Midterm 1

● Graded

Student

Sujith Potineni

Total Points

19 / 20 pts

Question 1

(no title)

4 / 5 pts

+ **5 pts** Correct

✔ + **1 pt** (a) Correct

+ **0.5 pts** (b) Got 13 for the inner product, but did not take the square root when normalizing.

✔ + **1 pt** (b) Correct

+ **0.5 pts** (c) Laid out the three goals for orthonormality, but failed to compute the relevant inner products.

✔ + **0.5 pts** (c) Checked the orthogonality, but did not check the unity

+ **1 pt** (c) Checked the unity, but did not check orthogonality.

+ **1.5 pts** (c) Correct

+ **0.5 pts** (d) Write ket 0 as linear combination of ket i and ket -i, but failed to rewrite ket 1 correctly.

+ **0.5 pts** (d) Expressed the coefficients as inner products, but failed to compute the inner products.

+ **1 pt** (d) Used the wrong inner product formula for the coefficients. The rest of the computation is right. Resulting in conjugated coefficients.

✔ + **1.5 pts** (d) Correct (even in case the wrong ket psi was used)

+ **0 pts** Incorrect

Question 2

(no title)

5 / 5 pts

✔ + **5 pts** Correct

+ **2.5 pts** (a) Correct

+ **2.5 pts** (b) Correct

+ **0 pts** Incorrect

Question 3

(no title)

5 / 5 pts

✔ **+ 5 pts** Correct

+ 1 pt (a) Answer correctly without the right justification

+ 2.5 pts (a) Correct

+ 1 pt (b) Answer correctly without the right justification

+ 2.5 pts (b) Correct

+ 0 pts Incorrect

Question 4

(no title)

5 / 5 pts

✔ **+ 5 pts** Correct

+ 2 pts Attempted to build a 2-qubit circuit using CNOT gates

+ 0 pts Incorrect

CSE 598 Quantum Computation
Midterm #1, Fall 2024

Instructor: Zilin Jiang

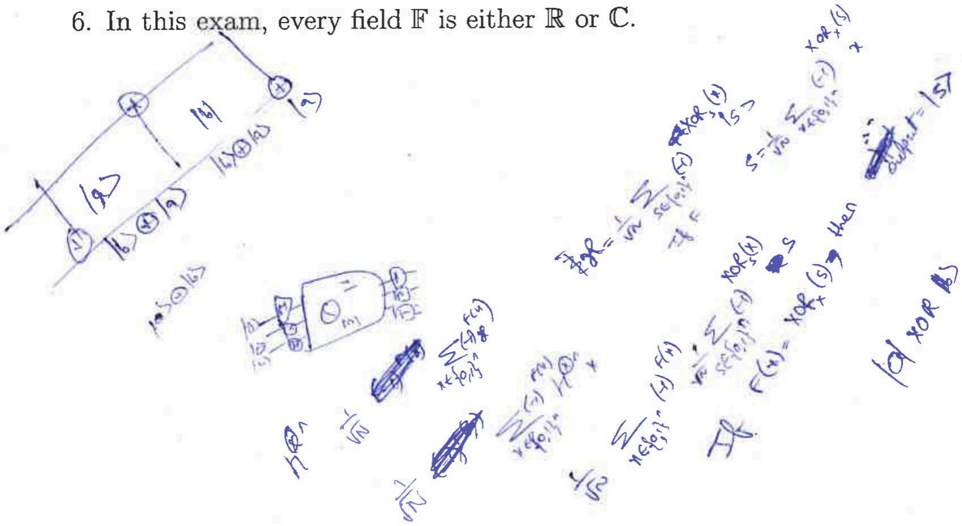
September 24, 2024

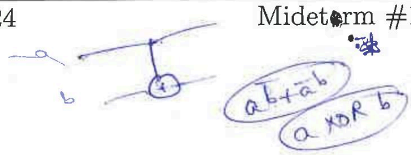
Full name: Bala Sujith Potineni

Time: 75 minutes. **Four problems** worth 5 points each.

