# CS 715: Advanced Topics in Algorithmic Game Theory and Mechanism Design

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Lecture 2

# Agenda

- Game Theory
  - Mixed Strategies
  - Utility Theory
  - Von Neumann and Morgenstern Theorem for zero sum games
  - N Player games and Nash Theorem
  - Games with Incomplete Information

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- Game Theory
  - Mixed Strategies
  - Utility Theory
  - Von Neumann and Morgenstern Theorem for zero sum games
  - N Player games and Nash Theorem
  - Games with Incomplete Information
- Mechanism Design Theory
  - Introduction
  - Desirable Properties of a Mechanism
  - Gibbard-Satterthwaite Impossibility Theorem
  - Quasi-linear Environment

# Mixed Strategies (1)

- In matching pennies game, row player tosses a coin and if H, then play H, else T.
- Similarly column player plays her action.
- Row player expected payoff = Pr(H,H) - Pr(H,T) - Pr(T,H) + Pr(T,T) = 0
- Column Player expected utility = -Pr(H,H) + Pr(H,T) + Pr(T,H) - Pr(T,T)
- Such randomization over actions is called as mixed strategy

# Mixed Strategies (2)

- Say for player $i$, there are $i_k$ actions, $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.
- She decides to play these actions with probabilities $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ with $p_{i_1} + p_{i_2} + \ldots + p_{i_k} = 1$
- $\Delta(S_i) = i_k$ dimensional simplex, representing all possible randomization over $S_i$.
- For player $i$, expected payoff =
$$\sum_{s_{-i} \in S_{-i}} p_{i_1} * p(s_{-i}) * U(a_{i_1}, s_{-i}) + p_{i_2} * p(s_{-i}) * U(a_{i_2}, s_{-i}) + \ldots + p_{i_k} * p(s_{-i}) * U(a_{i_k}, s_{-i})$$
- This leads to Utility Theory

# Utility Theory (1)

Let $X$ be the set of outcomes. $\succ$ be the preference of a player over the set of outcomes.

Axioms

- Completeness: every pair of outcomes is ranked
- Transitivity: If $x_1 \succ x_2$ and $x_2 \succ x_3$ then $x_1 \succ x_3$.
- Substitutability: If $x_1 \sim x_2$ then any lottery in which $x_1$ is substituted by $x_2$ is equally preferred.
- Decomposability: two different lotteries assign same probability to each outcome, then player is indifferent between these two lotteries
- Monotonicity: If $x_1 \succ x_2$ and $p > q$ then $[x_1 : p,\ x_2 : 1-p] \succ [x_1 : q,\ x_2 : 1-q]$
- Continuity: If $x_1 \succ x_2 \succ x_3$, $\exists p \ni x_2 \sim [x_1 : p,\ x_3 : 1-p]$

# Utility Theory (2)

Von Neumann and Morgenstern

**Theorem**

*Given a set of outcomes $X$ and a preference relation on $X$ that satisfies above six axioms, there exists a utility function $u : X \to [0, 1]$ with the following properties:*

1. $u(x_1) \geq u(x_2) iff \;\; x_1 \succ x_2$
2. $u([x_1 : p_1; x_2 : p_2; \ldots; x_m : p_m]) = \sum_{j=1}^{m} p_j u(x_j)$

# Zero Sum Games

- Recall: Zero sum games where one player's gain = other player's loss.
- We studied saddle points and pure strategy equilibrium
- Matching pennies: no pure strategy equilibrium
- Can it have a mixed strategy equilibrium?
- **Yes**.
- Let $p$ and $q$ be the mixed strategies of row and column player respectively.

# Equilibrium in Zero Sum Games

Von Neumann and Morgenstern showed:

**Theorem**

*For every $(m \times n)$ matrix $A$, there is a stochastic row vector $p^* = (p_1^*, \ldots, p_n^*)$ and a stochastic column vector ${q^*}^T = (q_1^*, \ldots, q_n^*)$ such that*

$$\min_{q \in \Delta(S_2)} p^* A q = \max_{p \in \Delta(S_1)} p A q^*$$

$(p^*, q^*)$ is equilibrium. Matching Pennies Game: $p^* = (0.5, 0.5) = {q^*}^T$

# $n$-player Games

Recall, $n$ player game in strategic form is represented as
$\Gamma =< N, (S_i)_{i \in N}, (U_i)_{i \in N} >$

