Collaborators : None

Sources : Lecture Notes

Q3) The Uncertainty Principle

Let $X, Y \in \mathbb{C}^{d \times d}$ be observables, i.e Hermitian matrices.

$\Rightarrow X^\dagger = X \; ; \; Y^\dagger = Y$

(a) $(X^2)^\dagger = (X \cdot X)^\dagger = X^\dagger \cdot X^\dagger = X \cdot X = X^2$ $\quad [\because (BA)^\dagger = A^\dagger \cdot B^\dagger \; ; \; X^\dagger = X]$

$(Y^2)^\dagger = (Y \cdot Y)^\dagger = Y^\dagger \cdot Y^\dagger = Y \cdot Y = Y^2$ $\quad [\because Y^\dagger = Y]$

$\boxed{\therefore X^2 \;\&\; Y^2 \text{ are Hermitian matrices}}$

(b) Case-1 ($\Rightarrow$): If $XY = YX$ (Commute), then

$(XY)^\dagger = Y^\dagger \cdot X^\dagger = YX = XY$ (Satisfies Hermitian property)

Case-2 ($\Leftarrow$): If XY is Hermitian, i.e, $(XY)^\dagger = XY$

$\Rightarrow Y^\dagger \cdot X^\dagger = XY$

$\Rightarrow Y \cdot X = XY$ (Satisfies Commutation property)

$$\boxed{\text{From Case-1 \& Case-2: } \underbrace{\{XY = YX\}}_{\text{(Commutation)}} \Leftrightarrow \underbrace{\{(XY)^\dagger = XY\}}_{\text{(Hermitian)}} \text{ if } X, Y \text{ are hermitian}}$$

(c) Let $]X,Y[$ denote $XY + YX$, then

$$\left(\tfrac{1}{2}]X,Y[\right)^\dagger = \left(\tfrac{1}{2}(XY+YX)\right)^\dagger = \left(\tfrac{1}{2}\left((XY)^\dagger + (YX)^\dagger\right)\right) = \tfrac{1}{2}\left(Y^\dagger X^\dagger + X^\dagger Y^\dagger\right)$$

$$= \tfrac{1}{2}(YX + XY) = \tfrac{1}{2}(XY + YX) = \left(\tfrac{1}{2}]X,Y[\right)$$

$\boxed{\therefore \left(\tfrac{1}{2}]X,Y[\right) \text{ is Hermitian}}$

(d) Let $[X;Y]$ denote $(XY - YX)$, then

$$\left(\tfrac{1}{2i}[X,Y]\right)^\dagger = \left(\tfrac{1}{2i}\right)^\dagger (XY - YX)^\dagger = \left(\tfrac{-1}{2i}\right)\left((XY)^\dagger - (YX)^\dagger\right) = \left(\tfrac{-1}{2i}\right)\left(Y^\dagger X^\dagger - X^\dagger Y^\dagger\right)$$

$$= \left(\tfrac{-1}{2i}\right)(YX - XY) = \left(\tfrac{1}{2i}\right)(XY - YX) = \left(\tfrac{1}{2i}[X,Y]\right)$$

$$\boxed{\therefore \ \frac{1}{2i}[X,Y] \text{ is Hermitian}}$$

(e) $\frac{1}{2}\,]X,Y[+ i\cdot\frac{1}{2i}[X,Y] = \frac{1}{2}\left(]X,Y[+ [X,Y]\right) = \frac{1}{2}\left((XY+\cancel{YX})+(XY-\cancel{YX})\right)$

$$= XY$$

$$\boxed{\therefore \ XY = \frac{1}{2}\,]X,Y[+ i\cdot\frac{1}{2i}[X,Y]}$$

(f) Let $X' = X - E_\rho[X]\,I$ & $Y' = Y - E_\rho[Y]\,I$.

