

The University of Melbourne  
School of Mathematics and Statistics  
Semester 2, 2020

# MAST20004 Probability



STUDENT NAME:

EMAIL ADDRESS:

**Cover image:** Jakob Bernoulli (1654 - 1705, Basel)

Jakob Bernoulli was a Swiss mathematician who made significant contributions to algebra, calculus, mechanics, calculus of variations, geometry, infinite series, and probability. His most renowned work was the derivation of the law of large numbers, which put simply, states that the average proportion of times an event occurs approaches the theoretical probability of the event, as the number of trials increases.

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For use of students of the University of Melbourne enrolled in the second year subject MAST20004 Probability.

# *Table of Contents*

• Problem Sheets .....	1
Topic 1 – Axioms of Probability .....	1
Topic 2 – Conditional Probability and Independence .....	3
Topic 3 – Random Variables and Distribution Functions .....	7
Topic 4 – Special Probability Distributions .....	14
Topic 5 – Bivariate Random Variables .....	23
Topic 6 – Sums of Independent Random Variables and Limit Theorems .....	33
Topic 7 – Stochastic Processes .....	38
• Answers to Problem Sheets .....	41
Topic 1 – Axioms of Probability .....	41
Topic 2 – Conditional Probability and Independence .....	43
Topic 3 – Random Variables and Distribution Functions .....	46
Topic 4 – Special Probability Distributions .....	51
Topic 5 – Bivariate Random Variables .....	58
Topic 6 – Sums of Independent Random Variables and Limit Theorems .....	64
Topic 7 – Stochastic Processes .....	67



# ***MAST20004 Probability***

## ***Semester 2, 2020***

### ***Problem Sheet 1***

#### ***Axioms of Probability***

##### **1. Sample Space**

Define a sample space for the experiment of putting three different books on a shelf in random order. If two of these three books are a two-volume dictionary, describe the event that these volumes stand in increasing order side-by-side (i.e., volume I precedes volume II and next to each other).

##### **2. Events**

Let  $E$ ,  $F$ , and  $G$  be three events; explain the meaning of the two relations  $E \cup F \cup G = G$  and  $E \cap F \cap G = G$ .

##### **3. More Events**

Prove that the event  $B$  is impossible if and only if for every event  $A$ ,

$$A = (B \cap A^c) \cup (B^c \cap A).$$

##### **4. Cards**

In an experiment, cards are drawn, one by one, at random and successively from an ordinary deck of 52 cards. Let  $A_n$  be the event that no face card or ace appears on the first  $n - 1$  draws, and the  $n$ th draw is an ace. In terms of  $A_n$ s, find an expression for the event that an ace appears before a face card, if

- (a) the cards are drawn with replacement;
- (b) they are drawn without replacement.

##### **5. Event Identities**

Let  $A$  and  $B$  be two events. Prove the following relations by the elementwise method.

- (a)  $(A \setminus (A \cap B)) \cup B = A \cup B$ ;
- (b)  $(A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (A^c \cap B)$ .

## 6. Infinite Sequence of Sets

Let  $\{A_1, A_2, A_3, \dots\}$  be a sequence of events of a sample space  $S$ . Find a sequence  $\{B_1, B_2, B_3, \dots\}$  of mutually exclusive events such that for all  $n \geq 1$ ,  $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$ .

## 7. Hiring

A company has only one position with three highly qualified applicants: John, Barbara, and Marty. However, because the company has only a few women employees, Barbara's chance to be hired is 20% higher than John's and 20% higher than Marty's. Find the probability that Barbara will be hired.

## 8. Probability Statements

Which of the following statements is true? If a statement is true, prove it. If it is false, give a counterexample.

- (a) If  $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = 1$ , then  $A$ ,  $B$ , and  $C$  are mutually exclusive events;
- (b) If  $\mathbb{P}(A \cup B \cup C) = 1$ , then  $A$ ,  $B$ , and  $C$  are mutually exclusive events.

## 9. Probability Identity

Let  $A$ ,  $B$ , and  $C$  be three events. Prove that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

## 10. Random Numbers

A number is selected randomly from the set  $\{1, 2, \dots, 1000\}$ . What is the probability that

- (a) it is divisible by 3 but not by 5;
- (b) it is divisible neither by 3 nor by 5?

## 11. Voting

For a Democratic candidate to win an election, she must win districts I, II, and III. Polls have shown that the probability of winning I and III is 0.55, losing II but not I is 0.34, and losing II and III but not I is 0.15. Find the probability that this candidate will win all three districts. (Draw a Venn diagram.)

## 12. Boole's Inequality

Let  $A_1, A_2, A_3, \dots$  be a sequence of events of a sample space. Prove that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

This is called Boole's inequality.

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 2***

### ***Conditional Probability and Independence***

#### **13. Dice**

Suppose that two fair dice have been tossed and the total of their top faces is found to be divisible by 5. What is the probability that both of them have landed 5?

#### **14. Movies**

The cinemas of a town are showing seven comedies and nine dramas. Brian has seen five of the movies. If the first three movies he has seen are dramas, what is the probability that the last two are comedies? Assume that Brian chooses the shows at random and sees each movie at most once.

#### **15. Conditional Probability Properties**

Let  $\Omega$  be the sample space of an experiment, and let  $B$  be an event of  $\Omega$  with  $\mathbb{P}(B) > 0$ . Prove that

- (a)  $\mathbb{P}(A|B) \geq 0$  for any event  $A \subseteq \Omega$ ;
- (b)  $\mathbb{P}(\Omega|B) = 1$ ;
- (c) If  $A_1, A_2, \dots$  is a sequence of mutually exclusive events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i | B\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i | B).$$

#### **16. Defective and Nondefective Items**

If eight defective and 12 nondefective items are inspected one by one, at random and without replacement, what is the probability that

- (a) the first four items inspected are defective;
- (b) from the first three items at least two are defective?

## 17. Credit Rating

Suppose that 75% of all people with credit records improve their credit ratings within three years. Suppose that 18% of the population at large have poor credit records, and of those only 30% will improve their credit ratings within three years. What percentage of the people who will improve their credit records within the next three years are the ones who currently have good credit ratings?

## 18. Cards

From an ordinary deck of 52 cards, cards are drawn one by one, at random and without replacement. What is the probability that the fourth heart is drawn on the tenth draw?

*Hint:* Let  $F$  denote the event that in the first nine draws there are exactly three hearts, and  $E$  be the event that the tenth draw is a heart. Use  $\mathbb{P}(F \cap E) = \mathbb{P}(F)\mathbb{P}(E|F)$ .

## 19. Proof

- (a) Show that if  $\mathbb{P}(A) = 1$ , then  $\mathbb{P}(A \cap B) = \mathbb{P}(B)$ ;
- (b) Prove that *any* event  $A$  with  $\mathbb{P}(A) = 0$  or  $\mathbb{P}(A) = 1$  is independent of *every* event  $B$ .

## 20. More Proofs

- (a) Show that if an event  $A$  is independent of itself, then  $\mathbb{P}(A) = 0$  or  $1$ ;
- (b) Show that if  $A$  and  $B$  are independent and  $A \subseteq B$ , then either  $\mathbb{P}(A) = 0$  or  $\mathbb{P}(B) = 1$ .

## 21. Matching

A fair die is rolled six times. If on the  $i$ th roll,  $1 \leq i \leq 6$ , the outcome is  $i$ , we say that a match has occurred. What is the probability that at least one match occurs?

## 22. Families

From the set of all families with three children a *family* is selected at random. Let  $A$  be the event that “the family has children of both sexes” and  $B$  be the event that “there is at most one girl in the family”. Are  $A$  and  $B$  independent?

Answer the same question for families with two children and families with four children. Assume that for any family size all sex distributions have equal probabilities.

## 23. Independent Events

Let  $\{A_1, A_2, \dots, A_n\}$  be an independent set of events and  $\mathbb{P}(A_i) = p_i, 1 \leq i \leq n$ .

- (a) What is the probability that at least one of the events  $A_1, A_2, \dots, A_n$  occurs?
- (b) What is the probability that none of the events  $A_1, A_2, \dots, A_n$  occurs?



## 24. Urns and Balls

An urn contains two red and four white balls. Balls are drawn from the urn successively, at random and with replacement. What is the probability that exactly three whites occur in the first five trials?

## 25. $A$ Before $B$

Let  $\Omega$  be the sample space of a repeatable experiment. Let  $A$  and  $B$  be mutually exclusive events of  $\Omega$ . Prove that, in independent trials of this experiment, the event  $A$  occurs before the event  $B$  with probability  $\mathbb{P}(A)/(\mathbb{P}(A) + \mathbb{P}(B))$ .

## 26. Guns

A person has six guns. The probability of hitting a target when these guns are properly aimed and fired is 0.6, 0.5, 0.7, 0.9, 0.7, and 0.8, respectively. What is the probability of hitting a target if a gun is selected at random, properly aimed, and fired?

## 27. Strokes

Of the patients in a hospital, 20% of those with, and 35% of those without myocardial infarction have had strokes. If 40% of the patients have had myocardial infarction, what percent of the patients have had strokes?

## 28. Lost Spades

- (a) One of the cards of an ordinary deck of 52 cards is lost. What is the probability that a random card drawn from this deck is a spade?
- (b) Two cards from an ordinary deck of 52 cards are missing. What is the probability that a random card drawn from this deck is a spade?

## 29. Proof

Let  $B$  be an event of a sample space  $\Omega$  with  $\mathbb{P}(B) > 0$ . For a subset  $A$  of  $\Omega$ , define  $Q(A) = \mathbb{P}(A|B)$ . By Question 15 we know that  $Q$  is a probability function. For  $E$  and  $F$ , events of  $\Omega$  (with  $\mathbb{P}(F \cap B) > 0$ ), show that  $Q(E|F) = \mathbb{P}(E|F \cap B)$ .

## 30. Another Proof

Let  $A, B, C \subseteq \Omega$  with  $\mathbb{P}(B) > 0$ ,  $\mathbb{P}(B \cap C) > 0$ , and  $\mathbb{P}(B \cap C^c) > 0$ . Show that

$$\mathbb{P}(A|B) = \mathbb{P}(A|B \cap C)\mathbb{P}(C|B) + \mathbb{P}(A|B \cap C^c)\mathbb{P}(C^c|B).$$

## 31. Married on Campus

Suppose that 40% of the students on a campus, who are married to students on the same campus, are female. Moreover, suppose that 30% of those who are married, but not to students at this campus, are also female. If one third of the married students on this campus are married to other students on this campus, what is the probability that a randomly selected married student from this campus is a woman?

### **32. Batteries**

Suppose that 10 good and three dead batteries are mixed up. Jack tests them one by one, at random and without replacement. But before testing the fifth battery he realizes that he does not remember whether the first one tested is good or is dead. All he remembers is that the last three that were tested were all good. What is the probability that the first one is also good?

### **33. Multiple Choice**

On a multiple choice exam with four choices for each question, a student either knows the answer to a question or marks it at random. If the probability that he or she knows the answers is  $2/3$ , what is the probability that an answer that was marked correctly was not marked randomly?

### **34. Cards**

A stack of cards consists of six red and five blue cards. A second stack of cards consists of nine red cards. A stack is selected at random and three of its cards are drawn. If all of them are red, what is the probability that the first stack was selected?

### **35. More Cards**

There are three identical cards that differ only in color. Both sides of one are black, both sides of the second one are red, and one side of the third card is black and its other side is red. These cards are mixed up and one of them is selected at random. If the upper side of this card is red, what is the probability that its other side is black?

### **36. Horses**

There are two stables on a farm, one that houses 20 horses and 13 mules, the other with 25 horses and eight mules. Without any pattern, animals occasionally leave their stables and then return to their stables. Suppose that during a period when all the animals are in their stables, a horse comes out of a stable and then returns. What is the probability that the next animal coming out of the same stable will also be a horse?

### **37. Chips**

An urn contains five red and three blue chips. Suppose that four of these chips are selected at random and transferred to a second urn, which was originally empty. If a random chip from this second urn is blue, what is the probability that two red and two blue chips were transferred from the first urn to the second urn?

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 3***

### ***Random Variables and Distribution Functions***

#### **38. More Chips**

From an urn that contains five red, five white, and five blue chips, we draw two chips at random. For each blue chip we win \$1, for each white chip we win \$2, but for each red chip we lose \$3. If  $X$  represents the amount that we either win or we lose, what are the possible values of  $X$  and probabilities associated with them?

