Hence the general solution of the given equation is

$$(x^3 - 3y + c)(cy + e^{\frac{1}{2}x^2})(xy + cy + 1) = 0,$$

where c is an arbitrary constant.

Ex. 3. *Solve* : $x^2(p^2 - y^2) + y^2 = x^4 + 2xyp$.

The given equation can be written as

$$x^2p^2 - 2xyp + y^2 = x^4 + x^2y^2$$

or, $$(px - y)^2 = x^2(x^2 + y^2)$$

or, $$px - y = \pm x\sqrt{x^2 + y^2}$$

or, $$p = \frac{y}{x} \pm \sqrt{x^2 + y^2}.$$

Now we substitute $y = vx$, so that $p = \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

The equation then becomes

$$v + x\frac{dv}{dx} = v \pm \sqrt{x^2 + v^2x^2}$$

or, $$\frac{dv}{\sqrt{1 + v^2}} = \pm\, dx.$$

The solution of the equation with the positive sign on the right is

$$v = \sinh(x + c_1), \text{ that is, } y = x\sinh(x + c_1)$$

and that with the negative sign on the right is

$$v = \sinh(c_2 - x), \text{ that is, } y = x\sinh(c_2 - x).$$

Hence the general solution of the equation is

$$\{y - x\sinh(x + c)\}\{y - x\sinh(c - x)\} = 0,$$

where c is an arbitrary constant.

Ex. 4. *Solve* : $x^2\left(\dfrac{dy}{dx}\right)^2 + xy\dfrac{dy}{dx} - 6y^2 = 0$.

Writing p for $\dfrac{dy}{dx}$, we put the given equation as

$$p^2x^2 + pxy - 6y^2 = 0$$

or, $$(px + 3y)(px - 2y) = 0,$$

from which $px + 3y = 0$ gives $\log y + 3\log x + c_1 = 0$

and $px - 2y = 0$ gives $\log y - 2\log x + c_2 = 0$.

Hence the general solution of the equation is

$$(\log y + 3\log x + c)(\log y - 2\log x + c) = 0,$$

where c is an arbitrary constant.

Note. This solution may be written in a simpler form as $(x^3y + c)\left(\dfrac{x^2}{y} + c\right) = 0$, where c is an arbitrary constant.

Examples III (A)

Solve the following equations :

1. $p^2 + p - 6 = 0$.
2. $p^2 - p(e^x + e^{-x}) + 1 = 0$.
3. $\left(\dfrac{dy}{dx}\right)^3 = bx^4$.
4. $p^2 + 2py\cot x = y^2$.
5. $p(p - y) = x(x + y)$.
6. $xp^2 + (y - x)p - y = 0$. [T. H. 1991]
7. $p^2(2 - 3y)^2 = 4(1 - y)$.
8. $xy^2(p^2 + 2) = 2py^3 + x^3$.
9. $x^2p^2 + 3xyp + 2y^2 = 0$.
10. $x + yp^2 = p(1 + xy)$.
11. $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$.
12. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.
13. $p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$. [V. H. 1990]
14. (a) $y - (1 + p^2)^{-\frac{1}{2}} - b = 0$.
    (b) $(x^2 + 2ax)(1 + p^2) = (x + a)^2$.
15. $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0$.
16. $(p + y + x)(xp + y + x)(p + 2x) = 0$.
17. $(a^2 - x^2)p^3 + bx(a^2 - x^2)p^2 - p - bx = 0$.
18. $x^2\left(\dfrac{dy}{dx}\right)^2 - 2xy\dfrac{dy}{dx} + 2y^2 - x^2 = 0$.
19. (a) $xy\left\{\left(\dfrac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\dfrac{dy}{dx}$. [C. H. 1985]
    (b) $y\left\{x(2x + 1)\dfrac{dy}{dx} - y\left(\dfrac{dy}{dx}\right)^2\right\} = 2x^3$. [C. H. 1986]
20. $p^3 - (y + 2x - e^{x-y})p^2 + (2xy - 2xe^{x-y} - ye^{x-y})p + 2xye^{x-y} = 0$.

