Collaborators : None

Sources : Lecture Notes

Q5) Product Probability Spaces:

(a) As $p \in R^d$ be a probability distribution on $[d] = \{1, 2, \ldots, d\}$

Let $p = [p_1, p_2, \ldots, p_d]$ where p_i = Probability[Event when index $d = i$]

As $q \in R^e$ be a probability distribution on $[e] = \{1, 2, \ldots e\}$

Let $q = [q_1, q_2, \ldots, q_e]$ where q_i = Probability[Event when index $e = i$]

The Kronecker product $p \otimes q$ creates a new vector where each element is the product of elements from p & q.

$\Rightarrow \quad p \otimes q = [p_1q_1, p_1q_2, \ldots, p_1q_e, p_2q_1, \ldots, p_2q_e, \ldots\ldots, p_dq_1, \ldots, p_dq_e]_{1 \times de} \rightarrow (1)$

Assuming $(p \otimes q)_n$ represents event where

$\underbrace{\text{d-series index} = \lceil \frac{n}{e} \rceil}_{i}$ & $\underbrace{\text{e-series index}}_{j} = \begin{cases} e, & \text{if } (n - e\lfloor \frac{n}{e} \rfloor) = 0 \\ (n - e\lfloor \frac{n}{e} \rfloor), & \text{else} \end{cases}$

If we assume Each element of $p \otimes q$ corresponds to the joint probability of the 2 independent events described by p & q. $\rightarrow (2)$

As $n = (i-1) \times e + j$

$\Rightarrow (p \otimes q)_n = p_i q_j$; From (2),

To prove: $p_i q_j$ = Probability[Event p involving drawing i & Event q involving drawing j]

Proof: As P & Q draws are independent events

$Pr[\text{P\_event(drawing } i) \text{ \& Q\_event(drawing } j)]$

$= Pr[\text{P\_event(drawing } i)] \times Pr[\text{Q\_event(drawing } j)]$

$= p_i \times q_j$.

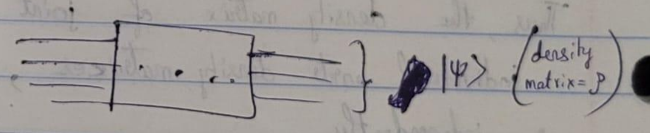
Thus, because of independent nature of events involving P & Q, the Kronecker product $p \otimes q$ represents joint probability distribution on $[d] \times [e]$.

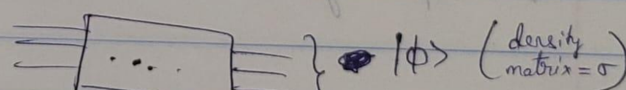
(b) <u>Given:</u> Mixed state $\rho \in \mathbb{C}^{d \times d} = \begin{cases} p_1 : \text{probability of being } |\psi_1\rangle \\ p_2 : \quad " \quad " \quad " \quad |\psi_2\rangle \\ \vdots \\ p_m : \text{probability of being } |\psi_i\rangle \end{cases}$

$$\Rightarrow \boxed{\rho = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|}$$

Mixed state $\sigma \in \mathbb{C}^{e \times e} = \begin{cases} q_1 : \text{probability of being } |\phi_1\rangle \\ \vdots \\ q_n : \text{probability of being } |\phi_n\rangle \end{cases}$

$$\Rightarrow \boxed{\sigma = \sum_{j=1}^{n} q_j |\phi_j\rangle\langle\phi_j|}$$





If these particles are created completely separately & independently, & then viewed as de-dimensional state,

If $|\psi\rangle = |\psi_i\rangle$ & $|\phi\rangle = |\phi_j\rangle$

then $J.S. = |\psi_i\rangle \otimes |\phi_j\rangle$. This occurs with probability $(p_i \times q_j)$. ($\because$ Independent events as they are unentangled).

After similarly considering probabilities for other possibilities of Joint state:

$\begin{cases} p_1 q_1 : \text{probability of being } |\psi_1\rangle \otimes |\phi_1\rangle \\ p_1 q_2 : \quad " \quad " \quad " \quad |\psi_1\rangle \otimes |\phi_2\rangle \\ \vdots \\ p_m q_n : \quad " \quad " \quad " \quad |\psi_m\rangle \otimes |\phi_n\rangle \end{cases}$

$\Rightarrow$ J.S. Density matrix $= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \left(|\psi_i\rangle \otimes |\phi_j\rangle \right) \left(|\psi_i\rangle \otimes |\phi_j\rangle \right)^{\dagger}$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \left(|\psi_i\rangle \otimes |\phi_j\rangle \right) \left(\langle\psi_i| \otimes \langle\phi_j| \right) \quad \left[\because (A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger} \right]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \left(|\psi_i\rangle \langle\psi_i| \right) \otimes \left(|\phi_j\rangle \langle\phi_j| \right) \quad \left[\because (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \right]$$

$$= \left[\sum_{i=1}^{m} p_i \left(|\psi_i\rangle \langle\psi_i| \right) \right] \otimes \left[\sum_{j=1}^{n} q_j \left(|\phi_j\rangle \langle\phi_j| \right) \right]$$

$$= \rho \otimes \sigma$$

Thus, the density matrix of joint state is kronecker product of individual parts density matrices, if they are created separately & independently.