#### **39. Plastic Die**

The side measurement of a plastic die, manufactured by factory A, is a random number between 1 and 1.25 centimetres. What is the probability that the volume of a randomly selected die manufactured by this company is greater than 1.424? Assume that the die will always be a cube.

#### **40. Families**

From families with three children a family is chosen at random. Let  $X$  be the number of girls in the family. Calculate and sketch the distribution function of  $X$ . Assume that in a three-child family all gender distributions are equally probable.

#### **41. Distribution Functions**

Determine if the following are distribution functions.

(a)

$$F_X(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{1+t}, & t \geq 0. \end{cases}$$

(b)

$$F_X(t) = \begin{cases} \frac{e^t}{2}, & t < 0 \\ 1 - \frac{3e^{-t}}{4}, & t \geq 0. \end{cases}$$

#### 42. Random Points

Let  $X$  be a randomly selected point from the interval  $(0, 3)$ . What is the probability that  $X^2 - 5X + 6 > 0$ ?

#### 43. New Car

Let the time until a new car breaks down be denoted by  $X$ , and let

$$Y = \begin{cases} X, & \text{if } X \leq 5 \\ 5, & \text{if } X > 5. \end{cases}$$

Then  $Y$  is the life of the car, if it lasts less than 5 years, and is 5 if it lasts longer than 5 years. Calculate the distribution function of  $Y$ ,  $F_Y$ , in terms of  $F_X$ , the distribution function of  $X$ .

#### 44. Minimum Face Value

In the experiment of rolling a fair die twice, let  $X$  be the minimum of the two numbers obtained. Determine the probability mass function and the distribution function of  $X$ , and sketch their graphs.

#### 45. Probability Mass Function

The distribution function of a random variable  $X$  is given by

$$F_X(x) = \begin{cases} 0, & x < -2 \\ 1/2, & -2 \leq x < 2 \\ 3/5, & 2 \leq x < 4 \\ 8/9, & 4 \leq x < 6 \\ 1, & x \geq 6. \end{cases}$$

Determine the probability mass function of  $X$  and sketch its graph.

#### 46. Jury

From 18 potential women jurors and 28 potential men jurors, a jury of 12 is chosen at random. Let  $X$  be the number of women selected. Find the probability mass function of  $X$ .

#### 47. First Six

In successive rolls of a fair die, let  $X$  be the number of rolls until the first 6 appears. Determine the probability mass function and the distribution function of  $X$ .

#### 48. More Probability Mass Functions

For each of the following, determine the value(s) of  $k$  for which  $p$  is a probability mass function. Note that in Parts (d) and (e),  $n$  is a positive integer.

- (a)  $p(x) = kx, x = 1, 2, 3, 4, 5;$
- (b)  $p(x) = k(1+x)^2, x = -2, 0, 1, 2;$
- (c)  $p(x) = k(1/9)^x, x = 1, 2, 3, \dots;$
- (d)  $p(x) = kx, x = 1, 2, 3, \dots, n;$
- (e)  $p(x) = kx^2, x = 1, 2, 3, \dots, n.$

*Hint:* Recall that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

#### 49. Soap Opera

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(x) = \begin{cases} 1 - \frac{16}{x^2}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

- (a) Calculate  $f$ , the probability density function of the soap opera.
- (b) Sketch the graphs of  $F$  and  $f$ .
- (c) What is the probability that the soap opera takes at most 50 hours? At least 60 hours? Between 50 and 70 hours? Between 10 and 35 hours?

#### 50. Tyres

The lifetime of a tyre selected randomly from a used tyre shop is  $10,000X$  kilometres, where  $X$  is a random variable with the density function

$$f(x) = \begin{cases} \frac{2}{x^2}, & 1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) What percentage of the tyres of this shop last fewer than 15,000 kilometres?
- (b) What percentage of those having lifetimes fewer than 15,000 kilometres last between 10,000 and 12,500 kilometres?

### 51. Another Density Function

Let  $X$  be a continuous random variable with density and distribution functions  $f$  and  $F$ , respectively. Assuming that  $\alpha \in \mathbb{R}$  is a point at which  $\mathbb{P}(X \leq \alpha) < 1$ , prove that

$$h(x) = \begin{cases} \frac{f(x)}{1 - F(\alpha)}, & x \geq \alpha \\ 0, & x < \alpha \end{cases}$$

is also a probability density function.

### 52. Investment

Suppose that the loss in a certain investment, in thousands of dollars, is a continuous random variable  $X$  that has a density function of the form

$$f(x) = \begin{cases} k(2x - 3x^2), & -1 < x < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Calculate the value of  $k$ .
- (b) Find the probability that the loss is at most \$500.

### 53. Convex Combination

Prove that if  $f$  and  $g$  are two probability density functions, then for  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $\alpha + \beta = 1$ ,  $\alpha f + \beta g$  is also a probability density function.

### 54. Fuses

A box contains 20 fuses, of which five are defective. What is the expected number of defective items among three fuses selected randomly?

### 55. Nonexistent Expected Value

It is well known that  $\sum_{x=1}^{\infty} 1/x^2 = \pi^2/6$ .

- (a) Show that  $p(x) = 6/(\pi x)^2$ ,  $x = 1, 2, 3, \dots$  is the probability mass function of a random variable  $X$ .
- (b) Prove that  $\mathbb{E}(X)$  does not exist.

### 56. Distribution Function

The distribution function of a random variable  $X$  is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ 3/8, & -3 \leq x < 0 \\ 1/2, & 0 \leq x < 3 \\ 3/4, & 3 \leq x < 4 \\ 1, & x \geq 4. \end{cases}$$

Calculate  $\mathbb{E}(X)$ ,  $\mathbb{E}(X^2 - 2|X|)$ , and  $\mathbb{E}(X|X|)$ .

### 57. Children

A newly married couple decides to continue having children until they have one of each sex. If the events of having a boy and a girl are independent and equiprobable, how many children should this couple expect?

*Hint:* Note that  $\sum_{i=1}^{\infty} ir^i = r/(1-r)^2$ ,  $|r| < 1$ .

### 58. Probability Mass Function

(a) Show that

$$p(n) = \frac{1}{n(n+1)}, \quad n \geq 1,$$

is a probability mass function.

(b) Let  $X$  be a random variable with probability mass function  $p$  given in Part (a). Find  $\mathbb{E}(X)$ .

### 59. Variance

Find the variance and the standard deviation of a random variable  $X$  with distribution function

$$F(x) = \begin{cases} 0, & x < -3 \\ 3/8, & -3 \leq x < 0 \\ 3/4, & 0 \leq x < 6 \\ 1, & x \geq 6. \end{cases}$$

### 60. Random Integer

Let  $X$  be a random integer from the set  $\{1, 2, \dots, N\}$ . Find  $\mathbb{E}(X)$ ,  $\mathbb{V}(X)$ , and  $\sigma_X$ .

### 61. Another Variance

Suppose that  $X$  is a discrete random variable with  $\mathbb{E}(X) = 1$  and  $\mathbb{E}(X(X-2)) = 3$ . Find  $\mathbb{V}(-3X+5)$ .

### 62. Soap Opera

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(x) = \begin{cases} 1 - \frac{16}{x^2}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

- (a) Calculate  $\mathbb{E}(X)$ .
- (b) Show that  $\mathbb{V}(X)$  does not exist.

### 63. Aptitude Test

The time it takes for a student to finish an aptitude test (in hours) has the density function

$$f(x) = \begin{cases} 6(x-1)(2-x), & 1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the mean and standard deviation of the time it takes for a randomly selected student to finish the aptitude test.

### 64. Expected Value

A random variable  $X$  has the density function

$$f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Calculate  $\mathbb{E}(e^X)$ .

### 65. Computer Network

Let  $Y$  be a continuous random variable with probability distribution function

$$F(y) = \begin{cases} e^{-k(\alpha-y)/A}, & y \leq \alpha \\ 1, & y > \alpha, \end{cases}$$

where  $A$ ,  $k$ , and  $\alpha$  are positive constants. (Such distribution functions arise in the study of local computer network performance.) Find  $\mathbb{E}(Y)$ .

### 66. Logarithm of a Random Variable

Let  $X$  be a continuous random variable with probability density function

$$f(x) = \begin{cases} 2/x^2, & 1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find  $\mathbb{E}(\log X)$ .



### 67. Double Exponential Distribution

Let  $X$  be a continuous random variable with probability density function

$$f(x) = \frac{e^{-|x|}}{2}, \quad -\infty < x < \infty.$$

Calculate  $\mathbb{V}(X)$ .

### 68. Absolute Moment

Let  $X$  be a discrete random variable. The *absolute moment of  $X$  of order  $t$*  is  $\mathbb{E}(|X|^t)$ . If  $0 < s < r$ , show that if the absolute moment of order  $r$  of  $X$  exists, then the absolute moment of order  $s$  also exists.

### 69. Recursive Moments

Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x \sin x}{\pi}, & 0 < x < \pi \\ 0, & \text{otherwise.} \end{cases}$$

Prove that  $\mathbb{E}(X^{n+1}) + (n+1)(n+2)\mathbb{E}(X^{n-1}) = \pi^{n+1}$ .

### 70. Computing Moments using Tail Probabilities

If  $\mathbb{P}(X \leq 0) = 1$  and  $\mathbb{E}[|X|^n] < \infty$ , show that for  $n > 0$ ,

$$\mathbb{E}[X^n] = -n \int_{-\infty}^0 x^{n-1} F_X(x) dx.$$

### 71. Post Office

Suppose that  $X$ , the interarrival time between two customers entering a certain post office, satisfies

$$\mathbb{P}(X > t) = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \quad t \geq 0,$$

where  $\alpha + \beta = 1$ ,  $\alpha, \beta \geq 0$ ,  $\lambda, \mu > 0$ . Calculate

(a)  $\mathbb{E}[X]$ ;

(b)  $\mathbb{V}(X)$ .

### 72. Bounded Random Variable

Let  $X$  be a continuous random variable with set of possible values  $\{x : 0 < x < \alpha\}$  (where  $\alpha > 0$ ) and distribution function  $F_X$ . Using integration by parts, prove that

$$\mathbb{E}[X] = \int_0^\alpha [1 - F_X(x)] dx.$$

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 4***

### ***Special Probability Distributions***

#### **73. Number Plates**

In a state where license plates contain six digits, what is the probability that the license number of a randomly selected car has two 9s? Assume that each digit of the license number is randomly selected from  $\{0, 1, \dots, 9\}$ .

#### **74. Nails**

A manufacturer of nails claims that only 3% of its nails are defective. A random sample of 24 nails is selected, and it is found that two of them are defective. Is it fair to reject the manufacturers claim based on this observation? (Hint: assuming 3% of its nails are defective, is the probability of getting a sample of at least 2 defectives out of 24 small?)

#### **75. Five Points**

From the interval  $(0, 1)$ , five points are selected at random and independently. What is the probability that

- (a) at least two of them are less than  $1/3$ ;
- (b) the first decimal point of exactly two of them is 3?

#### **76. Rare Blood Type**

A certain rare blood type can be found in only 0.05% of people. If the population of a randomly selected group is 3000, what is the probability that at least two persons in the group have this rare blood type?

#### **77. Children**

A woman and her husband want to have at least a 95% chance for at least one boy and at least one girl. What is the minimum number of children that they should plan to have? Assume that the events that a child is a girl and a boy are equiprobable and independent of the gender of other children born in the family.

### 78. Chuck-a-Luck

A game often played in carnivals and gambling houses is called *chuck-a-luck*, where a player bets on any number 1 through 6. Then three fair dice are tossed. If one, two, or all three land the same number as the player's, then he or she receives one, two, or three times the original stake plus his or her original bet, respectively. Otherwise, the player loses his or her stake. Let  $X$  be the net gain of the player per unit of stake. First find the probability mass function of  $X$ ; then determine the expected amount that the player will lose per unit of stake.

### 79. Aircraft Engines

Suppose that an aircraft engine will fail in flight with probability  $1 - p$  independently of the plane's other engines. Also suppose that a plane can complete the journey successfully if at least half of its engines do not fail. Is it true that a four-engine plane is always preferable to a two-engine plane? Explain.

### 80. Even Number of Successes

What is the probability of an even number of successes in  $n$  independent Bernoulli trials?

*Hint:* Let  $r_n$  be the probability of an even number of successes in  $n$  Bernoulli trials. By conditioning on the first trial and using the law of total probability, show that for  $n \geq 1$ ,

$$r_n = p(1 - r_{n-1}) + (1 - p)r_{n-1}.$$

Then prove that  $r_n = \frac{1}{2}(1 + (1 - 2p)^n)$ .

