# Department of Computer Science and Engineering (CSE) BRAC University

Fall 2023

CSE250 - Circuits and Electronics

### FIRST ORDER CIRCUITS

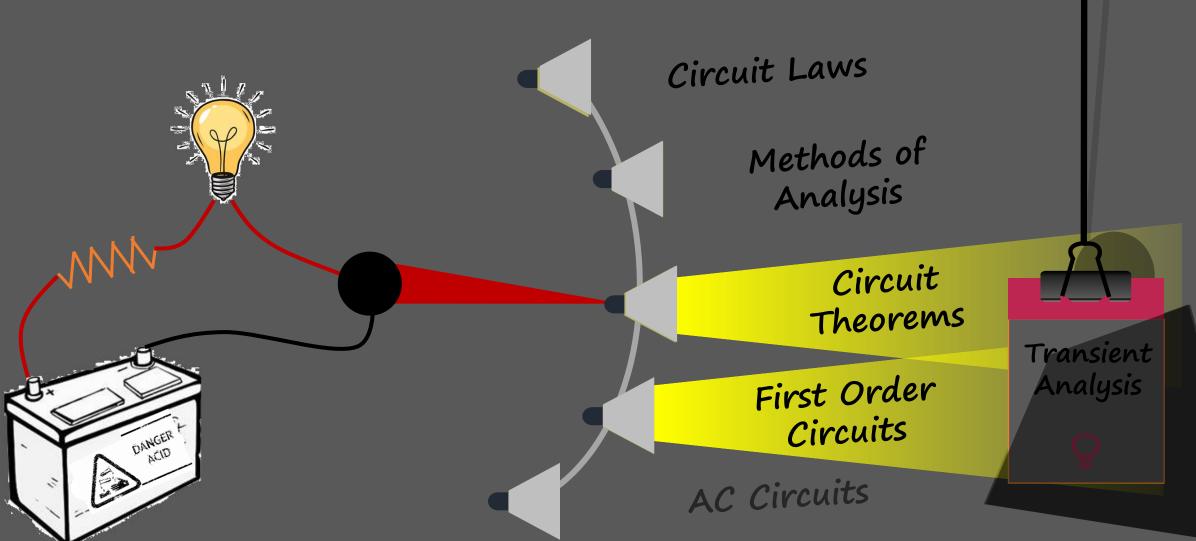


PRITHU MAHMUD, LECTURER

Department of Computer Science and Engineering (CSE)

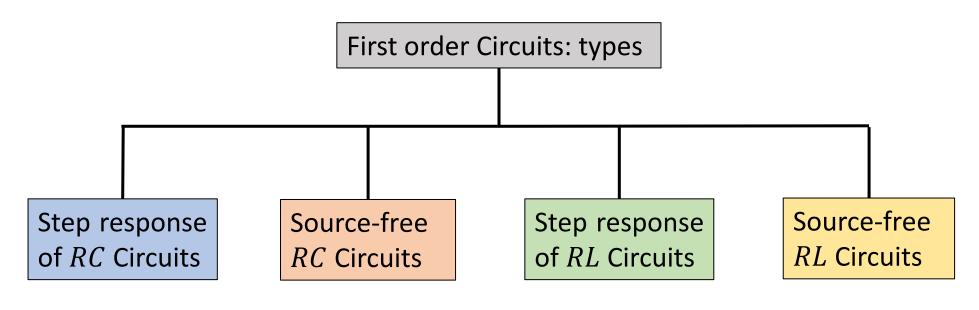
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# Course Outline: broad themes



## First Order Circuits

- A first-order circuit is characterized by a first-order differential equation.
- We shall examine two types of differential circuits: circuit comprising resistors and capacitors (RC circuit) and circuit comprising resistors and inductors (RL circuit).
- Two ways to excite the circuits: (i) by initial conditions of storage elements (source free circuits) and (ii) by independent sources (DC for this course).





### Circuit Elements

#### Active element

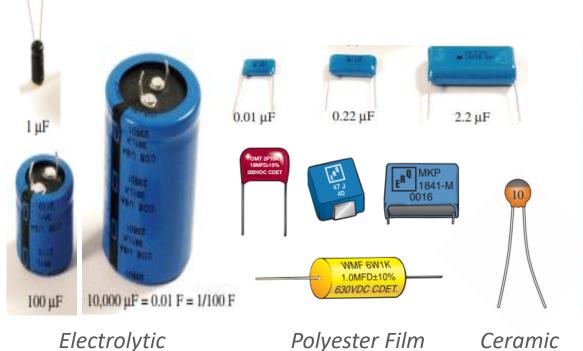
- > An *active element* is capable of generating energy.
- In other words, an element is said to be active if it can add some gain (in terms of voltage or current) to a circuit.
- > Active elements can absorb energy if they are forced to do so by other active elements.
- > Examples: Voltage/current sources, generators, transistors, operational amplifiers.

#### Passive element

- > Passive elements cannot supply energy. They can only consume/dissipate/store energy.
- > Examples: Resistors, capacitors, inductors, transformers.
- > Transformers change the voltage or current levels, but the power is unchanged. This is why transformers are passive element.

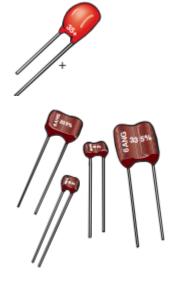
# Capacitors

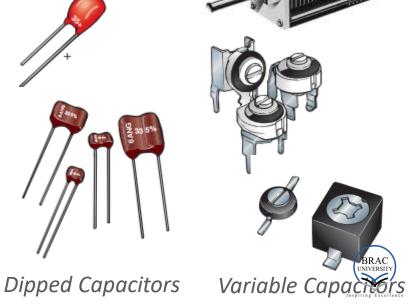
- A capacitor is a passive circuit element designed to store energy in its electric field.
- Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.





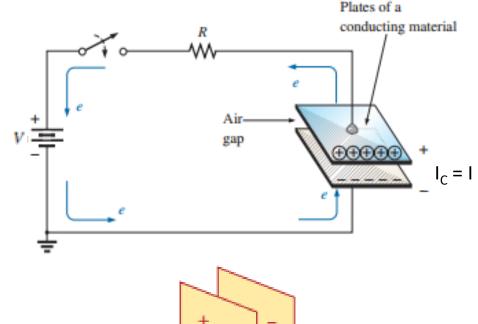
Mica Capacitors

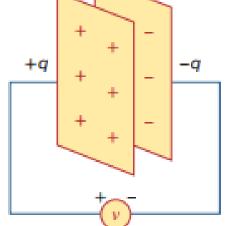




# Parallel Plate Capacitor

- Most widely used configuration is the two conducting surfaces (aluminium mainly) separated by a dielectric (air, ceramic, paper, or mica).
- The switch is open initially (no net charge).
- Closing the switch causes electrons to flow from and to the upper and lower plates respectively as shown by the arrows.
- Electron flow continues until the potential difference between the plates equals the applied potential.
- The final result is a net positive charge on the top plate and a negative charge on the bottom plate.

