Electrodynamics I Term Paper

The Electrostatic Bowl: Charged Hemispherical Shell Field Visualization and Particle Trajectory Estimation

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Contents:

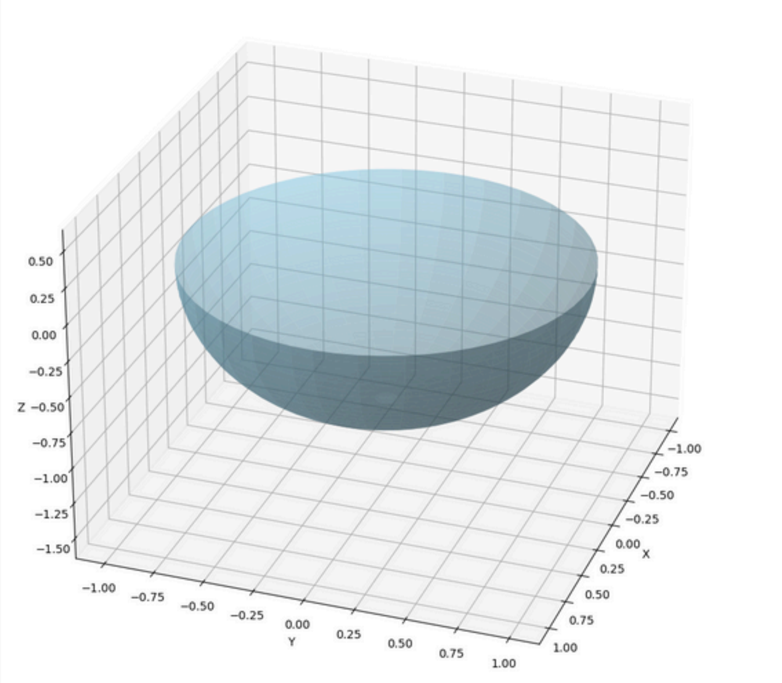
1. Brief Description:

The coulomb law says that the force between two charged particles is,

$$\overrightarrow{F_e} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

Now using the principle of superposition, we can find the force acting between charged objects any shapes. But every configuration is not completely solvable analytically.

In this term paper, I will discuss about the electric fields created due to an uniformly charged hemespherical shell and the distortion of the fields due to the presence of another charge near it. It is very hard to solve it analytically. So I'll use both the analytical and numerical methods.



Also if we introduce gravity, then the particle will feel both the electrostatic force and gravitational force. Then if we release it from a point, we expect to see beautiful trajectories of the particle. Some of these conditions are also discussed.

2. Analytical Approach:

We first discuss about the easiest case. Also this is one of the cases which are easily solvable by hand for this configuration.

Let's find the electric field along the axis of the shell. But first we find the electrostatic potential.

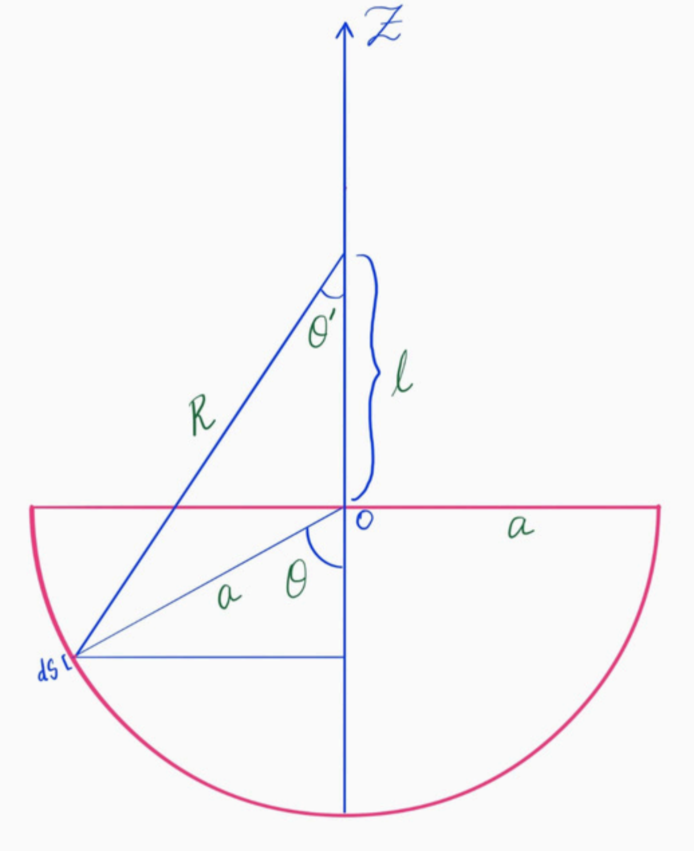
Potential at the test point due to a small surface element 'dS' is,

$$d\phi = \frac{1}{4\pi\epsilon_0}\frac{\sigma dS}{R}$$

By changing variables and solving it, we get the potential as,

$$\phi = \frac{\sigma a}{2\epsilon_0 l}\left[l + a - \sqrt{a^2 + l^2}\right] \qquad \text{for x > -a}$$

$$\phi = -\frac{\sigma a}{2\epsilon_0 l}\left[l + a + \sqrt{a^2 + l^2}\right] \qquad \text{for x < -a}$$



Then we can find the electric field by using the relation, $\vec{E} = -\nabla(\phi)$

That is,

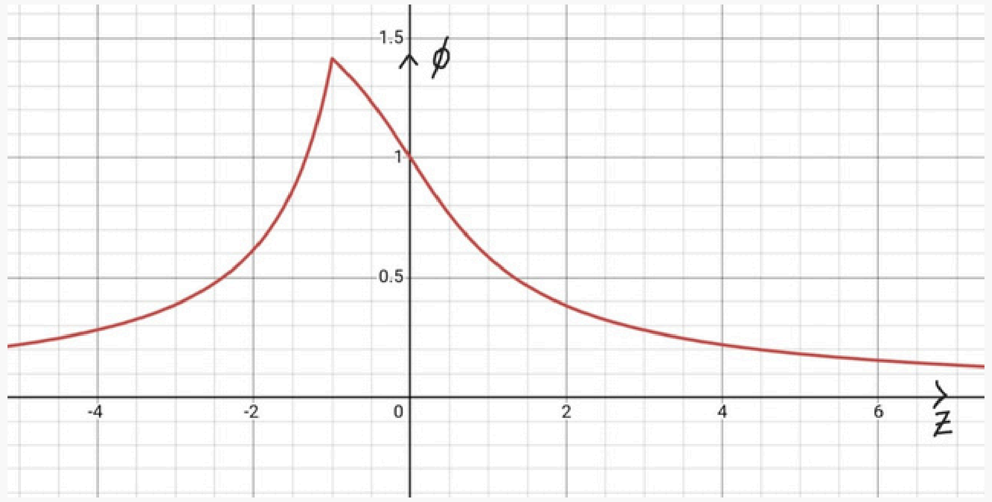
$$\vec{E} = \frac{\sigma a^2}{2\epsilon_0 l^2}\left[1 - \frac{a}{\sqrt{a^2 + l^2}}\right]\hat{k} \qquad \text{for x > -a}$$

$$\vec{E} = \frac{\sigma a^2}{2\epsilon_0 l^2}\left[-1 - \frac{a}{\sqrt{a^2 + l^2}}\right]\hat{k} \qquad \text{for x < -a}$$

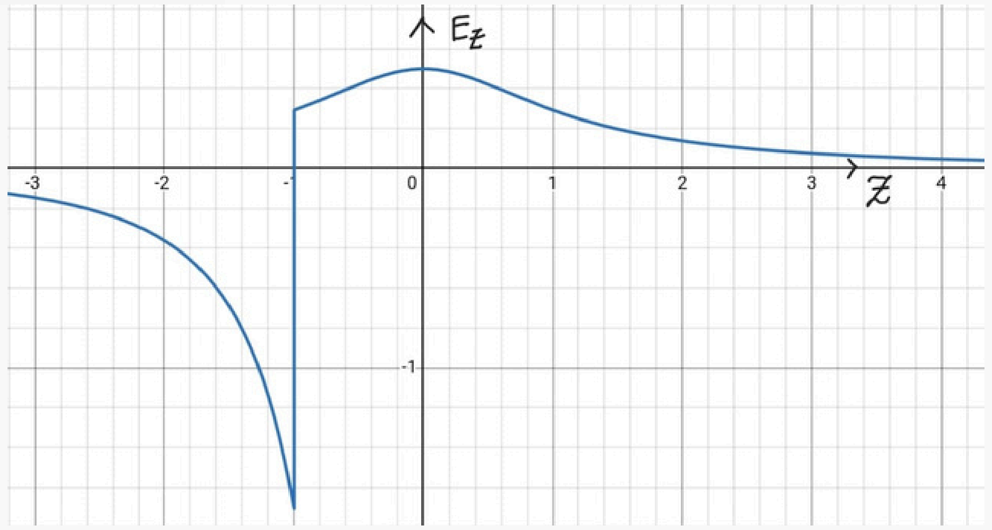
Now we put a test charge at the test point. We also introduce gravity along the -ve z direction. Then if the particle has charge ‘q’ and mass ‘m’, then the total force on the particle will be,

$$\vec{F} = \frac{q\sigma a^2}{2\epsilon_0 l^2}\left[1 - \frac{a}{\sqrt{a^2+l^2}}\right]\hat{k} - mg\hat{k}$$ for x > -a

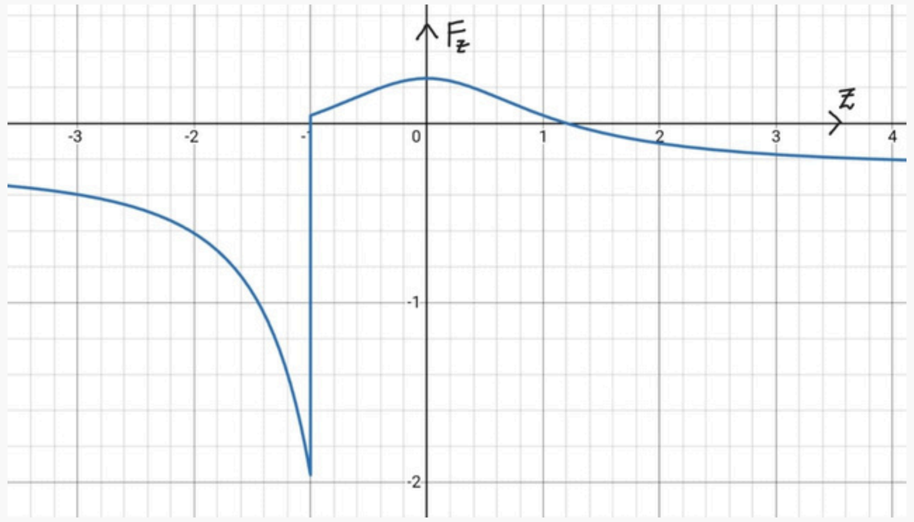
$$\vec{F} = \frac{q\sigma a^2}{2\epsilon_0 l^2}\left[-1 - \frac{a}{\sqrt{a^2+l^2}}\right]\hat{k} - mg\hat{k}$$ for x < -a



