CS 715: Advanced Topics in Algorithmic Game Theory and Mechanism Design

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Lecture 3

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Agenda

- Mechanism Design Theory
  - Gibbard-Satterthwaite Impossibility Theorem
  - Quasi-linear Environment
  - Vickrey-Clarke-Groves Mechanisms
  - Green-Laffont Impossibility Theorem
  - Bayesian Implementation: d'AGVA Mechanisms
  - Optimal Mechanism Design
  - Mechanism Design: Various Attributes

Mechanism Design Framework

$\mathscr{N} = \{1, 2, \ldots, n\}$

$\Theta_1, \ldots, \Theta_n$

X : Set of Outcomes

$u_1, u_2, \ldots, u_n :$
$X \times \Theta_i \rightarrow \mathbb{R}$

SCF

$f : \Theta_1 \times \ldots \Theta_n \rightarrow X$

- $\mathcal{M} = (S_1(), S_2(), \cdots, S_n(), g(\cdot))$ where, $g() : \prod_i S_i \rightarrow X$

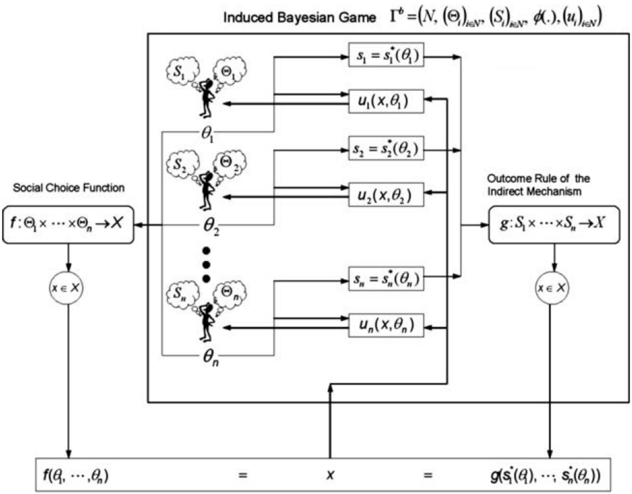


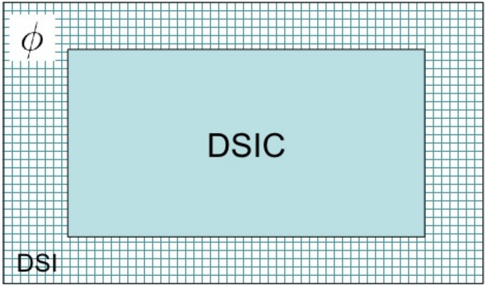
Figure : The idea behind implementation by an indirect mechanism

Definition (Dominant Strategy Incentive Compatibility (DSIC))

A social choice function $f : \Theta_1 \times \ldots \times \Theta_n \rightarrow X$ is said to be dominant strategy incentive compatible (or truthfully implementable in dominant strategies) if the direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ has a *weakly dominant strategy equilibrium* $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ in which $s_i^*(\theta_i) = \theta_i, \forall \theta_i \in \Theta_i, \forall i \in N$.

Theorem

Suppose that there exists a mechanism $\mathscr{M} = (S_1, \ldots, S_n, g(\cdot))$ that implements the social choice function $f(\cdot)$ in dominant strategy equilibrium. Then $f(\cdot)$ is dominant strategy incentive compatible.



DSI: Dominant Strategy Implementable
DSIC: Dominant Strategy Incentive Compatible
DSI $\setminus$ DSIC $= \phi$

Figure : Revelation principle for dominant strategy equilibrium

Properties of Mechanisms

DSIC

Dominant Strategy Incentive Compatibility Reporting truth is always good

AE

Allocative Efficiency Allocate item to those who value them most

Non-Dictatorship

No single agent is favored all the time

BIC

Bayesian Incentive Compatibility Reporting truth is good in expectation whenever others report truth

BB

Budget Transfer Payments are balanced and net transfer is zero

IR

Individual Rationality Payments participate voluntarily. (No losses)

Acronyms

DSIC	Dominant Strategy Incentive Compatible
BIC	Bayesian Nash Incentive Compatible
AE	Allocative Efficiency (Allocatively Efficient)
BB	Budget Balance
IR	Individual Rationality
VCG	Vickrey-Clarke-Groves Mechanisms
dAGVA	d'Aspremont and Gérard-Varet mechanisms

