Department of Electrical Engineering
Indian Institute of Technology Kharagpur

Digital Signal Processing Laboratory (EE39203)

Autumn, 2022-23

Experiment 4 **Sampling and Reconstruction of Continuous-Time Signals, Interpolation and Decimation**

Slot: Date:

Student Name: Roll No.:

Grading Rubric

	Tick the best applicable per row			Points
	Below Expectation	Lacking in Some	Meets all Expectation	
Completeness of the report				
Organization of the report (5 pts) *With cover sheet, answers are in the same order as questions in the lab, copies of the questions are included in report, prepared in LaTeX*				
Quality of figures (5 pts) *Correctly labelled with title, x-axis, y-axis, and name(s)*				
Understanding of sampling and reconstruction with an impulse generator (15 pts) *Plots of signals and their frequency spectrum, 'explain why' questions*				
Understanding of sampling and reconstruction with sample and hold (25 pts) *Plots, questions, analytical expression*				
Understanding of discrete-time interpolation (25 pts) *Plots of signals and their frequency spectrum, questions*				
Understanding of discrete-time decimation (25 pts) *Matlab code, questions*				
			TOTAL (100 pts)	

Total Points (100): TA Name: TA Initials:

Digital Signal Processing Laboratory (EE39203)
Experiment 4: Sampling and Reconstruction of Continuous-Time Signals, Interpolation and Decimation

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August 28th 2024

1. Learning Objective

To analyze and process continuous-time signals on a computer, the signals must first be digitized by sampling and quantization. This creates a digital signal that can be stored and processed. A digital system processes the discrete-time signals, producing discrete time outputs that can be converted back to analog using a digital-to-analog converter. When designing such a system, it is crucial to understand and manage distortions like low-pass filtering, aliasing, and quantization, ensuring they remain within acceptable limits or are compensated for through additional processing. Discrete-time systems play a crucial role in digital signal processing (DSP), as they facilitate the manipulation and transformation of signals across various applications, such as communications, control systems, and multimedia processing.

2. Overview of Sampling

Sampling is simply the process of measuring the value of a continuous-time signal at certain instants of time. These measurements are uniformly separated by the sampling period, Ts.

$$y[n] = x(t)|_{t=Ts}$$

$x(t)$ is input signal and $y[n]$ is sampled signal $Y(\omega)$ can be written in terms of $X(f)$:

$$Y(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(f) \Bigg|_{f=\frac{\omega-2\pi k}{2\pi T_s}} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T_s}\right)$$

To prevent aliasing, most sampling systems first low pass filter the incoming signal to ensure that its frequency content is below the Nyquist frequency. In that case $Y(\omega)$ can be related to X(f) by taking k = 0 in the above expression.

$$Y(\omega) = \frac{1}{T_s} X\left(\frac{\omega}{2\pi T_s}\right) \quad \forall \omega \in [-\pi, \pi]$$

$Y\left(e^{j\omega}\right)$ ia periodic with period $2\pi, Y\left(e^{j\omega}\right)$ and X(f) are related by a simple scaling of the frequency and magnitude axes. Also, $\omega = \pi$ in $Y\left(e^{j\omega}\right)$ corresponds to the Nyquist frequency, $f = \frac{1}{T_*}$

After digitally processing a sampled signal, it sometimes needs to be converted back to an analog form. This can theoretically be done by turning the discrete-time signal into a series of continuous-time impulses weighted by the sample values. When this impulse train is passed through an ideal low-pass filter with a cutoff at the Nyquist frequency, it produces a scaled version of the original low-pass filtered signal. The spectrum of this reconstructed signal, S(f), is then obtained by

$$S(f) = \begin{cases} \mathrm{Y}\left(2\pi f T_s\right), & \text{if } \forall|f| < \frac{1}{T_s} \\ 0, & \text{otherwise} \end{cases}$$

i) Sampling and Reconstruction Using Sample-and-Hold :

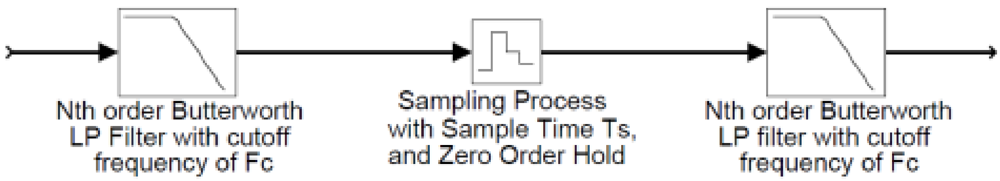


Figure 2: Sampling and reconstruction using a sample-and-hold.

(a) Magnitude Response of System for both types

Ideal Case: Impulse Train Sampling The frequency spectrum of the impulse train $S(f)$ can be computed by combining the sampling equation and the reconstruction equation:

$$S(f) = Y\left(2\pi f T_s\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(f - k f_s\right), \quad \forall|f| \leq \frac{1}{2T_s}$$

$$S(f) = 0, \quad \forall|f| > \frac{1}{2T_s}$$

If we assume that $f_s > 2f_e$ then the infinite sum reduces to one term. In this case, the reconstructed signal is given by:

$$S(f) = \frac{1}{T_s}X(f)$$

Sample-and-Hold Effect The sample-and-hold device generates a pulse of width T, and magnitude equal to the input sample. The signal out of the sample-and-hold is equivalent to the impulse train convolved with the pulse:

$$P(t) = \text{rect}\left(\frac{t}{T_s} - \frac{1}{2}\right)$$

Convolution in the time domain is equivalent to multiplication in the frequency domain, so this convolution with $p(t)$ is equivalent to multiplying by the Fourier transform $P(f)$, where

$$P(f) = T_s \cdot \left|\text{sinc}\left(\frac{f}{f_n}\right)\right|$$

Butterworth Filter Effect The magnitude of the frequency response of the N-th order Butterworth filter is given by:

$$|H_B(f)| = \frac{1}{\sqrt{1 + (t)^{2N}}}$$

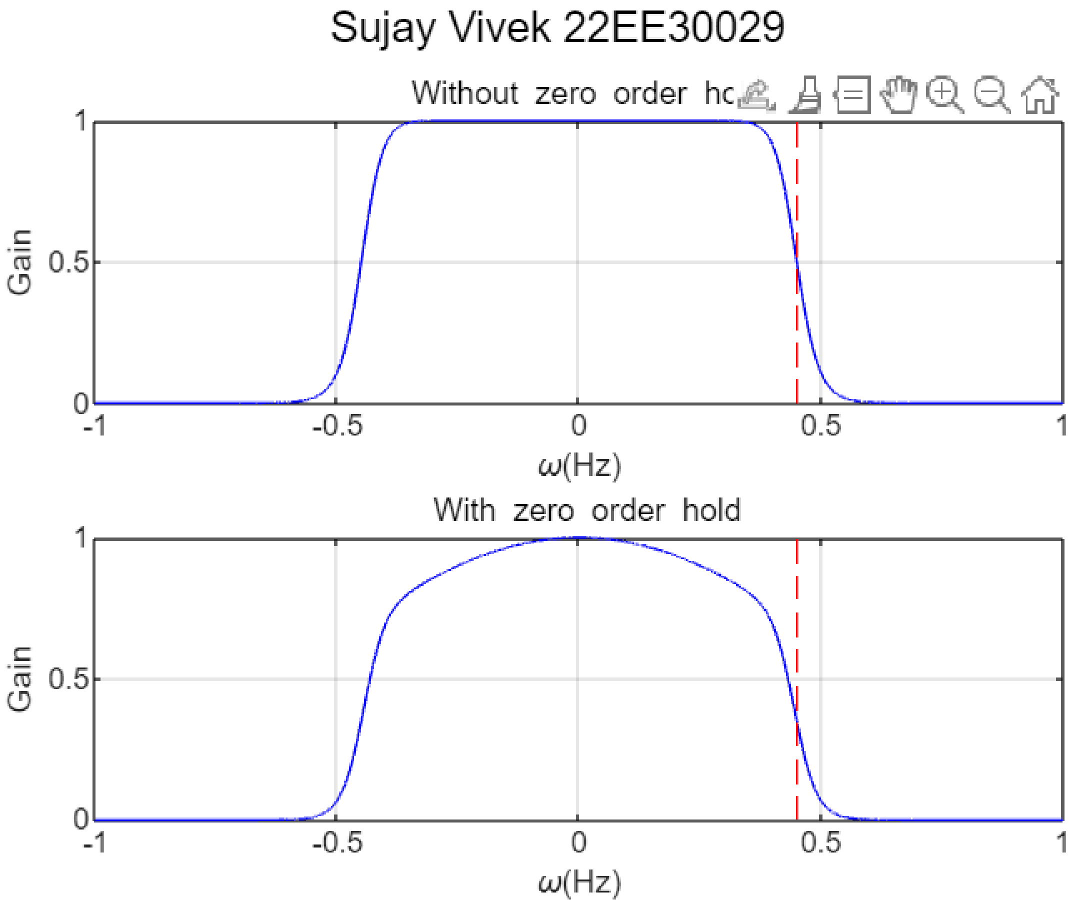
Complete Magnitude Response The complete magnitude response of the sample-and-hold system, including the effects of the Butterworth filters, the ideal sampling system, and the sample-and-hold pulse width, is given by:

$$|H(f)| = \left|H_b(f) \cdot P(f) \cdot \frac{1}{T_s} \cdot H_b(f)\right|$$

This can be substituted as

$$|H(f)| = \left(\frac{1}{\sqrt{1 + \left(\frac{f}{f_n}\right)^{2v}}}\right) \cdot \left|\text{sinc}\left(\frac{f}{f_*}\right)\right|$$