Answers

1. $(y + 3x - c)(y - 2x - c) = 0$.
2. $(y - e^x - c)(y + e^{-x} - c) = 0$.
3. $343(y + c)^3 = 27bx^7$.
4. $y(1 \pm \cos x) = c$.

5. $(2y + x^2 - c)(y + x + 1 - ce^x) = 0$. 6. $(y - x + c)(xy + c) = 0$.
7. $y^2(1 - y) = (\pm x + c)^2$. 8. $(x^2 - y^2 + c)(x^2 - y^2 + cx^4) = 0$.
9. $(xy - c)(x^2 y - c) = 0$. 10. $(2y - x^2 - c)(2x - y^2 - c) = 0$.
11. $(y - c)(y + x - c)(x^2 + xy + y^2 - c) = 0$.
12. $(y - c)(y + x^2 - c)(1 + xy + cy) = 0$.
13. $(2y - x^2 - c)(y - ce^x)(y + x - 1 - ce^{-x}) = 0$.
14. (a) $(x + c)^2 + (y - b)^2 = 1$. (b) $y = \pm a \log(x + a + \sqrt{x^2 + 2ax}) + c$.
15. $(y - cx^2)(y^2 + 3x^2 + c) = 0$.
16. $(y + x - 1 + ce^{-x})(2xy + x^2 + c)(y + x^2 + c) = 0$.
17. $(y + \frac{1}{2}bx^2 - c)\left\{\frac{x^2}{a^2} - \sin^2(y - c)\right\} = 0$.
18. $\left(\sin^{-1}\frac{y}{x} + \log cx\right)\left(\sin^{-1}\frac{y}{x} - \log cx\right) = 0$.
19. (a) $(x^2 - y^2 - 2c)(xy - c) = 0$. (b) $(y^2 - x^2 - 2c)(3y^2 - 4x^3 - 6c) = 0$.
20. $(y - ce^x)(y - x^2 - c)(e^y + e^x - c) = 0$.

3.4. Equations solvable for y.

If the differential equation be solvable for y, then it may be put in the form

$$y = f(x, p). \qquad \ldots \quad (1)$$

Differentiating both sides of (1) with respect to x, we get an equation of the form

$$p = F\left(x, p, \frac{dp}{dx}\right).$$

This is an equation in two variables x and p and it can be solved to get a solution of the form

$$\phi(x, p, c) = 0. \qquad \ldots \quad (2)$$

Eliminating p between (1) and (2), we shall get the required solution, which will be a relation connecting x, y and an arbitrary constant c.

Sometimes we write down the solution by expressing x and y separately as functions of p, treating p as a parameter. This happens when the elimination of p between (1) and (2) cannot be easily done.

3.5. Equations containing no x.

Consider the equation of the form $f(y, p) = 0$.

If it be solvable for p, then it can be put as

$$p = \frac{dy}{dx} = F(y).$$

Its solution is $\displaystyle\int \frac{dy}{F(y)} = x + c$.

On the other hand, if it be solvable for y, let it be

$$y = f(p).$$

This can be easily integrated by the previous method.

3.6. Equations solvable for x.

If the differential equation be solvable for x, then it may be put in the form

$$x = f(y, p). \qquad \ldots \quad (1)$$

Differentiating both sides of (1) with respect to y, we get an equation of the form

$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right).$$

This is an equation in two variables y and p and it can be solved to get a solution of the form

$$\phi(y, p, c) = 0. \qquad \ldots \quad (2)$$

Eliminating p between (1) and (2), we shall get the required solution, which will be a relation connecting x, y and an arbitrary constant c.

In case the elimination is not easily possible, x and y may be expressed in terms of p, treating p as a parameter.

3.7. Equations containing no y.

Consider the equation of the form $f(x, p) = 0$.

If it be solvable for p, then it can be put as

$$p = \frac{dy}{dx} = F(x)$$

and its solution is $\displaystyle\int F(x)\,dx = y + c$.

On the other hand, if it be solvable for x, let it be

$$x = f(p).$$

This can be easily integrated by the previous method.