Notation

N	Set of agents: $\{1, 2, \ldots, n\}$
Θ_i	Type set of Agent i
Θ	A type profile $= (\Theta_1 \times \ldots \times \Theta_n)$
Θ_{-i}	A profile of types of agents other than i $= (\Theta_1 \times \ldots \times \Theta_{i-1} \times \Theta_{i+1} \times \ldots \times \Theta_n)$
θ_i	Actual type of agent i, $\theta_i \in \Theta_i$
θ	A profile of actual types $= (\theta_1, \ldots, \theta_n)$
θ_{-i}	A profile of actual types of agents other than i $= (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$
$\hat{\theta}_i$	Reported type of agent i, $\hat{\theta}_i \in \Theta_i$
$\hat{\theta}$	A profile of reported types $= (\hat{\theta}_1, \ldots, \hat{\theta}_n)$
$\hat{\theta}_{-i}$	A profile of reported types of agents other than i $= (\hat{\theta}_1, \ldots, \hat{\theta}_{i-1}, \hat{\theta}_{i+1}, \ldots, \hat{\theta}_n)$

Notation

$\Phi_i(.)$	A cumulative distribution function (CDF) on Θ_i
$\phi_i(.)$	A probability density function (PDF) on Θ_i
X	Outcome Set
x	A particular outcome, $x \in X$
$u_i(.)$	Utility function of agent i
$f(.)$	A social choice function
F	Set of social choice functions
$\mathscr{D}$	A direct revelation mechanism
K	A Set of project choices
k	A particular project choice, $k \in K$
t_i	Monetary transfer to agent i
$v_i(.)$	Valuation function of agent i

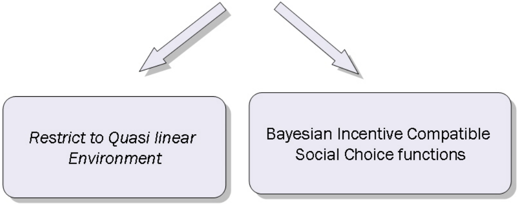
Gibbard-Satterthwaite Impossibility Theorem

Theorem

If

1. *The outcome set X is such that, $3 \leq |X| < \infty$*
2. $\mathscr{R}_i = \mathscr{S} \; \forall \; i$
3. *$f(\Theta) = X$, that is, the image of SCF $f(\cdot)$ is the set X.*

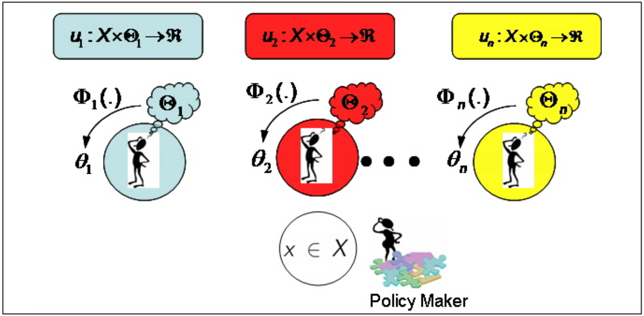
then the social choice function SCF $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.



Quasi Linear Environment

$$u_i(x, \theta_i) = v_i(k, \theta_i) + t_i$$

Valuation function of agent i

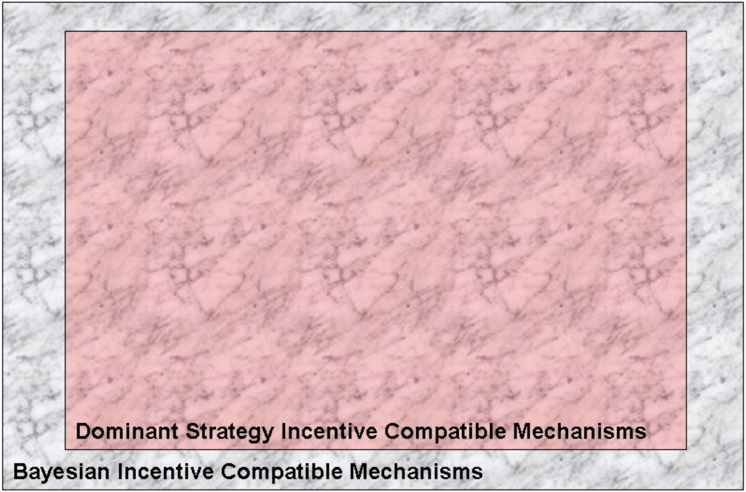


$$X = \left\{ (k, t_1, \ldots, t_n) : k \in K, t_i \in \mathbb{R} \; \forall \; i = 1, \ldots, n, \sum_i t_i \leq 0 \right\}$$