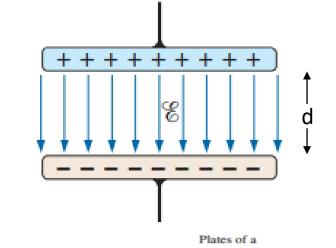






# Capacitance

- Capacitance is a measure of a capacitor's ability to store charge.
- Increasing V increases E as  $E \propto \frac{V}{d}$  as long as d is constant. An increase in E field causes increased charge separation i.e. increases q.

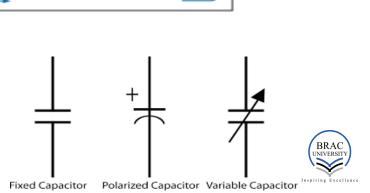


conducting material

- So,  $q \propto V$
- $\Rightarrow q = CV [C \text{ is a proportionality constant} \equiv Capacitance]$

$$\Rightarrow$$
  $C = \frac{q}{v}$  [F (Farad), mF,  $\mu$ F]

- $\Rightarrow$  For a particular capacitor  $\uparrow V$ ,  $\uparrow q$  but  $\frac{q}{v} = \text{const. So, } C$  does not depend on q or v. It depends on the physical dimension of the capacitor.
- $\Rightarrow$  For the parallel plate capacitor,  $C = \frac{\mathcal{E}A}{d}$



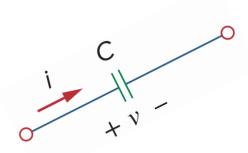
# I-V relation of a Capacitor

From the definition of the capacitance,

$$C = \frac{q}{v}$$

$$\Rightarrow q = Cv$$

Differentiating with respect to time,



$$\frac{dq}{dt} = C \frac{dv}{dt}$$

$$i = C \frac{dv}{dt}$$

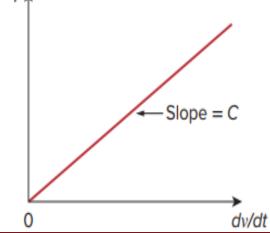
- This is the characteristic equation of a capacitor.
- Integrating with respect to time,

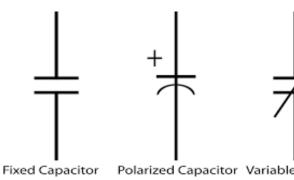
$$v(t) = \frac{1}{C} \int i(t) \, dt$$

If the voltage of the capacitor at any time  $t_0$  is  $v(t_0) =$  $q(t_0)/C$ , then,

$$v(t) = \frac{1}{C} \int_{t_0}^{t} i(t) dt + v(i_0)$$

It shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.





# Energy & Power of a Capacitor

• The instantaneous power delivered to a capacitor according to the passive sign convention is,

$$p = v(t)i(t) = Cv(t)\frac{dv(t)}{dt}$$

The energy stored in the capacitor is therefore

$$w(t) = \int_{-\infty}^{t} p(t) dt = \int_{-\infty}^{t} Cv(t) \frac{dv(t)}{dt} dt = \int_{v(-\infty)}^{v(t)} Cv(t) dv$$

$$\Rightarrow w(t) = \frac{1}{2} Cv^{2} \Big|_{v(-\infty)=V_{0}}^{v(t)=V}$$



$$\Rightarrow w(t) = \frac{1}{2}CV^2 - \frac{1}{2}CV_0^2$$

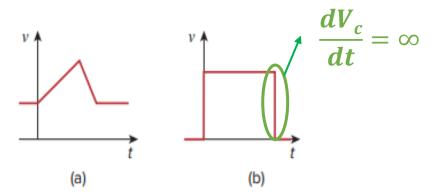


$$w(t) = \frac{1}{2}Cv(t)^2 = \frac{1}{2}CV^2$$



# Capacitor: important properties

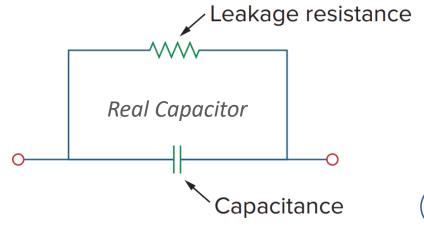
- 1. A capacitor is an <u>open circuit</u> to dc. At dc,  $i_C = C \frac{dV_{c-dc}}{dt} = 0$  [Open circuit]
- 2. The voltage on a capacitor cannot change abruptly.



Voltage change across a capacitor (a) allowed and (b) not allowed

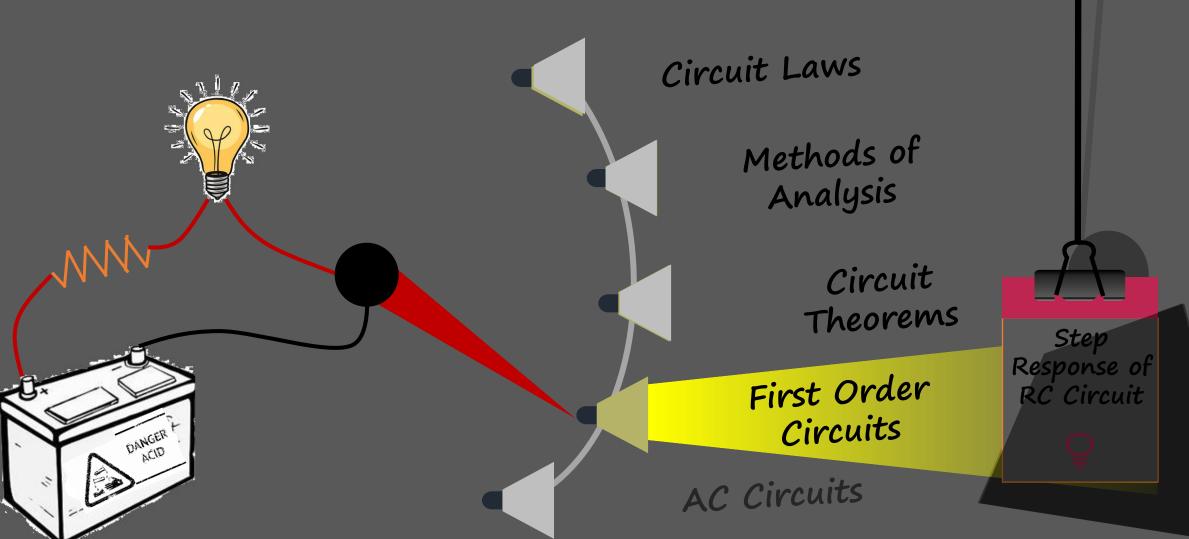
3. An ideal capacitor does not dissipate energy.

