Collaborators : None

Sources : Lecture Notes

Q2 Projectors and reflections

Let $|\psi\rangle$ and $|\phi\rangle$ be two unit vectors in C^d. We will be interested in $Q = |\phi\rangle\langle\psi|$, which is a $d \times d$ matrix, and can therefore be thought of as a transformation on d –dimensional vectors.

(a) Explicitly work out the matrix Q in the case $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |+\rangle$, and also in the opposite case $|\psi\rangle = |+\rangle$ and $|\phi\rangle = |0\rangle$.

Ans: $|\psi\rangle = |0\rangle = [1,0]^T$; $|\phi\rangle = |+\rangle = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$

$$|\phi\rangle\langle\psi| = [1,0]^T\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; 0,0\right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$$

(b) What does the transformation Q map the vector $|\psi\rangle$ to, and what does it map every vector orthogonal to $|\psi\rangle$ to?

Ans: As $Q = |\phi\rangle\langle\psi|$, mapping vector $|\psi\rangle$ after transformation Q, the resultant vector

$= (|\phi\rangle\langle\psi|)\,|\psi\rangle = |\phi\rangle(\langle\psi||\psi\rangle)$ $(\because \textit{Associtivity of tensors multiplication})$

$= |\phi\rangle(1) = |\phi\rangle$ $(\because \textit{unit} - \textit{vector dot} - \textit{product with itself gives } 1 \textit{ in real space.})$

$(\ Or\ \ |\phi\rangle * k,\ \ if\ \ |\psi\rangle \textit{ lies in non} - \textit{real space.})$

Similarly, mapping orthogonal vector $|\psi'\rangle$ after transformation Q, the resultant vector

$= (|\phi\rangle\langle\psi|)\,|\psi'\rangle = |\phi\rangle(\langle\psi||\psi'\rangle)$ $(\because \textit{Associtivity of tensors multiplication})$

$= |\phi\rangle(0) = [0,0,..(d - times)]^T$ $(\because \textit{unit} - \textit{vector dot} -$
$\textit{product with its orthogonal vector gives } 0.)$

(c) Suppose now that $|\psi\rangle = |\phi\rangle$. Let $P = |\psi\rangle\langle\psi|$. Describe in (geometric) words the transformation P.

Ans: Let $|\psi\rangle = |\phi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ $s.t.\ \ |a|^2 + b|^2 = 1$

$$P = | \psi\rangle\langle\psi | = \begin{bmatrix} a \\ b \end{bmatrix} [a\ b] = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

Let $| \chi\rangle = \begin{bmatrix} c \\ d \end{bmatrix} \Rightarrow P | \chi\rangle = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a^2c + abd \\ abc + b^2d \end{bmatrix} = (ac + bd) \begin{bmatrix} a \\ b \end{bmatrix} = (ac + bd) | \psi\rangle$

Similarly for n-dimensional vector, $P | \chi\rangle = | \psi\rangle(\langle\psi || \chi\rangle) = (\langle\psi || \chi\rangle) * | \psi\rangle$. Here $(\langle\psi || \chi\rangle)$ is a constant.

So, $P = |\psi\rangle\langle\psi|$ is a projection operator. Geometrically, it projects any vector onto the one-dimensional subspace spanned by $|\psi\rangle$.

(d) Let I denote the identity matrix in R^d. Describe in (geometric) words the transformation I−2P. Your description should include the words “hyperplane perpendicular to”. Prove that this transformation is unitary.

Ans: $(I - 2P) | \chi\rangle = I | \chi\rangle - 2(P | \chi\rangle) = | \chi\rangle - 2(\langle\psi || \chi\rangle) * | \psi\rangle$

The resultant vector is a negative of reflection of $| \chi\rangle$ with respect to subspace spanned by $|\psi\rangle$.

To prove it's unitary, we need to show $(I - 2P)(I - 2P)^\dagger = I$:

$$(I - 2P)(I - 2P)^\dagger = (I - 2P)(I - 2P) = I - 4P + 4P^2$$

$$(\because (I - 2P)^\dagger = I^\dagger - 2P^\dagger = I - 2P \quad because \ (| \psi\rangle\langle\psi |)^\dagger = | \psi\rangle\langle\psi| \quad)$$

$I - 4P + 4P^2 = I - 4P + 4P = I$ $(\because P^2 = | \psi\rangle\langle\psi| * | \psi\rangle\langle\psi| = | \psi\rangle\langle\psi| = P$

Thus, $(I - 2P)$ is unitary transformation.

(e) Suppose we are interested in the change-of-(orthonormal-)basis operation U that takes the orthonormal basis $| \psi_1\rangle, \ldots, | \psi_d\rangle$ to the orthonormal basis $| \phi_1\rangle, \ldots, | \phi_d\rangle$. Show that U can be written as $U = | \phi_1\rangle\langle\psi_1 | + \cdots + | \phi_d\rangle\langle\psi_d |$.

Ans: we want to show that the change-of-basis operation UUU, which takes the orthonormal basis $| \psi_1\rangle, \ldots, | \psi_d\rangle$ to the orthonormal basis $| \phi_1\rangle, \ldots, | \phi_d\rangle$, can be written as:

$$U = \sum_{i=1}^{d} | \phi_i\rangle\langle\psi_i |$$

Let v be any vector in C^d. We can expand v in the basis $\langle\psi_i |$:

$$v = \sum_{i=1}^{d} \langle \psi_i \mid v \rangle \mid \psi_i \rangle$$

Applying U to v, we get:

$$Uv = \sum_{i=1}^{d} \langle \psi_i \mid v \rangle U \mid \psi_i \rangle$$

$$= \sum_{i=1}^{d} \langle \psi_i \mid v \rangle \mid \phi_i \rangle$$

$$= \left(\sum_{i=1}^{d} \mid \phi_i \rangle \langle \psi_i \mid \right) \mid v \rangle$$

$$Thus, \qquad U = \left(\sum_{i=1}^{d} \mid \phi_i \rangle \langle \psi_i \mid \right)$$