$N$ : Set of players
$N = \{1, 2, \ldots, n\}$

$S_1$ : Strategies available to player 1
$S_2$ : Strategies available to player 2
$\vdots$
$S_n$ : Strategies available to player $n$
$S = S_1 \times S_2 \times \ldots \times S_n$
Strategy space of all the players

$U_1 : S \to \mathbb{R}$
$U_2 : S \to \mathbb{R}$
$\vdots$
$U_n : S \to \mathbb{R}$
Utility Functions

- Note that we denote the space of strategies include randomization, that is, it is set of **mixed strategies** by $\Delta(S_i)$ and represent a mixed strategy for a player $i$ as $\sigma_i$.

# Nash Equilibrium

**Definition (Pure Strategy Nash Equilibrium)**

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is called as Pure Strategy Nash Equilibrium, if for each player $i$, $s_i^*$ is a best response strategy to $s_{-i}*$.

That is, $\forall i$

$$U_i(s_i^*, s_{-i}^*) \geq U_i(s_i, s_{-i}^*) \; \forall \, s_i \; \in \; S_i$$

**Definition (Mixed Strategy Nash Equilibrium)**

A strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is called as Mixed Strategy Nash Equilibrium, if for each player $i$, $\sigma_i^*$ is a best response strategy to $\sigma_{-i}*$.

That is, $\forall i$

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \; \forall \, \sigma_i \; \in \; \Delta(S_i)$$

# Nash Theorem

- Nash Theorem: Every finite game has at least one Nash Equilibrium (NE).
  In NE player plays his/her best response to the strategy played by the remaining agents
- Interpretations
  - Prescription
  - Prediction
  - Self enforcing agreement
  - Evolution and Steady State


[1]

[1] Image Credits: Elke Wetzig (Elya) - Own work.

# Batter of Sexes

- You and your friend want to meet for coffee
- You both enjoy being with each other than being alone. (Say utility 1 for being with each other)
- You enjoy Esplande (Say additional utility of 1) where as your friend enjoys BC more.

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| | E | BC |
|---|---|---|
| E | (2,1) | (0,0) |
| BC | (0,0) | (1,2) |

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# Batter of Sexes

- You and your friend want to meet for coffee
- You both enjoy being with each other than being alone. (Say utility 1 for being with each other)
- You enjoy Esplande (Say additional utility of 1) where as your friend enjoys BC more.
  - Pure strategy equilibria (E,BC) and (BC,E)
  - Mixed Strategy equilibrium $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$

| | E | BC |
|---|---|---|
| E | (2,1) | (0,0) |
| BC | (0,0) | (1,2) |

# Games with Incomplete Information: Auction

- Two players A and B compete for an item.
- Value it 2 and 1 respectively
- Strategies: bid either 0 or 1.
- Outcome: A wins if Bid(A) $>$ Bid(B), B wins otherwise
- Payoff: (Value - payment) if a player wins and 0 otherwise

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- Strategies: bid either 0 or 1.
- Outcome: A wins if Bid(A) $>$ Bid(B), B wins otherwise
- Payoff: (Value - payment) if a player wins and 0 otherwise
  - (1,0) is pure strategy Nash equilibrium. (Note it is weakly dominant strategy equilibrium)
  - However the valuation of the other players are not known
  - This leads to mechanism design theory

| | | Player B | |
|---|---|---|---|
| | | 0 | 1 |
| Player | 0 | (0,1) | (0,0) |
| A | 1 | (1,0) | (0,0) |

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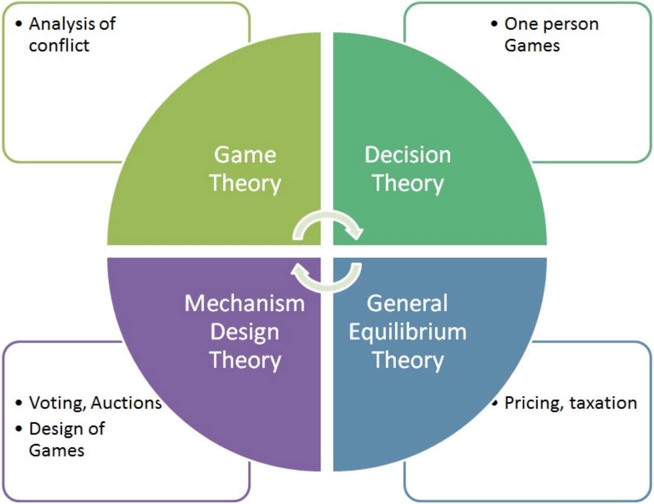
# Game Theory and Other Related Theories