4. A real, nonideal capacitor has a parallel-model leakage resistance.





# Course Outline: broad themes



# Step Response of a RC circuit

- The *step response* of a circuit is its behaviour under the sudden application of dc voltage or current source. We assume the circuit response to be the capacitor voltage.
- Since the voltage of a capacitor cannot change instantaneously,

$$\Rightarrow v(0^-) = v(0^+) = V_0$$

$$\Rightarrow$$
 Using KCL (for  $t > 0$ ),

$$\Rightarrow C \frac{dv}{dt} + \frac{v - V_S}{R} = 0$$

$$\Rightarrow \frac{dv}{dt} = -\frac{v - V_S}{RC}$$

$$\Rightarrow \frac{dv}{v - V_S} = -\frac{1}{RC} dt$$

Integrating both sides,

$$\Rightarrow v(0^{-}) = v(0^{+}) = V_{0} \quad \Rightarrow \quad \left[\ln(v - V_{S})\right]_{V_{0}}^{v(t)} = -\left[\frac{t}{RC}\right]_{0}^{t}$$

$$\Rightarrow$$
 Using KCL (for  $t>0$ ),  $\Rightarrow$   $\ln(v(t)-V_S)-\ln(V_0-V_S)=-\frac{t}{RC}+0$ 

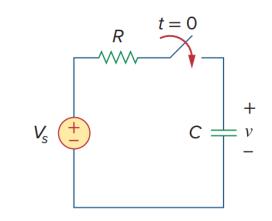
$$\Rightarrow ln \frac{v - V_S}{V_0 - V_S} = -\frac{t}{RC}$$

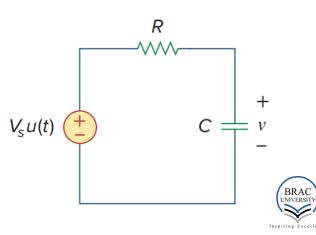
$$\Rightarrow \frac{v - V_S}{V_0 - V_S} = e^{-t/RC}$$

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$$\Rightarrow v - V_S = (V_0 - V_S)e^{-t/RC}$$

$$\Rightarrow v(t) = V_S + (V_0 - V_S)e^{-t/RC}$$





# Time Constant (charging) for RC circuit

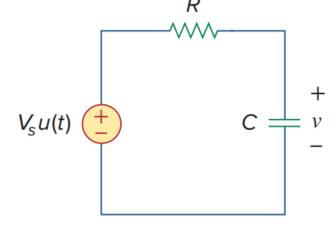
$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{RC}}, & t > 0 \end{cases}$$

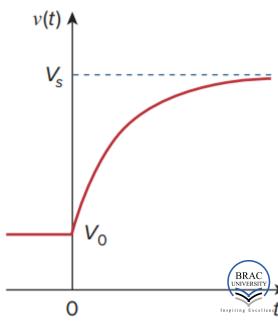
• This is known as the complete response (or total response) of the RC circuit to a sudden application of a dc voltage source. It is assumed that the capacitor was initially charged to  $V_0$ .

$$\Rightarrow v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & t > 0 \end{cases}$$

- $\Rightarrow$  where  $\tau = RC$  is the *time constant* (unit in sec).
- Notice that, we write  $\tau = RC$  for the circuit consisting of only a resistor R in series with the capacitor. As we know, all the linear two terminal circuits can be reduced to this form by Thevenin's Theorem, so the resistor R is actually the Thevenin Resistance  $R_{Th}$ . Therefore,

$$\tau = R_{Th}C$$





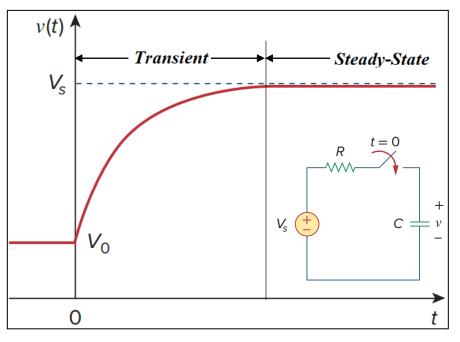
# Transient and Steady-State Response

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & t > 0 \end{cases}$$

• The *complete response* can be broken into two parts—one temporary and the other permanent, that is,

$$v(t) = v_{ss} + v_t$$
, where,  
 $v_{ss} = V_s$  &  $v_t = (V_0 - V_s)e^{-\frac{t}{\tau}}$ 

• The transient response  $(v_t)$  is the circuit's temporary response that will die out with time.



- The steady-state response  $(v_{ss})$  is the behaviour of the circuit a long time after an external excitation is applied.
- The complete response can be written as,

$$v(t) = V_{final} + \left[V_{initial} - V_{final}\right]e^{-\frac{t}{\tau}}$$

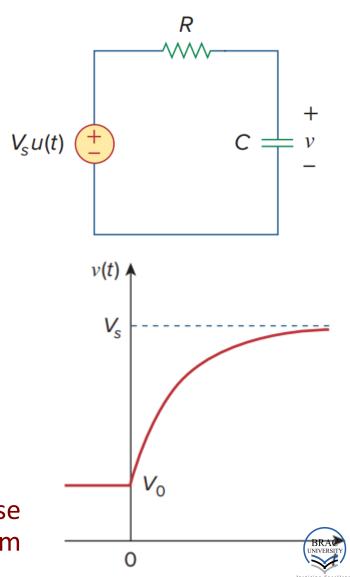
$$or, \qquad v(t) = V(\infty) + \left[V(0) - V(\infty)\right]e^{-\frac{t}{\tau}}$$



# Definition of $\tau$ (charging)

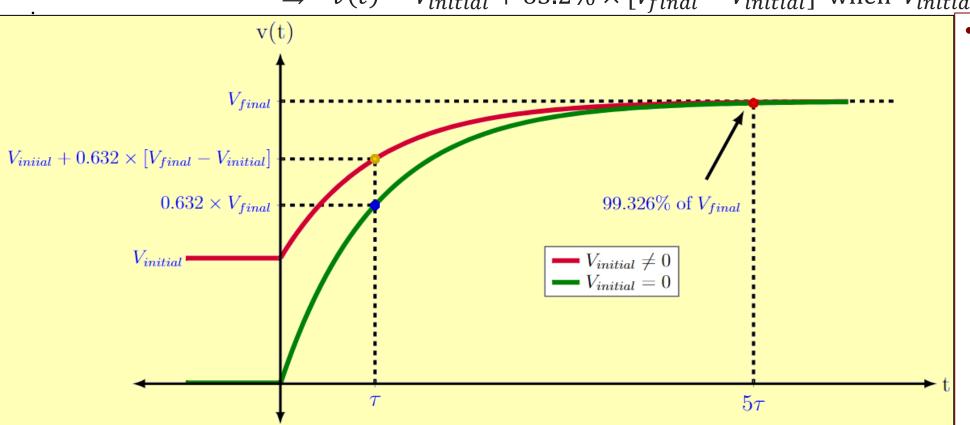
$$v(t) = V_{final} + \left[V_{initial} - V_{final}\right]e^{-\frac{t}{\tau}}$$

- At  $t = \tau$ ,  $v(t) = V_{final} + \left[V_{initial} V_{final}\right]e^{-1}$   $\Rightarrow v(t) = V_{final}\left(1 \frac{1}{e}\right) + V_{initial}\left(\frac{1}{e}\right)$   $\Rightarrow v(t) = V_{final}\left(1 \frac{1}{e}\right) V_{initial}\left(1 \frac{1}{e}\right) + V_{initial}$   $\Rightarrow v(t) = V_{initial} + \left[V_{final} V_{initial}\right]\left(1 \frac{1}{e}\right)$
- We can define the time constant in this way,
- The *charging time constant* is the time required for the response to reach to a factor of (1-1/e) or 63.2% towards  $V_{final}$  from an initial response  $V_{initial}$ .



