Lecture 1: Random experiments, sample spaces, events

Probability theory is a mathematical area which studies randomness in a quantitative way. There are several important reasons for learning probability. On the applied side, it provides effective tools for studying real world problems that have a nature of randomness in various scientific areas such as economics, finance, physics, biology, engineering, computer science etc. On the theoretical side, it provides new ideas and tools for solving problems in a wide range of mathematical areas such as number theory, algebra, differential equations, differential geometry etc. Probability theory is also the foundation of statistics and other advanced topics such as stochastic processes, stochastic calculus etc.

1 Some entertaining problems

We begin our journey with a few classical and entertaining problems. These problems give us a first taste of the subject before going into further mathematical developments.

The Monty Hall problem

Consider the following game in which Monty Hall is the host and you are the participant. There are three closed doors in front of you, behind exactly one of which is placed a prize:

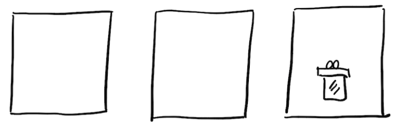

Figure 1: The Monty Hall prolem.

Firstly, you are asked to choose a door. Then Monty Hall opens a door which is not chosen by you and does not contain the prize. Such a door apparently exists. After that, Monty Hall offers you the opportunity to change your mind to select the other unopened door. Should you stick to your original choice or change your mind?

This problem can be solved easily without knowing any sophisticated probability theory.

Let us first consider the strategy of *not* changing your mind. In this case, you will get the prize eventually if and only if your original selection is the correct one. Therefore, your probability of winning the prize is 1/3.

Now let us consider the strategy of changing your mind. In this case, if your original selection is correct, after changing your mind you will miss the prize. On the other hand, suppose that your original selection misses the prize. According to the game rule, the door opened by Monty Hall must be the one which does not contain the prize. In other words, the remaining unopened door is the one containing the prize, and in your new selection you will hit this door. Therefore, you will get the prize eventually if and only if your original selection is the wrong one, namely your winning probability is 2/3.

From the above analysis, it is clear that you should change your mind!

The St. Petersburg Paradox

Consider the game of tossing a fair coin repeatedly, and you are betting on seeing a Tail. At the first toss, you bet $1, and if you win (i.e. if the outcome is a Tail) you end this game. On the other hand, if you lose the first game (i.e. if the outcome of the first toss is a Head) you bet $2 in the second toss. If you win, you end the game, while if you lose, you bet $4 in the third toss, and so forth (with this stake-doubling strategy).

Let us take a naive look at this game. If you win in the first toss, you win $1. If you win in the second toss, your winning is $2 but you have lost $1 in the first round. Therefore, you net winning is $1. If you win in the third toss, your winning is \$4 but you have lost $1 + 2 = 3$ dollars in the first two rounds. In other words, your net winning is also \$1. This argument clearly carries on, and by simple arithmetics, it seems to suggest that, your net winning is always \$1. This looks like a free-lunch game – is it true?

We will come back to re-examine this problem from a more mature perspective after developing some probabilistic tools.

The bus-stop paradox

Buses of a particular route arrive at a particular bus-stop randomly throughout the day, and on average they are arriving at the rate of one bus per hour. Suppose that you decide to go to the bus-stop at a random instant of time. What is your expected wait time for a bus?

Let us draw a picture and consider the following suggested solution:

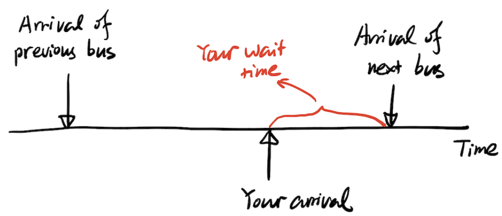

Figure 2: The bus-stop paradox.

