Linear Algebra

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Outline

Vectors

Matrices

Eigen values and vectors

Functions

Vectors

- Science and Maths are used to describe the world around us. There are many quantities which require only 1 measurement to describe them. e.g Length of a string, or area of any shape or temperature of any surface. Such quantities are called scalars. Any quantity which can be represented as a number (positive or negative) is called scalar. This value is known as magnitude.
- On the other hand, there are quantities which require at least 2 measurements to describe them. Along with the magnitude, they have a “direction" associated e.g velocity or force. These quantities are known as “Vectors".
- When we say that a person ran for 2 Km, its a scalar but when we say that a person ran for 2 Km, North-east from his initial position, its a vector.

Operations on vectors

Consider two vectors $\vec{A} = (a_1, a_2, ...a_n)$ and $\vec{B} = (b_1, b_2, ...b_n)$

- Vector Addition
  - $\vec{C} = (a_1 + b_1, a_2 + b_2, ...a_n + b_n)$
- Vector Subtraction
  - $\vec{C} = (a_1 - b_1, a_2 - b_2, ...a_n - b_n)$
- Vector Multiplication
- Vector Subtraction
  - $\vec{C} = (a_1 * b_1, a_2 * b_2, ...a_n * b_n)$
- Scalar Multiply to vector
  - $K.(\vec{A}) = K * a_1, K * a_2......K * a_n$

Consider two vectors $\vec{A} = (a_1, a_2, ...a_n)$ and $\vec{B} = (b_1, b_2, ...b_n)$

- Magnitude of Vectors
  - $|A| = \sqrt{a_1^2 + a_2^2 + ...a_n^2}$
- A vector of magnitude, or length, 1 is called a unit vector.

For all vectors u, v, and w, and for all scalars b and c:

1. u + v = v + u.

2. u + (v + w) = (u + v) + w.

3. v + O = v.

4 1.v = v; 0.v = O.

5. v + (- v) = O.

6. b(cv) = (bc)v.

7. (b + c)v = bv + cv.

8. b(u + v) = bu + bv.

Component form of vectors

- unit vectors can have any direction, the unit vectors parallel to the x - and y - axes are particularly useful. They are defined as i = < 1, 0 > and j = < 0, 1 >.
- Any vector can be expressed as a linear combination of unit vectors i and j. For example, let $\vec{v} =< v_1, v_2 >$. Then $\vec{v} =< v_1, v_2 >=< v_1, 0 > + < 0, v_2 >= v_1 < 1, 0 > +v_2 < 0, 1 >= v_1 i + v_2 j$.

Directions in vectors

- The terminal point P of a unit vector in standard position is a point on the unit circle denoted by (cosθ, sinθ). Thus the unit vector can be expressed in component form, $\vec{u} =< cos\theta, sin\theta >$, or as a linear combination of the unit vectors i and j, $\vec{u} = (cos\theta)i + (sin\theta)j$, where the components of u are functions of the direction angle θ measured counterclockwise from the x - axis to the vector. As θ varies from 0 to 2π, the point P traces the circle $x^2 + y^2 = 1$. This takes in all possible directions for unit vectors so the equation $\vec{u} = (cos\theta)i + (sin\theta)j$ describes every possible unit vector in the plane.

Angle between vector

- The dot product of two vectors is a real number, or scalar. This product is useful in finding the angle between two vectors and in determining whether two vectors are perpendicular.
- The dot product of two vectors $\vec{u} =< u_1, u_2 >$ and $\vec{v} =< v_1, v_2 >$ is $u.v = u_1.v_1 + u_2.v_2$
- If θ is the angle between two nonzero vectors u and v, then

$$cos\theta = \frac{u.v}{|u|\,|v|} \tag{1}$$

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Examples

- Find a unit vector that has the same direction as the vector w = < - 3, 5 >.
- Find the angle between u = < 3, 7 > and v = < - 4, 2 >.

Application of vectors

- Cosine Similarity
  - Cosine Similarity is a metric that gives the cosine of the angle between vectors. It signfies the similarity and dissimilarity between two vectors.
- Text Vectorization
  - The process of converting or transforming a data set into a set of Vectors is called vectorization. It’s easier to represent data set as vectors where attributes are already numeric

Matrix

- Conventionally, the number of rows in a matrix is denoted by m and the number of columns by n. Since a rectangle's area is height x width, we denote a matrix's size by m x n. Thus is the matrix was to be called A, it would be written notationally as

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \tag{2}$$

- A matrix with just one row is called a row matrix and a matrix with just one column is called a column matrix.

