{y/2}^{\sqrt{y}} f(x,y)dxdy + \int_1^2 \int_{y/2}^{1} f(x,y)dxdy$ (B) $\int_0^2 \int_{y}^{y/2} f(x,y)dxdy$

(C) $\int_0^1 \int_{y/2}^{\sqrt{y}} f(x,y)dxdy + \int_1^2 \int_{y}^{2y} f(x,y)dxdy$ (D) $\int_0^2 \int_{y}^{2\sqrt{y}} f(x,y)dxdy.$

[Do It Yourself] 2.66. *Let* $I = \int_0^1 \int_{x^2}^{2-x} xydydx$. *The change of order of integration in the integral gives* I *as*

(A) $I = \int_0^1 \int_0^{\sqrt{y}} xydxdy + \int_1^2 \int_0^{2-y} xydxdy$ (B) $I = \int_0^1 \int_0^{2-y} xydxdy + \int_1^2 \int_0^{2-y} xydxdy$

(C) $I = \int_0^1 \int_0^{\sqrt{y}} xydxdy + \int_0^1 \int_0^{2-y} xydxdy$ (D) $I = \int_0^1 \int_0^{2-y} xydxdy + \int_1^2 \int_0^{\sqrt{y}} xydxdy.$

[Do It Yourself] 2.69. *Evaluate* $\int_0^1 \int_y^1 x^2 e^{xy} dx\, dy.$

[Do It Yourself] 2.70. *Evaluate the integral* $\int\int_R \sin(x+y)\, dxdy$ *Where* $R: \{0 \le x \le \frac{\pi}{2};\ 0 \le y \le \frac{\pi}{2}\}.$

[Do It Yourself] 2.71. *Evaluate the integral* $\int\int_R (4 - x^2 - y^2)\, dxdy$ *Where* R *is the region bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 3/2.$

[Do It Yourself] 2.72. *Evaluate the integral* $\int\int_R \frac{dxdy}{\sqrt{x^2+y^2}}$ *Where* R *is the region bounded by* $|x| \le 1,\ |y| \le 1.$

[Do It Yourself] 2.73. *Change the order of integration* $\int_0^1 dx \int_{x^2}^{\sqrt{x}} f(x,y)dy.$

[Do It Yourself] 2.74. *Evaluate* $\int_0^1 \int_y^1 e^{x^2}\, dxdy.$

[Do It Yourself] 2.75. *Show that* $\int\int_R e^{y/x}\, dxdy$ *Where* R *is the triangle bounded by* $y = x,\ y = 0,\ x = 1$ *is* $\frac{e-1}{2}$.

2.4 Transformation of variables

▶ Let f be a bounded function of x and y over a closed region S. If the transformation $x = \phi(u,v)$, $y = \psi(u,v)$ represents a continuous bijection between the closed region S of the $xy-plane$ onto a region S_1 on $uv-plane$ and if the functions ϕ, ψ have continuous first order partial derivatives then $\boxed{\int\int_S f(x,y)dx\ dy = \int\int_{S_1} f[\phi(u,v),\psi(u,v)] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}$.

▶ Here $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian and is non-zero for transformation.

Example 2.14. *The value of* $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)}dx\ dy$ *is*

(A) $\frac{\pi}{4}$ *(B)* $\frac{1}{2\pi}$ *(C)* $\frac{1}{4}$ *(D)* $\frac{1}{2}$.

$\Rightarrow$ *Let* $x = r\cos\theta,\ y = r\sin\theta$.

Therefore, $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$

So, $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)}dx\ dy = \dfrac{1}{2\pi}\int_{r=0}^{\infty}\int_{\theta=\pi/4}^{5\pi/4} e^{-\frac{r^2}{2}} r dr\ d\theta$

$= \dfrac{1}{2\pi}\Big[\displaystyle\int_{r=0}^{\infty} e^{-\frac{r^2}{2}} r dr\Big]\Big[\int_{\theta=\pi/4}^{5\pi/4} d\theta\Big] = \dfrac{1}{2\pi}\times 1\times\pi = \dfrac{1}{2}.$

2.4.1 Surface Area using Integral

Example 2.15. *Let* $S = \{(x,y)\in\mathbb{R}^2 : x,y \geq 0,\ \sqrt{4-(x-2)^2} \leq y \leq \sqrt{9-(x-3)^2}\}$ *then find the area of* S.