# Time Constant ( $\tau$ ): graphically

At 
$$t = \tau$$
,  $v(t) = V_{initial} + [V_{final} - V_{initial}](1 - \frac{1}{e})$   
 $\Rightarrow v(t) = 63.2\% \times V_{final} \text{ when } V_{initial} = 0$   
 $\Rightarrow v(t) = V_{initial} + 63.2\% \times [V_{final} - V_{initial}] \text{ when } V_{initial} \neq 0$ 

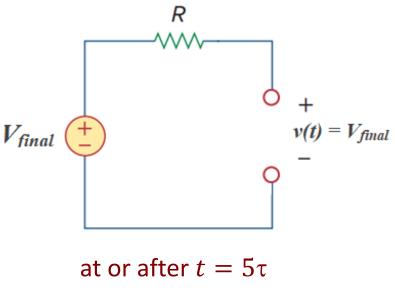


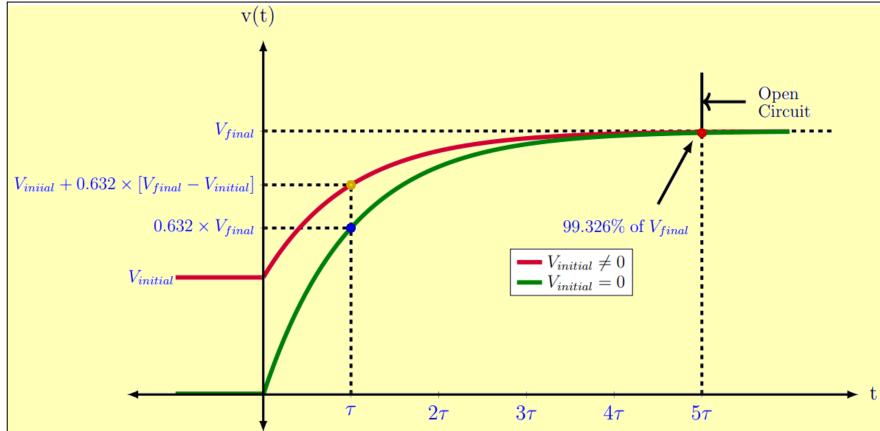
As  $\tau$  only depends on  $R_{Th}$  and C  $(\tau =$  $R_{Th}C$ ), for a given circuit, that is, for a fixed  $R_{Th}$  and C, the time needed for the capacitor voltage to rise to the final value  $(V_{final})$  is the same whether or not the capacitor is initially charged ( $V_{initial}$  zero or nonzero).

# Significance of $\tau$ (charging): $5\tau$ Time

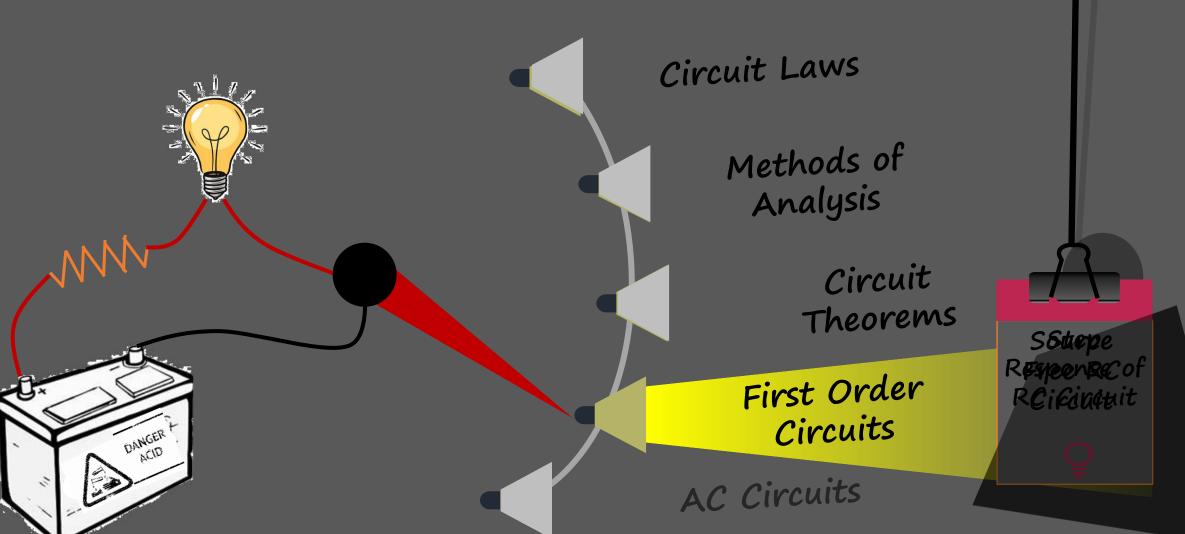
• As can be seen from the following plot, the capacitor voltage reaches the final voltage approximately after 5 times the Time Constant  $(\tau)$ . The capacitor is fully charged and acts as open circuit from  $5\tau$  time onward. So, when designing circuits, the charging time of a capacitor under the application of a

certain dc supply can be set by choosing  $R_{Th}$ .





# Course Outline: broad themes



### Source-Free RC circuit

- A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.
- $\Rightarrow$  Assume that a capacitor is charged to  $V_0$  and then it is connected to a resistor as shown. The capacitor starts to discharge the stored energy to the resistor.
- $\Rightarrow$  Initially stored charge,  $w(0) = \frac{1}{2}CV_0^2$
- $\Rightarrow$  From the figure using KCL,  $i_C + i_R = 0$

$$\Rightarrow C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$\Rightarrow \frac{dv}{dt} + \frac{v}{RC} = 0$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{RC}dt$$

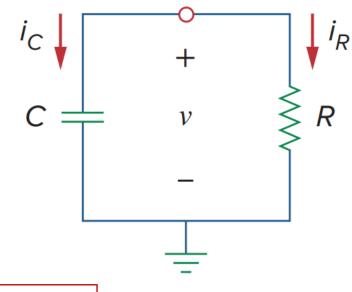
Integrating both sides,

$$\Rightarrow lnv = -\frac{t}{RC} + lnA$$

$$\Rightarrow ln \frac{v}{A} = -\frac{t}{RC}$$

$$\Rightarrow v = Ae^{-\frac{t}{RC}}$$

At 
$$t = 0$$
,  $v(0) = A = V_0$ . So,  $v(t) = V_0 e^{-\frac{t}{RC}}$ 



$$v(t) = V_0 e^{-\frac{t}{RC}}$$



# Time Constant (discharging) for RC circuit

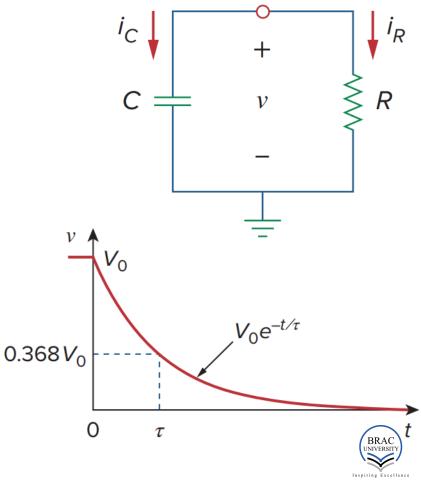
$$v(t) = V_0 e^{-\frac{t}{RC}}$$

 This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. It is called the natural response of the circuit.

