Quantum mechanics

Law #3

- A qubit state (in general, a joint state) can be changed by any linear transformation that preserves lengths.
- Unitary transformation
  - A linear transformation U is unitary if it preserves lengths.
  - Proposition: U is unitary if and only if $U^\dagger U = I$
  - Theorem: U is unitary if and only if $U^\dagger = U^{-1}$ if and only if $UU^\dagger = I$
  - Corollary: Columns of U form an orthonormal basis. Same holds for rows.
  - Theorem: Unitary transformations preserve inner products

Examples of unitary transformations

- Unitary transformations map an orthonormal basis to another orthonormal basis
- Rotations R_θ
- Hadamard H
- NOT gate
- Phase shift $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- Pauli-Z gate $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- SWAP $= \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$

Square root of a unitary transformation

- Fact: For every unitary transformation U, there exists a unitary transformation W such that $W^2 = U$.
- Examples: $\sqrt{Z} = S$, $\sqrt{R_\theta} = R_{\theta/2}$, $\sqrt{\text{NOT}} = \begin{bmatrix} (1+i)/2 & (1-i)/2 \\ (1-i)/2 & (1+i)/2 \end{bmatrix}$

Quantum state tomography

General question

- Given an unknown state $|\varphi\rangle$, you are promised that $|\phi\rangle \in \{|u\rangle, |v\rangle\}$
- You are asked to guess $|\varphi\rangle$. For simplicity, suppose that $|u\rangle, |v\rangle \in \mathbb{R}^2$, and the angle between them is θ.
- Idea: Rotate and measure.
- Observation: may assume that the measurement is in standard basis (why?)
- Option 1: two-sided error algorithm
- Option 2: one-sided error algorithm
- Option 3: zero-sided error algorithm

Option 1

Measure in $|u'\rangle, |v'\rangle$

- Let $|u'\rangle, |v'\rangle$ be an orthonormal basis such that their angle bisector overlaps with the angle bisector of $|u\rangle, |v\rangle$
- Two-sided error algorithm
  - Measure $|\varphi\rangle$ in the $|u'\rangle, |v'\rangle$ basis
  - Guess $|u\rangle$ when the readout is $|u'\rangle$, and guess $|v\rangle$ when the readout is $|v'\rangle$
- In either case, error probability is $|\langle u \mid v'\rangle|^2 = \cos^2(\pi/4 + \theta/2) = (1 - \sin\theta)/2$

Option 2

Measure in $|u\rangle, |u^\perp\rangle$

- Let $|u\rangle, |u^\perp\rangle$ be an orthonormal basis that contains $|u\rangle$
- One-sided error algorithm
  - Measure in the $|u\rangle, |u^\perp\rangle$ basis
  - Guess $|u\rangle$ when the readout is $|u\rangle$, and guess $|v\rangle$ when the readout is $|u^\perp\rangle$
- Only in the case where $|\varphi\rangle = |v\rangle$, the error probability is $|\langle u \mid v\rangle|^2 = \cos^2\theta$

Option 3

Measure in $|u\rangle, |u^{\perp}\rangle$ or $|v\rangle, |v^{\perp}\rangle$ randomly

- Flip a coin to choose a basis from $|u\rangle, |u^{\perp}\rangle$ and $|v\rangle, |v^{\perp}\rangle$
- Measure in that basis
  - If the basis is $|u\rangle, |u^{\perp}\rangle$, guess $|v\rangle$ only when the readout is $|u^{\perp}\rangle$
  - If the basis is $|v\rangle, |v^{\perp}\rangle$, guess $|u\rangle$ only when the readout is $|v^{\perp}\rangle$
  - Otherwise, say “I don’t know”
- Probability of correct guess is $(1/2)|\langle u \mid v^{\perp}\rangle|^2 = (1/2)\sin^2\theta$
- Not optimal when $\theta = \pi/2$

Multi-qubit systems

4-dimensional quantum systems

- Two photons
- Basic states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$
- Joint state: $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$

Multi-qubit systems

Three questions

- Alice $|\psi\rangle$, Bob $|\varphi\rangle$
- Question 1: What is the joint 4-dimensional state?
- Question 2: How will the joint state change if Bob applies unitary U to his qubit?
- Question 3: What is the readout if Alice measures her qubit?

What is the joint 4-dimensional state?

Question 1

- Alice: $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$
- Bob: $|\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$
- Joint state $\begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$
- What happens if Alice (or Bob) measures her (or his) qubit?

In general…

Alice and Bob

- $a = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{d-1} \end{bmatrix} \in \mathbb{C}^d, b = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_{e-1} \end{bmatrix} \in \mathbb{C}^e$
- Joint state is $a \otimes b = ?$
- Example:
  - $|0\rangle \otimes |0\rangle$
  - $|0\rangle \otimes |+\rangle$
  - $|+\rangle \otimes |0\rangle$
- Tensor product of two matrices
  - $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$
  - $A \otimes B = ?$

Properties of tensor products

Distributivity, associativity, conjugate transpose and multiplication

- $(A + B) \otimes C = A \otimes C + B \otimes C$
- $A \otimes (B + C) = A \otimes B + A \otimes C$
- $A \otimes (B \otimes C) = (A \otimes B) \otimes C =: A \otimes B \otimes C$
- $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$