Collaborators : None

Sources : Lecture Notes

Q2) The basics of quantum random variables

Let $\rho \in \mathbb{C}^{d\times d}$ be a density matrix.

(a) As ρ is a density matrix, ~~Then for the mixed state being measured in standard basis~~ (∵ It doesn't change, in which orthonormal basis you measure)

~~Mixed state (σ) = $\begin{cases} p_1 : \text{probability of being } \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} \text{ state } (|\psi_1\rangle) \\ p_2 : \text{ " " } \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix} \text{ state } (|\psi_2\rangle) \\ \vdots \\ p_m : \text{probability of being } \begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix} \text{ state } (|\psi_m\rangle) \end{cases}$~~

~~And $\rho = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j| = \sum_{j=1}^{m} p_j \begin{bmatrix}0\\ \vdots\\1\\ \vdots\\0\end{bmatrix} [00\ldots 1000]$~~ ($j^{th}$ position)

~~$= \sum_{j=1}^{m} p_j \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & p_3 & \\ 0 & & & \ddots \; p_m \end{bmatrix}$~~

From properties of density matrix, $\boxed{\sum_{j=1}^{m} \rho_{jj} = 1}$

$$E_\rho[I] = tr(\rho I) = tr(\rho) = \sum_{j=1}^{m} \rho_{jj} = 1 \quad [~~\text{Sum of probabilities} = 1~~]$$

$$\boxed{\therefore E_\rho[I] = 1}$$

(b) $E_\rho[X^\dagger] = tr(\rho X^\dagger) = tr\left(\left(\sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|\right) X^\dagger\right) = tr\left(\left(\left(\sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|\right)^\dagger\right)^\dagger X^\dagger\right)$

$= tr\left(\left(\sum_{j=1}^{m} p_j (\langle\psi_j|)^\dagger (|\psi_j\rangle)^\dagger\right)^\dagger X^\dagger\right) \quad [\because (AB)^\dagger = B^\dagger A^\dagger]$

$= tr\left(\left(\sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|\right)^\dagger X^\dagger\right) = tr(\rho^\dagger X^\dagger) = tr((X\rho)^\dagger)$

$= [tr(X\rho)]^* \quad [\because tr(A^\dagger) = [tr(A)]^* \text{ when } A \text{ is square matrix}]$

$= [tr(\rho X)]^* \quad [\because tr(AB) = tr(BA)]$

$= (E_\rho[X])^*$

$$\boxed{\therefore E_\rho[X^\dagger] = E_\rho[X]^*}$$

(c) Let $\alpha, \beta \in \mathbb{C}$, then

$$E_\rho[\alpha X + \beta Y] = tr(\rho(\alpha X + \beta Y)) = tr(\alpha \rho X + \beta \rho Y)$$
$$= \alpha\, tr(\rho X) + \beta\, tr(\rho Y)$$
$$= \alpha E_\rho[X] + \beta E_\rho[Y] \qquad \left[\because \begin{array}{l} tr(A+B) = tr(A) + tr(B) \\ tr(\alpha A) = \alpha\, tr(A) \end{array}\right]$$

$$\boxed{\therefore E_\rho[\alpha X + \beta Y] = \alpha E_\rho[X] + \beta E_\rho[Y]}$$

Let $X, Y \in \mathbb{C}^{d \times d}$ be Hermitian, then

$$[\alpha X + \beta Y]^\dagger = [\alpha X]^\dagger + [\beta Y]^\dagger = \alpha^* X^\dagger + \beta^* Y^\dagger \qquad [\because \alpha, \beta \text{ are scalars}]$$
$$= \alpha^* X + \beta^* Y \qquad [\because X, Y \text{ are Hermitian}],$$

If $\alpha, \beta \in \mathbb{R}$, then $\alpha^* = \alpha$; $\beta^* = \beta$; then

$$[\alpha X + \beta Y]^\dagger = [\alpha X + \beta Y] \Rightarrow [\alpha X + \beta Y] \text{ is Hermitian}$$

Else, there maybe a chance that $\forall$ elements (x, y) of $[\alpha X + \beta Y]$,

$$\alpha x + \beta y = \alpha^* x + \beta^* y$$
$$\Rightarrow x(\alpha - \alpha^*) = y(\beta^* - \beta) \Leftrightarrow y = \frac{x(\alpha - \alpha^*)}{(\beta^* - \beta)}$$

So, if Y is such that all its elements, $y = \frac{x(\alpha - \alpha^*)}{(\beta^* - \beta)}$ where x is corresponding index elements of X, then also, $[\alpha X + \beta Y]$ is Hermitian.