Since your arrival time at the bus-stop is random, it is natural to say that you are arriving the bus-stop at a time that is "uniformly random" between the arrival of the previous bus (before you arrive) and the arrival of the next bus (after your arrival). We use I to denote this inter-arrival interval of the two buses. By assumption, this interval as average length of one hour, and your average arrival time is apparently the mid-point of I. In other words, your average wait time should be 30 minutes. Do you think this is a reasonable solution?

We let the curious reader to wait until the later stage of our study before answering this question.

Monkey typing Shakespeare

A monkey type one letter on the keyboard randomly at each time. Do you think the monkey will eventually produce an exact copy of Shakespeare's "Hamlet"? If yes, how long does it take on average for the monkey to produce such a copy?

The first question can be answered easily by using what we will learn soon. However, the second question goes beyond the scope of elementary probability, and we will leave its solution to the more advanced topic of martingale theory in future study.

2 Random experiment and the sample space

The very first mathematical task of probability theory is to make precise the notion of "the probability of an event", e.g. the event that "tomorrow will rain". Mathematically, it is expressed under the abstract mathematical notation $\mathbb{P}(A)$, where A denotes the event and $\mathbb{P}(\cdot)$ denotes a "probability function" acting on events. Therefore, before attempting to understand this probability function, we should first investigate the concept of events in a mathematically precise way. And before studying events, we must first introduce the notion of random experiments, outcomes and sample spaces.

Definition 2.1. A *random experiment* is a process leading to a number (could either be finite or infinite) of possible outcomes and the actual outcome that occurs depends on factors that cannot be predicted beforehand. The *sample space* (sometimes known as the outcome space), usually denoted as Ω, is the set of all possible outcomes of the given experiment.

Mathematically, the sample space is a *set* whose elements are the outcomes of a random experiment. We give several examples to illustrate this concept.

Example 2.1. Consider the random experiment of tossing a coin. Apparently, there are precisely two possible outcomes. Therefore, the sample space can be written as

$$\Omega = \{H, T\},$$

where H denotes the outcome of a Head and T denotes the outcome of a Tail.

Example 2.2. An Australian roulette wheel has 37 numbers on it (0 to 36). Consider the random experiment of spinning a roulette wheel. The sample space is then given by

$$\Omega = \{0, 1, 2, \cdots, 36\},$$

where each number represents a possible outcome.

It is important to point out that, given a random experiment there may be different ways of defining a sample space depending on what we choose to observe. We use the following example to illustrate this.

Example 2.3. A typical horse race contains 8 participating horses and there are 3 winners among them (the first three places). A horse race is considered as a random experiment since its outcome cannot be predicted beforehand.

(i) Suppose that we are only interested in the champion horse. Since the champion must be one of the 8 horses, the sample space is given by the set of all the 8 horses:

$$\Omega = \{\text{all horses in the race}\}.$$

If we label the 8 horses by $1, 2, \cdots, 8$, then we can alternatively write

$$\Omega = \{1, 2, \cdots, 8\}.$$

(ii) Suppose that we observe the placing of the 3 winners. Then the sample space should be given by

$$\Omega = \{\text{all possible ordered sets of 3 horses in the race}\}.$$

Using the previous labelling, we can also write

$$\Omega = \{(i, j, k) : i, j, k \text{ are distinct numbers chosen from } 1, 2, \cdots, 8\}.$$

For a generic outcome (i, j, k), the first component represents the champion, the second component represents the silver and the third represents the bronze.
(iii) Now suppose that we observe the entire finishing order of the 8 horses. In this case, the sample space should be given by the set of all possible finishing orders. Mathematically, it consists of all 8-tuples $(i_1, \cdots, i_8)$ where $i_1, \cdots, i_8$ are all distinct chosen from the labels 1 to 8. The first component represents the champion, the second for the silver and the eighth for the slowest horse. Equivalently, Ω is the set of all permutations of $1, 2, \cdots, 8$.

Remark 2.1. The concept of a random experiment is only defined heuristically – it is not a mathematically well defined object. It is the sample space that has a concrete mathematical meaning, namely it is given by a *set*.