# Mechanism Design

- Game Theory: Analysis of strategic interaction among players

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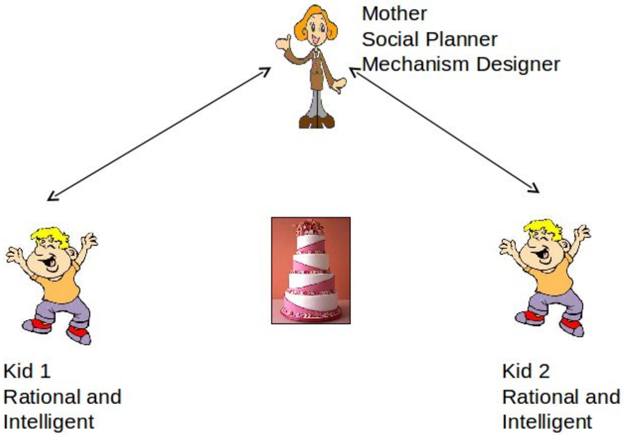
## Mechanism Design

Mechanism Design is the art of designing rules of a game to achieve a specific outcome in presence of *multiple self-interested agents*, each with *private information* about their preferences

# Mechanism Design Example: Fair Division of Cake




- This is an example of a mechanism without money

# Mechanism Design Example: Vickrey Auction

- Second Price Auction (SPA) for selling a single item.
  - The bidder with the highest bid wins
  - She pays as much as the second highest bid
- Vickrey[a] showed : The truth revelation is dominant strategy

[a] W Vickrey, "Counter-speculation, Auctions, and Competitive Sealed Tenders", Journal of Finance 1961.

**Example**

- Say there are four bidders
- They value the object as 80, 100, 60, and 40
- Suppose they bid truthfully
- The bidder 2 gets the object and pays 80

- This is an example of a mechanism with money

# Mechanism Design Framework

$$\mathcal{N} = \{1, 2, \ldots, n\}$$

- $\mathcal{M} = (S_1(), S_2(), \cdots, S_n(), g(\cdot))$ where, $g() : \prod_i S_i \to X$

# Mechanism Design Framework

$\mathcal{N} = \{1, 2, \ldots, n\}$

$\Theta_1, \ldots, \Theta_n$

- $\mathcal{M} = (S_1(), S_2(), \cdots, S_n(), g(\cdot))$ where, $g() : \prod_i S_i \to X$

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$\mathcal{N} = \{1, 2, \ldots, n\}$ $\Theta_1, \ldots, \Theta_n$

$X$ : Set of Outcomes

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# Mechanism Design Framework

$\mathcal{N} = \{1, 2, \ldots, n\}$

$\Theta_1, \ldots, \Theta_n$

$X$ : Set of Outcomes

$u_1, u_2, \ldots, u_n : X \times \Theta_i \to \mathbb{R}$

- $\mathcal{M} = (S_1(), S_2(), \cdots, S_n(), g(\cdot))$ where, $g() : \prod_i S_i \to X$

# Mechanism Design Framework

$\mathcal{N} = \{1, 2, \ldots, n\}$

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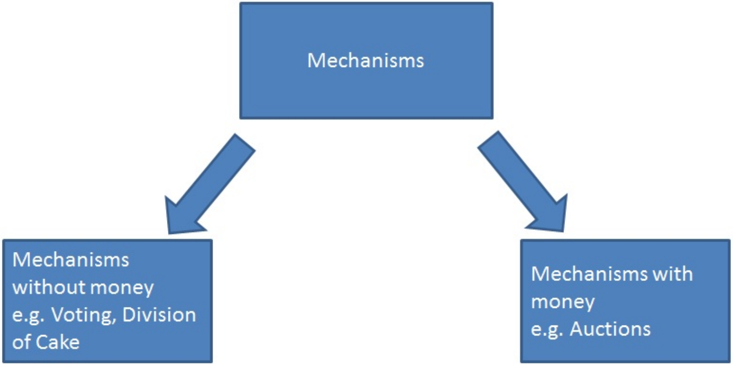
$u_1, u_2, \ldots, u_n : X \times \Theta_i \rightarrow \mathbb{R}$

