Assignment 6

Graded

Student

Sujith Potineni

Total Points

40 / 40 pts

Question 1

Read me first

0 / 0 pts

+ 0 pts Incorrect

✔ **+ 0 pts** Correct

Question 2

The basics of quantum random variables

10 / 10 pts

✔ **+ 10 pts** Correct

+ 0 pts Incorrect

Question 3

The Uncertainty Principle

10 / 10 pts

✔ **+ 10 pts** Correct

+ 0 pts Incorrect

+ 6 pts Only solved (a-e)

+ 8 pts Only solved (a-f)

Question 4

The SWAP test

10 / 10 pts

✔ **+ 10 pts** Correct

+ 8 pts Only solved (a-d), but solution to (e) is wrong.

+ 0 pts Incorrect

Question 5

Zero-error state discrimination

10 / 10 pts

✔ **+ 10 pts** Correct

− 1 pt Did not show the optimal c in (d)

+ 0 pts Incorrect

Q1 Read me first
0 Points

- Tests show that the people who get the most out of this assignment are those who read the "read me first" like this one.
- Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.
- Acknowledgment Requirements:

1. Acknowledge, individually for every problem at the beginning of each solution, a list of all collaborators and sources consulted other than the course notes. Examples include: names of people you discussed homework with, books, other notes, Wikipedia, and other websites.
2. If you consulted any online sources, please specify the exact webpages by including their links. Omission of links or any other required citations will result in a loss of grades and be considered a failure to acknowledge appropriately.
3. If no additional sources are consulted, you must write "sources consulted: none" or equivalent.
4. **Failure to acknowledge sources will lead to an automatic 1pt penalty.**

- Late policy: In general **no late homework** will be accepted unless there is a genuine emergency backed up by official documents.
- All steps should be justified.
- Formatting and Submission Requirements:

1. Separate Solutions: Ensure that solutions for each problem are separated clearly.
2. PDF Submissions: If you are submitting a LaTeX PDF, use the "fullpage" package to set the margins to 1 inch. Do not include additional information such as the title, date, your name, the problem statement, or any rough work—only include your final solution.
3. Typed Solutions: If typing directly in the provided textbox, please use LaTeX formatting for formulas.
   Images: Rotated images will not be graded. Ensure all images are properly oriented.
4. Scanning Quality: Use proper scanning software to scan your handwritten solutions. Avoid casual photos of your work.
5. **Failure to meet these formatting and submission requirements may result in up to a 2-point penalty for each problem.**

- You are encouraged to be **type in LaTeX**. To learn how to use LaTeX, I recommend the tutorials on Overleaf. It is ok to draw diagrams by hand and insert them as pictures in your TeX files.
- For each question below, upload a PDF file and/or type in the box (see Gradescope x LaTeX tutorial). Each submission should contain (1) the acknowledgement of all collaborators and sources consulted and (2) your solution.

Q2 The basics of quantum random variables

10 Points

Let $\rho \in \mathbb{C}^{d\times d}$ be a density matrix. Recall that for an observable (i.e., Hermitian matrix) $X \in \mathbb{C}^{d\times d}$, we define

$$E_\rho[X] = \langle \rho, X\rangle = \mathrm{tr}(\rho^\dagger X) = \mathrm{tr}(\rho X) = \sum_{i,j=1}^{d} \rho_{ij} X_{ij}.$$

In this problem, we will extend the above notation to allow for a non-Hermitian matrix X. This is not "physically meaningful" (since there is no measurement instrument corresponding to a non-Hermitian matrix X), but it will be mathematically convenient to let us reason about observables.

(a) Prove that $E_\rho[I] = 1$, where I denotes the $d \times d$ identity matrix.

(b) Prove that $E_\rho[X^\dagger] = E_\rho[X]^*$.

(c) Let $X, Y \in \mathbb{C}^{d\times d}$ be Hermitian and let $\alpha, \beta \in \mathbb{C}$. Prove "linearity of expectation":

$$E_\rho[\alpha X + \beta Y] = \alpha E_\rho[X] + \beta E_\rho[Y].$$

Also, show that $\alpha X + \beta Y$ is Hermitian if $\alpha, \beta \in \mathbb{R}$ (otherwise, we can't be sure).

(d) Prove that $E_\rho[A^\dagger A] \geq 0$ for any matrix $A \in \mathbb{C}^{k\times d}$.

(e) Let $\sigma \in \mathbb{C}^{d\times d}$. Prove that $E_{\rho\otimes\sigma}[X \otimes Y] = E_\rho[X]E_\sigma[Y]$.
(This generalizes the classical probability fact that if x and y are independent random variables, then $E[xy] = E[x]E[y]$.)