More examples of sample spaces are given as follows.

Example 2.4. (i) Consider the random experiment of tossing a coin repeatedly until a head occurs. Suppose that we observe the number of tosses required. Since the first head could possible appear at any toss, the sample space is given by

$$\Omega = \{1, 2, 3, \cdots\},$$

namely the set of positive integers. This example shows that a sample space need not contain only finitely many outcomes.
(ii) A machine automatically files a one litre bottle with fluid, and the actual quantity (measured in litres) in the bottle is observed. Since the amount of fluid in the bottle can be any real number between 0 and 1, the sample space is given by

$$\Omega = [0, 1].$$

This example shows that a sample space need not be countable.
(iii) A person is driving a car until it runs out of fuel, and the total distance traveled is observed. Theoretically this distance can be any non-negative real number. Therefore, the sample space is given by

$$\Omega = [0, \infty).$$

3 Events and their basic operations

The next important concept is the notion of events.

Definition 3.1. Let Ω be a given sample space associated with some random experiment. An *event* is a set of possible outcomes, namely, a subset of Ω.

The reason why events should be understood as subsets of Ω is easy to describe. When performing the random experiment, it produces an outcome ω (an element in the sample spaceΩ). The relationship between ω and an event A should apparently be given by

the outcome ω either triggers A or not.

The most natural way to interpret this relationship is through "whether ω belongs to A or not". In other words, an event A should be considered as a subset of Ω. We say that A *occurs* if the observed outcome ω belongs to A, in which case we denote it as $\omega \in A$. Otherwise we say that A *does not occur* and denote this case as $\omega \notin A$. The relationship between outcomes and events is the relationship of *belongingness.*

Example 3.1. Consider the random experiment of tossing a die. The sample space is given by

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

The event "the number on the die is even" is given by the collection of the outcomes $2, 4, 6$, namely, this is the event

$$A = \{2, 4, 6\}.$$

Example 3.2. Consider the random experiment of spinning a roulette wheel. The sample space is given by

$$\Omega = \{0, 1, 2, \cdots, 36\}.$$

The event "one of the first three numbers occurs" is given by

$$A = \{1, 2, 3\}.$$

The event "the number 0 appears" is given by

$$B = \{0\},$$

which is a subset containing only one outcome.

Notation. We often use capital letters A, B, C etc. to denote events, and use lower-case letters ω, x, y, z etc. to denote elements in a set (i.e. outcomes).

Given a sample space Ω, there are two obvious events one can consider. The set Ω itself is an event called the *certain event*. To justify its name, we know that $\omega \in \Omega$ is always true. Therefore, whatever outcome ω we observe, it always triggers Ω. The other extreme is the empty set $\emptyset$, which is also an event and it is known as the *impossible event*.

Since events are subsets of Ω, they are naturally subject to the usual relations and operations for sets. Let A, B be two events.

(i) *Union.* $A \cup B$ is the event that at least one of A, B occurs (i.e. A or B or both occur). Mathematically, this is the subset formed by putting the outcomes in A, B together, which is exactly the union of A, B in the language of sets. We can also consider the union of finitely many events $A_1, \cdots, A_n$ (denoted as $\cup_{i=1}^n A_i$) or more generally the union of a sequence of events $A_1, A_2, \cdots, A_n, \cdots$ (denoted as $\cup_{n=1}^\infty A_n$). This is the event that at least one of the given events occurs.
(ii) *Intersection.* $A \cap B$ is the event that both A and B occur. This is the subset

formed by all common outcomes in A, B, namely, the intersection of A, B. We can also consider the intersection of finitely many (or a sequence of) events.
(iii) *Complement.* A^c is the event that A does not occur. It is the subset formed by the outcomes that do not belong to A.
(iv) Another commonly used notation is $B \backslash A$. This is the event that B occurs but A does not. It is the subset formed by the outcomes in B that do not belong to A. This is also the same as the event $B \cap A^c$.