CS 715: Advanced Topics in Algorithmic Game Theory and Mechanism Design

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Lecture 7

Agenda

Part I

Dynamic Two Sided Matching with Dynamic Women

- Introduction
- Stable Matching and Deferred Acceptance
- Our Approach and Results
- Summary

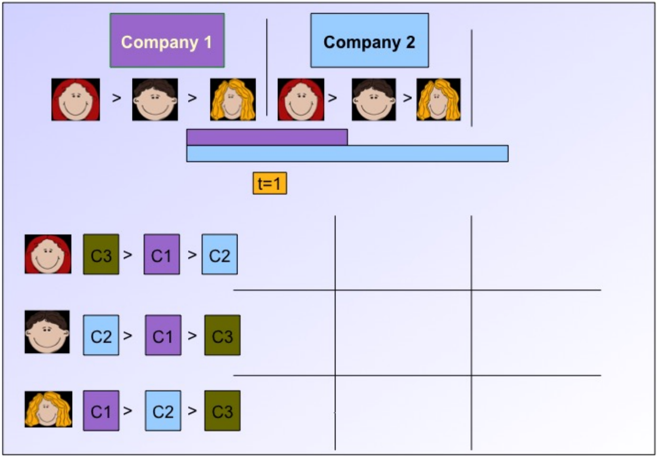
Part II

Dynamic Two Sided Matching with Dynamic Men

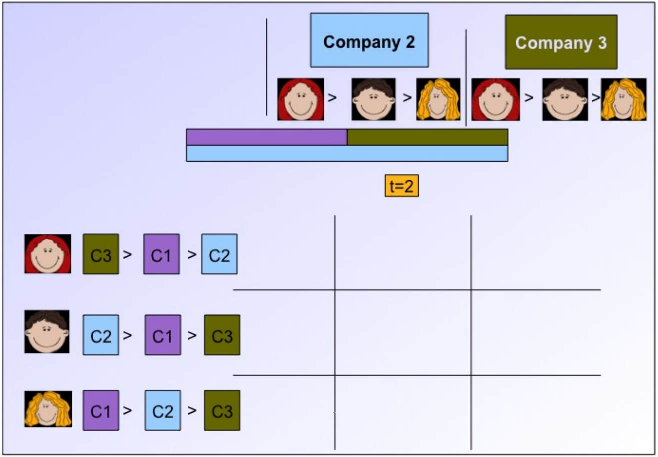
- Introduction
- Progressive Stability
- Our Approach and Results
- Summary

PartI: Dynamic Mechanism Design with Dynamic Women

Motivating Example



Motivating Example



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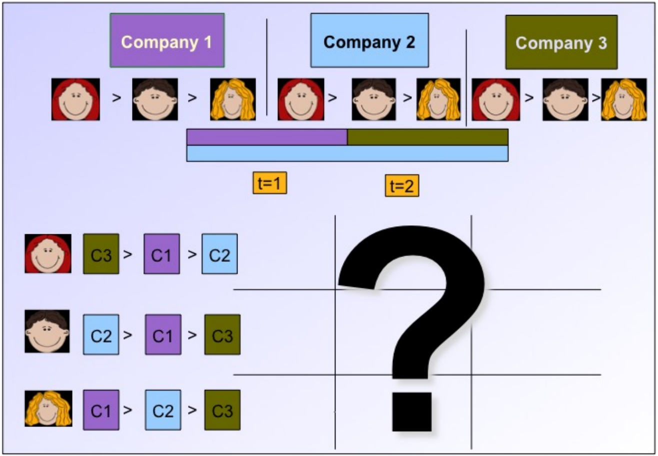

Observations

- This can be considered as a matching problem
- Students are static (= Men, M)
  - The preferences of the students are private
- Companies arrive-depart dynamically (= Women, W)
- We can assume that a company won't lie about its preferences
  - Typically it would be known what grades, skill-sets are required for a particular position in the company
- We focus on incentive properties on static side

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Model and Notation

n	Total number of men (women)
M	Set of Men
W	Set of Women
a_i, d_i	Arrival time and departure time for a woman w_i
a_j, d_j	Arrival time and departure time for a man m_j
ρ	$= \{(a_i, d_i)_{w_i \in W}\}$ Arrival-Departure Schedule of Women
ρ	$= \{(a_j, d_j)_{m_j \in M}\}$ Arrival-Departure Schedule of Men
$\succ_i$	Preference of $i \in M \cup W$
$\succ$	$= (\succ_i, \succ_{-i})$ Preference profile of all agents
$W(t)$	Women that are not matched till t.
$M(t)$	$\{m_j \ni a_j \leq t \leq d_j$ and m_j is not matched.$\}$
$AM(t)$	$\{m_j \mid a_j = t\}$ Set of men arriving in time slot t
$DM(t)$	$\{m_j \mid d_j = t\}$ Set of men departing in time slot t
f	Matching mechanism
μ	$= f(\succ, \rho)$. A matching

