Department of Electrical Engineering
Indian Institute of Technology Kharagpur

Digital Signal Processing Laboratory (EE39203)

Autumn, 2022-23

Experiment 3 **Frequency Analysis**

Slot: Date:

Student Name: Roll No.:

Grading Rubric

	Tick the best applicable per row			Points
	Below Expectation	Lacking in Some	Meets all Expectation	
Completeness of the report				
Organization of the report (5 pts) *With cover sheet, answers are in the same order as questions in the lab, copies of the questions are included in report, prepared in LaTeX*				
Quality of figures (5 pts) *Correctly labelled with title, x-axis, y-axis, and name(s)*				
Ability to compute Fourier series expansion and synthesize periodic signals using the expansion (15 pts) *Derivation and sketch, plots of synthesized signals, questions*				
Implementation of DTFT (25 pts) *Matlab function*				
Magnitude and Phase Response of DTFT (25 pts) *DTFT's magnitude and phase plots*				
Discrete time system analysis (25 pts) *Exercises in 3.3, completed block diagram, table of measurements, impulse and frequency response*				
			TOTAL (100 pts)	

Total Points (100): TA Name: TA Initials:

Digital Signal Processing Laboratory (EE39203)
Experiment 3: Frequency Analysis

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1. Learning Objective

In this experiment, we will use Fourier series and Fourier transforms to analyze continuous-time and discrete-time signals and systems. The Fourier representations of signals involve the decomposition of the signal in terms of complex exponential functions. These decompositions are very important in the analysis of linear time-invariant (LTI) systems, due to the property that the response of an LTI system to a complex exponential input is a complex exponential of the same frequency! Only the amplitude and phase of the input signal are changed. Therefore, studying the frequency response of an LTI system gives complete insight into its behaviour

2. Synthesis of Periodic Signals

To Compute the Fourier Series expansion of the given periodic signals and synthesize the signals using Fourier Series representation and draw the resulting waveform for the time interval [0,To]

$$s(t) = \mathrm{rect}\left(t - \frac{1}{2}\right)$$

Period $T_0 = 2$. For $t \in [0, 2]$:

$$s(t) = \mathrm{rect}(2t) - \frac{1}{2}$$

Period $T_0 = 1$. For $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$:
The Fourier series expansion is :-

$$\begin{aligned} g(t) &= a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \\ &= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \end{aligned}$$

The coefficients are :-

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t) dt \\ a_m &= \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi mt}{T}\right) dt \\ b_n &= \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi nt}{T}\right) dt \end{aligned}$$

Signal 1:-

Given the signal $s(t) = \text{rect}\left(t - \frac{1}{2}\right)$ for $t \in [0, 2]$ with a period $T_0 = 2$, the Fourier series can be expressed as:

$$s(t) = a_0 + \sum_{k=1}^{\infty} \left(C_k \cos\left(2\pi k f_0 t\right) + B_k \sin\left(2\pi k f_0 t\right)\right)$$

where the fundamental frequency f_0 is given by:

$$f_0 = \frac{1}{T_0} = \frac{1}{2}$$

The Fourier series is expressed as:

$$s(t) = a_0 + \sum_{k=1}^{\infty} \left(C_k \cos\left(2\pi k f_0 t\right) + B_k \sin\left(2\pi k f_0 t\right)\right)$$

where the DC component a_0 is computed as follows:

$$a_0 = \frac{1}{T_0} \int_0^{T_0} s(t) dt$$

Substituting the given signal $s(t)$:

$$a_0 = \frac{1}{2} \int_0^2 \text{rect}\left(t - \frac{1}{2}\right) dt$$

The rectangular function rect $\left(t - \frac{1}{2}\right)$ is non-zero only in the interval $l \in \left[\frac{1}{2}, \frac{3}{2}\right]$. Thus, the integral simplifies to:

$$a_0 = \frac{1}{2} \int_{\frac{1}{2}}^{\frac{2}{2}} 1 dt$$

$$a_0 = \frac{1}{2} \times t\big|_0^1 = \frac{1}{2} \times (1 - 0) = \frac{1}{2}$$

The cosine coefficient C_k is given by:

$$C_k = \frac{2}{T_0} \int_0^{T_0} s(l) \cos\left(2\pi k \int_0 l\right) dl$$

Substituting the signal and $f_0 = \frac{1}{2}$:

$$C_k = \int_0^2 \text{rect}\left(\ell - \frac{1}{2}\right) \cos(\pi k l) d\ell$$

