Information Theory

Assignment 1

Q. 1

$X \sim \text{Geometric}(p)$

$$\Rightarrow P(X=k) = (1-p)^{k-1}p, \quad k \in \mathbb{N}.$$

$$\Rightarrow H(X) = \sum_{x \in \operatorname{supp} X} \left(-p(x) \log_2 p(x)\right)$$

$$= -\sum_{k=1}^{\infty} (1-p)^{k-1} p \log_2\left((1-p)^{k-1} p\right)$$

$$= -\sum_{k=1}^{\infty} (1-p)^{k-1} p \left[(k-1)\log_2(1-p) + \log_2 p \right]$$

$$= \left(-p \log_2 (1-p)\right) \underbrace{\sum_{k=1}^{\infty} (k-1)(1-p)^{k-1}} + \left(-p \log_2 p\right) \underbrace{\sum_{k=1}^{\infty} (1-p)^{k-1}}_{= \frac{1}{1-(1-p)} = \frac{1}{p}}$$

$$S = (1-p) + 2(1-p)^2 + \dots$$

$$(1-p)S = \qquad (1-p)^2 + \dots$$

$$pS = (1-p) + (1-p)^2 + \dots$$

$$= (1-p)\left(\frac{1}{1-(1-p)}\right)$$

$$= \frac{1-p}{p} \Rightarrow S = \frac{1-p}{p^2}$$

$$\Rightarrow H(X) = \left(-p \log_2 (1-p)\right)\left(\frac{1-p}{p^2}\right)$$

$$+ \left(-p \log_2 p\right)\left(\frac{1}{p}\right)$$

$$= -\frac{(1-p)\log_2(1-p)}{p} - \log_2 p$$

$$= \frac{-(1-p)\log_2(1-p) - p\log_2 p}{p}$$

$$= \boxed{\frac{H_2(p)}{p}} \quad \{p \neq 0 \text{ is given}\}$$

☆ Answers assuming $\text{supp } X = \mathbb{N} \cup \{0\}$ and

$$P(X=k) = (1-p)^k p, \; k \in \mathbb{N} \cup \{0\}$$

will also be accepted. The entropy is anyways going to be the same since it depends only on the pmf.

Q.2 a) $X \sim \text{Poisson}(\lambda)$

$$\Rightarrow P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k \in \mathbb{N} \cup \{0\}$$

$$\Rightarrow H(X) = \sum_{k=0}^{\infty} \left(-\frac{\lambda^k e^{-\lambda}}{k!}\right) \log_2\left(\frac{\lambda^k e^{-\lambda}}{k!}\right)$$

$$= \frac{-1}{\ln 2} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \ln\left(\frac{\lambda^k e^{-\lambda}}{k!}\right)$$

$$= \frac{-1}{\ln 2} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \left(k \ln\lambda - \lambda - \ln(k!)\right)$$

$$= \frac{-e^{-\lambda} \ln\lambda}{\ln 2} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k k}{k!}}_{= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{\lambda}}$$

$$+ \frac{\lambda e^{-\lambda}}{\ln 2} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{= e^{\lambda}}$$

$$+ \frac{e^{-\lambda}}{\ln 2} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k \ln(k!)}{k!}}_{= f(\lambda)}$$

$$= \left(-e^{-\lambda} \frac{\ln\lambda}{\ln 2}\right) \lambda e^{\lambda} + \frac{\lambda e^{-\lambda}}{\ln 2} e^{\lambda}$$

$$+ \frac{e^{-\lambda}}{\ln 2} f(\lambda)$$

$$= \boxed{\frac{-\lambda \ln\lambda}{\ln 2} + \frac{\lambda}{\ln 2} + \frac{e^{-\lambda} f(\lambda)}{\ln 2}}$$

b) The ratio test states ~~that~~ (among other things) that for a sequence $\{a_n\}_{n\geq 1}$, if

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then the series $\sum a_n$ converges.

$$\cancel{\text{Put } a_k = \frac{\lambda^k \ln(k!)}{k!},\ k\in\mathbb{N}}$$

$$\cancel{\left|\frac{a_{k+1}}{a_k}\right| = \frac{\lambda^{k+1}\ln((k+1)!)}{(k+1)!} \times \frac{k!}{\ln(k!)\lambda^k}}$$

$$\cancel{= \frac{\lambda}{k+1}}$$

Put $a_k = \dfrac{\lambda^{k+1}\ln((k+1)!)}{(k+1)!}$, $k\in\mathbb{N}$

$[\lambda\in\mathbb{R}^+]$

$$\Rightarrow \left|\frac{a_{k+1}}{a_k}\right| = \frac{a_{k+1}}{a_k}$$

$$= \frac{\lambda^{k+2}\ln((k+2)!)}{(k+2)!} \times \frac{(k+1)!}{\lambda^{k+1}\ln((k+1)!)}$$