Table: Notation

Desirable Properties

- Blocking pair: (m, w) blocks the matching if they prefer to match with each other than their current match
- Stability: no blocking pair
- Strategyproof: For each agent, for any arrival-departure schedule, for any preferences, it is a best response to report preferences truthfully
- Good Rank Efficiency. Lower the $Rank(f)$, better[1]

[1] Average rank assigned by the agents to their match

Definitions

Definition (Strategy-Proof)

Online mechanism f is *strategy-proof* (or *truthful*) for men if for each man m, for all arrival-departure schedules ρ, and for all preferences $\prec_{-m}$, and for all $\succ'_m \neq \succ_m$,

$$\mu'(m) \not\succ \mu(m),$$

where $\mu' = f(\succ'_m, \succ_{-m}, \rho)$.

Definition (Stability)

We say a pair $(m, w) \in M \times W$ blocks a matching μ if, $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. If there is no blocking pair, we say the matching is stable. And if a matching mechanism always produce stable matching, we say the mechanism is stable.

The rank of a matching μ is $rank(\mu) = \frac{1}{2n} \sum_{i \in M \cup W} rank_i(\mu)$.
Assume a distribution function Φ on $(\succ, \rho)$

Definition (Rank-efficiency)

The rank-efficiency of an online mechanism f, given distribution function Φ, is

$$rank^f = \mathbb{E}_{(\succ,\rho)\sim\Phi}[rank(f(\succ, \rho))].$$

Male-Proposal Deferred Acceptance

Gale-Shapley[2] proposed,

- Each man proposes to his most preferred woman
- Each woman keeps her most preferred man among the proposals received and rejects all the others
- The men who are rejected in the above step propose to their next preferred woman
- If a woman receives a match better than her current match, she is matched with the new man and the previous man is not matched
- The process continues till all the men are matched

[2] D. Gale and L. S. Shapley, "College admissions and the stability of marriage", The American Mathematical Monthly, 69(1), 9-15, (January 1962) [1]

Important Static Case Results

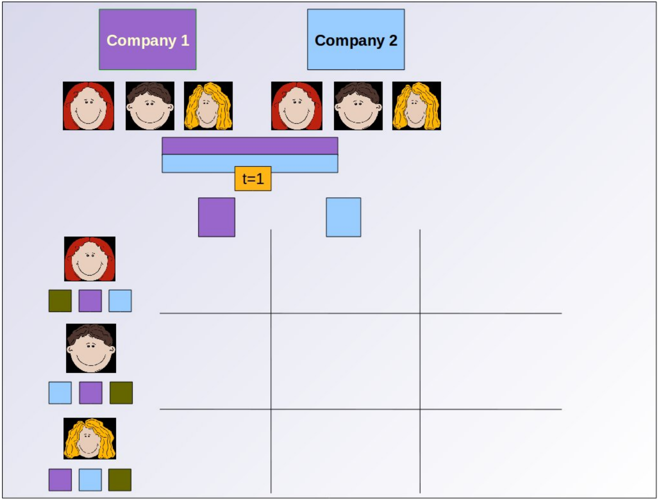
- Deferred acceptance is stable [1]
- Male-Proposal Deferred acceptance is strategyproof for men[3]
- Male Proposing Deferred Acceptance is male-optimal and Female Proposing Deferred Acceptance is female-optimal [1]
- No stable algorithm is strategyproof

[3] A. E. Roth, "The economics of matching: Stability and incentives", Mathematics of Operations Research, 7(4), 617-628, (1982). [4]

On-line Deferred Acceptance

1. In each period, run Male-Proposal Deferred Acceptance on current population. All these matches are temporary
2. If any woman is departing in a period, the man with whom she is currently matched is made final

But It Fails to Be Strategyproof



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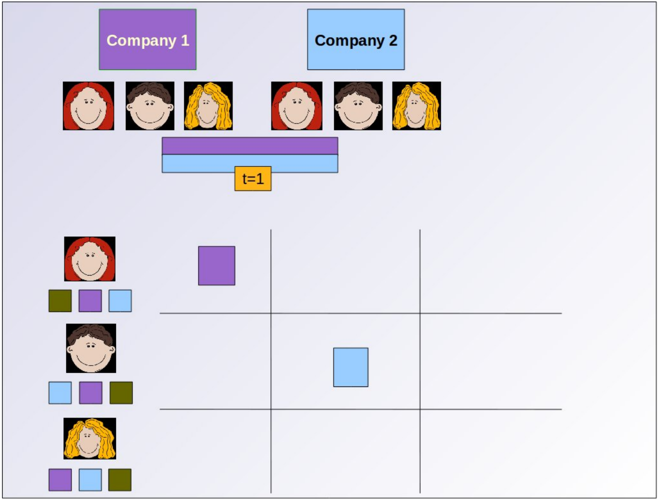