### 81. Targets

The probability is  $p$  that Marty hits target  $M$  when he fires at it. The probability is  $q$  that Alvie hits target  $A$  when he fires at it. Marty and Alvie fire one shot each at their targets. If both of them hit their targets, they stop; otherwise, they will continue.

- (a) What is the probability that they stop after each has fired  $r$  times?
- (b) What is the expected value of the number of turns taken before the turn where they both hit the target?

### 82. Basketball

A certain basketball player makes a foul shot with probability 0.45. What is the probability that

- (a) his first basket occurs on the sixth shot?
- (b) his first and second baskets occur on his fourth and eighth shots, respectively?

### 83. Light Bulbs

The probability is  $p$  that a randomly chosen light bulb is defective. We screw a bulb into a lamp and switch on the current. If the bulb works, we stop; otherwise, we try another and continue until a good bulb is found. What is the probability that at least  $n$  bulbs were defective?

#### 84. Bridge

On average, how many games of bridge are necessary until a player is dealt three aces? A bridge hand is 13 randomly selected cards from an ordinary deck of 52 cards.

#### 85. Senior Citizens

Suppose that 15% of the population of a town are senior citizens. Let  $X$  be the number of nonsenior citizens who enter a mall before the tenth senior citizen arrives. Find the probability mass function of  $X$ . Assume that each customer who enters the mall is a random person from the entire population.

#### 86. Geometric Distribution

Suppose  $X \stackrel{d}{=} G(p)$ . Calculate the probability that  $X$  is even.

#### 87. Negative Binomial

For  $r = 1, 2, \dots$ , show that,

$$\binom{-r}{z} = (-1)^z \binom{z+r-1}{r-1},$$

using the extended binomial coefficient definition on Slide 166.

#### 88. Negative Binomial Distribution

For  $r \in (0, \infty)$  and  $0 < p \leq 1$ , let  $Z \stackrel{d}{=} \text{Nb}(r, p)$ . For  $z = 0, 1, 2, \dots$ , show that

$$p_Z(z) = \binom{-r}{z} p^r (p-1)^z \geq 0.$$

#### 89. Professors' Cars

Of the 28 professors in a certain department, 18 drive foreign and 10 drive domestic cars. If five of these professors are selected at random, what is the probability that at least three of them drive foreign cars?

#### 90. Bernoulli Trials

Suppose that independent Bernoulli trials with parameter  $p$  are performed successively. Let  $N$  be the number of trials *until*  $x$  successes, and  $X$  be the number of successes in the first  $n$  trials. Show that

$$\mathbb{P}(N = n) = \frac{x}{n} \mathbb{P}(X = x).$$

*Remark:* By this relation, in coin tossing, for example, we can state that the probability of getting a fifth head on the seventh toss is  $5/7$  of the probability of five heads in seven tosses.

#### 91. Charity

In an annual charity drive, 35% of a population of 560 make contributions. If, in a statistical survey, 15 people are selected at random and without replacement, what is the probability that at least two persons have contributed?

## 92. Defective Items

The policy of the quality control division of a certain corporation is to reject a shipment if more than 5% of its items are defective. A shipment of 500 items is received, 30 of them are randomly tested, and two have been found defective. Should that shipment be rejected? That is, assuming 5% of items are defective, is the probability of getting a sample of at least two defectives small?

## 93. Trout

To estimate the number of trout in a lake, we caught 50 trout, tagged and returned them. Later we caught 50 trout and found that four of them were tagged. From this experiment estimate  $n$ , the total number of trout in the lake.

*Hint:* Let  $p_n$  be the probability of four tagged trout among the 50 trout caught. Find the value of  $n$  that maximizes  $p_n$ .

## 94. Annual Income

Suppose that 3% of the families in a large city have an annual income of over \$60,000. What is the probability that, of 60 random families, at most three have an annual income of over \$60,000?

## 95. Misprints

Misprints in a particular book occur independently, and on average, there are three misprints in every 10 pages. If every chapter of the book contains 35 pages, what is the probability that Chapters 1 and 5 have 10 misprints each?

## 96. No Crimes

In a certain town, crimes occur at a Poisson rate of five per month. What is the probability of having exactly two months (not necessarily consecutive) with no crimes during the next year?

## 97. Bookstore

Customers arrive at a bookstore at a Poisson rate of six per hour. Given that the store opens at 9:30am, what is the probability that exactly one customer arrives by 10:00am and 10 customers by noon?

## 98. Lottery Tickets

Suppose that in Melbourne, on a certain day,  $N$  lottery tickets are sold and  $M$  win, with  $M/N$  very small. To have a probability of at least  $\alpha$  of at least one win on that day, approximately how many tickets should be purchased?

### 99. Poisson Distribution

Let  $X$  be a Poisson random variable with parameter  $\lambda$ . Show that the maximum of  $\mathbb{P}(X = i)$  occurs at  $\lfloor \lambda \rfloor$ , where  $\lfloor \lambda \rfloor$  is the greatest integer less than or equal to  $\lambda$ .

*Hint:* Let  $p_X$  be the probability mass function of  $X$ . Prove that

$$p_X(i) = \frac{\lambda}{i} p_X(i-1).$$

Use this to find the values of  $i$  at which  $p$  is increasing and the values of  $i$  at which it is decreasing.

### 100. Line Segment

A point is selected at random on a line segment of length  $l$ . What is the probability that

- (a) the longer segment is at least twice as long as the shorter segment?
- (b) What is the probability that none of the two segments is smaller than  $l/3$ ?

### 101. Random Angle

Let  $\Theta$  be a random number between  $-\pi/2$  and  $\pi/2$ . Find the probability density function of  $X = \tan \Theta$

### 102. Expectation of a Logarithm

Let  $X$  be a uniform random variable over the interval  $(0, 1)$ . Calculate  $\mathbb{E}(-\log X)$ .

### 103. Chicken Pen

A farmer who has two pieces of lumber of lengths  $a$  and  $b$  ( $a < b$ ) decides to build a pen in the shape of a triangle for his chickens. He sends his foolish son out to cut the longer piece and the boy, without taking any thought as to the ultimate purpose, makes a cut on the lumber of length  $b$ , at a point selected randomly. What are the chances that the two resulting pieces and the piece of length  $a$  can be used to form a triangular pen?

*Hint:* Three segments form a triangle if and only if the length of any one of them is less than the sum of the lengths of the remaining two.

### 104. Heart Attack

The time between the first and second heart attacks for a certain group of people is an exponential random variable. If 50% of those who have had a heart attack will have another one within the next five years, what is the probability that a person who had one heart attack five years ago will not have another one in the next five years?

### 105. Exponential Random Variable

Let  $X$  be an exponential random variable with parameter  $\lambda$ . Find

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq 2\sigma_X).$$

**106. Telephone Call**

Mr. Jones is waiting to make a phone call at a train station. There are two public telephone booths next to each other, occupied by two persons, say  $A$  and  $B$ . If the duration of each telephone call is an exponential random variable with  $\lambda = 1/8$ , what is the probability that among Mr. Jones,  $A$ , and  $B$ , Mr. Jones will not be the last to finish his call?

**107. Double Exponential Random Variable**

The random variable  $X$  is called double exponentially distributed if its density function is given by

$$f(x) = ce^{-|x|}, \quad -\infty < x < \infty.$$

- (a) Find the value of  $c$ ;
- (b) Prove that  $\mathbb{E}(X^{2n}) = (2n)!$  and  $\mathbb{E}(X^{2n+1}) = 0$ .

**108. Radio Tubes**

Let  $X$ , the lifetime (in years) of a radio tube, be exponentially distributed with mean  $1/\lambda$ . Prove that  $\lfloor X \rfloor$ , the integer part of  $X$ , which is the complete number of years that the tube works, is a geometric random variable.

**109. Gamma Distribution**

Let  $X \stackrel{d}{=} \gamma(r, \alpha)$ . If  $c > 0$ , show that  $cX \stackrel{d}{=} \gamma(r, \alpha/c)$ .

**110. Defective Light Bulbs**

A manufacturer produces light bulbs at a Poisson rate of 200 per hour. The probability that a light bulb is defective is 0.015. During production, the light bulbs are tested one by one, and the defective ones are put in a special can that holds up to a maximum of 25 light bulbs. On average, how long does it take until the can is filled?

**111. Born on Christmas Day**

A small college has 1095 students. What is the approximate probability that more than five students were born on Christmas day? Assume that the birthrates are constant throughout the year and that each year has 365 days.

**112. Beta Mean and Variance**

Let  $X \stackrel{d}{=} \text{Beta}(\alpha, \beta)$ . Calculate  $\mathbb{E}[X]$  and  $\mathbb{V}(X)$ .

**113. Beta Random Variable**

For which value of  $c$  is the following a probability density function of some random variable  $X$ ? Find  $\mathbb{E}[X]$  and  $\mathbb{V}(X)$ .

$$f(x) = \begin{cases} cx^4(1-x)^5, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

**114. Blood Pressure Medicines**

Suppose that new blood pressure medicines introduced are effective on  $100P\%$  of the patients, where  $P$  is beta random variable with parameters  $\alpha = 20$  and  $\beta = 13$ . What is the probability that a new blood pressure medicine is effective on at least 60% of the hypertensive population?

**115. Resistors**

The proportion of resistors a procurement office of an engineering firm orders every month, from a specific vendor, is a beta random variable with mean  $1/3$  and variance  $1/18$ . What is the probability that next month, the procurement office orders at least  $7/12$  of its purchase from the vendor?

**116. Pareto Random Variable**

Let  $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$ . Find the distribution function  $F_X$ , and then calculate  $\mathbb{E}[X]$  and  $\mathbb{V}(X)$ .

**117. Pareto Random Variable**

Let  $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$ . Evaluate  $\mathbb{E}[X^n]$  and state the values of  $n$  for which it is defined.

**118. Newspaper Subscribers**

The ages of subscribers to a certain newspaper are normally distributed with mean 35.5 years and standard deviation 4.8. What is the probability that the age of a random subscriber is

- (a) more than 35.5 years;
- (b) between 30 and 40 years?

**119. IQ**

Suppose that the IQ of a randomly selected student from a university is normal with mean 110 and standard deviation 20. Determine the interval of values that is centered at the mean and includes 50% of the IQs of the students at that university.

**120. Manual Dexterity**

Suppose that the scores on a certain manual dexterity test are normal with mean 12 and standard deviation 3. If eight randomly selected individuals take the test, what is the probability that none will make a score less than 14?



**121. Normal Density Function**

Determine the value(s) of  $k$  for which the following is the probability density function of a normal random variable.

$$f(x) = \sqrt{k}e^{-k^2x^2-2kx-1}, \quad -\infty < x < \infty.$$

**122. Density Function of  $Y$** 

Let  $X \stackrel{d}{=} N(0, 1)$ . Calculate the density function of  $Y = \sqrt{|X|}$ .

**123. Another Normal Density Function**

Prove that for some constant  $k$ ,  $f(x) = ka^{-x^2}$ ,  $a \in (1, \infty)$ , is a normal probability density function.

**124. Skeletons**

At an archaeological site 130 skeletons are found and their heights are measured and found to be approximately normal with mean 172 centimetres and standard deviation 9 centimetres. At a nearby site, five skeletons are discovered and it is found that the heights of exactly three of them are above 185 centimetres. Based on this information is it reasonable to assume that the second group of skeletons belongs to the same family as the first group of skeletons?

**125. Weibull Mean and Variance**

Let  $X \stackrel{d}{=} \text{Weibull}(\beta, \gamma)$ . Calculate  $\mathbb{E}[X]$  and  $\mathbb{V}(X)$ .

**126. Weibull and exponential**

Let  $X \stackrel{d}{=} \text{Weibull}(\beta, \gamma)$  and  $Y = [X/\beta]^\gamma$ , derive the pdf of  $Y$ .

**127. Exponential of a Random Variable**

Let  $X$  be a continuous random variable with distribution function  $F$  and density function  $f$ . Calculate the density function,  $g$ , of the random variable  $Y = e^X$ .

**128. Logarithm of a Random Variable**

Let  $X$  be a continuous random variable with the density function

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Find the probability density function of  $Y = \log_2 X$ .

**129. Log of a Pareto Random Variable**

Let  $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$ . Find the distribution and density functions of  $Y = \log X$ .

**130. Square of a Random Variable**

Let  $f$  be the probability density function of a random variable  $X$ . In terms of  $f$ , calculate the probability density function of  $Y = X^2$ ,  $g$ .