Instructions

1. Closed book. No notes or any electronic devices during the exam.
2. You must provide justification in your solutions (not just answers).
3. You may quote theorems and facts proved in class, course textbook/notes, or homework, provided that you state the facts that you are using.
4. This exam will be scanned before grading, so please ensure your writing is clear and legible.
5. If you have questions, raise your hand. The proctor will come to you. Do not ask out loud.
6. In this exam, every field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.





"About your cat, Mr. Schrödinger — I have good news and bad news."

Problem 1

(a) Let $|\psi\rangle = \frac{|0\rangle+2|1\rangle}{\sqrt{5}}$ and $|\phi\rangle = \frac{2i|0\rangle+3|1\rangle}{\sqrt{13}}$. What's $\langle\psi|\phi\rangle$?

(b) Usually quantum states are normalized: $\langle\psi|\psi\rangle = 1$. The state $|\phi\rangle = 2i|0\rangle - 3i|1\rangle$ is not normalized. What constant A makes $|\psi\rangle = \frac{|\phi\rangle}{A}$ a normalized state?

(c) Define $|i\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ and $|-i\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$. Show explicitly that the vectors $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$.

(d) Write the normalized vector $|\psi\rangle$ from part b in the $|i\rangle, |-i\rangle$ basis.

a) As $\underbrace{|\psi\rangle = \frac{|0\rangle+2|1\rangle}{\sqrt{5}}}_{\text{(column vector)}} \Rightarrow \underbrace{\langle\psi| = \overline{\frac{|0\rangle+2|1\rangle}{\sqrt{5}}}}_{\text{(row vector)}} = \frac{|0\rangle+2|1\rangle}{\sqrt{5}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

$\Rightarrow \langle\psi|\phi\rangle = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2i}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$ (in basis $\{|0\rangle, |1\rangle\}$)

$= \frac{2i}{\sqrt{65}} + \frac{6}{\sqrt{65}} = \frac{2i+6}{\sqrt{65}}$

b) As $|\psi\rangle = \frac{|\phi\rangle}{A} = \frac{1}{A}\left[2i|0\rangle - 3i|1\rangle\right] \Rightarrow \langle\psi| = \frac{1}{A}\left[\overline{2i} \quad \overline{-3i}\right] = \frac{1}{A}\left[-2i \quad 3i\right]$

$= \frac{1}{A}\begin{bmatrix} 2i \\ -3i \end{bmatrix}$

$\Rightarrow \langle\psi|\psi\rangle = \frac{1}{A}\left[-2i \quad 3i\right]\frac{1}{A}\begin{bmatrix} 2i \\ -3i \end{bmatrix} = \frac{1}{A^2}(4+9) = \frac{13}{A^2}$ → This should be 1.

$\Rightarrow \frac{13}{A^2} = 1 \Rightarrow A = \pm\sqrt{13} \Rightarrow |\psi\rangle$ → $\left[\frac{2i|0\rangle - 3i|1\rangle}{\sqrt{13}}\right]$ (or) $\left(\frac{-2i|0\rangle + 3i|1\rangle}{\sqrt{13}}\right)$

c) $\langle i||-i\rangle = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}}_{\text{after applying conjugate \& transpose}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{2} - \frac{1}{2}\right) = 0$

As $\langle i|-i\rangle = 0 \Rightarrow$ vectors $|i\rangle$ & $|-i\rangle$ are orthonormal.

Thus they form an orthonormal basis for $\mathbb{C}^2$, as there should be exactly 2 basis vectors which are orthonormal to each other for a group in $\mathbb{C}^2$.

d) Let $|\psi\rangle = \frac{2i|0\rangle - 3i|1\rangle}{\sqrt{13}}$.

