Collaborators : None

Sources : Lecture Notes

Q2 Indistinguishable States

(a) Let $|\psi\rangle$ and $|\psi^{\perp}\rangle$ be orthonormal real qubit states. Show

$$\frac{1}{\sqrt{2}}|\psi\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|\psi^{\perp}\rangle \otimes |\psi^{\perp}\rangle$$

is precisely equal to the Bell state,

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

Sol: Let's express the given state:

$$\frac{1}{\sqrt{2}}|\psi\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|\psi^{\perp}\rangle \otimes |\psi^{\perp}\rangle$$

in terms of the Bell state:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

To begin, we express $|\psi\rangle$ and $|\psi^{\perp}\rangle$ in a real basis: $\frac{c}{d} = \left(\frac{-b}{a}\right)^*$

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |\psi \perp\rangle = -kb^*|0\rangle + ka^*|1\rangle$$

$$Let\ \ |\psi^{\perp}\rangle = c|0\rangle + d|1\rangle \ \Rightarrow ac^* + bd^* = 0 \Rightarrow \frac{c}{d} = \left(\frac{-b}{a}\right)^*$$

$$c = -kb^* \ ;\ d = ka^* \quad ,where\ |k| = 1$$

where a and b are real numbers such that $a^2 + b^2 = 1$.

Now, compute the tensor products:

$$|\psi\rangle \otimes |\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) = a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle$$

$$|\psi^{\perp}\rangle \otimes |\psi^{\perp}\rangle = (-kb^*|0\rangle + ka^*|1\rangle) \otimes (-kb^*|0\rangle + ka^*|1\rangle) = k^2 b^{*2}|00\rangle - k^2 b^* a^* |01\rangle - k^2 b^* a^* |10\rangle + k^2 a^{*2}|11\rangle$$

As $|\psi\rangle$ and $|\psi^{\perp}\rangle$ are real qubit states, $k = 1\ or\ (-1) \Rightarrow k^2 = 1\ \ and\ a^* = a\ \ and\ b^* = b$

Now sum the two terms:

$$|\psi\rangle \otimes |\psi\rangle + |\psi^{\perp}\rangle \otimes |\psi^{\perp}\rangle = (a^2 + b^2)\,|00\rangle + (a^2 + b^2)\,|11\rangle = |00\rangle + |11\rangle$$

Thus, the original state becomes:

$$\sqrt{2} * (\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle).$$

which is indeed the Bell state. Dividing both sides by $\sqrt{2}$ should give the required result.

(b) Let $|u\rangle \in C^2$ be a qubit state and let $|v\rangle = c\,|u\rangle$, where c is a complex number of magnitude 1 (for example, $c = -1$ or $c = i$). (In this scenario, c is called a "global phase".)

Suppose someone hands you a qubit $|\psi\rangle$ and promises you that $|\psi\rangle$ is either $|u\rangle$ or $|v\rangle$. (You know, mathematically, exactly what $|u\rangle$ and $|v\rangle$ are; but you do not know whether $|\psi\rangle$ is $|u\rangle$ or $|v\rangle$.) Show, to the best of your abilities, that there is nothing you can possibly do to tell the difference. You should at least show that applying 1-qubit unitaries and 1-qubit measurements in any combination does not help. (If you want to be even more sophisticated, show that it doesn't help even if you introduce additional qubits in known states, and then apply unitaries and measurements to this larger-dimensional system.)

Sol: A global phase factor does not change any measurement outcome. To demonstrate this:

1. Measurement in any basis: The probability of measuring a particular outcome in quantum mechanics is given by the square of the amplitude of the state in that basis. Since the global phase c has a magnitude of 1, it does not affect the absolute value of the amplitude, meaning the measurement outcomes are identical for both $|u\rangle\ and\ |v\rangle$.
2. Applying unitaries: Applying a unitary transformation U to both $|u\rangle\ and\ |v\rangle$ results in:

$$U\,|u\rangle \quad and \quad U\,|v\rangle = U(c\,|u\rangle) = c * U\,|u\rangle$$

The global phase c is preserved, so no unitary transformation can distinguish between $|u\rangle\ and\ |v\rangle$.