$$\Rightarrow v(t) = V_0 e^{-\frac{t}{\tau}}$$

- where  $\tau = RC$  is the time constant (unit in sec).
- Notice that, we write  $\tau = RC$  for the circuit consisting of only a resistor R in series with the capacitor. As we know, all the linear two terminal circuits can be reduced to this  $0.368V_0$  form by Thevenin's Theorem, so the resistor R is actually the Thevenin Resistance  $R_{Th}$ . Therefore,

$$\tau = R_{Th}C$$



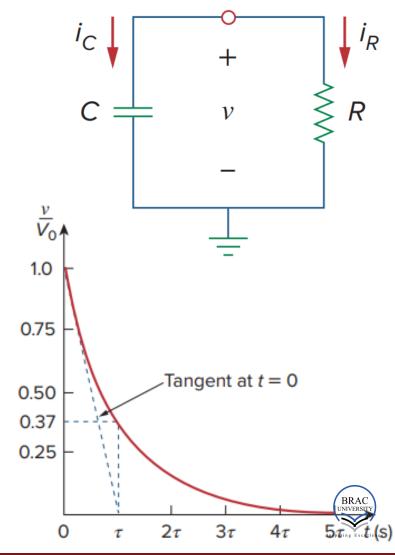
# Definition of $\tau$ (discharging)

$$v(t) = V_0 e^{-\frac{t}{\tau}}$$

• At t= au,  $v(t)=V_0e^{-1}$ 

$$\Rightarrow v(t) = 0.368 \times V_0$$

- We can define the discharging time constant in this way,
- The discharging time constant is the time required for the response to fall to a factor of  $^1/_e$  or 36.8% from an initial response  $V_{initial}$  or  $V_0$ .
- Recall that the *charging time constant* is the time required for the response to reach to a factor of (1-1/e) or 63.2% towards  $V_{final}$  from an initial response  $V_{initial}$ .

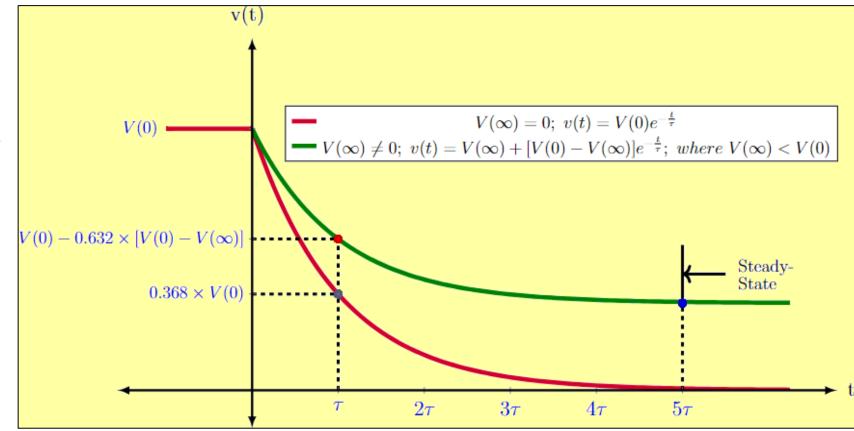


# Significance of $\tau$ (discharging): $5\tau$ Time

• As can be seen from the following plot, the capacitor voltage decreases to the final voltage approximately after 5 times the Time Constant  $(\tau)$ . In case where  $V(\infty) = 0$ , the capacitor is fully discharged from  $5\tau$  time onward. So, when designing circuits, the discharging time of a capacitor can

be set by choosing  $R_{Th}$ .

• In the case that a capacitor is subjected to a final voltage lower than its initial voltage, the discharging  $\tau$  is the time required for the response to decay to 63.2% from V(0) towards  $V(\infty)$ . See Problem 6



### Procedure

$$v(t) = V(\infty) + [V(0) - V(\infty)]e^{-t/\tau}$$

Determine the initial voltage of the capacitor  $V_{initial}$  or  $\underline{V(0)}$ 

Consider only the active<sup>‡</sup> portion of the circuit before switching. For example, if switching occurs at t=0, consider the circuit for t<0.

If the circuit includes any dc source (current or voltage), open the capacitor and determine the voltage at the open terminal. This is the V(0). V(0)=0 if there is no independent source in the circuit.

 $\frac{\text{Determine the final}}{\text{voltage of the capacitor}}$   $\frac{V_{final} \text{ or } V(\infty)$ 

Now consider the active<sup>‡</sup> portion of the circuit after switching. For example, for t > 0.

Repeat the step. This time, the voltage across the capacitor is  $V(\infty)$ . Circuits with  $V(\infty) = 0$  are called source free.

Determine the time constant  $(\tau)$ 

Again, only consider the active<sup>‡</sup> portion after switching. For example, for t > 0.

Determine the Thevenin resistance  $(R_{Th})$  as seen from the capacitor terminals

 $\tau = R_{Th}C$ 

 $\frac{\text{Determine}}{v(t)}$ 

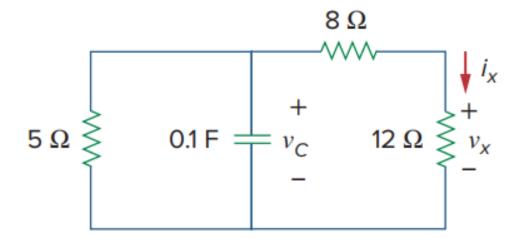
Plug in V(0),  $V(\infty)$ , and  $\tau$  into the equation for v(t)

Determine any other voltages or currents in the circuit using v(t) and the circuit laws.

‡ active portion of the circuit excludes everything that has no influence on the capacitor

# Example 1

• Let  $V_{\mathcal{C}}(0)=15\,V$  , Determine  $v_{\mathcal{C}},v_{x}$ , and  $i_{x}$  for t>0.



#### Solution

The equivalent resistance as seen from the capacitor terminal is,

$$R_{eq} = (8 + 12) \mid |5 = 4 \Omega$$

Time constant,  $\tau = R_{eq}C = 4 \times 0.1 = 0.4 s$ 

Thus, for a source-free RC circuit,  $V(\infty) = 0$ . So,

$$v_C(t) = V(0)e^{-\frac{t}{\tau}} = 15e^{-2.5t} (V)$$

The voltage  $v_x$  can be found by simple voltage division.

$$v_x(t) = \frac{12}{12 + 8} \times v_C(t) = 9e^{-2.5t} (V)$$

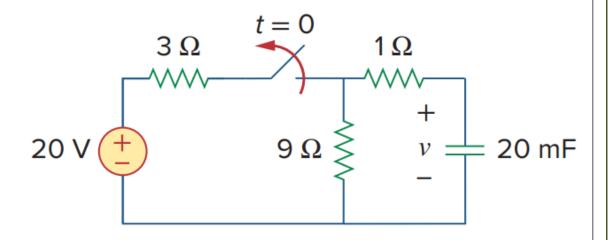
According to the Ohm's law,

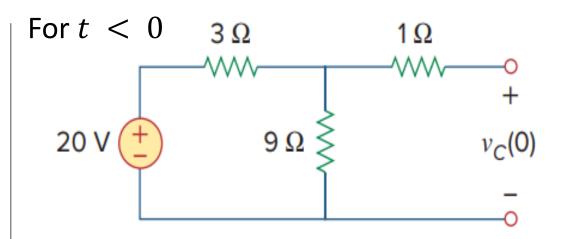
$$i_x = \frac{v_x}{12} = \frac{9e^{-2.5t}}{12} = 0.75e^{-2.5t} \text{ (A)}$$



# Example 2

• The switch in the circuit has been closed for a long time, and it is opened at t=0. Find v(t) for t>0. Calculate the initial energy stored in the capacitor.





