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(36) Spectral theorem.
(37) Spectral Decomposition.
(38) Quadratic eqns.
(39) Principal Axis Theorem.
(40) Singular Value Decomposition.

free variable = n - rank (A) fv $= 0$: unique sl
fv > 0 : ∞ sln

Date / /

Linear Algebra

- Infinite solutions } Consistent — intersecting pt
- One solution } 2 lines 1 point of intersection
- No solution — Inconsistent — || lines

Homogenous system : $Ax + By = 0$
$Cx + Dy = 0$

Equivalent system : same soln for 2 sets of eqns.

Gaussian elimination

① write augmented matrix

ementary ansformations ② augmented matrix $\rightarrow$ row echelon form

③ back substitution, solve.

$$AX = B$$

$$[A|B] = \begin{bmatrix} * & * & * & - \\ 0 & * & * & - \\ 0 & 0 & * & - \end{bmatrix} \rightarrow \text{solve using back substitution}$$

Reduced row echelon form - Gauss Jordan.

Leading entries 1, others 0

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

no of eqns < no of variables, free variables shd be written in the form of s & t. Parametric equations.

$w = 2 + x - z$ $\qquad x = s \;\&\; z = t$

$y = 1 + z$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+s-t \\ s \\ 1+t \\ t \end{bmatrix}$$

Inverse of a matrix

$AX = B$

$AA^{-1} = I$

Only if $|A| \neq 0$, inverse exists

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\boxed{\begin{aligned} (A^{-1})^{-1} &= A \\ (cA)^{-1} &= \frac{1}{c}A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1} \\ (A^{T})^{-1} &= (A^{-1})^{T} \\ A^{n} &= (A^{-1})^{n} \end{aligned}}$$

Gauss Jordan

$[A \mid B]$ - Augmented matrix

$[A \mid I]$ - super augmented

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$

$\Rightarrow$ Finally, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & 5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right]$

Matrix factorization

[ ] = [ ] * [ ]

LU Decomposition

$A = L * U$

$$\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \quad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

unit lower
△ar matrix

① $AX = B$

② $(LU)X = B$

③ $L(UX) = B \quad , \quad LY = B$

④ $UX = Y \longrightarrow$ Backward substitution

⑤ ~~$LY = B \longrightarrow$ forward substitution.~~

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} * \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ U_{11}l_{31} & U_{12}U_{31} + U_{22}l_{32} & U_{13}l_{31} + U_{23}l_{32} + U_{33} \end{bmatrix}$$

Equate to find all l & u elements

- $LY = B$

Equate & find Y.

$UX = Y$

Equate & find X.

Inverse of matrix using Gauss Jordan.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ Row operations.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

A^{-1}

existence of identity element: $\vec{u} + \vec{0} = \vec{0} + \vec{u}$
existence of additive inverse: $\vec{u} + (-\vec{u}) = \vec{0}$, $\vec{0}, \vec{u} \in V$, $-\vec{u} \in V$

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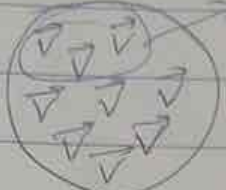
Vector Space. (can't be NULL)



vector subspace

built on scalar field (F)

a b
c d

Abelian group
→ 5 properties.

$u, v, w \in V$.
$u = u_1, u_2$
$v = v_1, v_2$
$w = w_1, w_2$

1) **Associative:**

$$u + (v + w) = (u_1, u_2) + [(v_1, v_2) + (w_1, w_2)]$$
$$= \quad (u + v) + w = [(u_1, u_2) + (v_1, v_2)] + (w_1, w_2)$$
$$= u_1 + v_1 + w_1, \; u_2 + v_2 + w_2.$$

2) **Commutative:** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

3) **Existence of identity:** $\vec{0} + \vec{u} = \vec{u} + \vec{0}$

4) **Existence of inverse:** $\vec{u} + (-\vec{u}) = 0$

5) **Closure form:** $\vec{u} + \vec{v} = \vec{w}$, $\boxed{u, v, w \in V}$

Spanning sets and Linear Independence

If vector $[a] = [b]$ & $[c]$ vector, a is linearly dependent on b & c

$(v_1, v_2, \dots v_k)$

$c_1v_1 + c_2v_2 + c_3v_3 + \dots c_kv_k = 0$

Linear combination of k vectors.

If atleast one $\underset{(\text{scalar})}{c}$ is not zero – system linearly dependent.

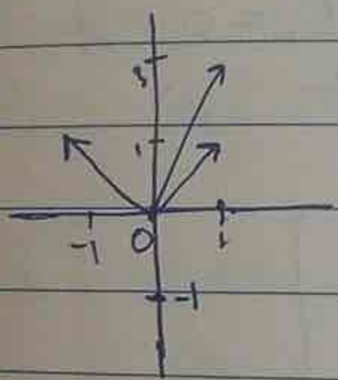
$$x - y = 1$$
$$x + y = 3$$

x	0	1
y	-1	0

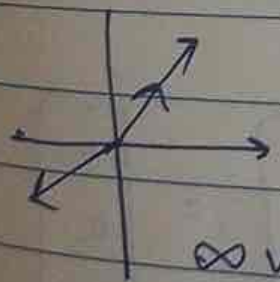
x	0	3
y	3	0

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

$(1,1)$ $(-1,1)$ $(1,3)$

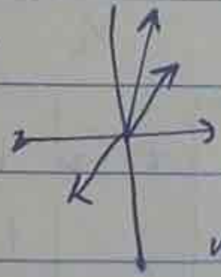


If



∞ values

If



no solutions.

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Linear dependence.

1) $\begin{bmatrix}1\\4\end{bmatrix}$ & $\begin{bmatrix}-1\\2\end{bmatrix}$

$\quad V_1 \qquad\quad V_2$

$$c_1\begin{bmatrix}1\\4\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_1 - c_2 = 0$$
$$c_1 = c_2.$$
$$4c_1 + 2c_2 = 0.$$
$$6c_1 = 0$$
$$c_1 = 0 = c_2 = 0$$

Linearly independent.

