PS-4

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Representative household whose goal is to maximize their utility from consumption and leisure over time:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

Constraints:

$$y_t = z_t k_t^{\alpha} n_t^{1-\alpha}$$

$$y_t = c_t + i_t$$

$$1 = n_t + l_t$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$u(c_t, l_t) = \frac{(c_t^{\phi} l_t^{1-\phi})^{1-\rho}}{1-\rho}$$

Removing unnecessary constraints:

$$n_t = 1 - l_t$$

$$c_t + i_t = z_t k_t^{\alpha} n_t^{1-\alpha}$$

$$c_t + i_t = z_t k_t^{\alpha} (1-l_t)^{1-\alpha}$$

Making it unconstrained optimization problem:

$$\max_{\substack{c_t, k_t, i_t, \\ \lambda_t, \eta_t, l_t}} U = E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\phi} l_t^{1-\phi})^{1-\rho}}{1-\rho} + \lambda_t (c_t + i_t - z_t k_t^{\alpha} (1-l_t)^{1-\alpha}) + \eta_t (k_{t+1} - i_t - (1-\delta) k_t) \right]$$

(i) $\dfrac{dU}{di_t} = 0$

$$\Rightarrow E_0\left[\sum_{t=0}^{\infty} \cancel{\frac{d}{di_t}\beta^t \frac{(c_t^{\phi} l_t^{1-\phi})^{1-\rho}}{1-\rho}} + \frac{d}{di_t}\lambda_t\left(c_t + i_t - z_t k_t^{\alpha}(1-l_t)^{1-\alpha}\right)\right.$$

$$\left. + \frac{d}{di_t}\eta_t\left(k_{t+1} - i_t - (1-\delta)k_t\right)\right] = 0$$

$$\Rightarrow \lambda_t - \eta_t = 0$$

$$\Rightarrow \boxed{\lambda_t = \eta_t}$$

(ii) $\dfrac{dU}{d\lambda_t} = 0$

$$\Rightarrow c_t + i_t - z_t k_t^{\alpha}(1-l_t)^{1-\alpha} = 0$$

$$\Rightarrow \boxed{c_t + i_t = z_t k_t^{\alpha} n_t^{1-\alpha}} \quad \text{since } n_t + l_t = 1$$

(iii) $\dfrac{dU}{d\eta_t} = 0$

$$\Rightarrow k_{t+1} - i_t - (1-\delta)k_t = 0$$

$$\Rightarrow \boxed{k_{t+1} = i_t + (1-\delta)k_t}$$

(iv) $\frac{dU}{dk_t} = 0$

$$\Rightarrow \frac{d}{dk_t}\left[E_0 \sum_{t=0}^{\infty} \lambda_t \left(- z_t k_t^{\alpha} (1-l_t)^{1-\alpha} \right) + \right.$$

$$\left. \eta_t \left(k_{t+1} - (1-\delta) k_t \right) \right] = 0$$

$$\Rightarrow E_{t-1}\left[\lambda_t \left(- z_t \alpha k_t^{\alpha-1} (1-l_t)^{1-\alpha} \right) + \eta_{t-1} - (1-\delta)\eta_t \right]$$

$$\Rightarrow E_{t-1}\left[\left(\lambda_t \left(- z_t \alpha \left(\frac{k_t}{\eta_t} \right)^{\alpha-1} \right) + \eta_{t-1} - \eta_t (1-\delta) \right] = 0$$

(v) $\frac{dU}{dl_t} = 0$

$$\Rightarrow E_0\left[\sum_{t=0}^{\infty} \frac{d}{dl_t} \cdot \beta^t \frac{e_t^{\phi(1-\rho)} \, l_t^{(1-\phi)(1-\rho)}}{1-\rho} + \frac{d}{dl_t} \lambda_t \left(- z_t k_t^{\alpha} (1-l_t)^{1-\alpha} \right) \right]$$

$$\Rightarrow E_0\left[\sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{\phi(1-\rho)} \, l_t^{(1-\phi)(1-\rho)-1} \, (1-\phi)\cancel{(1-\rho)}}{\cancel{1-\rho}} \right.$$