Plots



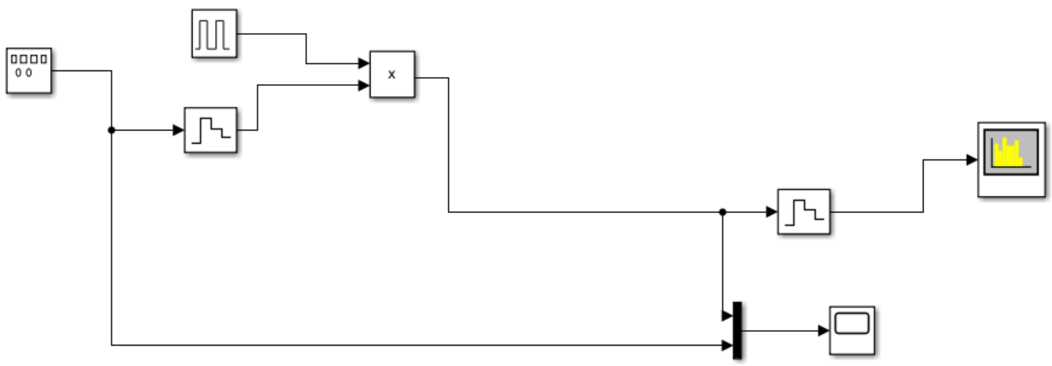
(a) Magnitude Response of System for both types

Effect of Filter on CD Player

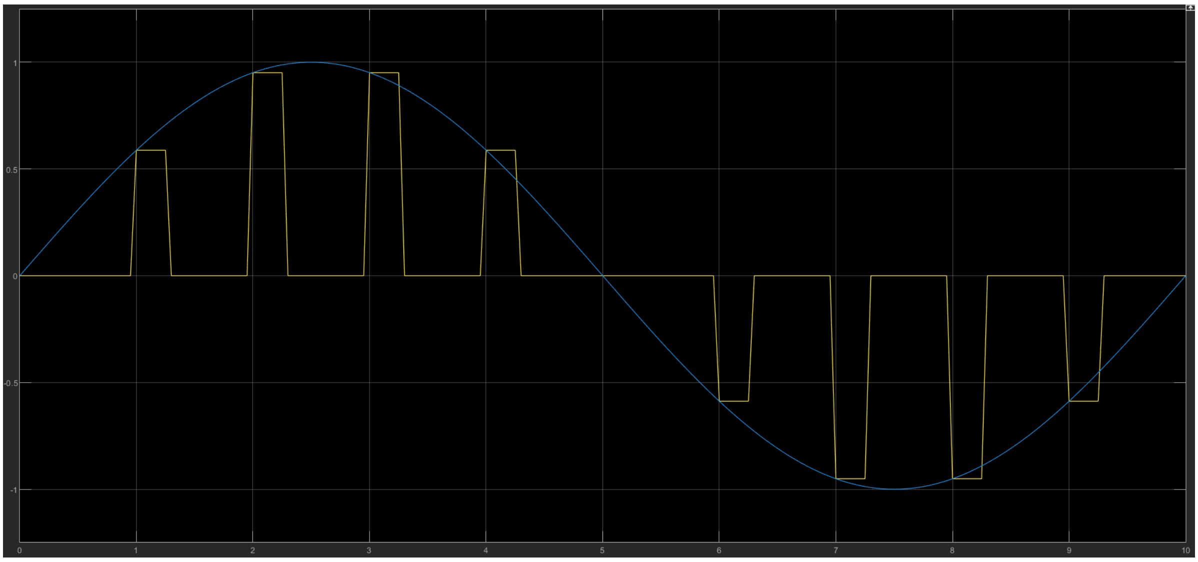
The shape of the magnitude responses from the Butterworth filter and the sample-and-hold operation significantly impacts the design of a high-quality audio CD player. The Butterworth filter has a flat passband with a sharp roll-off at the cutoff frequency, ensuring minimal distortion in the passband and effective attenuation of unwanted higher frequencies. The magnitude response without the sample-and-hold shows a smooth roll-off, while the response with the sample-and-hold exhibits additional attenuation near the Nyquist frequency due to the sinc function, leading to a drooping effect in the high-frequency range. This roll-off can attenuate higher frequencies within the audio range, potentially affecting the clarity and fidelity of the reconstructed audio signal.

ii) Sampling and Reconstruction with an Impulse Generator

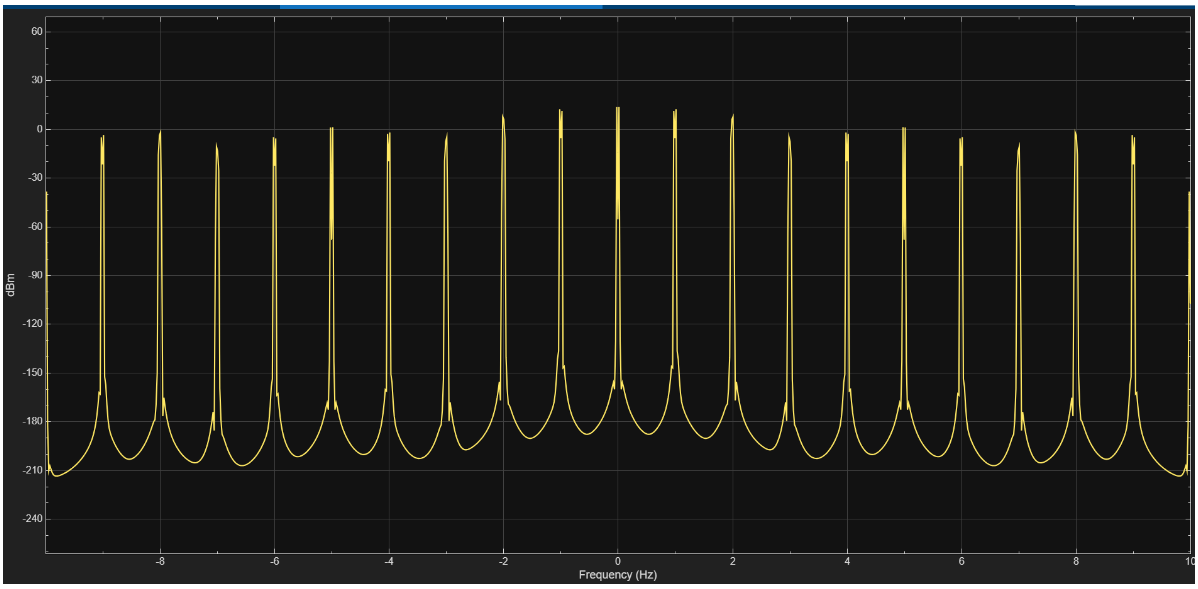
In this section, we will experiment with the sampling and reconstruction of signals using a pulse generator. This pulse generator is the combination of an ideal impulse generator and a perfect zero-order-hold device.