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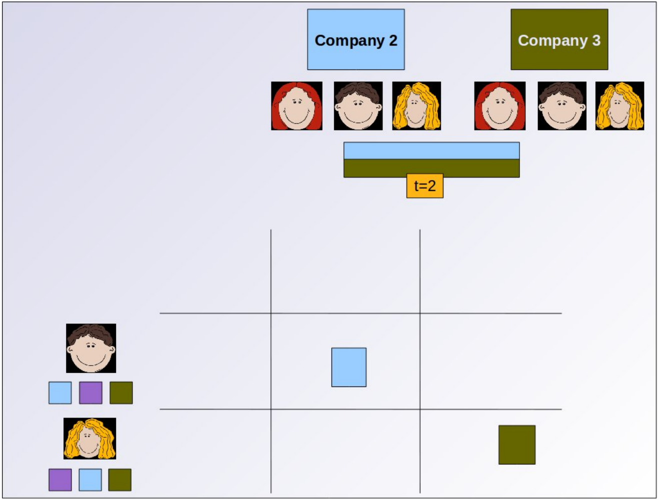

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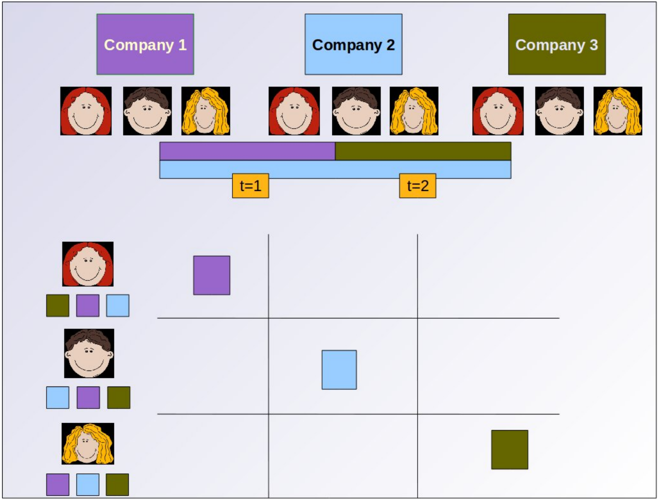

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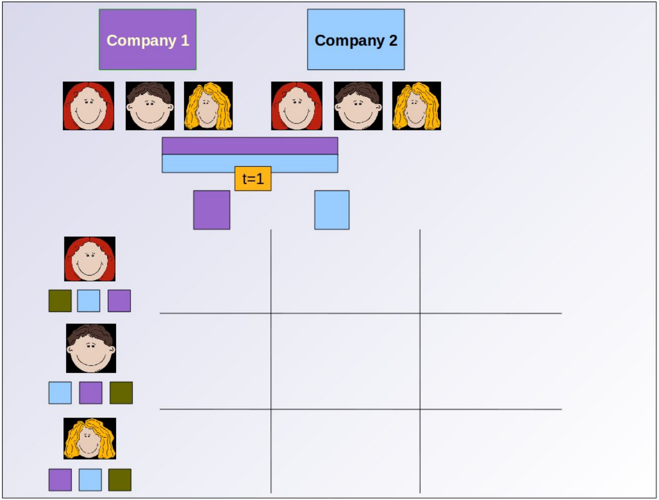

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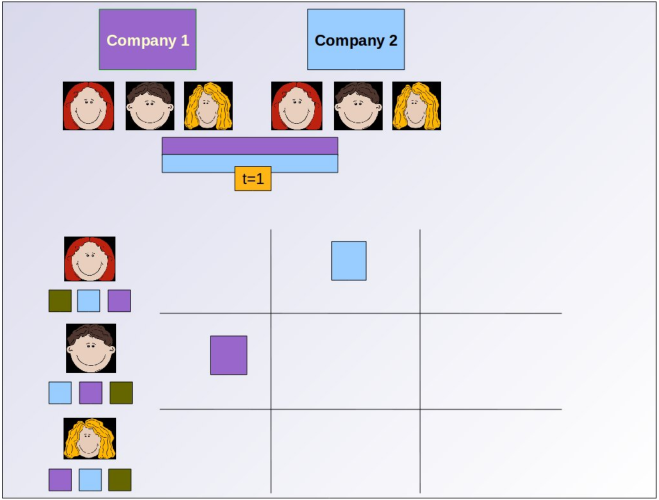

