Recurrence Relation

A sequence can be written by finding a relationship among its terms. Such a relationship is called <u>recurrence relation</u>.

Ex1: $S = \{5, 8, 11, 14, 17, \cdots\}$ ← Sequence

$S_n = S_{n-1} + 3,\ n \geq 2$, with initial condition $S_1 = 5$. ← Recurrence Relation

Ex2: $S = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$ ← Sequence

$S_n = S_{n-1} + S_{n-2},\ n \geq 3$ with initial conditions $S_1 = S_2 = 1$

↑ Recurrence Relation.

Problem: Find the first four terms of the following recurrence relation.

$a_k = a_{k-1} + 3a_{k-2}$, for all integers $k \geq 2$, $a_0 = 1$, $a_1 = 2$.

Solution: $\left.\begin{array}{l} a_0 = 1 \\ a_1 = 2 \end{array}\right\}$(Given) $a_2 = a_1 + 3a_0 = 2 + 3 \cdot 1 = 5$, $a_3 = a_2 + 3a_1 = 5 + 3 \cdot 2 = 11$

Problem: Show that the sequence $\{2, 3, 4, 5, \cdots\cdots, 2+n, \cdots\cdots\}$ for $n \geq 0$ satisfies the recurrence relation $a_k = 2a_{k-1} - a_{k-2},\ k \geq 2$.

Solution: Let, n^{th} term of the sequence, $a_n = 2 + n$

$$a_k = 2 + k$$
$$a_{k-1} = 2 + (k-1) = 1 + k$$
$$a_{k-2} = 2 + (k-2) = k$$

So, $2a_{k-1} - a_{k-2} = 2(1+k) - k = 2 + k = a_k$

$$\therefore a_k = 2a_{k-1} - a_{k-2}$$

Problem: A person invests Rs. 10000 @ 12% interest compounded annually. How much will be there at the end of 15 years.

Solution: Let, A_n represents the amount at the end of n years. So, at the end of $n-1$ years, the amount is A_{n-1}. Since, the amount after n years equals the amount after $n-1$ years plus interest for the n^{th} year. Thus, the sequence $\{A_n\}$ satisfies the following recurrence relation.

$A_n = A_{n-1} + (0.12) A_{n-1}$

or, $A_n = (1.12) A_{n-1}$, $n \geq 1$, with initial condition $A_0 = 10000$.

$A_1 = (1.12)^1 A_0$

$A_2 = (1.12) A_1 = (1.12)^2 A_0$

$A_3 = (1.12) A_2 = (1.12)^3 A_0$

$\vdots$

$A_n = (1.12)^n A_0$

So, $A_{15} = (1.12)^{15} \cdot 10000$ Rs. (Ans.)

Solution of Recurrence Relation: An explicit formula for a_n which satisfy the recurrence relation with initial conditions is called a solution of recurrence relation.

Linear Recurrence Relation with Constant Coefficients: Following is its general form.

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n), \text{ where } c_i\text{'s} \quad \cdots\cdots (1)$$

are constants.

Order of the relation is k (i.e. difference between highest and lowest subscript of a).

Degree of a relation is the highest power of a_n. (Here degree = 1).

A linear recurrence relation is an equation that defines each term in a sequence as a linear combination of previous terms. It allows to express the next term in the sequence as a function of preceding terms using constant coefficients (c_i's). It does not contain terms like a_{n-1}^2, $a_{n-1} \times a_{n-2}$ and other non-linear terms.

If the R.H.S of eqn-(1) is zero, then the recurrence relation is called homogeneous, otherwise, it is non-homogeneous.

Examples:

i) $a_n = 2a_{n-1}$ is a linear homogeneous relation with constant coefficients of order 1 and degree 1.

ii) $a_n = 2a_{n-1} a_{n-2}$ is not a linear homogeneous relation because of the presence of the product term.

iii) $a_n - a_{n-1} = 3$ is not a linear homogeneous relation because R.H.S $\neq 0$.

iv) $a_n = a_{n-1} + a_{n-2}$ is a linear homogeneous relation with constant coefficients of order 2 and degree 1.

Methods of solving linear recurrence relation:

1. Iterative method 2. Characteristic roots 3. Generating Function

1. **Iterative method:** In this method the recurrence relation for a_n is used repeatedly to look for a pattern to find a general expression for a_n in terms of n.

Prob: Solve the recurrence relation $a_n = a_{n-1} + 2$, $n \geq 2$ subject to initial condition $a_1 = 3$ using iterative method.