**131. Reciprocal of a Random Variable**

Let  $X$  be a random variable with the probability density function given by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Let

$$Y = \begin{cases} X, & \text{if } X \leq 1 \\ 1/X, & \text{if } X > 1. \end{cases}$$

Find the probability density function of  $Y$ .

**132. Transformation by  $\psi(x) = \min(x, M)$** 

Let  $X \stackrel{d}{=} \exp(1)$ , find the cdf of  $Y = \min(X, 5)$  and derive  $\mathbb{E}(Y)$  and  $\mathbb{V}(Y)$ .

**133. Lognormal Distribution**

Let  $X$  be a lognormal random variable with parameters  $\mu$  and  $\sigma^2$ .

- (a) Find  $\mathbb{E}(X^r)$ ;
- (b) Calculate  $\mathbb{E}(X)$  and  $\mathbb{V}(X)$ .

**134. Large Fires**

In 1977 a British researcher demonstrated that if  $X$  is the loss from a large fire, then  $X$  is a lognormal random variable. Suppose that the expected loss due to fire in the buildings of a certain industry, in thousands of dollars, is 120 with standard deviation 36. What is the probability that the loss from a fire in such an industry is less than \$100,000?

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 5***

### ***Bivariate Random Variables***

#### **135. Joint Distribution**

Let the joint probability mass function of discrete random variables  $X$  and  $Y$  be given by

$$p(x, y) = \begin{cases} c(x + y), & \text{if } x = 1, 2, 3, \quad y = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine

- (a) the value of the constant  $c$ ;
- (b) the marginal probability mass functions of  $X$  and  $Y$ ;
- (c)  $\mathbb{P}(X \geq 2|Y = 1)$ ;
- (d)  $\mathbb{E}(X)$  and  $\mathbb{E}(Y)$ .

#### **136. Another Joint Distribution**

Let the joint probability mass function of discrete random variables  $X$  and  $Y$  be given by

$$p(x, y) = \begin{cases} \frac{x^2 + y^2}{25}, & \text{if } x = 1, 2, \quad y = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find

- (a)  $\mathbb{P}(X > Y)$ ;
- (b)  $\mathbb{P}(X + Y \leq 2)$ ;
- (c)  $\mathbb{P}(X + Y = 2)$ .

#### **137. Cards**

From an ordinary deck of 52 cards, seven cards are drawn at random and without replacement. Let  $X$  and  $Y$  be the number of hearts and the number of spades drawn, respectively.

- (a) Find the joint probability mass function of  $X$  and  $Y$ ;
- (b) Calculate  $\mathbb{P}(X \geq Y)$ .

**138. Marginal Densities**

Let the joint probability density function of random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} 8xy, & \text{if } 0 \leq y \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Calculate the marginal probability density functions of  $X$  and  $Y$ ;
- (b) Calculate  $\mathbb{E}(X)$  and  $\mathbb{E}(Y)$ .

**139. Probability Calculations**

Let the joint probability density function of random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate

- (a)  $\mathbb{P}(X + Y \leq 1/2)$ ;
- (b)  $\mathbb{P}(X - Y \leq 1/2)$ ;
- (c)  $\mathbb{P}(XY \leq 1/4)$ ;
- (d)  $\mathbb{P}(X^2 + Y^2 \leq 1)$ .

**140. Line Segment**

On a line segment  $AB$  of length  $l$ , two points  $C$  and  $D$  are placed at random and independently. What is the probability that  $C$  is closer to  $D$  than to  $A$ ?

**141. Expected Values**

Let  $X$  and  $Y$  be two continuous random variables with finite expectations. Show that if  $\mathbb{P}(X \leq Y) = 1$ , then  $\mathbb{E}(X) \leq \mathbb{E}(Y)$ .

**142. Cards**

From an ordinary deck of 52 cards, eight cards are drawn at random and without replacement. Let  $X$  and  $Y$  be the number of clubs and spades, respectively. Are  $X$  and  $Y$  independent?

**143. Maximum and Minimum of  $X$  and  $Y$** 

Let  $X$  and  $Y$  be two independent random variables with distribution functions  $F$  and  $G$ , respectively. Find the distribution functions of  $U = \max(X, Y)$  and  $V = \min(X, Y)$ .

**144. Independent or Not**

Let the joint probability density function of random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} 8xy, & \text{if } 0 \leq y \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine if  $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ .

**145. Probability Density of  $X/Y$** 

Let  $X$  and  $Y$  be two independent random variables with the same probability density function given by

$$f(x) = \begin{cases} e^{-x}, & \text{if } 0 < x < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Show that  $g$ , the probability density function of  $X/Y$ , is given by

$$g(t) = \begin{cases} \frac{1}{(1+t)^2}, & \text{if } 0 < t < \infty \\ 0, & t \leq 0. \end{cases}$$

**146. Expected Value of Maximum and Minimum**

Let  $X$  and  $Y$  be independent random points from the interval  $(-1, 1)$ . Find  $\mathbb{E}(\max(X, Y))$  and  $\mathbb{E}(\min(X, Y))$

**147. Product of Two Functions**

For  $x, y \in \mathbb{R}$ , let the joint probability density function of two random variables  $X$  and  $Y$  satisfy

$$f_{(X,Y)}(x, y) = g(x)h(y),$$

where  $g$  and  $h$  are two functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Show that  $X$  and  $Y$  are independent.

**148. Conditional Density Function**

Let the joint probability density function of continuous random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} 2, & \text{if } 0 < x < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find  $f_{X|Y}(x|y)$ .

**149. Conditional Probability**

Let the conditional probability density function of  $X$  given that  $Y = y$  be given by

$$f_{X|Y}(x|y) = \frac{3(x^2 + y^2)}{3y^2 + 1}, 0 < x < 1, 0 < y < 1.$$

Find  $\mathbb{P}(1/4 < X < 1/2 | Y = 3/4)$ .

### 150. Another Conditional Density Function

Let the joint probability density function of continuous random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find  $f_{X|Y}(x|y)$ .

### 151. Random Points

First a point  $Y$  is selected at random from the interval  $(0, 1)$ . Then another point  $X$  is selected at random from the interval  $(Y, 1)$ . Find the probability density function of  $X$ .

### 152. Conditional Mean and Variance

Let the joint probability density function of continuous random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} ce^{-x}, & \text{if } x \geq 0, \quad |y| < x \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Determine the constant  $c$ ;
- (b) Find  $f_{X|Y}(x|y)$  and  $f_{Y|X}(y|x)$ ;
- (c) Calculate  $\mathbb{E}(Y|X = x)$  and  $\mathbb{V}(Y|X = x)$ .

### 153. Yet Another Conditional Density Function

A point is selected at random and uniformly from the region  $R = \{(x, y) : |x| + |y| \leq 1\}$ . Find the conditional probability density function of  $X$  given  $Y = y$ .

### 154. Sum of Two Random Numbers

From the interval  $(0, 1)$ , two random numbers are selected independently. Show that the probability density function of their sum is given by

$$g(t) = \begin{cases} t, & \text{if } 0 \leq t < 1 \\ 2 - t, & \text{if } 1 \leq t < 2 \\ 0, & \text{otherwise.} \end{cases}$$

**155. Conditional Probability**

Let  $-1/9 < c < 1/9$  be a constant. Let  $p_{(X,Y)}(x, y)$ , the joint probability mass function of the random variables  $X$  and  $Y$ , be given by the following table

$x$	$y$		
	$-1$	$0$	$1$
$-1$	$1/9$	$1/9 - c$	$1/9 + c$
$0$	$1/9 + c$	$1/9$	$1/9 - c$
$1$	$1/9 - c$	$1/9 + c$	$1/9$

- (a) Show that the probability mass function of  $X + Y$  is the convolution function of the probability mass functions of  $X$  and  $Y$  for all  $c$ .
- (b) Show that  $X$  and  $Y$  are independent if and only if  $c = 0$ .

**156. Conditional Mean and Variance**

Let the joint probability density function of random variables  $X$  and  $Y$  be given by

$$f(x, y) = \begin{cases} 2e^{-(x+2y)}, & \text{if } x \geq 0, \quad y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find

- (a)  $\mathbb{E}(X)$ ;
- (b)  $\mathbb{E}(Y)$ ;
- (c)  $\mathbb{E}(X^2 + Y^2)$ .

**157. Letters**

An absentminded professor wrote  $n$  letters and sealed them in envelopes without writing the addresses on the envelopes. Having forgotten which letter he had put in which envelope, he wrote the  $n$  addresses on the envelopes at random. What is the expected number of the letters addressed correctly?

*Hint:* For  $i = 1, 2, \dots, n$ , let

$$X_i = \begin{cases} 1, & \text{if the } i\text{th letter is addressed correctly} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate  $\mathbb{E}(X_1 + X_2 + \dots + X_n)$ .

**158. Empty Boxes**

Suppose that 80 balls are placed into 40 boxes at random and independently. What is the expected number of empty boxes?

*Hint:* For  $i = 1, 2, \dots, 40$ , let

$$X_i = \begin{cases} 1, & \text{if the } i\text{th box is empty} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate  $\mathbb{E}(X_1 + X_2 + \dots + X_{40})$ .

**159.  $X$  and  $Y$** 

Let  $X$  and  $Y$  be nonnegative random variables with an arbitrary joint probability distribution function. Let

$$I(x, y) = \begin{cases} 1, & \text{if } X > x, \quad Y > y \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that

$$\int_0^\infty \int_0^\infty I(x, y) dx dy = XY.$$

(b) Prove that

$$\mathbb{E}(XY) = \int_0^\infty \int_0^\infty \mathbb{P}(X > x, Y > y) dx dy.$$

**160. Discrete Random Variable**

Let  $N$  be a discrete random variable with set of possible values  $\{1, 2, 3, \dots\}$ . Prove that

$$\mathbb{E}(N) = \sum_{i=1}^{\infty} \mathbb{P}(N \geq i).$$

*Hint:* For  $i = 1, 2, \dots$ , let

$$X_i = \begin{cases} 1, & \text{if } N \geq i \\ 0, & \text{otherwise.} \end{cases}$$

Then show that

$$N = \sum_{i=1}^{\infty} X_i.$$

**161. Covariance**

Let the joint probability mass function of random variables  $X$  and  $Y$  be given by

$$p(x, y) = \begin{cases} \frac{x(x+y)}{70}, & \text{if } x = 1, 2, 3, \quad y = 3, 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Find  $\text{Cov}(X, Y)$ .



### 162. Covariance Proofs

For random variables  $X$ ,  $Y$ , and  $Z$ , prove that

- (a)  $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$ ;
- (b)  $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$ .

### 163. More Covariance Proofs

For random variables  $X$ , and  $Y$ , prove that

- (a)  $\text{Cov}(X + Y, X - Y) = \mathbb{V}(X) - \mathbb{V}(Y)$ ;
- (b)  $\mathbb{V}(X - Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\text{Cov}(X, Y)$ .

### 164. Independent and/or Uncorrelated?

Let  $X$  and  $Y$  be the coordinates of a random point selected uniformly from the unit disk  $\{(x, y) : x^2 + y^2 \leq 1\}$ . Are  $X$  and  $Y$  independent? Are they uncorrelated? Why or why not?

### 165. Investment

Mr. Ingham has invested money in three assets; 18% in the first asset, 40% in the second one, and 42% in the third one. Let  $r_1$ ,  $r_2$ , and  $r_3$  be the annual rate of returns for these three investments, respectively. For  $1 \leq i, j \leq 3$ ,  $\text{Cov}(r_i, r_j)$  is the entry in the  $i$ th row and the  $j$ th column of the following table. (Note that  $\mathbb{V}(r_i) = \text{Cov}(r_i, r_i)$ .)

	$r_1$	$r_2$	$r_3$
$r_1$	0.064	0.03	0.015
$r_2$	0.03	0.0144	0.021
$r_3$	0.015	0.021	0.01

Find the standard deviation of the annual rate of return for Mr. Inghams total investment.

### 166. Covariance of Linear Combinations

Let  $X_1, X_2, \dots, X_n$ ,  $Y_1, Y_2, \dots, Y_m$  be random variables, and  $a_1, a_2, \dots, a_n$ ,  $b_1, b_2, \dots, b_m$  be constants. Prove that

$$\text{Cov} \left( \sum_{i=1}^n a_i X_i, \sum_{j=1}^m b_j Y_j \right) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(X_i, Y_j).$$

### 167. Variance of a Linear Combination

Let  $X_1, X_2, \dots, X_n$  be random variables, and  $a_1, a_2, \dots, a_n$  be constants. Prove that

$$\mathbb{V} \left( \sum_{i=1}^n a_i X_i \right) = \sum_{i=1}^n a_i^2 \mathbb{V}(X_i) + 2 \sum_{i=1}^n \sum_{j>i}^n a_i a_j \text{Cov}(X_i, X_j).$$

**168. Correlation**

Let  $X$  and  $Y$  be jointly distributed with  $\rho(X, Y) = 1/2$ ,  $\sigma_X = 2$ , and  $\sigma_Y = 3$ . Find  $\mathbb{V}(2X - 4Y + 3)$ .

