Final Exam

Graded

Student

Sujith Potineni

Total Points

16 / 25 pts

Question 1

(no title) **5** / 5 pts

✔ **+ 5 pts** Correct

+ 1 pt Applied H to both ket 0 and ket 1.

+ 2 pts Computed the joint state after the application of H.

+ 0 pts Incorrect

Question 2

(no title) **5** / 5 pts

✔ **+ 5 pts** Correct

+ 4 pts Showed for AND, OR, NOT.

+ 0 pts Incorrect

Question 3

(no title) **5** / 5 pts

✔ **+ 5 pts** Correct

+ 0 pts Incorrect

Question 4

(no title) **1** / 5 pts

+ 5 pts Correct

+ 3 pts Answered (a-c) correctly

✔ **+ 1 pt** Recalled correctly all the Pauli matrices, but did not answer any questions correctly.

+ 1 pt Answered (b) correctly.

– 0.5 pts Some Pauli matrices were wrong.

+ 0 pts Incorrect

+ 0.5 pts (e) Proved the "if" statement.

Question 5

(no title)

0 / 5 pts

+ 5 pts Correct

+ 2 pts (c) correct

✔ **+ 0 pts** Incorrect

CSE 598 Quantum Computation
Final Exam, Fall 2024

Instructor: Zilin Jiang

December 10, 2024

Full name: Bala Sujith Potineni

Time: 110 minutes. **Five problems** worth 5 points each.

Instructions

1. Closed book. No notes or any electronic devices during the exam.
2. You must provide justification in your solutions (not just answers).
3. You may quote theorems and facts proved in class, course textbook/notes, or homework, provided that you state the facts that you are using.
4. This exam will be scanned before grading, so please ensure your writing is clear and legible.
5. If you have questions, raise your hand. The proctor will come to you. Do not ask out loud.

This is scratch paper. Material written on this page will NOT be graded.

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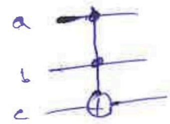


$(a \cdot b) \cdot \bar{c} + (\bar{a} + \bar{b}) c$

$= a b \bar{c} + \bar{a} c + \bar{b} c$

$b = 1;\ c = 0$

$U \otimes U$

This is scratch paper. Material written on this page will NOT be graded.

$\frac{1}{4}\left(|\phi\rangle + |\psi\rangle\right)^{*}\left(|\phi\rangle + |\psi\rangle\right)$

$\langle\phi|\phi\rangle + \langle\phi|\psi\rangle + \langle\psi|\phi\rangle +$

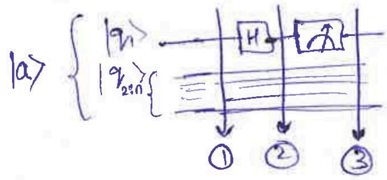
BSP

Problem 1 Consider the quantum state

$$|a\rangle = \frac{|0\rangle|\phi\rangle + |1\rangle|\psi\rangle}{\sqrt{2}},$$

where $|\phi\rangle$ and $|\psi\rangle$ are unknown normalized quantum states with the same number of qubits. Suppose a Hadamard gate is applied to the first qubit, followed by a measurement of the first qubit in the standard basis (that is, the computational basis). Determine the probability of observing the measurement outcome $|1\rangle$ as a function of the states $|\phi\rangle$ and $|\psi\rangle$.

Ans:



Let $|a\rangle = \dfrac{|0\rangle|\phi\rangle + |1\rangle|\psi\rangle}{\sqrt{2}}$; Let 1^{st} qubit be $|q_1\rangle$ & other qubits be $|q_{2:n}\rangle$ ($(n-1)$-qubits)

At ①, Joint state $= |a\rangle$

At ②, Joint state $= \dfrac{|+\rangle|\phi\rangle + |-\rangle|\psi\rangle}{\sqrt{2}} = \dfrac{|0\rangle\left(\frac{|\phi\rangle+|\psi\rangle}{\sqrt{2}}\right) + |1\rangle\left(\frac{|\phi\rangle-|\psi\rangle}{\sqrt{2}}\right)}{\sqrt{2}}$

$= \left[|0\rangle\left(\dfrac{|\phi\rangle+|\psi\rangle}{2}\right) + |1\rangle\left(\dfrac{|\phi\rangle-|\psi\rangle}{2}\right)\right]$