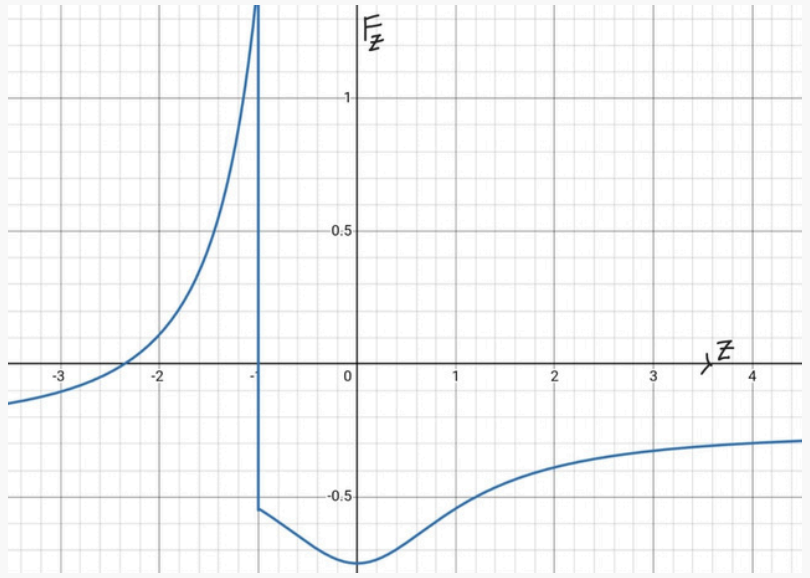
Potential vs z



Electric field vs z



Total force vs z on +ve charge



Total force vs z on -ve charge

3. Numerical Approach:

Using numerical methods, we can solve (atleast approximate) this problem for different initial cases. Here I have used Python to solve these problems. All the code can be found at,

https://github.com/Sujoy7471/GitFiles/tree/main/Electrostatic%20Bowl

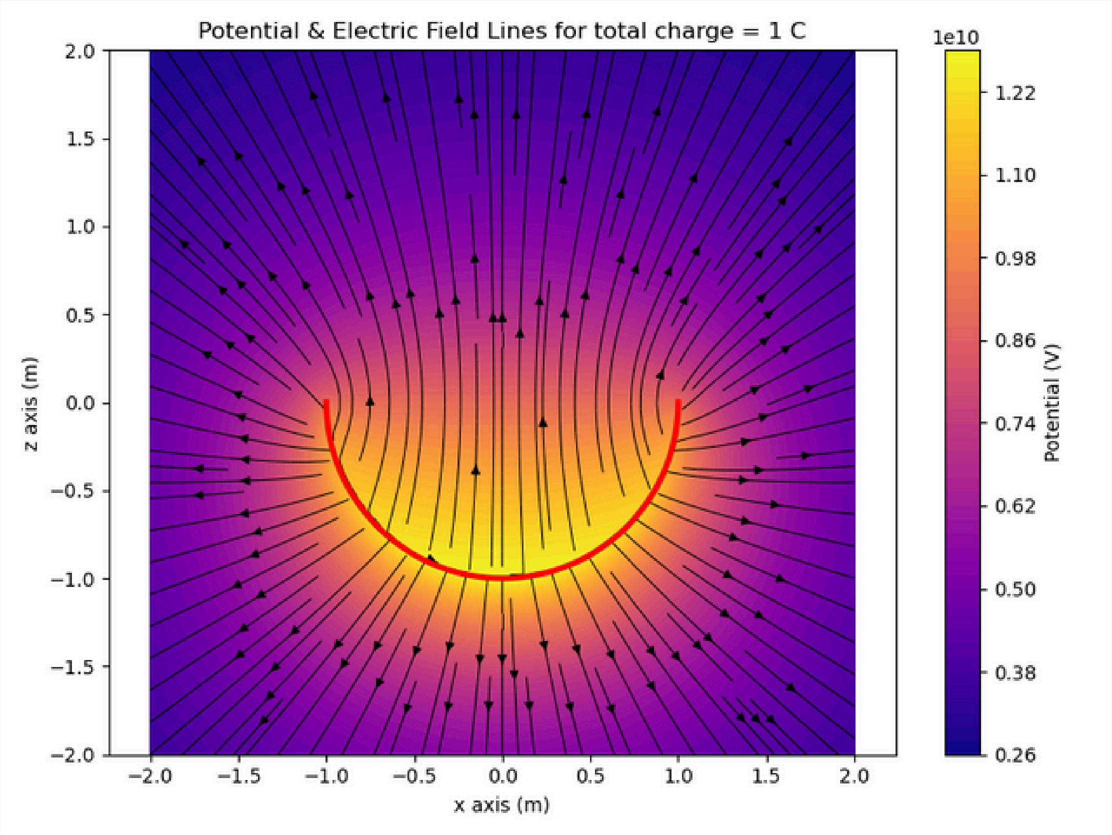
Brief description of the method: Here I did the followwing,

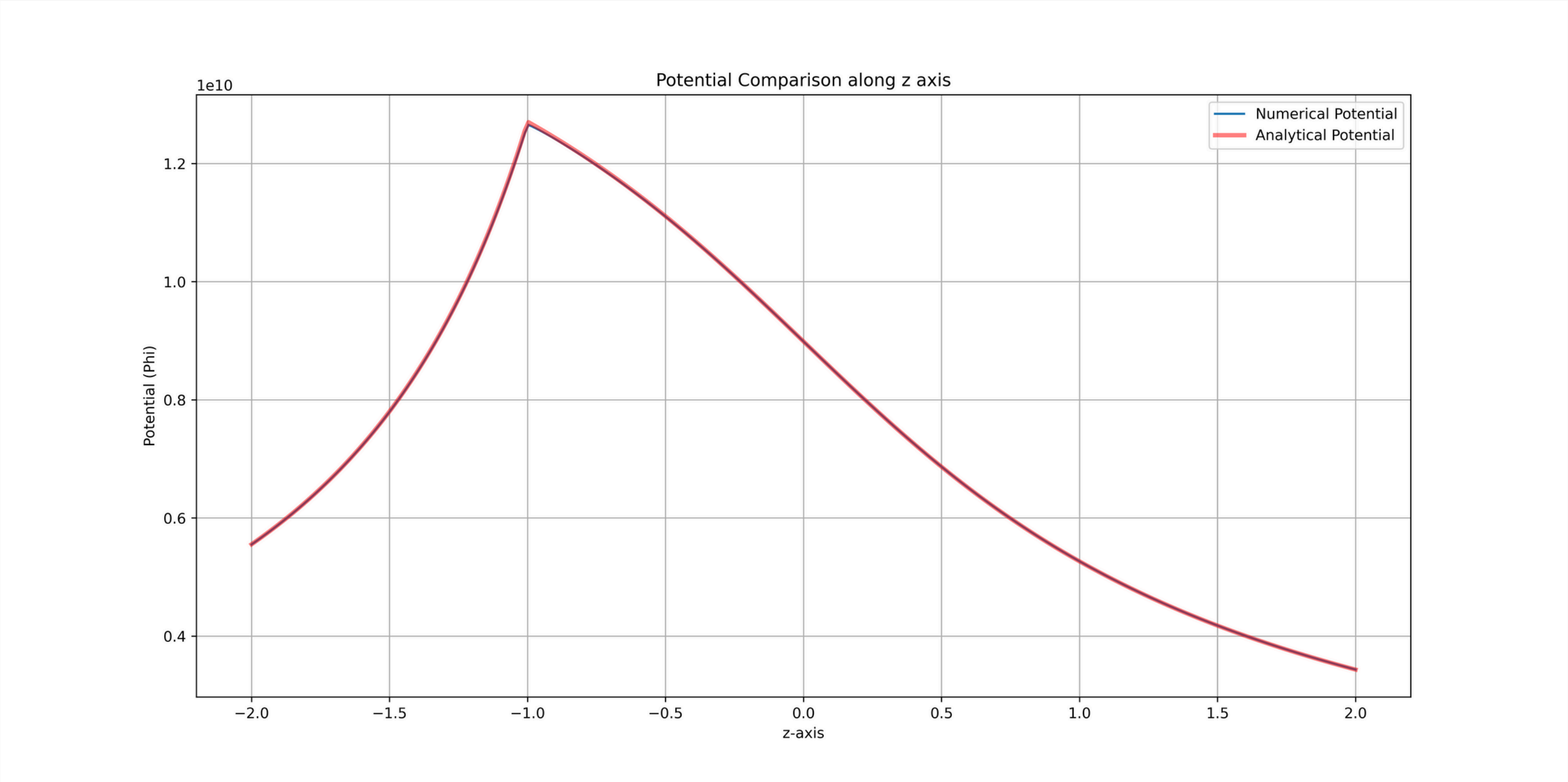
- I sampled 500,000 random points from the surface of the sphere and saved them. I used them for evaluating the integrals numerically.
- Then I divided the whole space in small blocks.
- I calculated potential and fields at each block using those points and plotted them in heatmap and streamplot.
- For the trajectories of the particle, I used RK4 method to find the next position (differ by a small time interval). I continued that for a certain time to get a good representation of the path.
- Then I plotted the trajectories in 3D plot.

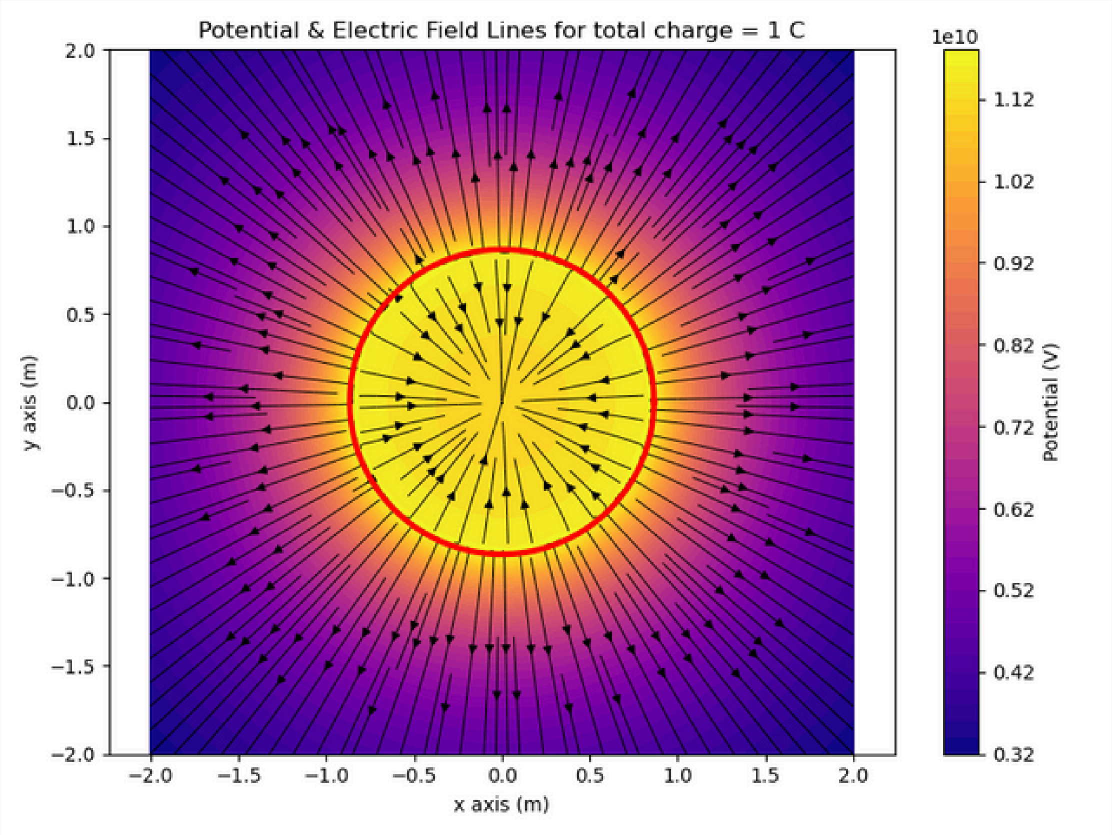
$$\vec{E} = \int_{-a}^{0} dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2-y^2}}^{\sqrt{a^2-z^2-y^2}} dx \frac{\sigma}{4\pi\epsilon_0} \frac{[x_0 - x]\hat{i} + [y_0 - y]\hat{j} + [z_0 - z]\hat{k}}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}$$

$$\phi = \frac{\sigma}{4\pi\epsilon_0} \int_{-a}^{0} dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2-y^2}}^{\sqrt{a^2-z^2-y^2}} dx \frac{1}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}}$$