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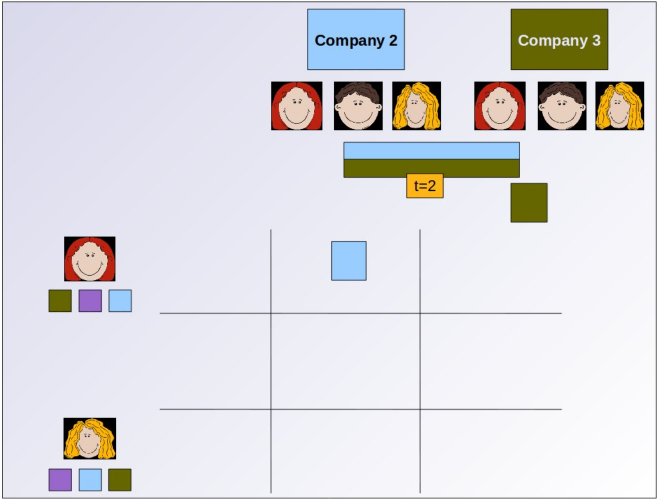

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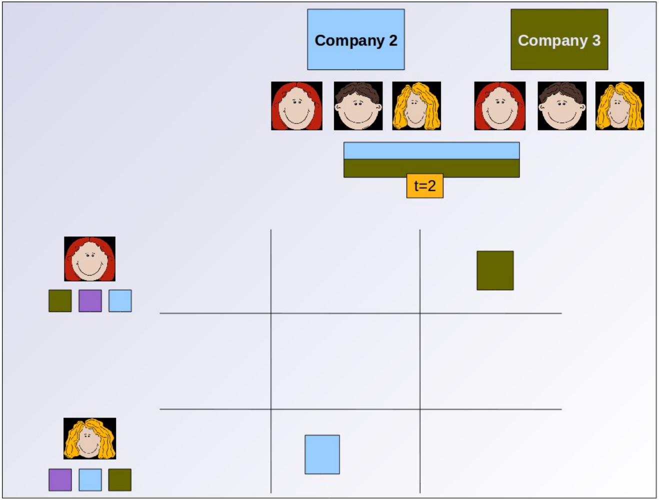

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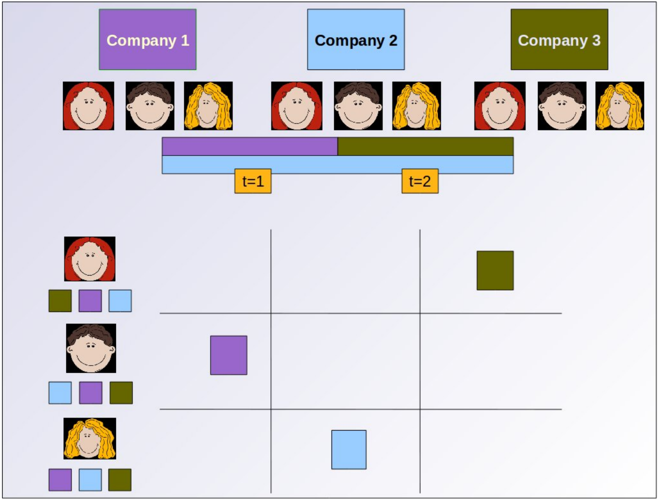
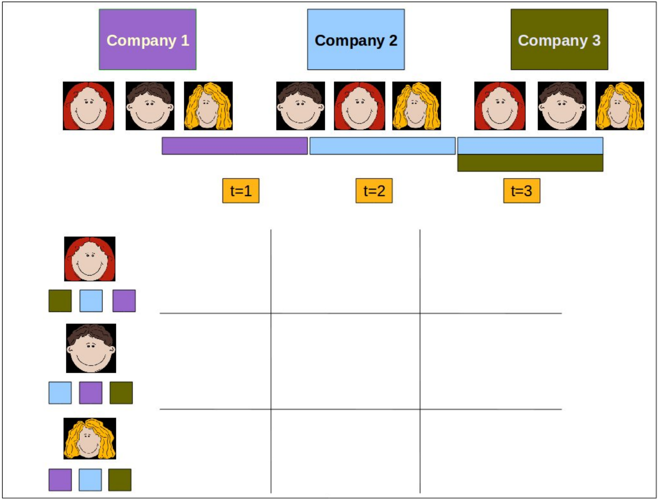