For t < 0, the switch is closed. With the capacitor open at dc, the circuit transforms into the one shown above.

No current flows through the  $1 \Omega$ . So, the voltage across the  $9 \Omega$  is the  $v_C(t)$  for t < 0,

$$v_C(t) = \frac{9}{9+3} \times 20 = 15 V, \qquad t < 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0) = v_C(0^-) = 15 V$$



# Example 2: t > 0

For t > 0, the switch is open. The circuit transforms into the one shown above. As there is no independent source in the circuit,  $V(\infty) = 0$ .

The Thevenin resistance as seen from the capacitor terminal,

$$R_{Th} = 1 + 9 = 10 \Omega$$

The time constant is,

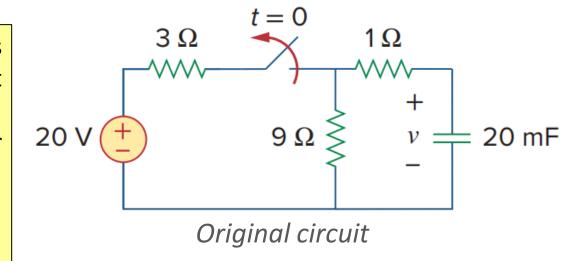
$$\tau = R_{Th}C = 10 \times 20 \times 10^{-3} = 0.2 s$$

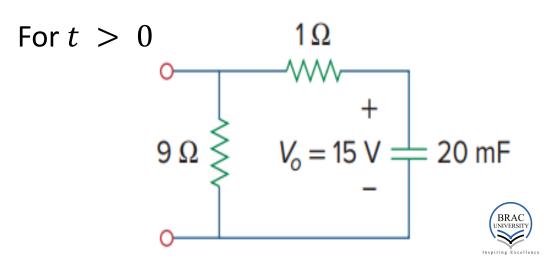
So, the voltage across the capacitor for t > 0 is,

$$v_C(t) = V(0)e^{-\frac{t}{\tau}}$$
  
= 15e<sup>-5t</sup> (V)

The initial energy stored in the capacitor is,

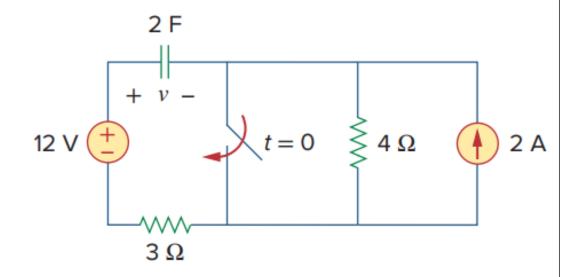
$$w_C(t) = \frac{1}{2}CV(0)^2$$
$$= \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 J$$

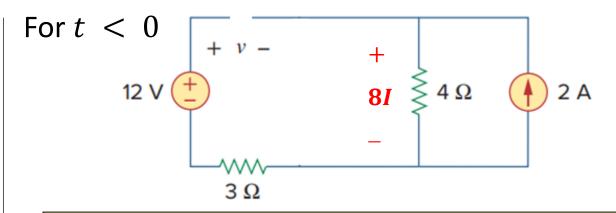




# Example 3

• Calculate the capacitor voltage v(t) for t < 0 and for t > 0.





For t < 0, the switch is open. With the capacitor open at dc, the circuit transforms into the one shown above.

The  $2\,A$  current from the current source will flow only through the  $4\,\Omega$  resistance. The voltage drop across the  $4\,\Omega$  resistance is,  $4\times 2=8\,V$ .

There is no voltage drop across the  $3 \Omega$  (i=0 at open circuit). So,

$$v(t) = 12 - 8 = 4 V$$
,  $t < 0$ 

Since the voltage across the capacitor cannot change instantaneously,

$$v(0) = v(0^-) = 4 V$$



# Example 3: t > 0

For t > 0, the switch is closed. With the capacitor again open at dc, the circuit transforms into the one shown above.

Again, there is no voltage drop across the 3  $\Omega$  (i=0 at open circuit). So,

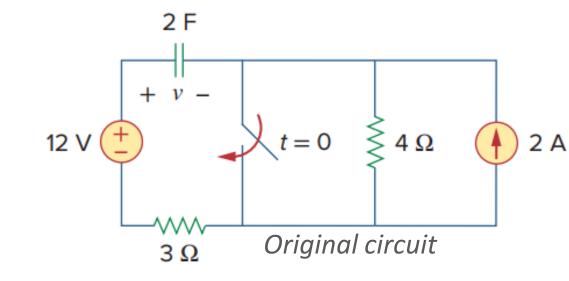
$$v(t) = 12 V, t > 0$$

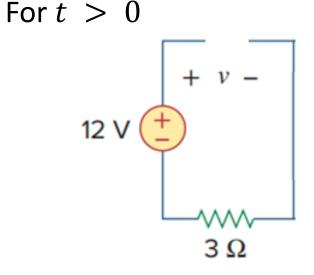
This is the steady-state voltage across the capacitor for t > 0.

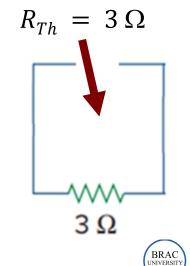
$$v(\infty) = 12 V$$

The time constant is,  $\tau = R_{Th}C = 3 \times 2 = 6 s$ So,

$$v(t) = V(\infty) + [V(0) - V(\infty)]e^{-t/\tau}$$
$$= 12 + [4 - 12]e^{-\frac{t}{6}} = 12 - 8e^{-\frac{t}{6}}$$

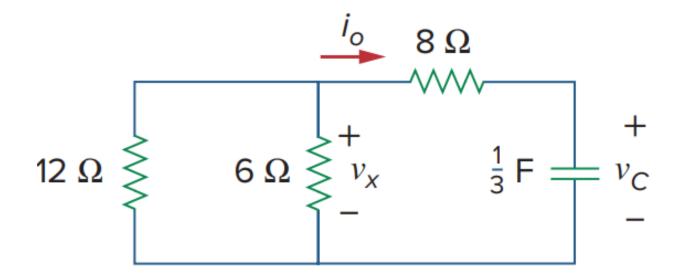






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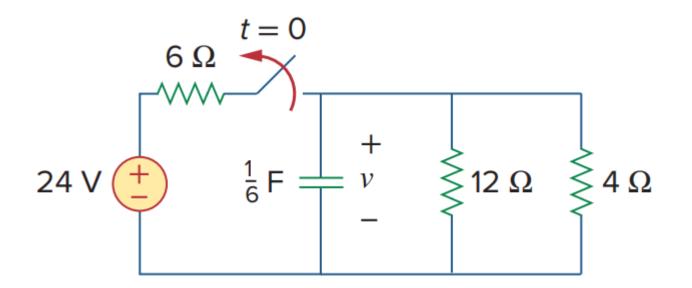
• Let  $V_C(0) = 60 V$ , Find  $v_C, v_X$ , and  $i_X$  for t > 0.



