Assignment 4

Graded

Student

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Total Points

24 / 24 pts

Question 1

Read me first

0 / 0 pts

✔ **+ 0 pts** Correct

Question 2

Implausible consequences of superstrong nonlocality

8 / 8 pts

✔ **+ 8 pts** Correct

+ 0 pts Incorrect

Question 3

A perfect magic trick

8 / 8 pts

✔ **+ 8 pts** Correct

+ 4 pts (a) Correct

+ 2 pts Only fully justified the case where all the challenges are "down".

+ 0 pts Incorrect

Question 4

Hardy's Paradox

8 / 8 pts

✔ **+ 8 pts** Correct

+ 1 pt (a) Only correctly analyzed the TT case.

+ 2 pts (a) Only correctly analyzed 2 out of the 4 cases.

+ 4 pts (b)

+ 0 pts Incorrect

Q1 Read me first

0 Points

- Tests show that the people who get the most out of this assignment are those who read the "read me first" like this one.
- Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.
- Acknowledgment Requirements:

1. Acknowledge, individually for every problem at the beginning of each solution, a list of all collaborators and sources consulted other than the course notes. Examples include: names of people you discussed homework with, books, other notes, Wikipedia, and other websites.
2. If you consulted any online sources, please specify the exact webpages by including their links. Omission of links or any other required citations will result in a loss of grades and be considered a failure to acknowledge appropriately.
3. If no additional sources are consulted, you must write "sources consulted: none" or equivalent.
4. **Failure to acknowledge sources will lead to an automatic 1pt penalty.**

- Late policy: In general **no late homework** will be accepted unless there is a genuine emergency backed up by official documents.
- All steps should be justified.
- Formatting and Submission Requirements:

1. Separate Solutions: Ensure that solutions for each problem are separated clearly.
2. PDF Submissions: If you are submitting a LaTeX PDF, use the "fullpage" package to set the margins to 1 inch. Do not include additional information such as the title, date, your name, the problem statement, or any rough work—only include your final solution.
3. Typed Solutions: If typing directly in the provided textbox, please use LaTeX formatting for formulas.
   Images: Rotated images will not be graded. Ensure all images are properly oriented.
4. Scanning Quality: Use proper scanning software to scan your handwritten solutions. Avoid casual photos of your work.
5. **Failure to meet these formatting and submission requirements may result in up to a 2-point penalty for each problem.**

- You are encouraged to be **type in LaTeX**. To learn how to use LaTeX, I recommend the tutorials on Overleaf. It is ok to draw diagrams by hand and insert them as pictures in your TeX files.
- For each question below, upload a PDF file and/or type in the box (see Gradescope x LaTeX tutorial). Each submission should contain (1) the acknowledgement of all collaborators and sources consulted and (2) your solution.

Q2 Implausible consequences of superstrong nonlocality

8 Points

The usual terminology for the CHSH game is as follows:

- Alice's referee's challenge is called x and is either 1 (Red) or 0 (Yellow);
- Bob's referee's challenge is called y and is either 1 (Green) or 0 (Orange);
- Alice's response is called $a \in \{0, 1\}$ (rather than Solid/Dotted);
- Bob's response is called $b \in \{0, 1\}$ (rather than Solid/Dotted);
- the "success condition" is $a + b = x \cdot y \pmod 2$.

Now suppose that Alice and Bob could build magic "non-local boxes" that would allow them to succeed at the CHSH game with 100% probability. That is, even though Alice and Bob are spatially distant: Alice can put a bit $x \in \{0, 1\}$ into the box and get back a bit $a \in \{0, 1\}$; Bob can put a bit $y \in \{0, 1\}$ into the box and get back a bit $b \in \{0, 1\}$; and, these bits will always satisfy $a + b = x \cdot y \pmod 2$.

(a) Assume that Alice and Bob are spatially distant, but they have access to N of these magic "non-local boxes". Assume also that Alice knows N bits $x_1, \dots, x_N \in \{0, 1\}$, Bob knows N bits $y_1, \dots, y_N \in \{0, 1\}$, and they have a desire to compute the "inner product mod 2" function of their bits,

$$\mathrm{IP}_2(x_1, \dots, x_N, y_1, \dots, y_N) = x_1 \cdot y_1 + \dots + x_N \cdot y_N \pmod 2.$$

Show that by using the non-local boxes, and then allowing one classical bit of communication from Alice to Bob, Bob can learn the value $\mathrm{IP}_2(x_1, \dots, x_N, y_1, \dots, y_N)$.

(b) Recall that every Boolean function $f : \{0, 1\}^m \to \{0, 1\}$ can be computed by a Boolean circuit using AND gates and NOT gates. Show that every Boolean function $f : \{0, 1\}^m \to \{0, 1\}$ can also be computed by a polynomial modulo 2.

(c) Suppose that we have a Boolean function on $2n$ inputs, $f : \{0, 1\}^{2n} \to \{0, 1\}$, where we use the notation x_i for the first n input variables and the notation y_i for the second n. Prove that it is possible to express f as

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{j=1}^{N} A_j(x) \cdot B_j(y) \pmod 2,$$

where $A_1(x), \dots, A_N(x)$ are each products of zero or more x_i's, and similarly $B_1(y), \dots, B_N(y)$ are each products of zero or more y_i's. (The product of zero terms is considered to be 1.)