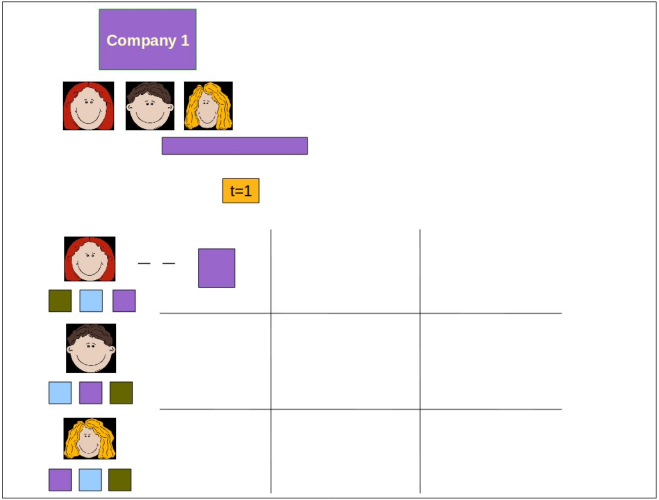
We propose: GSODAS

- Why does the On-line Deferred Acceptance fail?
- What if a man has option to decommit from a match if he gets a better match later?
- **G**eneralized **S**table, **O**n-line **D**eferred **A**cceptance with **S**ubstitutes
- Algorithm:
  - Run Male-Proposal Deferred Acceptance in each period.
  - If a man receives a better match, he can decomit from his current match
    - If this women has already left, she receives a substitute for him
  - Match a departing women with her current match

GSODAS: An Example



GSODAS: An Example



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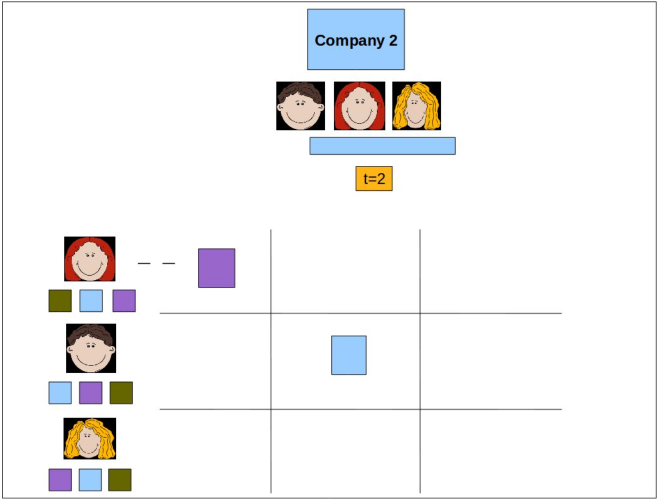

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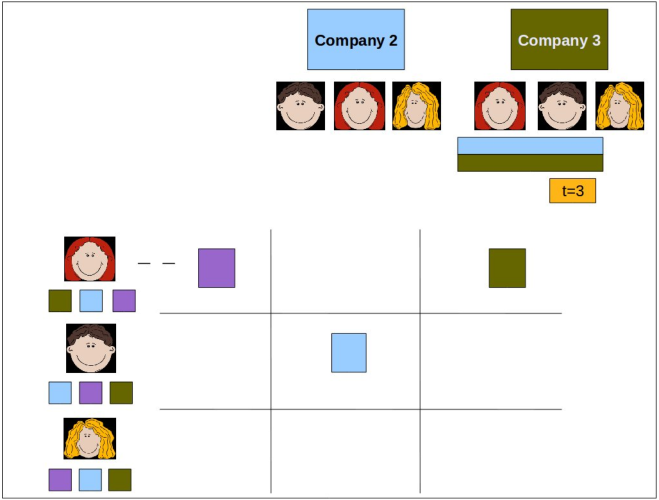
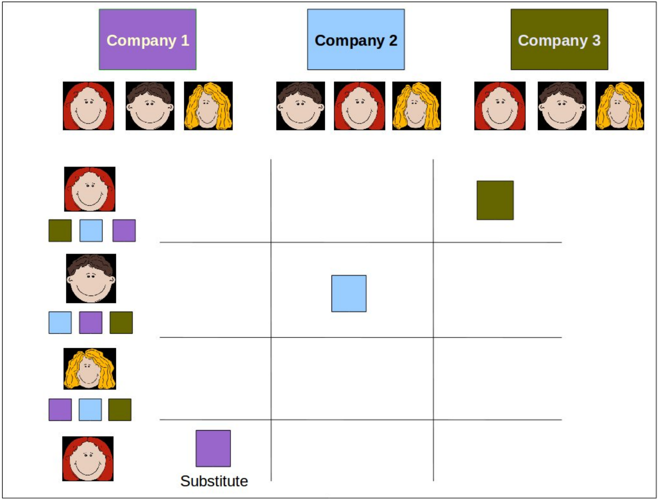


GSODAS: An Example



Properties of GSODAS

- Stable
- Strategyproof for men
- What about # substitutes required?