2) $\begin{bmatrix}1\\2\\0\end{bmatrix}$ $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ & $\begin{bmatrix}1\\4\\2\end{bmatrix}$

$\quad v_1 \qquad v_2 \qquad\quad v_3$

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\cancel{c_1 + c_2 + c_3 = 0} \qquad \cancel{\begin{bmatrix}1 & 1 & 1\\2 & 1 & 4\\0 & -1 & 2\end{bmatrix}}$$
$$\cancel{2c_1 + c_2 + 4c_3 = 0}$$
$$\cancel{-c_2 + 2c_3 = 0}.$$

$$c_1 + c_2 + c_3 = 0$$
$$-c_2 + 2c_3 = 0$$
$$6c_3 = 0 \quad \boxed{c_3 = 0} \quad c_1 = 0$$
$$c_2 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} c_1 + 3c_3 = 0 \\ c_2 - 2c_3 = 0 \end{array}\right\} \infty \text{ solns}$$

Linearly dependent

Bases:
Set of vectors that spans the space & is linearly independent

Space $\vec{V}_s$ — vectors in basis can be combined to represent any vector in space.

basis $\vec{V}_1 \vec{V}_2$

R^3 $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Null space + Dim (row space) = no of cols.

2 + 3 = 5

Subspaces : $W(F)$

Subset : 3 conditions.

1. Closure under vector addition.
   if $u, v \in W$
   $u + v \in W$
2. Closure under scalar addition
   if $u \in W$
   $c \rightarrow$ scalar.
   $u \cdot c \in W$
3. Contains zero vector
   $\vec{0}$ must be in W.

Fundamental subspaces

- Row space
- Column space
- Null space
- Null space of transpose.

Row space : Vector space spanned by rows of matrix

- All possible linear combs of the rows of the matrix.

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$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 8 & 10 & 12 \\ 1 & -2 & 3 & -4 & 5 & -6 \\ -1 & -2 & -3 & -4 & -5 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$
$R_4 \to R_4 + R_1$

Row echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -4 & 0 & -8 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

space

Basis: span $(1,2,3,4,5,6), (0,-4,0,-8,0,-12)$
(span) (non zero rows)

$R(A) = Row(A)$

Basis of $Row(A) = \{(1,2,3,4,5,6), (0,-4,0,-8,0,-12)\}$

$Dim(Row(A)) = 2$.

Column space:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Reduced row echelon,

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix}$$

Column space $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Null space: $\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ = eqn

$$x_1 + x_2 + 4x_3 + x_4 + 2x_5 = 0$$
$$x_2 + 2x_3 + x_4 + x_5 = 0$$
$$x_4 + 2x_5 = 0.$$

$x_3 = s, \quad x_5 = t$.

$$x_1 = -x_2 - 4x_3 - x_4 - 2x_5 = -2x_3 - x_5$$
$$x_2 = -2x_3 - x_4 - x_5 = -2x_3 + x_5$$
$$x_4 = -2x_5$$

Put $x_3 = s, x_5 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t \\ -2s + t \\ s + 0t \\ 0s - 2t \\ 0s + t \end{bmatrix}$$

$$= s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Null space = 2 .

Date

CO-3 : Linear transformation

$$x \longrightarrow \boxed{\text{function } f(x)} \longrightarrow y$$

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

→ For Linear transformation :

conditions:

1. $\vec{a}, \vec{b} \in \mathbb{R}^n$, $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$
2. $T(c\vec{a}) = c \cdot T(\vec{a})$

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$

$\vec{a} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \end{bmatrix}$

Check for the 2 conditions, It is LT.

→ Image of Linear Transformation.

Domain → Codomain Range

Domain: set of all vectors before transformation

Codomain: Set of vectors that the transformation could potentially produce

(Image)

Range: Set of actual output vectors obtained after applying transformation to every vector in the domain.

(Range)

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$T: R^2 \to R^2$

$T(\vec{x}) = A(\vec{x})$

$A = \begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix}$.

$A \cdot \vec{x}$

$\begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 2x_2 \\ -2x_1 + 3x_2 \end{bmatrix}$

$x_1, x_2 \in R$.

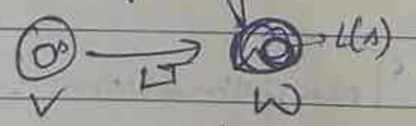
Image example: $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$.

~~If (1, 0)~~ :

Vector (1,0) will be (2,0).

Vector (0,1) will be (0,1)

Image of LT



s is subspace

L(s) in image of s.

Image of entire vector space V is called the range of L.

Ex: $L: R^3 \to R^2$.

$L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$.

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\vec{s} = \begin{bmatrix} c \\ 2c \\ 0 \end{bmatrix}$ $L(\vec{s}) = \begin{bmatrix} c \\ 2c - 0 \end{bmatrix}$ =

Image of sub

Date / /

$L: R^3 \to R^2$

Domain of L is R^3 & co-domain of L is R^2.

Q) Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$ $\quad x = \begin{bmatrix} x \\ y \end{bmatrix}$

$$Ax = \begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix} \qquad \therefore T_A : R^2 \to R^3.$$

$$= x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Linear Combinations of column vector $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$

$\therefore$ Range of T is column space of A.

Q) Find image of the LT $T: R^3 \to R^2$ defined by $T(\vec{x}) = A(\vec{x})$ with $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\vec{x}$$

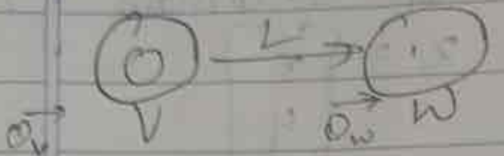
$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} x_1 - x_2 - 2x_3 &= 0. \\ x_2 + x_3 &= 0. \end{aligned}$$

Col space $= \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

$\mathrm{im}(T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$

Kernel.



Ker(L) is the set of vectors in W.

$L : R^3 \to R^2 \qquad L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$

$L(\vec{v}) = \vec{0}_W$

Now, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ so, $L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$L\begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_1 = 0 \quad v_2 = v_3$

$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} \to$ Kernel of L.