Project Choice

Monetary transfer to agent i

Bayesian Incentive Compatible Mechanisms


Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms

Definition (Allocative Efficiency (AE))

A SCF $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is allocatively efficient if $\forall\ \theta \in \Theta$, $k(\theta)$,

$$k(\theta) \in \underset{k \in K}{\arg\max} \sum_{i=1}^{n} v_i(k, \theta_i) \tag{1}$$

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Assumption: $\forall\ \theta$, the function $\sum_{i=1}^{n} v_i(., \theta_i) : K \to \mathbb{R}$ attains a maximum over the set K.

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Definition (Budget Balance (BB))

A SCF $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is budget balanced if $\forall\ \theta \in \Theta$,

$$\sum_{i=1}^{n} t_i(\theta) = 0 \tag{2}$$

Lemma

A SCF $f(\cdot)$ is ex-post efficient in quasi-linear environment if and only if it is allocatively efficient and budget balanced.

Lemma

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Lemma

All SCFs in quasi-linear environment are non-dictatorial.

Groves' Theorem

Theorem (Groves' Theorem)

An allocatively efficient SCF f can be truthfully implemented in dominant strategies if,

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j) \right] + h_i(\theta_{-i}); \quad i = 1, 2, \ldots, n \tag{3}$$

where $h_i(\cdot)$ is any arbitrary function of θ_{-i}.

[1] T. Groves. Incentives in teams. *Econometrica*, 41:617-631, 1973

Groves' Theorem

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The above payment is Groves payment scheme. [1]

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Definition (Groves Mechanisms)

A direct revelation mechanism, $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ in which $f(\cdot)$ satisfies allocative efficiency (1) and Groves payment scheme (3) is known as Groves mechanism.

Clarke's Mechanism

- In Groves Mechanism, let

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j); \;\; \forall\, \theta_{-i} \in \Theta_{-i}, \; i = 1, \ldots, n \tag{4}$$

- That is, each agent i pays,

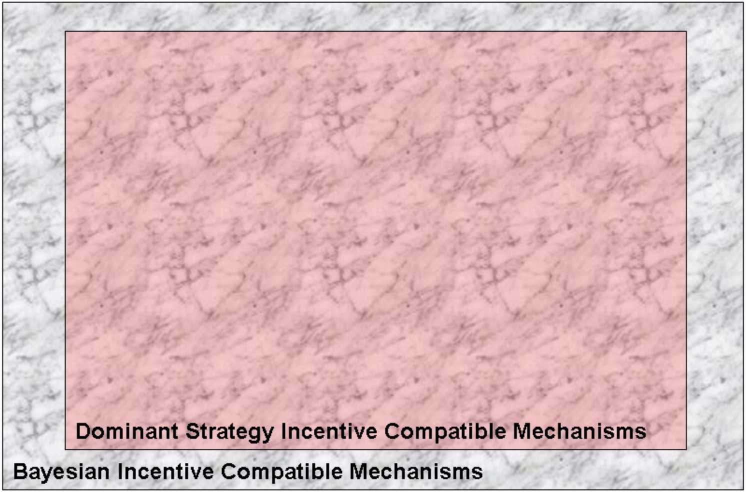
$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] - \left[\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right] \tag{5}$$

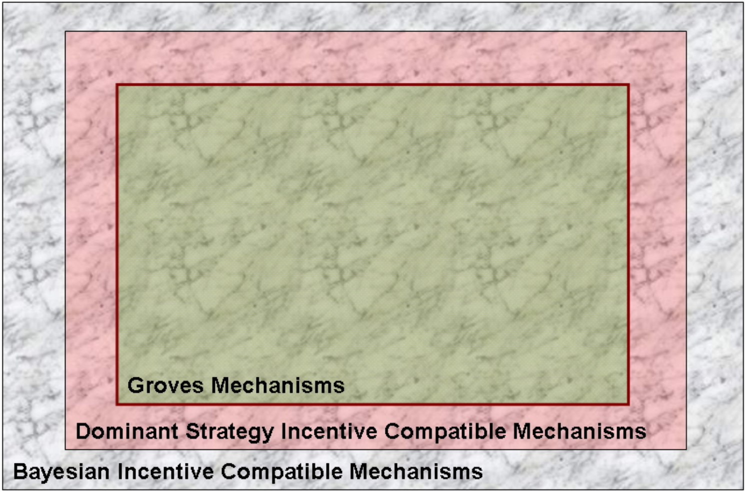
- This mechanism is called Clarke's[2] Pivotal Mechanism