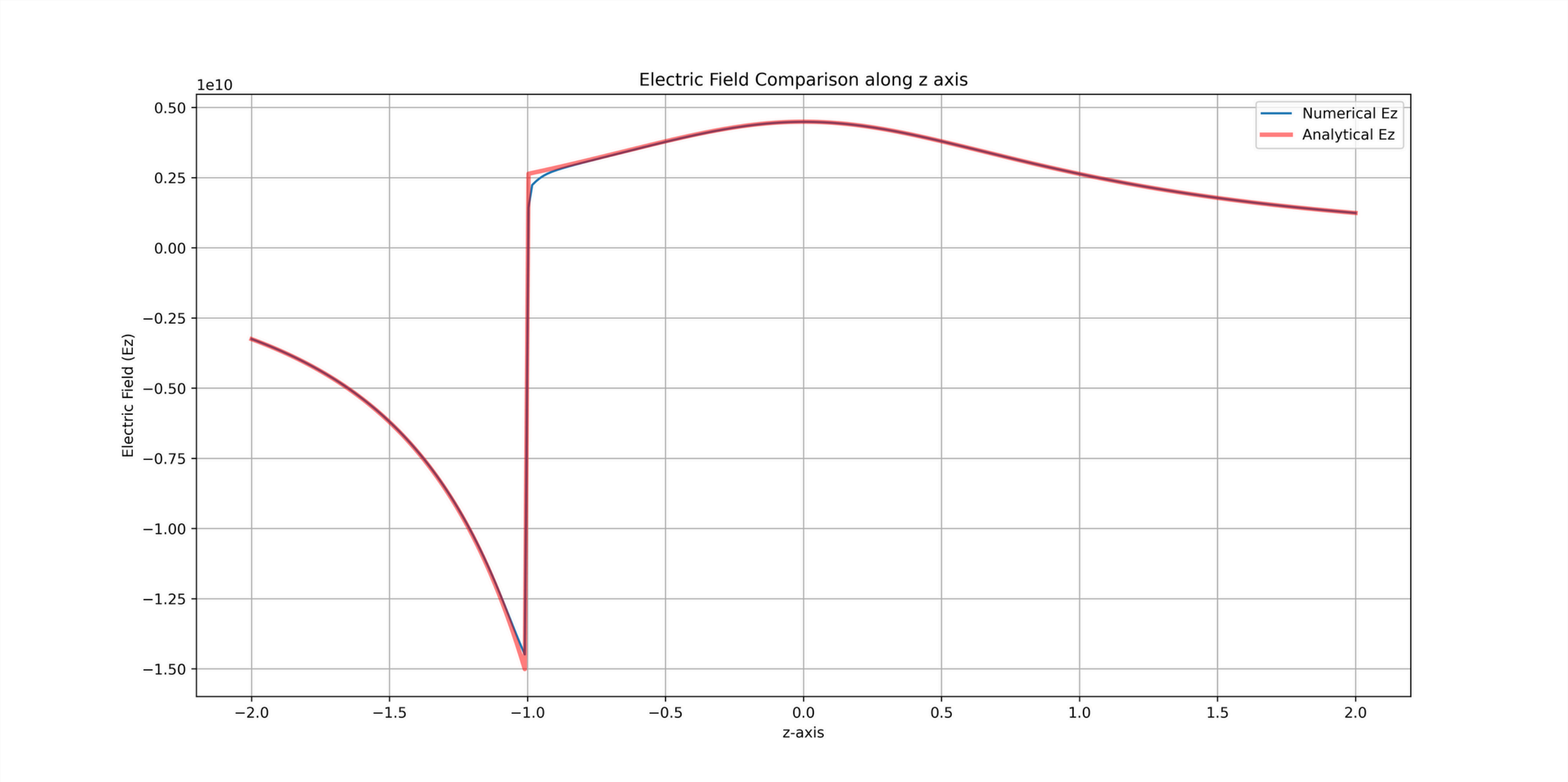
3.1 Achieving Consistent Results with the Analytical Method