To find kernel :

(1) Matrix will be given.

$T(\vec{x}) = A(\vec{x}) = 0$

A will be given.

Reduce A – Echelon form.

(2) Solve like solving for null space

Ex: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(3) Now, $Ker(T) = span\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

→ $T: R^2 \to R^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 25x - 10y \\ -10x + 4y \end{bmatrix}$.

(a) $(5, -2)$

$$\left[\begin{array}{cc|c} 25 & -10 & 5 \\ -10 & 4 & -2 \end{array}\right] \xrightarrow{op^n s} \left[\begin{array}{cc|c} 5 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So it is a consistent solution

Steps: (Range of T)

1. Given pair in augmented form.
2. One row 0 (LHS & RHS = 0 in atleast 1 row)
3. Augmented - 0 (all 0)
4. Then vectors said to be on the range of codomain

→ Kernel of T

$$T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 25(2) - 10(5) \\ -10(2) + 4(5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ vector is in Kernel of T

Kernel = Null Space

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Inverse of Linear Transformation

$T: R^2 \to R^2$ is LT defined

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 7x_1 + 9x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$$

Steps:

1. Identify the matrix.
2. Find inverse.
3. Multiply with the x matrix

Composition of Linear Transformation

$$X \xrightarrow{S} Y \xrightarrow{T} Z$$

$S: X \to Y \quad X \subseteq R^n, \; Y \subseteq R^m$

$S(\vec{x}) = A(\vec{x})$

$\downarrow$

Transformation matrix of S

$T: Y \to Z, \; Z \in R^l$.

$T(\vec{y}) = B\vec{y}$

$T \circ S : x \to z$

Composition of T with S

$$x \xrightarrow{S} y \xrightarrow{T} z$$

$S \quad S(\vec{x}) \quad T(S(\vec{x}))$

$T \circ S = T(S(\vec{x}))$

To check if $T \circ S$ is linear or not

① $T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x} + \vec{y}))$

$= T(S(\vec{x}) + S(\vec{y}))$ [L·T rule]

$= T(S(\vec{x})) + T(S(\vec{y}))$

$$\boxed{T \circ S(\vec{x} + \vec{y}) = T \circ S(\vec{x}) + T \circ S(\vec{y})}$$

② $T \circ S(c\vec{x}) = c \cdot T \circ S(\vec{x})$

$$\boxed{\begin{array}{l} T \circ S(\vec{x}) = T(S(\vec{x})) = T(A\vec{x}) = B(A(\vec{x})) \\ \text{So, } T \circ S = [T][S] \end{array}}$$

<u>Example:</u>

$T : R^2 \to R^3$ & $S : R^3 \to R^4$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$$S\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_3 \\ 3y_2 - y_3 \\ y_1 - y_2 \\ y_1 + y_2 + y_3 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$S \circ T = [S][T]$

Find $S \circ T : R^2 \to R^4$

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$$SoT(x) = \begin{bmatrix} 5 & 4 \\ 3 & -7 \\ -1 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$SoT(x) = \begin{bmatrix} 5x_1 + 4x_2 \\ 3x_1 - 7x_2 \\ -x_1 + x_2 \\ 6x_1 + 3x_2 \end{bmatrix}$$

Inverse of Composite Linear Transformation

$S: R^n \to R^n$ $\quad T: R^n \to R^n$.

If $SoT = I_n$ & $ToS = I_n$

S & T are inverse transformation

Composition of L.T

Combine 2 or more LT to single transform

Rotation & Reflection.

Rotation:

clockwise $R(\theta)$

anticlockwise $R(-\theta)$

Reflection:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Ex: Rotate $[3, 4]$ counter clockwise by $45°$

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

now applying the vector $[3, 4]$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 & -4\sqrt{2}/2 \\ 3\sqrt{2}/2 & 4\sqrt{2}/2 \end{bmatrix}$$

Ex: Show that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ is L.T.

$$\begin{bmatrix} x \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$F\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rank Nullity Theorem:

→ $\dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = \dim V$

→ $\mathrm{rank}(T) + \mathrm{null}(T) = \dim V$.

Ex: Verify Rank Nullity thm

$$T(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z)$$

$$T\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

→ reduce this using echelon form

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

$T(0,0,1) = (-1,1,-2)$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + 2y - z = 0$

$y + z = 0$

$z = s$

Rank = 2 (no of non zero rows)

& no of free variables = 1 = null T

$$\text{rank}(T) + \text{null}(T) = \dim V$$

$$2 + 1 = 3$$

$\therefore$ Verified

One - one transformation : Nullity = 0

Onto transformation : $T : R^n \to R^m$; if Rank = m

Qn : $T : R^2 \to R^2$

$T(x, y) = (2x + 3y, 5x + 7y)$

$T(1, 0) = (2, 5)$

$T(0, 1) = (3, 7)$

$2 = m$

$T_A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ Rank $T_A = 2$

Nullity = 0

$\therefore$ T is one one & onto.

Qn : $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$ Check if LT or not

① $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $T[\vec{a} + \vec{b}] = \begin{bmatrix} |a_1 + b_1| \\ |a_2 + b_2| \end{bmatrix}$

$T(\vec{a}) + T(\vec{b}) = \begin{bmatrix} |a_1| + |b_1| \\ |a_2| + |b_2| \end{bmatrix}$

$\boxed{|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|}$ $\therefore$ Not LT

CO-4

Eigen values & Eigen vectors

$$Ax = \lambda x$$

A: $n \times n$ matrix; x: vector; λ: scalar eigen values; x: vector

→ Show that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \lambda x$$

$$= \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boxed{\lambda = 4}$$

Characteristic Eqn: To find Eigen values.

$$|A - \lambda I| = 0$$

1. A will be given.
2. I is the Identity matrix of dimension of A.
3. ~~Find~~ Do λI. Then find determinant of .
4. Equate that eqn to 0.