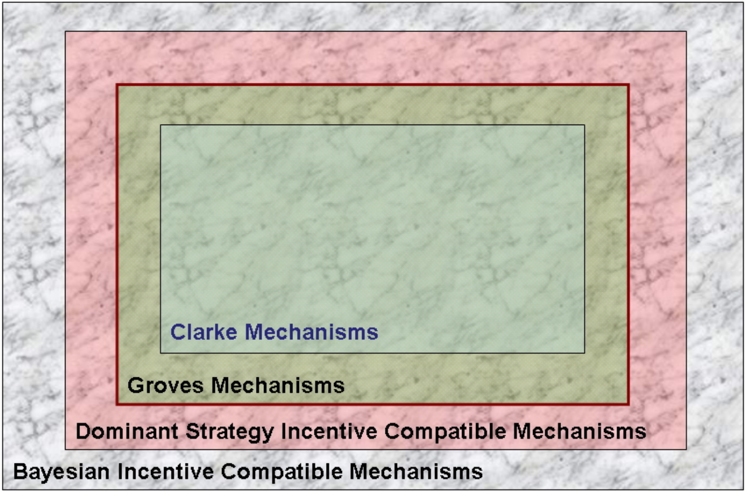
[2] E. Clarke. Multi-part pricing of public goods. *Public Choice*, 11:17-23, 1971

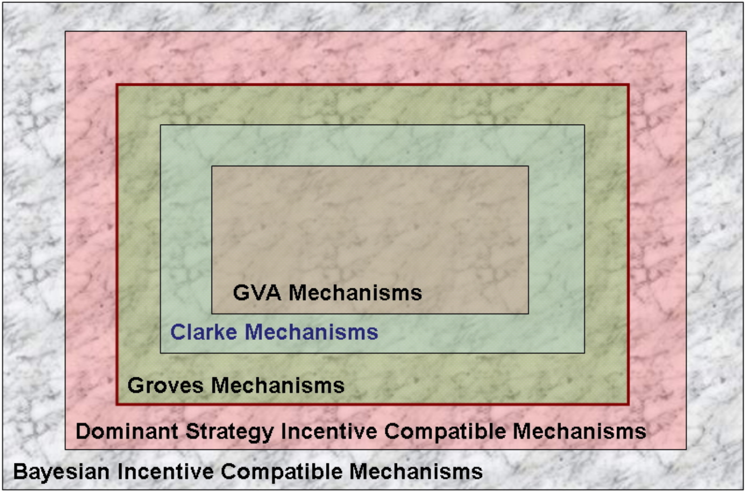
Generalized Vickrey Auction

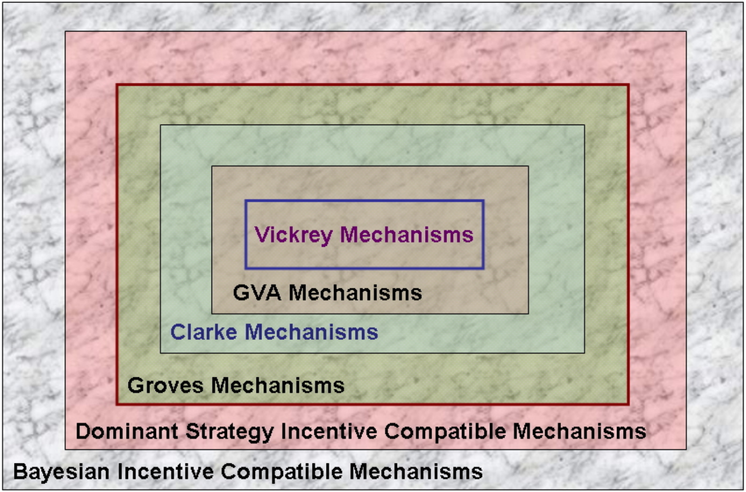
- Generalized Vickrey Auction
  - The Clarke's pivotal mechanism when applied to combinatorial auction setting


Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms


Groves Mechanisms
Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms


Clarke Mechanisms
Groves Mechanisms
Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms


GVA Mechanisms
Clarke Mechanisms
Groves Mechanisms
Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms


Vickrey Mechanisms
GVA Mechanisms
Clarke Mechanisms
Groves Mechanisms
Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms

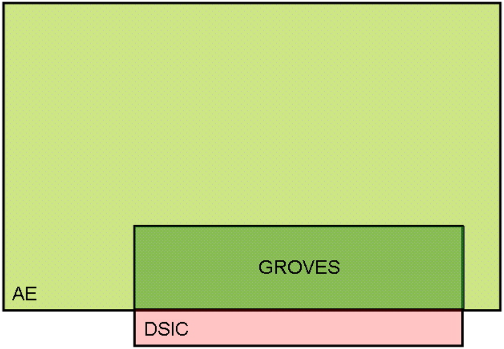
Green-Laffont Theorems

Theorems by J. Green and J. Laffont[3]

Theorem (Green-Laffont Theorem)

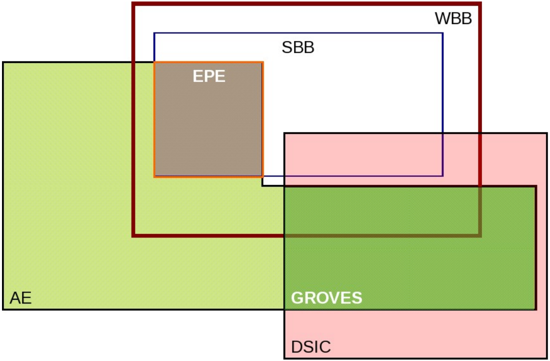
If $\forall\ i$, $\{v_i(.,\theta_i) : \theta_i \in \Theta_i\} = \mathscr{F}$; then any allocatively efficient (AE) SCF $f(\cdot)$ is dominant strategy incentive compatible (DSIC) if and only if it satisfies the Groves payment scheme (3).

[3] J. R. Green and J. J. Laffont. *Incentives in Public Decision Making*. North-Holland Publishing Company, Amsterdam, 1979



AE : Allocative Efficient
DSIC : Dominant Strategy Incentive Compatible

Green-Laffont Impossibility Theorem



AE : Allocative Efficient
DSIC : Dominant strategy Incentive Compatible
WBB : Weak Budget Balanced
SBB: Strict Budget Balanced
EPE: Ex-post efficient

Green-Laffont Impossibility Theorem: AE + SBB + DSIC is not possible.

J. R. Green and J. J. Laffont. Incentives in Public Decision Making. North-Holland Publishing Company, Amsterdam, 1979.

Gibbard-Satterthwaite Impossibility Theorem

Theorem

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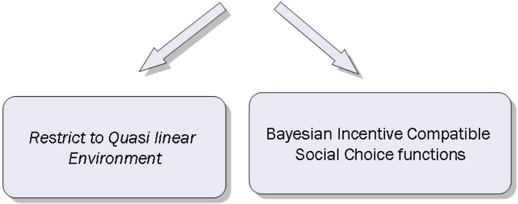

Bayesian Incentive Compatible Implementation

Theorem (The dAGVA Theorem)

Let the SCF $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ be allocatively efficient and the agents' types be statistically independent of each other. This function can be truthfully implemented in Bayesian Nash equilibrium if,

$$t_i(\theta) = E_{\tilde{\theta}_{-i}} \left[\sum_{j \neq i} v_j(k^*(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \right] + h(\theta_{-i}); \quad \forall\, i, \quad \forall\, \theta \in \Theta \tag{6}$$

where $h_i(\cdot)$ is any arbitrary function of θ_{-i}.

The above payment scheme is **dAGVA payment (incentive) scheme**.

Definition (s)

A dAGVA/expected externality/expected Groves Mechanism is a direct revelation mechanism, $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ in which SCF $f(\cdot)$ is allocatively efficient (1) and satisfies dAGVA payment scheme (6).