Worst Case Optimality of GSODAS

- GSODAS achieves stability at the cost of substitutes
- There is a necessary trade-off between usage of substitutes and stability
- Let $n = \alpha T,\ \alpha \in \{1, 2, 3, \ldots\}$, where T is number of rounds in the game
- For matching μ use the metric
  $\boxed{S(\mu) = \#\text{ Unstable men in } \mu + \#\text{ Substitutes used in } \mu}$
- We show,

Theorem

1. *Any on-line algorithm for matching, in worst case analysis,* $S(\mu) \geq \alpha(T-1)$ *and*
2. *For GSODAS* $S(\mu) \leq \alpha(T-1)$ *with equality in the worst case.*

- This implies, $\boxed{\text{GSODAS achieves stability optimally}}$

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Other Strategyproof On-line Algorithm

- Strategyproof Randomized On-line Matching Algorithm
- ROMA1
  - Run Male-Proposal Deferred Acceptance on departing women and men chosen at random.
  - This match is final
- ROMA2
  - In each period, if # women at least τ, run Male-Proposal Deferred Acceptance on all women and men chosen at random.
  - This match is final

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Benchmark: Consensus (Non-Strategyproof)

- On-line stochastic optimization[4]
- In each period, simulate the future arrival of the women by sampling future possible scenarios
- For each present woman, find out which man is getting matched most frequently. Run Male-Proposal Deferred Acceptance with present women and men who are most often matched.
- Commit only those matches that involve departing women

[4] P. Van Hentenryck and R. Bent, Online Stochastic Combinatorial Optimization, MIT Press, 2006.

Number of Substitutes required by GSODAS

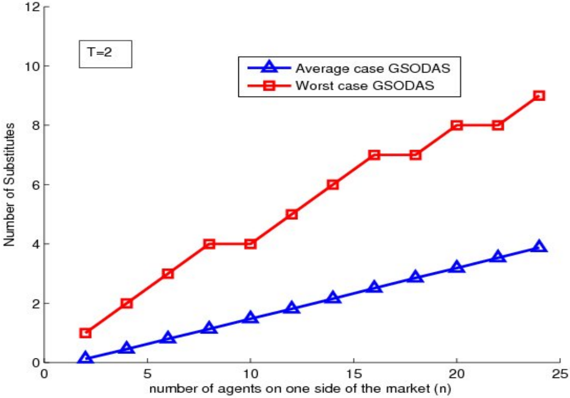


Figure: The number of substitutes required for men in GSODAS as n increases, fixing $T = 2$.

Number of Substitutes required by GSODAS

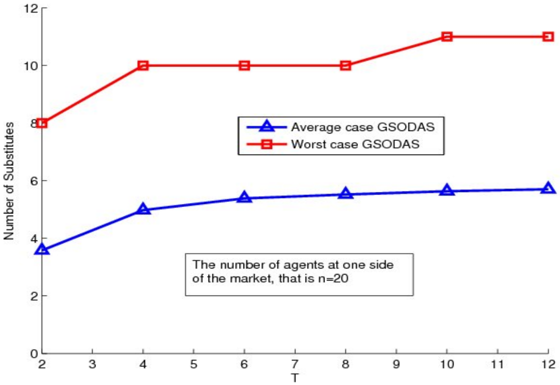


Figure: The number of substitutes required for men in GSODAS as T increases, fixing $n = 20$.

Stability vs Rank Efficiency

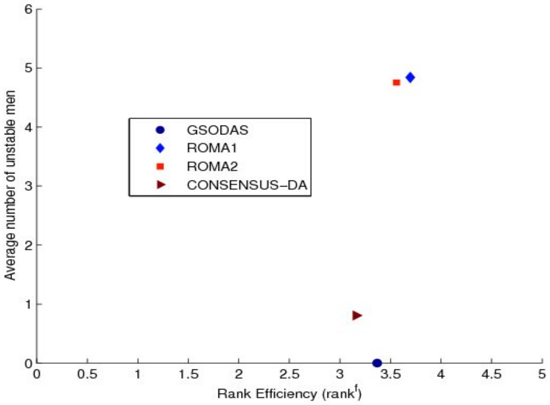


Figure: The rank-efficiency (x-axis) vs. the number of unstable men (y-axis) for $n = 10$ and $T = 2$.

Stability vs Rank Efficiency

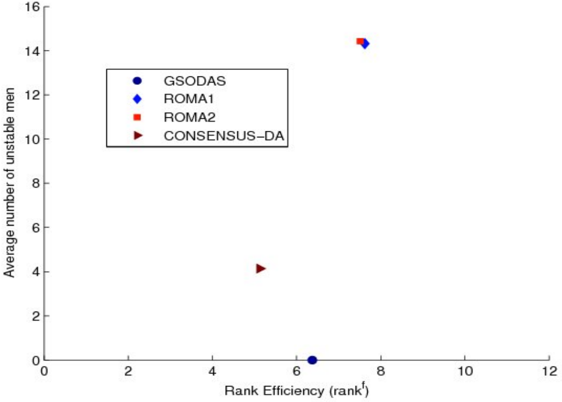


Figure: The rank-efficiency (x-axis) vs. the number of unstable men (y-axis) for $n = 20$ and $T = 4$.