Since this integral is only non-zero for $t \in [0,1]$, we get:

$$C_k = \int_0^1 \cos(\pi k t) dt$$

The integral for C_k is nonzero only for $t \in [0,1]$;

$$C_k = \int_0^1 \cos(\pi k \cdot t) dt$$

This integral evaluates as follows:

$$C_k = \left. \frac{\sin(\pi k t)}{\pi k} \right|_0^1 = \frac{\sin(\pi k)}{\pi k} - \frac{\sin(0)}{\pi k}$$

Since $\sin(0) = 0$, this simplifies to:

$$C_k = \frac{\sin(\pi k)}{\pi k}$$

Given that $\sin(\pi k) = 0$ for all integer k, we have:

$$C_k = \frac{0}{\pi k} = 0$$

Thus, $C_k = 0$ for all k. The sine coefficient B_k is given by:

$$B_k = \frac{2}{T_0} \int_0^{T_0} s(t) \sin\left(2\pi k f_0 t\right) dt$$

Substituting the signal and $f_0 = \frac{1}{2}$:

$$B_k = \int_0^2 \operatorname{rect}\left(t - \frac{1}{2}\right) \sin(\pi k t) dt$$

This integral is non-zero only for $t \in [0,1]$:

$$B_k = \int_0^1 \sin(\pi k t) dt$$

Evaluating this integral:

$$B_k = -\left. \frac{\cos(\pi k i)}{\pi k} \right|_0^1 = -\frac{\cos(\pi k)}{\pi k} + \frac{\cos(0)}{\pi k}$$

Since $\cos(0) = 1$, we get:

$$B_k = \frac{1 - \cos(\pi k)}{\pi k}$$

Given that $\cos(\pi k) = (-1)^k$, the fimal result is:

$$B_k = \frac{1 - (-1)^k}{\pi k}$$

Therefore:

$$B_k = \begin{cases} \frac{2}{\pi k} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

To express the Fourier series in sine-phase form:

$$s(t) = a_0 + \sum_{k=1}^{\infty} A_k \sin(2\pi k f_0 t + \theta_k)$$

where:

$$A_k = \sqrt{C_k^2 + B_k^2}$$

and

$$\theta_k = \tan^{-1}\left(\frac{C_k}{B_k}\right)$$

For this signal:

$$A_k = |B_k| = \frac{2}{\pi k} \text{ for odd } k$$

Since $C_k = 0$,

$$\theta_k = \frac{\pi}{2} \left(\text{or } -\frac{\pi}{2}\right) \text{ for odd } k$$

The final Fourier series expansion for Signal 1 is:

$$s(t) = \frac{1}{2} + \sum_{k=1,3 \text{ add}}^{\infty} \frac{2}{\pi k} \sin\left(\pi k t + \frac{\pi}{2}\right)$$

Fourier Series of Signal 2

Given the signal $s(t) = \mathrm{rect}(2t) - \frac{1}{2}$ for $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ with a period $T_0 = 1$, the fundamental frequency is:

$$f_0 = \frac{1}{T_0} = 1$$

The Fourier series is expressod as:

$$s(t) = a_0 + \sum_{k=1}^{\infty} \left(C_k \cos(2\pi k f_0 t) + B_k \sin(2\pi k f_0 t) \right)$$

where:

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) dt$$

Substituting the given signal $s(t)$:

$$a_0 = \int_{-1/2}^{1/2} \left(\text{rect}(2t) - \frac{1}{2} \right) dt$$

The rectangular function rect(2t) is non-zero only for $t \in \left[-\frac{1}{4}, \frac{1}{4}\right]$, where rect $(2t) = 1$:

$$a_0 = \int_{-1/4}^{1/4} \left(1 - \frac{1}{2} \right) dt$$

$$a_0 = \int_{-1/4}^{1/4} \frac{1}{2} dt$$

$$a_0 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

For the cosine coefficient C_k :

$$C_k = 2 \int_{-T_0/2}^{T_0/2} \left(\text{rect}(2t) - \frac{1}{2} \right) \cos(2\pi k f_0 t) \, dt$$