Solution: Given, $a_n = a_{n-1} + 2 \cdots\cdots (1)$

Replacing n by $n-1$ in (1),

$$a_{n-1} = a_{n-2} + 2$$

From eqn-(1), $a_n = a_{n-1} + 2$

$$= (a_{n-2} + 2) + 2$$

$$= a_{n-2} + 2\cdot 2 \cdots\cdots (2)$$

Replacing n by $n-2$ in (1),

$$a_{n-2} = a_{n-3} + 2$$

From eqn-(2), $a_n = (a_{n-3} + 2) + 2\cdot 2$

$$= a_{n-3} + 3\cdot 2 \cdots\cdots (3)$$

In general, $a_n = a_{n-k} + k\cdot 2$

For $k = n-1$, $a_n = a_{n-(n-1)} + (n-1)\cdot 2$

$$= a_1 + (n-1)\cdot 2$$

$$a_n = 3 + (n-1)\cdot 2 \quad [\because a_1 = 3]$$

$$a_n = n+1 \text{ (Ans.)}$$

2. **Characteristic roots method:** In this method, the solution is obtained as sum of two parts. The homogeneous solution satisfies the recurrence relation when the R.H.S of the relation is set to 0 i.e. $f(n) = 0$ and the particular solution satisfies the relation with $f(n)$ on the R.H.S.

i) **Homogeneous ~~Solution~~ Recurrence Relation:** Basic approach for solving homogeneous relation $[f(n) = 0]$ is to look for solution of the form $a_n = r^n$. Now, $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \quad [\text{Putting } a_n = r^n]$$

Dividing both sides by r^{n-k}, we get,

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0 \leftarrow \text{Characteristic eqn. of recurrence relation.}$$

The solutions of this equation are called characteristic roots of recurrence relation. A characteristic eqn. of k^{th} degree has k characteristic roots.

a) **Distinct roots:** If characteristic eqn. has distinct roots $r_1, r_2, \ldots, r_k$, then the general form of the solutions for homogeneous eqn. is $a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \cdots + b_k r_k^n$ where $b_1, b_2, b_3, \ldots, b_k$ are constants which may be chosen to satisfy any initial conditions.

Problem: Solve $a_n = a_{n-1} + 2a_{n-2}, n \geq 2$, with initial conditions $a_0 = 0, a_1 = 1$.

Solution: $a_n = a_{n-1} + 2a_{n-2}$

$a_n - a_{n-1} - 2a_{n-2} = 0$ is a second order linear homogeneous recurrence relation with constant coefficients.

$\hookrightarrow (1)$

Let, $a_n = r^n$ is a solution of eqn-(1).

The characteristic eqn. is $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

So, $r = 2, -1$ are distinct real roots.

So, the general solution is : $a_n = b_1(2)^n + b_2(-1)^n$

Now, $a_0 = 0$ implies $b_1 + b_2 = 0$

and $a_1 = 1$ implies $2b_1 - b_2 = 1$

The solution of these two eqns. are $b_1 = 1/3$, $b_2 = -1/3$

So, the final solution is : $a_n = (1/3)2^n - (1/3)(-1)^n$ (Ans.)

b) Multiple roots: If the characteristic eqn. of a homogeneous recurrence relation is $(r-2)^3 = 0$, then $r = 2$ is a required root of multiplicity 3. Then the general sol^n is :

$$a_n = (b_1 + nb_2 + n^2 b_3) \cdot 2^n$$

In general, if r is a root of the characteristic eqn. of m^{th} order of a given recurrence relation with multiplicity m, then the general form of the sol^n is :

$$a_n = (b_1 + nb_2 + n^2 b_3 + \cdots + n^{m-1} b_m) r^n,$$ where $b_1, b_2, \ldots, b_m$ are constants which may be chosen to satisfy any initial conditions.

Problem: Solve the recurrence relation

$$a_n = 4(a_{n-1} - a_{n-2})$$ with initial conditions $a_0 = a_1 = 1$.

Solution: The given relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$ ....(1)

Let, $a_n = r^n$ is a solution of eqn. (1).

Then, the characteristic eqn. is $r^2 - 4r + 4 = 0$ which gives $r = 2, 2$.

Thus the general solution is $a_n = (b_1 + nb_2) 2^n$

So, $a_0 = b_1$ and $a_1 = 2(b_1 + b_2)$

Now, $a_0 = 1$ gives $b_1 = 1$

and $a_1 = 1$ gives $2(b_1 + b_2) = 1 \Rightarrow b_2 = -1/2$

So, the final solution is : $a_n = (1 - \frac{1}{2}n) 2^n$

c) Mixed roots: A combination of distinct and multiple roots is also possible. If the characteristic equation of a homogeneous recurrence relation of 5^{th} order is $(r-2)(r-4)(r-3)^3 = 0$ then $r = 2, 4, 3, 3, 3$. Then the general solution is: $a_n = b_1 2^n + b_2 4^n + (b_3 + nb_4 + n^2 b_5) 3^n$.

Problem: Solve the recurrence relation

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$$

Solution: Let $a_n = r^n$ be a solution of the given equation.

The characteristic eqn. is $r^3 - 8r^2 + 21r - 18 = 0$

$r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 = 0$

$r^2(r-2) - 6(r-2) + 9(r-2) = 0$

$(r-2)(r^2 - 6r + 9) = 0$

$(r-2)(r-3)^2 = 0$ which gives $r = 2, 3, 3$.

So, the general solution is $a_n = (b_1 + b_2 n) 3^n + b_3 2^n$