# Stabilizer Code

## A quantum error-correcting code in which the encoding and decoding can be done entirely by stabilizer circuits

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# Key Definitions

- Stabilizer Gates:
    - CNOT, Hadamard, Phase Gate $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
- Stabilizer Circuits:
    - Quantum circuits made entirely of stabilizer gates.
- Stabilizer States:
    - States generated by a stabilizer circuit starting from $|00\ldots0\rangle$.

# What Does This All Mean?

- Key Concept:
  - A unitary $U$ stabilizes a pure state $|\Psi\rangle$ if $U|\Psi\rangle = |\Psi\rangle$.
    - $|\Psi\rangle$ is an eigenstate of $U$ with eigenvalue $+1$.
- Important Note:
  - Global phase matters!
    - If $U|\Psi\rangle = -|\Psi\rangle$, then $U$ does not stabilize $|\Psi\rangle$.
- Properties of Stabilizers:
  - If $U$ and $V$ stabilize $|\Psi\rangle$ then $UV, VU, U^{-1}, V^{-1}$ and $I$ (identity) also stabilize $|\Psi\rangle$.
  - The set of stabilizers forms a group under multiplication.

# The Pauli Matrices

- Definition: The four Pauli matrices, fundamental to quantum physics.
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Error Types in Quantum Error Correction:
  - No Error: $I|1\rangle = |1\rangle$
  - Bit Flip: $X|1\rangle = |0\rangle$
  - Phase Flip: $Z|1\rangle = -|1\rangle$
  - Both (Bit & Phase Flip): $Y|1\rangle = -i|0\rangle$

# Stabilizer Groups - Basics

- Definition: Given an $n$-qubit pure state $|\Psi\rangle$, its stabilizer group is:
  - The group of all tensor products of Pauli matrices that stabilize $|\Psi\rangle$.
- Properties:
  - Forms a group since:
    - Pauli matrices are closed under multiplication.
    - Stabilization of $|\Psi\rangle$ is closed under group operations.
  - Abelian: Stabilizer groups are always commutative.
- Examples:
  - Stabilizer group of $|0\rangle$: $\{I, Z\}$
  - Stabilizer group of $|+\rangle$: $\{I, X\}$
  - Stabilizer group of $|0\rangle \otimes |+\rangle$: $\{II, IX, ZI, ZX\}$ (Omitting $\otimes$ for simplicity)

# Generating Sets for Stabilizer Groups

- Key Question: How can we succinctly specify the stabilizer group $G$ of an $n$-qubit stabilizer state (size $2^n$)?
- Answer: The stabilizer group $G$ is always generated by $n$ elements, which are $\pm$ tensor products of Pauli matrices.
  - To specify $G$, it suffices to provide $n$ such generators.
- Example: Stabilizer group of the Bell pair $\{II, XX, -YY, ZZ\}$
  - Generating set 1: $\{XX, ZZ\}$
  - Generating set 2: $\{XX, -YY\}$

# The Gottesman–Knill Theorem

- Theorem: There is a polynomial-time classical algorithm to simulate any stabilizer circuit acting on an initial stabilizer state (e.g., $|00\ldots0\rangle$).
- Capabilities:
  - Compute probabilities of measurement outcomes.
  - Simulate measurement outcomes using random bits.
- Implication:
  - Positive: Efficient simulation of stabilizer circuits.
  - Negative: Stabilizer circuits alone cannot provide superpolynomial quantum speedups.
- How It Works:
  - Track the stabilizer group's generators.
  - Update the generators for each gate (CNOT, Hadamard, Phase) or measurement.

# The Gottesman–Knill Algorithm

- Initial Stabilizer Representation: For $|00\ldots0\rangle$:
  - Stabilizer group includes $II\ldots I$, implied by default.
  - Generating set: $ZIII\ldots I, IZII\ldots I, IIZI\ldots I, \ldots, IIII\ldots Z$
- Tableau Representation
  - Use two $n \times n$ binary matrices:
    - $X$ Matrix: Tracks $X$ or $Y$
    - $Z$ Matrix: Tracks $Z$ or $Y$

# Simplifying Stabilizer Formalism: Ignoring Signs

- Cheating a Little:
    - Keeping track of +'s and –'s is complex and often not illuminating.
    - Simplification: Ignore the signs in the stabilizer formalism.
- What Do We Lose?
    - Definite Outcome: +'s and –'s are needed to determine which outcome: $|0\rangle$ or $|1\rangle$
- Random vs. Definite:
    - If the goal is only to know whether a measurement gives a definite or random outcome:
    - Ignoring signs is sufficient.

# Gate Operations in Tableau Representation

- Rules for Gate Updates:
- Hadamard Gate ($H$) on $i$-th qubit:
  - Swap the $i$-th column of the $X$ and $Z$ matrices.
- Phase Gate ($P$) on $i$-th qubit:
  - XOR $i$-th column of the $X$ matrix into the $Z$ matrix.
- CNOT Gate:
  - Control $i$, target $j$:
    - XOR $i$-th column of $X$ into $j$-th column of $X$.
    - XOR $j$-th column of $Z$ into $i$-th column of $Z$.

# Measurements and Observations

- Measurement in $\{ |0\rangle, |1\rangle \}$ Basis:
  - Determinate outcome if $i$-th column of the $X$ matrix is all 0's.
- Why?
  - Columns indicate commutation relationships; all 0's in $X$ implies only $Z$ acts, ensuring definite outcomes.
- Efficiency:
  - Tracks stabilizers efficiently.
  - Skips signs unless the specific outcome is needed.

