Lecture 21. November 14, 2024

Last time:

any NP-complete problem is in BQP.

Open problem: $NP \subseteq BQP$? World-changing if possible.

If quantum computers can't solve NP-complete problems in poly time, can still ask "How close can they get ~~solved?~~ to solving?".

BBBV thm: Ignoring the structure of NP-complete problems will only yield the "Grover speedup" (from $N = 2^n$ to $\sqrt{N}$).

To do better than Grover, need to exploit problem structure in some way.

Since we're aiming from "small" speedups, the choice of NP-complete problem might matter.

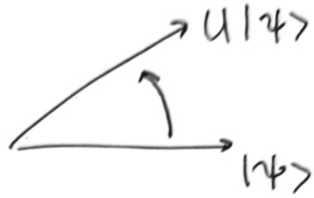
<u>The Adiabatic Algorithm</u> (Farhi, Goldstone, Gutmann, Sipser 2000).

- A famous attempt to get quantum speedup for NP-complete problems by actually exploiting their structure.
- Extremely important quantum algorithm.
- Unlike Shor's or Grover's algorithm, ~~no rigorous analysis~~ it does not run fast in all cases.
- No one knows how useful this algorithm will be in practice (b/c no reliable large-scale quantum computers).
- Huge speedup for some instances of optimization problems

Hamiltonians

Unitaries are "discrete" linear transformations of quantum states

$$|\psi\rangle \mapsto U|\psi\rangle.$$



A physicist would treat time as continuous, and say $|\psi\rangle$ rotate continuously to $U|\psi\rangle$ over some interval of time.

Hamiltonians are (1) the instantaneous time generators of unitary transformations.
(2) described by Hermitian matrices (don't need to be PSD or have trace 1 unlike density mat).

Physicist perspective:

- Shrödinger's Equation: $i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$, with H being some Hamiltonian. describes the evolution of an isolated quantum pure state in continuous time. (Actual S-eqn includes Planck's constant $\hbar$.)
- Can solve S-eqn: the state at time t is $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$.

Q. What does it mean to raise e to the power of a matrix?

A. Taylor series: $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

E.G.: $e^{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}} = \begin{bmatrix}e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n}\end{bmatrix}.$

In general, a Hermitian $A = UDU^{-1}$, where D is diagonal.

Then $e^A = \cdots = Ue^DU^{-1}$.

THM: If H is Hermitian, then e^{iHt} is unitary.

Pf: Recall Hermitian H has real eigenvalues.

By Spectral Theorem, $H = UDU^{+}$ where U is unitary and $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Thus $e^{iHt} = U e^{iDt} U^{+}$

$$= U \begin{bmatrix} e^{-it\lambda_1} & & \\ & \ddots & \\ & & e^{-it\lambda_n} \end{bmatrix} U^{+}$$

Since $|e^{-it\lambda_k}| = 1$, this matrix is unitary, and so is e^{iHt} $\square$

THM: Given unitary U and time t, there exists Hermitian H s.t. $U = e^{-iHt}$.

Pf: First $U = VDV^{+}$, where V is unitary and D ~~is diagonal with~~ $= \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$

with each $|\mu_i| = 1$. ~~diagon~~

Compute λ_i s.t. $e^{-it\lambda_i} = \mu_i$. Set $H = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V^{+}$ $\square$

RMK. By Euler's formula, $e^{2\pi i} = 1$, each λ_i is not uniquely determined.

E.g.: $e^{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{\begin{bmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{bmatrix}}$

Physicists' speak: Hamiltonian $H = UDU^{+}$, where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

These $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are called energies, and $\lambda_1 \leq \dots \leq \lambda_n$ are ordered from least to greatest.

To each energy λ_j, associate "eigenstate" $|v_j\rangle$ s.t. $H|v_j\rangle = \lambda_j |v_j\rangle$.