As $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ & $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$ $\Rightarrow$ $|0\rangle = \frac{|i\rangle + |-i\rangle}{\sqrt{2}}$; $|1\rangle = \frac{|i\rangle - |-i\rangle}{\sqrt{2}i}$

$$\Rightarrow |\psi\rangle = \frac{2i\left(\frac{|i\rangle + |-i\rangle}{\sqrt{2}}\right) - 3i\left(\frac{|i\rangle - |-i\rangle}{\sqrt{2}i}\right)}{\sqrt{13}} = \frac{\left(\sqrt{2}i - \frac{3}{\sqrt{2}}\right)|i\rangle + \left(\sqrt{2}i + \frac{3}{\sqrt{2}}\right)|-i\rangle}{\sqrt{13}}$$

$$= \frac{(2i-3)|i\rangle + (2i+3)|-i\rangle}{\sqrt{26}}$$

Problem 2 For the following circuits, calculate the output state before the measurement. Then calculate the measurement probabilities in the specified basis. Here we use:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = iXZ, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

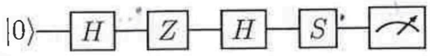
$\begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{bmatrix}$

$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

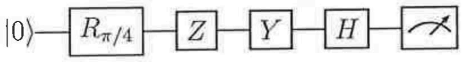
(Rotation)

$Y = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

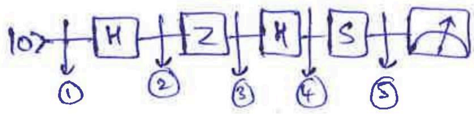
(a) Measure in the $\{|0\rangle, |1\rangle\}$ basis:



(b) Measure in the $\{|+\rangle, |-\rangle\}$ basis:



a)



At ①, $\Rightarrow |\phi\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

At ②, $\Rightarrow |\phi\rangle = |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

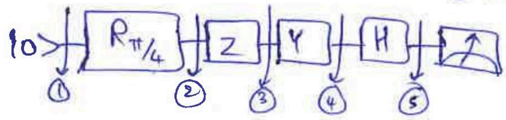
At ③, $\Rightarrow |\phi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

At ④, $\Rightarrow |\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(2)(|1\rangle) = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

At ⑤, $\Rightarrow |\phi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = \underline{i|1\rangle}$

After measurement, readout $= \begin{cases} |1\rangle \text{ with } 100\% \text{ probability.} \\ |0\rangle \text{ with } 0\% \text{ probability.} \end{cases}$

b)



At ①, $\Rightarrow |\phi\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

At ②, $\Rightarrow |\phi\rangle = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$

At ③, $\Rightarrow |\phi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = |-\rangle$

Extra space for Problem 2

At ④, $\Rightarrow |\phi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = \frac{i|0\rangle + i|1\rangle}{\sqrt{2}} = i|+\rangle$

At ⑤, $\Rightarrow |\phi\rangle = \frac{i|+\rangle + i|-\rangle}{\sqrt{2}}$ $\cancel{i|1\rangle}$ $= \frac{i}{\sqrt{2}}|+\rangle + \left(\frac{+i}{\sqrt{2}}\right)|-\rangle = i|0\rangle$

$p = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2} (= 50\%)$

~~After measurement, readout = { |0⟩ with 0% probability; |1⟩ with 1% probability~~

After measurement in $\{|+\rangle, |-\rangle\}$ basis, readout = $\begin{cases} |+\rangle & \text{with } 50\% \text{ probability} \\ |-\rangle & \text{with } 50\% \text{ probability} \end{cases}$

$\left(\frac{i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right)$ $\frac{-1}{2}$

Problem 3 Determine whether each of the following two qubit states are separable or entangled. If the state is separable, then provide its factorization into a pair of one qubit states. If the state is entangled, then explicitly prove that no factorization into one qubit states exists.

$$|\psi\rangle = \frac{|00\rangle + i|01\rangle + i|10\rangle - |11\rangle}{2}, \quad |\phi\rangle = \frac{3}{5}|01\rangle - \frac{4}{5}|10\rangle.$$

a) $|\psi\rangle = \dfrac{|00\rangle + i|01\rangle + i|10\rangle - |11\rangle}{2}$

Assume $|\psi\rangle$ is unentangled. Then it should be of the form

$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ where $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$

