Multi-qubit systems

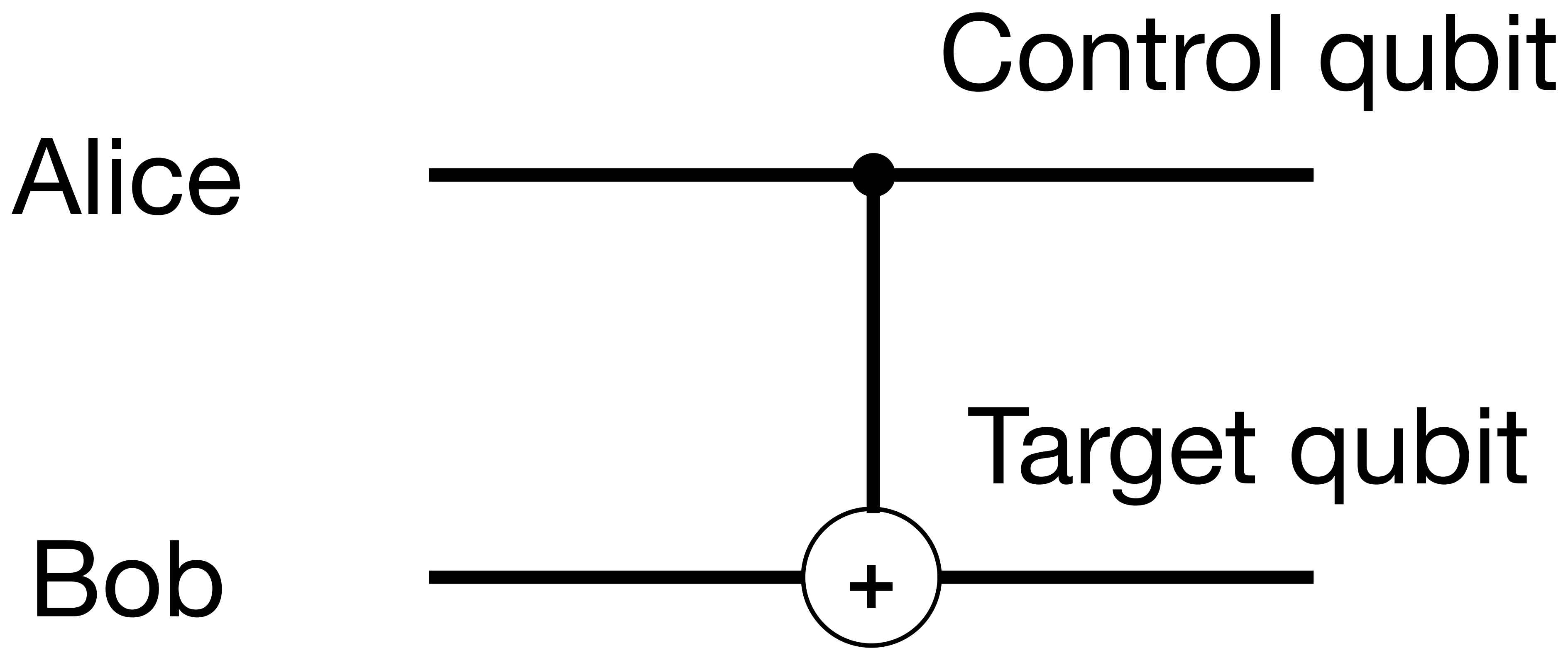
Three questions

- Alice $|\psi\rangle$, Bob $|\varphi\rangle$
- Question 1: What is the joint 4-dimensional state? Tensor product
- Question 2: How will the joint state change if Bob applies unitary U to his qubit?
- Question 3: What is the readout if Alice measures her qubit?

CNOT gate

A 4-dimensional unitary transformation

- Joint state $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$



- If the control qubit is 0, do nothing; if the control qubit is 1, apply NOT to target qubit.