(d) $E_\rho[A^\dagger A] = tr(\rho(A^\dagger A))$

As $A \in \mathbb{C}^{k \times d} \Rightarrow A^\dagger \in \mathbb{C}^{d \times k} \Rightarrow A^\dagger A\ \mathbb{C}^{d \times d}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & \cdots & & a_{2d} \\ \vdots & & & \vdots \\ a_{k1} & \cdots & & a_{kd} \end{bmatrix} \Longrightarrow A^\dagger = \begin{bmatrix} a_{11}^* & a_{21}^* & a_{31}^* & \cdots & a_{k1}^* \\ a_{12}^* & a_{22}^* & \cdots & & a_{k2}^* \\ \vdots & & & & \vdots \\ a_{1d}^* & \cdots & & & a_{kd}^* \end{bmatrix}$

$$\Rightarrow A^\dagger A = \begin{bmatrix} a_{11}^* & a_{21}^* & \cdots & a_{k1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{k2}^* \\ \vdots & & & \vdots \\ a_{1d}^* & \cdots & & a_{kd}^* \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & \cdots & & a_{2d} \\ \vdots & & & \vdots \\ a_{k1} & \cdots & & a_{kd} \end{bmatrix}$$

$$\Rightarrow tr(A^\dagger A) = (a_{11}^* a_{11} + a_{21}^* a_{21} + \cdots + a_{k1}^* a_{k1}) + (a_{12}^* a_{12} + a_{22}^* a_{22} + \cdots + a_{k2}^* a_{k2}) + \cdots$$
$$+ \cdots + (a_{1d}^* a_{1d} + a_{2d}^* a_{2d} + \cdots + a_{kd})$$

((i,i)th element of $A^\dagger A$)

$$[A^\dagger A]_{ii} = \sum_{j=1}^{k} a_{ji}^{*} a_{ji}$$

$$\Rightarrow tr\left(\rho(A^\dagger A)\right) = \sum_{n=1}^{d}\left[\rho_{nn}\left(\sum_{j=1}^{k}\right.\right.$$

$$E_\rho[A^\dagger A] = tr\left(\rho(A^\dagger A)\right) = tr\left(\left(\sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|\right)(A^\dagger A)\right)$$

$$= tr\left(\sum_{j=1}^{m} p_j\left[|\psi_j\rangle\langle\psi_j| A^\dagger A\right]\right)$$

$$= tr\left(\sum_{j=1}^{m} p_j\left[\left(\langle\psi_j|A^\dagger\right)\left(A|\psi_j\rangle\right)\right]\right) \quad [\because tr(AB) = tr(BA)]$$

$$= tr\left(\sum_{j=1}^{m} p_j\left[\left(A|\psi_j\rangle\right)^\dagger\left(A|\psi_j\rangle\right)\right]\right) \quad [\text{Here } A|\psi_j\rangle \text{ is } \mathbb{C}^{k\times 1}]$$

Let $|\phi_j\rangle = A|\psi_j\rangle \in \mathbb{C}^{k\times 1}$

Since $|\phi_j\rangle^\dagger|\phi_j\rangle = \langle\phi_j|\phi_j\rangle = \|\phi_j\|^2 \geq 0$

$$= tr\left(\sum_{j=1}^{m} p_j \|\phi_j\|^2\right) \geq 0 \quad [\because \forall j \in \{1,2,\ldots,m\}\ p_j \geq 0 \ \&\ \|\phi_j\|^2 \geq 0]$$

$$\boxed{\therefore E_\rho[A^\dagger A] \geq 0}$$

(e) $E_{\rho\otimes\sigma}[X\otimes Y] = tr\left((\rho\otimes\sigma)(X\otimes Y)\right) = tr(\rho X \otimes \sigma Y)$

$$= tr(\rho X)\cdot tr(\sigma Y) \quad \left[\because \begin{array}{l}(A\otimes B)\cdot(C\otimes D) = (AC)\otimes(BD)\\ tr(A\otimes B) = tr(A)\cdot tr(B)\end{array}\right]$$

$$= E_\rho[X]\cdot E_\sigma[Y]$$

$$\boxed{\therefore E_{\rho\otimes\sigma}[X\otimes Y] = E_\rho[X]\cdot E_\sigma[Y]}$$

(f) $Cov_\rho[X,Y] = E_\rho\left[(X-\mu_X I)^\dagger(Y-\mu_Y I)\right]$ where $\mu_X = E_\rho[X]$, $\mu_Y = E_\rho[Y]$

$$= E_\rho\left[X^\dagger Y - \mu_Y X^\dagger - \mu_X^{*} Y + \mu_X^{*}\mu_Y I\right]$$

$$= E_\rho[X^\dagger Y] - \mu_Y E_\rho[X^\dagger] - \mu_X^{*} E_\rho[Y] + \mu_X^{*}\mu_Y E_\rho[I]$$ ($\because$ Linear properties)

$$\Rightarrow E_\rho[X^\dagger Y] - \mu_Y \mu_X^* - \cancel{\mu_X^* \mu_Y} + \cancel{\mu_X^* \mu_Y} \quad \left[\because E_\rho[X^\dagger] = \mu_X^* \quad E_\rho[I] = 1\right]$$

$$\boxed{\therefore \mathrm{Cov}_\rho[X,Y] = E_\rho[X^\dagger Y] - \mu_X^* \mu_Y}$$