Now $e^{-iHt}|v_j\rangle = \underbrace{e^{-i\lambda_j t}}_{\text{a global phase}} |v_j\rangle$.

THM: If $|\psi\rangle = a_1|v_1\rangle + \cdots + a_n|v_n\rangle$ then

$$e^{-iHt}|\psi\rangle = a_1 e^{-i\lambda_1 t}|v_1\rangle + \cdots + a_n e^{-i\lambda_n t}|v_n\rangle. \qquad \square$$

RMK: • Boring picture of the universe: all that has ever happened and all that ever will happen is various eigenstates of the universe pick up global phases, each rotating around the unit circle at a speed prop to its energy.

• "Energy = speed at which a quantum state picks up a global phase".

• Energy is conserved: expected value of energy $= \sum_{j=1}^{n} |a_j|^2 \lambda_j$.

Notation: $|v_1\rangle$ – ground state (lowest energy).
λ_1 – ground state energy
$|v_2\rangle$ – first excited state
$|v_3\rangle$ – second excited state...

Standard game plan for much of modern physics:

1. Start with Hamiltonian H of your system
2. Diagonalize H and get out ~~energy~~ eigenstates.
3. As a first guess, see if your system is just in its ground state $|v_1\rangle$.

Q: Why are quantum systems often found in their ground states?

A: Thanks to 2nd Law of Thermodynamics... physical systems "like" to minimize their energy.

Addition of Hamiltonians $H = H_1 + H_2$ ← meaning two things going on at the same time.

Mathematical question: $e^{A+B} = e^A e^B$? No, in general.

However, if A, B commute, then $e^{A+B} = e^A e^B$ why?

Goal: Need to apply $e^{-it(H_p + H_2)}$ efficiently, given only the ability to apply H_1 and H_2, even if H_1 and H_2 don't commute.

Trotterization trick: $e^{A+B} \approx \underbrace{e^{\varepsilon A} e^{\varepsilon B} \cdots e^{\varepsilon A} e^{\varepsilon B}}_{1/\varepsilon \text{ times}}$

"Achieve the effect as A and B occuring simulteneously by repeatedly switching between doing a bit of A and a bit of B".

Mathematically. $e^{A+B} = \lim_{\varepsilon \to 0} (e^{\varepsilon A} e^{\varepsilon B})^{1/\varepsilon}$.

Q: How to simulate real-world quantum system using quantum coumputer?

A: ① Write the total Hamiltonian H acting on ~~the system~~ n qubits as a sum of many simple Hamiltonians acting on a few qubits each:

$$H = H_1 + \cdots + H_m$$

② Trotterize H. in order to simulate it by product of "simple" unitaries.

Q: ~~How to~~ Can I ~~~~ efficiently get the ground state by combining the ground state of each H_j?

A: No: can encode 3SAT problem into the ground state problem.

Sps $\varphi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$, where each C_i has 3 vars.

For each $C_i = x_{i_1}^{\varepsilon_{i_1}} \vee x_{i_2}^{\varepsilon_{i_2}} \vee x_{i_3}^{\varepsilon_{i_3}}$ $\quad (x^0 = x.\ x^1 = \bar{x})$

let H_i act on i_1, i_2, i_3-th qubit. and

$$H_i |s_{i_1}\rangle |s_{i_2}\rangle |s_{i_3}\rangle = \begin{cases} |s_{i_1}\rangle |s_{i_2}\rangle |s_{i_3}\rangle & \text{if } s_{i_1} = \varepsilon_{i_1},\ s_{i_2} = \varepsilon_{i_2},\ s_{i_3} = \varepsilon_{i_3}. \\ 0 & \text{if} \end{cases}$$

Check: ① the ground state $H = H_1 + \cdots + H_m$ must be of the form $|s_1\rangle \cdots |s_n\rangle$ for $s_i \in \{0, 1\}$.

② φ is satisfiable $\Leftrightarrow$ ground-state energy $= 0$.