# Quantum Error Correction

# Difficulties with Quantum Speedup

- D-Wave's devices are cooled to 10 milliKelvin.
- Even that might be too hot; lead to too much decoherence and error.
- You might have been wondering…
  - Why is a scalable quantum computer possible at all?
- Two separate issues
  - Engineering challenge
  - Everything else that prevents scalable quantum computer even in principle
    - Perhaps, quantum mechanics itself were to break down

- Assuming quantum mechanics does not break down, what are some arguments against scalable quantum computers?
  - Large entangled states are impossible to maintain.
  - Applying unitary gates may produce errors, and snowball over time.
  - The no-cloning theorem limits us.
- Fundamental discoveries in the mid-90s that addressed all the above concerns.
  - The difficulties of building a scalable quantum computer are “merely” difficulties of engineering, and not of principle.

# Classical Error Correction

- The 3-bit repetition code
  - Encode one logical bit using 3 physical bit
    - The logical bit $\overline{0}$ is encoded as 000, and $\overline{1}$ is seconded as 111
  - This code lets us both detect and correct an error in any one physical bit.
    - Show that any code that can both detect and correct a single bit-flip error must use at least 3 bits.
    - If we just want to detect a single bit-flip error, 2 bits suffice.
- More sophisticated codes are able to encode many logical bits at once, while detecting and correcting a large number of errors.

# Skepticism in the Early Days of Classical Computing

- Early skeptics doubted the scalability of classical computers.
  - Argument: “With enough vacuum tubes, some will fail, causing the entire computer to fail.”
- John von Neumann’s Response:
  - Annoyed by this reasoning, he proved the Classical Fault-Tolerance Theorem:
    - Reliable circuits can be built using unreliable parts.
    - Gates with independent failure probability $\varepsilon$ (as long as $\varepsilon$ is small) can still form reliable circuits.

# The Classical Fault-Tolerance Theorem

- Key Insights:
  - A circuit for any Boolean function $F$ can be made reliable, even if each gate has a failure probability $\varepsilon$
  - The circuit size increases only slightly compared to a circuit built with perfect gates.
- Output Reliability Challenge:
  - Single-bit outputs can't exceed a correctness probability of $1 - \varepsilon$.
  - Solution:
    - Use a redundant output: a long string of bits processed by a majority vote.
    - Assumes the final majority computation is error-free.
- Proof Strategy:
  - Recursive use of the 3-bit repetition code.
  - Majority of majorities approach (e.g., 9 bits → 27 bits → 81 bits) pushes error probability lower.

- Challenge:
  - Error-correction circuits are themselves prone to error.
  - Recursive majorities might seem to introduce more errors than they correct.
- Solution:
  - Von Neumann showed that if the physical error probability $\varepsilon$ is small enough, each round of error-correction is a “net win,” reducing errors overall.
- Von Neumann’s method became less critical with the advent of transistors.
- Transistors have extremely low error rates (e.g., one wrong result per year in billions of transistors, often due to cosmic rays).

# The Challenge of Error Correction in Quantum Computing

- Until error correction is achieved, quantum computers may not demonstrate practical utility.
- Example: The world record for Shor's algorithm is still factoring 21 into 3×7.
- Some experts believe meaningful speedups will only occur after surpassing this hurdle.
- Why is Error Correction Difficult?
  - Classical error correction (e.g., 3-bit repetition code) relies on binary states: errors are "yes-or-no."
  - Quantum Challenge:
    - Quantum errors exist on a continuum.
    - Example: A qubit might experience an error like: $|0\rangle \mapsto \sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|1\rangle$
    - It's not just about detecting an error but understanding how much of an error occurred.

# Introducing the Quantum Zeno Effect

- What is the Quantum Zeno Effect?
  - A phenomenon where frequent measurement can "freeze" a quantum system, preventing it from drifting away from a desired state.
- How it Works:
  - Continuously measure the qubit in the $\{|0\rangle, |1\rangle\}$ basis.
  - If measurement gives $|0\rangle$:
    - No error is present; the qubit remains in $|0\rangle$.
    - Analogous to verifying a situation and resetting it to the desired state.
  - If measurement gives $|1\rangle$:
    - Correct the qubit back to $|0\rangle$.
- Limitations: The Quantum Zeno Effect is only effective for:
  - Addressing continuous drift (not discrete errors like bit-flips).
  - Situations where the correct basis of the qubit is known.

# Going beyond continuous drift

- The Simple 3-Bit Repetition Code:
  - Encodes: $|\bar{0}\rangle \mapsto |000\rangle$ and $|\bar{1}\rangle \mapsto |111\rangle$
  - Effective for correcting bit-flip errors.
- The Challenge:
  - Quantum systems face more than just bit-flip errors.
  - Example: Consider how the code handles $|+\rangle$ and $|-\rangle$
  - Question: How many qubits do we need to act on to convert the encoding of $|+\rangle$ to $|-\rangle$?