(f) Let $X, Y \in \mathbb{C}^{d\times d}$, not necessarily Hermitian. Define their covariance with respect to ρ to be

$$\mathrm{Cov}_\rho[X, Y] = E_\rho[(X - \mu_X I)^\dagger (Y - \mu_Y I)],$$

where $\mu_X = E_\rho[X]$, $\mu_Y = E_\rho[Y]$. Prove that

$$\mathrm{Cov}_\rho[X, Y] = E_\rho[X^\dagger Y] - \mu_X^* \mu_Y.$$

(g) Prove that covariance is "translation-invariant" in each argument, meaning $\mathrm{Cov}[X + \alpha I, Y + \beta I] = \mathrm{Cov}[X, Y]$ for all $\alpha, \beta \in \mathbb{C}$. Prove also that

$\mathrm{Cov}[\alpha X, \beta Y] = \alpha^* \beta \, \mathrm{Cov}[X, Y]$.

(h) Let $X \in \mathbb{C}^{d \times d}$, not necessarily Hermitian. Define the variance of X with respect to ρ to be

$$\mathrm{Var}_\rho[X] = \mathrm{Cov}_\rho[X, X].$$

Show that $\mathrm{Var}_\rho[X] \geq 0$ always, that $\mathrm{Var}_\rho[X]$ is translation-invariant, and that $\mathrm{Var}_\rho[\alpha X] = |\alpha|^2 \mathrm{Var}_\rho[X]$.

Sources consulted:

Lecture Notes

Solution:

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Q3 The Uncertainty Principle

10 Points

Let $X, Y \in \mathbb{C}^{d \times d}$ be observables; i.e., Hermitian matrices.

(a) Prove that X^2 and Y^2 are Hermitian.

(b) Prove that XY is Hermitian if and only if X and Y commute (i.e., $XY = YX$).

(c) Let $]X, Y[$ denote $XY + YX$ (this is nonstandard notation). Prove that $\frac{1}{2}]X, Y[$ is Hermitian.

(d) Let $[X, Y]$ denote the matrix $XY - YX$, called the "commutator" of X and Y because it's 0 if and only if X and Y commute (this is standard notation). Prove that $\frac{1}{2i}[X, Y]$ is Hermitian.

(e) Prove that $XY = \frac{1}{2}]X, Y[+ i \cdot \frac{1}{2i}[X, Y]$.

(f) In 1927, Werner Heisenberg stated his famous Uncertainty Principle for two particular observables of a quantum particle, its "position" and "momentum." In 1928, Earle Kennard properly mathematically proved Heisenberg's Uncertainty Principle. In 1929, Bob Robertson generalized the Uncertainty Principle to a statement about any two observables. Specifically, he proved the following:

$$\sigma_\rho[X] \cdot \sigma_\rho[Y] \geq \left| E_\rho \left[\frac{1}{2i}[X, Y] \right] \right|,$$

where $\sigma_\rho[X] = \sqrt{\operatorname{Var}_\rho[X]}$ is the standard deviation of the observable X (and similarly for $\sigma_\rho[Y]$).

Show that if we want to establish Robertson's Uncertainty Principle, we can reduce to the case that $E_\rho[X] = E_\rho[Y] = 0$.

(g) Having made this reduction, prove the Uncertainty Principle.
Hint: Use the Cauchy–Schwarz inequality: For $X, Y \in \mathbb{C}^{d \times d}$,

$$|\operatorname{Cov}_\rho[X, Y]|^2 \leq \operatorname{Var}_\rho[X] \operatorname{Var}_\rho[Y].$$

You do not need to justify the Cauchy–Schwarz inequality.

Sources consulted:

Lecture Notes

Solution:

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Q4 The SWAP test

10 Points

We've previously discussed the SWAP gate operating on two qubits, but it also makes sense as an operator on two qudits. In general, a two-qudit state looks like

$$|\psi\rangle = \sum_{i,j=1}^{d} \alpha_{ij} |i\rangle \otimes |j\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d.$$

The SWAP operator is the linear transformation defined by

$$\mathrm{SWAP}|\psi\rangle = \sum_{i,j=1}^{d} \alpha_{ij} |j\rangle \otimes |i\rangle.$$

(a) Explicitly write the matrix for SWAP in the case of $d = 3$. Label the rows and columns using a natural order like

$$|11\rangle, |12\rangle, |13\rangle, |21\rangle, \ldots, |33\rangle.$$

(b) We're used to SWAP being a quantum gate and thus unitary. Prove that SWAP is also a Hermitian matrix, hence a valid observable for density matrices ϱ on $\mathbb{C}^d \otimes \mathbb{C}^d$.

