Collaborators : None

Sources : Lecture Notes

Q5 Tensor Product Practice

(a) Given an $m \times n$ matrix A, for every $1 \leq i \leq m, 1 \leq j \leq n$, show that the entry on the i^{th} row and j^{th} column of A equals $\langle i \mid A \mid j \rangle$. Here $\mid i \rangle$ represents the i^{th} vector in the standard basis.

Ans:

- $\langle i \mid$ is a column vector with dimension $1xm$ where the i^{th} element is 1, and all other elements are 0.
- $\mid j \rangle$ is a similar column vector with dimension $nx1$ where the j^{th} element is 1.

$$A \mid j \rangle = [s_{k1} \mid s_{k1} = \sum_{p=1}^{n}(a_{kp} * je_{p1}) \ \ ; a_{kp} \epsilon A \, ; j_{p1} \epsilon \mid j \rangle]_{\text{mx1}}$$

$$\Rightarrow A \mid j \rangle = [s_{k1} \mid s_{k1} = a_{kj} \ \ ; a_{kp} \epsilon A]_{\text{mx1}} \ (\because je_{k1} = 0 \ \forall k \neq j)$$

$$\langle i \mid A \mid j \rangle = (\langle i \mid)(A \mid j \rangle) = [\sum_{p=1}^{m}(ie_{1p} * s_{p1}) \ ; a_{pj} = s_{p1} \epsilon A \mid j \rangle \ ; ie_{1p} \epsilon \langle i \mid]_{\text{1x1}}$$

$$\Rightarrow \langle i \mid A \mid j \rangle = [s_{i1}]_{\text{1x1}} = \boldsymbol{a_{ij}} \ (\because ie_{1k} = 0 \ \forall k \neq i)$$

Thus, the entry on the i^{th} row and j^{th} column of A, $\boldsymbol{a_{ij}}$ equals $\langle i \mid A \mid j \rangle$.

(b) Show that if A and B are invertible matrices, then so is $A \otimes B$, and in fact $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Ans: Given that both A and B are invertible, we know that $A^{-1}$and B^{-1} exist.

Multiplying $(A \otimes B)$ with $(A^{-1} \otimes B^{-1})$,

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) \quad (\because (A \otimes B)(C \otimes D) = (AC) \otimes (BD))$$

$$\Rightarrow (A \otimes B)(A^{-1} \otimes B^{-1}) = I_m \otimes I_n = I_{mn}$$

As inverse is unique, thus $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(c) Verify that $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$.

Ans: Consider the matrix element $(A \otimes B)_{ij,kl}$ where (i,j) and (k,l) are the row and column indices in the tensor product matrix. This element can be expressed as:

$$(A \otimes B)_{ij,kl} = A_{ik}B_{jl}$$

Taking the conjugate transpose, we get:

$$((A \otimes B)^{\dagger})_{ij,kl} = \overline{A_{ki}B_{lj}} = \overline{A_{ki}}\,\overline{B_{lj}} = (A^{\dagger})_{ik}(B^{\dagger})_{jl} = (A^{\dagger} \otimes B^{\dagger})_{ij,kl}$$

This shows that the conjugate transpose of the tensor product is indeed the tensor product of the conjugate transposes:

$$(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$$

(d) Suppose $|\, u_1\rangle, \ldots, |\, u_d\rangle$ is an orthonormal basis for C^d, and $|\, v_1\rangle, \ldots, |\, v_e\rangle$ is an orthonormal basis for C^e. Show that the collection $|\, u_i\rangle \otimes |\, v_j\rangle$ (for all $1 \le i \le d, 1 \le j \le e$) is an orthonormal basis for C^{de}.

Ans: $(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = \sum_{p=1}^{e}\left(\sum_{q=1}^{d}((|\, v_n\,\rangle)_p * \left(|\, v_j\rangle\right)_p * \ (|\, u_m\rangle)_q * (|\, u_i\rangle)_q)\right)$

$$= \sum_{p=1}^{e}((|\, v_n\,\rangle)_p * \left(|\, v_j\rangle\right)_p) * \left(\sum_{q=1}^{d}((|\, u_m\rangle)_q * (|\, u_i\rangle)_q)\right)$$

If $m \ne i$, then as $|\, u_1\rangle, \ldots, |\, u_d\rangle$ is an orthonormal basis, $\sum_{q=1}^{d}((|\, u_m\rangle)_q * (|\, u_i\rangle)_q) = 0$,

$$(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = 0$$

Else if $m = i$, $(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = \sum_{p=1}^{e}((|\, v_n\,\rangle)_p * \left(|\, v_j\rangle\right)_p) * (1)$

Similarly, if $n \ne j$, then as $|\, v_1\rangle, \ldots, |\, v_e\rangle$ is an orthonormal basis, $\sum_{p=1}^{e}((|\, v_n\,\rangle)_p * \left(|\, v_j\rangle\right)_p) = 0$,

$$(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = 0$$

Else if $n = j$, $(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = 1$

Thus, if $(m, n) \ne (i, j)$, $(|\, u_m\rangle \otimes |\, v_n\rangle)^T(\,|\, u_i\rangle \otimes |\, v_j\rangle) = 0$

Therefore, the collection $|\, u_i\rangle \otimes |\, v_j\rangle$ (for all $1 \le i \le d, 1 \le j \le e$) is an orthonormal basis for C^{de}.