Collaborators : None

Sources : Lecture Notes; https://en.wikipedia.org/wiki/Bell_state ; https://www.cs.cmu.edu/~odonnell/quantum15/lecture03.pdf

Q6 1 ebit + 1 qubit ≥ 2 bits

(a) Alice and Bob share an EPR pair, a maximally entangled state of two qubits. The initial state of the system is:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This means that Alice has one qubit, and Bob has the other. Alice wants to send two classical bits, u and v, to Bob, where $u, v \in \{0,1\}$. She can send this information by applying certain quantum gates to her qubit and then handing her qubit to Bob. Here's the procedure Alice follows:

- If $u = 1$, she applies a NOT gate to her qubit. If $u = 0$, she does nothing.
- If $v = 1$, she applies a Z gate (which applies a phase flip). If $v = 0$, she does nothing.

The quantum gates transform Alice's qubit based on her message:

1. If $u = 0$ *and* $v = 0$: The state remains unchanged as $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
2. If $u = 1$ *and* $v = 0$: Alice applies a NOT gate to her qubit, flipping the 0 in her qubit to a 1, and 1 to a 0. The state becomes $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$
3. If $u = 0$ *and* $v = 1$: Alice applies a Z gate, introducing a phase flip, and the state becomes $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
4. If $u = 1$ *and* $v = 1$: Alice applies both the NOT gate and the Z gate, and the state becomes $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$

After these operations, Alice hands her qubit to Bob. Now, Bob has access to both qubits, and by measuring in the $Bell\ basis$, he can distinguish between the four possible states and thereby determine $u\ and\ v$.

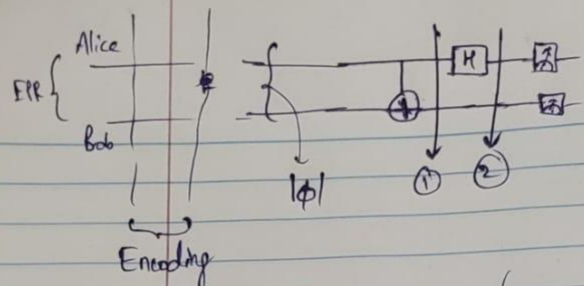
The four possible Bell states are:

- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

- $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$
- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$

By measuring the combined state of both qubits in this basis, Bob can distinguish which message Alice sent. Based on his measurement of the qubit pair into Bell state, he can decode it to {$|\Phi^+\rangle \rightarrow 00$; $|\Phi^-\rangle \rightarrow 01$; $|\Psi^+\rangle \rightarrow 10$; $|\Psi^-\rangle \rightarrow 11$ }

(b)



[Applying CNOT gate first, then H-gate (for Alice-bit) Finally measurement in $\{|0\rangle, |1\rangle\}$ basis.]

Case-1: If $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$, then

At ①, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$

At ②, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|{+}0\rangle + |{-}0\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle + \cancel{\frac{1}{\sqrt{2}}|10\rangle} + \frac{1}{\sqrt{2}}|00\rangle - \cancel{\frac{1}{\sqrt{2}}|10\rangle}\right)$$

$$= |00\rangle$$

Case-2: If $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)$

At ①, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|11\rangle + |01\rangle\right)$

At ②, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|{-}1\rangle + |{+}1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{|01\rangle}{\sqrt{2}} - \cancel{\frac{|11\rangle}{\sqrt{2}}} + \frac{|01\rangle}{\sqrt{2}} + \cancel{\frac{|11\rangle}{\sqrt{2}}}\right)$$

$$= |01\rangle$$

Case-3: If $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$

At ①, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right)$

At ②, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|{+}0\rangle - |{-}0\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\cancel{\frac{|00\rangle}{\sqrt{2}}} + \frac{|10\rangle}{\sqrt{2}} - \cancel{\frac{|00\rangle}{\sqrt{2}}} + \frac{|10\rangle}{\sqrt{2}}\right)$$

$$= |10\rangle$$

Case-4: If $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$

At ①, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|11\rangle - |01\rangle\right)$

At ②, $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|-1\rangle - |+1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\cancel{\frac{|01\rangle}{\sqrt{2}}} - \frac{|11\rangle}{\sqrt{2}} - \cancel{\frac{|01\rangle}{\sqrt{2}}} - \frac{|11\rangle}{\sqrt{2}}\right)$$

$$= -|11\rangle$$

(Measurement gives $|11\rangle$)