(c) Suppose $|u_1\rangle, \ldots, |u_d\rangle$ is any orthonormal basis for $\mathbb{C}^d$. This means that the set of all vectors $|u_i\rangle \otimes |u_j\rangle$ $(1 \leq i, j \leq d)$ is an orthonormal basis for $\mathbb{C}^{d^2}$. Show that SWAP is "basis-independent" in the sense that

$$\mathrm{SWAP} \sum_{i,j=1}^{d} \beta_{ij} |u_i\rangle \otimes |u_j\rangle = \sum_{i,j=1}^{d} \beta_{ij} |u_j\rangle \otimes |u_i\rangle.$$

(d) Suppose you have some quantum apparatus that produces a d-dimensional particle in a mixed state with density matrix $\rho \in \mathbb{C}^{d \times d}$. Write the eigenvalues of ρ as $\lambda_1, \ldots, \lambda_d$, with associated eigenvectors $|u_1\rangle, \ldots, |u_d\rangle$. Let $\varrho = \rho \otimes \rho$, which is the d^2-dimensional density matrix corresponding to the state you get if you run your quantum apparatus two times independently and then treat the two particles as a joint system. Prove that

$$\mathbb{E}_{\varrho}[\mathrm{SWAP}] = \sum_{i=1}^{d} \lambda_i^2.$$

(e) The quantity $\sum_{i=1}^{d} \lambda_i^2$ is called the purity of the mixed state ρ. Show that the maximum possible value of the purity is 1 and it occurs when ρ is a pure state. Show also that the minimum possible value of the purity is $1/d$, and it occurs when ρ is the maximally mixed state.

Sources consulted:

Lecture Notes

Solution:

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Q5 Zero-error state discrimination

10 Points

Consider the following task. There were two fixed qubit states $|u\rangle, |v\rangle \in \mathbb{R}^2$ which we assumed had real amplitudes for simplicity. We were given access to an unknown qubit state $|\psi\rangle \in \mathbb{R}^2$ (with real amplitudes) and were promised that either $|\psi\rangle = |u\rangle$ or $|\psi\rangle = |v\rangle$. Our goal was to try to guess which is the case.

In Lecture 4 we saw the optimal algorithm allowing for "two-sided error," and the optimal algorithm allowing for "one-sided error." We also saw a natural "zero-sided error" algorithm, but observed that it couldn't be optimal. In this problem we will see the optimal zero-sided error algorithm (though we won't prove its optimality). Assume henceforth that the angle between $|u\rangle$ and $|v\rangle$ is $0 < \theta < \frac{\pi}{2}$. Also, write $|u^{\perp}\rangle$ for a unit vector perpendicular to $|u\rangle$, and $|v^{\perp}\rangle$ for a unit vector perpendicular to $|v\rangle$.

(a) Let $\Pi_1 = |u^{\perp}\rangle\langle u^{\perp}|$, the linear operator on $\mathbb{R}^2$ that projects onto the $|u^{\perp}\rangle$ vector. Show that $\Pi_1 = I - |u\rangle\langle u|$ (where I denotes the 2×2 identity matrix) and that this is a positive operator. We'll similarly let $\Pi_2 = |v^{\perp}\rangle\langle v^{\perp}|$.

(b) The idea of the algorithm is to define $E_1 = \frac{1}{c}\Pi_1$ and $E_2 = \frac{1}{c}\Pi_2$, where c is a positive scalar that is just large enough such that $E_0 = I - E_1 - E_2$ is a positive operator. Having done this, $\{E_0, E_1, E_2\}$ becomes a valid POVM. Suppose we then measure the unknown state $\rho = |\psi\rangle\langle\psi|$ with this POVM. Show that when $|\psi\rangle = |u\rangle$, the probability of outcome 1 is 0, and similarly when $|\psi\rangle = |v\rangle$, the probability of outcome 2 is 0.

(c) In light of the previous problem, we see that if we get outcome 1 we can safely guess $|\psi\rangle = |v\rangle$, and if we get outcome 2 we can safely guess $|\psi\rangle = |u\rangle$. If we get outcome 0, we will guess "don't know." Our goal, therefore, is to minimize the probability of getting outcome 0. Show that this probability is $1 - \frac{1-\cos^2\theta}{c}$.

(d) In light of the previous problem, we clearly want c to be as small as possible. As mentioned, we have the restriction that E_0 must be a positive operator. Show that if $|w\rangle \in \mathbb{R}^2$ is any unit vector, $\langle w|E_0|w\rangle = 1 - \frac{\sin^2\theta_1 + \sin^2\theta_2}{c}$, where θ_1 is the angle from $|u\rangle$ to $|w\rangle$ and θ_2 is the angle from $|w\rangle$ to $|v\rangle$. We have the restriction $\theta_1 + \theta_2 = \theta$. Hence the least possible c for which E_0 is positive is the least c such that $1 - \frac{\sin^2\theta_1 + \sin^2\theta_2}{c} \geq 0$ whenever $\theta_1 + \theta_2 = \theta$. Show that this least c is $c = 1 + \cos\theta$.

(e) Deduce that there is a zero-sided error qubit discrimination algorithm with failure probability $\cos\theta$.

Sources consulted:

Lecture Notes

Solution:

HW6_Q5.pdf

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