Bayes' Theorem Example: Tossing a Fair Coin Three Times

We will define two events based on the outcomes of tossing a fair coin three times:

- **Event E:** The first coin shows head.
- **Event F:** At least two heads appear.

To calculate the probability of Event E given Event F using Bayes' Theorem, which is the probability that the first coin shows head given that at least two heads appear, follow these steps:

1. **Identify the Sample Space (S):** When you toss a coin three times, the sample space S is: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
2. **Define Event F (At least two heads appear):** The outcomes that satisfy this condition are: {HHH, HHT, HTH}
3. **Define Event E (First coin shows head):** The outcomes that satisfy this condition are: {HHH, HHT, HTH, HTT}
4. **Intersection of E and F (First coin is head and at least two heads appear):** The outcomes that satisfy both conditions are: {HHH, HHT, HTH}
5. **Calculate Probabilities:**
    - **P(F) (Probability of getting at least two heads):** Since there are three outcomes in F and eight possible outcomes, P(F)=3/8
    - **P(E) (Probability of the first coin showing head):** Since there are four outcomes in E and eight possible outcomes, P(E)=1/2
    - **P(F|E) (Probability of getting at least two heads given the first coin is head):** There are three favorable outcomes ({HHH, HHT, HTH}) and four possible outcomes when the first coin is a head, P(F|E)= 3/4
6. **Apply Bayes' Theorem to calculate P(E|F):**

- Bayes' Theorem Formula: $P(E|F) = \frac{P(F|E) \times P(E)}{P(F)}$
- Calculation: $P(E|F) = \frac{\frac{3}{4} \times \frac{1}{2}}{\frac{3}{8}} = \frac{0.75 \times 0.5}{0.375} = \frac{0.375}{0.375} = 1$

Thus, the probability that the first coin shows head given that at least two heads appear is 100%.

This example illustrates how Bayes' Theorem can be applied to update the probability of a prior event (first coin showing head) based on the outcome of related subsequent events (at least two heads appearing).

Bayes' Theorem Exercise Questions: Tossing a Fair Coin Three Times

1. Calculate P(F|E) where E is getting exactly one head, and F is the first toss is a head.
2. Calculate P(F|E) where E is getting no heads, and F is not getting a head on the third toss.
3. Calculate P(F|E) where E is getting all tails, and F is the first toss is a tail.
4. Calculate P(F|E) where E is getting at least one tail, and F is the second toss is a head.
5. Calculate P(F|E) where E is getting exactly two heads, and F is the third toss is a head.
6. Calculate P(F|E) where E is getting exactly two tails, and F is the second toss is a tail.
7. Calculate P(F|E) where E is the outcome TTH, and F is starting with a tail.
8. Calculate P(F|E) where E is getting at least one head, and F is the first toss is a tail.
9. Calculate P(F|E) where E is getting a sequence of HTH, and F is the first toss is a head.
10. Calculate P(F|E) where E is getting heads only on the first and third tosses, and F is not getting a tail on the second toss.
11. Calculate P(F|E) where E is getting tails only on the first and third tosses, and F is the second toss is a head.
12. Calculate P(F|E) where E is the outcome HTT, and F is the first toss is a head.
13. Calculate P(F|E) where E is the outcome HHH, and F is getting a head on the third toss.
14. Calculate P(F|E) where E is the outcome TTT, and F is getting a tail on the third toss.
15. Calculate P(F|E) where E is getting all heads, and F is getting at least two heads.

These questions are designed to apply Bayes' Theorem, recalculating the probability of the initial event (F) based on the outcome (E).