**169. Broken Stick**

A stick of length 1 is broken into two pieces at a random point. Find the correlation coefficient and the covariance of these pieces.

**170. Uncorrelated Result**

Prove that if  $\text{Cov}(X, Y) = 0$ , then

$$\rho(X + Y, X - Y) = \frac{\mathbb{V}(X) - \mathbb{V}(Y)}{\mathbb{V}(X) + \mathbb{V}(Y)}.$$

**171. No Linear Relation**

Show that if the joint probability density function of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} \frac{\sin(x+y)}{2}, & \text{if } 0 \leq x \leq \frac{\pi}{2}, \quad 0 \leq y \leq \frac{\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

then there exists no linear relation between  $X$  and  $Y$ .

**172. Two Successive Tails**

A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.

*Hint:* Let

$$M := \min\{i : i\text{th toss is a head}\},$$

find the distribution of  $M$  and then condition on  $M$ .

**173. Mixture of Two Random Variables**

For given independent random variables  $Y$  and  $Z$ , let

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1 - p. \end{cases}$$

Find  $\mathbb{E}(X)$  in terms of  $\mathbb{E}(Y)$  and  $\mathbb{E}(Z)$ .

**174. Verification**

Let  $X$  and  $Y$  be continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} \frac{3(x^2 + y^2)}{2}, & \text{if } 0 < x < 1, \quad 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Verify that  $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$ .

### 175. Eggs Hatching

Suppose an insect lays  $N$  eggs where  $N \stackrel{d}{=} \text{Pn}(\lambda)$ . Suppose each egg, independently, has a probability of  $p$  of hatching. Let  $X$  be the number of eggs that hatch. Evaluate  $\mathbb{E}[X]$  and  $\mathbb{V}(X)$ , and deduce the distribution of  $X$ .

### 176. Electronic Components

Suppose a machine produces  $N$  electronic components each day where  $N \stackrel{d}{=} \text{Bi}(m, q)$ . Suppose each component, independently, has a probability of  $p$  of working. Let  $Y$  be the number of components that work. Evaluate  $\mathbb{E}[Y]$  and  $\mathbb{V}(Y)$ , and deduce the distribution of  $Y$ .

### 177. Consecutive Zeros

What is the expected number of random digits that should be generated to obtain three consecutive zeros?

*Hint:* Let

$$M := \min\{i : i\text{th digit is nonzero}\},$$

find the distribution of  $M$  and then condition on  $M$ .

### 178. Fishing

A fisherman catches fish in a large lake with lots of fish, at a Poisson rate of two per hour. If, on a given day, the fisherman spends randomly anywhere between 3 and 8 hours fishing, find the expected value and the variance of the number of fish he catches.

### 179. Wallets

Suppose that  $X$  and  $Y$  represent the amount of money in the wallets of players A and B, respectively. Let  $X$  and  $Y$  be jointly uniformly distributed on the unit square  $[0, 1] \times [0, 1]$ . A and B each places his wallet on the table. Whoever has the smallest amount of money in his wallet wins all the money in the other wallet. Let  $W_A$  be the amount of money that player A will win. Show that  $E(W_A) = 0$ .

### 180. Heights

Let  $X$  be the height of a man and  $Y$  the height of his daughter (both in centimetres). Suppose that the joint probability density function of  $X$  and  $Y$  is bivariate normal with the following parameters:  $\mu_X = 180$ ,  $\mu_Y = 152$ ,  $\sigma_X = 7.6$ ,  $\sigma_Y = 6.9$ , and  $\rho = 0.45$ . Find the probability that the height of the daughter, of a man who is 178 centimetres tall, is at least 150 centimetres.

**181. Bivariate Normal**

The joint probability density function of  $X$  and  $Y$  is bivariate normal with  $\mu_X = \mu_Y = 0$ ,  $\sigma_X = \sigma_Y = 9$ , and  $\rho = 0$ . Find

(a)  $\mathbb{P}(X \leq 6, Y \leq 12)$ ;

(b)  $\mathbb{P}(X^2 + Y^2 \leq 36)$

*Hint: use polar coordinates.*

**182. Grade Point Average**

At a certain university, the joint probability density function of  $X$  and  $Y$ , the grade point averages of a student in his or her first and second years, respectively, is bivariate normal. From the grades of past years it is known that  $\mu_X = 60\%$ ,  $\mu_Y = 50\%$ ,  $\sigma_X = 10\%$ ,  $\sigma_Y = 8\%$ , and  $\rho = 0.4$ . Find the probability that a student with grade point average 70% in his or her first year will earn a grade point average of at least 64% in his or her senior year.

**183. Approximating Means and Variances**

For each of the following using Taylor series approximate  $\mathbb{E}[Y]$  and  $\mathbb{V}(Y)$ .

(a) Let  $X \stackrel{d}{=} R(0, 1)$  and  $Y = \sin X$ .

(b) Let  $X \stackrel{d}{=} \exp(2)$  and  $Y = \log X$ .

(c) Let  $X \stackrel{d}{=} G\left(\frac{1}{3}\right)$  and  $Y = X^2$ .

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 6***

### ***Sums of Independent Random Variables/Limit Theorems***

#### **184. Chebyshev's Inequality**

Let  $X$  be a nonnegative random variable with  $\mathbb{E}(X) = 5$  and  $\mathbb{E}(X^2) = 42$ . Find an upper bound for  $\mathbb{P}(X \geq 11)$  using Chebyshev's inequality.

#### **185. Accidents**

Suppose that the average number of accidents at an intersection is two per day. Let the variance of the number of accidents be two. Use Chebyshev's inequality to find a bound on the probability that tomorrow at least five accidents will occur.

#### **186. IQ Scores**

The average IQ score on a certain campus is 110. If the variance of these scores is 15, what can be said about the percentage of students with an IQ above 140?

#### **187. Chebyshev's Inequality Again**

Suppose that  $X$  is a random variable with  $\mathbb{E}(X) = \mathbb{V}(X) = \mu$ . What does Chebyshev's inequality say about  $\mathbb{P}(X > 2\mu)$ ?

#### **188. Multiple Choice**

In a multiple-choice test with false answers receiving negative scores, the mean of the grades of the students is 0 and its standard deviation is 15. Find an upper bound for the probability that a student's grade is at least 45.

#### **189. Probability Generating Function**

The probability generating function for a discrete random variable  $X$  is given by

$$P_X(z) = 0.3 + 0.2z + 0.1z^2 + 0.4z^3.$$

- (a) Write down the probability mass function for  $X$ ;
- (b) Using the probability generating function, calculate  $\mathbb{E}(X)$  and  $\mathbb{V}(X)$ ;
- (c) Write down the probability generating function for  $Y = X^3$ .

### 190. Sum of Negative Binomial Random Variables

For  $i = 1, 2, \dots, n$ , let  $X_i \stackrel{d}{=} \text{Nb}(r_i, p)$ , where the  $X_i$ s are independent, the  $r_i$ s are positive real numbers, and  $0 < p < 1$ . If

$$X = X_1 + X_2 + \dots + X_n,$$

write down the probability generating function for  $X$ .

### 191. Random Sum

Let  $\{X_n\}$  be a sequence of independent and identically-distributed random variables whose values are nonnegative integers, and  $N$  a random variable whose values are also nonnegative integers which is independent of  $\{X_n\}$ . Let

$$S_N = \sum_{i=1}^N X_i.$$

Denote by  $P_N(z)$  and  $P_X(z)$  the probability generating functions of  $N$  and the  $X_i$ s, respectively. Show that the probability generating function of  $S_N$  is given by

$$P_{S_N}(z) = P_N(P_X(z)),$$

for  $z \in [0, 1]$ .

### 192. Binomial Sum

Let  $Y = X_1 + X_2 + \dots + X_N$  where  $N \stackrel{d}{=} \text{Bi}(n, p)$ ,  $X_i \stackrel{d}{=} \text{Bi}(m, q)$ , and  $N, X_1, X_2, \dots$  are independent.

- (a) Find  $P_{Y|N}(z)$ , the conditional probability generating function of  $Y$  given  $N$ , and state the values of  $z$  for which it is defined;
- (b) Find  $P_Y(z)$ , the probability generating function of  $Y$ , and state the values of  $z$  for which it is defined;
- (c) Using  $P_Y(z)$ , evaluate  $\mathbb{E}(Y)$ .

### 193. Continuous Density Function

Let  $X$  be a continuous random variable with probability density function  $f(x) = 2x$ , if  $0 \leq x \leq 1$ , zero elsewhere. Find the moment generating function of  $X$ .

### 194. Uniform Random Variable

Let  $X$  be a uniform random variable over the interval  $(a, b)$ . Find the moment generating function of  $X$ .

**195. Probability Mass Function**

Let

$$M_X(t) = \frac{1}{21} \sum_{n=1}^6 n e^{nt}.$$

Find the probability mass function of  $X$ .

**196. Linear Function**

Let  $M_X(t) = 1/(1-t)$ ,  $t < 1$  be the moment generating function of a random variable  $X$ . Find the moment generating function of the random variable  $Y = 2X + 1$ .

**197. Moments**

Suppose that the moment-generating function of  $X$  is given by

$$M_X(t) = \frac{e^t + e^{-t}}{6} + \frac{2}{3}, -\infty < t < \infty.$$

Find  $\mathbb{E}(X^r)$ ,  $r \geq 1$ .

**198. Constant Random Variable**

Suppose that for a random variable  $X$ ,  $\mathbb{E}(X^n) = 2^n$ ,  $n = 1, 2, 3, \dots$ . Calculate the moment generating function and the probability mass function of  $X$ .

**199. Cumulant generating function for Poisson**

Calculate the cumulant generating function of  $X \stackrel{d}{=} \text{Poisson}(\lambda)$  and then find its skewness coefficient and kurtosis coefficient.

**200. Cumulant generating function for exponential**

Derive the cumulant generating function of  $X \stackrel{d}{=} \exp(\lambda)$  and use the cgf to compute its skewness coefficient and kurtosis coefficient.

**201. Inversion formula for Laplace transform**

For random variable  $X$  having pdf of the form  $f_X(x) = \begin{cases} 0.5e^{-x} + e^{-2x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$  calculate the Laplace transform of  $X$  and use the inversion formula for Laplace transforms to derive the cdf of  $X$ .

**202. Recognising Distributions**

In each of the following cases  $M_X(t)$ , the moment generating function of  $X$ , is given. Determine the distribution of  $X$ .

(a)  $M_X(t) = \left( \frac{e^t + 3}{4} \right)^7;$

$$(b) \quad M_X(t) = \frac{1}{2 - e^t};$$

$$(c) \quad M_X(t) = \left( \frac{2}{2 - t} \right)^r;$$

$$(d) \quad M_X(t) = \exp(3e^t - 3).$$

### 203. Poisson Probability

Let  $X$ ,  $Y$ , and  $Z$  be three independent Poisson random variables with parameters  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ , respectively. For  $y = 0, 1, 2, \dots, t$ , calculate  $\mathbb{P}(Y = y | X + Y + Z = t)$ .

### 204. Linear Combination of Normals

For  $i = 1, 2, \dots, n$ , let  $X_1, X_2, \dots, X_n$  be independent random variables such that  $X_i \stackrel{d}{=} N(\mu_i, \sigma_i^2)$ . Then for constants  $\alpha_1, \alpha_2, \dots, \alpha_n$ , show that

$$\sum_{i=1}^n \alpha_i X_i \stackrel{d}{=} N \left( \sum_{i=1}^n \alpha_i \mu_i, \sum_{i=1}^n \alpha_i^2 \sigma_i^2 \right)$$

### 205. Independent Normals

Let  $X \stackrel{d}{=} N(1, 2)$  and  $Y \stackrel{d}{=} N(4, 7)$  be independent random variables. Find the probability of the following events

$$(a) \quad X + Y > 0;$$

$$(b) \quad X - Y < 2;$$

$$(c) \quad 3X + 4Y > 20.$$

### 206. IQ

The distribution of the IQ of a randomly selected student from a certain college is  $N(110, 16)$ . What is the probability that the average of the IQs of 10 randomly selected students from this college is at least 112?