~~At ③, Joint state~~

~~After $|q_1\rangle$ measurement, $|0\rangle\left(\frac{|\phi\rangle+|\psi\rangle}{2}\right)$ w.p. 0.5~~

At ③, After $|q_1\rangle$ measurement ;

outcome $= |1\rangle$ with probability $\left\| \dfrac{|\phi\rangle - |\psi\rangle}{2} \right\|^2$

$\longrightarrow \left(\dfrac{\langle\phi| - \langle\psi|}{2}\right)\left(\dfrac{|\phi\rangle - |\psi\rangle}{2}\right) = \left(\dfrac{\langle\phi|\phi\rangle - \langle\phi|\psi\rangle - \langle\psi|\phi\rangle + \langle\psi|\psi\rangle}{4}\right)$

$= \boxed{\dfrac{2 - \langle\phi|\psi\rangle - \langle\psi|\phi\rangle}{4}}$ $\quad [\because \langle\phi|\phi\rangle = \langle\psi|\psi\rangle = 1]$

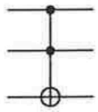
BSP

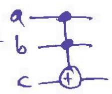
Extra space for Problem 1

$$\boxed{\therefore \text{ Probability of observing outcome } 1 = \left[\frac{1}{2} - \frac{\langle\phi|\psi\rangle + \langle\psi|\phi\rangle}{4}\right]}$$

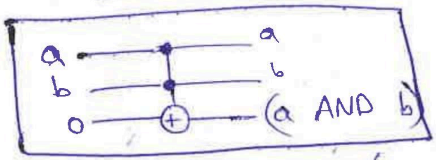
BSP

Problem 2 In class, we demonstrated how to use only CSWAP gates to simulate classical logic gates such as AND, OR, NOT, and FANOUT gates. Prove that it is also possible to simulate classical AND, OR, NOT, and FANOUT gates using only CCNOT gates (also known as Toffoli gates).



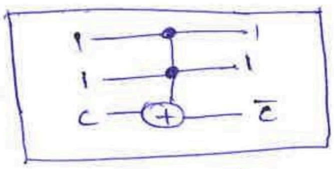
Ans: Assuming CCNOT ≡  s.t. [c gets flipped if a & b both are '1', else remains same]

Case-1 : (AND-gate)



If c=0, then when (a & b) are both 1, then c=0 flips to '1' else when (a, or b) is 0, then c=0 remains '0'. Effectively simulating (a AND b).

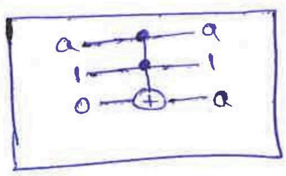
Case-2 : ~~(OR-gate)~~ (NOT-gate)



If a=1 & b=1, then c gets flipped to (NOT c); effectively simulating NOT-gate.

~~Case-3: (FANOUT-gate)~~

Case-3: (FANOUT-gate)



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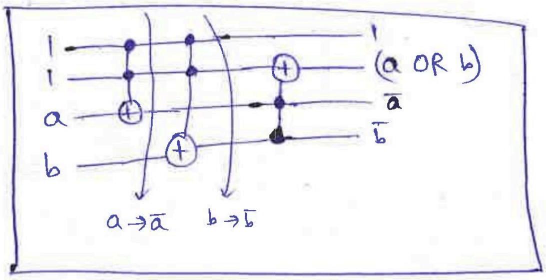
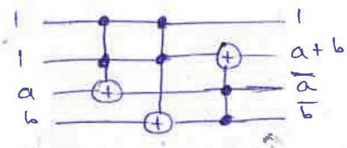
Extra space for Problem 2

~~[illegible]~~ When $(b=1 \ \& \ c=0)$; $\begin{cases} \text{if } a=1 \longrightarrow c \text{ flips from } 0 \text{ to } 1, \text{ which} = a \\ \text{if } a=0 \longrightarrow c \text{ doesnt flips \& remains } 0, \text{ which} = a \end{cases}$

Now, we have 1^{st} & 3^{rd} qubits both a.