For example, if $n = 2$ and f is the function EQ indicating equality of the two 2-bit strings formed by x and y, it holds that

$$\mathrm{EQ}(x_1, x_2, y_1, y_2) = 1 \cdot 1 + x_1 \cdot 1 + x_2 \cdot 1 + 1 \cdot y_1 + 1 \cdot y_2 + x_1 x_2 \cdot 1 + x_1 \cdot y_2 + x_2 \cdot y_1 + 1 \cdot y_1$$

(d) Return to the scenario from part (a), but instead Alice knows n bits $x_1, \ldots, x_n \in \{0, 1\}$, Bob knows n bits $y_1, \ldots, y_n \in \{0, 1\}$, and they have a desire to compute a certain Boolean function $f : \{0, 1\}^{2n} \to \{0, 1\}$ applied to their two inputs,

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

(They both know the function f.) Show that by using as many non-local boxes as they want, and then allowing one classical bit of communication from Alice to Bob, Bob can learn the value $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Remark: It seems very implausible that Alice and Bob should be able to remotely compute any joint function of arbitrarily long private input strings while only communicating one classical bit. This can be taken as evidence of the physical impossibility of succeeding at the CHSH game with 100% probability. In fact, Brassard–Buhrman–Linden–Méthot–Tapp–Unger showed that Alice and Bob could do this implausible task even if their magic non-local boxes only succeeded at the CHSH game with probability exceeding $\frac{1}{2} + \frac{1}{\sqrt{6}} \approx 91\%$.

Sources consulted:

Lecture Notes

Solution:

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Q3 A perfect magic trick

8 Points

Alice is on Earth, Bob is on Mars, Charlie is on Neptune. With each of them is a referee. While the referees were still on Earth, each secretly chooses a challenge randomly that is either "up" ($\uparrow$) or "down" ($\downarrow$), and they made sure that there are an odd number of $\downarrow$'s. At the stroke of midnight, Phoenix time, each referee reveals their challenge, and Alice, Bob, and Charlie are required to promptly respond to their challenges with a "0" or "1". They succeed under the following conditions:

- all three referee challenges were $\downarrow$, and three responses have an even number of 1's; or
- referee challenges were one $\downarrow$ and two $\uparrow$'s, and three responses have an odd number of 1's.

As usual, assume that the spatial distance between Alice, Bob, and Charlie prevents them from communicating at all, and Alice, Bob, and Charlie can't spy on the referees.

(a) Prove that if Alice, Bob, and Charlie respond deterministically to their challenges, the probability with which they can succeed in the magic trick is at most $\frac{3}{4}$. (Remark: as with the CHSH game, from this one can also easily conclude that if Alice, Bob, and Charlie can share classical random bits, they still cannot succeed with probability more than $\frac{3}{4}$.)

(b) Suppose that Alice, Bob, and Charlie prepare the following 3-qbit state on Earth before the magic trick begins:

$$\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle.$$

Alice takes the first qbit, Bob takes the second qbit, Charlie takes the third qbit. Now, when they receive their challenges, they each use the following strategy: If they are challenged with $\downarrow$, they measure their qbit and respond with the outcome. If they are challenged with $\uparrow$, they first apply a Hadamard gate to their qbit, and then they measure and respond with the outcome.

Prove that Alice, Bob, and Charlie succeed with 100\% probability.

Sources consulted:

Lecture Notes

Solution:

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Q4 Hardy's Paradox
8 Points

Alice and Bob prepare the following 2-qubit state:

$$|\psi\rangle = (H \otimes H)\left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle\right).$$

Alice now takes control of the first qubit and Bob takes control of the second qubit.

Each of Alice and Bob now flips a coin and does the following: If they flip Tails, they directly measure their qubit; if they flip Heads, they first apply a Hadamard to their qubit and then they measure.

(a) Prove the following statements:

- If Alice flips T and Bob flips T, it's possible A & B will measure 1, 1 respectively.
- If Alice flips T and Bob flips H, it's impossible A & B will measure 1, 0 respectively.
- If Alice flips H and Bob flips T, it's impossible A & B will measure 0, 1 respectively.
- If Alice flips H and Bob flips H, it's impossible A & B will measure 1, 1 respectively.

(b) Lucien says the following: "Let's consider the situation before any coin flips or measurement happens, and try to decide what outcomes the qubits are capable of producing when measured.

- On one hand, consider the first statement in (a). Since it's possible that Alice will flip Tails and Bob will flip Tails, we conclude that prior to any coin flips/measuring, it's possible for Alice's qubit to register 1 after being directly measured.

- Now consider the second statement in (a). Since Alice's qubit is capable of generating a 1 when she flips Tails, it must be impossible for Bob's qubit to produce a 0 when he flips Heads, and consequently Hadamards-then-measures.

- Let's repeat the previous two bullet points, interchanging 'Alice' and 'Bob'. By the first statement in (a), we conclude that prior to any coin flips/measuring, it's possible for Bob's qubit to register a 1 when directly measured. Hence by the third statement in (a), since Bob's qubit is capable of generating a 1 when directly measured, we conclude that it must be impossible for Alice's qubit to produce a 0 when she Hadamards-then-measures.

- We've concluded that in case of flipping Heads, for both Alice and Bob it's impossible for them to register a 0 when they Hadamard-and-measure; i.e., they must both register a 1 in this case. But this contradicts the fourth statement in (a)."

Critique the four bullet points above. Do you agree or disagree with Lucien?

Sources consulted:

Lecture Notes

Solution:

HW4_Q4.pdf

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