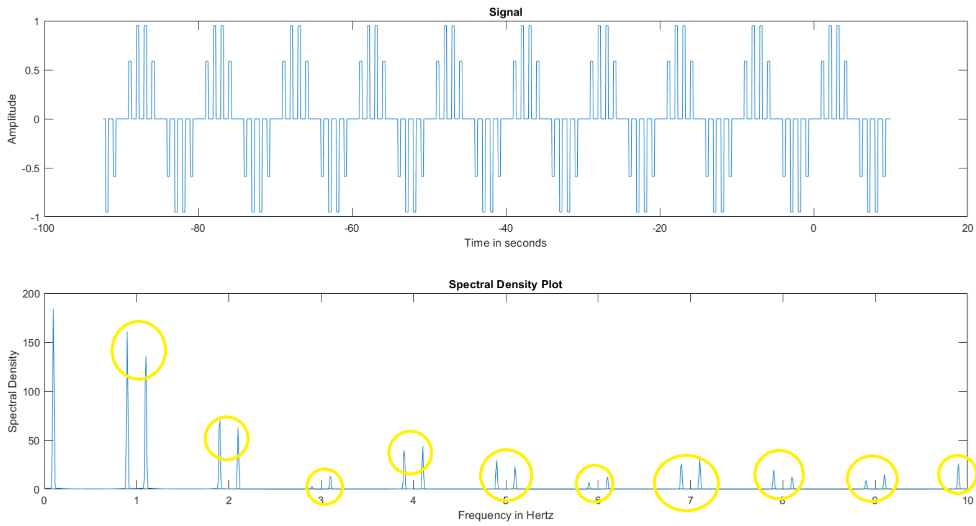
(a) Block Diagram



(a) Input and Output of signal of 0.1Hz and 0.3s pulse width

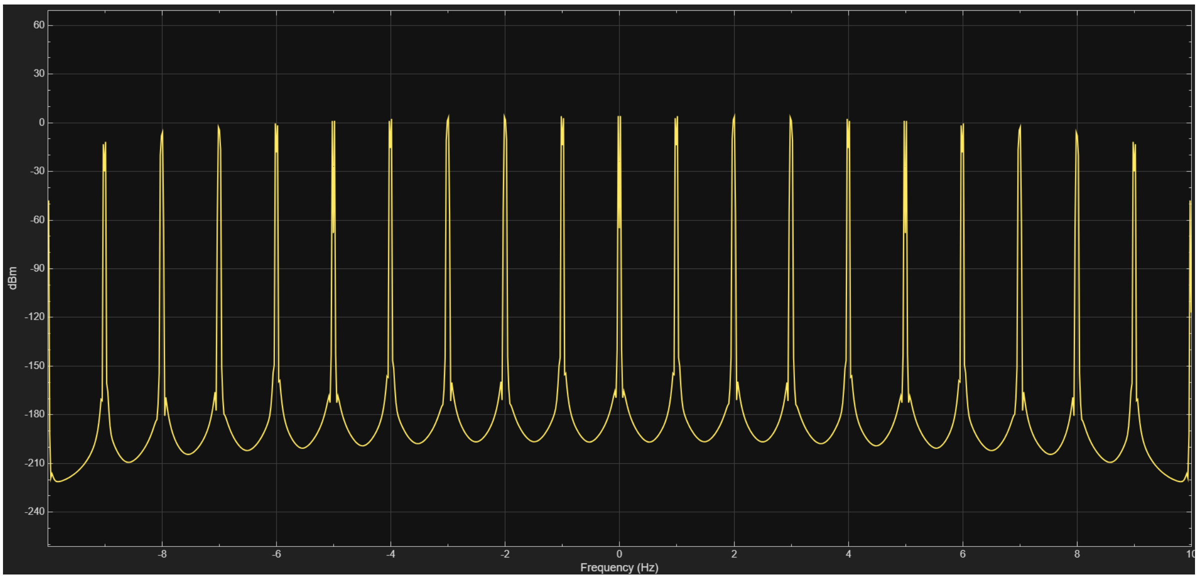


(a) Frequency Spectrum of Signal of 0.1Hz and 0.3s pulse width

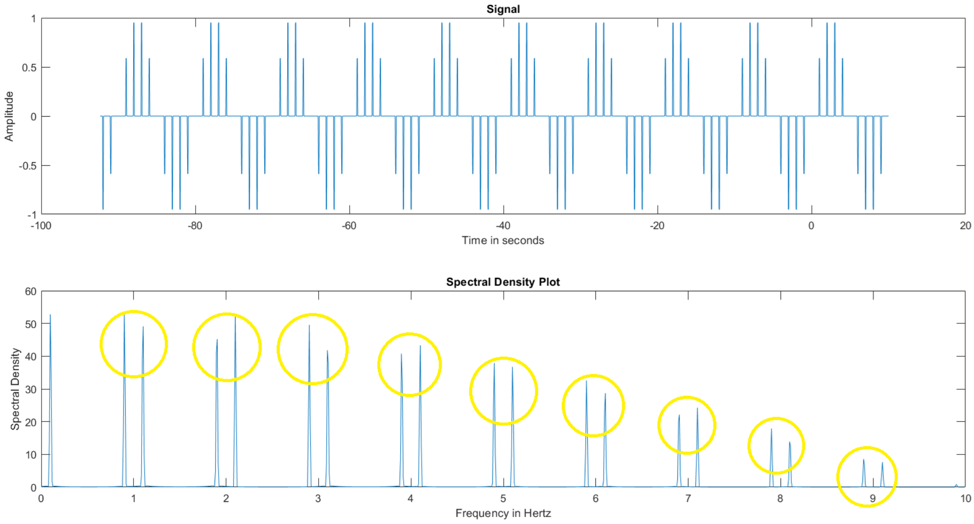


(a) Amplitude and Spectral Density of Signal of 0.1Hz and 0.3s pulse width

The Aliasing Frequencies are marked with Yellow Circles

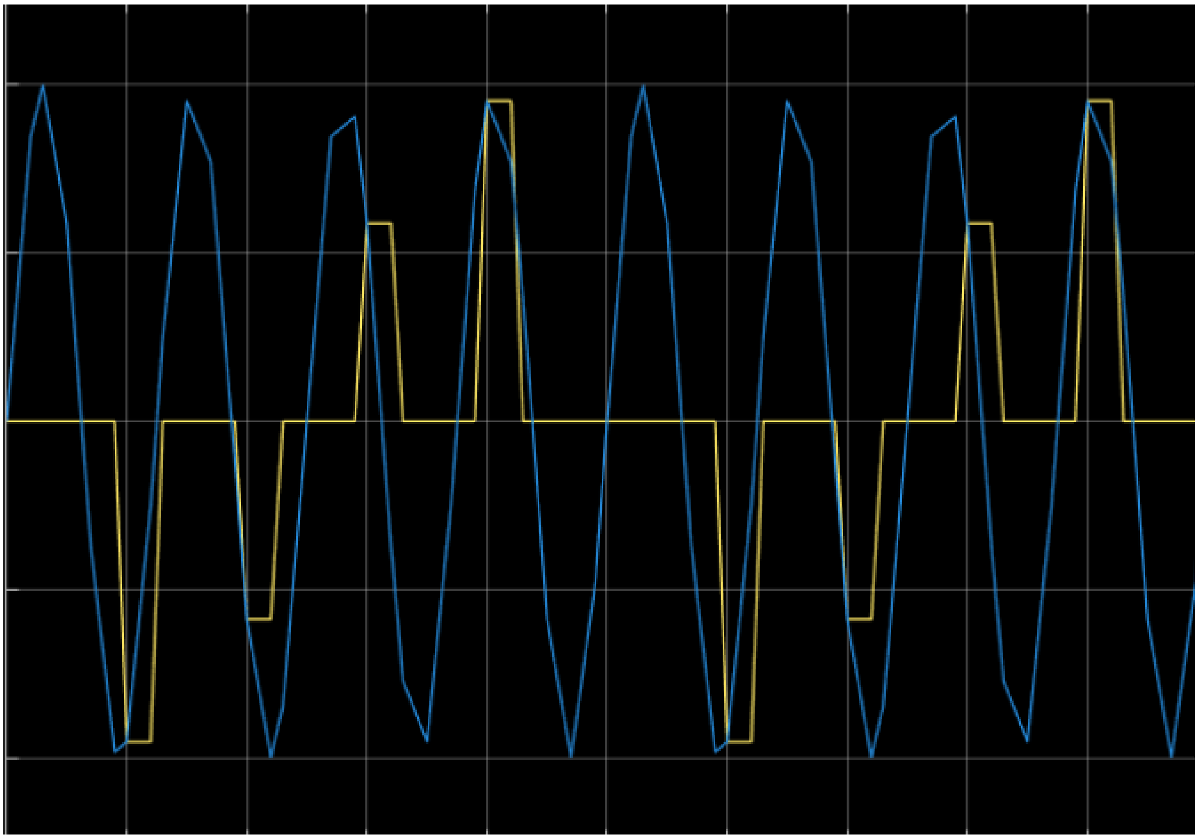


(a) Frequency Spectrum with 0.1Hz and 0.1s pulse width

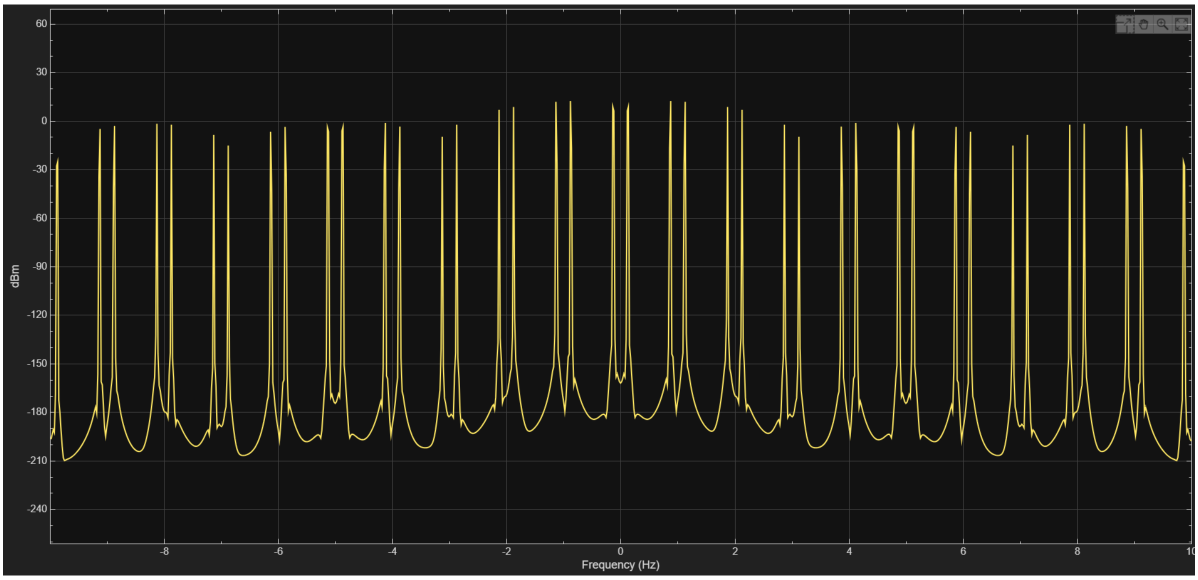


(a) Amplitude and Spectral Density Plots

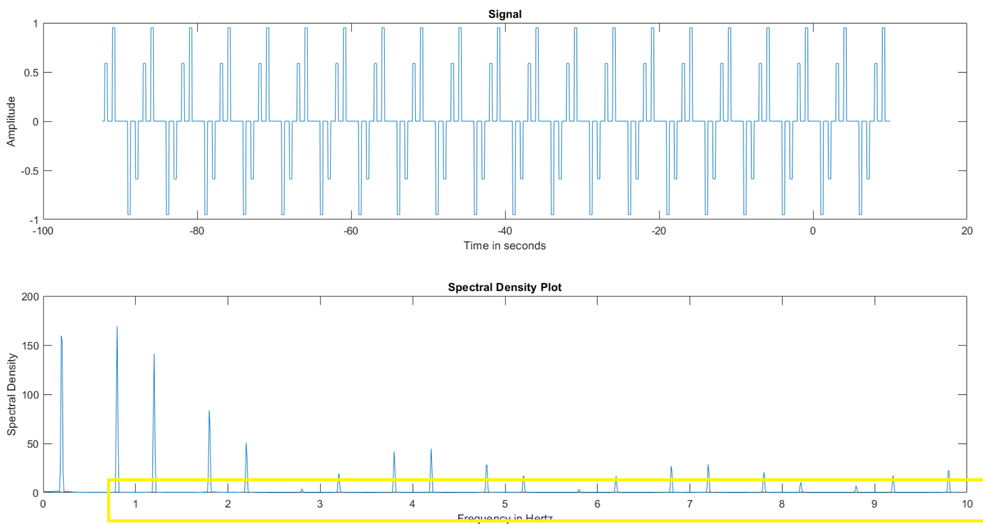
The Aliasing Frequencies are marked with Yellow Circles



(a) Input and Output Signal Diagram with 0.8 Hz and 0.3 s Pulse Width



(a) Frequency Spectrum with 0.8 Hz and 0.3 s Pulse Width

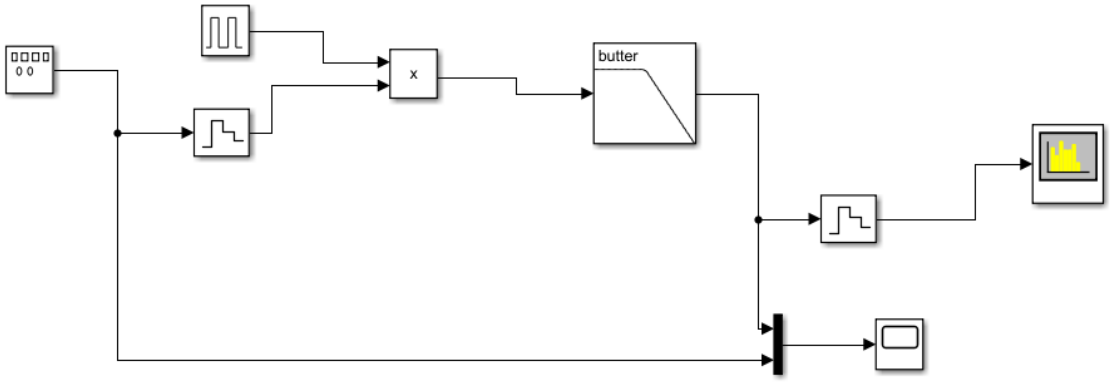


(a) Amplitude and Spectral Density curves

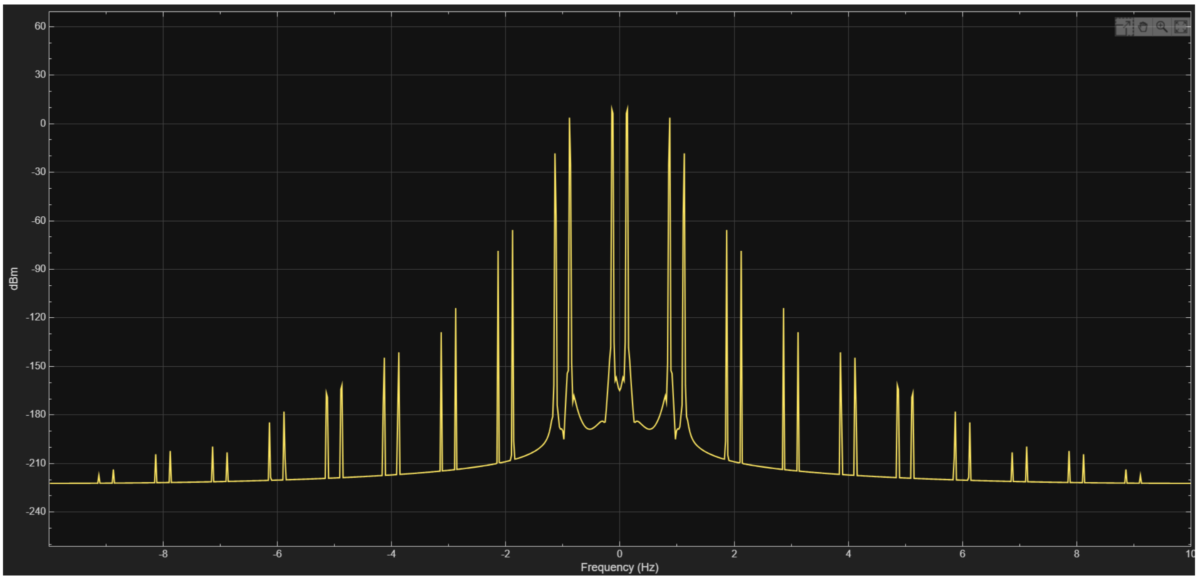
The aliasing frequencies are marked in Yellow

The frequency of the reconstructed signal differs from the frequency of the input sinusoid due to aliasing effects. Aliasing occurs when the sampling rate is not high enough to accurately capture the high-frequency components of the signal. In this experiment, although the sampling rate is 1 Hz, which theoretically should be sufficient for a sine wave of 0.8 Hz, the pulse width and the reconstruction process introduce distortions. The sinc function, which characterizes the reconstruction process, influences the frequency content of the output signal. Specifically, the convolution of the impulse train with the rectangular pulse (resulting from the sample-and-hold process) introduces additional frequency components and can shift the observed fundamental frequency. Consequently, the peak of the frequency spectrum of the reconstructed signal may not align with the input frequency, illustrating how the reconstruction process affects the signal's frequency characteristics.