$\Rightarrow$ *Draw the area carefully.*

Now $\sqrt{4-(x-2)^2} = \sqrt{x(4-x)} \Rightarrow 0 < x < 4$ *and this is an upper half of the circle* $(x-2)^2+y^2 = 2^2$.

Also $\sqrt{9-(x-3)^2} = \sqrt{x(6-x)} \Rightarrow 0 < x < 6$ *and this is an upper half of the circle* $(x-3)^2+y^2 = 3^2$.

Now draw the region and you can find the area is $5\pi/2$.

[Do It Yourself] 2.82. *Let* $S = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \le 1\}$ *then what is the area of* S*?*
$[Hint:\ Easy]$

[Do It Yourself] 2.83. *Find the value of the real number* m *for which*
$\int_0^1 \int_x^{\sqrt{2-x^2}} (x^2+y^2)dydx = \int_{m\pi}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^3 dr\ d\theta.$
$[Ans:\ m = 1/4,\ Transform\ and\ find\ the\ limits\ of\ integral]$

[Do It Yourself] 2.84. *Let* D *be the triangle bounded by the* y *axis , the line* $2y = \pi$ *and the line* $y = x$ *. Then the value of the integral* $\int\int_D \frac{\cos y}{y} dxdy$ *is*
(A) $1/2$ *(B)* 1 *(C)* $3/2$ *(D)* 2.
$[Hint:\ \int\int_D \frac{\cos y}{y} dxdy = \int_{y=0}^{\pi/2} \int_{x=0}^{y} \frac{\cos y}{y} dxdy]$

[Do It Yourself] 2.88. *Using the transformation* $x + y = u,\ y = uv$*, show that* $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}}\ dy = \frac{1}{2}(e-1).$
$[Hint:\ Draw\ the\ region:\ x = 0, y = 0, x + y = 1.\ Now,\ y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0;\ x = 0 \Rightarrow uv = u \Rightarrow v = 1;\ x + y = 1 \Rightarrow u = 1;\ 0 \le u \le 1,\ 0 \le v \le 1]$

[Do It Yourself] 2.89. *Show that* $\int_0^1 dx \int_0^x \sqrt{x^2+y^2}\ dy = \frac{1}{6}(\sqrt{2} + \ln(1+\sqrt{2})).$

[Do It Yourself] 2.90. *Show that* $\int\int_R (x^2+y^2)dx\ dy$*, where* E *is the region bounded by* $xy = 1,\ y = 0,\ y = x,\ x = 2$ *is* $47/24$.

[Do It Yourself] 2.91. *Show that* $\int\int_E (x^2+y^2)dx\ dy = 6/35$*, where* E *is the region bounded by* $y = x^2,\ y^2 = x$.

[Do It Yourself] 2.94. *Find range* W *of the transformation variable* (u,v) *for the transformation* $x = f_1(u,v),\ y = f_2(u,v)$ *and the range* R *of* (x,y) *are given below*

1. R *is bounded by a triangle with vertices* $(0,0),\ (1,0),\ (0,1)$*. Transformation* $x = \frac{v(1-u)}{2},\ y = \frac{v(1+u)}{2}$*. Draw* W *and find the integration limits.*
2. R *is bounded by a triangle with vertices* $(0,0),\ (1,0),\ (0,1)$*. Transformation* $x = v - u,\ y = v + u$*. Draw* W *and find the integration limits.*
3. R *is bounded by a triangle with lines* $x = 0,\ y = 0,\ x + y = 1$*. Transformation* $x = uv,\ y = u(1-v)$*. Draw* W *and find the integration limits.*

[Do It Yourself] 2.95. *Show that* $\int\int_R e^{\frac{y-x}{y+x}}\,dxdy$ *Where* R *is the triangle bounded by* $y+x=1,\ y=0,\ x=0$ *is* $\frac{1}{4}(e-\frac{1}{e})$.
$[Hint:\ Transformation\ y-x=2u,\ y+x=2v]$

[Do It Yourself] 2.96. *Show that* $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dxdy = \frac{1}{2}(e-1)$.
$[Hint:\ Transformation\ x+y=u,\ y=uv]$

[Do It Yourself] 2.98. *Find* $\int_0^1\int_0^x \sqrt{x^2+y^2}\,dxdy$.