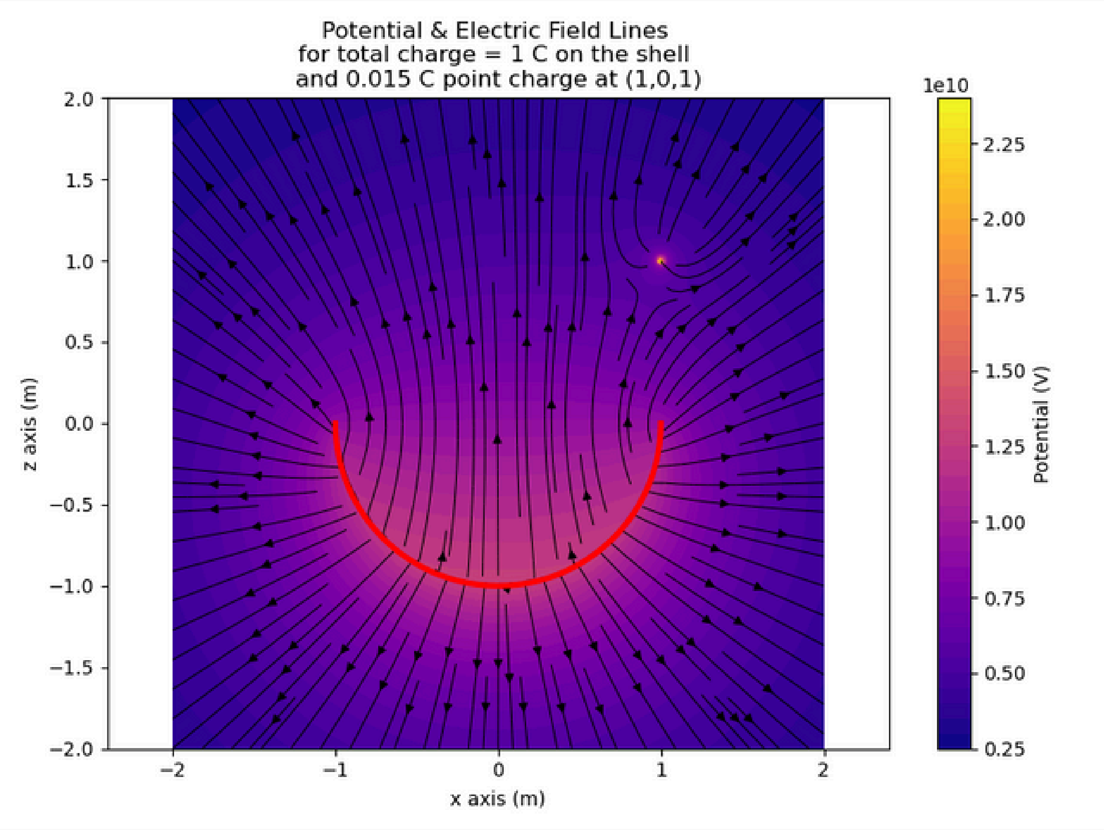


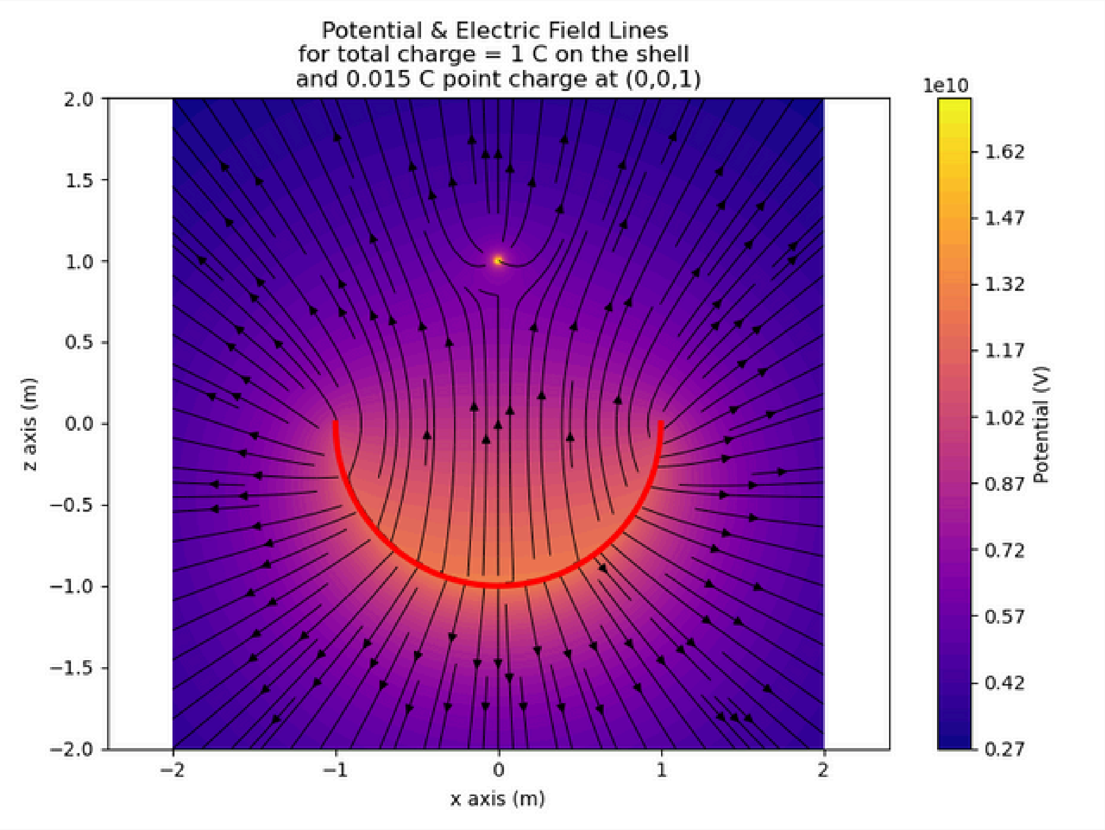


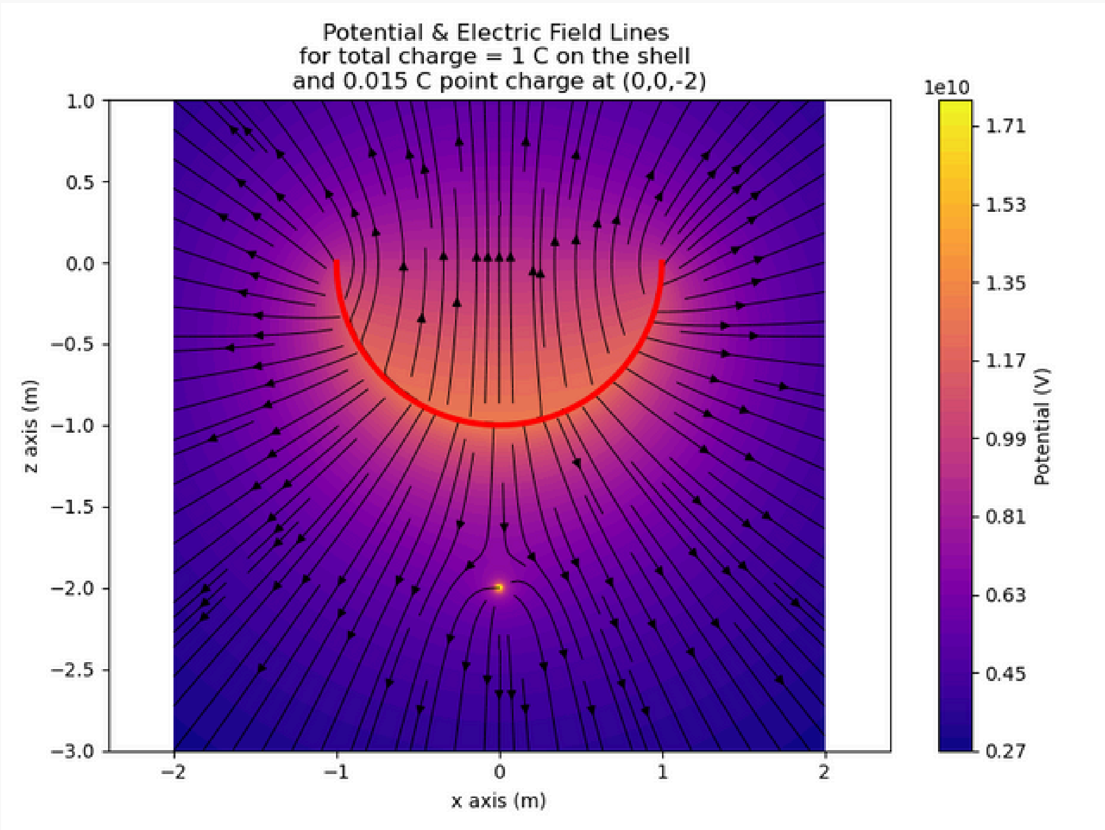
for z = -0.5

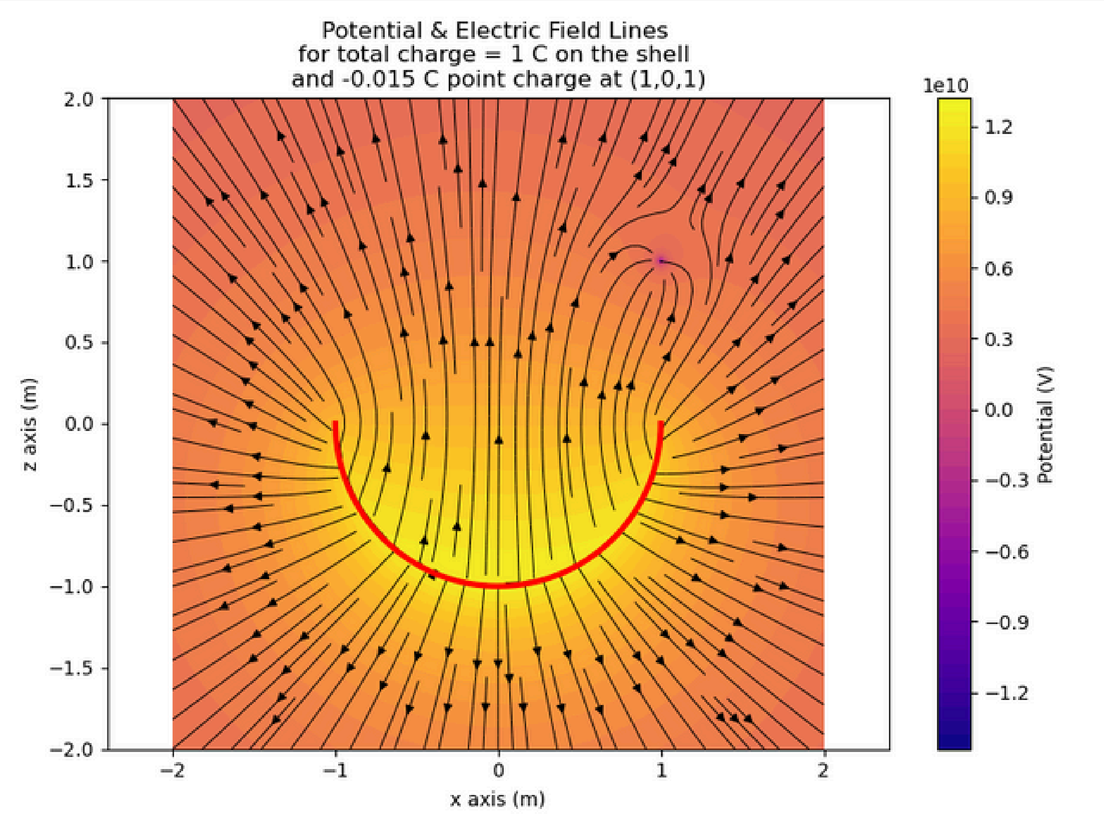


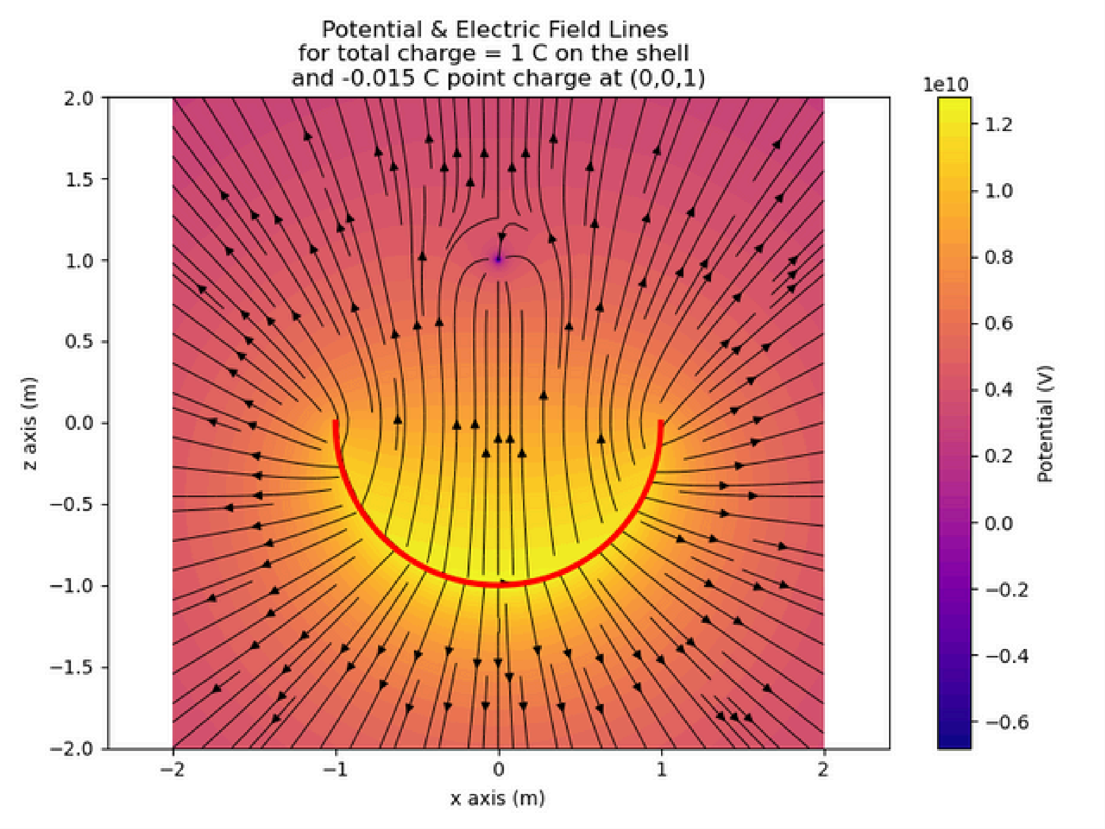
3.2 Beyond Basic Comparisons

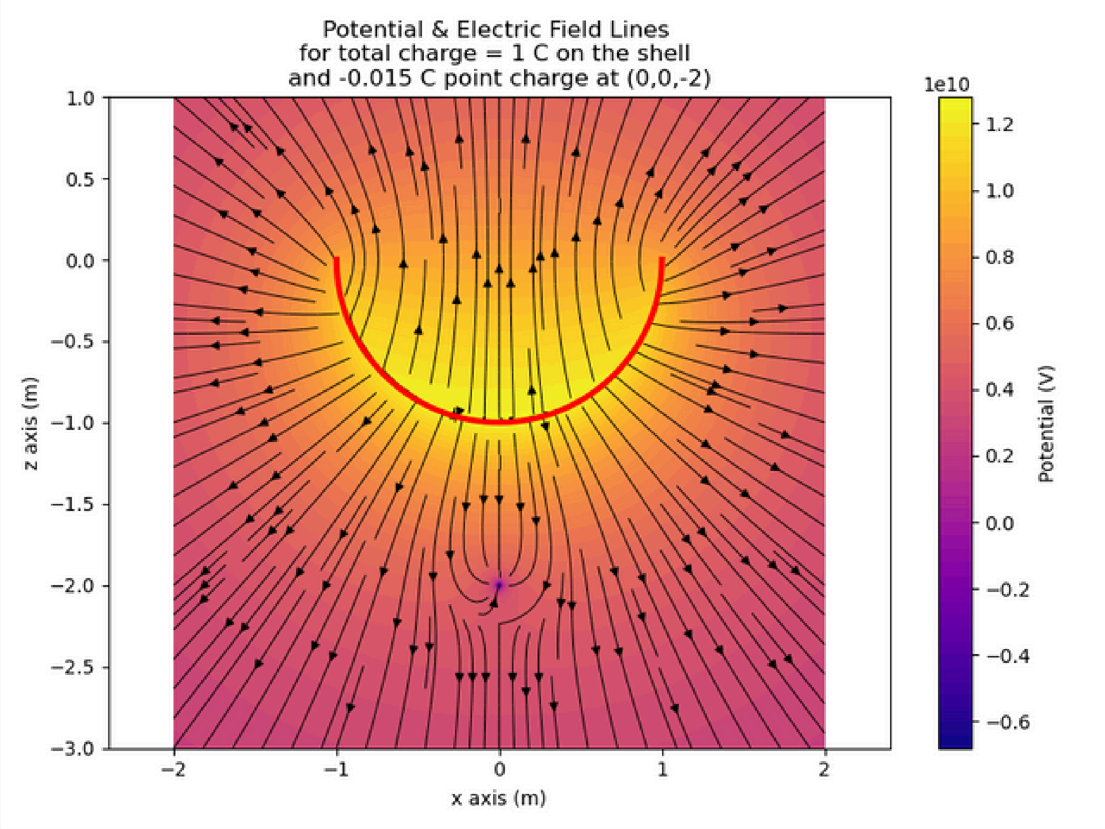












4. Estimation of Charged Particle Trajectories

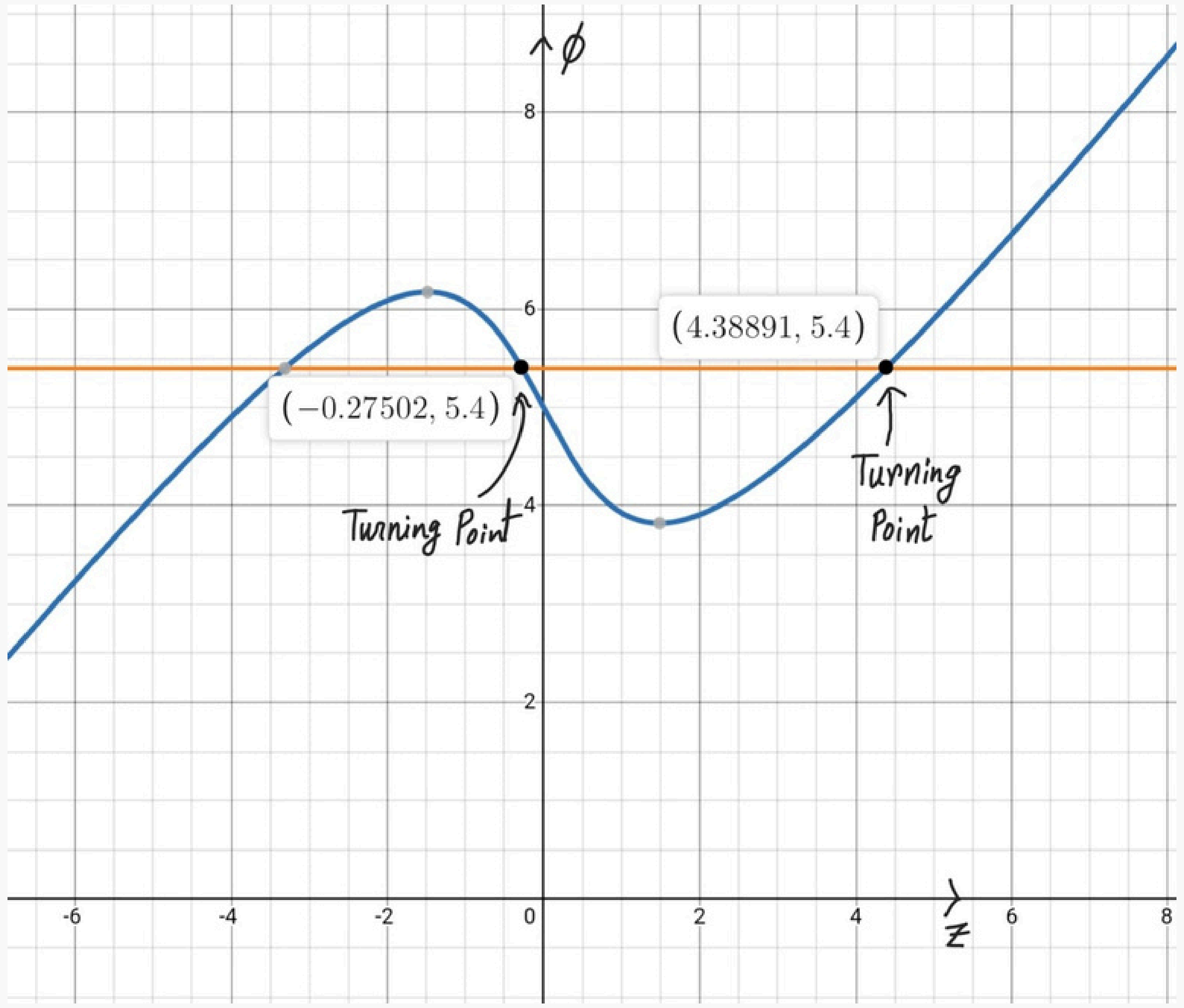
In this part, I will goanna discuss about some of the beautiful (at least I think :)) trajectories of the positively charged particle.

4.1. The Oscillator

The Potential of the charged particle will vary with ‘z’ as given in the equations below. That is given in the plot. Here we put all the quantities in natural units. The charge of the point is 10 units and the total charge of the shell is 1 unit. Also the mass is 1 unit.