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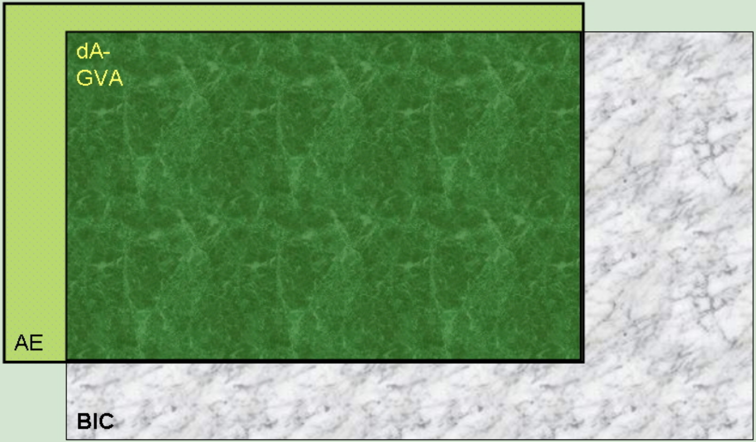
- C. d'Aspremont and L.A. Gérard-Varet. Incentives and incomplete information. *Journal of Public Economics*, 11:2545, 1979.
- K. Arrow. The property rights doctrine and demand revelation under incomplete information. In M. Boskin, editor, *Economics and Human Welfare*. Academic Press, New York, 1979.


dA-
GVA
AE
BIC

Notion of Individual Rationality

- By participating in mechanism utility to any agent should not decrease.

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  - Ex-ante individual rationality