Case-4 (OR-gate)





Initially, calculate (NOT a) & (NOT b) & then, if any other qubit is 1, ~~[illegible]~~ we can $\underbrace{\text{CCNOT}}$ it to (a OR b)

(~~[illegible]~~ If (NOT a) & (NOT b) ⇒ it flips to 0
else remains in 1

$\Rightarrow \begin{cases} \text{NOT}(a \text{ OR } b) \Rightarrow o/p = 0 \\ \text{else} \rightarrow o/p = 1 \end{cases}$

$\Rightarrow \begin{cases} a \text{ OR } b \Rightarrow o/p = 1 \\ \text{else} \Rightarrow o/p = 0 \end{cases}$

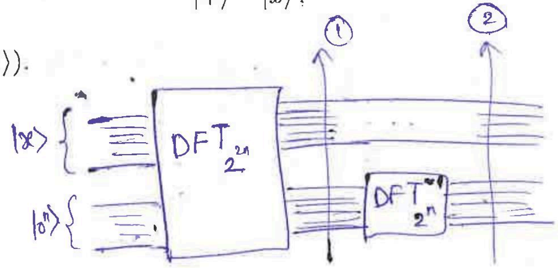
BSP

Problem 3 Let $x \in \{0,1\}^n$. Consider the $2n$-qubit quantum state $|x\rangle|0^n\rangle$, where $|x\rangle$ is an n-qubit computational basis state and $|0^n\rangle$ is an n-qubit zero state. Suppose we apply the $2n$-qubit discrete Fourier transform $\mathrm{DFT}_{2^{2n}}$ to the entire $2n$-qubit state, followed by the inverse discrete Fourier transform $\mathrm{DFT}_{2^n}^{-1}$ on the last n qubits (while applying the identity operation on the first n qubits). Show that the resulting state is

$$|+\rangle^{\otimes n}|x\rangle,$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Ans.



Since $x \in \{0,1\}^n$ & not a superposed state, then let 'x' be represent in numeral x_n s.t. $0 \le x_n \le 2^n - 1$

$\Rightarrow$ $\underbrace{\text{Numeral}\left(|x\rangle \otimes |0^n\rangle\right)}_{\text{or index}} = 2^n \times x_n \longrightarrow$ index of it in the vector form.

$$\Rightarrow \boxed{|x\rangle|0^n\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{N\times 1} \longrightarrow \text{index} = 2^n x_n}$$

where $N = 2^{2n}$

Applying $\mathrm{DFT}_{2^{2n}}$ on it,

$\Rightarrow$ At ①, $o/p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \cdots & \omega_N^{-2(N-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \cdots & \omega_N^{-(N-1)^2} \end{bmatrix}_{N\times N} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{N\times 1} = \begin{bmatrix} \omega_N^{-(2^n x_n)} \\ \omega_N^{-(2^n x_n)(2)} \\ \vdots \\ \omega_N^{-(2^n x_n)(N-1)} \end{bmatrix}_{N\times 1}$

Here, $\text{element}[i] = \omega_N^{-i(2^n x_n)}$

BSP

~~At ②, (after applying DFT)~~

This state $= \frac{1}{\sqrt{2^{2n}}} \sum_{i=0}^{2^{2n}-1} \left(\omega_N^{-2^n x_n \cdot i}\right) |i\rangle = \frac{1}{\sqrt{2^{2n}}} \sum_{i=0}^{2^{2n}-1} \left(e^{-\left(\frac{2\pi i}{2^{2n}}\right)\left(2^n x_n\right) \cdot i}\right) |i\rangle = \frac{1}{\sqrt{2^{2n}}} \sum_{i=0}^{2^{2n}-1} \left(e^{\frac{-2\pi i\, x_n}{2^n} \cdot i}\right) |i\rangle$

Now divide each 'i' into 2 parts s.t. $i = a_i 2^n + b_i$.

$\Rightarrow$ Then, state $= \frac{1}{\sqrt{2^{2n}}} \sum_{i=0}^{2^{2n}-1} \left(e^{\frac{-2\pi i}{2^n} x_n \cdot (2^n a_i + b_i)}\right) |a_i 2^n + b_i\rangle$

$= \frac{1}{\sqrt{2^{2n}}} \sum_{a_i=0}^{2^n-1} \sum_{b_i=0}^{2^n-1} \left[\underbrace{\left(e^{-2\pi i\, x_n a_i}\right)}_{1} \left(e^{\frac{-2\pi i\, x_n}{2^n} b_i}\right)\right] |a_i 2^n + b_i\rangle$