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Operations on Matrix

- Matrix addition and Subtraction
  - Number of Rows of A = Number of Rows of B
  - Number of Columns of A = Number of Columns of B

$$\begin{pmatrix} 2 & 5 & 7 \\ 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 \\ -4 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 7 \\ -3 & 5 & 5 \\ 4 & 7 & 1 \end{pmatrix}$$

Operations on Matrix

- Matrix addition and Subtraction Properties
  - Addition of matrices is commutative which means $A+B = B+A$
  - Addition of matrices is associative which means $A+(B+C) = (A+B)+C$
  - Subtraction of matrices is non-commutative which means $A-B \neq B-A$
  - Subtraction of matrices is non-associative which means $A-(B-C) \neq (A-B)-C$
  - The order of matrices A, B, A-B and A+B is always the same
  - If the order of A and B is different, A+B, A-B can't be computed
  - The complexity of addition/subtraction operation is $O(m*n)$ where $m*n$ is order of matrices

Operations on Matrix

- The multiplication of two matrices A(m*n) and B(n*p) gives a matrix C(m*p). Notice that for multiplication you do not need the rows/columns of A and B to be the same. You only need
  - No. of Columns of A = No. of Rows of B
  - Or, No. of Columns of B = No. of Rows of A.

-4*2 + 3*1 + 2*4

$$\begin{bmatrix} -4 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & 9 \end{bmatrix}$$

m x n : 2 x 3 X n x p : 3 x 2 = m x p : 2 x 2

Operations on Matrix

- Identity Matrix : It is the matrix equivalent of the number “1":
  - It is "square" (has same number of rows as columns),
  - It has 1s on the diagonal and 0s everywhere else.
  - Its symbol is the capital letter I.

$$I_{2X2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

$$I_{3X3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Determinant of Matrix

- The determinant is a special number that can be calculated from a matrix.

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

Inverse of a matrix

- Why inverse ? Because with matrices we don't divide! Seriously, there is no concept of dividing by a matrix.

The inverse of A is A^{-1} only when:

$$AA^{-1} = A^{-1}A = \mathbf{I}$$

Sometimes there is no inverse at all.

Example

- Find the inverse of the given 3X3 matrix as follows :

$$M_{3X3} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \tag{5}$$

Eigen values and vectors

- Special properties in matrix

Let A be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a **nonzero vector** for which

$$AX = \lambda X$$

for some scalar λ. Then λ is called an **eigenvalue** of the matrix A and X is called an **eigenvector** of A associated with λ, or a λ-eigenvector of A.

The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the **spectrum** of A.

Example

- Consider the matrix A and two column vector X_1 and X_2:

$$A_{3X3} = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} X_1 = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Functions

- Function, in mathematics, an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable). Functions are ubiquitous in mathematics and are essential for formulating physical relationships in the sciences.
- This relationship is commonly symbolized as $y = f(x)$ - which is said “f of x" - and y and x are related such that for every x, there is a unique value of y. That is, $f(x)$ can not have more than one value for the same x.

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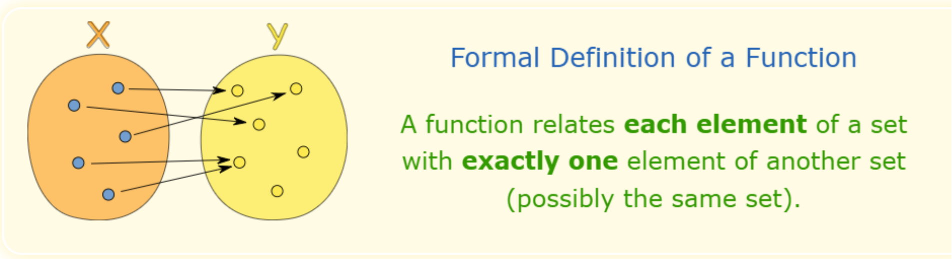
Functions

- Special rules of a function
  - It must work for every possible input value
  - And it has only one relationship for each input value



Functions

Example: $y = x^3$

- The input set "X" is all Real Numbers
- The output set "Y" is also all the Real Numbers

We can't show ALL the values, so here are just a few examples:

X: x	**Y: x^3**
-2	-8
-0.1	-0.001
0	0
1.1	1.331
3	27
and so on...	and so on...

Functions

Refer from the previous slide table

- the set "X" is called the Domain,
- the set "Y" is called the Codomain, and
- the set of elements that get pointed to in Y (the actual values produced by the function) is called the Range.

Thank You