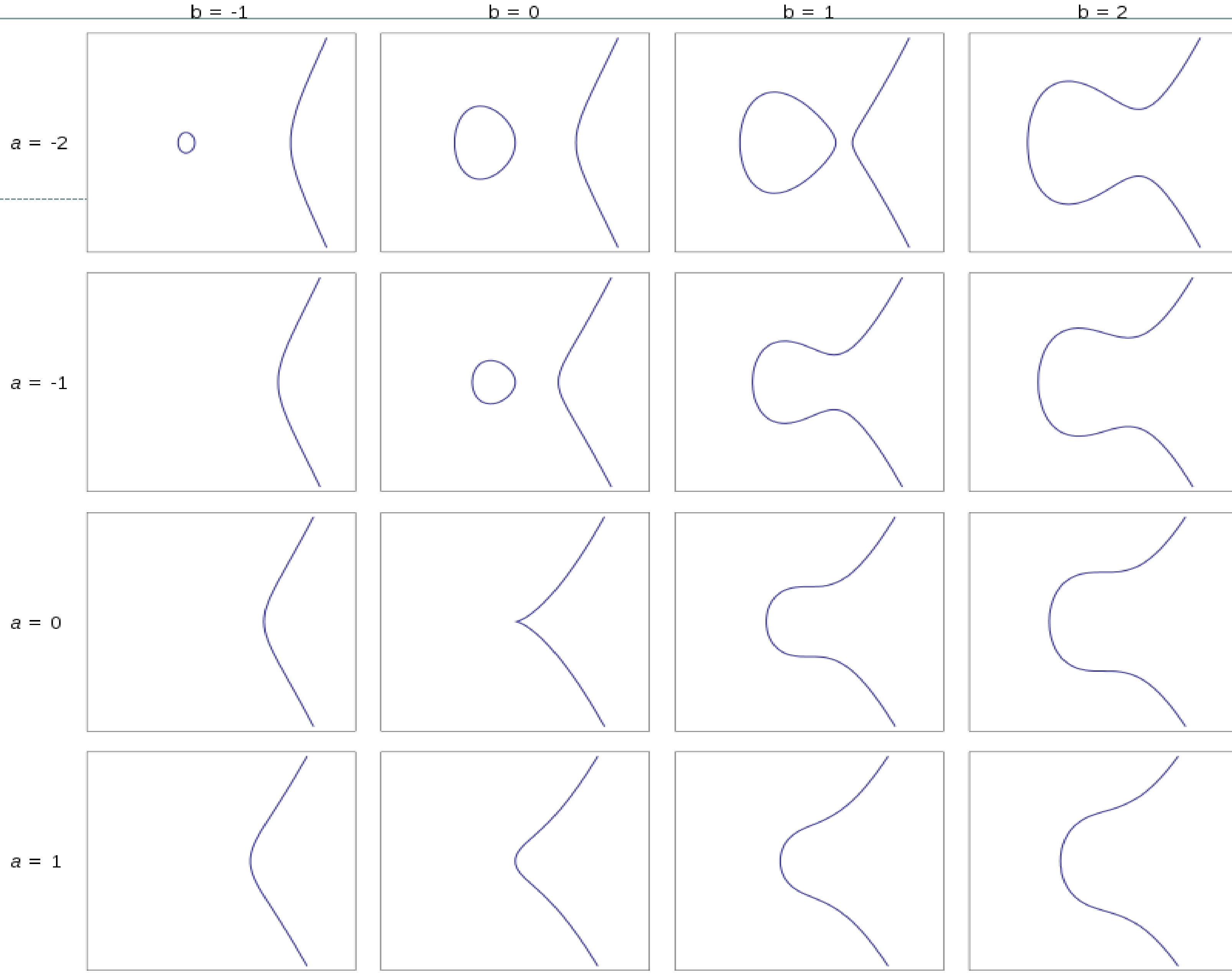
# General form of a EC

- An *elliptic curve* is a plane curve defined by an equation of the form

$$y^2 = x^3 + ax + b$$

where $\mathbf{4a^3 + 27b^2 \neq 0}$


b = -1
b = 0
b = 1
b = 2
a = -2
a = -1
a = 0
a = 1

# Characteristics of Elliptic Curve

- Forms an abelian group
  - Symmetric about the x-axis
  - *Point at Infinity* acting as the identity element

