Collaborators : None

Sources : Lecture Notes

Q2) Borromean entanglement Vs. Non-Borromean entanglement

(a) $|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

If $|GHZ\rangle$ is unentangled, it can be either/both of the forms

$$(a|0\rangle + b|1\rangle) \otimes (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \quad \text{(or)}$$

$$(c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \otimes (a|0\rangle + b|1\rangle)$$

Case-1: If $|GHZ\rangle = (a|0\rangle + b|1\rangle) \otimes (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle)$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) = \left[ac|000\rangle + ad|001\rangle + ae|010\rangle + af|011\rangle + bc|100\rangle + bd|101\rangle + be|110\rangle + bf|111\rangle\right]$$

$$\Rightarrow \underbrace{ac = bf = \tfrac{1}{\sqrt{2}}}_{a \neq 0;\ c \neq 0;\ b \neq 0;\ f \neq 0} \ ; \quad ad = ae = af = bc = bd = be = 0$$

$\Rightarrow af \neq 0$ but $af = 0 \rightarrow$ Contradiction

So, our initial assumption of Case-1 is wrong.

Case-2: If $|GHZ\rangle = (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \otimes (a|0\rangle + b|1\rangle)$

$$\Rightarrow \frac{1}{\sqrt{2}}\left[|000\rangle + |111\rangle\right] = \left[ac|000\rangle + bc|001\rangle + ad|010\rangle + bd|011\rangle + ae|100\rangle + be|101\rangle + af|110\rangle + bd|111\rangle\right]$$

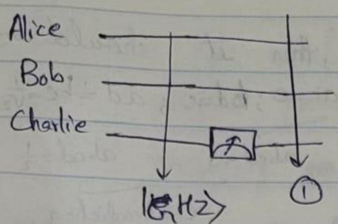
$$\Rightarrow \underbrace{ac = bd = \tfrac{1}{\sqrt{2}}}_{a \neq 0;\ d \neq 0} \ ; \quad bc = ad = bd = ae = be = af = 0$$

$\Rightarrow ad \neq 0$ but $ad = 0 \rightarrow$ Contradiction.

So, our initial assumption of Case-2 is wrong.

Thus, from conclusions of Case-1 & Case-2, we observe that $|GHZ\rangle$ can't be unentangled of '1-& 2-' qubit states, which implies that that it can't be unentangled at all.

(b)



$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

At ①,

$$\text{Joint state (after Readout)} = \begin{cases} |000\rangle \text{ with probability } 0.5 \longrightarrow |R_1\rangle = |00\rangle \\ |111\rangle \text{ with probability } 0.5 \longrightarrow |R_2\rangle = |11\rangle \end{cases}$$

Alice & Bob's qubits

Here $|R_1\rangle = |0\rangle \otimes |0\rangle$
$|R_2\rangle = |1\rangle \otimes |1\rangle$ $\longrightarrow$ ②

Whatever be the readout, either $|R_1\rangle$ or $|R_2\rangle$, according to ②, it's unentangled.

(c) $|W\rangle = \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$

$$= \frac{1}{\sqrt{3}}|00\rangle \otimes |1\rangle + \sqrt{\frac{2}{3}}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

If Charlies measures her qubit, then

$$\text{J.S.}\left(\text{after Readout}\right) = \begin{cases} |00\rangle \otimes |1\rangle \quad \text{with probability } \frac{1}{3} \\ \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) \otimes |0\rangle \quad \text{with probability } \frac{2}{3} \end{cases}$$

$\Rightarrow$ Alice & Bob's qubits $= \begin{cases} |00\rangle \quad \text{with probability } \frac{1}{3} \\ \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) \quad \text{with probability } \frac{2}{3} \end{cases}$

Here $|00\rangle = |0\rangle \otimes |0\rangle$ (unentangled) can be the state with probability $\frac{1}{3}$ (i.e, positive) $\longrightarrow$ ①

And $\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)$ which is Bell state is obviously entangled.

If $\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)$ is unentangled, then it should be of form $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \Rightarrow \underbrace{ac = 0 ; bd = 0}_{abcd = 0} ; \underbrace{ad = bc = \frac{1}{\sqrt{2}}}_{abcd = \frac{1}{2}}$

Contradiction

$\Rightarrow \frac{|10\rangle + |01\rangle}{\sqrt{2}}$ is entangled $\rightarrow$ (2)

$|W\rangle = \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$

If $|W\rangle$ is unentangled, then it should be either/both of form

$(a|0\rangle + b|1\rangle) \otimes (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle)$ (or)

$(c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \otimes (a|0\rangle + b|1\rangle)$

Case-1: If $|W\rangle = (a|0\rangle + b|1\rangle) \otimes (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle)$

$\Rightarrow \frac{1}{\sqrt{3}}\left[|001\rangle + |010\rangle + |100\rangle\right] = \left[\begin{array}{l} ac|000\rangle + ad|001\rangle + ae|010\rangle + af|011\rangle + \\ bc|100\rangle + bd|101\rangle + be|110\rangle + bf|111\rangle \end{array}\right]$

$\Rightarrow \underbrace{ad = ae = bc = \frac{1}{\sqrt{3}}} ; \; ac = af = bd = be = bf = 0$

$a \neq 0 ; d \neq 0 ; e \neq 0 ; b \neq 0 \Rightarrow ac \neq 0$ but $ac = 0 \Rightarrow$ Contradiction
$c \neq 0$

So our initial assumption of Case-1 is wrong.

Case-2: If $|W\rangle = (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \otimes (a|0\rangle + b|1\rangle)$

$\Rightarrow \frac{1}{\sqrt{3}}\left[|001\rangle + |010\rangle + |100\rangle\right] = \left[\begin{array}{l} ac|000\rangle + bc|001\rangle + ad|010\rangle + bd|011\rangle + \\ ae|100\rangle + be|101\rangle + af|110\rangle + bf|111\rangle \end{array}\right]$

$\Rightarrow \underbrace{bc = ad = ae = \frac{1}{\sqrt{3}}} ; \; ac = bd = be = af = bf = 0$

$a \neq 0 ; c \neq 0 \Rightarrow ac \neq 0$ but $ac = 0 \longrightarrow$ Contradiction

So, our initial assumption of Case-2 is wrong

From conclusions of Case-1 & Case-2, we conclude that $|W\rangle$ can't be unentangled of '1' & '2' qubit states, which implies that $|W\rangle$ can't be unentangled at all $\rightarrow$ (3)

From ②, we concluded that if third qubit is measured, then there's positive probability of ~~Alice & Bob's~~ other qubit's state remaining entangled. [i.e $Pr\left[J.S. = \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\right] = \frac{2}{3}$]

~~Now,~~ Let's say ~~we measure~~ instead of 3^{rd}-qubit, we measure 1^{st} or 2^{nd} qubit. Consider shifting positions (permutating) qubits. The resulting joint state is equal to $|W\rangle$. Hence we can conclude that even if we measure 1^{st} & 2^{nd} qubit, the other 2 qubits satisfy ②. $\longrightarrow$ ④

From ② , ④ & ③ , we can conclude that $|W\rangle$ is entangled & even if one of the qubit is measured there is positive probability that the remaining 2 qubits are still entangled.