Space of Mechanisms


BIC


DSIC
BIC


AE
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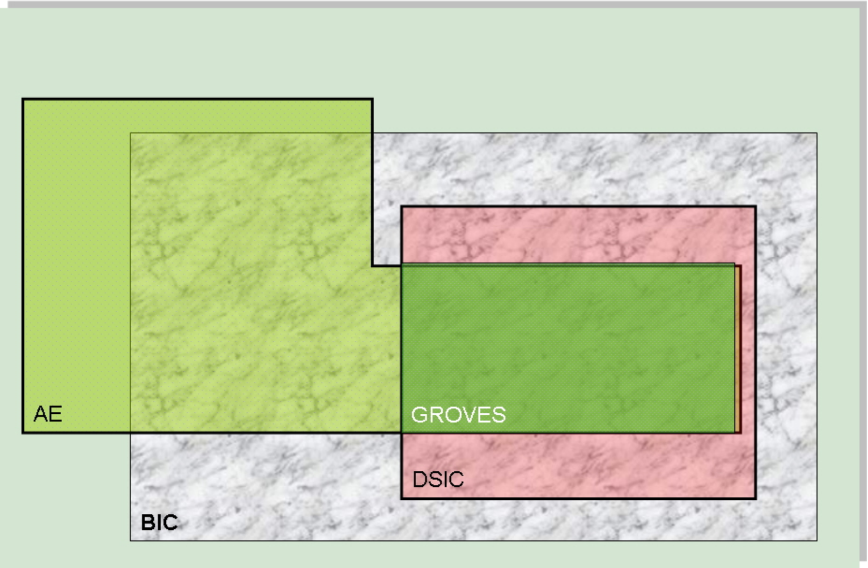
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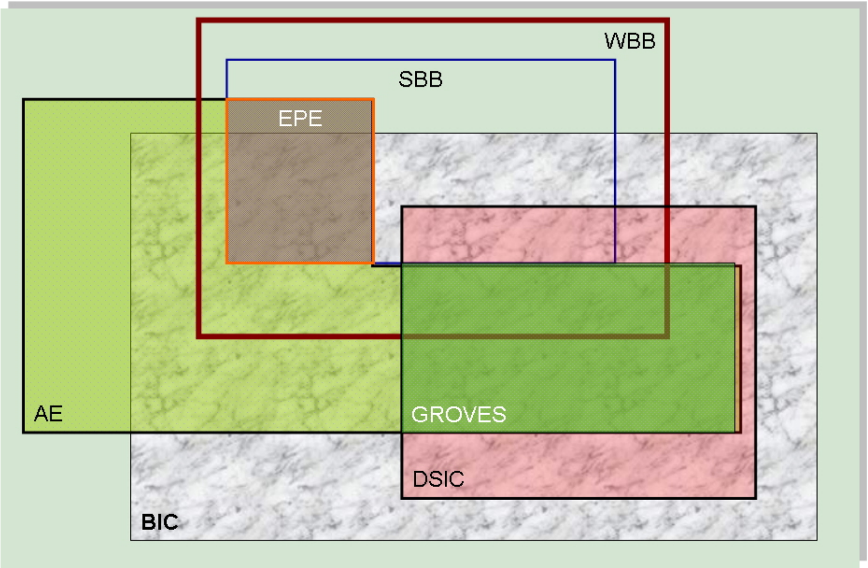
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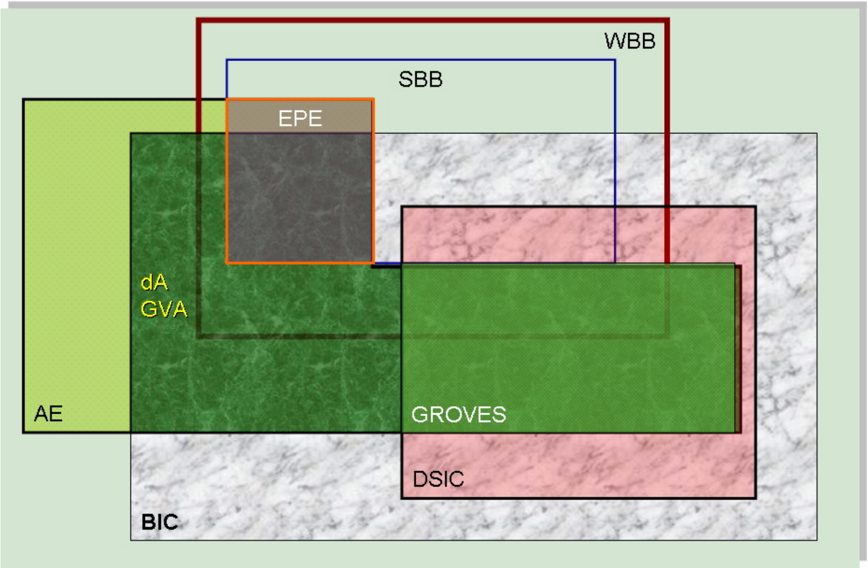
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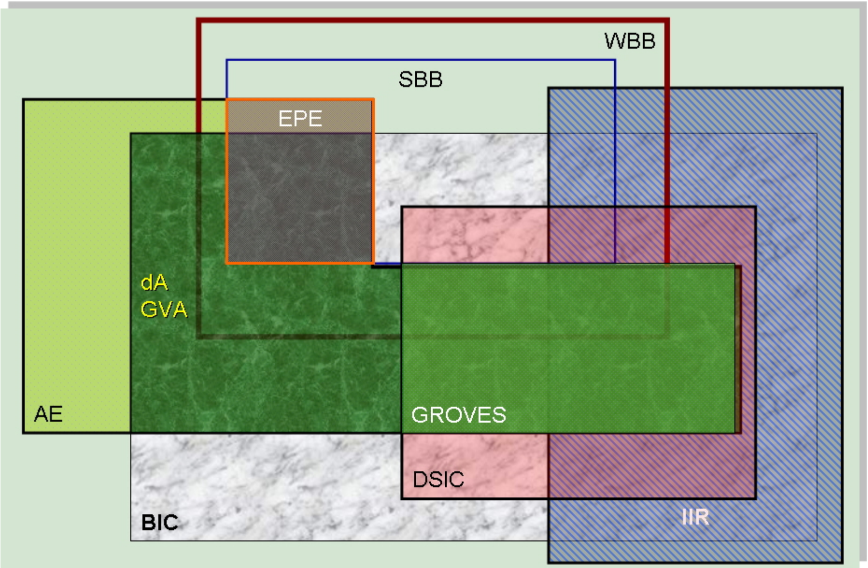
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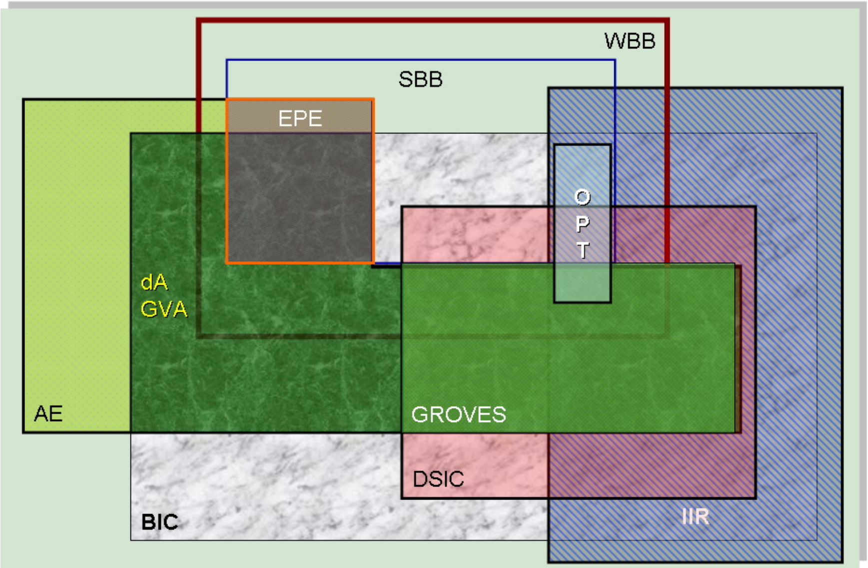
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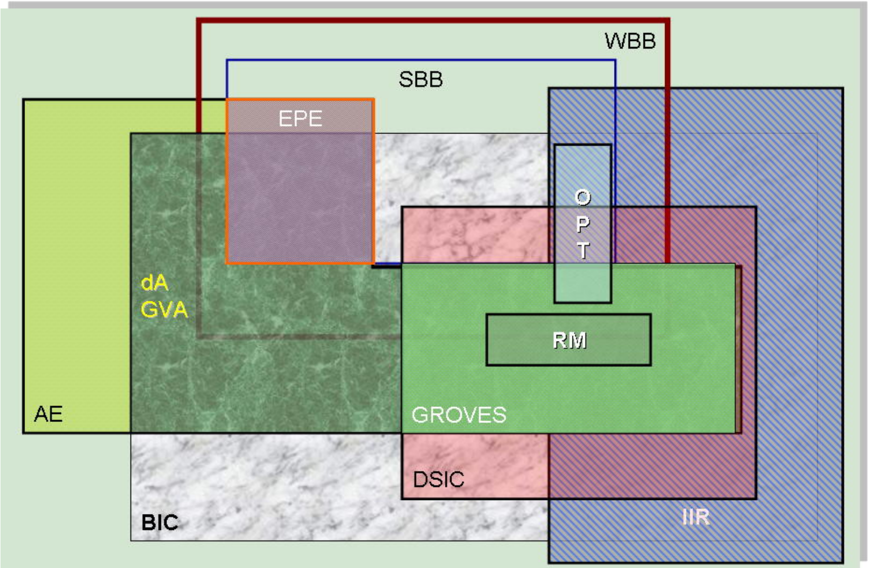
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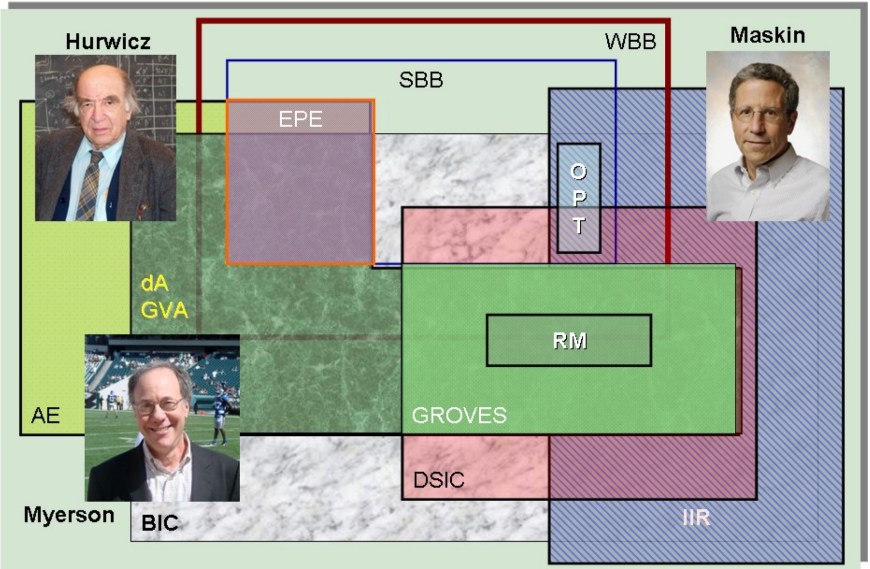
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Reading

- Game Theory by Roger Myerson. Harvard University press, 2013.
- Algorithmic Game Theory, edited by Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay Vazerani.
- Dinesh Garg, Y. Narahari, Sujit Gujar, "Foundations of Mechanism Design: A Tutorial - Part 1: Key Concepts and Classical Results." Sadhana - Indian Academy Proceedings in Engineering Sciences, Volume 33, Part 2, pp 83-130, April 2008.
- Dinesh Garg, Y. Narahari, Sujit Gujar, "Foundations of Mechanism Design: A Tutorial - Part 2: Advanced Concepts and Results." Sadhana - Indian Academy Proceedings in Engineering Sciences, Volume 33, Part 2, pp 131-174, April 2008.

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