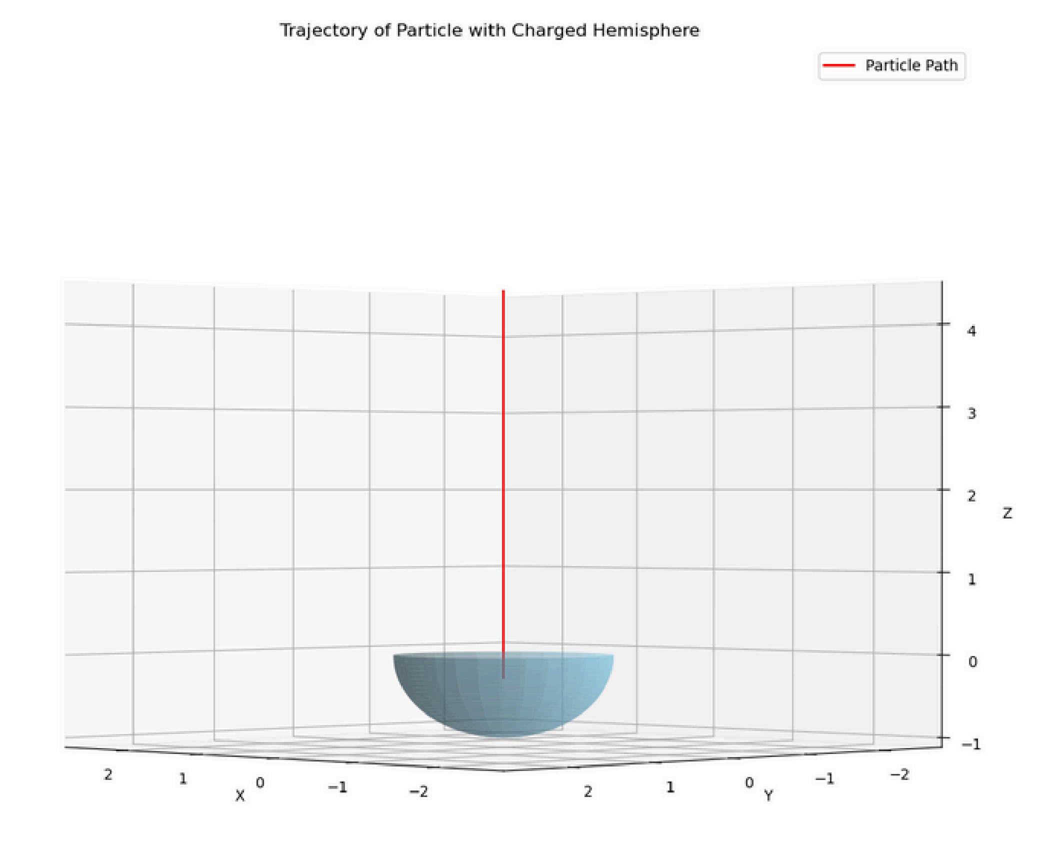
$$\phi = \frac{\sigma a}{2\epsilon_0 l}\left[l + a - \sqrt{a^2 + l^2}\right] + mgl$$ for x > -a

$$\phi = -\frac{\sigma a}{2\epsilon_0 l}\left[l + a + \sqrt{a^2 + l^2}\right] + mgl$$ for x < -a

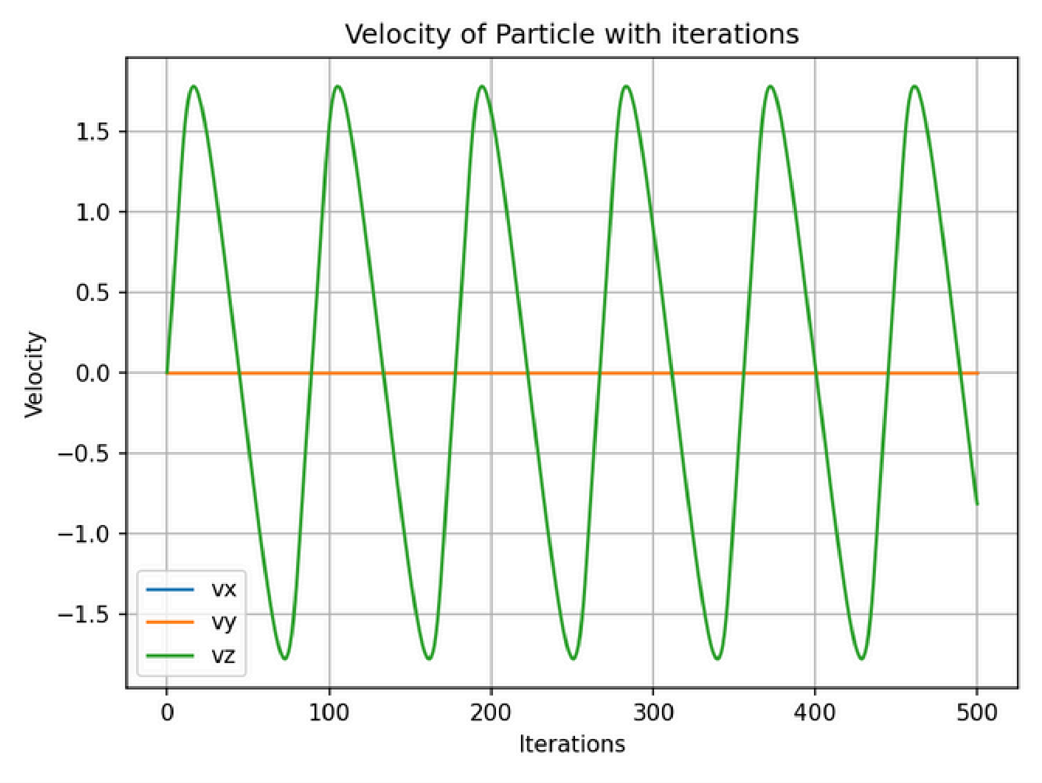


Total Potential Energy vs z-axis distance (l)

Here we can see an interesting thing. The curve has a local minima about 1.5 and a local maxima at about -1.5. Now If the energy of the particle lies between the values, It can have a bound state. One of the state has shown in the yellow line. If initially we place the particles at one of those turning points, then it will oscillate. That turning points in this case has shown in the plot. This specific plot is done in desmos. In the next slide I will show how the oscillation happens.