- In other words, $|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |01\rangle, |10\rangle \mapsto |11\rangle, |11\rangle \mapsto |10\rangle$

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A 4-dimensional unitary transformation

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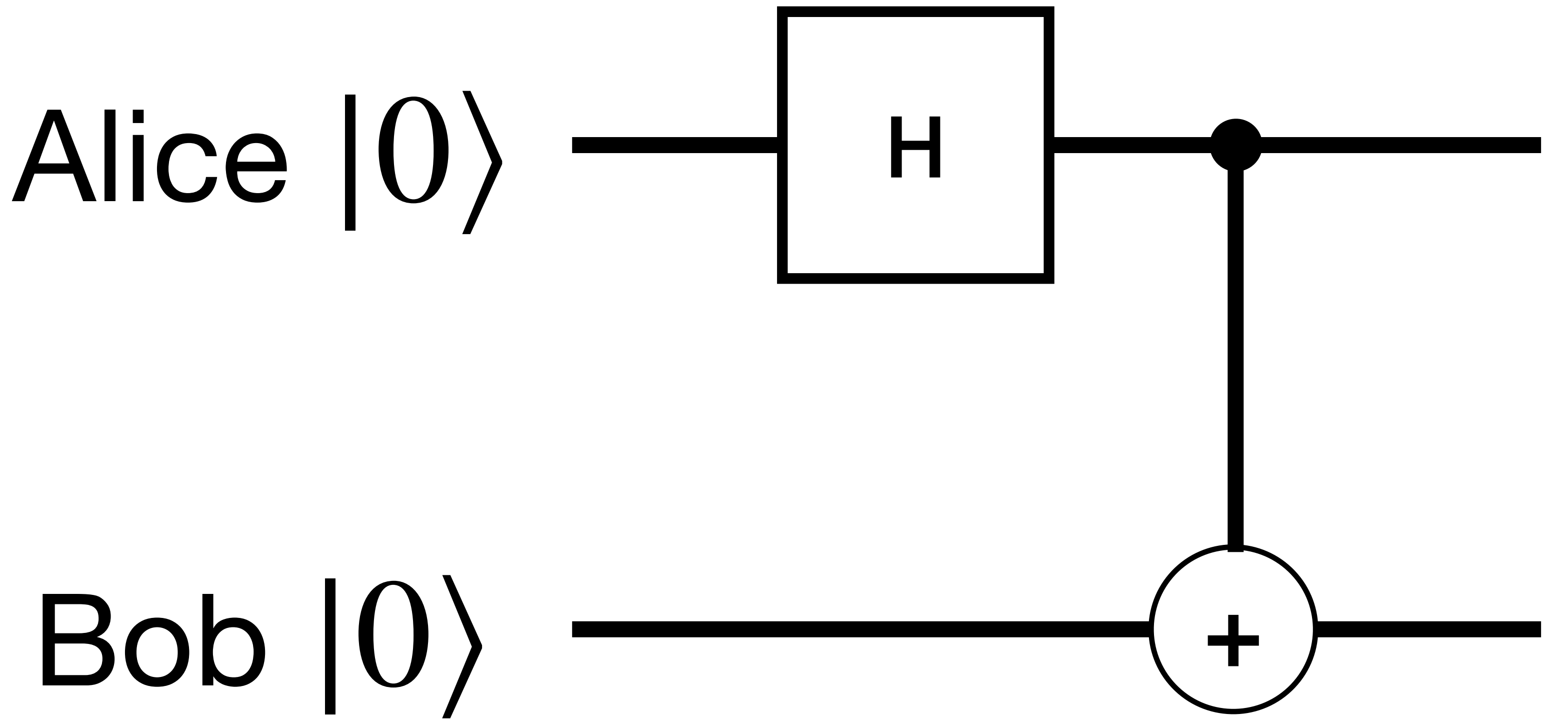
- In other words, $|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |01\rangle, |10\rangle \mapsto |11\rangle, |11\rangle \mapsto |10\rangle$
- CNOT = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- CNOT$(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle) = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|11\rangle + \alpha_{11}|10\rangle$

Hadamard followed by CNOT

What's the joint state?



