2. Characteristic roots method:

ii) Non-Homogeneous Recurrence Relation: A second order non-homogeneous linear recurrence relation with constant coefficients is of the form $a_{n-2}+5a_{n-1}+6a_n=f(n)$

Its solution a_n consists of two parts.

1. Homogeneous solution $a_n^{(h)}$ of the given recurrence relation by keeping $f(n)=0$
2. Particular solution $a_n^{(P)}$ of the given recurrence relation with $f(n)$ on the R.H.S. So the required general solution is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

Particular Solution: There is no general method for finding the particular solution of a recurrence relation for every function $f(n)$. So, method of undetermined co-efficients will be discussed here which is useful when $f(n)$ consists of special forms. Depending on certain forms of $f(n)$, a trial solution containing a number of unknown constant coefficients is considered which are to be determined by substitution in the recurrence relation.

Form of $f(n)$	Trial Function
b^n (if b is not a root of characteristics Eqn.)	$\rightarrow$ Ab^n
Polynomial $P(n)$ of degree m	$\rightarrow$ $A_0+A_1n+A_2n^2+\dots+A_mn^m$
$c^n P(n)$ (if c is not a root of characteristic eqn.)	$\rightarrow$ $c^n(A_0+A_1n+A_2n^2+\dots+A_mn^m)$
b^n (if b is a root of characteristic eqn. of multiplicity s)	$\rightarrow$ $A_0 n^s b^n$
$c^n P(n)$ (if c is a root of characteristic eqn. of multiplicity t)	$\rightarrow$ $n^t(A_0+A_1n+A_2n^2+\dots+A_mn^m)c^n$
$\sin bn$ or $\cos bn$	$\rightarrow$ $A_0\sin bn + A_1\cos bn$
$b^n\sin bn$ or $b^n\cos bn$	$\rightarrow$ $b^n(A_0\sin bn+A_1\cos bn)$

Note: i) If $f(n)$ is a constant i.e. polynomial of degree zero, the trial solution is taken as A.

ii) If $f(n)$ is a linear combination of the above forms, the trial solution is taken as the sum of corresponding trial functions with different unknown constant coefficients to be determined.

When $f(n) = \alpha$, a constant:

Problem: Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial condition $a_0 = 1$ and $a_1 = -1$.

Solution: The associated homogeneous recurrence relation is:

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \cdots (1)$$

Let, $a_n = r^n$ be a solution of (1).

The characteristic eqn. is $r^2 - 5r + 6 = 0$

$$\Rightarrow r = 3, 2$$

So, the solution of (1) is $a_n^{(h)} = c_1 3^n + c_2 2^n$.

To find the particular solution of the given equation, let $a_n = A$. Substituting in the given eqn.,

$$A - 5A + 6A = 2$$
$$\text{or, } A = 1$$

$\therefore$ Particular solution $a_n^{(P)} = 1$

So, the general solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = c_1 3^n + c_2 \cdot 2^n + 1 \cdots (2)$$

To find c_1 and c_2, put $n = 0$ and $n = 1$ in eqn-(2)

$$a_0 = c_1 + c_2 + 1$$
$$\text{or, } c_1 + c_2 + 1 = 1 \quad [\because a_0 = 1]$$
$$\text{or, } c_1 + c_2 = 0 \cdots (3)$$

Again, $a_1 = 3c_1 + 2c_2 + 1$

$$\text{or, } -1 = 3c_1 + 2c_2 + 1 \quad [\because a_1 = -1]$$
$$\text{or, } 3c_1 + 2c_2 = 2 \cdots (4)$$

Solving (3) and (4), we get, $c_1 = -2$, $c_2 = 2$.

So, the required solution is: $a_n = -2 \cdot 3^n + 2 \cdot 2^n + 1$ (Ans.)

[Putting values of c_1 and c_2 in eqn(2)]

When $f(n) = X$, a polynomial:

Problem: Solve the following.

$$y_{n+2} - y_{n+1} - 2y_n = n^2$$

Solution: Substituting $y_n = r^n$ in the associated homogeneous relation, the characteristic eqn. is $r^2 - r - 2 = 0$

$$\text{or, } r = -1, 2$$

The solution of the associated homogeneous recurrence relation is: $y_n^{(h)} = c_1 (-1)^n + c_2 \cdot 2^n$

Let the particular solution of the given equation be

$$y_n = A_0 + A_1 n + A_2 n^2 \text{ (Since } f(n) \text{ is a polynomial of degree 2)}$$

Substituting in the given eqn., we have,

$$A_0 + A_1(n+2) + A_2(n+2)^2 - [A_0 + A_1(n+1) + A_2(n+1)^2] - 2(A_0 + A_1 n + A_2 n^2) = n^2$$

$$\text{or, } (-2A_0 + A_1 + 3A_2) + (-2A_1 + 2A_2)n - 2A_2 n^2 = n^2$$

On comparing the coefficients of like powers of n, we have

$$-2A_0 + A_1 + 3A_2 = 0 \cdots (1)$$
$$-2A_1 + 2A_2 = 0 \cdots (2)$$
$$-2A_2 = 1 \cdots (3)$$

From eqn-(3), $A_2 = -1/2$

In eqn-(2), putting $A_2 = -1/2$, we get,

$$A_1 = A_2 = -1/2$$

From eqn-(1), $-2A_0 - 1/2 - 3/2 = 0$

$$\text{or, } A_0 = -1$$

So, the particular solution of given recurrence relation is: $y_n^{(P)} = -1 - (1/2)n - (1/2)n^2$

So, the general solution of the given recurrence relation is $y_n = y_n^{(h)} + y_n^{(P)}$

$$y_n = c_1 \cdot (-1)^n + c_2 \cdot 2^n - 1 - (1/2)n - (1/2)n^2 \text{ (Ans.)}$$

When $f(n) = \alpha^n$, α is a root of characteristic eqn:

Problem: Solve the following recurrence relation

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$$

Solution: Let, $a_n = r^n$ be a solution of the associated homogeneous relation $a_{n+2} - 4a_{n+1} + 4a_n = 0$

The characteristic eqn. is $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$$r = 2, 2$$

So, the solution of associated homogeneous relation is $a_n^{(h)} = (c_1 + c_2 n) 2^n$.

To find the particular solution of the given relation, we note $b = 2$ is a root of characteristic eqn. with multiplicity $s = 2$. So, the particular solution has the form $a_n = A_0 \cdot n^2 \cdot 2^n$.

Substituting in the given relation, we get,

$$A_0 \cdot (n+2)^2 \cdot 2^{(n+2)} - 4 \cdot A_0 (n+1)^2 \cdot 2^{n+1} + 4 A_0 n^2 \cdot 2^n = 2^n$$

$$\Rightarrow 4A_0 (n+2)^2 - 8 A_0 (n+1)^2 + 4 A_0 n^2 = 1$$

$$\Rightarrow A_0 = 1/8$$

So, Particular solution is: $a_n^{(P)} = \frac{1}{8} \cdot n^2 \cdot 2^n$.

$\therefore$ General solution is: $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = (c_1 + c_2 n) 2^n + \frac{1}{8} \cdot n^2 \cdot 2^n \text{ (Ans.)}$$