Generating Function

The generating function for the sequence $a_0, a_1, \dots, a_K, \dots$ of real numbers is infinite series.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_K x^K + \dots = \sum_{K=0}^{\infty} a_K x^K \dots (1)$$

Some special generating functions:

1. The generating function of the sequence $1, 1, 1, \dots$ is

$$G(x) = 1 + x + x^2 + \dots = \sum_{K=0}^{\infty} x^K$$

which can be written in closed form as

$$G(x) = (1-x)^{-1} = \frac{1}{1-x} \leftarrow \text{Closed form expression}$$

$$G(x) - 1 = x(1 + x + \dots)$$

or, $G(x) - 1 = x \cdot G(x)$ $[\because G(x) = 1 + x + x^2 + \dots]$

or, $G(x)[1-x] = 1$

or, $G(x) = \dfrac{1}{1-x}$

2. The generating function of the sequence $1, 2, 3, 4, \dots$ is

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{K=0}^{\infty} (K+1) x^K$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2} \text{ in closed form } |x| < 1.$$

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$= (1 + x + 2x^2 + 3x^3 + \dots) + (x + x^2 + x^3 + \dots)$$

$$= \frac{1}{(1-x)^2} - \frac{x}{1-x} + \frac{x}{1-x}$$

$$= \frac{1}{(1-x)^2}$$

Let, $G_1(x) = 1 + x + 2x^2 + 3x^3 + \dots$

$$= (1 + x + x^2 + x^3 + \dots) + (x^2 + 2x^3 + 3x^4 + \dots)$$

$$= \frac{1}{1-x} + x(x + 2x^2 + 3x^3 + \dots)$$

$$G_1(x) = \frac{1}{1-x} + x(G_1(x) - 1)$$

or, $(1-x) G_1(x) = \dfrac{1}{1-x} - x$

or, $G_1(x) = \dfrac{1}{(1-x)^2} - \dfrac{x}{1-x}$

Let, $G_2(x) = x + x^2 + x^3 + \dots$

$$= x(1 + x + x^2 + \dots)$$

$$= x \cdot \frac{1}{1-x} = \frac{x}{1-x}$$

3. The generating function of the sequence $0, 1, 2, 3, \dots$ is

$$G(x) = 0 + 1x + 2x^2 + 3x^3 + \dots = \sum_{K=0}^{\infty} K x^K$$

$$= x(1 + 2x + 3x^2 + \dots)$$

$$= \frac{x}{(1-x)^2} \text{ in closed form}$$

4. The generating function of the sequence $1, a, a^2, a^3, \dots$ is

$$G(x) = 1 + ax + a^2x^2 + a^3x^3 + \dots = \sum_{K=0}^{\infty} a^K x^K$$

$$= \frac{1}{1-ax} \text{ in closed form } |ax| < 1$$

General term of Sequence a_K		Generating Function $G(x)$
1	$\rightarrow$	$\frac{1}{1-x}$
$(-1)^K$	$\rightarrow$	$\frac{1}{1+x}$
$K+1$	$\rightarrow$	$\frac{1}{(1-x)^2}$
K	$\rightarrow$	$\frac{x}{(1-x)^2}$
$K(K+1)$	$\rightarrow$	$\frac{2x}{(1-x)^3}$
$(K+1)(K+2)$	$\rightarrow$	$\frac{2}{(1-x)^3}$
a^K	$\rightarrow$	$\frac{1}{1-ax}$
$(-a)^K$	$\rightarrow$	$\frac{1}{1+ax}$
$\frac{1}{K!}$	$\rightarrow$	e^x

Addition and Multiplication of two Generating Functions:

Arithmetic operations allow us to create new generating functions from old ones.

Let, $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{K=0}^{\infty} a_K x^K$

$G(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots = \sum_{K=0}^{\infty} b_K x^K$

$$F(x)+G(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots$$
$$= \sum_{K=0}^{\infty}(a_K+b_K)\cdot x^K \quad \text{(Ans)}$$

$\therefore$ $F(x)+G(x)$ is the generating function of a_K+b_K.

$$F(x)\cdot G(x) = (a_0+a_1x+a_2x^2+a_3x^3+\cdots)\cdot(b_0+b_1x+b_2x^2+b_3x^3+\cdots)$$
$$= a_0b_0 + (a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2$$
$$+ (a_0b_3+a_1b_2+a_2b_1+a_3b_0)x^3+\cdots+\left(\sum_{i=0}^{K} a_i b_{K-i}\right)x^K+\cdots$$
$$= \sum_{K=0}^{\infty}\left(\sum_{i=0}^{K} a_i b_{K-i}\right)x^K$$

$\therefore$ $F(x)\cdot G(x)$ is the generating function of $a_K \times b_K$.