Bell state / EPR pair

Einstein, Podolsky, Rosen

- $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Definition Joint state is unentangled if it is of the form $|\psi\rangle \otimes |\varphi\rangle$. It is entangled if it is not unentangled.

- Are the following joint states entangled?

  - $|0\rangle \otimes |0\rangle$ (unentangled)

  - $|+\rangle \otimes |0\rangle$ (unentangled)

  - EPR (entangled, proof by contradiction)

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Transform partially

- Alice and Bob share an EPR pair
- Bob applies NOT to his photon
- New state = $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$
- Alice applies H to her photon
- New state = $\frac{1}{\sqrt{2}}| + \rangle \otimes |1\rangle + \frac{1}{\sqrt{2}}| - \rangle \otimes |0\rangle$

In general…

Joint state under partial transformation

- $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- Apply $U = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ to the second qubit
- Overall transformation $I \otimes U$
- More generally, if U is applied to the first qubit, and then V to second qubit, then the overall transformation is $(I \otimes V)(U \otimes I) = U \otimes V$.
- If V is applied to second qubit, and then U is applied to the first qubit, then the overall transformation is the same.

Universal gates for quantum circuits

The power of CNOT

- Classical computation: NAND and NOR gates are universal for classical circuits

- Quantum computation: every $2^n \times 2^n$ unitary transformation on n qubits can be implemented by a quantum circuit of 1-qubit unitary gates and CNOTs

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Measurement

Applied partially

- $|\varphi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- If a measurement is carried out on both qubits, then the readout is $|00\rangle$ with probability $|\alpha_{00}|^2$, $|01\rangle$ with probability $|\alpha_{01}|^2$, $|10\rangle$ with probability $|\alpha_{10}|^2$, and $|11\rangle$ with probability $|\alpha_{11}|^2$.
- If a measurement is carried out on the first qubit, then the readout is $|0\rangle$ with probability $p_0 = |\alpha_{00}|^2 + |\alpha_{01}|^2$, and $|1\rangle$ with probability $p_1 = |\alpha_{10}|^2 + |\alpha_{11}|^2$
- If the readout is $|0\rangle$, then the joint state collapses to $|0\rangle \otimes (\alpha_{00}|0\rangle + \alpha_{01}|1\rangle)/\sqrt{p_0}$
- If the readout is $|1\rangle$, then the joint state collapses to $|1\rangle \otimes (\alpha_{10}|0\rangle + \alpha_{11}|1\rangle)/\sqrt{p_1}$
- The joint state is unentangled after the measurement.

Measurement

Applied partially

- Example 1
  - Alice and Bob share an EPR pair.
  - What happens if Alice measures her qubit in the $|0\rangle, |1\rangle$ basis?
- Example 2
  - Alice and Bob share an EPR pair.
  - What happens if Alice measures her qubit in the $|+\rangle, |-\rangle$ basis?
- Example 3
  - 3-qubit system: $|\varphi\rangle = \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \ldots + \alpha_{111}|111\rangle$
  - What happens if we measure the first two qubits?

Mixed states

Probability distribution over quantum states

- The “state” after Alice’s measurement is a probability distribution over quantum states, called a mixed state.
- We have seen mixed states for 1-qubit system.
- $|\varphi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- How to express the “state” of $|\varphi\rangle$ after a measurement is carried out on the first qubit?
- The mixed state is
  - $|0\rangle \otimes (\alpha_{00}|0\rangle + \alpha_{01}|1\rangle)/\sqrt{p_0}$ with probability $p_0 = |\alpha_{00}|^2 + |\alpha_{01}|^2$
  - $|1\rangle \otimes (\alpha_{10}|0\rangle + \alpha_{11}|1\rangle)/\sqrt{p_1}$ with probability $p_1 = |\alpha_{10}|^2 + |\alpha_{11}|^2$