SCF

$f : \Theta_1 \times \ldots \Theta_n \rightarrow X$

- $\mathcal{M} = (S_1(), S_2(), \cdots, S_n(), g(\cdot))$ where, $g() : \prod_i S_i \rightarrow X$

# Mechanisms: With Money and Without Money




Settings in which mechanisms allow monetary transfers are referred to as quasi-linear settings

# Properties of Mechanisms

**DSIC**

Dominant Strategy Incentive Compatibility Reporting truth is always good

**AE**

Allocative Efficiency Allocate item to those who value them most

**Non-Dictatorship**

No single agent is favored all the time

**BIC**

Bayesian Incentive Compatibility Reporting truth is good in expectation whenever others report truth

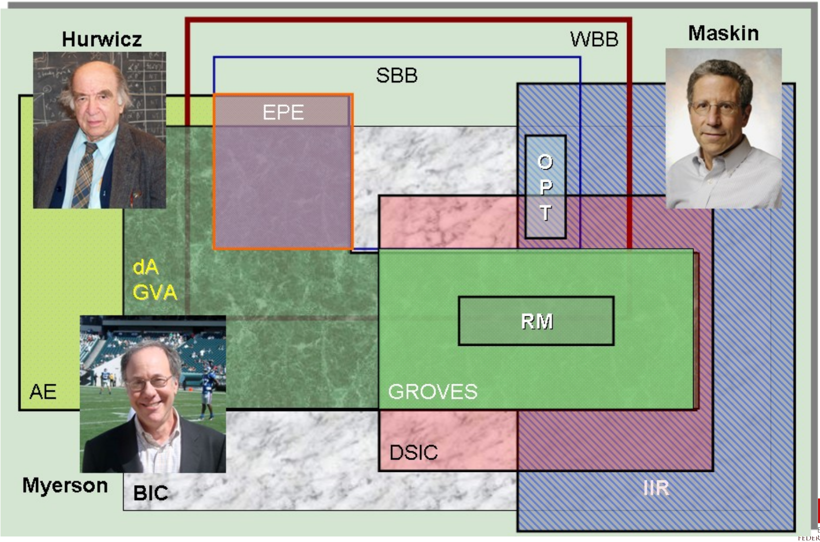
**BB**

Budget Transfer Payments are balanced and net transfer is zero

**IR**

Individual Rationality Payments participate voluntarily. (No losses)

# Space of Mechanisms in Quasi-Linear Settings



# Acronyms

| | |
|---|---|
| DSIC | Dominant Strategy Incentive Compatible |
| BIC | Bayesian Nash Incentive Compatible |
| AE | Allocative Efficiency (Allocatively Efficient) |
| BB | Budget Balance |
| IR | Individual Rationality |
| VCG | Vickrey-Clarke-Groves Mechanisms |
| dAGVA | d'Aspremont and Gérard-Varet mechanisms |

# Notation

| | |
|---|---|
| $N$ | Set of agents: $\{1, 2, \ldots, n\}$ |
| $\Theta_i$ | Type set of Agent $i$ |
| $\Theta$ | A type profile $= (\Theta_1 \times \ldots \times \Theta_n)$ |
| $\Theta_{-i}$ | A profile of types of agents other than $i$ $= (\Theta_1 \times \ldots \times \Theta_{i-1} \times \Theta_{i+1} \times \ldots \times \Theta_n)$ |
| $\theta_i$ | Actual type of agent $i$, $\theta_i \in \Theta_i$ |
| $\theta$ | A profile of actual types $= (\theta_1, \ldots, \theta_n)$ |
| $\theta_{-i}$ | A profile of actual types of agents other than $i$ $= (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$ |
| $\hat{\theta}_i$ | Reported type of agent $i$, $\hat{\theta}_i \in \Theta_i$ |
| $\hat{\theta}$ | A profile of reported types $= (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ |
| $\hat{\theta}_{-i}$ | A profile of reported types of agents other than $i$ $= (\hat{\theta}_1, \ldots, \hat{\theta}_{i-1}, \hat{\theta}_{i+1}, \ldots, \hat{\theta}_n)$ |

