Collaborators : None

Sources : Lecture Notes

Q4) The SWAP test

(a) $d=3$, The SWAP gate exchanges the states of 2 qudits.

For $|\psi\rangle = \left[\sum_{i,j=1}^{d} \alpha_{ij} |i\rangle \otimes |j\rangle\right] \in \mathbb{C}^d \otimes \mathbb{C}^d$, then

$$SWAP\,|\psi\rangle = \left[\sum_{i,j=1}^{d} \alpha_{ij} |j\rangle \otimes |i\rangle\right]$$

For $d=3$, we have 9 possible combinations for the states $|i\rangle \otimes |j\rangle$ where $i,j \in \{1,2,3\}$. So the total basis for this system is: $|11\rangle, |12\rangle, |13\rangle; |21\rangle, |22\rangle, |23\rangle, |31\rangle, |32\rangle, |33\rangle$.

The SWAP operation swaps the 2^{nd}-qudit with 1^{st}-qudit, so we can write the action of the SWAP operator on these basis states as:

$$\left\{ \begin{array}{l} SWAP|11\rangle = |11\rangle;\ SWAP|12\rangle = |21\rangle;\ SWAP|13\rangle = |31\rangle; \\ SWAP|21\rangle = |12\rangle;\ SWAP|22\rangle = |22\rangle;\ SWAP|23\rangle = |32\rangle; \\ SWAP|31\rangle = |13\rangle;\ SWAP|32\rangle = |23\rangle;\ SWAP|33\rangle = |33\rangle \end{array} \right\}$$

$\therefore$ Matrix Representation of the SWAP gate in this basis is a 9x9 matrix where each element M_{ij} corresponds to the inner-product $\langle i|SWAP|j\rangle$, which we can extract from the above relationships:

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b). In matrix representation, SWAP gate simply swaps the indices of the 2-qudits, & since it only permutes the basis states without any complex conjugation, it is clearly a real matrix. And SWAP operation is also symmetric.

Thus. $SWAP^* = SWAP$; $SWAP^T = SWAP$

$\Rightarrow SWAP^\dagger = SWAP \Rightarrow$ SWAP is Hermitian.

(c) To prove that SWAP is "basis-independent", we need to show that the SWAP operation applied to any arbitrary linear combination of tensor products of basis states results in the same linear combination but with swapped indices.

Suppose, we have a state in the form:

$$|\phi\rangle = \sum_{i,j=1}^{d} \beta_{ij} |u_i\rangle \otimes |u_j\rangle$$

Let $\{|b_1\rangle, |b_2\rangle, \ldots, |b_d\rangle\}$ be standard basis, i.e, $\{|1\rangle, |2\rangle, |3\rangle, \ldots, |d\rangle\}$.

Then $|u_i\rangle = \sum_{k=1}^{d} \langle b_k|u_i\rangle |b_k\rangle$ & $|u_j\rangle = \sum_{p=1}^{d} \langle b_p|u_j\rangle |b_p\rangle$

$$\Rightarrow |\phi\rangle = \sum_{i,j=1}^{d} \beta_{ij} \left[\sum_{k=1}^{d} \langle b_k|u_i\rangle |b_k\rangle\right] \otimes \left[\sum_{p=1}^{d} \langle b_p|u_j\rangle |b_p\rangle\right]$$

$$\Rightarrow |\phi\rangle = \sum_{i,j=1}^{d} \sum_{k=1}^{d} \sum_{p=1}^{d} \left[\left(\beta_{ij} \langle b_k|u_i\rangle \langle b_p|u_j\rangle\right) |b_k\rangle \otimes |b_p\rangle\right]$$

Applying SWAP, $\Rightarrow SWAP|\phi\rangle = \sum_{i,j=1}^{d} \sum_{k=1}^{d} \sum_{p=1}^{d} \left[\left(\beta_{ij} \langle b_k|u_i\rangle \langle b_p|u_j\rangle\right) SWAP\left(|b_k\rangle \otimes |b_p\rangle\right)\right]$

$$\Rightarrow SWAP|\phi\rangle = \sum_{i,j=1}^{d} \sum_{k=1}^{d} \sum_{p=1}^{d} \left(\beta_{ij} \langle b_k|u_i\rangle \langle b_p|u_j\rangle\right)\left(|b_p\rangle \otimes |b_k\rangle\right)$$ ($\because$ Linear property)

$$= \sum_{i,j=1}^{d} \beta_{ij} |u_j\rangle \otimes |u_i\rangle$$ (Here $|u_i\rangle$ & $|u_j\rangle$ qudits are swapped)