$= \frac{1}{\sqrt{2^{2n}}} \sum_{a=0}^{2^n-1} \sum_{b=0}^{2^n-1} \left[e^{\frac{-2\pi i\, x_n}{2^n} b}\right] |a 2^n + b\rangle \longrightarrow \frac{1}{\sqrt{2^{2n}}} \begin{bmatrix} \begin{pmatrix} \omega_{2^n}^{x_n b} \\ \vdots \end{pmatrix} \\ (\cdot) \\ (\ ) \end{bmatrix}$

$\Downarrow$

(Coefficients not dependent on a, but only on b i.e, (bottom n-qubits value)) (Every 2^n - states, the coefficients are periodic (repeat))

$\Downarrow$

Unentangled b/w 2-parts

$\frac{1}{\sqrt{2^{2n}}} \sum_{a=0}^{2^n-1} \sum_{b=0}^{2^n-1} \left(\omega_{2^n}^{-x_n b}\right) |a\rangle \otimes |b\rangle = \left[\sum_{a=0}^{2^n-1} |a\rangle\right] \otimes \left[\sum_{b=0}^{2^n-1} \left(\omega_{2^n}^{-x_n b} \cdot |b\rangle\right)\right] \left(\frac{1}{\sqrt{2^{2n}}}\right)$

Apply $DFT_{2^n}^{-1}$ here (on last n-qubits), i.e, [each term has $\sum_{b=0}^{2^n-1} \left(\omega_{2^n}^{-x_n b} |b\rangle\right)$]

$\Rightarrow \frac{1}{\sqrt{2^n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_{2^n}^1 & \omega_{2^n}^2 & \cdots & \omega_{2^n}^{2^n-1} \\ \vdots & & & & \vdots \\ 1 & \cdots & & & \omega_{2^n}^{(2^n-1)^2} \end{bmatrix}_{2^n \times 2^n} \frac{1}{\sqrt{2^{2n}}} \begin{bmatrix} \omega_{2^n}^{-x_n(0)} \\ \omega_{2^n}^{-x_n(1)} \\ \vdots \\ \omega_{2^n}^{-x_n(2^n-1)} \end{bmatrix}_{2^n \times 1} = \frac{1}{\sqrt{2^n}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \rightarrow x_n \\ \\ \\ \\ \end{matrix}$ (Here each element$[i] = \frac{\left(\omega_{2^n}^{(i-x_n)2^n} - 1\right)}{\left(\omega_{2^n}^{i-x_n} - 1\right)}$ when $i \neq x_n$, its 0 if $i = x_n$, its 2^n.)

$\hookrightarrow$ Last n-qubits $= |x_n\rangle \left(\frac{1}{\sqrt{2^n}}\right)$

Combining with $\left(\sum_{a=0}^{2^n-1} |a\rangle\right)$ of 1^{st} n-qubits $\longrightarrow \left(\frac{1}{\sqrt{2^n}} \sum_a |a\rangle\right) \otimes |x_n\rangle = |+^n\rangle |x_n\rangle$

BSP

bit flip

phase flip

bit + phase flip

Problem 4 Consider a qubit whose density matrix is ρ.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) Show that there exist real numbers r_0, r_1, r_2, r_3 such that

$$\rho = \frac{r_0}{2}I + \frac{r_1}{2}X + \frac{r_2}{2}Y + \frac{r_3}{2}Z,$$

where I, X, Y, Z are the Pauli matrices.

(b) Show that $r_0 = 1$.

(c) Show that

$$\mathrm{tr}(\rho^2) = \frac{1}{2}\left(r_0^2 + r_1^2 + r_2^2 + r_3^2\right).$$

(d) Show that $r_1^2 + r_2^2 + r_3^2 \leq 1$.

(e) Show that $r_1^2 + r_2^2 + r_3^2 = 1$ if and only if ρ represents a pure state.

$$\rho = \sum_{i=1}^{m} p_i |\psi_i\rangle\langle\psi_i| = \begin{bmatrix} p & q \\ r & s \end{bmatrix}_{2\times 2}$$

Here $|\psi_i\rangle$ is $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ (2-element column vector)

(a) Since I, X, Y, Z are 4-linearly-independent matrices, it is possible to find

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \frac{r_0}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{r_1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{r_2}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + \frac{r_3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $|\psi_1\rangle = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ & $|\psi_2\rangle = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$; $|a_1|^2 + |b_1|^2 = |a_2|^2 + |b_2|^2 = 1$

$$\Rightarrow \rho = \begin{bmatrix} (p_1 a_1 a_1^* + p_2 a_1 b_1^*) & (p_1 a_2 a_2^* + p_2 a_2 b_2^*) \\ (p_1 b_1 a_1^* + p_2 b_1 b_1^*) & (p_2 b_2 a_2^* + p_2 b_2 b_2^*) \end{bmatrix}$$

