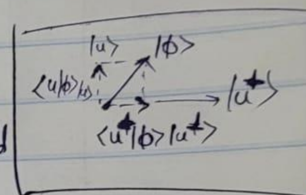
Collaborators : None

Sources : Lecture Notes

Q5) Zero-error state discrimination:

(a) Let $\Pi_1 = |u^\perp\rangle\langle u^\perp|$, the linear operator on $\mathbb{R}^2$ that projects onto the $|u^\perp\rangle$. Since $\{|u^\perp\rangle, |u\rangle\}$ form the orthormal basis, then sum of the projects in the orthormal basis vectors should give the original vector.



$$\Rightarrow \quad |u^\perp\rangle\langle u^\perp| \cdot |\phi\rangle + |u\rangle\langle u| \cdot |\phi\rangle = |\phi\rangle$$

$$\Rightarrow \left(I - |u\rangle\langle u| - |u^\perp\rangle\langle u^\perp|\right) \cdot |\phi\rangle = 0$$

As this is applicable for all $|\phi\rangle$, $\Rightarrow I - |u\rangle\langle u| - |u^\perp\rangle\langle u^\perp| = 0$

$$\Rightarrow \boxed{|u^\perp\rangle\langle u^\perp| = I - |u\rangle\langle u|} = \Pi_1$$

Similarly, $\Pi_2 = |v^\perp\rangle\langle u^\perp| = I - |v\rangle\langle v|$

(b) $E_1 = \frac{1}{c}\Pi_1 = \frac{1}{c}\left[I - |u\rangle\langle u|\right]$; $E_2 = \frac{1}{c}\Pi_2 = \frac{1}{c}\left[I - |v\rangle\langle v|\right]$

. When state $|\psi\rangle = |u\rangle$, the probability of outcome i is given by:

$$Pr(1) = \langle u|E_1|u\rangle = \frac{1}{c}\langle u|(I - |u\rangle\langle u|)|u\rangle$$

$$= \frac{1}{c}\left[\langle u|I|u\rangle - \langle u|u\rangle\langle u|u\rangle\right] = \frac{1}{c}\left[\langle u|u\rangle - \langle u|u\rangle\langle u|u\rangle\right]$$

Since $\langle u|u\rangle = 1 \Rightarrow Pr(1) = \frac{1}{c}[1-1] = 0$

$$\boxed{\therefore Pr(1) = 0 \text{ when } |\psi\rangle = |u\rangle}$$

Similarly, probability of outcome '2' is given by: (When $|\psi\rangle = |v\rangle$)

$$Pr(2) = \langle v|E_2|v\rangle = \frac{1}{c}\langle v|(I - |v\rangle\langle v|)|v\rangle$$

$$= \frac{1}{c}\left[\langle v|I|v\rangle - \langle v|v\rangle\langle v|v\rangle\right] = \frac{1}{c}\left[\langle v|v\rangle - \langle v|v\rangle\langle v|v\rangle\right]$$

Since $\langle v|v\rangle = 1 \Rightarrow Pr(2) = \frac{1}{c}[1-1] = 0$

$$\boxed{\therefore Pr(2) = 0 \text{ when } |\psi\rangle = |v\rangle}$$

(c) $E_0 = I - E_1 - E_2 = I - \frac{1}{c}\Pi_1 - \frac{1}{c}\Pi_2$

Probability of outcome '0' when $|\psi\rangle = |\psi\rangle$ is:

$$Pr(0) = \langle\psi|E_0|\psi\rangle = \langle\psi|(I - \tfrac{1}{c}\Pi_1 - \tfrac{1}{c}\Pi_2)|\psi\rangle$$

$$\Rightarrow Pr(0) = \langle\psi|I|\psi\rangle - \tfrac{1}{c}\langle\psi|\Pi_1|\psi\rangle - \tfrac{1}{c}\langle\psi|\Pi_2|\psi\rangle \longrightarrow (1)$$

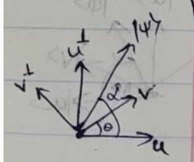


Fig-1

$$\left.\begin{array}{l}\langle\psi|\Pi_1|\psi\rangle = \langle\psi|u^\perp\rangle\langle u^\perp|\psi\rangle = \cos^2(90-\alpha) \quad \text{[Reference: Fig-1]} \\ \langle\psi|\Pi_2|\psi\rangle = \langle\psi|v^\perp\rangle\langle v^\perp|\psi\rangle = \cos^2(90^\circ-\alpha+\theta)\end{array}\right\} \rightarrow (2)$$

Using (2) in (1),

$$\Rightarrow Pr(0) = 1 - \tfrac{1}{c}\sin^2\alpha - \tfrac{1}{c}\sin^2(\alpha-\theta)$$

$$Pr(0) = 1 - \frac{\sin^2\alpha + \sin^2(\alpha-\theta)}{c}$$

$$\cancel{\frac{d\{Pr(0)\}}{d\alpha} = \left(\frac{-2}{c}\right)\left[\sin\alpha\cos\alpha + \sin(\alpha-\theta)\cos(\alpha-\theta)\right] = 0}$$

$$\cancel{\left(\frac{-1}{c}\right)\left[\sin(2\alpha) + \sin(2\alpha-2\theta)\right] = 0}$$

$$\cancel{\Rightarrow 2\theta = \pm(n+\tfrac{1}{2})(\pi)}$$

$$\cancel{\Rightarrow \theta = \pm(n+\tfrac{1}{2})\pi \text{ but } \theta \text{ is previously fixed}}$$