Differentiating both sides of the given equation with respect to y, we get

$$\frac{1}{p}+\frac{1}{\sqrt{1+p^2}}\frac{dp}{dy}-\frac{p^2}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dy}=0$$

or, $$\frac{1}{p}+\frac{1}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dy}=0$$

or, $$dy=-\frac{p}{(1+p^2)^{\frac{3}{2}}}\,dp.$$

Integrating, we get

$$y+c=\frac{1}{\sqrt{1+p^2}}$$

or, $$(y+c)^2=\frac{1}{1+p^2}. \qquad \ldots (1)$$

From the given equation, we have

$$(x-a)^2=\frac{p^2}{1+p^2}. \qquad \ldots (2)$$

Eliminating p from (1) and (2), we get the general solution as

$$(x-a)^2+(y+c)^2=1.$$

Examples III (B)

Find the general solutions of the following equations :

1. $y=p\sin x+\cos x-1$.
2. $y=2px+\tan^{-1}(xp^2)$.
3. $xp^2-2yp+ax=0$.
4. $y=2px+p^2y$.
5. $y=2px+y^2p^3$.
6. $y=2px+p^2$. [V. H. 1997]
7. $x^3p^2+x^2py+a^3=0$.
8. $x^2=a^2(1+p^2)$.
9. $y^2=a^2(1+p^2)$.
10. $x=4p+4p^3$.
11. $e^y-p^3-p=0$. [C. H. 1982]
12. $y=p^2x+p$. [C. H. 1980]
13. $p^3+mp^2=a(y+mx)$.
14. $y=(1+p)x+ap^2$.
15. $y=ap^2+bp^3$.
16. $x=py-p^2$.
17. $y=x+p^3$.
18. $y^2\log y=xyp+p^2$.
19. $p^3-4xyp+8y^2=0$.
20. $y=2px+f(xp^2)$.

Answers

1. $y=(x+c)\tan\frac{1}{2}x$.
2. $y=2\sqrt{cx}+\tan^{-1}c$.
3. $2y=cx^2+\frac{a}{c}$.
4. $y^2=2cx+c^2$.
5. $y^2=2cx+c^3$.
6. $4(x^2+y)^3=(3xy+2x^3+c)^2$.
7. $c^2+cxy+a^3x=0$.
8. $x=a\sqrt{1+p^2}$, $y=\frac{1}{2}a\{p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})\}+c$.
9. $y=a\sqrt{1+p^2}$, $x=a\log(p+\sqrt{1+p^2})+c$.
10. $x=4p+4p^3$, $y=2p^2+3p^4+c$.
11. $y=\log(p^3+p)$, $x=2\tan^{-1}p-p^{-1}+c$.
12. p - eliminant of $y=p^2x+p$ and $(p-1)^2x=\log p-p+c$.
13. $ax+c=\frac{3}{2}p^2-mp+m^2\log(p+m)$, with the given relation.
14. $x=2a(1-p)+ce^{-p}$, $y=2a-ap^2+c(1+p)e^{-p}$.
15. $y=ap^2+bp^3$, $x=2ap+\frac{3}{2}bp^2+c$.
16. p - eliminant of $x=py-p^2$ and $y=p+(c+\cosh^{-1}p)(p^2-1)^{-\frac{1}{2}}$.
17. $x=\frac{3}{2}p^2+3p+3\log(p-1)+c$, $y=p^3+\frac{3}{2}p^2+3p+3\log(p-1)+c$.
18. $\log y=cx+x^2$.
19. $64y=c(c-4x)^2$.
20. $y=2c\sqrt{x}+f(c^2)$.