(c) $\mathrm{tr}(\rho^2) = \mathrm{tr}\left[\left(\frac{r_0}{2}I + \frac{r_1}{2}X + \frac{r_2}{2}Y + \frac{r_3}{2}Z\right)^2\right] = \mathrm{tr}\left[\frac{r_0^2}{4}I + \frac{r_0 r_1}{2}IX + \ldots\right]$

$=$

BSP

Extra space for Problem 4

BSP

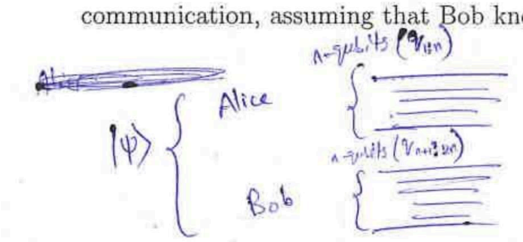
Problem 5 Alice and Bob share n EPR pairs. Denote their shared $2n$-qubit quantum state by $|\psi\rangle$, where Alice has the first n qubits, and Bob has the last n qubits.

(a) Let U be an arbitrary n-qubit unitary operator, and let $\overline{U}$ denote the complex conjugate of U (without transposition). Prove that

$$(U \otimes \overline{U})\,|\psi\rangle = |\psi\rangle .$$

(b) Suppose Alice receives a classical input $x \in \{0,1\}^n$ and applies an n-qubit unitary operation U_x to her part of the shared state. She then measures her qubits in the computational basis, obtaining a classical outcome $a \in \{0,1\}^n$. Derive the probability distribution of Alice's measurement outcomes and explain the reasoning.

(c) Suppose Bob receives the same input $x \in \{0,1\}^n$ as Alice. Describe a procedure that allows Bob to determine Alice's measurement outcome a from part (b) without any communication, assuming that Bob knows the mapping $x \mapsto U_x$.



Let Alice qubits be $|q_1\rangle, |q_2\rangle, \ldots |q_n\rangle$ & Bob's qubits be $|q_{n+1}\rangle, \ldots |q_{2n}\rangle$.
And $\forall\, i \in \{1,\ldots,n\}$, $|q_i q_{i+n}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ } EPR property.

And $|\psi\rangle = |q_1 q_2 \cdots q_{2n}\rangle \underset{\text{rearrange}}{=} |q_1 q_{1+n}\rangle \otimes |q_2 q_{2+n}\rangle \otimes \cdots$

$$= \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)_{i=1} \otimes \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)_{i=2} \otimes \cdots \left(\ \right)_{i=n}$$

(a) This results in states where $\forall\, i \in \{1,\ldots,n\}$ $|q_i\rangle$ term $= |q_{i+n}\rangle$ term (either 0 or 1)

Pattern

$|\psi\rangle = \frac{1}{\sqrt{2^n}} \begin{bmatrix} \vdots \end{bmatrix}_{2^{2n} \times 1}$ → In this exactly 2^n terms are 1 s.t.

BSP

Extra space for Problem 5

(b) Each of the qubit state canbe divided into individual terms where (first n-qubits = last n-qubits)

Applying U_x on each term [$\because$ U_x multiplication is linear $\Downarrow$ Doesnt affect on other) gives out same output $\Rightarrow U_x(a+b) = U_x(a) + U_x(b)$

O/p's of terms are distinct, since U_x is unitary & reversible.

$\Rightarrow$ Each term Alice qubit state has bijective relation to Bob's qubit.

Since there are 2^n - such terms in Alice qubits.

$\Rightarrow$ We get equal probability in outputs.

$$\Rightarrow \boxed{O/p = \begin{cases} 1/2^n & \text{for } |s\rangle = |00..0\rangle \\ 1/2^n & \text{for } |s\rangle = |00..1\rangle \\ \vdots & \\ 1/2^n & \text{for } |s\rangle = |11..1\rangle \end{cases}}$$

(c) Since each measurement outcome of Alice is uniquely mapped to Bob's state. ; Bob just has to apply U_x based on x-qubit to get the same outcome as Alice