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}.$$

$$|A - \lambda I| = \left| \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0.$$

$$\left| \begin{bmatrix} 5-\lambda & 3-0 \\ 2-0 & 10-\lambda \end{bmatrix} \right| = 0.$$

$\cancel{\lambda}\ (5-\lambda)(10-\lambda) - 6\cancel{\lambda} = 0$

$(5-\lambda)(10-\lambda) = 6$

$50 - 10\lambda - 5\lambda + \lambda^2 = 6$

$\lambda^2 - 15\lambda + 44 = 0$

$\lambda^2 - 11\lambda - 4\lambda + 44 = 0$

$\lambda(\lambda - 11) - 4(\lambda - 11) = 0$

$\boxed{\lambda = 4}$ & $\boxed{\lambda = 11}$

<u>To find eigen ~~values~~ vectors.</u>

1. Find eigen values
2. Substitute eigen values in $(A - \lambda I)X = 0$
3. Calculate the eigen vector X for <u>each</u> eigen vector.

For A

$$A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$$

$$|A - \lambda I|$$

$$= \left| \begin{bmatrix} 1-\lambda & 4 \\ -4 & -7-\lambda \end{bmatrix} \right|$$

$$(1-\lambda)(-7-\lambda) + 16$$

$$\boxed{\lambda = -3}$$

$$\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = 0$$

$$B = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$$

$$Bx = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Downarrow$ Reduce

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 + x_2 = 0$$

$$\boxed{x_2 = s} \qquad x_2 = -x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigen vector : $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for eigen value 3

Eigen space : span$\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

Date

To find eigen space:

① Find eigen value

② Find eigen vector

③ ~~Colle~~ [Span ([collection of all eigen vectors

is the eigen space

Another method to find eigen values

ⓐ **For 2x2 matrix**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

~~Another way of~~

Characteristic eqn : $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} -$

$= \lambda^2 - \text{trace}(A) + \text{determinant}(A) = 0$

ⓑ **For 3x3 matrix**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic eqn:

$\lambda^3 - \text{trace}(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda -$

cofactors of a_{11}, a_{22}, a_{33}

Cayley - Hamilton theorem

① Every matrix A is the root of its characteristic eqn.

X. In Δ^{lar} matrix $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$,
Eigen values : diagonal entries
$= a, d, f$.

② Sum of n eigen values = sum of n diagonal entries

③ Product of diagonal entries = det (A)

④ If all entries are same in nxn matrix, one of the eigen values is n(c) & rest are 0

$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\lambda_1 = 2(5) = 10$

$\lambda_2 = 0$

λ are 10, 0.

⑤ Eigen values of A = Eigen values of A^T

⑥ If sum of each row/column = c, then c is eigen value of A.

Ex: $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ → sum = 4, → sum = 4.

∴ One eigen value is 4.

$\lambda^2 - 3\lambda + 4 = 0$.

$\lambda = 4, -1$.

⑦ A square matrix is invertible if and only if 0 is not an eigen value of A.

Algebraic Multiplicity (AM)

Geometric Multiplicity (GM)

AM : No of times eigen values are repeated

GM : Dimension of eigen space

Find AM & GM of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

Sum of all rows = 0

$\boxed{0, 0, -2}$

① Find eigen values

$|A - \lambda I| = 0$

$$\left| \begin{bmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{bmatrix} \right| = 0$$

$(-1-\lambda)(-\lambda)(-1-\lambda) + \lambda = 0$

$(1-\lambda)(1-\lambda)(-\lambda) + \lambda = 0$

$(1 - 2\lambda + \lambda^2)(-\lambda) + \lambda = 0$

$\cancel{-\lambda} + 2\lambda^2 - \lambda^3 \cancel{+\lambda} = 0$

$-2\lambda^2 - \lambda^3 = 0$

$\lambda^3 + 2\lambda^2 = 0$

$\lambda = 0, 0, -2$

AM of -2 is 1

AM of 0 is 2

$[A - 0I | 0] = [A | 0] \xrightarrow[\text{then reduce}]{} [A]$

$x_1 - x_3 = 0$

$x_1 = x_3$

Distinct E. values : E. vectors are Linearly Independent

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So: $E_0 = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Thus GM for $\lambda = 0$ is 2.

Now for $\lambda = -2$.

$E_{-2} = \left\{ x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$

Thus GM for $\lambda = -2$ is 1.

1. If A is invertible, λ^n is E. val of A^n for any value n.
2. If A is invertible, $\frac{1}{\lambda}$ is an E. val of A^{-1}

Qn) Compute $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10}$ $x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\lambda = -1, 2$

E. vector for $\lambda = -1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = V_1$

E. vector for $\lambda = 2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = V_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ -1 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 3 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$x = 3V_1 + 2V_2$

$$\begin{aligned} A^{10} x &= A^{10}(3V_1 + 2V_2) \\ &= 3(A^{10} V_1) + 2(A^{10} V_2) \\ &= 3\lambda_1^{10} V_1 + 2\lambda_2^{10} V_2 \\ &= 3(-1)^{10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2(2)^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -3 \end{bmatrix} + 2^{11} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{11} \end{bmatrix}$$

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Determinants

$$|A| = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= \sum_{j=1}^{n} a_{1j} \underbrace{C_{1j}}_{\text{cofactor}}$$

Ex: Determinant of $A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$

max 0's

$$\det A = \cancel{a_{13}}C_{13} + a_{23}C_{23} + \cancel{a_{33}}C_{33} + \cancel{a_{43}}C_{43}$$

$$= 2\,C_{23}$$

$2+3=5$)

$$= -2 \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ -2 & 1 & 0 \end{vmatrix}$$ → expand using this

$$= -2\left(-2 \begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -2\left(-2(-8) - 1(5)\right)$$

$$= -2(16 - 5) = -22$$

- Δ^{lar} matrix : $|A|$ = product of diagonal entries
- $\det(KA) = K^n \det(A)$
- $\det(AB) = \det(A) \cdot \det(B)$
- If A is invertible, then $|A^{-1}| = \dfrac{1}{\det(A)}$

