Quantum Probability

- Recap of last lecture:
  - Mixed state: "probability p_i of $|\Psi_i\rangle \in \mathbb{C}^d$", $i = 1,\ldots,m$
  - Encoded by density matrix $\rho \in \mathbb{C}^{d\times d}$

$$\rho = \sum_{i=1}^{m} p_i |\Psi_i\rangle\langle\Psi_i|$$

- Hermitian matrix ρ is a density matrix if and only if $\rho \succeq 0$ and $\mathrm{tr}(\rho) = 1$

Actions on mixed states

- Applying unitary transformation $U \in \mathbb{C}^{d \times d}$: $\rho \mapsto U\rho U^\dagger$
- Measuring in basis $|u_1\rangle, \ldots, |u_d\rangle$ get outcome $|u_i\rangle$ with probability $\langle u_i|\rho|u_i\rangle$
- Adjoining two states: if ρ is density matrix of a d-dimensional mixed state, and σ is density matrix of an e-dimensional mixed state, and we decide to call the whole system one de-dimensional state, its density matrix is $\rho \otimes \sigma$ (homework)

Linear algebra interlude

- Proposition: Let $A \in \mathbb{C}^{d \times d'}, B \in \mathbb{C}^{d' \times d}$. Then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- Classic trick: $\langle u_i | \rho | u_i \rangle = \mathrm{tr}(\langle u_i | \rho | u_i \rangle) = \mathrm{tr}(\rho | u_i \rangle \langle u_i |)$
- Also $\mathrm{tr}(A^\dagger B)$ = …
- Definition: $\langle A, B \rangle = \mathrm{tr}(A^\dagger B)$. Inner product for matrices.

Quantum Probability 101

- Classical Definition #1: A d-outcome probability distribution is a vector $p \in \mathbb{R}^d$ with $p \geq 0$ and $\sum_{i=1}^{d} p_i = 1$
- Quantum Definition #1: A d-dimensional sate / density matrix is a Hermitian $\rho \in \mathbb{C}^{d \times d}$ with $\rho \succeq 0$ and $\sum_{i=1}^{d} \rho_{ii} = 1$
- Think of ρ as a “source of quantum randomness”
- Classical -> quantum, quantum -> classical

Example

- Recall: if state ρ measured in $\{|u_1\rangle, \ldots, |u_d\rangle\}$ basis, probability of $|u_i\rangle$ = …
- Let $E_i = |u_i\rangle\langle u_i|$
- Remark: $E_i \succeq 0$ and $E_1 + \ldots + E_d = I_{d\times d}$
- Compare: Given probability distribution $p \in \mathbb{R}^d$, let e_i be the standard basis
    - Then $p_i = \langle p, e_i \rangle$
    - And $e_i \geq 0$ and $e_1 + \ldots + e_d = (1,1,\ldots,1)$

Probability 101 Events

- Given probability distribution $p \in \mathbb{R}^d$, let's define some "mutually exclusive, collectively exhaustive" events: Event$_1$, …, Event$_m$
- Naivest: $m = d$, Event$_i$ $= i$ is drawn from p
- Lumping: some outcomes grouped together
  - Example: Event$_1$ = the draw from p is odd, Event$_2$ = …even
  - Identify with "indicator vectors": …
  - Probability of Event$_1$ = $p_1 + p_3 + \ldots = \langle p, ? \rangle$, and Probability of Event$_2$ = ?
  - Note: each e_i is non-negative and sum to $(1,1,\ldots,1)$

Using additional randomness

- Suppose $e_1, \ldots, e_m$ are any vectors in $\mathbb{R}^d$ with $e_i \geq 0$ and $e_1 + \ldots + e_m = (1,1,\ldots,1)$
- Example: $e_1 = (0,.2,.3,0), e_2 = (1,0.7,0.2,0), e_3 = (0,.1,.5,1)$
- What are Event$_1$, Event$_2$, Event$_3$?
- Probably of Event$_j$ equals $\sum_{i=1}^{d} p_i(e_j)_i = \langle p, e_j \rangle$

Quantum Generalization

- How can you measure a state $\rho \in \mathbb{C}^{d \times d}$?
  - In orthonormal basis -> d outcomes
  - Say $d = 2^q$, ρ has q qubits. Can do a “partial measurement” of r qubits: yields 2^r outcomes
  - Additional randomness: can add additional qubits, do a 2^{r+s}-dimensional unitary, partially / fully measure: up to 2^{r+s} outcomes

Quantum Generalization

- Turns out: the above yields m Hermitian $E_1, \dots, E_m \in \mathbb{C}^{d \times d}$ satisfying
  - $E_i \succeq 0$ and $E_1 + \ldots + E_m = I_{d \times d}$
  - And Pr[outcome is "i"] = $\langle \rho, E_i \rangle$
  - Check $\langle \rho, E_i \rangle \geq 0$ and $\langle \rho, E_1 \rangle + \ldots + \langle \rho, E_m \rangle = 1$
- Conversely: given Hermitian $E_1, \dots, E_m$ satisfying $E_i \succeq 0$ and $E_1 + \ldots + E_m = I_{d \times d}$, can in principle build a physical process with associated behavior.
- POVM (positive operator-valued measurement) $\{E_1, \dots, E_m\}$ is quantum generalization of mutually exclusive collectively exhaustive events

Probably 101 Random Variables

- Given probability distribution $p \in \mathbb{R}^d$, a random variable is just a real number x_i for each outcome $i \in \{1,\ldots,d\}$
  - Any vector $x \in \mathbb{R}^d$
  - Expected value is $\mathrm{E}_p[x] = \sum_{i=1}^{d} p_i x_i = \langle p, x \rangle$
- Quantum generalization: **observable** for d-dimensional state ρ
  - Any Hermitian $X \in \mathbb{C}^{d \times d}$
  - Recall: can express $X = \sum_{i=1}^{d} \ldots$

- Physically, could build an instrument that
  - Measured in $\{|u_1\rangle, \ldots, |u_d\rangle\}$ basis
  - On outcome $|u_i\rangle$, read out real number x_i
- If you applied this instrument to ρ, what's the expected value of readout?
- Notation: $\mathrm{E}_\rho[X]$
- Remark: X^2 operator stretches by x_i^2 factor in direction $|u_i\rangle$. So expected (readout)^2 = $\mathrm{E}_\rho[X^2]$

Quantum probability theory

- Sources of randomness: ρ
- Events: ?
- Random variable: ?
- Done.
- Warning: if X and Y are observables (that is, Hermitian matrices) $XY \neq YX$ necessarily.
- Indeed “quantum probability” often called “non-commutative probability”