Differentiating both sides of (1) with respect to X, we get

$$P = P + \left(X - \tfrac{1}{2}P\right)\frac{dP}{dX}$$

or, $$\frac{dP}{dX}\left(X - \tfrac{1}{2}P\right) = 0.$$

Therefore, either $\dfrac{dP}{dX} = 0$, which gives $P = C$, a constant ... (2)

or $X - \frac{1}{2}P = 0$, which gives $P = 2X$. ... (3)

Eliminating P between (1) and (2), we get the complete primitive as

$$Y = CX - \tfrac{1}{4}C^2.$$

Restoring the values of X and Y, we get the primitive as

$$y^2 = Cx - \tfrac{1}{4}C^2.$$

Eliminating P between (1) and (3), we get the singular solution as

$$Y = 2X^2 - X^2 = X^2.$$

Restoring the values of X and Y, we get the singular solution as

$$y^2 = x^2, \text{ that is, } y = \pm x.$$

Ex. 3. *Solve :* $x^2(y - px) = p^2 y$. [B. H. 1990]

Let us put $x^2 = u$ and $y^2 = v$,

so that $2x\,dx = du$ and $2y\,dy = dv$.

Therefore $$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

or, $$\frac{y}{x}p = \frac{dv}{du} = P \text{ (say)}.$$

Putting for P in the given equation, we get

$$x^2\left(y - \frac{Px^2}{y}\right) = \frac{x^2}{y^2}P^2 \cdot y$$

or, $$y^2 - Px^2 = P^2$$

or, $$v - Pu = P^2$$

or, $v = Pu + P^2$, which is in Clairaut's form.

The complete primitive is thus

$v = cu + c^2$, c being an arbitrary constant.

Restoring the values of u and v, we get

$$y^2 = cx^2 + c^2.$$

Ex. 4. *Reduce the equation* $(px^2 + y^2)(px + y) = (p + 1)^2$ *to Clairaut's form by using the substitutions* $u = xy$ *and* $v = x + y$ *and hence find its complete primitive.* [C. H. 1992]

Substituting $u = xy$ and $v = x + y$, we have

$$\frac{du}{dx} = y + x\frac{dy}{dx} \text{ and } \frac{dv}{dx} = 1 + \frac{dy}{dx}.$$

Therefore $$\frac{du}{dv} = \frac{y + px}{p + 1}, \text{ where } p = \frac{dy}{dx}.$$

Now, $$px^2 + y^2 = (px + y)(x + y) - xy(p + 1).$$

The given equation can be written as

$$\{(px + y)(x + y) - xy(p + 1)\}(px + y) = (p + 1)^2$$

or, $$\left(\frac{px + y}{p + 1}\right)^2 (x + y) - \frac{xy(px + y)}{p + 1} = 1$$

or, $$\left(\frac{du}{dv}\right)^2 v - u\frac{du}{dv} = 1$$

or, $$u = v\frac{du}{dv} - \frac{1}{\frac{du}{dv}}$$

or, $u = Pv - \dfrac{1}{P}$, where P stands for $\dfrac{du}{dv}$.

Thus the equation has been reduced to Clairaut's form, whose complete primitive is

$u = cv - \dfrac{1}{c}$, where c is an arbitrary constant.

Restoring the values of u and v in terms of x and y, we get

$$xy = c(x + y) - \frac{1}{c}$$

or, $$cxy - c^2(x + y) + 1 = 0.$$

Examples III (C)

1. Solve the equation $y = px + \dfrac{a}{p}$ and obtain the singular solution.

2. Solve the equation $y = px + \sqrt{a^2p^2 + b^2}$ and obtain the singular solution.

3. Solve the equation $py = p^2(x - b) + a$ and obtain the singular solution.

4. Solve the equation $(y+1)p - xp^2 + 2 = 0$ and obtain the singular solution.

5. Solve the equation $y = px + p - p^2$ and obtain the singular solution. [B. H. 1998; K. H. 2007, 2008]

Reduce the following equations to Clairaut's form and obtain the complete primitives (6 – 12) :

6. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$.

7. $\sin px \cos y = \cos px \sin y + p$. [V. H. 2004]

8. $e^{3x}(p-1) + p^3 e^{2y} = 0$.

9. $y = 2px + 4yp^2$. [Put $y^2 = Y$.]

10. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$. [N. B. H. 2007]

11. $y - 2xp + ayp^2 = 0$.

12. $xy(y - px) = x + py$. [Put $x^2 = u$, $y^2 = v$.]