Cramer's rule : (Invertible matrix only)

$$x_i = \frac{\det A_i(b)}{\det A} \qquad i = 1, 2, \ldots n$$

Qn: Use cramer's rule to solve.

$$x_1 + 2x_2 = 2$$
$$-x_1 + 4x_2 = 1$$

$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$|A| = 6$

$A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$ $\quad |A_1| = 6$

$\dfrac{|A_1|}{|A|} = \dfrac{6}{6} = 1$

$\boxed{x = 1}$

$A_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad |A_2| = 3$

$\dfrac{|A_2|}{|A|} = \dfrac{3}{6} = \dfrac{1}{2}$

$\boxed{y = 2}$

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Geometric Applications of determinants

(1) Vol. Parallelopiped : $= |A|$

$$V = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(2) Area of parallelogram : $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$

(3) Lines & planes :

* Points : (x_1, y_1) & (x_2, y_2)

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

The eqn of line passing through (x_1, y_1) & $($

is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

Ex: $(2, 3)$ & $(-1, 0)$ Find ~~[illegible]~~ eqn of line.

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$x(3) - y(3) + 1(3)$$
$$3x - 3y + 3 = 0.$$
$$\boxed{x - y + 1 = 0}$$

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(4) 3 points collinear if : $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(5) 4 points coplanar if $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$

(6) Curve fitting:

$y = a + bx + cx^2$ passing through:

(a_1, b_1) (a_2, b_2) & (a_3, b_3)

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Similarity:

A & B similar if: $P^{-1}AP = B$

Obs: We shd P such that $AP = PB$

Qn: $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$

$$AP = PB$$

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -c & -d \end{bmatrix}$$

$$PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} a-2b & -b \\ c-2d & -d \end{bmatrix}$$

$$\cancel{a} + 2c = \cancel{a} - 2b \qquad \boxed{b = -c}$$

$$b + 2d = -b \qquad \boxed{b = -d}$$

$$-c = c - 2d \qquad \boxed{c = d}$$

$$\cancel{-d = -d}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -d \\ d & d \end{bmatrix}$$

put $a = 1$ & $d = 1$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

To find similar matrices.

Steps:

A is Given.

① Find Eigen values.

② Eigen vectors (corresponding to all eigen values)

③ all eigen vectors combined into a matrix gives P.

Qn: $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

$$\lambda^2 - 0\lambda + (-1) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

• To find eigen vectors.

For $\cancel{+}1$: $\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2 \qquad \boxed{x_1 = s}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ -s \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For +1: $\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x_1 + 3x_2 = 0$$

$$-3x_2 = x_1 \qquad \boxed{x_2 = s}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ -3s \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$B = P^{-1} A P$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Theorem: (a) $A \sim A$ (reflexive.

$I^{-1} A I = A$ choose $P = I$

(b) $A \sim B$ then $B \sim A$

$B = P^{-1} A P$

$PB = \cancel{P}(\cancel{P^{-1}} A P)$

$PBP^{-1} = A\cancel{P}\cancel{P^{-1}}$

$PBP^{-1} = A$

(c) If $A \sim B$ & $B \sim C$, then $A \sim C$.

(Transitivity)

$B = P^{-1} A P$

$C = Q^{-1} B Q$

$B = Q^{-1} C Q$

$PAP^{-1} = Q^{-1} C Q$.

$Q(PAP^{-1}) = \cancel{Q}(\cancel{Q^{-1}} C Q)$

$Q(PAP^{-1})(Q^{-1}) = C\cancel{Q}\cancel{Q^{-1}}$

$(QP) A (QP)^{-1} = C$.

Hence Proved

Theorem: If $A \sim B$

① $|A| = |B|$

② A & B have same rank

③ A & B have same characteristic polynomial

④ A & B have same eigen values

⑤ A is invertible iff B is invertible

DIAGONALIZATION

$P^{-1} A P = D$. or $\boxed{AP = DP}$

sum = 1

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \quad \lambda = 1$$

char eqn : $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$$\lambda^2 - 3\lambda + 2$$

$$(\lambda - 1) \overline{) \lambda^3 - 4\lambda^2 + 5\lambda - 2}$$

$$\underline{\overset{(-)}{\lambda^3} \overset{(+)}{-} \lambda^2}$$

$$-3\lambda^2 + 5\lambda$$

$$\underline{\overset{(+)}{-3\lambda^2} \overset{(-)}{+} 3\lambda}$$

$$2\lambda - 2$$

$$\underline{\overset{(-)}{2\lambda} \overset{(+)}{-} 2}$$

$$0$$

$$\lambda^2 - 2\lambda - \lambda + 2$$

$$\lambda(\lambda - 2) - 1(\lambda - 2)$$

$$\boxed{\lambda = 2, 1, 1}$$

For $\lambda = 1$.

$$|A - \lambda I| x = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}}_{\text{Reduce}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x + y = 0$
$-y + z = 0$
$2x - 5y + 3z = 0$.

Reduce.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$-x + y = 0$

$-y + z = 0$. $-y = -z$.

$\boxed{z = s}$. $\boxed{y = z}$

$x = y = z$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

E. Vector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 1$.

For $\lambda = 2$, E. vector $= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

A has only 2 E. vectors.
$\therefore$ NOT diagonalizable.

Here AM $\neq$ GM $\therefore$ not diagonalizable.
(Don't ask how, I DONT KNOW)

$\boxed{GM \leq AM}$

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<u>Theorem</u>: If all E. values are distinct, matrix is diagonalizable.

<u>Diagonalization thm</u>:
If A is a matrix with distinct E values
(a) A is diagonalizable
(b) $AM = GM$.