$$Pr(0) = 1 + \frac{1-2\sin^2\alpha + 1-2\sin^2(\alpha-\theta)}{2c} - \frac{1}{c}$$

$$= \left(1-\frac{1}{c}\right) + \frac{\cos(2\alpha) + \cos(2\alpha-2\theta)}{2c}$$

$$= \left(1-\frac{1}{c}\right) + \frac{\cos(2\alpha-\theta)\cos(\theta)}{c}$$

To minimize $Pr(0)$, then $\cos(2\alpha-\theta) = -1$

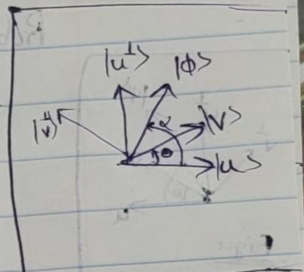
$$\Rightarrow \alpha = \frac{270^\circ+\theta}{2} = 135^\circ + \frac{\theta}{2}$$

$$\boxed{\begin{array}{l}\text{Then } Pr(0) = \left[1 - \frac{1}{c} - \frac{\cos\theta}{c}\right] \text{ if } \frac{\cos\theta}{c} > 0 \\ \cancel{\text{If } \frac{\cos\theta}{c} < 0 \rightarrow Pr(0) = 1 - \frac{1}{c} + \frac{\cos\theta}{c}}\end{array}}$$

→ as is because $c>0$ & $\cos\theta > 0$ [$\because 0<\theta<\frac{\pi}{2}$]

(d) From (c), I derived that

$$Pr(0) = 1 - \frac{\sin^2(\alpha) + \sin^2(\alpha-\theta)}{c}$$

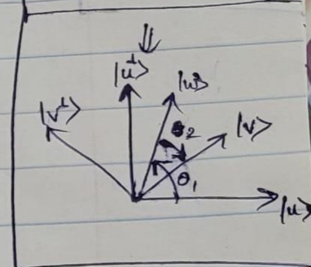


Here $(\alpha) + (\theta - \alpha) = \theta$

Just replace $|\phi\rangle$ with $|w\rangle$!

$\theta_1 =$ Angle from $|u\rangle$ to $|w\rangle \to \alpha$

$\theta_2 =$ Angle from $|w\rangle$ to $|v\rangle \to (\theta-\alpha)$



Substituting these,

$$\Rightarrow Pr(0) = 1 - \left[\frac{\sin^2(\theta_1) + \sin^2(\theta_2)}{c}\right] \quad \& \quad \theta_1 + \theta_2 = \alpha + \theta - \alpha = \theta$$

[Similar to (c)] $\Rightarrow$ $$Pr(0) = \left(1 - \frac{1}{c}\right) + \left[\frac{\cos(\theta - 2\theta_2)\cos(\theta)}{c}\right]$$

$$= \left(1 - \frac{1}{c}\right) + \left[\frac{\cos(\theta_1 - \theta_2)\cos(\theta)}{c}\right]$$

$\cos\theta > 0$ since $0 < \theta < \frac{\pi}{2}$

Assuming $c > 0$, then $\text{Max}[Pr(0)] = 1 - \frac{1}{c} + \frac{(1)\cos\theta}{c}$ [when $\theta_1 - \theta_2 = 0$]

$= \left[1 - \frac{1 - \cos\theta}{c}\right] \to (1)$

Assuming $c < 0$, then $\text{Max}[Pr(0)] = \left[1 - \frac{1}{c} - \frac{\cos\theta}{c}\right]$ [when $\theta_1 - \theta_2 = 180^\circ$] $\to (2)$

From (1), if $Pr(0) \geq 0$

$\Rightarrow \left[1 - \frac{1 - \cos\theta}{c}\right] \geq 0$

$\Rightarrow c \geq (1 - \cos\theta)$ [$\because c > 0$]

From (2), if $Pr(0) \geq 0$

$\Rightarrow \left[1 - \frac{1 + \cos\theta}{c}\right] \geq 0 \Rightarrow c \leq$

If $Pr(0) \geq 0 \Rightarrow 1 - \left[\dfrac{1 - \cos(\theta_1 - \theta_2)\cos(\theta)}{c}\right] \geq 0$

$\Rightarrow \quad c \geq 1 - \cos(\theta_1 - \theta_2)\cos(\theta) \quad [\because c > 0]$

Since θ_1, θ_2 canbe anything dependent on $|\omega\rangle$, then consider the extreme limit.

~~$\Rightarrow$ For $\cos(\theta_1$~~

$\Rightarrow (1 - \cos\theta) \leq [1 - \cos(\theta_1 - \theta_2)\cos\theta] \leq [1 + \cos\theta] \quad [\because -1 \leq \cos(\theta_1 - \theta_2) \leq 1]$

$\Rightarrow$ c should always be greater than $(1 + \cos\theta)$

$\boxed{\therefore \text{ c should be atleast } (1 + \cos\theta)}$

(e) The failure probability corresponds to the probability of outcome '0'. ~~Substi~~ If minimum failure probability $= 1 - \frac{1 - \cos^2\theta}{c}$ then substituting $c = (1 + \cos\theta)$ from (d) to calculate minimum positive failure probability,

$$\Rightarrow Pr(0) = 1 - \frac{1 - \cos^2\theta}{1 + \cos\theta} = 1 - \frac{(1 - \cos\theta)\cancel{(1 + \cos\theta)}}{\cancel{(1 + \cos\theta)}} = \cos\theta$$

Thus, there is zero-sided error qubit discrimination algorithm with failure probability $\cos\theta$.