Substituting $T_0 = 1$ and $f_0 = 1$:

$$C_k = 2 \int_{-1/2}^{1/2} \left(\text{rect}(2t) - \frac{1}{2} \right) \cos(2\pi k t) dt$$

The integral is non-zero only for $t \in \left[-\frac{1}{4}, \frac{1}{4}\right]$;

$$C_k = 2 \int_{-1/4}^{1/4} \left(1 - \frac{1}{2} \right) \cos(2\pi k t) dt$$

$$C_k = \int_{-1/4}^{1/4} \cos(2\pi k t) dt$$

$$C_k = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

Given the signal $s(t) = \text{rect}(2t) - \frac{1}{2}$ for $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ with a period $T_0 = 1$, the sine coeflicient B_k is calculated as follows:

$$B_k = 2 \int_{-T_0/2}^{T_0/2} \left(\text{rect}(2l) - \frac{1}{2}\right) \sin\left(2\pi k \int_0 t\right) dl$$

Substituting $T_0 = 1$ and $f_0 = 1$:

$$B_k = 2 \int_{-1/2}^{1/2} \left(\text{rect}(2t) - \frac{1}{2}\right) \sin(2\pi kl) dt$$

The rectangular function rect(2t) is non-zero only for $t \in \left[-\frac{1}{4}, \frac{1}{4}\right]$, so the integral simplifies to:

$$B_k = 2 \int_{-1/4}^{1/4} \left(1 - \frac{1}{2}\right) \sin(2\pi kl) dl$$

$$B_k = \int_{-1/4}^{1/4} \sin(2\pi kl) dl$$

Since the integrand $\sin(2\pi kt)$ is an odd function and the limits of integration are symmetric about zero, the integral evaluates to zero:

$$B_k = 0 \text{ for all } k$$

To express the Fourier series in sine-phase form:

$$s(t) = a_0 + \sum_{k=1}^{\infty} A_k \sin\left(2\pi k f_0 t + \theta_k\right)$$

where:

$$A_k = \sqrt{C_k^2 + B_k^2}$$

and

$$\theta_k = \tan^{-1}\left(\frac{C_k}{B_k}\right)$$

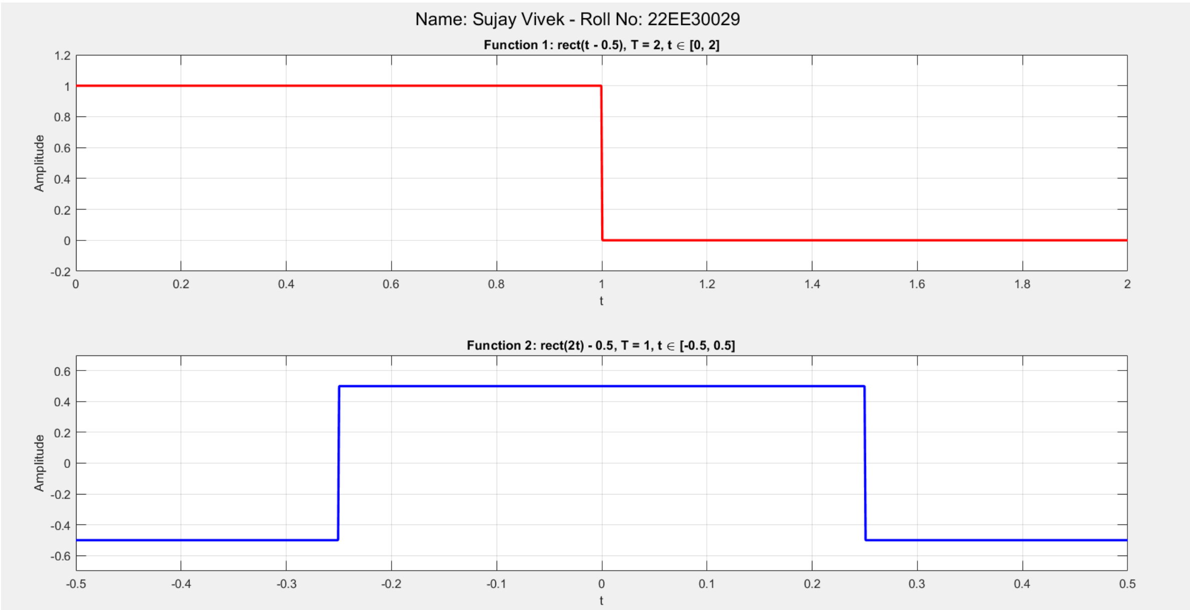
For this signal, since $B_k = 0$:

$$A_k = |C_k| = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

Since $B_k = 0$ and $\theta_k = 0$, the final Fourier series expansion for Signal 2 is:

$$s(t) = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k} \sin(2\pi kt)$$

Plots



(a) Plots of both the Functions

3 Discrete-Time Fourier Transform

The DTFT (Discrete-Time Fourier Transform) is the Fourier representation used for finite energy discrete-time signals. For a discrete-time signal, x(n), we denote the DTFT as the:

$$X(w) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-jwn}$$

Through DTFT we want to investigate how different signals and digital filters shape their frequency spectra, with a detailed analysis conducted using MATLAB simulations.

Signals :-

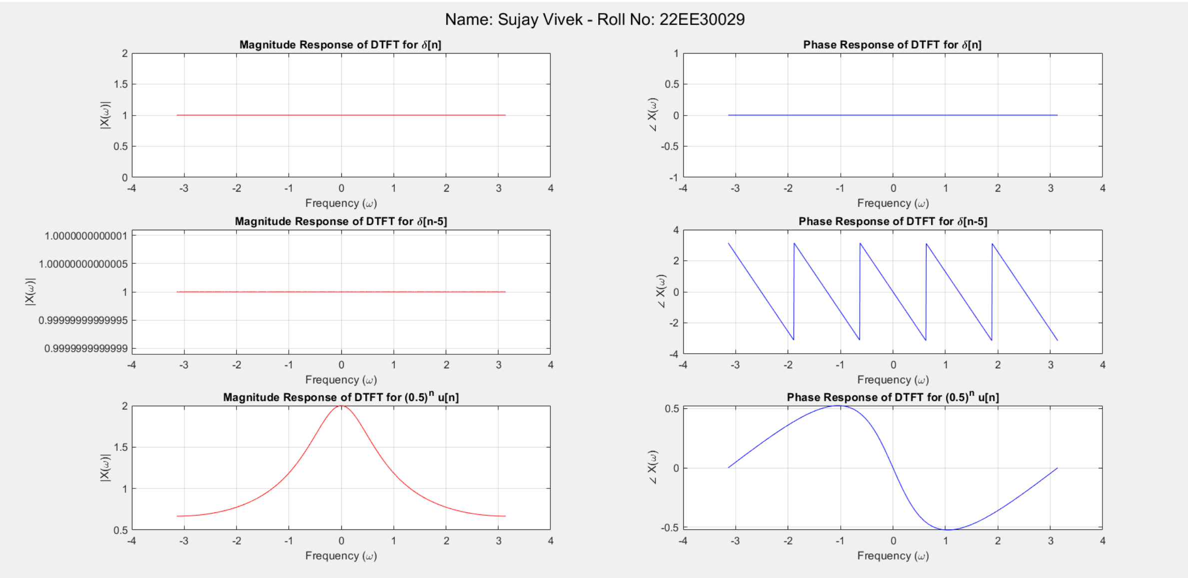
$$1.x[n] = \delta[n]$$

$$2.x[n] = \delta[n-5]$$

$$3.x[n] = (0.5)^n u[n]$$

Plots

Plotting with the help of MATLAB:



(a) Magnitude and Phase Response Plots of DTFT of the signal

Observations :-

a) DTFT of $\delta[n]$

Magnitude Response: The magnitude response is constant at 1 across all frequencies. This is expected because the DTFT of the unit impulse $\delta[n]$ is a constant function, meaning it has an equal contribution at all frequencies.

Phase Response: The phase response is 0 across all frequencies. This is because the impulse signal is non-shifted (at n = 0), so its phase is zero.

b) DTFT of $\delta[n-5]$

Magnitude Response: Similar to the $\delta[n]$ case, the magnitude response is constant at 1 across all frequencies. This is because shifting an impulse in time does not change its magnitude response.