Ans: 
$$v_C = 60e^{-0.25t} V$$
;  $v_x = 20e^{-0.25t} V$ ;  $i_x = -5e^{-0.25t} A$ 



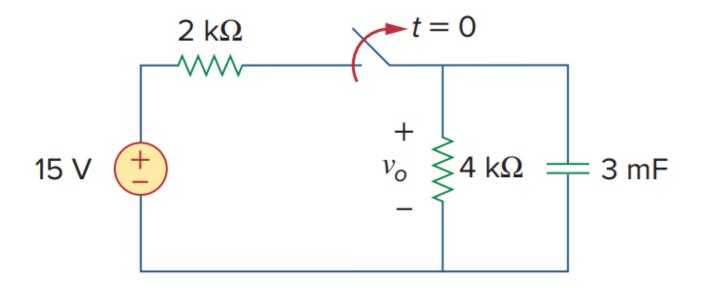
• The switch in the circuit has been closed for a long time, and it is opened at t=0. Find v(t) for t>0. Calculate the initial energy stored in the capacitor.



Ans:  $v(t) = 8e^{-2t} V$ ;  $w_c(0) = 5.333 J$ 



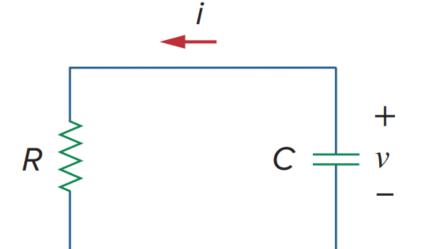
• The switch opens at t = 0. Find  $v_0(t)$  for t > 0.



$$\underline{\text{Ans}} : \boldsymbol{v}(t) = \mathbf{10} e^{-t/12} V$$



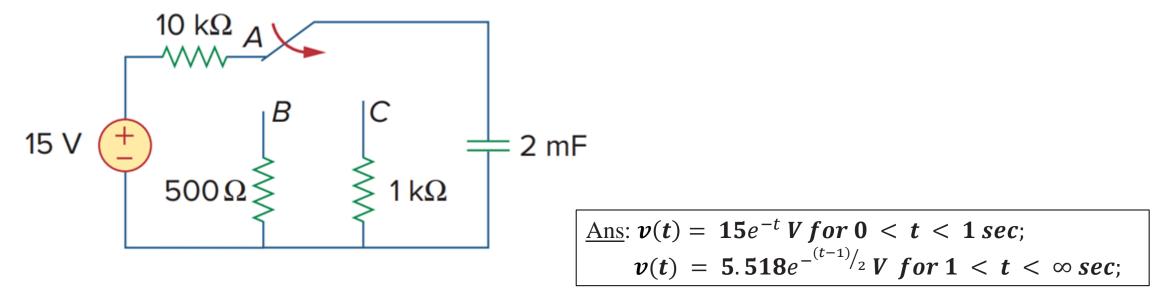
- For the circuit below,  $v = 10e^{-4t} V$  and  $i = 0.2e^{-4t} A$ 
  - (a) Find R and C.
  - (b) Determine the time constant.
  - (c) Calculate the initial energy in the capacitor.
  - (d) Obtain the time it takes to dissipate 50% of the initial energy.



Ans:  $R = 50 \Omega$ ; C = 5 mF;  $\tau = 0.25 s$ ;  $w_{c(0)} = 0.25 J$ ; t = 86 ms

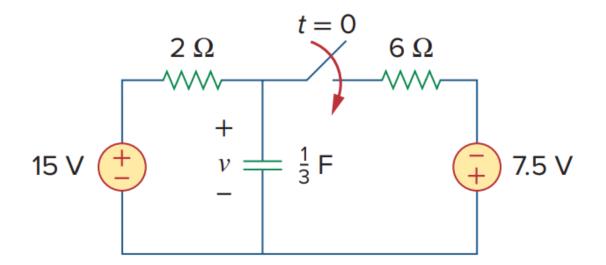


• Assume that the switch has been in position A for a long time and is moved to position B at t=0. Then at t=1s, the switch moves from B to C. Find  $I_{\mathcal{C}}(t)$  for t>0.





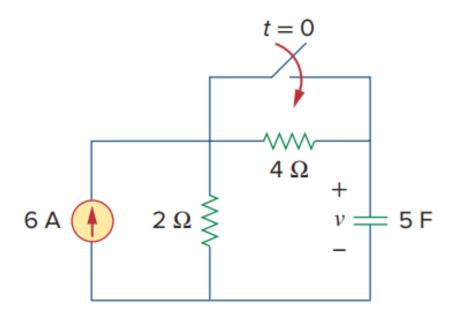
• Find v(t) for t>0 in the circuit shown below. Assume the switch has been open for a long time and is closed at t=0. Calculate v(t) at t=0.5s.



Ans:  $v_c(t) = 9.375 + 5.625e^{-2t} V for t > 0$ ;  $v_c(0.5) = 11.444 V$ 



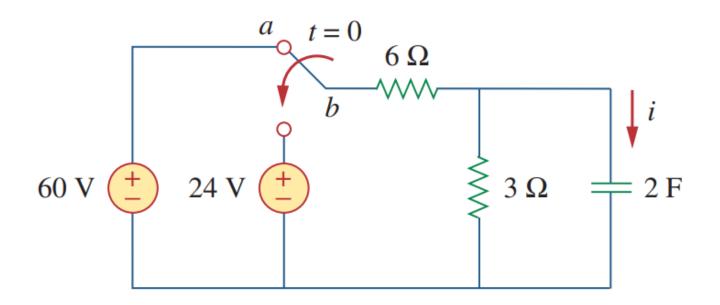
• Calculate the capacitor voltage for t < 0 and for t > 0.



Ans: v(t) = 12 V for t < 0; v(t) = 12 V for t > 0



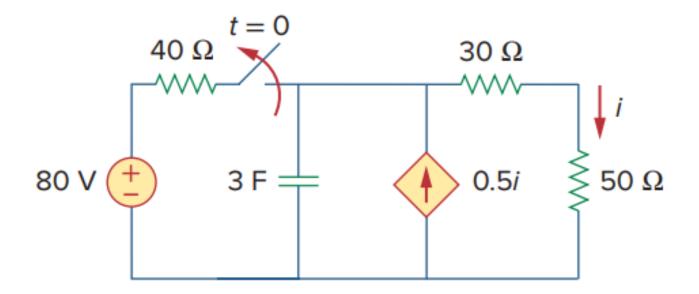
• The switch has been in position a for a long time. At t = 0 it moves to position b. Calculate i(t) for all t > 0.



Ans:  $i(t) = -6e^{-0.25t} A for t > 0$ 



• Consider the circuit shown below. Find i(t) for t < 0 and t > 0.