Qn) Compute A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

E-val $= -1, 2$.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = [v_1 \; v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$P^{-1} = \dfrac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

thus $P^{-1}AP = D \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = D$

Now, $A = P^{-1}DP$

$A^{10} = P^{-1}D^{10}P^{-1}$

$= \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$

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Inner product

$\langle x, y \rangle$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$x^T y = [x_1 y_1 + x_2 y_2]$ = scalar

$x^T y$ is called inner product of x & y

written as: $\langle x, y \rangle$

Properties of inner product

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $\langle cu, v \rangle = c\langle u, v \rangle = \langle u, cv \rangle$
4. $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$
   Linearity property
5. If $u = 0$, $\langle u, u \rangle \geq 0$ Positive definite property

$|u| = \sqrt{u^2} = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2}$

$\left.\hat{u} = \dfrac{\vec{u}}{|u|}\right\}$ normalizing

Orthogonal Set

Set of vectors $(v_1, v_2, v_3, \cdots, v_k)$ - orthogonal set

if $\langle v_i \cdot v_j \rangle = 0$ for $i \neq j$

Vectors - mutually perpendicular

Orthogonal basis

① $\langle v_i, v_j \rangle = 0$ for $i \neq j$

② $v_1, v_2, \ldots, v_n$ are Linearly Independent

$(c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0)$

③ Any w, $w = c_1v_1 + c_2v_2 + \cdots + c_kv_k$

$\langle w, v_1 \rangle = c_1 \langle v_1, v_1 \rangle$

$$c_1 = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} \qquad c_2 = \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle}$$

$$\boxed{c_k = \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle}}$$

$$w_B = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \cdots + \frac{\langle w, v_k \rangle}{\langle v_k, v_k \rangle} v_k$$

Example

$v_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Find w_B

① Prove v_1 & v_2 are orthogonal & linearly ind.

② $c_1 = \frac{\langle w, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{1}{2}$

$c_2 = \frac{\langle w, v_2 \rangle}{\langle v_2, v_2 \rangle} = -1$

$w = \frac{1}{2} v_1 - 1 v_2$

$w_B = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$

$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Date / /

$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ on R^3

Find w_B $B = \{v_1, v_2, v_3\}$

① Prove v_1, v_2, v_3 are orthogonal. & linearly independant

② Find c_1, c_2, c_3 = $0, 2/3, 1/3$

③ $(w_B) = \begin{pmatrix} 0 \\ 2/3 \\ 1/3 \end{pmatrix}$

$\rightarrow w = 0v_1 + \frac{2}{3}v_2 + \frac{1}{3}v_3$

3) Find an orthogonal basis for subspace w of R^3 given by $w = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$

$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - 2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $v = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

but u & v are not orthogonal.

· Find another non-zero vector w that is orthogonal to either u or v

$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\langle u \cdot w \rangle = 0$ $x = -z$

$x + y = 0$ $y = z$

$x - y + 2z = 0$ $z = z$

$w = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\{u, w\}$ is an orthogonal set

Orthonormal set

① Orthogonal

② $|q_i| = 1$

$\langle q_1, q_2 \rangle = 0$

$|\langle q_1, q_2 \rangle| = 1$

★ If $\{q_1, q_2, \ldots q_n\}$ be orthonormal basis

$w = \langle w, q_1 \rangle q_1 + \langle w, q_2 \rangle q_2 + \ldots + \langle w, q_n \rangle q_n$

1) Construct an orthonormal basis for R^3

$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$q_1 = \frac{v_1}{|v_1|}$, $q_2 = \frac{v_2}{|v_2|}$, $q_3 = \frac{v_3}{|v_3|}$

$\{q_1, q_2, q_3\}$ forms an orthogonal basis for R^3

2) unit vector in the same direction $v = (1, -2, \ldots)$

$u = \frac{v}{|v|} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 1/3 \\ \cdot \end{pmatrix}$

Date / /

Orthogonal matrices.

$$A = \begin{bmatrix} c_1 & c_2 \end{bmatrix}_{n\times n}$$ c_1 & c_2 are orthonormal

If $A^T A = I$ - orthogonal.

Sq matrix: both row & col ~~norms~~ magnitude will be 1
Non-sq matrix: col norm will be 1.

→ To determine if matrix is orthogonal & to find inverse.

1. Check if $A^T A = I$ & $c_1 \cdot c_1 = 1$ $c_1 \cdot c_2 = 0$
2. A^T is only the inverse if condition 1 is true

→ Find missing entries - orthogonal

$$\begin{bmatrix} a_1 & b_1 & - \\ a_2 & b_2 & - \\ a_3 & b_3 & - \end{bmatrix}$$

$$a_1^2 + b_1^2 + c_1^2 = 1$$
$$a_2^2 + b_2^2 + c_2^2 = 1$$
$$a_3^2 + b_3^2 + c_3^2 = 1$$

$$a_1^2 + a_2^2 + a_3^2 = 1$$
$$b_1^2 + b_2^2 + b_3^2 = 1$$
$$c_1^2 + c_2^2 + c_3^2 = 1$$

Rotation & Reflection

Diagonal values are same: Rotation
Diagonal values are different: Reflection

Rotation: clockwise $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

anticlockwise $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Reflection $\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$

$\boxed{y = mx}$ $\boxed{m = \frac{y}{x}}$

Orthogonal Complement

$\vec{x}, \vec{v}, \vec{b}$ orthogonal to subspace W
$\vec{x}, \vec{v}, \vec{b}$ is orthonormal to all the vectors

$W^{\perp}$ $\vec{v}, \vec{b}$
$(W^{\perp})^{\perp} = W$
$W \cap W^{\perp} = $ null .

$V \cdot w_1 = V \cdot w_2 = 0$

$A x = 0$
matrix → vector

$(\text{Row } A)^{\perp} = \text{Null}(A)$
$(\text{Col } A)^{\perp} = (\text{Null } A^T)$

Date / /

1) W = subspace spanned by $\{w_1, w_2\}$. Find a basis for $W^{\perp}$, where $w_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $w_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \delta \begin{bmatrix} -1/4 \\ 5/4 \\ 1 \end{bmatrix} = 4_1 \begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix}.$$

Basis for $W^{\perp}$ = $\left\{ \begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix} \right\}$

2) W is subspace spanned by $u = [1, 2, 3, -1, 2]$

$v = [2, 4, 7, 2, -1]$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \Rightarrow \text{row echelon form, null space}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -17 \\ 0 \\ 5 \\ 9 \end{bmatrix} \right\} = \text{null}(A)$$

3> Find orthogonal complement of $W^\perp$ given for given basis for $W^\perp$: $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 2x - y = 0 \right\}$

$2x - y = 0$

$2x = y$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x \cdot w = 0$ $\quad V = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \quad a + 2b = 0$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

<u>Projection</u>

$proj_u v = \dfrac{\langle \vec{u} \cdot \vec{v} \rangle}{\langle \vec{u} \cdot \vec{u} \rangle} \vec{u}$

1> $\vec{v} = [3, 4]$ $\quad \vec{u} = [1, 2]$

$proj_u v = \begin{bmatrix} 11/5 \\ 22/5 \end{bmatrix}$.