# Overcoming Infinite Quantum Errors

- The Concern:
  - Even with a code that corrects both bit-flip and phase-flip errors, there seem to be infinitely many other ways for a qubit to err.
  - Wouldn’t this require an infinite amount of redundancy?
- The Solution – A Bit of Magic:
  - A quantum error-correcting code that handles both bit-flip and phase-flip errors is automatically sufficient to correct all possible single-qubit errors.
- Why?

- Without loss of generality, let's assume that we have an error on the first qubit of the entangled state $|\Psi_0\rangle$:
    - $|\Psi_0\rangle = \alpha|0\rangle|v\rangle + \beta|1\rangle|w\rangle$
- A bit -flip error on the first qubit would result in
    - $|\Psi_1\rangle$
- A phase-flip on the first qubit would result in
    - $|\Psi_2\rangle$
- And both would result in
    - $|\Psi_3\rangle$

# The Goal of Quantum Error Correction

- Objective: Recover the original state $|\Psi_0\rangle$ when an error affects the first qubit, within the 4-dimensional "error subspace."
- Why Not Measure All Qubits?
  - Measuring destroys the quantum state.
- The Approach: Perform a measurement that projects onto one of four basis vectors:
  - Outcome $|\Psi_0\rangle$: Do nothing.
  - Outcome $|\Psi_1\rangle$: Apply a bit-flip.
  - Outcome $|\Psi_2\rangle$: Apply a phase-flip.
  - Outcome $|\Psi_3\rangle$: Apply both a bit-flip and a phase-flip.
- Result: Restore the qubit to $|\Psi_0\rangle$ without destroying the quantum state.

# Towards a Quantum Error-Correcting Code

- Challenge: How can we detect and correct both bit-flip and phase-flip errors on any qubit?
- Detecting Phase-Flip Errors:
  - Use the 3-qubit repetition code in the Hadamard basis:
  - $|+\rangle \rightarrow |+\rangle|+\rangle|+\rangle$ and $|-\rangle \rightarrow |-\rangle|-\rangle|-\rangle$
- Limitation:
  - This code fails to protect against bit-flip errors
- Next Step: These observations lead to the development of the first serious quantum error-correcting code, capable of addressing both types of errors.

# The Shor 9-Qubit Code (1995)

- Background:
  - Proposed by Peter Shor (yes, the same Shor!).
  - Combines two codes: repetition codes for bit-flip and phase-flip error correction.
- Structure:
  - Logical qubit encoded as a 3×3 grid of physical qubits:
    - Each row: 3-qubit repetition code for bit-flip errors.
    - Each column: 3-qubit repetition code for phase-flip errors.
- Encoding: Logical $|0\rangle \mapsto \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$ and Logical $|1\rangle \mapsto \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$
- Corrects any single-qubit error (bit-flip, phase-flip, or any combination) among the 9 qubits.

# Detecting and Correcting Bit-Flip Errors

- How It Works:
  - Use a quantum circuit to check if all 3 qubits in a row have the same value.
  - If one qubit differs, set it to match the majority value in the row.
  - Repeat this process for all 3 rows independently.

# Detecting and Correcting Phase-Flip Errors

- How It Works:
  - Build a quantum circuit to compute the relative phase (+ or –) between $|000\rangle$ and $|111\rangle$ in each row.
  - Check if all 3 phases are the same within the row.
  - If one phase is different, adjust it to match the majority phase.
- Why It Works:
  - These operations correct errors not only for $|0\rangle$ and $|1\rangle$ states but also for arbitrary superpositions of the form $\alpha|0\rangle+\beta|1\rangle$.

# Handling Unitary Errors

- What We've Achieved:
  - The code corrects any stray unitary transformations applied to a single qubit.
  - This includes both bit-flip and phase-flip errors.
- Next Challenge:
  - What about errors involving decoherence or measurement?

# Why All Errors Are Covered

- Key Insight:
    - Once unitary errors are corrected, we automatically handle all errors, including decoherence and measurement.
- Reason:
    - Arbitrary errors may transform pure states into mixed states but keep the system within the same 4-dimensional error subspace.
    - Measuring the error syndrome still projects the state to one of four orthogonal states: $|\Psi_0\rangle, |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle$
- Recovery:
- Apply bit flips and phase flips as needed to return to the original state $|\Psi_0\rangle$.

# Handling Errors on All Qubits

- What If All Qubits Are Off?
  - If all qubits are slightly off, it's equivalent to having a superposition of many configurations, most of which involve only a few qubits being significantly off.
- Solution:
  - Apply a standard quantum error-correcting code designed to handle a few qubits being significantly off.
  - This allows us to restore the state to nearly its original form.

# Quantum Error-Correcting Code Evolution

- Shor’s 9-Qubit Code:
  - The first quantum error-correcting code.
- Andrew Steane’s 7-Qubit Code:
  - A shorter code that detects and corrects any 1-qubit error.
- Raymond Laflamme’s 5-Qubit Code:
  - A code using only 5 qubits that can detect and correct a 1-qubit error.
  - 5 qubits is the minimum for detecting and correcting an arbitrary 1-qubit error, similar to how 3 bits is the minimum for classical error correction.

# Next Steps

- Upcoming Topic:
  - The stabilizer formalism, which provides an efficient and compact way to manipulate quantum error-correcting codes.