### 207. Achievement Test

For the scores on an achievement test given to a certain population of students, the expected value is 500 and the standard deviation is 100. Let  $\bar{X}$  be the mean of the scores of a random sample of 35 students from the population. Estimate  $\mathbb{P}(460 < \bar{X} < 540)$ .

### 208. Credit Card

Each time that Jim charges an item to his credit card, he rounds the amount to the nearest dollar in his records. If he has used his credit card 300 times in the last 12 months, what is the probability that his record differs from the total expenditure by, at most, 10 dollars?

*Hint:* For  $i = 1, 2, \dots, 300$ , let  $X_i$  be the amount of the  $i$ th expenditure minus Jim's  $i$ th record. Then  $X_i \stackrel{d}{\approx} R\left(-\frac{1}{2}, \frac{1}{2}\right)$ .



### 209. Party Guests

Suppose that, whenever invited to a party, the probability that a person attends with his or her guest is  $1/3$ , attends alone is  $1/3$ , and does not attend is  $1/3$ . A company has invited all 300 of its employees and their guests to a Christmas party. What is the probability that at least 320 will attend?

*Hint:* For  $i = 1, 2, \dots, 300$ , let  $X_i = 0$  if employee  $i$  does not attend;  $X_i = 1$  if employee  $i$  attends alone;  $X_i = 2$  if employee  $i$  attends with a guest.

### 210. Heads Before Tails

A fair coin is tossed successively. Using the central limit theorem, find an approximation for the probability of obtaining at least 25 heads before 50 tails.

*Hint:* For  $i = 1, 2, \dots, 50$ , let  $X_i$  be the number of heads between the  $(i - 1)$ th and the  $i$ th tails. Then  $X_i \stackrel{d}{=} G\left(\frac{1}{2}\right)$ .

### 211. Poisson Identity

Let  $\{X_1, X_2, \dots\}$  be a sequence of independent Poisson random variables, each with parameter 1. By applying the central limit theorem to this sequence, prove that

$$\lim_{n \rightarrow \infty} \frac{1}{e^n} \sum_{k=0}^n \frac{n^k}{k!} = \frac{1}{2}.$$

*Hint:* Let  $Y_n = \sum_{i=0}^{\infty} X_i$ . Write down the exact expression for  $\mathbb{P}(Y_n \leq n)$ , and the approximate expression using a normal approximation. They will be equal in the limit by the central limit theorem.

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 7***

### ***Stochastic Processes***

#### **212. Wire**

A wire manufacturing company has inspectors to examine the wire for fractures as it comes out of a machine. The number of fractures is distributed in accordance with a Poisson process, having one fracture on the average for every 60 metres of wire. One day an inspector has to take an emergency phone call and is missing from his post for ten minutes. If the machine turns out 7 metres of wire per minute, what is the probability that the inspector will miss more than one fracture?

#### **213. Even and Odd Numbers of Events**

Let  $\{N(t), t \geq 0\}$  be a Poisson process with rate  $\lambda$ . What is the probability of an

- (a) an even number of events in  $(t, t + \alpha)$ ;
- (b) odd number of events in  $(t, t + \alpha)$ ?

#### **214. Trees**

In a forest, the number of trees that grow in a region of area  $R$  has a Poisson distribution with mean  $\lambda R$ , where  $\lambda$  is a given positive number.

- (a) Find the probability that the distance from a certain tree to the nearest tree is more than  $d$ .
- (b) Find the probability that the distance from a certain tree to the  $n$ th nearest tree is more than  $d$ .

#### **215. Poisson Process**

For a Poisson process with parameter  $\lambda$ , show that, for all  $\epsilon > 0$ ,

$$\mathbb{P} \left( \left| \frac{N(t)}{t} - \lambda \right| \geq \epsilon \right) \rightarrow 0,$$

as  $t \rightarrow \infty$ . This shows that, for a large  $t$ ,  $N(t)/t$  is a good estimate for  $\lambda$ .

**216. Bank**

Customers arrive at a bank at a Poisson rate of  $\lambda$ . Let  $M(t)$  be the number of customers who enter the bank by time  $t$  only to make deposits to their accounts. Suppose that, independent of other customers, the probability is  $p$  that a customer enters the bank only to make a deposit. Show that  $\{M(t) : t \geq 0\}$  is a Poisson process with parameter  $\lambda p$ .

**217. Absorbing Markov Chain**

Let  $\{X_n : n = 0, 1, \dots\}$  be a Markov chain with state space  $\{1, 2, 3\}$  and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Starting from state 1, what is the probability that the process never enters state 2?

**218. Die**

A fair die is tossed repeatedly. The maximum of the first  $n$  outcomes is denoted by  $X_n$ . Is  $\{X_n : n = 1, 2, \dots\}$  a Markov chain? Why or why not? If it is a Markov chain, calculate its transition probability matrix.

**219. Trout**

An observer at a lake notices that when fish are caught, only 1 out of 9 trout is caught after another trout, with no other fish between, whereas 10 out of 11 nontrout are caught following nontrout, with no trout between. Assuming that all fish are equally likely to be caught, what fraction of fish in the lake is trout?

**220. Emmett**

On a given day, Emmett drives to work (state 1), takes the train (state 2), or hails a taxi (state 3). Let  $X_n = 1$  if he drives to work on day  $n$ ,  $X_n = 2$  if he takes the train on day  $n$ , and  $X_n = 3$  if he hails a taxi on that day. Suppose that  $\{X_n : n = 1, 2, \dots\}$  is a Markov chain, and depending on how Emmett went to work the previous day, the probability of choosing any one of the means of transportation is given by the transition probability matrix

$$P = \begin{pmatrix} 1/6 & 2/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 2/5 & 1/2 & 1/10 \end{pmatrix}.$$

- (a) Given that Emmett took the train today and every day in the last five days, what is the probability that he will not take the train to work tomorrow?
- (b) If Emmett took the train to work today, what is the probability that he will not take the train to work tomorrow and the day after tomorrow?

### 221. Product of Transition Matrices

Show that if  $\mathbf{P}$  and  $\mathbf{Q}$  are two transition probability matrices with the same number of rows, and hence columns, then  $\mathbf{PQ}$  is also a transition probability matrix. Note that this implies that if  $\mathbf{P}$  is a transition probability matrix, then so is  $\mathbf{P}^n$  for any positive integer  $n$ .

### 222. Vacation

On a given vacation day, a sportsman either goes horseback riding (activity 1), or sailing (activity 2), or scuba diving (activity 3). For  $1 \leq i \leq 3$ , let  $X_n = i$ , if the sportsman devotes vacation day  $n$  to activity  $i$ . Suppose that  $\{X_n : n = 1, 2, \dots\}$  is a Markov chain, and depending on which of these activities the sportsman chooses on a certain vacation day, the probability of engagement in any one of the activities on the next vacation day is given by the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.20 & 0.30 & 0.50 \\ 0.32 & 0.15 & 0.53 \\ 0.60 & 0.13 & 0.27 \end{pmatrix}.$$

We know that the sportsman did not go scuba diving on the first day of his vacation, and it was equally likely that he went either riding or sailing on the first day. What is the probability that he did not go scuba diving on the second and third vacation days either?

### 223. Another Vacation

On a given vacation day, Francesco either plays golf (activity 1) or tennis (activity 2). For  $i = 1, 2$ , let  $X_n = i$ , if Francesco devotes vacation day  $n$  to activity  $i$ . Suppose that  $\{X_n : n = 1, 2, \dots\}$  is a Markov chain, and depending on which of the two activities he chooses on a certain vacation day, the probability of engagement in any one of the activities on the next vacation day is given by the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.30 & 0.70 \\ 0.58 & 0.42 \end{pmatrix}.$$

Find the long-run probability that, on a randomly selected vacation day, Francesco plays tennis.

***MAST20004 Probability***  
***Semester 2, 2020***  
***Problem Sheet 1 Answers***  
***Axioms of Probability***

**1. Sample Space**

Denote the dictionaries by  $d_1, d_2$ ; the third book by  $a$ .

The answers are  $\{d_1d_2a, d_1ad_2, d_2d_1a, d_2ad_1, ad_1d_2, ad_2d_1\}$  and  $\{d_1d_2a, ad_1d_2\}$ .

**2. Events**

If  $E$  or  $F$  occurs, then  $G$  occurs; If  $G$  occurs, then  $E$  and  $F$  occur.

**3. More Events**

If  $B = \emptyset$ , the relation is obvious; To show the reverse implication, let  $A = \Omega$  to show that  $B = \emptyset$ .

**4. Cards**

(a)  $\bigcup_{n=1}^{\infty} A_n$ ;

(b)  $\bigcup_{n=1}^{37} A_n$ .

**5. Event Identities**

(a) Let  $x \in (A \setminus (A \cap B)) \cup B$  and show that  $x \in A \cup B$ , which shows  $(A \setminus (A \cap B)) \cup B \subseteq A \cup B$ .  
Similarly show that  $(A \setminus (A \cap B)) \cup B \supseteq A \cup B$

(b) Use the method explained for Part (a).

**6. Infinite Sequence of Sets**

Let  $B_1 = A_1$ ,  $B_2 = A_2 \setminus A_1$ ,  $B_3 = A_3 \setminus (A_1 \cup A_2)$ ,  $\dots$ ,  $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ ,  $\dots$

**7. Hiring**

7/15 or 3/8, depending on your interpretation of the question.

## 8. Probability Statements

- (a) False. Consider rolling a die to give a counterexample;
- (b) False. Consider rolling a die to give a counterexample.

## 9. Probability Identity

Apply property (9) on Slide 37 twice.

## 10. Random Numbers

- (a) 267/1000;
- (b) 533/1000.

## 11. Voting

A Venn diagram shows that the answer is 0.36.

## 12. Boole's Inequality

Define the sequence of mutually exclusive events  $B_1, B_2, B_3, \dots$  as in the solution to Question 6, and use the fact, for  $n \geq 1$ ,  $B_n \subseteq A_n$ .

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 2 Answers***

### ***Conditional Probability and Independence***

#### **13. Dice**

1/7

#### **14. Movies**

$$\binom{7}{2} / \binom{13}{2} = 0.269$$

#### **15. Conditional Probability Properties**

- (a) Use the definition of conditional probability;
- (b) Use the definition of conditional probability;
- (c) Use the definition of conditional probability and Axiom 3.

#### **16. Defective and Nondefective Items**

- (a) 0.0144;
- (b) 0.344.

#### **17. Credit Rating**

92.8%

#### **18. Cards**

$$\frac{\binom{13}{3} \binom{39}{6}}{\binom{52}{9}} \times \frac{10}{43} = 0.059$$

#### **19. Proof**

- (a) Use Property (9) on Slide 37;
- (b) For the case when  $\mathbb{P}(A) = 0$  use the definition of independence; For the case when  $\mathbb{P}(A) = 1$  use Part (a) and the definition of independence.

## 20. More Proofs

- (a) Use the definition of independence;
- (b) Recognise that if  $A \subseteq B$ , then  $\mathbb{P}(A \cap B) = \mathbb{P}(A)$ . Then use the definition of independence.

## 21. Matching

$$1 - (5/6)^6 = 0.6651$$

## 22. Families

$A$  and  $B$  are independent when there are three children, but not independent when there are two or four children.

## 23. Independent Events

- (a)  $1 - (1 - p_1)(1 - p_2) \dots (1 - p_n)$ ;
- (b)  $(1 - p_1)(1 - p_2) \dots (1 - p_n)$ .

## 24. Urns and Balls

$$\binom{5}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = 0.329$$

## 25. $A$ Before $B$

Let  $\mathbb{P}(A) = p$  and  $\mathbb{P}(B) = q$ . Define  $A_n$  to be the event that neither  $A$  nor  $B$  occurs in the first  $n$  trials, but  $A$  occurs on the  $n$ th trial. Calculate  $\mathbb{P}(\cup_{n=1}^{\infty} A_n)$ .

## 26. Guns

$$0.7$$

## 27. Strokes

$$29\%$$

## 28. Lost Spades

- (a) Condition on the two events: a spade is missing, a spade is not missing. The answer is  $1/4$ ;
- (b) Condition on the three events: two spades are missing, one spade is missing, no spade is missing. The answer is  $1/4$ ;

## 29. Proof

Use the definition of conditional probability.



**30. Another Proof**

Use the definition of conditional probability and the law of total probability.

**31. Married on Campus**

Use the result from Question 31. The answer is  $1/3$ .