$$= \left(\frac{\lambda}{k+2}\right) \frac{\ln((k+1)!) + \ln(k+2)}{\ln((k+1)!)}$$

$$= \frac{\lambda}{k+2} + \frac{\lambda \ln(k+2)}{(k+2)\ln((k+1)!)}$$

Since $\lim\limits_{k\to\infty} \frac{\lambda}{k+2} = 0$,

$\lim\limits_{k\to\infty} \frac{\lambda \ln(k+2)}{k+2} = 0$, and

$\lim\limits_{k\to\infty} \frac{1}{\ln((k+1)!)} = 0$, we get

$$\lim_{k\to\infty} \left(\frac{\lambda}{k+2} + \frac{\lambda \ln(k+2)}{(k+2)\ln((k+1)!)}\right) = 0$$

$$\Rightarrow \limsup_{k\to\infty} \left(\frac{\lambda}{k+2} + \frac{\lambda \ln(k+2)}{(k+2)\ln((k+1)!)}\right) = 0$$

$$< 1 \ \forall \lambda \in \mathbb{R}^+$$

$\Rightarrow f(\lambda)$ converges $\forall \lambda \in \mathbb{R}^+$.

Q.3 a)

The condensation test states that for a non-increasing sequence of real numbers $\{a_n\}_{n\geq 1}$, the series $\sum a_n$ converges iff the series $\sum 2^n a_{2^n}$ converges.

~~The sequence $\{n \ln n\}_{n\geq 2}$ is strictly increasing~~

~~$\Rightarrow \left\{\frac{1}{n \ln n}\right\}$~~

The sequence $\{n(\ln n)^{\beta}\}_{n\geq 2}$ is strictly increasing if $\beta > 1$

$$\Rightarrow \left\{\underbrace{\frac{1}{n(\ln n)^{\beta}}}_{\text{call this } a_n}\right\}_{n\geq 2}$$ is strictly decreasing.

So consider the sequence $\{b_n\}_{n\geq 2}$

where $b_n = 2^n a_{2^n}$

$$= 2^n \frac{1}{2^n(\ln(2^n))^{\beta}}$$

$$= \frac{1}{n^{\beta}(\ln 2)^{\beta}} = \frac{1}{(\ln 2)^{\beta}}\left(\frac{1}{n^{\beta}}\right)$$

We are given that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

converges iff $\alpha > 1$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{(\ln 2)^{\beta}}\left(\frac{1}{n^{\beta}}\right)$$

converges iff $\beta > 1$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\beta}}$ converges iff

$\underline{\beta > 1}$

b) $$\gamma = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

For a random var X with

$$P(X=n) = \frac{1}{\gamma(n+1)(\ln(n+1))^2},$$

$$H(X) = \sum_{n=1}^{\infty} -p(n)\log_2(p(n))$$

$$= \sum_{n=2}^{\infty} \frac{-1}{\gamma n(\ln n)^2} \log_2\left(\frac{1}{\gamma n(\ln n)^2}\right)$$

$$= \frac{1}{\gamma}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}\left(\log_2 \gamma + \log_2 n + 2\log_2(\ln n)\right)$$

$$= \frac{\log_2 \gamma}{\gamma} \underbrace{\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}}_{=\gamma}$$

$$+ \frac{1}{\gamma}\sum_{n=2}^{\infty} \frac{\log_2 n}{n(\ln n)^2}$$

$$+ \frac{2}{\gamma}\sum_{n=2}^{\infty} \frac{\log_2(\ln n)}{n(\ln n)^2}$$

Now see that

$$\sum_{n=2}^{\infty} \frac{\log_2 n}{n(\ln n)^2} = \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$$

which we know diverges to $+\infty$.

Further, all but finitely many terms of the sequence

$$\left\{ \frac{\log_2(\ln n)}{n(\ln n)^2} \right\}_{n \geq 2}$$

are negative

$\Rightarrow$ The series $\displaystyle\sum_{n=2}^{\infty} \frac{\log_2(\ln n)}{n(\ln n)^2}$

either converges or diverges to $+\infty$. In any case, $H(X)$ diverges to $+\infty$.

9.4 Easily programmed. I get 9.71626 bits/word. Error of $\pm 10^{-4}$ will be tolerated.

Q.5 A succinct & clear proof has been given in Cover and Thomas, on pages 59 and 60. (theorem 3.1.2)

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