$\therefore$ SWAP is basis-independent

(d) Let ρ be the density matrix for a d-dimensional state & let its eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_d$ with corresponding eigenvectors $|u_1\rangle, |u_2\rangle, \ldots, |u_d\rangle$. The density matrix ρ is:

$$\boxed{\rho = \sum_{i=1}^{d} \lambda_i |u_i\rangle\langle u_i|}$$

The state $\rho \otimes \rho$ is the density matrix for the joint system of 2 particles, each in state ρ, & it is:

$$\boxed{\rho \otimes \rho = \sum_{i,j=1}^{d} \lambda_i \lambda_j |u_i\rangle\langle u_i| \otimes |u_j\rangle\langle u_j|}$$

Now the expected value of SWAP with respect to the density matrix $\rho \otimes \rho$ is:

$$E_{\rho}[SWAP] = Tr((\rho \otimes \rho)\, SWAP)$$
$$= Tr(SWAP(\rho \otimes \rho))$$

Using the fact that $SWAP\, |u_i\rangle \otimes |u_j\rangle = |u_j\rangle \otimes |u_i\rangle$, we can compute:

$$E_{\rho}[SWAP] = Tr\left(\sum_{i,j=1}^{d} \lambda_i \lambda_j \left(|u_i\rangle\langle u_i| \otimes |u_j\rangle\langle u_j|\right) \cdot SWAP\right)$$

$$= Tr\left(\sum_{i,j=1}^{d} \lambda_i \lambda_j \left((|u_i\rangle \otimes |u_j\rangle) \cdot (\langle u_i| \otimes \langle u_j|) \cdot SWAP\right)\right)$$

$$\cancel{= Tr\left(\sum_{i,j=1}^{d} \lambda_i \lambda_j \left((|u_i\rangle \otimes |u_j\rangle) \cdot SWAP \cdot (\langle u_i| \otimes \langle u_j|)\right)\right)}$$

$\because Tr(AB) = Tr(BA)$

$$= Tr\left(\sum_{i,j=1}^{d} \lambda_i \lambda_j \left((\langle u_i| \otimes \langle u_j|) \cdot SWAP \cdot (|u_i\rangle \otimes |u_j\rangle)\right)\right)$$

$$= Tr\left(\sum_{i,j=1}^{d} \lambda_i \lambda_j \left((\langle u_i| \otimes \langle u_j|) \cdot (|u_j\rangle \otimes |u_i\rangle)\right)\right)$$

$$= Tr\left[\sum_{i,j=1}^{d} \lambda_i \lambda_j \left[\underbrace{(\langle u_i | u_j\rangle) \otimes (\langle u_j | u_i\rangle)}_{\text{scalar}}\right]\right]$$

$$= \mathrm{Tr}\left[\sum_{i,j=1}^{d} \lambda_i \lambda_j \langle u_i|u_j\rangle \langle u_j|u_i\rangle\right]$$

$\forall\, i=j$, $\boxed{\langle u_i|u_j\rangle\langle u_j|u_i\rangle} = 1$, else 0

$$\rightarrow \sum_{i=1}^{d} \lambda_i^2$$

$$\boxed{\therefore\ E_\rho[\mathrm{SWAP}] = \sum_{i=1}^{d} \lambda_i^2}$$

(e) $\displaystyle\sum_{i=1}^{d} \lambda_i = \mathrm{Tr}(\rho) = 1$ $\left(\because \text{Sum of eigen values of } A = \mathrm{Trace}(A)\right.$
$\left.\mathrm{Trace}(\rho) = 1\right)$

$$\mathrm{Purity}(\rho) = \left[\sum_{i=1}^{d} \lambda_i^2\right]$$

Since ρ is positive semi-definite, $\forall i \in \{1,\dots,d\}$ $\lambda_i \geq 0$

$\Rightarrow \left[\displaystyle\sum_{i=1}^{d} \lambda_i^2\right]$ is maximum for one value of λ_i being 1 & others 0.

This means a pure state $(\because \lambda_k = 1)$. Here $\displaystyle\sum_{i=1}^{d} \lambda_i^2 = 1$

$\left[\displaystyle\sum_{i=1}^{d} \lambda_i^2\right]$ is minimum, when $\forall\, i \in \{1,2,\dots,d\}$ $\lambda_i = \frac{1}{d}$.

This means a maximally mixed state.

& $\displaystyle\sum_{i=1}^{d} \lambda_i^2 = d\left(\frac{1}{d}\right)^2 = \frac{1}{d}$