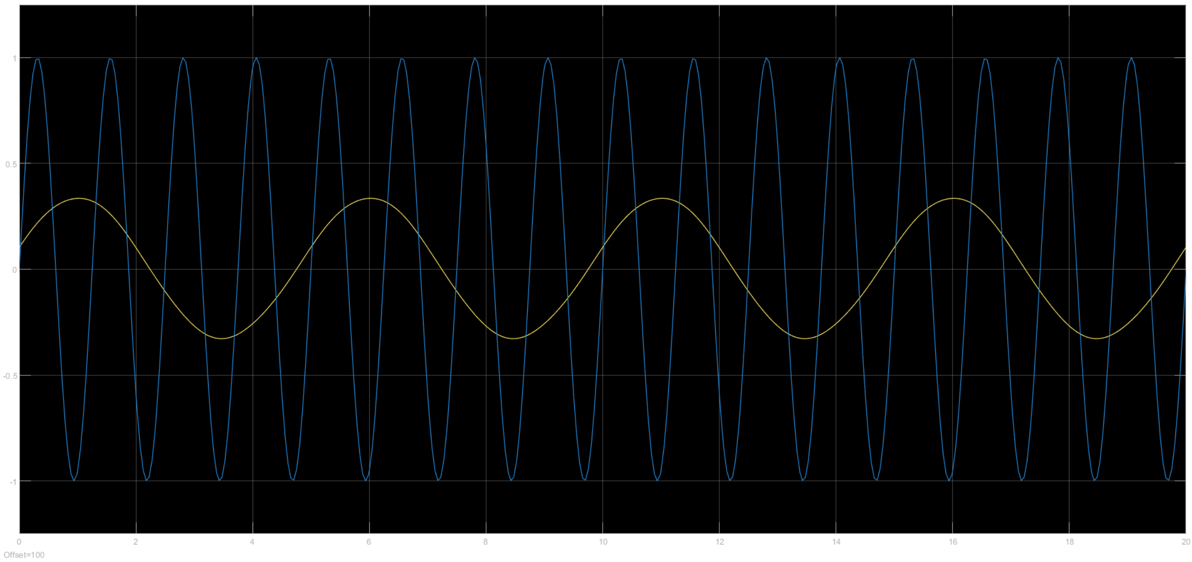
After adding a Butterworth filter of 10th order with a cutoff frequency of 0.5Hz The block diagram of the circuit looks like :