$proj_v u = \frac{11}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33/25 \\ 44/25 \end{bmatrix}$

2> Find orthogonal projection of $v = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$ on subspace spanned by $u_1 = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$

$u_1 = v_1$

$u_2 = v_2 - \dfrac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$

QR decomposition

$$A = \begin{bmatrix} & \\ & \end{bmatrix}$$

Q — orthogonal R — upper Δ^{lar}

1. Extract column vectors of A.
2. Gram Schmidt procedure - orthonormalized form.
3. $R_{ij} = Q_i^T A_j$
4. Verify $A = QR$.

1) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$ $a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$u_1 = a_1$$

$$u_2 = a_2 - \frac{\langle a_1 \cdot a_2 \rangle}{\langle a_1 \cdot a_1 \rangle} a_1$$

$$q_1 = \frac{u_1}{|u_1|} \qquad q_2 = \frac{u_2}{|u_2|}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix}$$

$$R_{ij} = \begin{bmatrix} 3/\sqrt{3} & 0 \\ 0 & 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$A = QR$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A$$

$A = QR$

Hence Proved.

Orthogonal Diagonalization

Matrix A is orthogonall diagonalizable where Q is orthogonal, D is diagonalizable

$$A \begin{cases} Q \text{ (orthogonal)} \\ D \text{ (diagonal)} \Rightarrow \begin{bmatrix} \text{Eval} & 0 \\ 0 & \text{Eval} \end{bmatrix} \end{cases}$$

$$\boxed{A = QDQ^T} \qquad D = Q^TAQ$$

- Matrix should be symmetrical to be orthogonally diagonalizable

① $A = A^T$ (checking of symmetrical)
② E. values
③ E. vectors
④ Q matrix $\rightarrow P = [v_1\ v_2]$ $\rightarrow$ normalize $= [u_1, u_2]$
⑤ $D = Q^T A Q$.

Qn

1) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ $\qquad A - \lambda I = 0$

① $A = A^T$ ✓
② E. values: $-3, 2$.
③ $\lambda = -3 : \begin{bmatrix} 1 \\ -2 \end{bmatrix} = v_1$; $\lambda = 2 : \begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$

thus $P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \Rightarrow [u_1\ u_2]$

$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$Q^TAQ = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = D$.

$\therefore$ A is orthogonally diagonalizable

Gram-Schmidt Orthonormalization.

$$\begin{array}{ccc} v_1, & v_2, & v_3 \\ \downarrow & \downarrow & \downarrow \\ Q_1 & Q_2 & Q_3 \end{array}$$

Orthogonalization

$$\left\{ \begin{aligned} Q_1 &= V_1 \\ Q_2 &= V_2 - \frac{\langle V_2, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} Q_1 \\ Q_3 &= V_3 - \frac{\langle V_3, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} Q_1 - \frac{\langle V_3, Q_2 \rangle}{\langle Q_2, Q_2 \rangle} Q_2 \end{aligned} \right.$$

Orthonormalization

$$\left\{ \hat{Q}_1 = \frac{Q_1}{|Q_1|} \qquad \hat{Q}_2 = \frac{Q_2}{|Q_2|} \qquad \hat{Q}_3 = \frac{Q_3}{|Q_3|} \right.$$

$$\text{proj}_u v = \frac{\langle v \cdot u_1 \rangle}{\langle u_1 \cdot u_1 \rangle} u_1 + \frac{\langle v \cdot u_2 \rangle}{\langle u_2 \cdot u_2 \rangle} u_2$$

- If A is symmetric, then any 2 eigen vectors corresponding to distinct eigenvalues of A are orthogonal.

Spectral theorem

- Set of E. values = spectrum

Spectral thm: A is orthogonally diagonalizable iff A is symmetric matrix

Spectral decomposition

$$A = QDQ^T = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} Q^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 q_1 & \cdots & \lambda_n q_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \boxed{\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T}$$

Spectral decomposition of A.

This is done ONLY with E. values & [illegible]

Qn ⟩ $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

① E. values

$(7-\lambda)(4-\lambda) - 4 = 0$

$28 - 4\lambda - 7\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 11\lambda + 24 = 0$ $\boxed{\lambda = 8, 3}$

② E. vectors

$\lambda = 8$ $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $-x + 2y = 0$

$2y = x$

[illegible]

$\lambda = 3$ $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $2x + y = 0$

[illegible]

$q_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad q_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ → orthogonal matrix.

$A = Q\Lambda Q^T$

$A = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}_{2\times2} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

$A = \begin{bmatrix} 8(2/\sqrt{5}) & 3(-1/\sqrt{5}) \\ 8(1/\sqrt{5}) & 3(2/\sqrt{5}) \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$A = \begin{bmatrix} 32/5 + 3/5 & \frac{16}{5} - \frac{6}{5} \\ \frac{16}{5} - \frac{6}{5} & \frac{8}{5} + \frac{12}{5} \end{bmatrix}$

$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

$$\boxed{A = 8 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} + 3 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}$$

$$\boxed{A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T}$$

Q. E. values & E. vectors given.

$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$ . . .

① Find q from E. vectors (normalise)

② Find q transpose

③ Multiply with E. vectors & add.