| | |
|---|---|
| **(A1) Closure:** | If $a$ and $b$ belong to $G$, then $a \cdot b$ is also in $G$. |
| **(A2) Associative:** | $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c$ in $G$. |
| **(A3) Identity element:** | There is an element e in $G$ such that $a \cdot \mathrm{e} = \mathrm{e} \cdot a = a$ for all $a$ in $G$. |
| **(A4) Inverse element:** | For each $a$ in $G$ there is an element $a'$ in $G$ such that $a \cdot a' = a' \cdot a = \mathrm{e}$. |
| **(A5) Commutative:** | $a \cdot b = b \cdot a$ for all $a, b$ in $G$. |

# Discrete Logarithm Problem (DLP)

- Let P and Q be two points on the elliptic curve
  - Such that Q = kP, where k is a scalar value
- DLP: Given P and Q, find k?
  - If k is very large, it becomes computationally infeasible
- The security of ECC depends on the difficulty of DLP
- Main operation in ECC is Point Multiplication

# Point Multiplication

- Point Multiplication is achieved by two basic curve operations:

1. Point Addition, $L = J + K$
2. Point Doubling, $L = 2J$

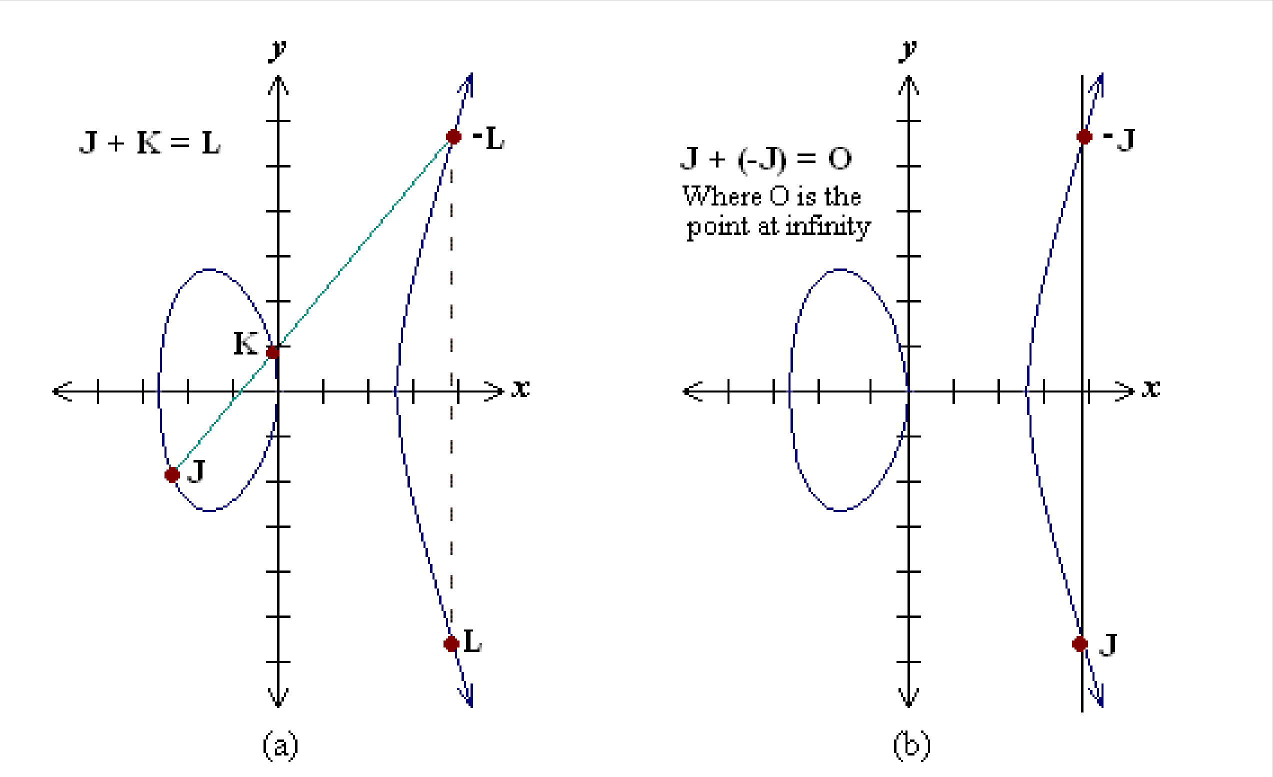
Example:

If $k = 23$; then, $kP = 23*P$

$= 2(2(2(2P) + P) + P) + P$

# Point Addition

**Geometrical explanation:**



# Point Addition

**Analytical explanation:**

- Consider two distinct points J and K such that $J = (x_J, y_J)$ and $K = (x_K, y_K)$
- Let $L = J + K$ where $L = (x_L, y_L)$, then

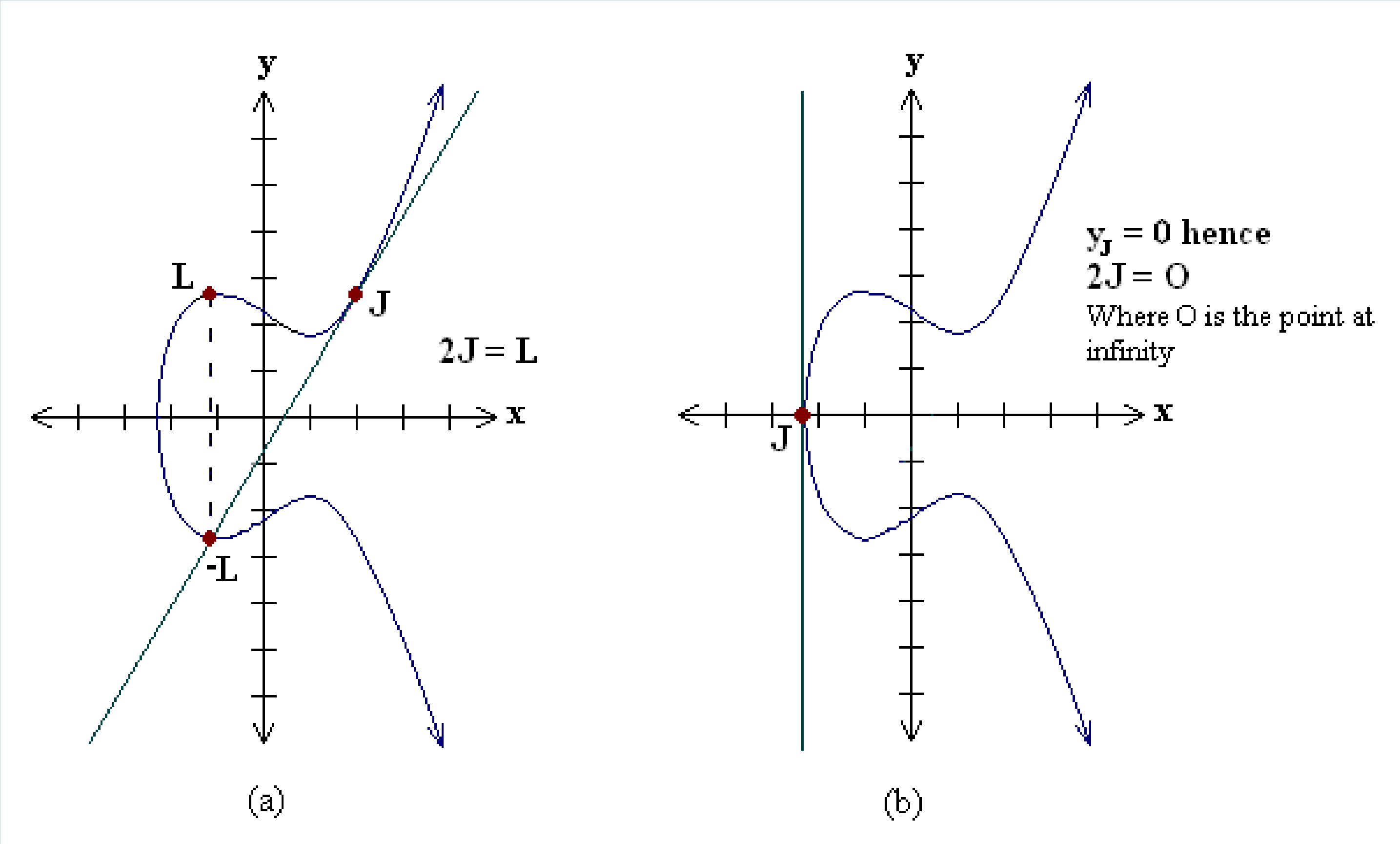
$$x_L = s^2 - x_J - x_K$$

$$y_L = -y_J + s\,(x_J - x_L)$$

$s = (y_J - y_K)/(x_J - x_K)$, s is slope of the line through J and K.