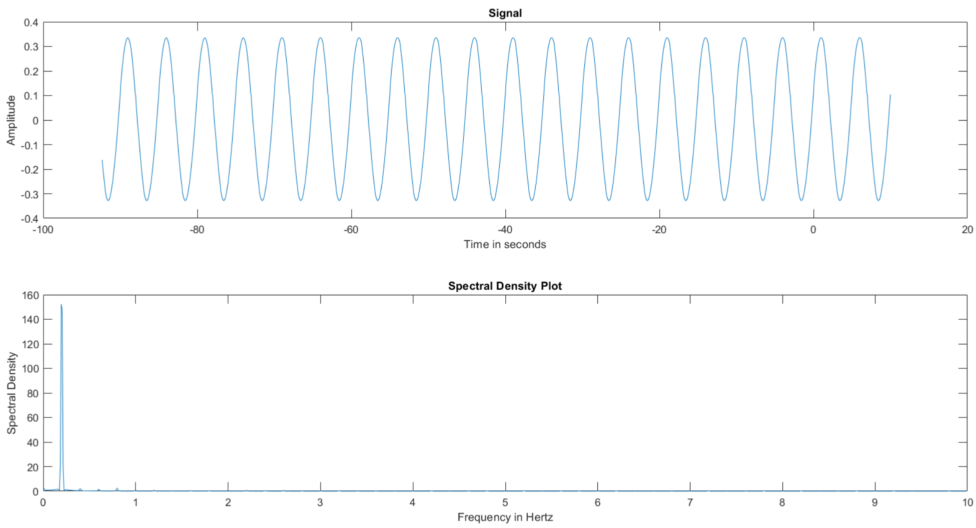
(a) Block Diagram Including Filter



(a) Frequency Spectrum using Butterworth Filter

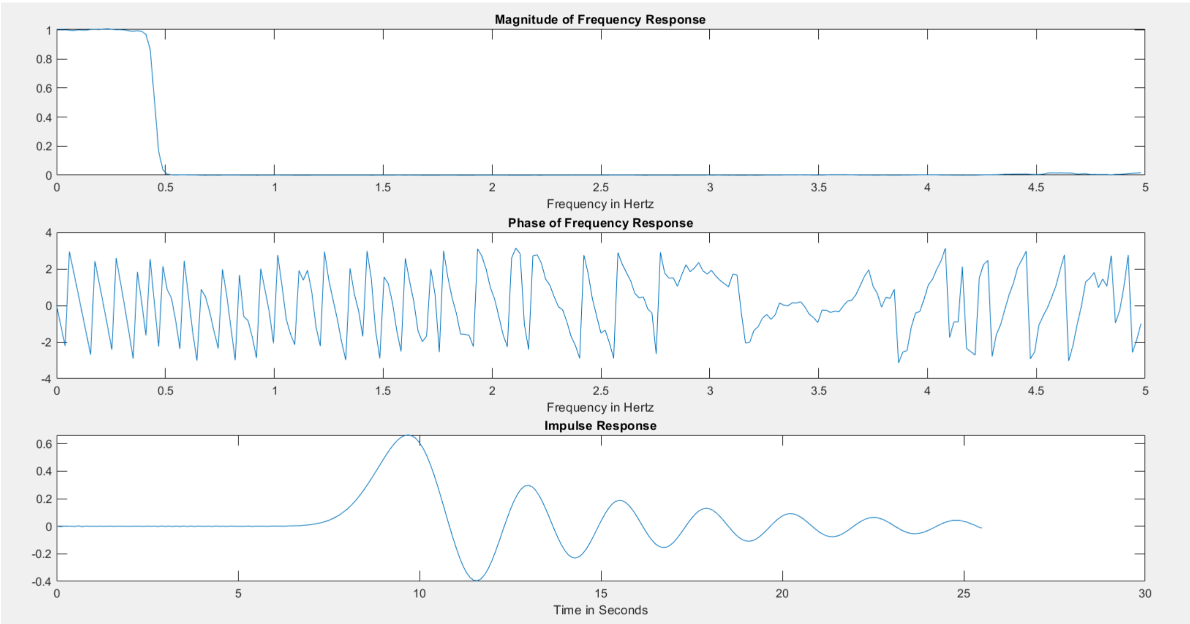


(a) Plot of Input and Output Signals while using Butterworth Filter

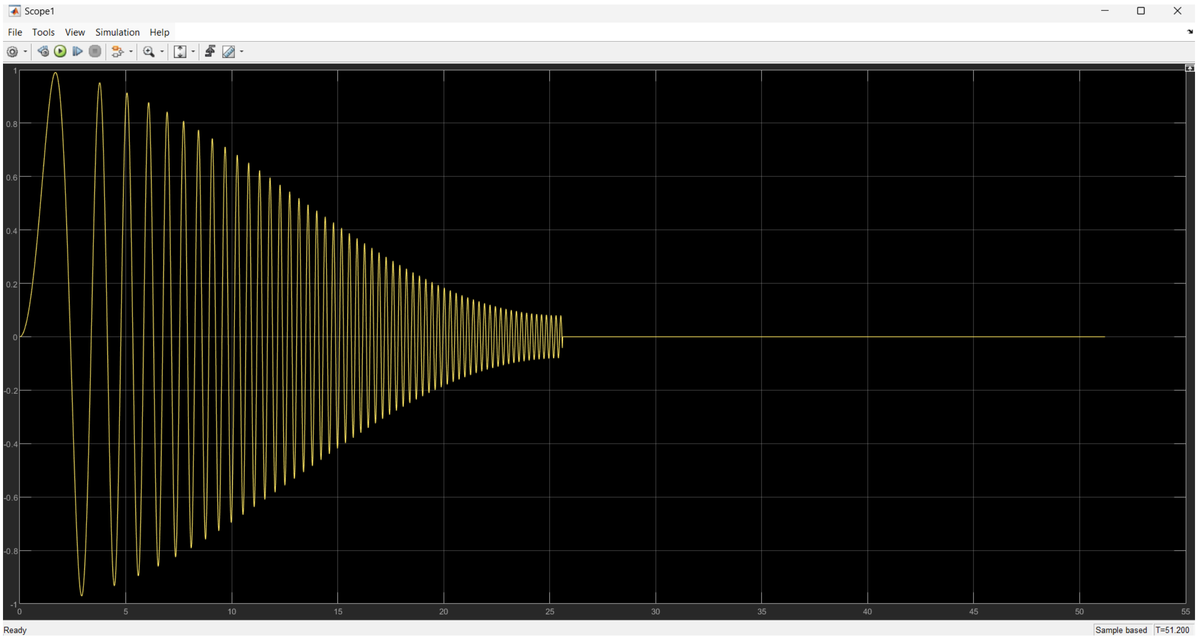


(a) Amplitude and Spectral Density Curve with Butterworth filter

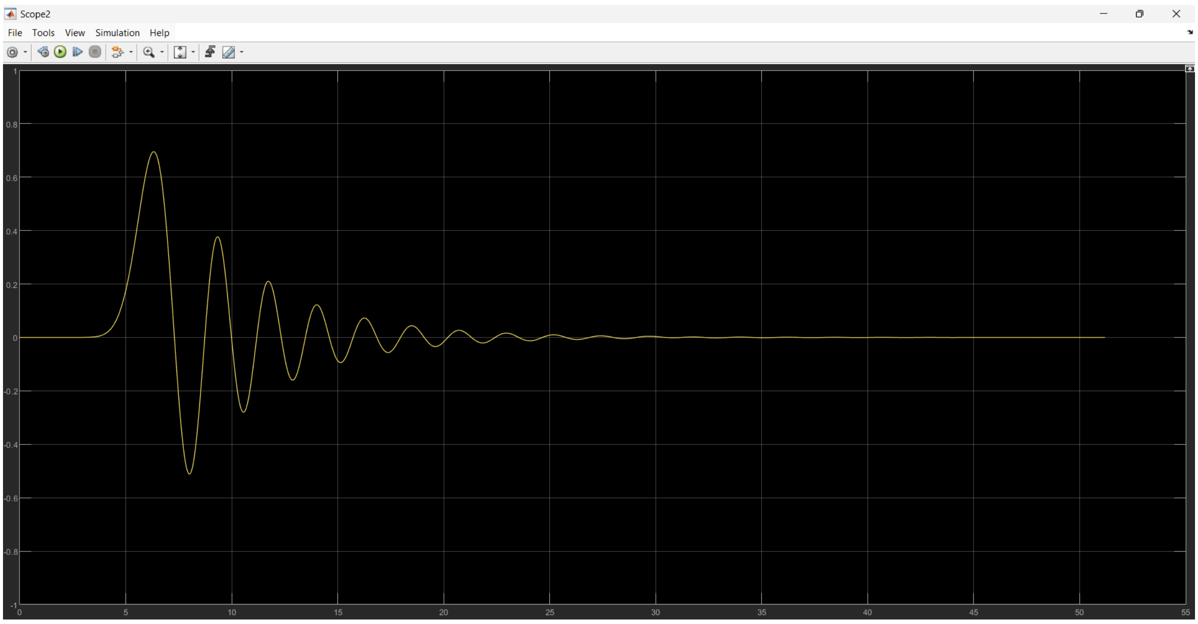
iii) Sampling and Reconstruction with Sample and Hold



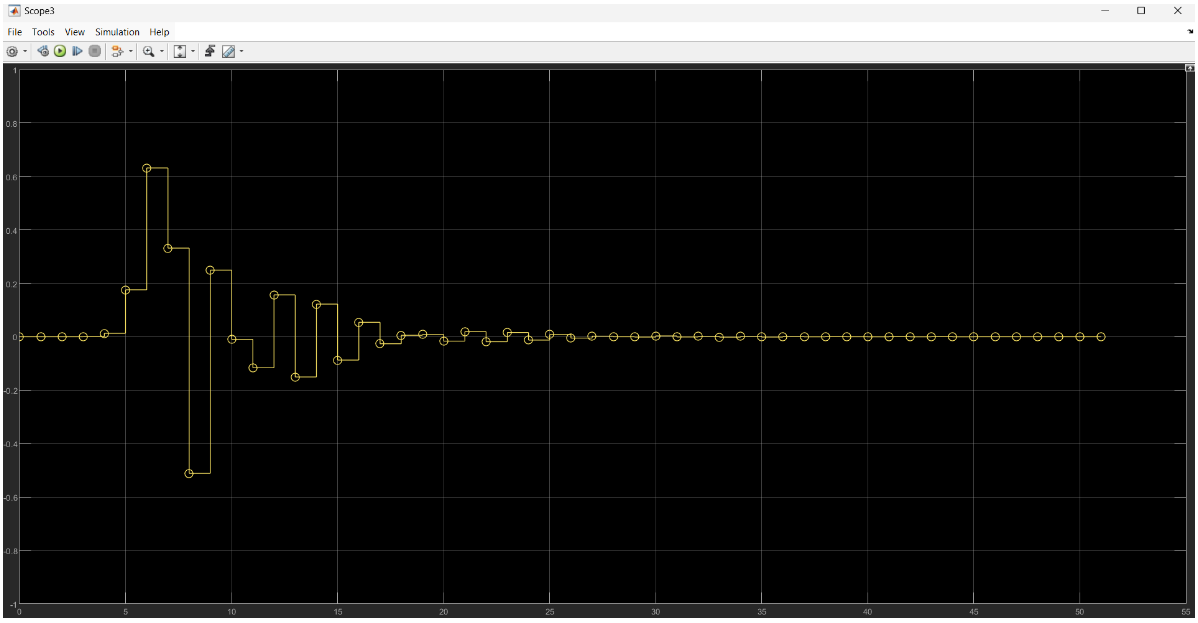
(a) Magnitude and Phase response of the system without sample and hold



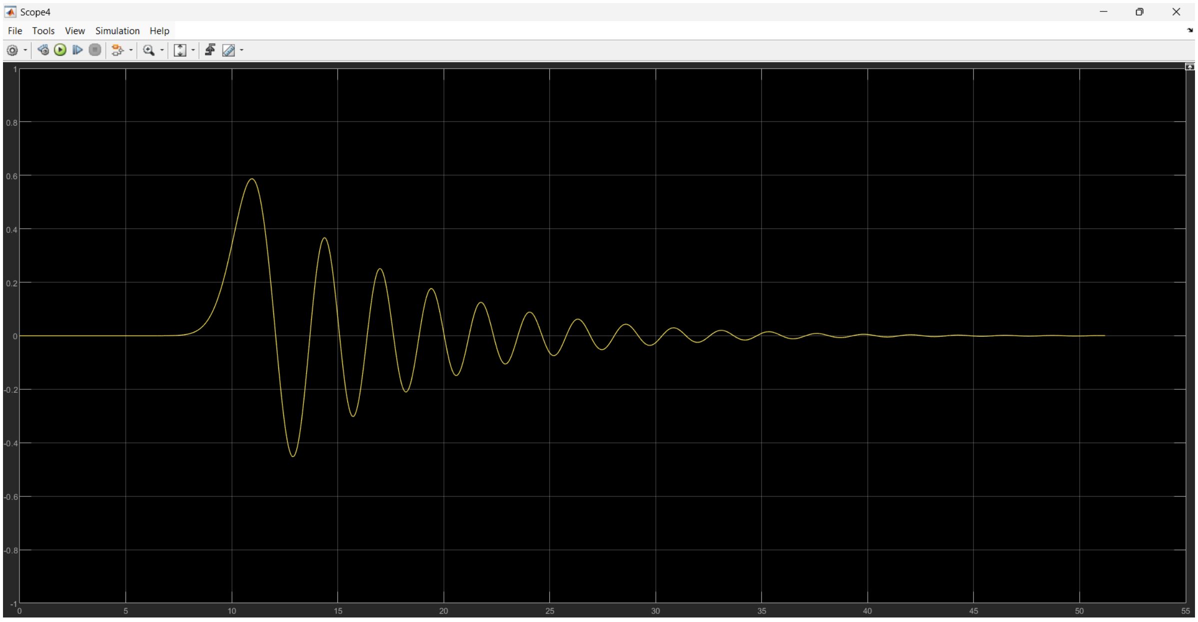
(a) Weighted Chirp Signal



(a) Output after Passing through Filter without Sample and Hold



(a) Signal wave after passing through sample and hold



(a) Signal Wave after Passing through the filter, with Sample and Hold

Difference in Magnitude Response Shapes

The difference in the shape of the magnitude response with and without the sample-and-hold device arises due to the impact of the sample-and-hold process on the frequency spectrum of the signal.