Phase Response: The phase response is a linear function of frequency. The phase changes linearly with frequency, illustrating that the signal is delayed.

c) DTFT of $0.5^n * u[n]$

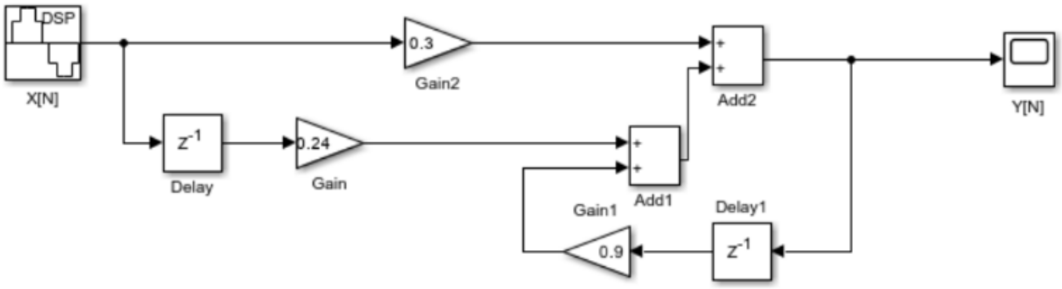
Magnitude Response: The magnitude response is not constant. It decreases with frequency, which is characteristic of an exponentially decaying signal. The peak magnitude occurs at ω=0, and it decreases as the frequency increases.

Phase Response: The phase response is non-linear and increases with frequency. This is due to the complex nature of the exponential signal, which introduces a varying phase shift across different frequencies

3.1 Magnitude and Phase of the Frequency Response of a Discrete-Time Systems

To examine the magnitude and phase characteristics of a discrete-time system's frequency response. This involves calculating the frequency response from the system's difference equation or transfer function and assessing how the system's output amplitude and phase change with frequency. The analysis will include visualizing these characteristics using MATLAB or similar tools to understand how the system influences signal strength and timing across various frequencies.

System Design



(a) Design of the System

Method I

Given the difference equation:

$$y[n] = 0.9y[n-1] + 0.3x[n] + 0.24x[n-1]$$

To find the impulse response $h[n]$, substitute $x[n] = \delta[n]$

$$h[n] = 0.9h[n-1] + 0.3\delta[n] + 0.24\delta[n-1]$$

For $n = 0$:

$$h[0] = 0.9 \cdot h[-1] + 0.3\delta[0] + 0.24\delta[-1] = 0 + 0.3 + 0 = 0.3$$

For $n = 1$:

$$h[1] = 0.9 \cdot h[0] + 0.3\delta[1] + 0.24\delta[0] = 0.9 \cdot 0.3 + 0 + 0.24 = 0.27 + 0.24 = 0.51$$

For $n = 2$:

$$h[2] = 0.9 \cdot h[1] + 0.3\delta[2] + 0.24\delta[1] = 0.9 \cdot 0.51 + 0 + 0.24 \cdot 0.9 = 0.459 + 0.216 = 0.675$$

For $n = 3$:

$$h[3] = 0.9 \cdot h[2] + 0.3\delta[3] + 0.24\delta[2] = 0.9 \cdot 0.675 + 0 + 0.24 \cdot 0.51 = 0.6075 + 0.1224 = 0.7299$$

The general expression for $h[n]$ for $n \geq 1$ is:

$$h[n] = 0.9^n \cdot 0.3 + 0.24 \cdot 0.9^{n-1}$$

Method II

Difference equation:

$$y[n] = 0.9y[n-1] + 0.3x[n] + 0.24x[n-1]$$

Taking the DTFT of both sides:

$$\begin{aligned}
Y\left(e^{j\omega}\right) &= 0.9e^{-j\omega}Y\left(e^{j\omega}\right) + 0.3X\left(e^{j\omega}\right) + 0.24e^{-j\omega}X\left(e^{j\omega}\right) \\
Y\left(e^{j\omega}\right) - 0.9e^{-j\omega}Y\left(e^{j\omega}\right) &= \left(0.3 + 0.24e^{-j\omega}\right)X\left(e^{j\omega}\right) \\
Y\left(e^{j\omega}\right)\left(1 - 0.9e^{-j\omega}\right) &= \left(0.3 + 0.24e^{-j\omega}\right)X\left(e^{j\omega}\right)
\end{aligned}$$

Frequency response $H\left(e^{j\omega}\right)$:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{0.3 + 0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}$$

Magnitude Response: The magnitude response $\left|H\left(e^{j\omega}\right)\right|$ is given by:

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{0.3 + 0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}\right|$$

Phase Response: The phase response $\angle H\left(e^{j\omega}\right)$ is given by:

$$\angle H\left(e^{j\omega}\right) = \arg\left(\frac{0.3 + 0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}\right)$$

