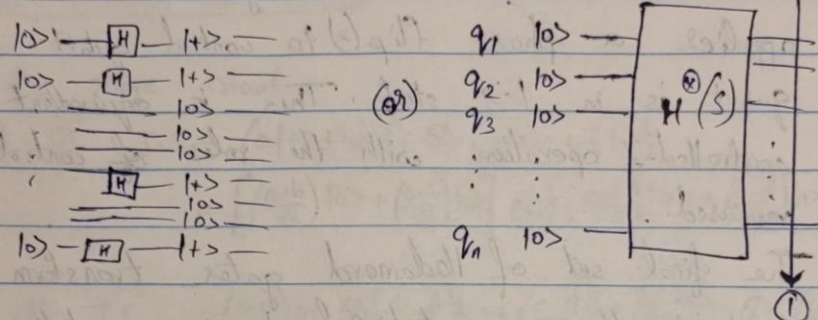
Collaborators : None

Sources : Lecture Notes

Q3) Hadamard Transform - I

Start with 'n' qubits in state $|000\ldots0\rangle$. For certain subset $S \subseteq \{1,2,\ldots n\}$, we apply Hadamard gate H to qubit i $\forall i \in S$.



At ①, Joint state $= |\phi_1\rangle \otimes |\phi_2\rangle \otimes \ldots \otimes |\phi_n\rangle$

where $|\phi_i\rangle = \begin{cases} |0\rangle, & \text{if } i \notin S \\ |+\rangle, & \text{if } i \in S \end{cases}$

Introducing indicator string $y \in \{0,1\}^n$ for S.

If $i \in S$, then $y_i = 1$, else $y_i = 0$

$\Rightarrow |\phi_i\rangle = |+\rangle^{y_i} |0\rangle^{(1-y_i)}$ (or) $y_i|+\rangle + (1-y_i)|0\rangle$

$= [y_i/\sqrt{2} + 1 - y_i]|0\rangle + [y_i/\sqrt{2}]|1\rangle$

~~Joint State~~

As Joint state $= \bigotimes_{i=1}^{n} \left[(y_i/\sqrt{2} + 1 - y_i)|0\rangle + (y_i/\sqrt{2})|1\rangle \right]$

We can see that if i^{th} qubit is $|1\rangle$, then it definetely resulted from $y_i = 1$.

Consider the coefficient of $|z\rangle$ (some $|0100\ldots1\ldots\rangle$) in the final joint state. Then, its coefficient =

$$\prod_{i=1}^{n}\left[(1-z_i)\left[\frac{y_i}{\sqrt{2}}+1-y_i\right]+z_i\left[\frac{y_i}{\sqrt{2}}\right]\right]$$

$$=\prod_{i=1}^{n}\left[\frac{y_i}{\sqrt{2}}+1-y_i-\cancel{\frac{z_i y_i}{\sqrt{2}}}-z_i+z_i y_i+\cancel{\frac{z_i y_i}{\sqrt{2}}}\right]$$

$$=\prod_{i=1}^{n}\left[1-y_i\left(1-\frac{1}{\sqrt{2}}\right)-z_i+z_i y_i\right]$$

Thus,

$$\text{Joint state} = \sum_{z\in\{0,1\}^n}\left[\prod_{i=1}^{n}\left(1-y_i\left(1-\frac{1}{\sqrt{2}}\right)-z_i+z_i y_i\right)\right]|z\rangle$$

$$=\sum_{z\in\{0,1\}^n}\left[\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2}}+(1-y_i)\left(1-\frac{1}{\sqrt{2}}\right)-z_i(1-y_i)\right]\right]|z\rangle$$

$$=\sum_{z\in\{0,1\}^n}\left[\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2}}+\left(1-z_i-\frac{1}{\sqrt{2}}\right)(1-y_i)\right]\right]|z\rangle$$

(OR)

y_i	z_i	coefficient contribution
0	0	1
0	1	0
1	0	$1/\sqrt{2}$
1	1	$1/\sqrt{2}$

$$(y_i \text{ NOR } z_i)+\frac{1}{\sqrt{2}}(y_i \text{ XOR } z_i)$$

$$\Downarrow$$

$$=\sum_{z\in\{0,1\}^n}\left[\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2}}(y_i \text{ XOR } z_i)+(y_i \text{ NOR } z_i)\right]\right]|z\rangle$$