Summary So Far

- The naive deferred acceptance fails in on-line settings
- We introduce the fall-back option
- GSODAS is stable and strategyproof (for men) at the cost of substitutes
- GSODAS achieves stability with minimal number of substitutes (worst case analysis)
- Experiments show GSODAS requires significantly lower substitutes than the worst case bounds
- GSODAS performs quite well for rank efficiency

5

[5] This work appeared as: Sujit Gujar and David Parkes, "Dynamic matching with a fall-back option", European Conference on Artificial Intelligence ECAI 2010. [3]

Possible Extensions

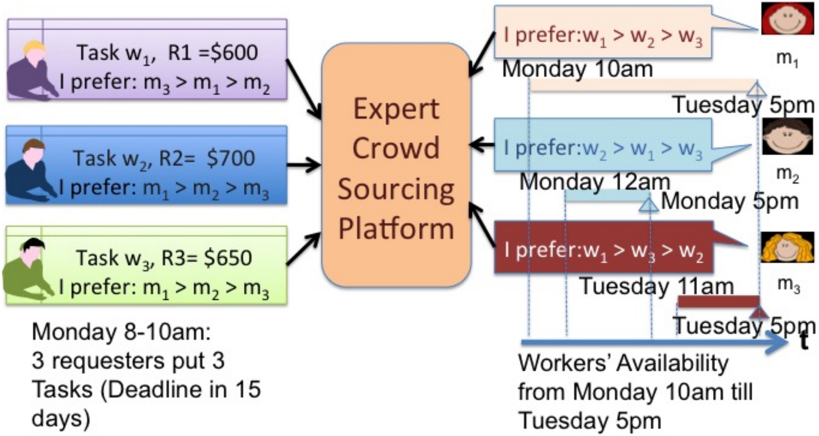
- In experiments: as time horizon (T) increase as many as 55% men use substitutes in the worst case
- Likely to be unacceptable in many practical situations

$\Rightarrow$ Relax the stability notion.

- Look for algorithms where strategic agents are dynamic

PartII: Dynamic Mechanism Design with Dynamic Men

Motivating Example



- Substitutes are not feasible
- So No Stability due to the above Theorem by Gujar and Parkes
- Similar to the previous settings, the repeated usage of Deferred Acceptance fails
- Partition Online Deferred Acceptance (PODA)
- PODA mechanisms are strategyproof for the men

Partition Mechanisms

1. Inputs: Preferences of Men and Women ($\succ$), ρ, Π $\{t_1, \ldots, t_k\}$
2. Output: A matching μ
3. $t = 1$, $W(1) = W$
4. If $\{t \in \{t_1, \ldots, t_k\}\}$
   - $\mu^{t_i} = DA(M_i, W(t))$
5. $t \leftarrow t + 1$
6. $W(t) \leftarrow W(t-1) \setminus \{w : \mu^t(w) \neq \phi\}$
7. If $\{W(t) == \phi\}$
   - $\mu = \cup_{t_i} \mu^{t_i}$
   - STOP.
8. GO TO STEP 4.

Examples of Partition Mechanisms

- Partition induced by the arrival period (f^{APODA}) or departure period
- Partition induced by threshold (f^{ThODA})

Progressive Stability

Do not allow women, that are matched with men who are no longer available for matching, to form a blocking pair.

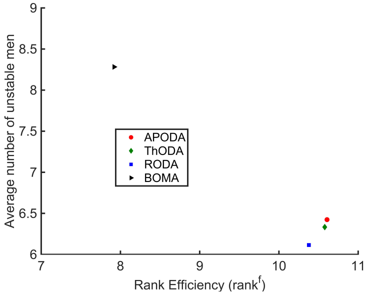
Definition (Progressive Stability)

A pair (m, w) is said to be blocking pair at time t if (i) m, w both are present in the system at t, and not matched with each other, (ii) prefer to match with each other than their current match, (iii) their current matches are also present in the system. In each time period, if no such pair exists, we say a matching is progressively stable.