Impulse Response

$$H(\omega) = \frac{0.3 + 0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}$$

$$H(\omega) = \frac{0.3}{1 - 0.9e^{-j\omega}} + \frac{0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}$$

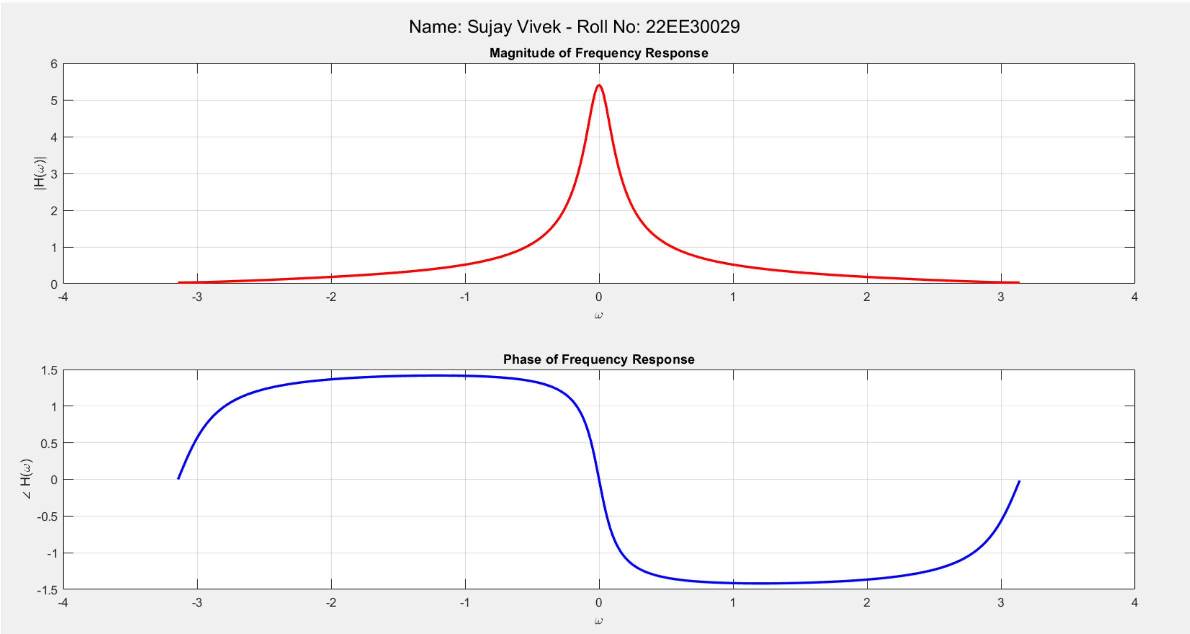
$$\frac{1}{1 - 0.9e^{-j\omega}} \text{ is the DTFT of } 0.9^n u[n]$$

$$\frac{e^{-j\omega}}{1 - 0.9e^{-j\omega}} \text{ is the DTFT of } 0.9^{n-1} u[n-1]$$

Therefore, the impulse response $h[n]$ is:

$$h[n] = 0.3 \cdot 0.9^n u[n] + 0.24 \cdot 0.9^{n-1} u[n-1]$$

Plots



(a) Phase and Magnitude Response

Observations :-

Frequency Response Calculation:

he frequency response H() is defined as:

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{0.3 + 0.24e^{-j\omega}}{1 - 0.9e^{-j\omega}}\right|$$

This expression represents a rational function where the numerator has a zero at a specific location in the z-plane, and the denominator defines the poles of the system.

Magnitude Response:

The magnitude of the frequency response H(ω) is calculated using 'abs(H)'. The plot shows how the system amplifies or attenuates signals at different frequencies. The shape of the magnitude plot indicates the system's filtering characteristics, such as whether it acts as a low-pass, high-pass, band-pass, or band-stop filter

Phase Response:

The phase of the frequency response H(ω) is calculated using 'angle(H)'. • The phase plot illustrates how the system shifts the phase of input signals across different frequencies. A linear phase response would indicate a constant time delay, while a non-linear phase response suggests varying delays across frequencies.