**32. Batteries**

$7/10$

**33. Multiple Choice**

$8/9$

**34. Cards**

$4/37$

**35. More Cards**

$1/3$

**36. Horses**

$205/297 = 0.69$

**37. Chips**

$4/7 = 0.571$

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 3 Answers***

### ***Random Variables and Distribution Functions***

**38. More Chips**

$x$	$-6$	$-2$	$-1$	$2$	$3$	$4$
$p_X(x)$	$0.095$	$0.238$	$0.238$	$0.095$	$0.238$	$0.095$

**39. Plastic Die**

$1/2$

**40. Families**

$$F_X(t) = \begin{cases} 0, & t < 0 \\ 1/8, & 0 \leq t < 1 \\ 1/2, & 1 \leq t < 2 \\ 7/8, & 2 \leq t < 3 \\ 1, & t \geq 3. \end{cases}$$

**41. Distribution Functions**

- (a)  $F$  is a distribution function.  $F(-\infty) = 0$ ;  $F(\infty) = 1$ ;  $F$  is right continuous; and since, for  $t > 0$ ,  $F'(t) > 0$ , it is nondecreasing.
- (b)  $F$  is not a distribution function.  $\lim_{t \rightarrow 0^-} F(t) = 1/2 > \lim_{t \rightarrow 0^+} F(t) = 1/4$ , and so  $F$  is not nondecreasing.

**42. Random Points**

$2/3$

**43. New Car**

$$F_Y(t) = \begin{cases} F_X(t), & t < 5 \\ 1, & t \geq 5. \end{cases}$$

#### 44. Minimum Face Value

$x$	1	2	3	4	5	6
$p_X(x)$	11/36	9/36	7/36	5/36	3/36	1/36

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 11/36, & 1 \leq x < 2 \\ 20/36, & 2 \leq x < 3 \\ 27/36, & 3 \leq x < 4 \\ 32/36, & 4 \leq x < 5 \\ 35/36, & 5 \leq x < 6 \\ 1, & x \geq 6. \end{cases}$$

#### 45. Probability Mass Function

$x$	-2	2	4	6
$p_X(x)$	1/2	1/10	13/45	1/9

#### 46. Jury

$$p_X(i) = \mathbb{P}(X = i) = \frac{\binom{18}{i} \binom{28}{12-i}}{\binom{46}{12}}, \quad i = 0, 1, 2, \dots, 12.$$

#### 47. First Six

For  $i = 1, 2, \dots$ ,

$$p_X(i) = \left(\frac{5}{6}\right)^{i-1} \frac{1}{6}.$$

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 1 - \left(\frac{5}{6}\right)^{\lfloor x \rfloor}, & x \geq 1. \end{cases}$$

#### 48. More Probability Mass Functions

(a)  $k = 1/15$ ;

(b)  $k = 1/15$ ;

(c)  $k = 8$ ;

(d)  $k = \frac{2}{n(n+1)}$ ;

(e)  $k = \frac{6}{n(n+1)(2n+1)}$ .

**49. Soap Opera**

(a)

$$f(x) = \begin{cases} \frac{32}{x^3}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

(b) Graphs required.

(c)  $9/25$ ;  $4/9$ ;  $0.313$ ;  $0$

**50. Tyres**

(a)  $66.67\%$

(b)  $60\%$

**51. Another Density Function**

Show, for  $x \in \mathbb{R}$ ,  $h(x) \geq 0$ , and  $\int_{-\infty}^{\infty} h(x)dx = 1$ .

**52. Investment**

(a)  $k = -1/2$ .

(b)  $3/16$

**53. Convex Combination**

Show, for  $x \in \mathbb{R}$ ,  $\alpha f(x) + \beta g(x) \geq 0$ , and  $\int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x)) dx = 1$ .

**54. Fuses**

$0.75$

**55. Nonexistent Expected Value**

(a) Show that sum of the probabilities equals 1.

(b) Show that the sum diverges.

**56. Distribution Function**

$5/8$ ;  $31/8$ ;  $23/8$

**57. Children**

$3$

**58. Probability Mass Function**

- (a) Show that sum of the probabilities equals 1 by expressing the series as a telescoping series.
- (b)  $\mathbb{E}(X)$  does not exist as the sum diverges.

**59. Variance**

$$\mathbb{V}(X) = 12.234; \text{sd}(X) = 3.498$$

**60. Random Integer**

$$\mathbb{E}(X) = (N + 1)/2; \mathbb{V}(X) = (N^2 - 1)/12; \sigma_X = \sqrt{(N^2 - 1)/12}$$

**61. Another Variance**

$$36$$

**62. Soap Opera**

- (a) 8
- (b) Show that  $\mathbb{E}(X^2)$  does not exist.

**63. Aptitude Test**

$$\mathbb{E}(X) = 3/2; \text{sd}(X) = 1/\sqrt{20}$$

**64. Expected Value**

$$3/2$$

**65. Computer Network**

$$\alpha - A/k$$

**66. Logarithm of a Random Variable**

$$1 - \log 2$$

**67. Double Exponential Distribution**

$$2$$

**68. Absolute Moment**

Use  $|x|^s \leq \max(1, |x|^r) \leq 1 + |x|^r$ ,  $x \in \mathbb{R}$ , and show that  $\mathbb{E}(|x|^s) < \infty$ .

**69. Recursive Moments**

Use integration by parts twice.

## 70. Computing Moments using Tail Probabilities

See Slides 149–152.

## 71. Post Office

$$(a) \mathbb{E}[X] = \frac{\alpha}{\lambda} + \frac{\beta}{\mu};$$

$$(b) \mathbb{V}(X) = \frac{2\alpha - \alpha^2}{\lambda^2} + \frac{2\beta - \beta^2}{\mu^2} - \frac{2\alpha\beta}{\lambda\mu}.$$

## 72. Bounded Random Variable

Proof required.

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 4 Answers***

### ***Special Probability Distributions***

#### **73. Number Plates**

$$\binom{6}{2} \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^4 = 0.098$$

#### **74. Nails**

Let  $X$  be the number of defective nails in the sample of 24.  $\mathbb{P}(X \geq 2) = 0.16124$  which is relatively high, and so it is unfair to reject the company's claim. If  $\mathbb{P}(X \geq 2) < 0.05$  then it would be reasonable to question the manufacturer's claim.

#### **75. Five Points**

(a) 0.539

(b) 0.073

#### **76. Rare Blood Type**

0.442

#### **77. Children**

6

#### **78. Chuck-a-Luck**

$\mathbb{P}(X = -1) = 125/216$ ;  $\mathbb{P}(X = 1) = 75/216$ ;  $\mathbb{P}(X = 2) = 15/216$ ;  $\mathbb{P}(X = 3) = 1/216$ ;  
 $\mathbb{E}(X) = -0.08$

#### **79. Aircraft Engines**

No. A four-engine plane is preferable to a two-engine plane if and only if  $p > 2/3$ .

#### **80. Even Number of Successes**

To find  $r_n$ , use the recursive relation,  $r_0 = 1$ , and induction.

**81. Targets**

(a)  $(1 - pq)^{r-1}pq$

(b)  $(1 - pq)/pq$

**82. Basketball**

(a)  $(0.55)^5(0.45) = 0.023$

(b)  $(0.55)^6(0.45)^2 = 0.0056$

**83. Light Bulbs**

$$p^n$$

**84. Bridge**

The probability a bridge hand contains three aces is  $p = \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}}$ . Therefore, the average number of hands until one has three aces is  $1/p = 24.27$ .

**85. Senior Citizens**

$X \stackrel{d}{=} \text{Nb}(10, 0.15)$ , therefore, for  $i = 0, 1, 2, \dots$ ,  $p_X(i) = \binom{i+9}{9}(0.85)^i(0.15)^{10}$ .

**86. Geometric Distribution**

$$1/(2 - p)$$

**87. Negative Binomial**

Proof required.

**88. Negative Binomial Distribution**

See Slides 166–169.

**89. Professors' Cars**

$$0.772$$

**90. Bernoulli Trials**

Note that, if  $M \stackrel{d}{=} \text{Nb}(x, p)$ , then  $N = M + x$ .

**91. Charity**

$$0.987$$



**92. Defective Items**

No. If 5% of items are defective then the shipment should contain 25 defective items. The probability that there are at least 2 defective items is  $1 - \binom{25}{0}\binom{475}{30}/\binom{500}{30} - \binom{25}{1}\binom{475}{29}/\binom{500}{30} = 0.4512$ , which is quite high.

**93. Trout**

The probability of four tagged trout were among the second 50 caught is

$$p_n = \frac{\binom{50}{4}\binom{n-50}{46}}{\binom{n}{50}}.$$

Now,

$$\frac{p_n}{p_{n-1}} = \frac{(n-50)^2}{n(n-96)},$$

and  $p_n \geq p_{n-1}$  if and only if  $n \leq 625$ . Since the ratio is 1 at  $n = 625$ , the estimated trout population size is either 624 or 625.

**94. Annual Income**

0.8943

**95. Misprints**

0.0153

**96. No Crimes**

0.0028

**97. Bookstore**

0.013

**98. Lottery Tickets**

If  $X$  is the number of wins in a day then  $X \stackrel{d}{\approx} \text{Pn}(\alpha M/N)$ . The answer is the least integer greater than or equal to  $-N \log(1 - \alpha)/M$ .

**99. Poisson Distribution**

Proof required.

**100. Line Segment**

(a)  $2/3$

(b)  $1/3$

**101. Random Angle**

$$f_X(x) = 1/(\pi(x^2 + 1))$$

**102. Expectation of a Logarithm**

1

**103. Chicken Pen**

a/b

**104. Heart Attack**

1/2. Use the memoryless property.

**105. Exponential Random Variable**

$$e^{-3}$$

**106. Telephone Call**

1/2. Use the memoryless property.

**107. Double Exponential Random Variable**

$$(a) \ c = 1/2$$

(b) Note that  $x^{2n+1}e^{-|x|}$  is an odd function, and  $x^{2n}e^{-|x|}$  is an even function. Use the gamma function defined on Slide 239.

**108. Radio Tubes**

$$\lfloor X \rfloor \stackrel{d}{=} G(1 - e^{-\lambda})$$

**109. Gamma Distribution**

Proof required.

**110. Defective Light Bulbs**

Let  $X$  be the time until 25 defective light bulbs are produced. Then  $X \stackrel{d}{=} \gamma(25, 200 \times 0.015 = 3)$  and  $\mathbb{E}(X) = 25/3$ .

**111. Born on Christmas Day**

$X \stackrel{d}{=} \text{Bi}(1095, 1/365)$  and  $X \approx \text{Pn}(3)$ . The answer is 0.08392.

**112. Beta Mean and Variance**

$$\mathbb{E}[X] = \frac{\alpha}{\alpha + \beta}; \quad \mathbb{V}(X) = \frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$$

**113. Beta Random Variable**

$X \stackrel{d}{=} \text{Beta}(5,6)$ ;  $c = 1,260$ ;  $\mathbb{E}[X] = 5/11$ ;  $\mathbb{V}(X) = 5/242$

**114. Blood Pressure Medicines**

0.538

**115. Resistors**

$\alpha = 1$ ;  $\beta = 2$ ;  $\mathbb{P}(X \geq 7/12) = 0.174$

**116. Pareto Random Variable**

$F_X(x) = \begin{cases} 1 - \left(\frac{\alpha}{x}\right)^\gamma, & x \geq \alpha \\ 0, & \text{otherwise} \end{cases}$ ;  $\mathbb{E}[X] = \frac{\gamma\alpha}{\gamma-1}$ ,  $\gamma > 1$ ;  $\mathbb{V}(X) = \frac{\gamma\alpha^2}{(\gamma-1)^2(\gamma-2)}$ ,  $\gamma > 2$

**117. Pareto Random Variable**

$\mathbb{E}[X^n] = \frac{\gamma\alpha^n}{\gamma-n}$  which is defined for  $n < \gamma$ .

**118. Newspaper Subscribers**

(a) 0.5

(b) 0.7013

**119. IQ**

(96.6, 123.4)

**120. Manual Dexterity**

0.000016

**121. Normal Density Function**

$k = \pi$  and  $f$  is the probability function of  $X \stackrel{d}{=} N(-1/\pi, 1/(2\pi^2))$ .

**122. Density Function of  $Y$** 

$f(y) = \begin{cases} \frac{4y}{\sqrt{2\pi}} e^{-y^4/2}, & y \geq 0 \\ 0, & y < 0. \end{cases}$

**123. Another Normal Density Function**

$k = \sqrt{\log a/\pi}$  and  $f$  is the density function of  $X \stackrel{d}{=} N(0, 1/(2 \log a))$ .

