Collaborators : None

Sources : Lecture Notes

Q3) A Perfect Magic Trick:

a) To prove: Maximum probability of success in the magic trick if Alice, Bob & Charlie respond deterministically $= 3/4$

Proof:

1) There are 8 possible challenge combinations: $(\downarrow\downarrow\downarrow)$, $(\downarrow\downarrow\uparrow)$, $(\downarrow\uparrow\downarrow)$, $(\downarrow\uparrow\uparrow)$, $(\uparrow\downarrow\downarrow)$, $(\uparrow\downarrow\uparrow)$, $(\uparrow\uparrow\downarrow)$, $(\uparrow\uparrow\uparrow)$.

2) For any deterministic strategy, Alice, Bob & Charlie can have a fixed response (0 or 1) for each challenge ($\uparrow$ or $\downarrow$).

3) Let's consider the success conditions:

- For $(\downarrow\downarrow\downarrow)$: They need an even number of 1's. [0 or 2 no. of 1's]
- For $(\downarrow\uparrow\uparrow)$, $(\uparrow\downarrow\uparrow)$, $(\uparrow\uparrow\downarrow)$: They need an odd number of 1's. [1 or 3 no. of 1's]
- For $(\uparrow\uparrow\uparrow)$, $(\downarrow\downarrow\uparrow)$, $(\downarrow\uparrow\downarrow)$, $(\uparrow\downarrow\downarrow)$: Any number of 1's $\rightarrow$ (Repeat)

* It is to be noted that all these ~~conditions~~ cases for success cannot be true simultaneously, because if we get even no. of 1's for $(\downarrow\downarrow\downarrow)$, then we should get also even number of 1's for $(\downarrow\uparrow\uparrow)$ or $(\uparrow\downarrow\uparrow)$ or $(\uparrow\uparrow\downarrow)$ because 2 of these have been flipped & thus difference b/w their number of 1's should also be even.

Next best case is if 3 conditions result in success with one failure, i.e $\rightarrow$ [Success for $(\downarrow\uparrow\uparrow)$, $(\uparrow\downarrow\uparrow)$, $(\uparrow\uparrow\downarrow)$ & failure for $(\downarrow\downarrow\downarrow)$]

This can be achieved by

Sample algorithm: Alice, Bob & Charlie matches $\downarrow \rightarrow 1$ & $\uparrow \rightarrow 0$

$\therefore$ Probability $= \dfrac{\text{Number of success}}{\text{Total cases}}$ (given (all $\downarrow$'s) or (1 $\downarrow$ & 2 $\uparrow$'s))

$$= \frac{\Pr(\{\downarrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\uparrow\downarrow\})}{\Pr(\{\downarrow\downarrow\downarrow, \downarrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\uparrow\downarrow\})} = \frac{3}{4}$$

b) To prove: Quantum strategy succeeds with 100% probability.

1) Initial state: $|\psi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$

2) For $\downarrow$ challenge: Measure in Computational basis
For $\uparrow$ challenge: Apply (H) gate, then measure.

3) Lets analyze each case:

Case-1: $\downarrow\downarrow\downarrow$

$\Rightarrow$ Direct measurement of $|\psi\rangle$

$\Rightarrow$ Readout $= \left\{ \begin{array}{l} |000\rangle \\ |011\rangle \\ |101\rangle \\ |110\rangle \end{array} \right.$ with 1/4 probability each.

Even number of 1's $\rightarrow$ Satisfying success condition

Case-2: $\uparrow\uparrow\uparrow$

$\Rightarrow$ Apply $H \otimes H \otimes H$ to $|\psi\rangle$:

Transforms to: $\frac{1}{2}(|+++\rangle - |+--\rangle - |-+-\rangle - |--+\rangle)$

$$= \frac{1}{2}\Big(|+\rangle \otimes \big(\tfrac{1}{2}|00\rangle + \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle + \tfrac{1}{2}|11\rangle - \tfrac{1}{2}|00\rangle - \tfrac{1}{2}|11\rangle + \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle\big) -$$
$$|-\rangle \otimes \big(\tfrac{1}{2}|00\rangle - \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle - \tfrac{1}{2}|11\rangle + \tfrac{1}{2}|00\rangle + \tfrac{1}{2}|01\rangle - \tfrac{1}{2}|10\rangle - \tfrac{1}{2}|11\rangle\big)\Big)$$

$$= \frac{1}{2}\Big(|+\rangle \otimes (|01\rangle + |10\rangle) - |-\rangle \otimes (|00\rangle - |11\rangle)\Big)$$

$$= \frac{1}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|010\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2\sqrt{2}}|110\rangle - \frac{1}{2\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}|011\rangle + \frac{1}{2\sqrt{2}}|100\rangle - \frac{1}{2\sqrt{2}}|111\rangle$$

Indeterministic of no. of 1's.

Case-3: $\downarrow\uparrow\uparrow$

Transforming $|\psi\rangle \Rightarrow \frac{1}{2}|0\rangle \otimes (|++\rangle - |--\rangle) - \frac{1}{2}|1\rangle \otimes (|+-\rangle + |-+\rangle)$

$= \frac{1}{2}|0\rangle \otimes (|01\rangle + |10\rangle) - \frac{1}{2}|1\rangle \otimes (|00\rangle - |11\rangle)$

$\Rightarrow$ Readout $= \left\{ \begin{array}{l} |001\rangle \\ |010\rangle \\ |100\rangle \\ |111\rangle \end{array} \right.$ with $pr = \frac{1}{4}$ } Odd no. of 1's.

Case-4: $\uparrow\downarrow\uparrow$

Transforming $|\psi\rangle \Rightarrow \frac{1}{2}|+\rangle \otimes |0\rangle \otimes |+\rangle - \frac{1}{2}|+\rangle \otimes |1\rangle \otimes |-\rangle - \frac{1}{2}|-\rangle \otimes |0\rangle \otimes |-\rangle - \frac{1}{2}|-\rangle \otimes |1\rangle \otimes |+\rangle$

$= \frac{1}{2}[|001\rangle + |100\rangle] - \frac{1}{2}[|010\rangle - |111\rangle]$ } Readout outcomes all have Odd no. of 1's.

Case-5 : ↑↑↓

Transforming $|\psi\rangle \Rightarrow \frac{1}{2}\left(|++0\rangle - |+-1\rangle - |-+1\rangle - |--0\rangle\right)$

$= \frac{1}{2}\left((|++\rangle - |--\rangle) \otimes |0\rangle - (|+-\rangle - |-+\rangle) \otimes |1\rangle\right)$

$= \frac{1}{2}\left(|010\rangle + |100\rangle - |001\rangle + |111\rangle\right)$

Readout outcomes have odd number of 1's.

* From Case-1, 3, 4, 5 ; we observe that for all given conditions considered, success conditions are satisfied.

Thus, this strategy succeeds with 100% probability.