# Point Doubling

**Geometrical explanation:**



# Point Doubling

**Analytical explanation**

- Consider a point J such that J = ($x_J$, $y_J$), where $y_J \neq 0$
- Let L = 2J where L = ($x_L$, $y_L$), Then

$$x_L = s^2 - 2x_J$$

$$y_L = -y_J + s(x_J - x_L)$$

$s = (3x_J^2 + a) / (2y_J)$, s is the tangent at point J and a is one of the parameters chosen with the elliptic curve.

# Finite Fields

- The Elliptic curve operations shown were on real numbers
  - Issue: operations are slow and inaccurate due to round-off errors

- To make operations more efficient and accurate, the curve is defined over two finite fields

  1. Prime field GF(p) and

  2. Binary field $GF({}_2{}^{m})$

- The field is chosen with finitely large number of points suited for cryptographic operations

# EC on Prime field GF(p)

- Elliptic Curve equation:

  $$y^2 \bmod p = x^3 + ax + b \bmod p$$

  where $4a^3 + 27b^2 \bmod p \neq 0.$
- Elements of finite fields are integers between 0 and p-1
- The prime number p is chosen such that there is finitely large number of points on the elliptic curve to make the cryptosystem secure.
- SEC (Standard for Efficient Cryptography) specifies curves with p ranging between 112-521 bits

# EC on Binary field GF($_2{}^m$)

- Elliptic Curve equation:

  $$y^2 + xy = x^3 + ax^2 + b,$$

  where $\mathbf{b \neq 0}$
- Here the elements of the finite field are integers of length at most **m** bits.
- In binary polynomial the coefficients can only be 0 or 1.
- The **m** is chosen such that there is finitely large number of points on the elliptic curve to make the cryptosystem secure.
- SEC specifies curves with **m** ranging between 113-571 bits

**Global Public Elements**

$E_q(a, b)$ elliptic curve with parameters $a$, $b$, and $q$, where $q$ is a prime or an integer of the form $2^m$

$G$ point on elliptic curve whose order is large value $n$

**User A Key Generation**

Select private $n_A$ $n_A < n$

Calculate public $P_A$ $P_A = n_A \times G$

**User B Key Generation**

Select private $n_B$ $n_B < n$

Calculate public $P_B$ $P_B = n_B \times G$

**Calculation of Secret Key by User A**

$K = n_A \times P_B$

**Calculation of Secret Key by User B**

$K = n_B \times P_A$

Figure 10.7 ECC Diffie-Hellman Key Exchange

# Elliptic Curve Encryption/Decryption

As with the key exchange system, an encryption/decryption system requires a point $G$ and an elliptic group $E_q(a, b)$ as parameters. Each user A selects a private key $n_A$ and generates a public key $P_A = n_A \times G$.

To encrypt and send a message $P_m$ to B, A chooses a random positive integer $k$ and produces the ciphertext $C_m$ consisting of the pair of points:

$$C_m = \{kG, P_m + kP_B\}$$

Note that A has used B's public key $P_B$. To decrypt the ciphertext, B multiplies the first point in the pair by B's secret key and subtracts the result from the second point:

$$P_m + kP_B - n_B(kG) = P_m + k(n_BG) - n_B(kG) = P_m$$

Table 10.3 Comparable Key Sizes in Terms of Computational Effort for Cryptanalysis

| Symmetric Scheme (key size in bits) | ECC-Based Scheme (size of $n$ in bits) | RSA/DSA (modulus size in bits) |
|---|---|---|
| 56 | 112 | 512 |
| 80 | 160 | 1024 |
| 112 | 224 | 2048 |
| 128 | 256 | 3072 |
| 192 | 384 | 7680 |
| 256 | 512 | 15360 |

*Source:* Certicom