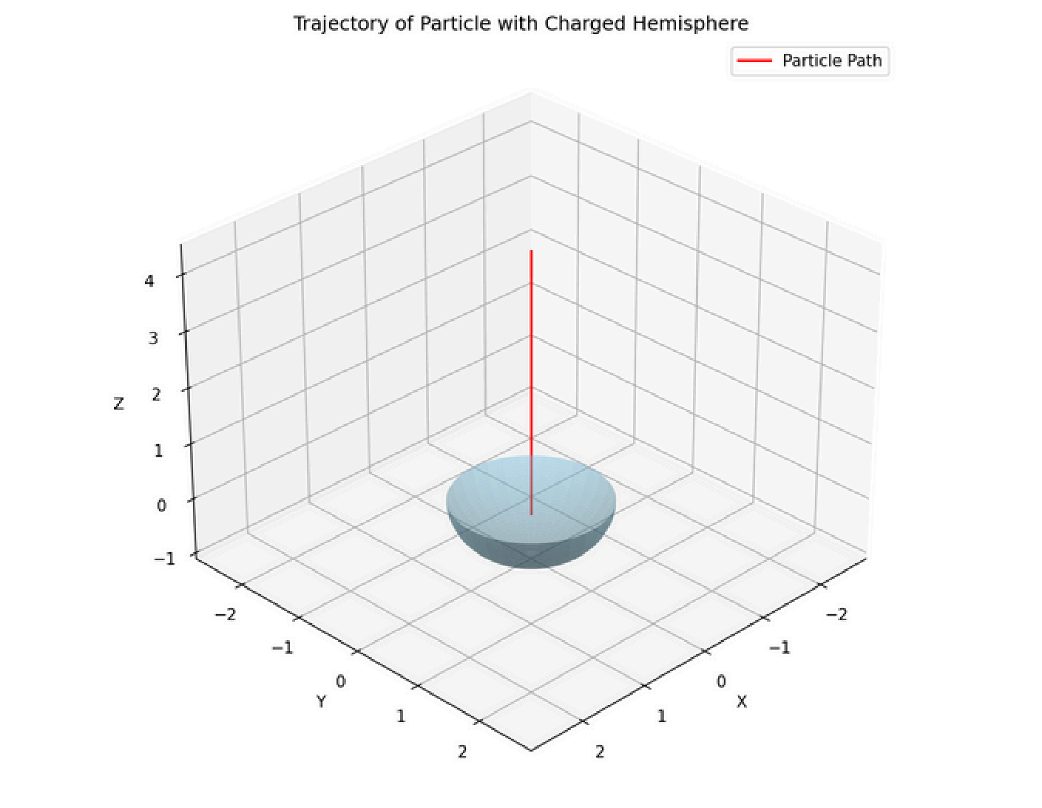


Here we show some interesting results obtained from the python simulation. In this case we have used same parameters as in the last slide.

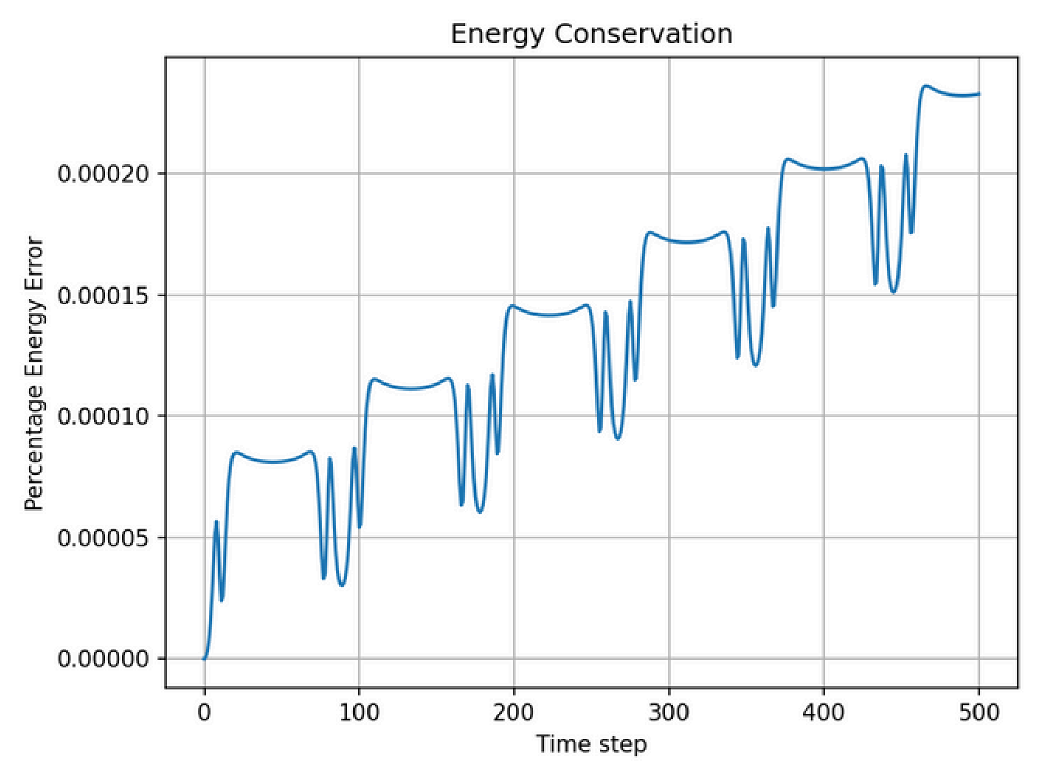


1. The motion is bounded between -0.27 and 4.38, as expected.

2. There is no deflection in the x-y direction. Which is also expected as the system is cylindrically symmetric about that axis.



3. The z-velocity is oscillating. But as the field is not uniform, it is not purely sinusoidal. So the motion is oscillatory but not SHM.

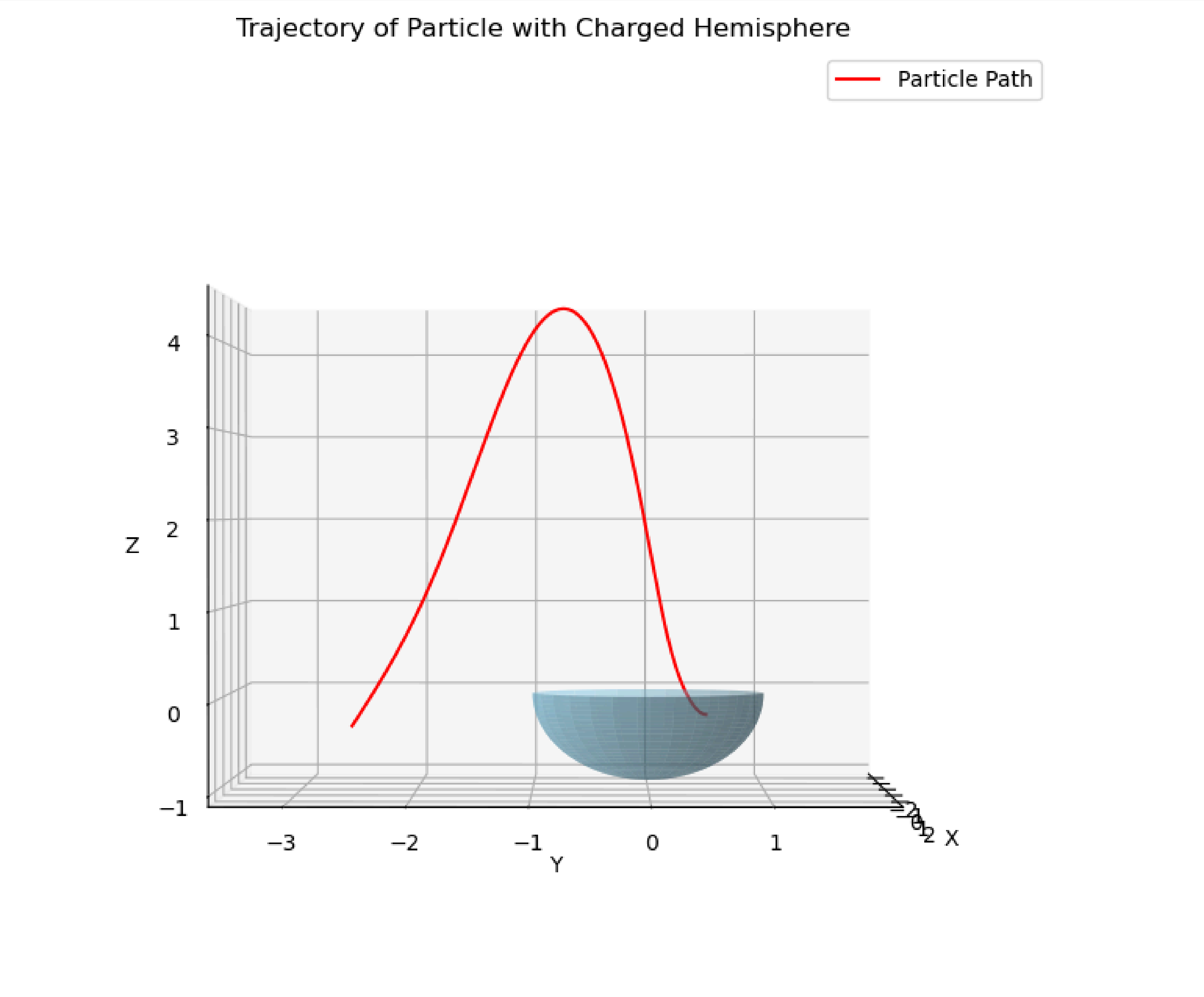


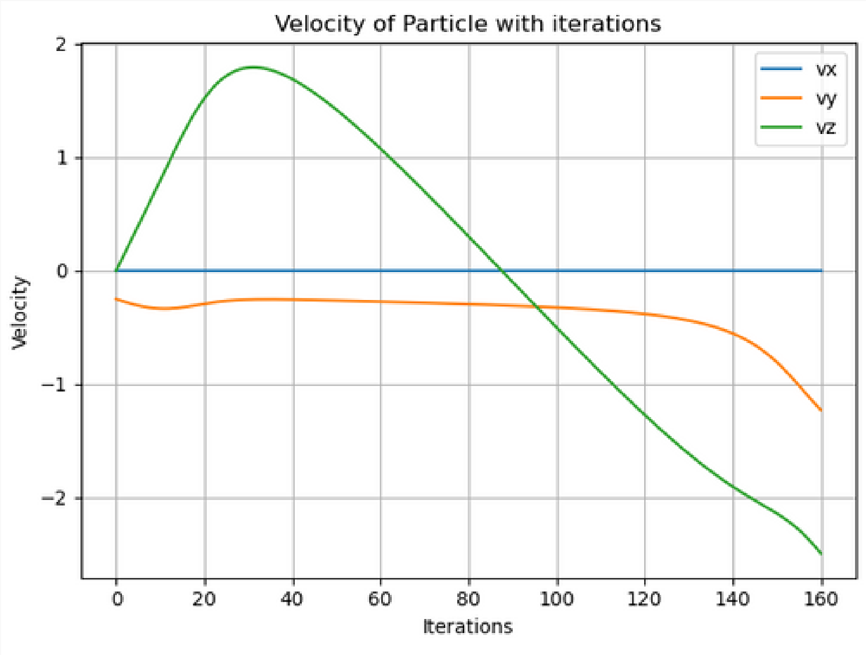
Here the energy error is very (very) small. So the trajectory is quite accurate.