Without Sample-and-Hold In the absence of the sample-and-hold device, the system consists of two cascaded low-pass Butterworth filters. The magnitude response of this system is given by:

$$|H_b(f)| = \frac{1}{\sqrt{1 + (t)^{2N}}}$$

where fc is the cutoff frequency and N is the order of the filter. This response has a smooth roll-off around the cutoff frequency fc and effectively attenuates frequencies above this cutoff.

With Sample-and-Hold The sample-and-hold process introduces a rectangular pulse with width equal to the sampling period Ts. In the frequency domain, this pulse corresponds to a sinc function:

$$P(f) = T_s \cdot |\text{sinc}\,(fT_s)|$$

The combined effect of the sample-and-hold device, along with the Butterworth filters, modifies the overall magnitude response of the system. The complete magnitude response becomes:

$$|H(f)| = |H_0(f)| \cdot \frac{1}{T_s} \cdot |\text{sinc}\,(fT_s)|$$

This modification causes additional attenuation and distortion, particularly evident near the Nyquist frequency (half of the sampling frequency) and in the lower frequency range.

Analytical Expression for Frequencies Below 0.45 Hz

For frequencies below 0.45 Hz, the behavior of the magnitude plot can be described by the combined effects of the Butterworth filters and the sample-and-hold sinc function. function. Given that $f = 0.45$ Hz and $T_s = 1\text{sec}$

1. Butterworth Filter Response:

$$|H_k(f)| = \frac{1}{\sqrt{1 + (f/fc)^{2N}}}$$

For $f < 0.45$He, the Butterworth filter response is close to 1, meaning it passes frequencies in this range with minimal attenuation.

2. Sample-and-Hold Effect:

For $f < 0.45$ Hz, the sine function's effect is relatively small because T_s is large compared to f. Thus, the magnitude response in this range primarily reflects the Butterworth filter's response with minor modulation from the sinc function.

Combining these:

$$|H(f)| = |H_b(f)| \cdot \frac{1}{T_s} \cdot \bigg| \operatorname{sinc}(fT_s)$$

For frequencies well below 0.45 Hz, the sinc function's effect is minimal, so:

$$|H(f)| \approx \frac{1}{T_n} \cdot |H_b(f)|$$

With $T_s = 1$ sec:

$$|H(f)| \approx |H_i(f)|$$

Thus, the magnitude response closely follows the Butterworth filter's response for frequencies below 0.45 Hz, but with an adjustment based on the sample-and-hold sinc function.

4) Discrete Time Interpolation

Discrete-time interpolation is a signal-processing technique used to reconstruct or estimate the values of a discrete-time signal between its known samples, effectively increasing the sampling rate of the signal. This process generates intermediate sample points that help smooth the signal and reduces disturbances and distortions introduced by Sampling, such as aliasing or distortion. The block diagram of the circuit in the simulink is