The magnitude response plot reveals the system's gain or attenuation across the frequency spectrum, while the phase response plot shows how the system delays or advances the phase of different frequency components. This analysis is crucial in understanding the behavior of the system in the frequency domain, which is essential for designing and analyzing filters and other signal processing systems

4. System Analysis

The goal of system analysis is to explore the dynamic behavior and characteristics of discrete-time systems. This includes studying the system's response to different input signals, assessing its stability, and evaluating its performance in both the time and frequency domains. The analysis offers insights into properties like linearity, time-invariance, and causality, and aids in designing systems that fulfill specific performance requirements.

$$y[n] = 0.9y[n-1] + 0.3x[n] + 0.24x[n-1]$$

Block Diagram

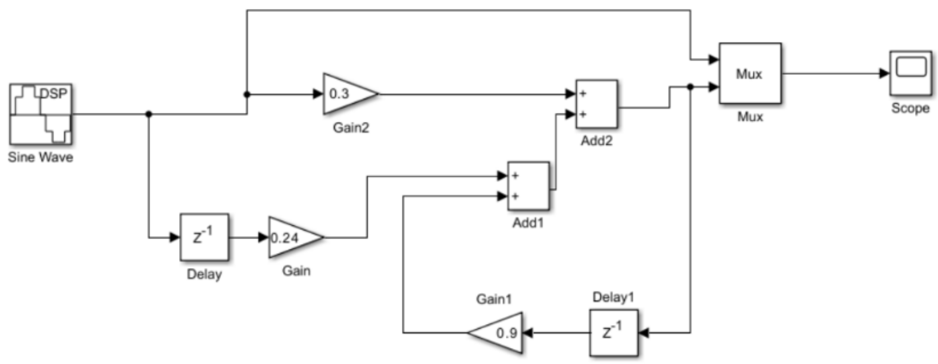


Figure 5: Plotting the Impulse Response of S1

The system was tested with sine wave inputs at the following frequencies:

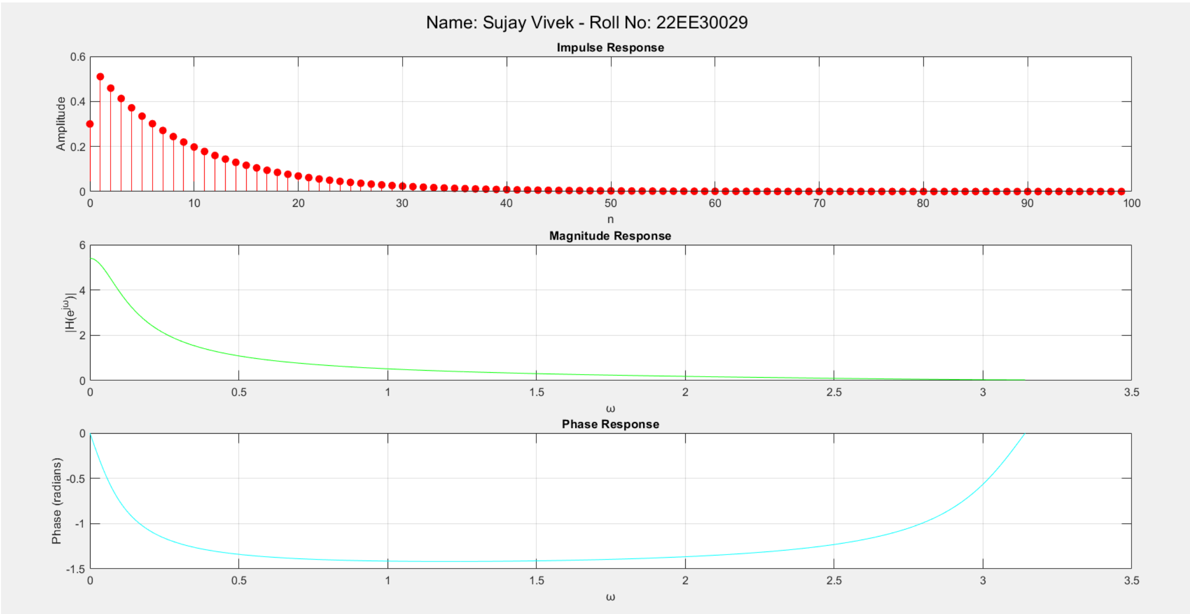
$$\omega = \pi/16$$

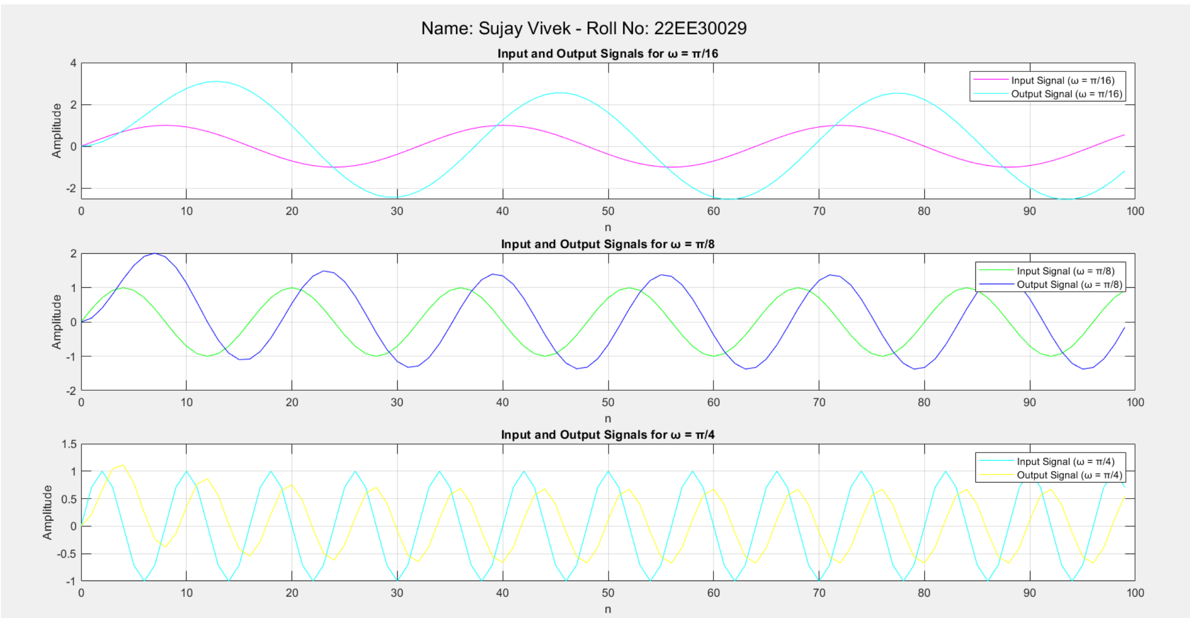
$$\omega = \pi/8$$

$$\omega = \pi/4$$

The magnitude response measurements were compared with the theoretical values of H($e^j w$)

Plots


Name: Sujay Vivek - Roll No: 22EE30029
Impulse Response
Amplitude
n
Magnitude Response
$|H(e^{j\omega})|$
ω
Phase Response
Phase (radians)
ω


Name: Sujay Vivek - Roll No: 22EE30029
Input and Output Signals for ω = π/16
Input Signal (ω = π/16)
Output Signal (ω = π/16)
Amplitude
n
Input and Output Signals for ω = π/8
Input Signal (ω = π/8)
Output Signal (ω = π/8)
Amplitude
n
Input and Output Signals for ω = π/4
Input Signal (ω = π/4)
Output Signal (ω = π/4)
Amplitude
n

4.1 Table of the amplitude measurements through Simulink and their theoretical value

5.3 Table of the amplitude measurements through Simulink and their theoretical values

Angular Frequency (rad/sec)	$\lvert H(\omega)\rvert$ Actual	$\lvert H(\omega)\rvert$ Theoretical
$\frac{\pi}{16}$	2.541	2.55
$\frac{\pi}{8}$	1.373	1.36
$\frac{\pi}{4}$	0.672	0.67

Conclusion

In this experiment, we explored frequency analysis techniques applied to discrete-time systems, both theoretically and practically. Through the computation and visualization of magnitude and phase responses across different input frequencies, we gained a clear understanding of the system's behavior. Our findings showed that the system's actual response aligned well with theoretical expectations, particularly in the magnitude measurements for various frequencies. By deriving the frequency response from the difference equation and comparing it with the impulse response, we confirmed the accuracy and consistency of our system analysis. This experiment also demonstrated the practical effectiveness of using tools like MATLAB to simulate and analyze complex systems. A deep understanding of frequency response is vital for the design and evaluation of digital filters and communication systems, underlining the importance of this knowledge in real-world engineering