$\rightarrow E_\rho[X'] = E_\rho[X] - E_\rho[X](1) = 0$; $E_\rho[Y'] = 0$

$\sigma_\rho[X'] = \sqrt{E_\rho[(X-E_\rho[X]I)^\dagger(X-E_\rho[X]I-0)} = \sigma_\rho[X]$; Similarly $\sigma_\rho[Y'] = \sigma_\rho[Y] \rightarrow$ ①

$[X,Y] = XY - YX = (X' + E_\rho[X]I)(Y' + E_\rho[Y]I) - (Y' + E_\rho[Y]I)(X' + E_\rho[X]I)$

$= \left(X'Y' + E_\rho[X]\,Y' + E_\rho[Y]\,X' + E_\rho[X]E_\rho[Y]\,I\right) - \left(Y'X' + E_\rho[Y]\,X' + E_\rho[X]\,Y' + E_\rho[X]E_\rho[Y]\,I\right)$

$= X'Y' - Y'X' = [X',Y'] \rightarrow$ ②

From ① & ②,

$$\left(\sigma_\rho[X]\cdot\sigma_\rho[Y] \geq \left|E_\rho\left[\frac{1}{2i}[X,Y]\right]\right|\right) \Leftrightarrow \left(\sigma_\rho[X']\cdot\sigma_\rho[Y'] \geq \left|E_\rho\left[\frac{1}{2i}[X',Y']\right]\right| \ \& \ E_\rho[X'] = E_\rho[Y'] = 0\right)$$

Therefore without loss of generality, proving inequality for (X', Y') where their E_ρ values are 0, means proving the inequality for (X,Y).

(g) Using Cauchy-Schwarz inequality:

$$|Cov_\rho[X,Y]|^2 \leq Var_\rho[X]\,Var_\rho[Y]. \rightarrow ①$$

$|Cov_\rho[X,Y]|^2 = |E_\rho[X^\dagger Y] - (E_\rho[X])^* E_\rho[Y]|^2$ [From Q2]

$= |E_\rho[XY] - 0|^2$ [$\because X$ is Hermitian; $E_\rho[X] = E_\rho[Y] = 0$ (considered)]

$= |E_\rho[XY]|^2 = \left|E_\rho\left[\frac{1}{2}\,]X,Y[+ \frac{1}{2i}(i\,[X,Y])\right]\right|^2$

$= \left|\frac{1}{2}E_\rho[\,]X,Y[\,] + \frac{1}{2i}E_\rho\left[i[X,Y]\right]\right|^2 \rightarrow$ ②

$$E_\rho\left[\,]X,Y[\,\right]^* = tr\left(\rho(]X,Y[)\right)^* = tr\left((]X,Y[)\rho\right)^* \quad \left[\because tr(AB) = tr(BA)\right]$$

$$= tr\left(\left[(]X,Y[)\rho\right]^\dagger\right) = tr\left(\rho^\dagger(]X,Y[)^\dagger\right) = tr\left(\rho(]X,Y[)\right)$$

$$= E_\rho\left[\,]X,Y[\,\right]$$

$$\Rightarrow \therefore E_\rho\left[\,]X,Y[\,\right] \text{ is Real} \rightarrow (3)$$

$$E_\rho\left[i[X,Y]\right]^* = tr\left(\rho(i[X,Y])\right)^* = tr\left((i[X,Y])\rho\right)^* ;$$

$$= tr\left(\left[(i[X,Y])\rho\right]^\dagger\right) = tr\left(\rho^\dagger(i[X,Y])^\dagger\right) = tr\left(\rho(i[X,Y])\right)$$

$$\therefore E_\rho\left[i[X,Y]\right] \text{ is Real} \rightarrow (4)$$

Using (3) & (4) in (2),

$$\left|Cov_\rho[X,Y]\right|^2 = \left|\frac{1}{2}E_\rho\left[\,]X,Y[\,\right] + \frac{1}{2i}E_\rho\left[i[X,Y]\right]\right|^2 \geq \left|\frac{1}{2i}\right|^2\left|E_\rho\left[i[X,Y]\right]\right|^2$$

$$\geq \left|E_\rho\left[\frac{1}{2i}[X,Y]\right]\right|^2 \longrightarrow (5)$$

Using (1) & (5),

$$Var_\rho[X]\,Var_\rho[Y] \geq \left|Cov_\rho[X,Y]\right|^2 \geq \left|E_\rho\left[\frac{1}{2i}[X,Y]\right]\right|^2$$

$$\Rightarrow \boxed{\sigma_\rho[X]\cdot\sigma_\rho[Y] \geq \left|E_\rho\left[\frac{1}{2i}[X,Y]\right]\right|}$$