Problem: Find the generating function of a sequence $\{a_k\}$ if $a_k = 2+3k$.

Solution: The generating function of a sequence whose general term is 2 is $F(x) = \frac{2}{1-x}$

The generating function of a sequence whose general term is $3k$ is $G(x) = \frac{3x}{(1-x)^2}$

$\therefore$ The required generating function is
$$F(x)+G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2} \text{ (Ans).}$$

Problem: Find the sequences corresponding to the generating function $(3+x)^3$.

Solution: $(3+x)^3 = 27+27x+9x^2+x^3$

The sequence is $(27, 27, 9, 1, 0, 0, 0, \ldots)$

Shifting properties of generating function:

1. If $G(x) = \sum_{n=0}^{\infty} a_n x^n$ generates the sequence $(a_0, a_1, a_2, \ldots)$, then $xG(x)$ generates the sequence $(0, a_0, a_1, a_2, \ldots)$, $x^2G(x)$ generates $(0, 0, a_0, a_1, a_2, \ldots)$ and in general $x^k G(x)$ generates $(0, 0, \ldots, 0, a_0, a_1, a_2, \ldots)$ where there are k zeroes before a_0.

For instance, we know that, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ generates the sequence $(1, 1, 1, \ldots)$, that is the sequence $\{a_n\}$ where $a_n = 1$ for each $n \geq 0$.

Thus, $\frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} = \sum_{r=1}^{\infty} x^r$ generates $(0, 1, 1, 1, \ldots)$

and $\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2} = \sum_{r=2}^{\infty} x^r$ generates $(0, 0, 1, 1, 1, \ldots)$

2. If $G(x) = \sum_{n=0}^{\infty} a_n x^n$ generates $(a_0, a_1, a_2, \ldots)$, then $G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$ generates $(0, a_1, a_2, \ldots)$, $G(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$ generates $(0, 0, a_2, a_3, \ldots)$ and in general $G(x) - a_0 - a_1 x - \cdots - a_{k-1} x^{k-1}$ generates $(0, 0, \ldots 0, a_k, a_{k+1}, \ldots)$, where there are k zeroes before a_k.

3. Dividing by powers of x shifts the sequence to the left. For instance, $(G(x) - a_0)/x = \sum_{n=1}^{\infty} a_n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} x^n$ generates the sequence $(a_1, a_2, a_3, \ldots)$; $(G(x) - a_0 - a_1 x)/x^2 = \sum_{n=2}^{\infty} a_n x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} x^n$ generates the sequence $(a_2, a_3, a_4, \ldots)$; and in general, for $k \geq 1$, $(G(x) - a_0 - a_1 x - \cdots - a_{k-1} x^k)/x^k$ generates $(a_k, a_{k+1}, a_{k+2}, \ldots)$.

Solution of Linear Recurrence Relation using Generating Function:

Problem: Use generating functions to solve the recurrence relation.

i) $a_n = 3a_{n-1} + 2$, $a_0 = 1$ ii) $a_n - 9a_{n-1} + 20a_{n-2} = 0$, $a_0 = -3$, $a_1 = -10$

Solution: i) Let, $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 1 to ∞, we get,

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3xG(x) + 2\left[\frac{1}{1-x} - 1\right] \quad \left[\because xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\right]$$

$$G(x) - 3xG(x) = 1 + \frac{2x}{1-x} \quad [\because a_0 = 1]$$

$$G(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-x}$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n \quad \left[\because G(x) = \sum_{n=0}^{\infty} a_n x^n\right]$$

So, $a_n = 2\cdot 3^n - 1$ (Ans.)

Solution-ii: Let, $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 2 to ∞, we get,

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1} x^n + 20\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

or, $\left[G(x) - a_0 - a_1 x\right] - 9x\left[G(x) - a_0\right] + 20x^2 G(x) = 0$

or, $G(x)\left[1 - 9x + 20x^2\right] = a_0 + a_1 x - 9a_0 x$

or, $G(x) = \dfrac{a_0 + a_1 x - 9a_0 x}{1 - 9x + 20x^2} = \dfrac{-3 - 10x + 27x}{1 - 9x + 20x^2}$ $[\because a_0 = -3, a_1 = -10]$

or, $G(x) = \dfrac{-3 + 17x}{(1-5x)(1-4x)}$

or, $G(x) = \dfrac{2}{1-5x} - \dfrac{5}{1-4x}$

$$\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 5^n x^n - 5\sum_{n=0}^{\infty} 4^n \cdot x^n$$

$\therefore a_n = 2\cdot 5^n - 5\cdot 4^n$ (Ans.)