3. Introducing additional qubits: If we introduce ancillary qubits, apply joint unitary operations, and perform measurements on the combined system, the global phase c still cannot be detected, as the phase only affects the overall state and not the probabilities of measurement outcomes.

$$U\,|u\rangle \otimes |\psi\rangle \quad and \quad U\,|v\rangle = U(c\,|u\rangle \otimes |\psi\rangle) = c * U\,|u\rangle \otimes |\psi\rangle$$

Thus, it is impossible to distinguish between $|u\rangle\ and\ |v\rangle = c\,|u\rangle$ using any combination of unitaries, measurements, and additional qubits.

(c) Suppose someone hands you a qubit $|\psi\rangle$ and promises you that they prepared it according to one of the following two scenarios:

Scenario 1: They flipped a fair coin and set $|\psi\rangle = |0\rangle$ if the result was Heads and set $|\psi\rangle = |1\rangle$ if the result was Tails.

Scenario 2: They flipped a fair coin and set $|\psi\rangle = |+\rangle$ if the result was Heads and set $|\psi\rangle = |-\rangle$ if the result was Tails.

Show, to the best of your abilities, that there is nothing you can possibly do to tell whether they employed Scenario 1 or Scenario 2. (Same comments as in (b) about what you should at least do, and what you can further strive to do.)

Sol:

- Scenario 1: The qubit is $|0\rangle$ *or* $|1\rangle$, each with probability ½ .
- Scenario 2: The qubit is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ *or* $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ each with probability ½.

We need to show that no operation can distinguish between these two preparation methods.

1. Measurement in the computational basis: In Scenario 1, measuring in the computational basis yields $|0\rangle$ with probability ½ and $|1\rangle$ with probability ½. In Scenario 2, the state is either $|+\rangle$ or $|-\rangle$, which, when measured in the computational basis, also yields $|0\rangle$ *or* $|1\rangle$ with equal probability. Hence, measurements in this basis cannot distinguish between the two scenarios.
2. Measurement in the Hadamard basis: In Scenario 2, measuring in the Hadamard basis would yield $|+\rangle$ or $|-\rangle$ with probability ½. However, in Scenario 1, both $|0\rangle$ *and* $|1\rangle$ are superpositions in the Hadamard basis and would also yield $|+\rangle$ or $|-\rangle$ with equal probability. Therefore, this measurement also fails to distinguish between the two scenarios.

   Not only in this computational basis but in any other computational basis with angle θ to standard basis, the probability of measurement in each basis=

   $½(Cos^2\theta + Sin^2\theta)\{Scenario - 1\} = ½(Cos^2(45 - \theta) + Cos^2(\theta + 45))\{Scenario - 2\} = ½$
3. Applying unitaries: Outcomes $\{|0\rangle$ *and* $|1\rangle\}$ with each probability ½ in scenario 1. Outcomes $\{|+\rangle$ *and* $|-\rangle\}$ with each probability ½ in scenario 2.

$U\{|0\rangle\ and\ |1\rangle\} = U|0\rangle\ and\ U|1\rangle$ each with probability ½. $They\ are\ orthonormal\ as\ (U|1\rangle)^\wedge T * U|0\rangle = |1\rangle^\wedge T * U^\wedge T * U * |0\rangle = 0$

Similarly in scenario 2,

$U\{|+\rangle and\ |-\rangle\} = U|+\rangle\ and\ U|-\rangle$ each with probability ½. $They\ are\ orthonormal\ as\ (U|+\rangle)^\wedge T * U|-\rangle = |+\rangle^\wedge T * U^\wedge T * U * |-\rangle = 0$

But $(U|+\rangle)^\wedge T * U|0\rangle = |+\rangle^\wedge T * U^\wedge T * U * |0\rangle = \frac{1}{\sqrt{2}}$. So is it for other pairs. Hence they cant be orthonormal and each outcome has 0.5 probability for each scenario in any basis.

4. <u>Introducing ancillary qubits:</u> Similar to part (b), introducing additional qubits and performing joint operations does not provide any additional information that could distinguish between Scenario 1 and Scenario 2, as the probabilities of outcomes remain the same. (Just apply tensor product to above equations which give same orthonormal result.)

Thus, it is impossible to distinguish between the two scenarios using any combination of operations or measurements.