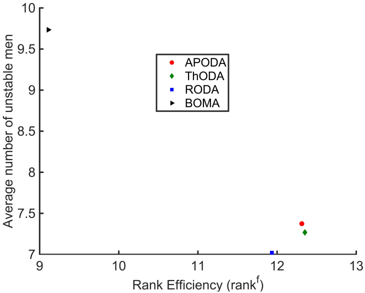
Improving Progressive Stability and Rank Efficiency

- PODA cannot achieve progressive stability
- Repeated Online Deferred Acceptance, RODA
  - Run the DA in every period and only the matches involving the departing men are final
- RODA is progressively stable but not strategyproof
- Bipartite Online Matching Algorithm, BOMA
  - f^{BOMA} is similar to f^{ThODA} except matching is done using Max-wt-Bipartite$(M(t), W(t))$
  - An edge between a man m and woman w has weight $2n + 2 - rank_m(w) - rank_w(m)$

Comparison of f^{APODA}, f^{ThODA}, f^{RODA}, f^{BOMA}



Scatter Plot for rank-efficiency and stability of the four mechanisms for $n = 20, \lambda = 3, \mu = .05$



Scatter Plot for rank-efficiency and stability of the four mechanisms for $n = 24, \lambda = 5, \mu = .05$

Comparison of The Proposed Mechanisms

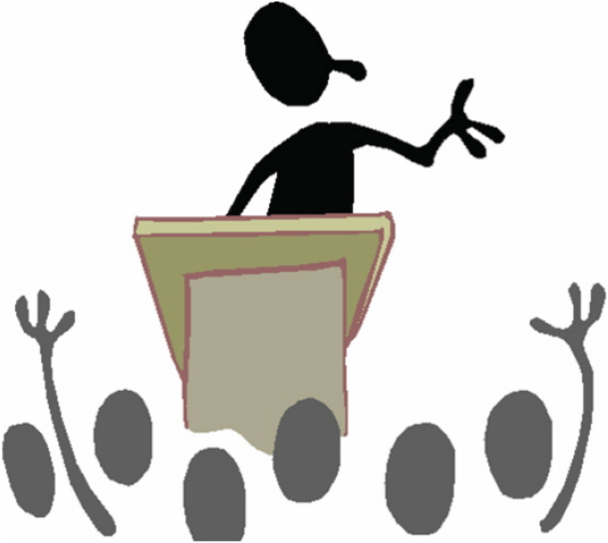
	f^{APODA}	f^{ThODA}	f^{RODA}	f^{BOMA}
Strategy-proof	Y	Y	N	N
Stability	3	2	1	4
Progressive Stability	N	N	Y	N
Rank-efficiency	3	4	2	1

Table: Comparison

This work is going to appear in MATCHUP 2015[6]

[6] Sujit Gujar and Boi Falting, Dynamic Task Assignments: An Online Two Sided Matching Approach". To Appear in 3rd International Workshop on Matching Under Preferences (MATCHUP 2015)" [2]

Questions?



David Gale and Lloyd S Shapley.
College admissions and the stability of marriage.
American Mathematical Monthly, pages 9–15, 1962.

Sujit Gujar and Boi Faltings.
Dynamic task assignments: An online two sided matching approach.
In *In 3rd International Workshop on Matching Under Preferences (MATCHUP 2015)*.
To appear.

Sujit Gujar and David C Parkes.
Dynamic matching with a fall-back option.
In *ECAI*, pages 263–268, 2010.

Alvin E Roth.
The economics of matching: Stability and incentives.
Mathematics of operations research, 7(4):617–628, 1982.

Thank You!!!

Proof of Claim 1

Yes this one and next one are extra slides.

- For GSODAS, # Unstable men $= 0$ and # Substitutes $\leq \alpha(T-1)$
- Construct preference profile for which for any on-line matching algorithm, $S(\mu) = \alpha(T-1)$

Preferences

m_1	$(T, \ldots, 2, 1, w)$	w_1	$(1, 2, \ldots, T, m),\ \ a = d = 1$
m_2	$(T, \ldots, 2, 1, w)$	w_2	$(\mu(w_1), 1, 2, \ldots, T, m),\ \ a = d = 2$
$\vdots$			
m_T	$(T, \ldots, 2, 1, w)$	w_T	$(\mu(w_{T-1}), 1, 2, \ldots, T, m),\ \ a = d = T$
m_{T+1}	$(2T, \ldots, T+2, T+1, w)$	w_{T+1}	$(T+1, T+2, \ldots, 2T, m),\ \ a = d = 1$
m_{T+2}	$(2T, \ldots, T+2, T+1, w)$	w_{T+2}	$(\mu(w_{T+1}), T+1, \ldots, 2T, m),\ \ a = d = 2$
$\vdots$			
m_{2T}	$(2T, \ldots, T+2, T+1, w)$	w_{2T}	$(\mu(w_{2T-1}), T+1, \ldots, 2T, m),\ \ a = d = T$
$\vdots$		$\vdots$	
$m_{(\alpha-1)T+1}$	$(\alpha T, \ldots, (\alpha-1)T+2, (\alpha-1)T+1, w)$	$w_{(\alpha-1)T+1}$	$(\mu(w_{(\alpha-1)T+1}), \ldots, \alpha T, m),\ \ a = d = 1$
$\vdots$		$\vdots$	

Table: Construction of Agent Preferences Used for Worst-case Substitutes Requirement in Online Matching Mechanisms