Solve the following equations (13 – 18) :

13. $(px - y)(x - py) = 2p$. [Put $x^2 = u$, $y^2 = v$.] [C. H. 1995]

14. $\left(1 - y^2 + \frac{y^4}{x^2}\right)p^2 - 2\frac{y}{x}p + \frac{y^2}{x^2} = 0$. [Put $y = vx$.]

15. $x^2p^2 + yp(2x + y) + y^2 = 0$. [Put $y = u$, $xy = v$.]

16. $(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{\frac{3}{2}}$. [Put $x = r\cos\theta$, $y = r\sin\theta$.]

17. $(xp - y)^2 = p^2 - 2\frac{y}{x}p + 1$.

18. $y^2(y - px) = x^4p^2$. [Put $x^{-1} = u$ and $y^{-1} = v$.]

19. Show that the general solution of the equation $x^2 - \frac{xy}{p} = f(y^2 - xyp)$ is $y^2 + \frac{cx^2}{f(c)} = c$, where c is an arbitrary constant.

20. Use the transformations $u = x^2$ and $v = y^2$ to solve the equation

$$(px - y)(py + x) = h^2p.$$

21. Use the transformations $u = x^2$ and $v = y^2$ to solve the equation

$$axyp^2 + (x^2 - ay^2 - b)p - xy = 0.$$

22. Use the transformations $u = x^2$ and $v = y^2$ to solve the equation

$xyp^2 - (x^2 + y^2 - 1)p + xy = 0$. [V. H. 1987 ; B. H. 1998]

23. Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by using the substitutions $x^2 = u$ and $y - x = v$ and then solve it. [C. H. 2000, 2006; V. H. 2006; K. H. 2008]

24. (*a*) Transform the differential equation

$$(2x^2 + 1)p^2 + (x^2 + y^2 + 2xy + 2)p + 2y^2 + 1 = 0$$

to Clairaut's form by using the transformations $x + y = u$ and $xy - 1 = v$ and then solve it.

(*b*) Using the transformations $u = x + y$, $v = x^2 + y^2$, reduce the equation $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$ into Clairaut's form and hence solve it. [C. H. 2005]

25. Reduce the equation $x^3p^2 + x^2yp + a^3 = 0$ to Clairaut's form by the substitutions $y = u$ and $x = \frac{1}{v}$ and obtain its complete primitive. [T. H. 2009]

Answers

1. $y = cx + \frac{a}{c}$; $y^2 = 4ax$.
2. $y = cx + \sqrt{a^2c^2 + b^2}$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
3. $cy = c^2(x - b) + a$; $y^2 = 4a(x - b)$.
4. $(y+1)c - c^2x + 2 = 0$; $(y+1)^2 + 8x = 0$.
5. $y = cx + c - c^2$; $4y = (x+1)^2$.
6. $(y - cx)^2 = a^2c^2 + b^2$.
7. $y = cx - \sin^{-1}c$.
8. $e^y = ce^x + c^3$.
9. $y^2 = cx + c^2$.
10. $(y - cx)^2 = a^2c^2 + b^2$.
11. $y^2 = cx - \frac{1}{4}ac^2$.
12. $y^2 - 1 = c(x^2 + 1)$.
13. $c^2x^2 - c(x^2 + y^2 - 2) + y^2 = 0$.
14. $c \pm y = \log\frac{y}{x + \sqrt{x^2 - y^2}}$.
15. $xy = cy + c^2$.
16. $\tan^{-1}\frac{y}{x} + c = \sin^{-1}\left(2a\sqrt{x^2 + y^2} - 1\right)$.
17. $\sin^{-1}\frac{y}{x} + \sin^{-1}\frac{1}{x} = c$.
18. $c^2xy + cy - x = 0$.
20. $y^2 = cx^2 - \frac{c}{c+1}h^2$.
21. $y^2 = cx^2 - \frac{bc}{ac+1}$.
22. $y^2 = cx^2 + \frac{c}{c-1}$.
23. $2c^2x^2 - 2c(y - x) + 1 = 0$.
24. (*a*) $xy - 1 = c(x+y) + c^2$ (*b*) $x^2 + y^2 = c(x+y) - \frac{1}{4}c^2$.
25. $cxy - a^3c^2x = 1$