$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = \frac{|00\rangle}{2} + \frac{i|01\rangle}{2} + \frac{i|10\rangle}{2} - \frac{|11\rangle}{2}$

$\Rightarrow ac = \frac{1}{2}$; $ad = \frac{i}{2}$; $bc = \frac{i}{2}$; $bd = \frac{-1}{2}$

$abcd = \frac{-1}{4}$

$abcd = \frac{-1}{4}$ ✓

$\boxed{c = \frac{1}{2a}}$ $\boxed{d = \frac{i}{2a}} \Rightarrow b = ia$

Let $a = k$ (some complex constant). Then

$|\psi\rangle = (k|0\rangle + ik|1\rangle) \otimes \left(\frac{1}{2k}|0\rangle + \frac{i}{2k}|1\rangle\right)$

Here $|k|^2 + |ik|^2 = \left|\frac{1}{2k}\right|^2 + \left|\frac{i}{2k}\right|^2 = 1 \Rightarrow |k| = \frac{1}{\sqrt{2}}$

So $\boxed{|\psi\rangle = (k|0\rangle + ik|1\rangle) \otimes \left(\frac{1}{2k}|0\rangle + \frac{i}{2k}|1\rangle\right) \text{ where } |k| = \frac{1}{\sqrt{2}}}$

is unentangled.

b) $|\phi\rangle = \frac{3}{5}|01\rangle - \frac{4}{5}|10\rangle$

Applying similar method from (a),

$\Rightarrow ac = 0$; $ad = \frac{3}{5}$; $bc = \frac{-4}{5}$; $bd = 0$

$abcd = 0$

$abcd = \frac{-12}{25}$

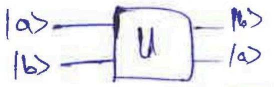
Dont match (Contradiction)

So, $|\phi\rangle$ is entangled.

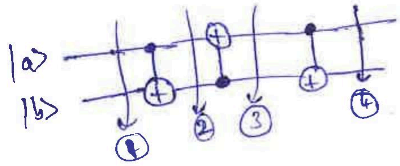
Extra space for Problem 3

Problem 4 The following question concerns a 2-qubit circuit. Using only CNOT gates, show how to build a SWAP gate.

If we have to consider only a 2-qubit circuit for SWAP gate but not a 3-qubit circuit of controlled SWAP gate. Then



~~Here first output bit = |b> = XOR01 |ab>.~~
~~(Here s = |01>)~~



Let $|a\rangle = p|0\rangle + q|1\rangle$ & $|b\rangle = r|0\rangle + s|1\rangle$

At ①, $|a\rangle \otimes |b\rangle = pr|00\rangle + ps|01\rangle + qr|10\rangle + qs|11\rangle$

At ②, $|\phi\rangle = pr|00\rangle + ps|01\rangle + qr|11\rangle + qs|10\rangle$

At ③, $|\phi\rangle = pr|00\rangle + ps|11\rangle + qr|01\rangle + qs|10\rangle$

~~r(p|0>+q|1>)⊗|0> + (p|1>+q ... s|1>⊗(p|1>+q|0>)~~
~~r|0>⊗(p|0>+q|1>) + s|1>⊗(p|1>+q|0>)~~

At ④, $|\phi\rangle = pr|00\rangle + ps|10\rangle + qr|01\rangle + qs|11\rangle$

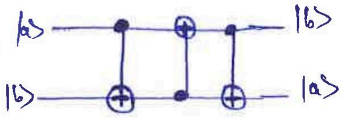
$= p(r|0\rangle + s|1\rangle) \otimes |0\rangle + q(r|0\rangle + s|1\rangle) \otimes |1\rangle$

$= (r|0\rangle + s|1\rangle) \otimes (p|0\rangle + q|1\rangle)$

$= |b\rangle \otimes |a\rangle$

$|a\rangle \otimes |b\rangle \longmapsto |b\rangle \otimes |a\rangle$

So, the circuit is



~~So, SWAP = CNOT·CNOT·CNOT = (CNOT)^3~~

Extra space for Problem 4