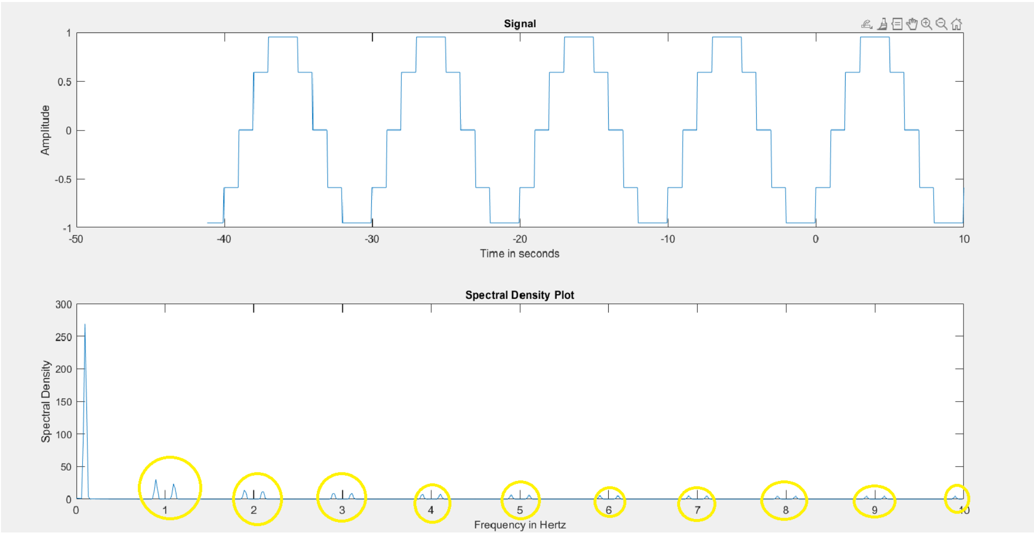


Figure 21: Signal Plot and Spectral Density Diagram

Aliases are Marked with Yellow Circles

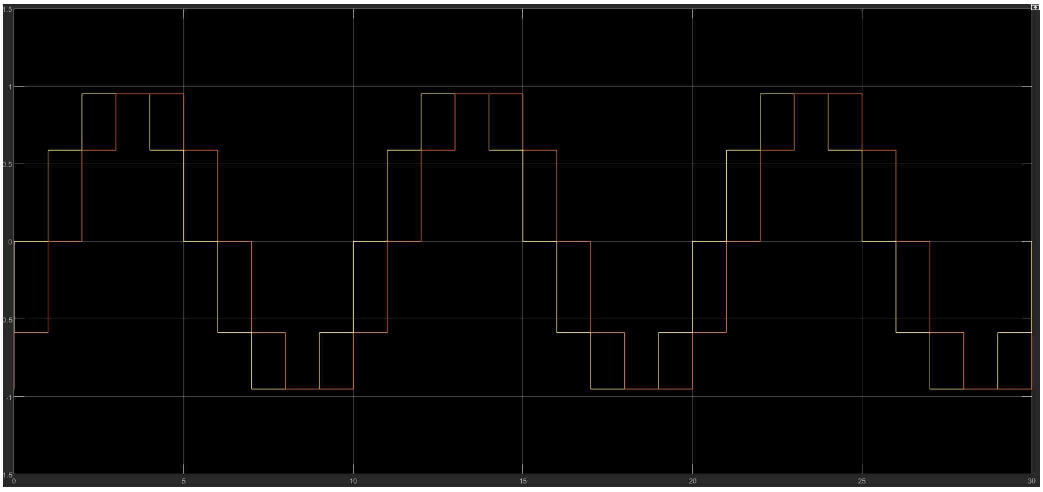


Figure 22: Output on Scope

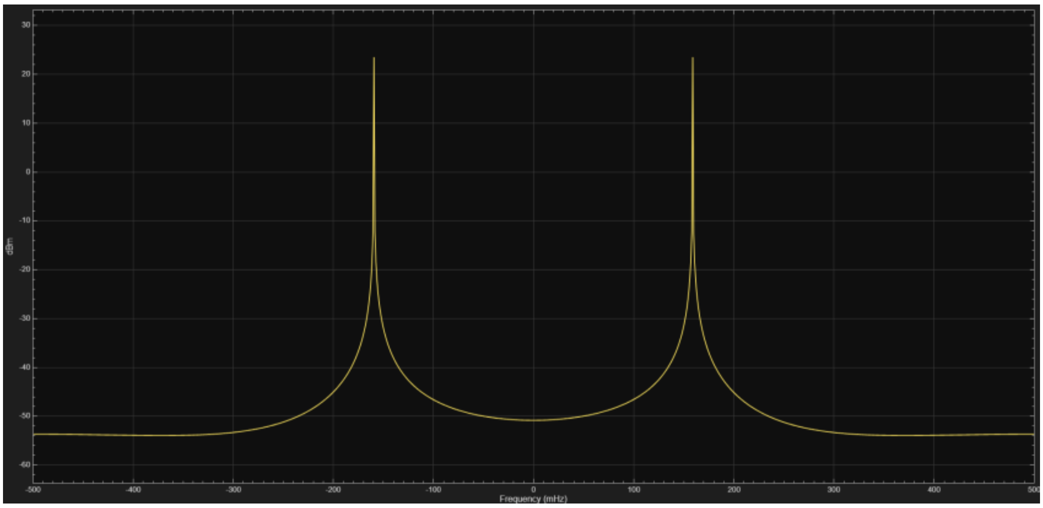


Figure 23: Frequency Spectrum

After Upsampling the System by a factor of 4

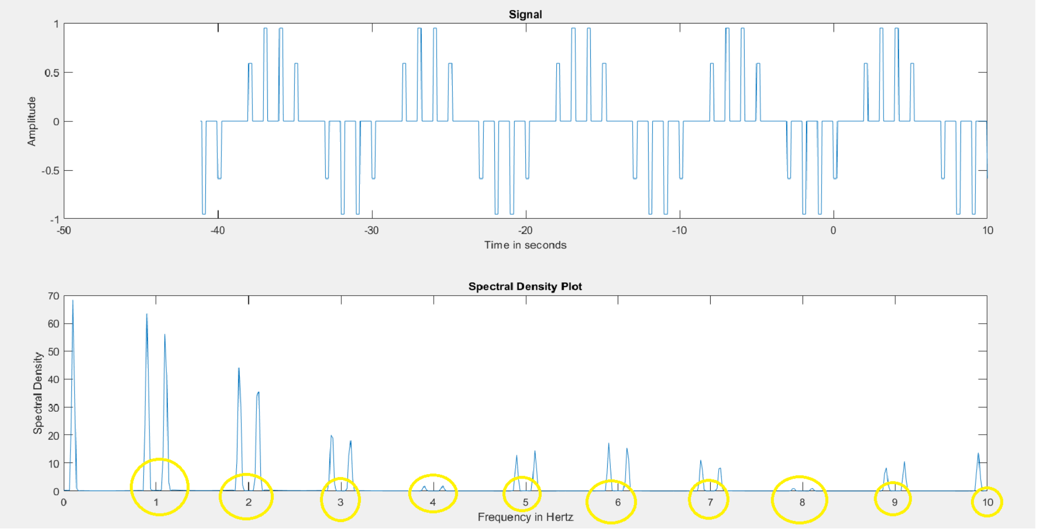


Figure 24: Signal Plot and Spectral Density Diagram

Aliases are Marked with Yellow Circles

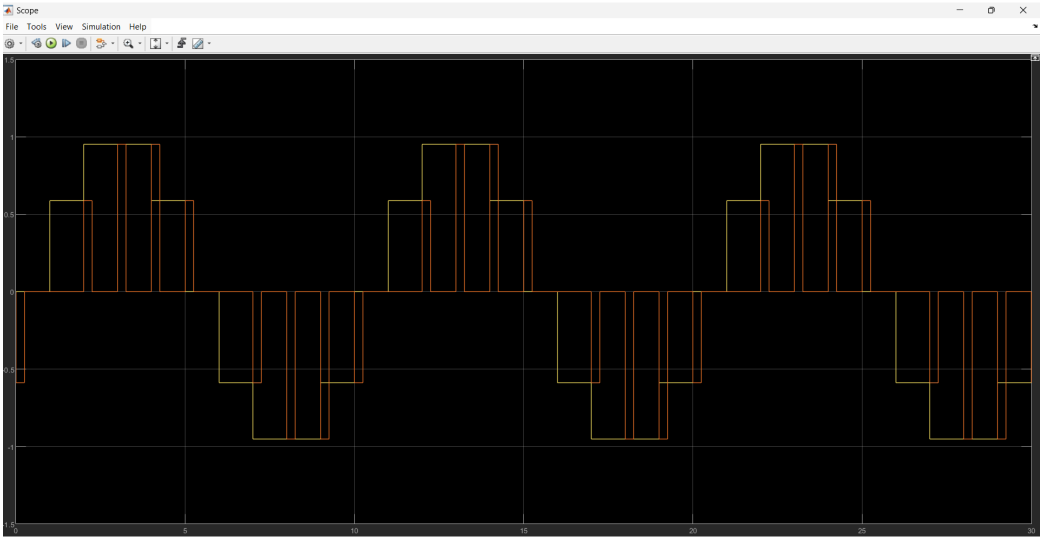


Figure 25: Output on Scope

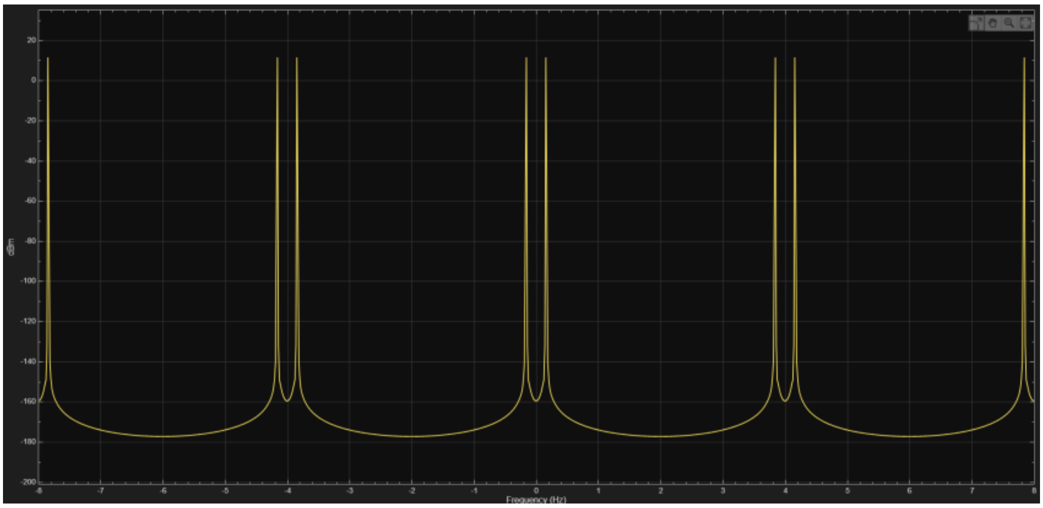


Figure 26: Frequency Spectrum

The frequency spectrum of the signal shows a repeated pattern of peaks centered around integer multiples of the sampling frequency, demonstrating the effects of aliasing. The envelope of the spectrum follows a characteristic sinc-like shape, with sharp, narrow peaks and significant attenuation between them. This shape is indicative of the sample-and-hold effect in the reconstruction process, where the amplitude of higher frequencies diminishes rapidly due to the sinc envelope. The aliasing components, represented by these replicated peaks, arise because the sampling rate is not sufficiently high to preserve all original frequency content without overlap. These components can introduce distortion, especially in the vicinity of the first aliased component, emphasizing the importance of proper filtering in avoiding undesired spectral components in reconstructed signals.

5) Discrete-Time Decimation

In this experiment, we explore decimation, the process of reducing the sampling rate of a discrete-time signal. Decimating a signal $x(n)$ by a factor of D involves creating a new signal $y(n)$ by taking every D-th sample of $x(n)$. Therefore, $y(n)$ is given by:

$$y(n) = x(Dn)$$

The frequency domain relationship between $y(n)$ and $x(n)$ can be expressed as:

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

This shows that $Y(\omega)$ is obtained by expanding $X(\omega)$ in frequency by a factor of D, repeating it every 2π, and scaling it in amplitude by $1/D$. Aliasing is prevented if $X\left(e^{j\omega}\right)$ is zero outside the interval $\left[-\frac{\pi}{D}, \frac{\pi}{D}\right]$, simplifying the expression to:

$$Y(\omega) = \frac{1}{D} X\left(\frac{\omega}{D}\right), \quad \forall \omega \in [-\pi, \pi]$$

Comment on the two methods used of Sampling audio

When using simple decimation to sample audio, disturbances often arise due to the introduction of aliasing. Decimation involves selecting every D-th sample from the original signal, which effectively reduces the sampling rate. However, if this process is performed without prior filtering, high-frequency components in the signal that are above the new Nyquist frequency can alias into lower frequencies. This aliasing occurs because these 15 high-frequency components are not adequately removed before sampling, leading to distortion and unwanted artifacts in the signal. As a result, the audio may sound distorted or noisy, as the high-frequency information that folds back into the signal creates interference and degrades the overall quality. To avoid these issues, a low-pass filter is typically applied before decimation to remove high-frequency components, ensuring that only frequencies within the range of the new sampling rate are preserved. This filtering step prevents aliasing and maintains better signal integrity, resulting in a cleaner and more accurate audio output.