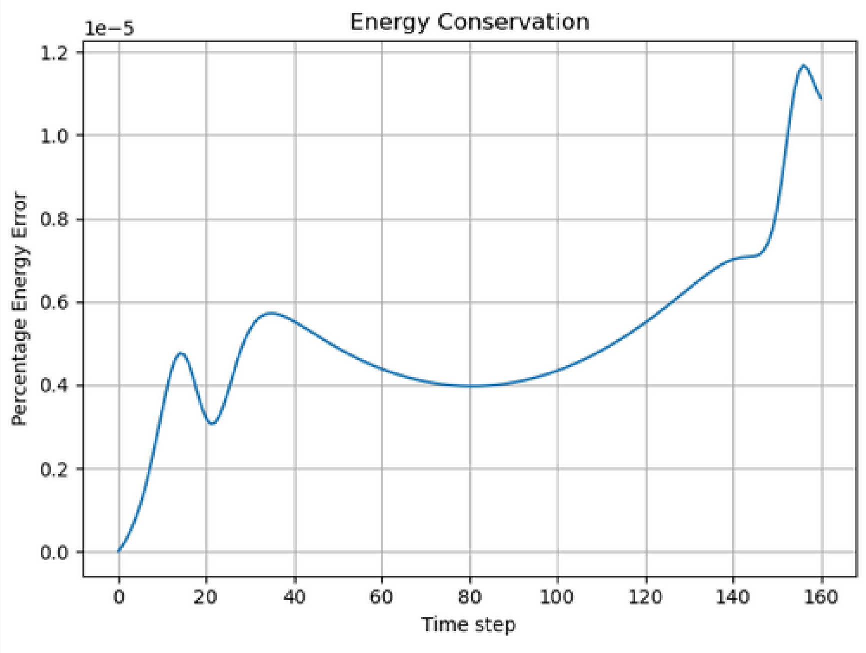
4.2. The Charged Splash

Here I have visualized a condition for which the particle, initially was inside the shell, moves like a splash of droplet from a bowl of liquid under the gravity.

All the quantities are in the natural units. The initial position of the charged particle with charge 63 units is (0, 0.5, -0.25). The initial velocity is (0, -0.25, 0).



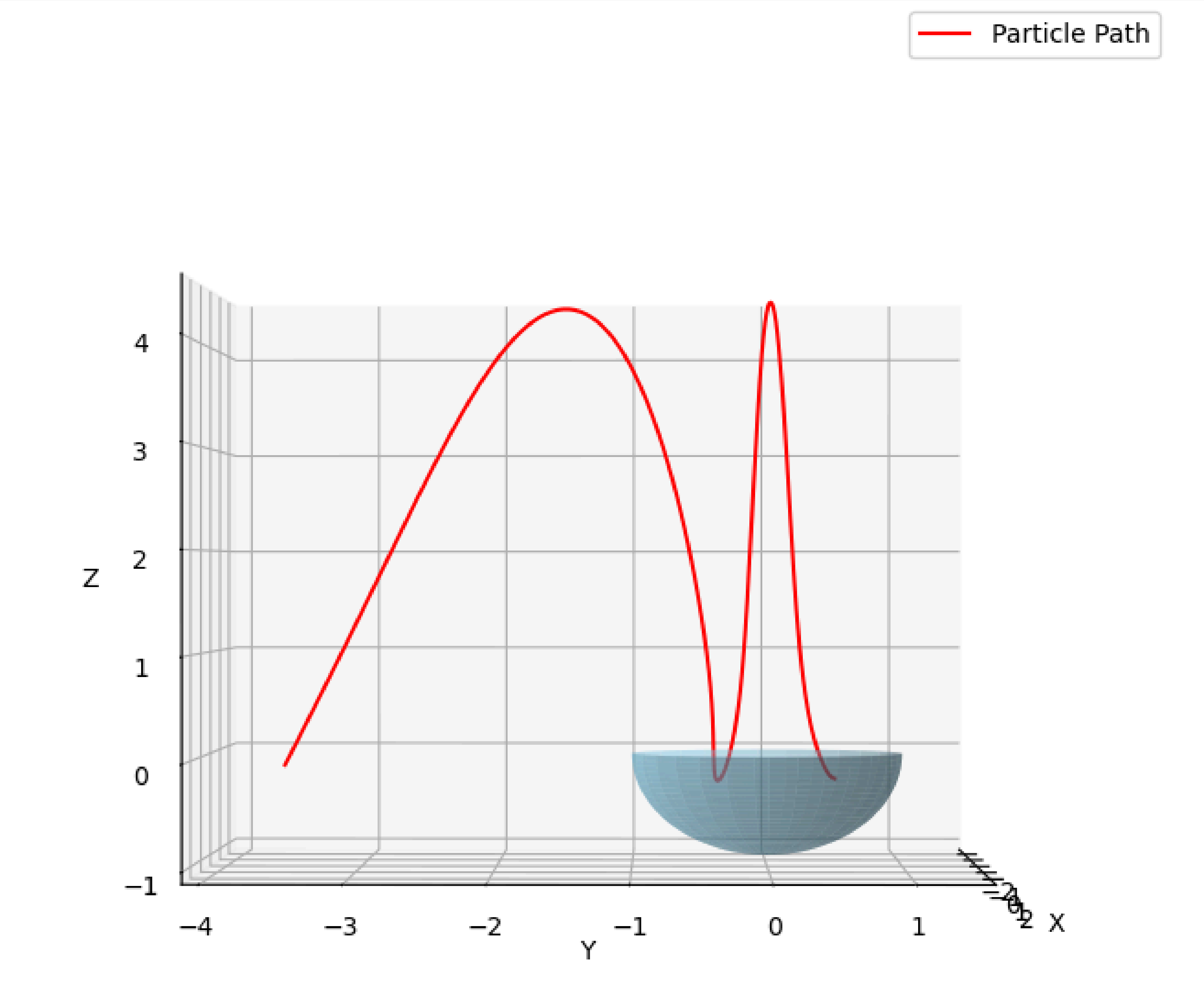


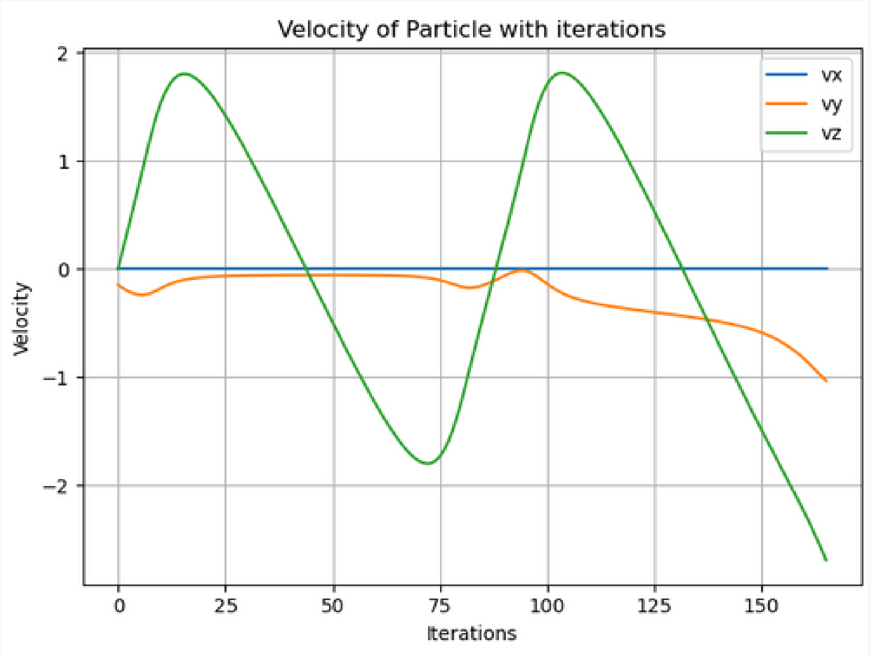


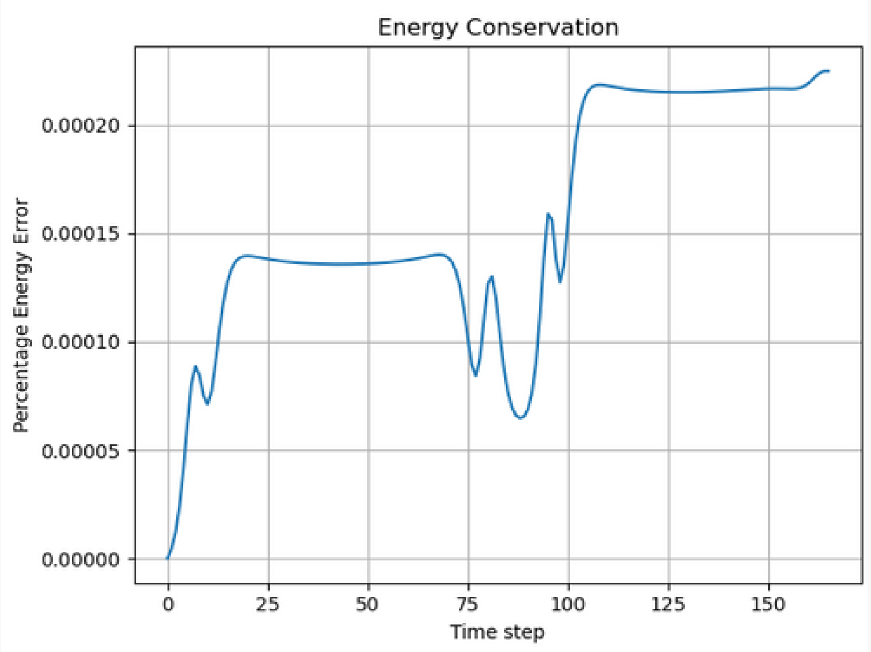
4.3. The Double Bouncer

By twicking the parameters, we can find a condition where the particle bounces from the boul not only one but two times. So I named it the double bouncer :)

All the quantities are in the natural units. The initial position of the charged particle with charge 63 units is (0, 0.5, -0.25). The initial velocity is (0, -0.15, 0).