# Notation

| | |
|---|---|
| $\Phi_i(.)$ | A cumulative distribution function (CDF) on $\Theta_i$ |
| $\phi_i(.)$ | A probability density function (PDF) on $\Theta_i$ |
| $X$ | Outcome Set |
| $x$ | A particular outcome, $x \in X$ |
| $u_i(.)$ | Utility function of agent $i$ |
| $f(.)$ | A social choice function |
| $F$ | Set of social choice functions |
| $\mathscr{D}$ | A direct revelation mechanism |
| $K$ | A Set of project choices |
| $k$ | A particular project choice, $k \in K$ |
| $t_i$ | Monetary transfer to agent $i$ |
| $v_i(.)$ | Valuation function of agent $i$ |

# Gibbard-Satterthwaite Impossibility Theorem

## Theorem

*If*

1. *The outcome set $X$ is such that, $3 \leq |X| < \infty$*
2. $\mathscr{R}_i = \mathscr{S} \ \forall \ i$
3. $f(\Theta) = X$*, that is, the image of SCF* $f(\cdot)$ *is the set* $X$*.*

*then the social choice function SCF $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.*

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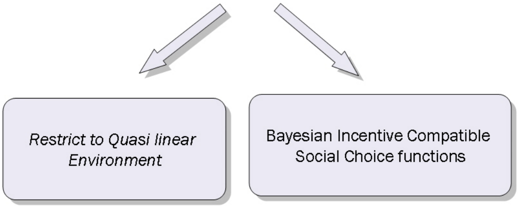
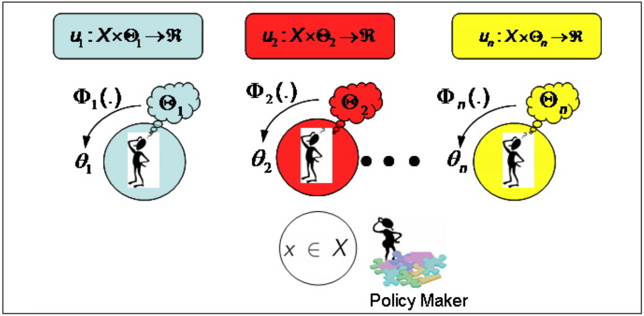


# Quasi Linear Environment

$$u_i(x, \theta_i) = v_i(k, \theta_i) + t_i$$

Valuation function of agent $i$

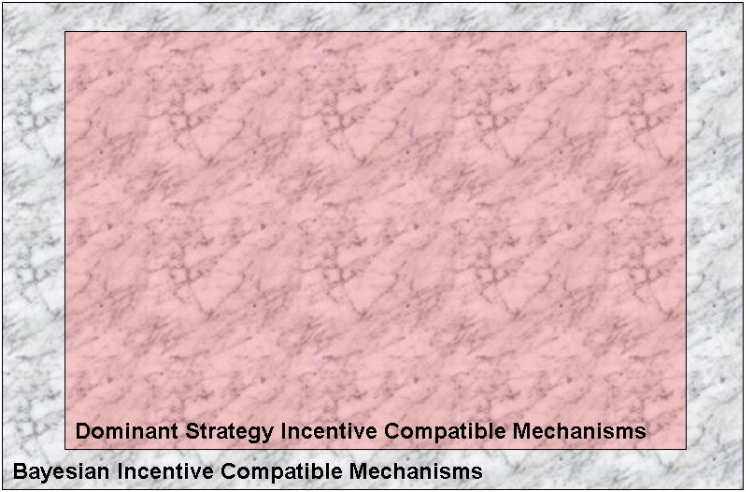



$$X = \left\{ (k, t_1, \ldots, t_n) : k \in K, t_i \in \mathbb{R} \; \forall \, i = 1, \ldots, n, \sum_i t_i \leq 0 \right\}$$

Project Choice

Monetary transfer to agent $i$

# Bayesian Incentive Compatible Mechanisms


Dominant Strategy Incentive Compatible Mechanisms
Bayesian Incentive Compatible Mechanisms

# Reading


- Game Theory by Roger Myerson. Harvard University press, 2013.
- Algorithmic Game Theory, edited by Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay Vazerani.
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- Dinesh Garg, Y. Narahari, Sujit Gujar, "Foundations of Mechanism Design: A Tutorial - Part 2: Advanced Concepts and Results." Sadhana - Indian Academy Proceedings in Engineering Sciences, Volume 33, Part 2, pp 131-174, April 2008.


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