Ans: i(t) = 0.8 A for t < 0;  $i(t) = 0.8e^{-t/480} A for t > 0$ 

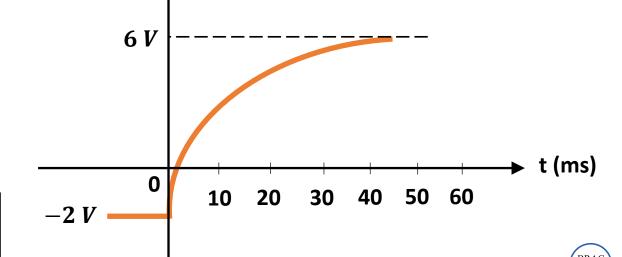


The figure below shows the voltage response of an RC circuit to a sudden DC voltage applied through an equivalent resistance of 4  $k\Omega$ .

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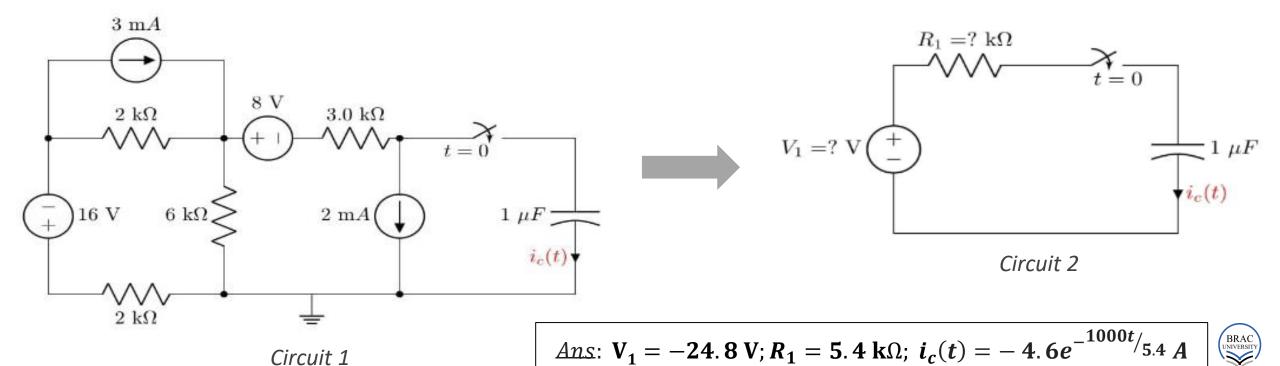
**v(t)** 

- Define time constant.
- Determine the approximate time constant from the figure.
- Find the mathematical expression of v(t) for t > 0.
- What is the initial energy stored in the capacitor?
- Draw the circuit diagram.

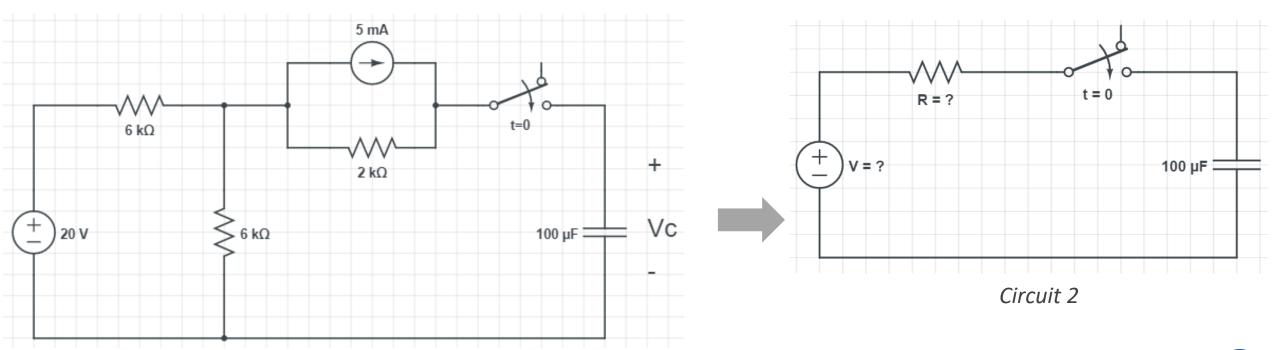


Ans: (ii) 
$$\tau = 9 ms$$
; (iii)  $v(t) = 6 - 8e^{-1000t/9} I for t > 0$ ; (iv)  $w = 4.5 \times 10^{-6} J$ 

- I. Simplify the circuit 1 below so that it takes the form of the circuit 2. Determine the values of  $V_1$  and  $R_1$ .
- II. Perform transient analysis to determine  $i_c(t)$  through the capacitor for t > 0.



- Simplify the Circuit 1 below so that it takes the form of the Circuit 2. Determine the values of *I* and *R*.
- Perform transient analysis to determine  $V_c(t)$  across the capacitor for t>0.



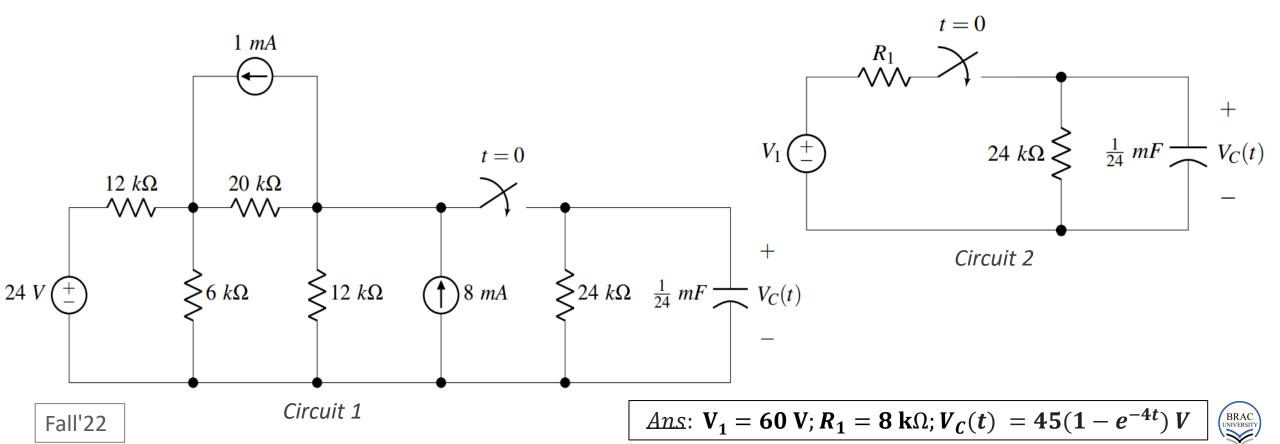
Circuit 1

Ans: V = 20 V;  $R = 5 \text{ k}\Omega$ ;  $V_c(t) = 20(1 - e^{-2t}) V$ 



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• Simplify the Circuit 1 below so that it takes the form of the Circuit 2. Determine the values of  $V_1$  and  $R_1$ . Perform transient analysis to determine  $V_c(t)$  across the capacitor for t>0.



- Reduce the left portion with respect to the dashed grey line of Circuit 1 so that it takes the form of Circuit 2 as shown. Write down the values of  $V_1$  and  $R_1$ .
- Now, analyse the Transient Behaviour of the circuit assuming that the switch moves from position x to position y at t=0. Determine v(t) for t>0.