Quadratic forms

for x, y, z: $ax^2 + by^2 + cz^2 + dxy + exz + fyz$

for x, y: $ax^2 + by^2 + cxy$

1) $Q(x, y) = 2x^2 + 3xy + 4y^2$

coeff of $\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3/2 \\ 3/2 & 4 \end{bmatrix}$

$= [x \; y] \begin{bmatrix} 2 & 3/2 \\ 3/2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\boxed{X^T A X}$

$= 2x^2 + \frac{3}{2}xy + \frac{3}{2}xy + 4y^2$

$= \boxed{2x^2 + 3xy + 4y^2}$

2) $Q(x, y, z) = 2x^2 + 3xy - 4yz + z^2$

coeff of $\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz/2 \\ xz & yz & z^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3/2 & 0 \\ 3/2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$

$= [x \; y \; z] \begin{bmatrix} 2 & 3/2 & 0 \\ 3/2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

3) $Q(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2$

$[x_1 \; x_2 \; x_3] \begin{bmatrix} 2 & 3 & -3/2 \\ 3 & -1 & 0 \\ -3/2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

4) $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ $4x^2 + 3y^2$

5) $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ $2x^2 - 6xy + 5y^2$

(Do with steps : $X^T A X$)

6) $f(x_1, x_2, x_3) = 8x_1^2 - 3x_2^2 + 2x_3^2 - x_1 x_2 + 8x_2 x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 8 & -1/2 & 0 \\ -1/2 & -3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Classification of Quadratic forms :

$f(x) = x^T A x$

1. +ve definite $f(x) > 0 \ \forall \ x \neq 0$
2. +ve semidefinite $f(x) \geq 0 \ \forall \ x$
3. -ve definite $f(x) < 0 \ \forall x \neq 0$
4. -ve semidefinite $f(x) \leq 0 \ \forall \ x$
5. Indefinite if $f(x)$ takes on both +ve & -ve values.

Same classification can be done based on E. values .

Principal Axis Theorem

$x = QY \qquad Y = Q^T x$

① E values, E vectors $A^T A = D$;

② Check if orthogonal - make orthonormal

③ Form D using E values :

④ $y^T D y$ for final equation .

Ex: $f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$

change of variables - no cross product in

Matrix : $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

① E values :

$(5-\lambda)(2-\lambda) - 4 = 0$

$10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 1)(\lambda - 6)$.

$\boxed{\lambda_1 = 6 \; , \lambda_2 = 1}$

E vectors

$V_2 : \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \quad -x + 2y = 0$

$2y = x$

$y = t$

$x = 2t$

$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad V_1 : \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \lambda = 1$

Orthogonal

$V_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$Q = [q_1 \ \ q_2] = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q^T A Q = D.$$

Change of variables $x = Qy$.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \& \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

converts f into $f(y) = f(y_1, y_2)$

$$[y_1 \ \ y_2] \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 6y_1^2 + y_2^2$$

Given $x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\boxed{y = Q^T X}$

$$f(x) = x^T A x$$

$$f(-1,3) = 5(-1)^2 + 4(-1)(3) + 2(3)^2 = 11$$

$$f(y_1, y_2) = 6y_1^2 + y_2^2 = 6\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-7}{\sqrt{5}}\right)^2$$

$$y = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}_{2\times 2} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$y = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-7}{\sqrt{5}} \end{bmatrix} \begin{matrix} \to y_1 \\ \to y_2 \end{matrix}$$

Singular Value Decomposition (SVD)

Matrix A.

$A^TA \Rightarrow$ symmetric matrix

sq root of eigen values : singular value

Every symmetric matrix can be factored as $A = PDP^T$

orthogonal   diagonal

Singular value of a matrix

Matrix A^TA

① E-values of A^TA

② $\sqrt{\text{E-value}} = \sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$

Qn $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

① E-values : $\lambda_1 = 3$, $\lambda_2 = 1$

② $\sigma_1 = \sqrt{3}$ $\sigma_2 = 1$

$\sqrt{3}$, 1 are singular values of A

$A = m \times n$

SVD:

$A = U \Sigma V^T$

$A \to m \times n$.

$U \to m \times m$

$V^T \to m \times n$.

① $U = \frac{1}{\sigma_1} A v_1 + \frac{1}{\sigma_2} A v_2 \ldots$

② $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$

③ $V^T = [v_1 \; v_2]^T$

$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ $m-r$ no of rows, $n-r$ no of cols

Eg. $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ Find SVD.

$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 40 & 200 \end{bmatrix}$

① E value: $\lambda_1 = 360 \quad \lambda_2 = 90 \quad \lambda_3 = 0$.

② E. vectors: $v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

③ Arrange E. values in descending order & find S values.

$\sigma_1 = \sqrt{360} = 6\sqrt{10}$

$\sigma_2 = \sqrt{90} = 3\sqrt{10}$

$\sigma_3 = 0 = 0$.

$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$ (consider only non-zero eigen values)

Rows : $m - r = 0$
$(2 - 2)$
Cols : $n - r = 1$
$(3 - 2)$

$m - r$ no of rows
$n - r$ no of cols

Σ : $A = 2\times3$ $\quad D = 2\times2$ $\quad \boxed{\begin{matrix} r = 2 \text{ since } m = 2 \\ r = 2 \quad n = 3 \end{matrix}}$ $\begin{bmatrix} D & 0 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$\rightarrow$ $\boxed{U_1 = \frac{1}{\sigma_1} . A . V_1} = \frac{1}{6\sqrt{10}} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times3} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}_{3\times1}$

$$U_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$\rightarrow$ $\boxed{U_2 = \frac{1}{\sigma_2} . A . V_2} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

Now SVD of A is

$$A = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}_{V^T}$$

- Matrix Σ is the same size as A. Diagonal entries are σ values 0's elsewhere.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad V_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\lambda_1 = 3, \quad \lambda_2 = 1$

$\sigma_1 = \sqrt{3}, \quad \sigma_2 = 1$

$$U_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U_2 = \cancel{\tfrac{1}{\sqrt{3}}}\ 1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

U should ALWAYS be sq. matrix.
Here 3x3.
But we have only U_1 & U_2.
$\{u_1, u_2, e_3\}$.
Find u_3 using Gram Schmidt procedure.

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$A = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$