$$+ \lambda_t z_t k_t^{\alpha} (1-\alpha) (1-l_t)^{-\alpha}$$

$$\Rightarrow E_0\left[\sum_{t=0}^{\infty} \beta^t c_t^{\phi(1-\rho)} \, l_t^{(1-\phi)(1-\rho)-1} \, (1-\phi) \right.$$

$$+ \lambda_t z_t \left(\frac{k_t}{\eta_t} \right)^{\alpha} (1-\alpha) = 0$$

(vi) $\frac{dU}{dc_t} = 0$

$$\Rightarrow \frac{\beta^t}{1-\rho} l_t^{(1-\phi)(1-\rho)} (1-\rho)(\phi) c_t^{\phi(1-\rho)-1} + \lambda_t = 0$$

$$\boxed{\lambda_t = -\beta^t \phi c_t^{\phi(1-\rho)-1} l_t^{(1-\phi)(1-\rho)} = \eta_t}$$

Solving for equation to reveal the consumer's inner-temporal choice between consumption and leisure.

from (vi) & (v) → since we're working in time t

$$\beta^t c_t^{\phi(1-\rho)} l_t^{(1-\phi)(1-\rho)-1} (1-\phi) = \left(\beta^t \phi c_t^{\phi(1-\rho)-1} l_t^{(1-\phi)(1-\rho)}\right) \times \left(z_t\left(\frac{k_t}{n_t}\right)^{\alpha}(1-\alpha)\right)$$

$$\Rightarrow \frac{1-\phi}{\phi} \frac{c_t^{\phi(1-\rho)}}{c_t^{\phi(1-\rho)-1}} \frac{l_t^{(1-\phi)(1-\rho)-1}}{l_t^{(1-\phi)(1-\rho)}} = (1-\alpha) z_t \left(\frac{k_t}{n_t}\right)^{\alpha}$$

First equation $\Rightarrow$ $\boxed{\frac{1-\phi}{\phi} \cdot \frac{c_t}{l_t} = (1-\alpha) z_t \left(\frac{k_t}{n_t}\right)^{\alpha}}$

Solving for equation to reveal the consumer's inter temporal Choice between consuming today & consuming tomorrow

from (iv) & (vi)

$$E_t\left[\lambda_{t+1}(-z_{t+1}\alpha k_{t+1}^{\alpha-1}\underbrace{(1-l_{t+1})}_{n_{t+1}}^{1-\alpha}) + \eta_t - \eta_{t+1}(1-\delta) = 0\right.$$

we know that $\lambda_t = \eta_t$

$$E_t\left[\lambda_{t+1}(-z_{t+1}\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}) + \lambda_t - \lambda_{t+1}(1-\delta) = 0\right.$$

$$\Rightarrow E_t\left[\lambda_{t+1}\left(-z_{t+1}\alpha\left(\frac{n_{t+1}}{k_{t+1}}\right)^{1-\alpha} - \lambda_{t+1}(1-\delta)\right)\right] = E_t(-\lambda_t)$$

$$\Rightarrow E_t\left[\lambda_{t+1}\underbrace{\left(-z_{t+1}\alpha\left(\frac{n_{t+1}}{k_{t+1}}\right)^{1-\alpha} - (1-\delta)\right)}_{= -\lambda_t}\right.$$ → lets say Ω

$$\Rightarrow E_t\left[+\beta^{t+1}\phi\, c_{t+1}^{\phi(1-\rho)-1}\, l_{t+1}^{(1-\phi)(1-\rho)}\,\Omega\right] = -\beta^t\phi\, c_t^{\phi(1-\rho)-1}\, l_t^{(1-\phi)(1-\rho)}$$

$$\Rightarrow \boxed{c_t^{\phi(1-\rho)-1}\, l_t^{(1-\phi)(1-\rho)} = \beta E_t\left[c_{t+1}^{\phi(1-\rho)-1}\, l_{t+1}^{(1-\phi)(1-\rho)} \times \Omega\right]}$$

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Second equation