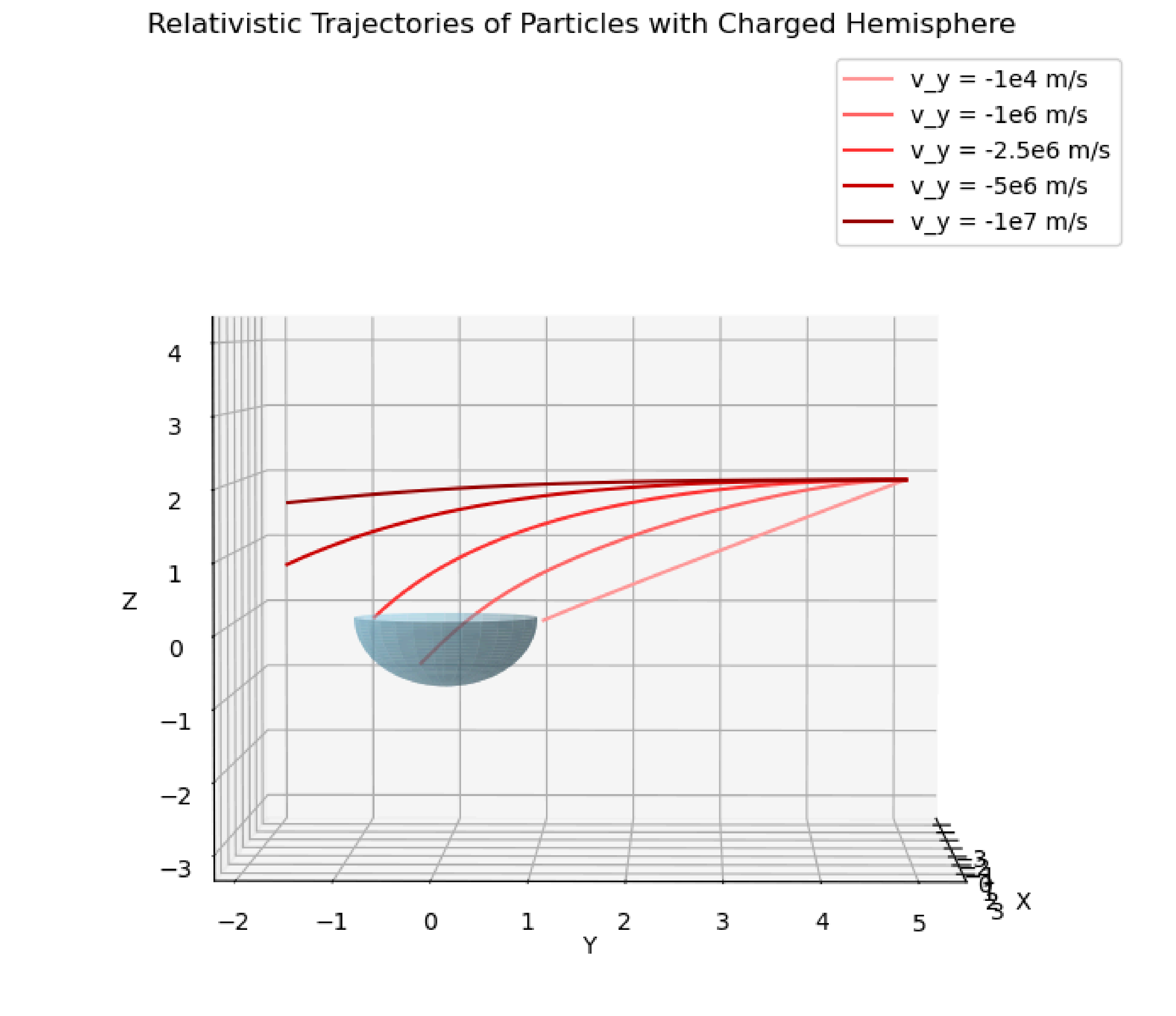


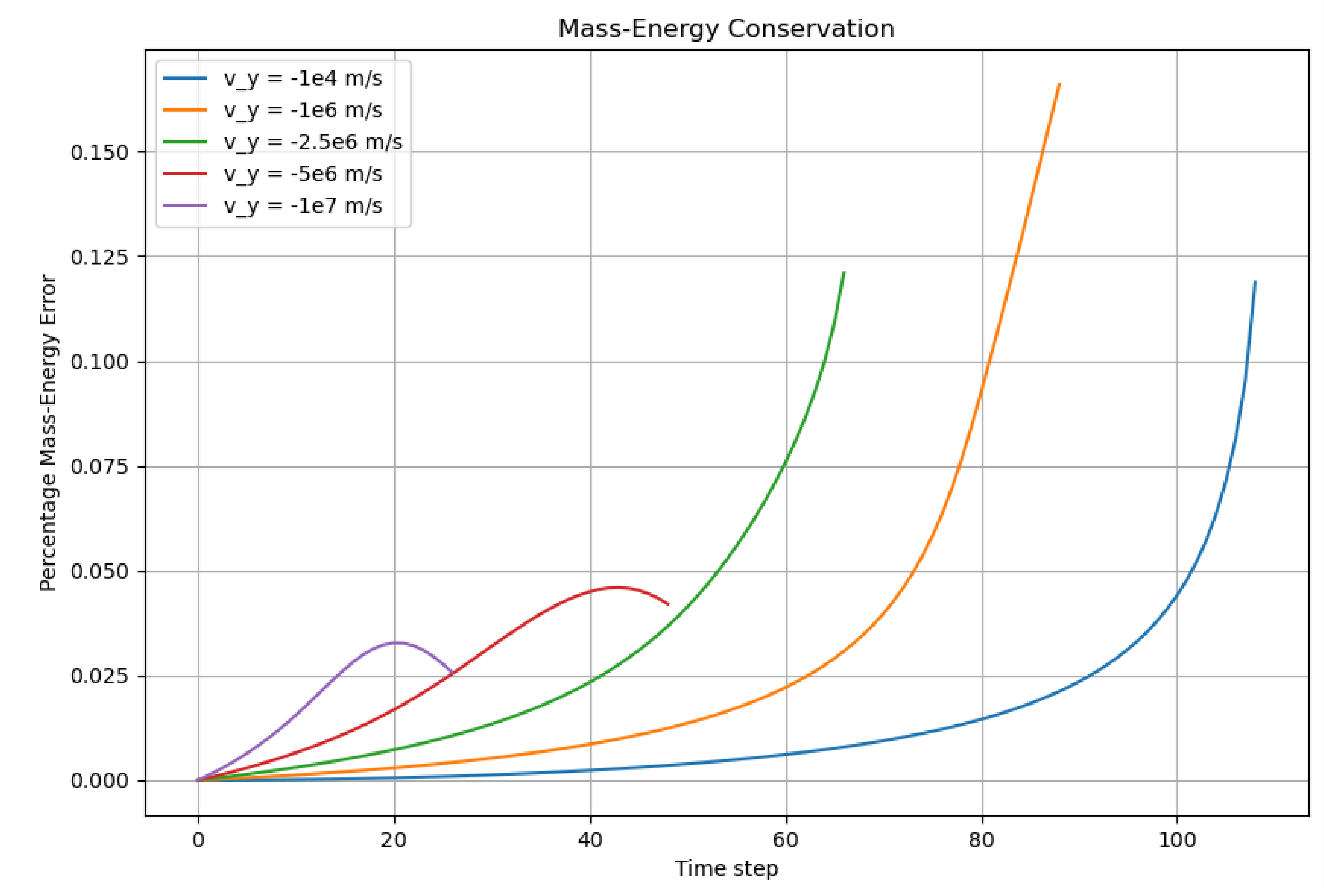




4.4. The Charged Beam Collector

Now let's try to see what happend for a beam of charged particles. Here for simplicity, we assume that the beam starts from (0,5,2). We take 5 electrons (yes this time all the quantities are in SI units) initially moving with different velocities along y axis. Then we visualize their trajectories due to the shell. The velocities are labelled in the plot. The bowl contains 0.1 μC of charge. Additionally, here I have taken relativistic effects into account :)



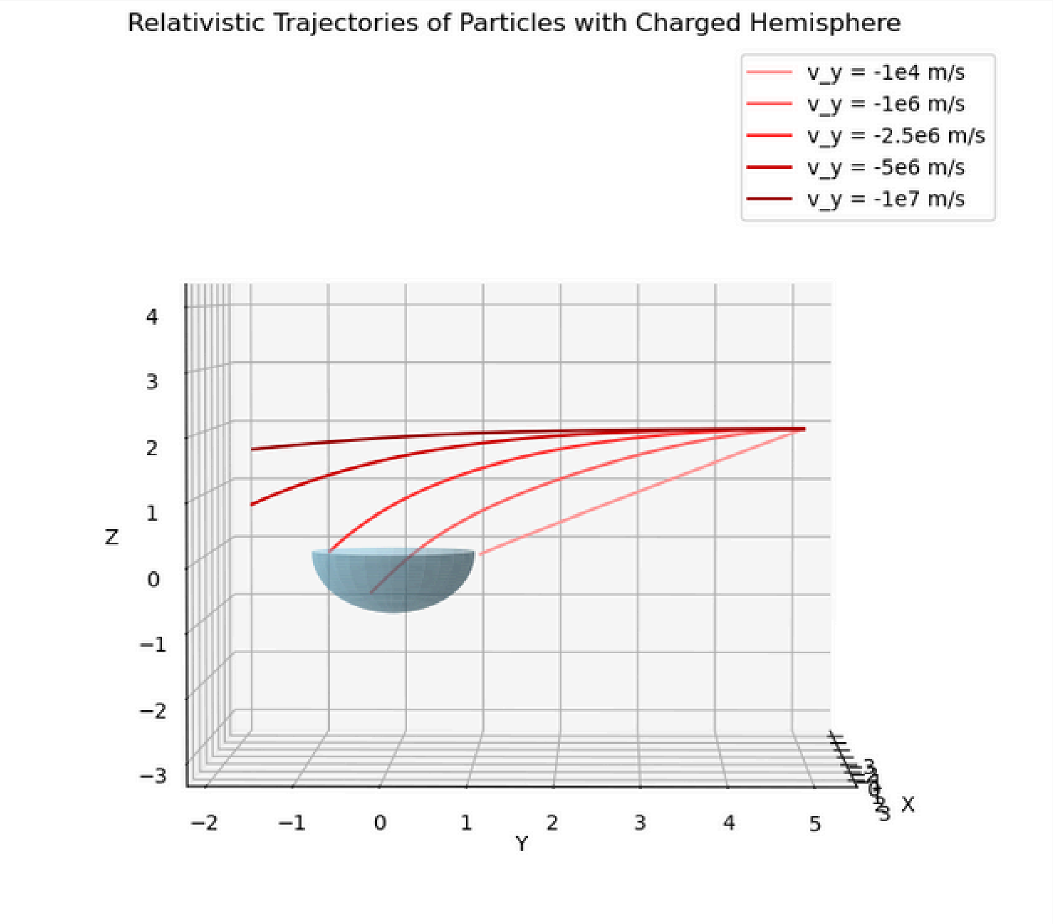


Potential use cases:

1. For studying the electrostatic properties of the shell.

2. Demonstrating the trajectories of particles under non-uniform electric field.

3. To make a electrostatic particle counter for measuring the number of particles, having velocity below a certain value, in a particle beam having various velocities. This is similar to the case that we have discussed just now.



References:

1. “Foundations of Electricity and Magnetism” by B. Ghosh
2. “Introduction to Electrodynamics” by David J. Griffiths
3. “Computational Physics” by Mark Newman
4. Wikipedia
5. Used ChatGPT for correcting grammer :)

Thank you for your attention.

Have a nice day!