# Rank of the X Matrix and Basis States

- Fact: For a stabilizer state, the number of basis states with nonzero amplitudes is $2^k$, where $k$ is the rank of the $X$ matrix.
- Example 1: If rank$(X) = 0$, only a single basis state (e.g., $|0000\rangle$) has a nonzero amplitude.
- Example 2…

# Stabilizer Codes in Quantum Error Correction

- Prevalence in Research:
  - Most quantum error-correcting codes studied are stabilizer codes.
- Why Stabilizer Codes Dominate:
  - Simplifies Calculations: Easier to compute and reason about.
  - No Trade-offs: Retain desired error-correcting properties.
- Comparison to Classical Codes:
  - Analogous to the central role of linear codes in classical error correction.

# The Stabilizer Formalism and Shor's 9-Qubit Code

- Importance of Stabilizer Formalism:
  - *Common language* of quantum error correction.
  - Indispensable in the field.
- Example: Shor's 9-Qubit Code
  - States $\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$ and $\left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$

# Generating set for the states' stabilizer group

- `Z Z I I I I I I I`
- `I Z Z I I I I I I`
- `I I I Z Z I I I I`
- `I I I I Z Z I I I`
- `I I I I I I Z Z I`
- `I I I I I I I Z Z`
- `X X X X X X I I I`
- `I I I X X X X X X`
- ± `X X X X X X X X X`
- The last line is + for $|0\rangle$, and - for $|1\rangle$

# Why Are These Elements in the Stabilizer Group?

- Phase-Flips (Z Operators):
  - Phase-flips on any pair of qubits within the same block cancel each other out.
- Bit-Flips (X Operators):
  - Bit-flips return the system to its initial state, potentially with a global -1 phase.
- Linearly Independent Elements:
  - Verify that the 9 stabilizer elements are linearly independent.
  - Ensures there are no additional stabilizer elements in the group.

# The 5-Qubit Code: An Optimal Error-Correcting Code

- Purpose:
  - Detects and corrects any single-qubit error.
- Codeword States:
  - Explicit representation: Superpositions over 32 different 5-bit strings!
  - Simplified with stabilizer formalism:

- Stabilizer Group (Generators):
- `X Z Z X I`
- `I X Z Z X`
- `X I X Z Z`
- `Z X I X Z`
- ± `X X X X X`
- Sign of Last Generator:
  - +: $|0\rangle$ (logical 0).
  - –: $|1\rangle$ (logical 1).

# Quantum Computation with Encoded Qubits

- Scenario:
  - Perform a gate on one or two logical qubits encoded using stabilizer codes.
- The “Obvious” Approach:
  - Decode the qubits.
  - Apply the gate to the bare, unencoded qubits.
  - Re-encode the result.
- Problems with This Approach:
  - Costly: Decoding and re-encoding are computationally expensive.
  - Error-Prone: During decoding, qubits are unprotected, making them vulnerable to decoherence and other quantum noise sources

# Transversal Gates and Encoded Qubits

- Ideal Scenario:
  - Applying gates to encoded qubits is as simple as applying them to unencoded qubits.
- Definition: Transversal Gates
  - A gate G is transversal for a code C if:
    - To apply G to encoded qubits:
      - Apply G to the first qubits of the codewords.
      - Apply G to the second qubits of the codewords.
      - Continue for all qubits in the codewords.

- Transversality of Hadamard:
    - A logical Hadamard is achieved by applying Hadamard to each physical qubit in the codeword individually.
- Check that the Hadamard gate is transversal for Shor’s 9-qubit code.
- Motivation:
    - Transversal gates reduce complexity and error potential when operating on encoded qubits.

# Transversal Gates and Universal Quantum Computation

- Codes with Transversal Stabilizer Gates:
    - CNOT, Hadamard, and Phase gates are transversal for certain quantum codes.
    - Enables cheap and efficient implementation of stabilizer circuits on encoded qubits.
- Limitation of Stabilizer Gates:
    - Stabilizer gates are not universal for quantum computation.
- Theorem: Non-stabilizer gates (e.g., Toffoli, $R_{\pi/8}$) cannot all be transversal.
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- Implication for Universal Quantum Computing:
  - Non-stabilizer gates are essential for universality.
  - These gates require sequences of expensive operations, increasing complexity and resource use.
- Key Takeaway:
  - While transversal stabilizer gates simplify quantum error correction, achieving universality necessitates costly non-stabilizer gates.

# The Role of Non-Stabilizer Gates in Quantum Computing

- Stabilizer Gates:
  - Considered “free” due to their low implementation cost.
  - Complexity of a quantum circuit is often defined by the number of non-stabilizer gates.
- Challenge:
  - Non-Stabilizer Gates (e.g., Toffoli, $R_{\pi/8}$):
    - Significantly more expensive to implement.
    - Dominates the running time of quantum computations.

- Engineering Focus:
  - Design methods to efficiently incorporate non-stabilizer gates into circuits.
  - Example Technique: Magic State Distillation
    - Magic States: Special non-stabilizer states (e.g., $\cos(\pi/8)|0\rangle+\sin(\pi/8)|1\rangle$).
    - Stabilizer operations and measurements adaptively simulate non-stabilizer gates.
    - Breaks out of the Gottesman-Knill prison to achieve universality.
- Practical Implication:
  - A large portion of the quantum computer's effort is focused on "magic state factories."
  - Actual computation on magic states becomes a secondary task.