(g) $$\mathrm{Cov}_\rho[X+\alpha I, Y+\beta I] = E_\rho\left[\left((X+\alpha I) - \mu_{X+\alpha I} I\right)^\dagger \left((Y+\beta I) - \mu_{Y+\beta I} I\right)\right] \to (1)$$

$$\mu_{X+\alpha I} = E_\rho[X+\alpha I] = \mu_X + \alpha \ ; \ \mu_{Y+\beta I} = \mu_Y + \beta \quad \left[\because \text{Linear property} \quad E_\rho[I] = 1\right]$$

Substituting in (1),

$$\Rightarrow \mathrm{Cov}_\rho[X+\alpha I, Y+\beta I] = E_\rho\left[(X+\cancel{\alpha I} - \mu_X I - \cancel{\alpha I})^\dagger (Y + \cancel{\beta I} - \mu_Y I - \cancel{\beta I})\right]$$

$$= E_\rho\left[(X-\mu_X I)^\dagger (Y - \mu_Y I)\right]$$

$$= \mathrm{Cov}_\rho[X,Y]$$

$$\boxed{\therefore \mathrm{Cov}_\rho[X+\alpha I, Y+\beta I] = \mathrm{Cov}_\rho[X,Y]}$$

$$\mathrm{Cov}_\rho[\alpha X, \beta Y] = E_\rho\left[(\alpha X - \mu_{\alpha X} I)^\dagger (\beta Y - \mu_{\beta Y} I)\right]$$

Since, $\mu_{\alpha X} = E_\rho[\alpha X] = \alpha E_\rho[X] = \alpha \mu_X \ ; \ \mu_{\beta Y} = \beta \mu_Y$ $\left[\because \text{Linear property}; \ E_\rho[I] = 1\right]$

$$\Rightarrow \mathrm{Cov}[\alpha X, \beta Y] = E_\rho\left[(\alpha X - \alpha \mu_X I)^\dagger (\beta Y - \beta \mu_Y I)\right]$$

$$= E_\rho\left[\alpha^* (X - \mu_X I) \beta (Y - \mu_Y I)\right] \quad \left(\because \alpha^\dagger = \alpha^*; \ \beta^\dagger = \beta^*\right)$$

$$= (\alpha^* \beta) E_\rho\left[(X - \mu_X I)(Y - \mu_Y I)\right] = (\alpha^* \beta) \ \mathrm{Cov}_\rho[X,Y]$$

$$\boxed{\therefore \mathrm{Cov}_\rho[\alpha X, \beta Y] = \alpha^* \beta \ \mathrm{Cov}_\rho[X,Y]}$$

(h) Let $X \in \mathbb{C}^{d \times d}$ be Hermitian.

$$Var_\rho[X] = Cov_\rho[X, X] = E_\rho\left[(X - \mu_X I)^\dagger (X - \mu_X I)\right]$$

$$\cancel{= E_\rho[X^\dagger X] - \mu_X^* \mu_X \quad [\because \text{From } (f)]}$$

$$\cancel{\text{As } X^\dagger = X \Rightarrow Var_\rho[X] = E_\rho[X^2] - |\mu_X|^2}$$

Consider $A = (X - \mu_X I)$. We know from (d) that $E_\rho[A^\dagger A] \geq 0$.

$$\Rightarrow E_\rho\left[(X - \mu_X I)^\dagger (X - \mu_X I)\right] \geq 0 \Rightarrow \boxed{Var_\rho[X] \geq 0}$$

$$Var_\rho[X + \alpha I] = Cov_\rho[X + \alpha I, X + \alpha I]$$

From (g), $\Rightarrow Cov_\rho[X + \alpha I, X + \alpha I] = Cov_\rho[X, X] = Var_\rho[X]$

$$\boxed{\therefore Var_\rho[X + \alpha I] = Var_\rho[X]}$$

$$Var_\rho[\alpha X] = Cov_\rho[\alpha X, \alpha X] = \alpha^* \alpha \, Cov_\rho[X, X] \quad [\because \text{From } (g)]$$

$$= |\alpha|^2 \, Var_\rho[X]$$

$$\boxed{\therefore Var_\rho[\alpha X] = |\alpha|^2 Var_\rho[X]}$$