**124. Skeletons**

Let  $X$  be the height of a randomly selected skeleton from the first group. Then  $\mathbb{P}(X > 185) = 0.0749$ . Let  $Y$  be the number in the second group that are taller than 185 centimetres. If the second group belongs to the same family as the first, then  $Y \stackrel{d}{=} \text{Bi}(5, 0.0749)$ , and  $\mathbb{P}(Y = 3) = 0.0036$ . Therefore, the chance of finding exactly three or more skeletons taller than 185 centimetres is very small. Thus, it is unlikely that the second group belonged to the same family as the first.

**125. Weibull Mean and Variance**

$$\mathbb{E}[X] = \beta \Gamma\left(\frac{\gamma+1}{\gamma}\right); \mathbb{V}(X) = \beta^2 \Gamma\left(\frac{\gamma+2}{\gamma}\right) - \left(\beta \Gamma\left(\frac{\gamma+1}{\gamma}\right)\right)^2$$

**126. Weibull and exponential**

$$\exp(1)$$

**127. Exponential of a Random Variable**

$$g(y) = \begin{cases} \frac{f(\log y)}{y}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

**128. Logarithm of a Random Variable**

$$f(y) = (3 \log 2) 2^y e^{-3(2^y)}, y \in \mathbb{R}.$$

**129. Log of a Pareto Random Variable**

$$F_Y(y) = \begin{cases} 1 - \left(\frac{\alpha}{e^y}\right)^\gamma, & y \geq \log \alpha \\ 0, & \text{otherwise} \end{cases}; f_Y(y) = \begin{cases} \gamma \alpha^\gamma e^{-\gamma y}, & y \geq \log \alpha \\ 0, & \text{otherwise} \end{cases}$$

**130. Square of a Random Variable**

$$g(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f(\sqrt{y}) + f(-\sqrt{y})), & y > 0 \\ 0, & y < 0. \end{cases}$$

**131. Reciprocal of a Random Variable**

Let  $F_Y$  and  $f_Y$  be distribution and density functions of  $Y$ , respectively. Then use the law of total probability to get  $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y \leq y, X \leq 1) + \mathbb{P}(Y \leq y, X > 1)$ .

$$\text{The density function of } Y \text{ is } f_Y(y) = \begin{cases} e^{-y} + \frac{e^{-1/y}}{y^2}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

**132. Transformation by  $\psi(x) = \min(x, M)$**

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y}, & 0 \leq y < 5, \\ 1, & y \geq 5; \end{cases} \quad \text{using the formula of moments via tail probabilities, } \mathbb{E}(Y) = 1 - e^{-5}, \mathbb{E}(Y^2) = 2(1 - 6e^{-5}), \mathbb{V}(Y) = 1 - 10e^{-5} - e^{-10}.$$

### 133. Lognormal Distribution

(a)  $\mathbb{E}(X^r) = e^{\mu r + \frac{1}{2}\sigma^2 r^2}$

(b)  $\mathbb{E}(X) = e^{\mu + \sigma^2/2}$  and  $\mathbb{V}(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$ .

### 134. Large Fires

0.3192

***MAST20004 Probability***  
***Semester 2, 2020***  
***Problem Sheet 5 Answers***  
***Bivariate Random Variables***

**135. Joint Distribution**

- (a)  $c = 1/21$
- (b)  $p_X(x) = (2x + 3)/21, x = 1, 2, 3; p_Y(y) = (3y + 6)/21, y = 1, 2$
- (c)  $7/9$
- (d)  $\mathbb{E}(X) = 46/21; \mathbb{E}(Y) = 11/7.$

**136. Another Joint Distribution**

- (a)  $2/5$
- (b)  $7/25$
- (c)  $6/25$

**137. Cards**

- (a) For  $0 \leq x \leq 7, 0 \leq y \leq 7, 0 \leq x + y \leq 7,$

$$p_{(X,Y)}(x, y) = \frac{\binom{13}{x} \binom{13}{y} \binom{26}{7-x-y}}{\binom{52}{7}}.$$

- (b)  $\mathbb{P}(X \geq Y) = \sum_{y=0}^3 \sum_{x=y}^{7-y} p_{(X,Y)}(x, y) = 0.61107$

**138. Marginal Densities**

- (a) For  $0 \leq x \leq 1, f_X(x) = 4x^3$ ; For  $0 \leq y \leq 1, f_Y(y) = 4y(1 - y^2)$
- (b)  $\mathbb{E}(X) = 4/5; \mathbb{E}(Y) = 8/15$

**139. Probability Calculations**

Note that if  $A \in [0, 1] \times [0, 1]$ , then  $\mathbb{P}((X, Y) \in A) = \text{Area}(A)$ .

- (a)  $1/8$
- (b)  $7/8$
- (c)  $0.597$
- (d)  $\pi/4$

**140. Line Segment**

$3/4$

**141. Expected Values**

Proof required. Note that, for  $x > y$ ,  $f_{(X,Y)}(x, y) = 0$ .

**142. Cards**

$X$  and  $Y$  are not independent.

**143. Maximum and Minimum of  $X$  and  $Y$** 

$$F_U(u) = F(u)G(u); F_V(v) = F(v) + G(v) - F(v)G(v)$$

**144. Independent or Not**

$$\mathbb{E}(XY) = 4/9 \neq \mathbb{E}(X)\mathbb{E}(Y) = 32/75.$$

**145. Probability Density of  $X/Y$** 

Proof required.

**146. Expected Value of Maximum and Minimum**

$$\mathbb{E}(\max(X, Y)) = 1/3; \mathbb{E}(\min(X, Y)) = -1/3$$

**147. Product of Two Functions**

Show that  $f_{(X,Y)}(x, y) = f_X(x)f_Y(y)$ .

**148. Conditional Density Function**

$$f_{X|Y}(x|y) = 1/y, 0 < x < y, 0 < y < 1$$

**149. Conditional Probability**

$$f_{X|Y}(x|3/4) = (48x^2 + 27) / 43 \text{ and } \mathbb{P}(1/4 < X < 1/2 | Y = 3/4) = 17/86.$$

**150. Another Conditional Density Function**

$$f_{X|Y}(x|y) = \begin{cases} \frac{x+y}{1/2+y}, & \text{if } 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**151. Random Points**

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{1-y}, & \text{if } 0 < y < 1, \quad y < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_X(x) = \begin{cases} -\log(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**152. Conditional Mean and Variance**

(a)  $c = 1/2$

(b)  $f_{X|Y}(x|y) = e^{-x+|y|}$ ,  $x > |y|$  and  $f_{Y|X}(y|x) = 1/(2x)$ ,  $|y| < x$

(c)  $\mathbb{E}(Y|X = x) = 0$  and  $\mathbb{V}(Y|X = x) = x^2/3$ .

**153. Yet Another Conditional Density Function**

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & \text{if } |x| + |y| \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = 1 - |y|, \quad -1 \leq y \leq 1$$

$$f_{X|Y}(x|y) = \frac{1}{2(1-|y|)}, \quad -1 + |y| \leq x \leq 1 - |y|, \quad -1 \leq y \leq 1$$

**154. Sum of Two Random Numbers**

Proof required. Use the convolution formula.

**155. Conditional Probability**

(a) Verify the convolution formula

(b) Show that  $p_{(X,Y)}(x,y) = p_X(x)p_Y(y)$  if and only if  $c = 0$ .

**156. Conditional Mean and Variance**

Note that  $X$  and  $Y$  are independent.

(a)  $\mathbb{E}(X) = 1$

(b)  $\mathbb{E}(Y) = 1/2$

(c)  $\mathbb{E}(X^2 + Y^2) = 5/2$



**157. Letters**

1

**158. Empty Boxes**

5.28

**159.  $X$  and  $Y$**

Proofs required.

**160. Discrete Random Variable**

Proof required.

**161. Covariance**

$-1/245$

**162. Covariance Proofs**

Proofs required.

**163. More Covariance Proofs**

Proofs required.

**164. Independent and/or Uncorrelated?**

$X$  and  $Y$  are not independent but are uncorrelated.

**165. Investment**

0.1407

**166. Covariance of Linear Combinations**

Use, if  $X_0 = X - \mu_X$  and  $Y_0 = Y - \mu_Y$ , then  $\text{Cov}(X, Y) = \mathbb{E}(X_0, Y_0)$ .

**167. Variance of a Linear Combination**

Use the result from Question 153

**168. Correlation**

112

**169. Broken Stick**

$-1/12$

**170. Uncorrelated Result**

Proof required.

**171. No Linear Relation**

$\rho(X, Y) = -0.245 \neq \pm 1$ , so there exists no linear relation between  $X$  and  $Y$ .

**172. Two Successive Tails**

6

**173. Mixture of Two Random Variables**

$\mathbb{E}(X) = p\mathbb{E}(Y) + (1 - p)\mathbb{E}(Z)$ .

**174. Verification**

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}\left(\frac{6Y^2 + 3}{12Y^2 + 4}\right) = \mathbb{E}(X) = 5/8.$$

**175. Eggs Hatching**

$\mathbb{E}[X] = \lambda p$ ;  $\mathbb{V}(X) = \lambda p$ ;  $X \stackrel{d}{=} \text{Pn}(\lambda p)$

**176. Electronic Components**

$\mathbb{E}[X] = mpq$ ;  $\mathbb{V}(Y) = mpq(1 - pq)$ ;  $X \stackrel{d}{=} \text{Bi}(m, pq)$

**177. Consecutive Zeros**

*Hint:* Let

$$M := \min\{i : i\text{th digit is nonzero}\},$$

find the distribution of  $M$  and then condition on  $M$ .

**178. Fishing**

The expected value is 11 and the variance is 19.33

**179. Wallets**

Suppose Player A carries  $\$x$  in his wallet. First show that  $\eta(x) = \mathbb{E}(W_A|X = x) = \frac{1}{2} - \frac{3}{2}x^2$ . Then use  $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(W_A|X))$ .

**180. Heights**

0.5753

**181. Bivariate Normal**

(a) 0.6799

(b) 0.1993

**182. Grade Point Average**

0.0708

**183. Approximating Means and Variances**

(a)  $\mathbb{E}[Y] \approx 0.4594$ ;  $\mathbb{V}(Y) \approx 0.0648$

(b)  $\mathbb{E}[Y] \approx -1.1931$ ;  $\mathbb{V}(Y) \approx 1$

(c)  $\mathbb{E}[Y] \approx 10$ ;  $\mathbb{V}(Y) \approx 96$

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 6 Answers***

### ***Sums of Independent Random Variables/Limit Theorems***

**184. Chebyshev's Inequality**

0.472

**185. Accidents**

0.222

**186. IQ Scores**

Less than 1.7% of students have IQ scores above 140.

**187. Chebyshev's Inequality Again**

$$\mathbb{P}(X > 2\mu) \leq 1/\mu$$

**188. Multiple Choice**

1/9

**189. Probability Generating Function**

(a)

$$p_X(x) = \begin{cases} 0.3, & x = 0 \\ 0.2, & x = 1 \\ 0.1, & x = 2 \\ 0.4, & x = 3. \end{cases}$$

(b)  $P'_X(z) = 0.2 + 0.2z + 1.2z^2; \mathbb{E}(X) = P'_X(1) = 1.6$

$$P''(z) = 0.2 + 2.4z; \mathbb{V}(X) = P''_X(1) + P'_X(1) - P'_X(1)^2 = 1.64$$

(c)  $P_Y(z) = 0.3 + 0.2z + 0.1z^8 + 0.4z^{27}$

**190. Sum of Negative Binomial Random Variables**

$$P_X(z) = p^{r_1+r_2+\dots+r_n} (1 - (1-p)z)^{-(r_1+r_2+\dots+r_n)}$$

**191. Random Sum**

Use the expression for conditional probability generating functions.

**192. Binomial Sum**

- (a) For  $z \in \mathbb{R}$ ,  $P_{Y|N}(z) = (1 - q + qz)^{mN}$ .
- (b) For  $z \in \mathbb{R}$ ,  $P_Y(z) (1 - p + p(1 - q + qz)^m)^n$ .
- (c)  $\mathbb{E}(Y) = mnpq$

**193. Continuous Density Function**

$$M_X(t) = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}$$

**194. Uniform Random Variable**

$$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{(b-a)t}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

**195. Probability Mass Function**

$$p_X(x) = x/21, x = 1, 2, 3, 4, 5, 6$$

**196. Linear Function**

$$M_Y(t) = e^t / (1 - 2t), t < 1/2$$

**197. Moments**

If  $r$  is odd,  $\mathbb{E}(X^r) = 0$ ; If  $r$  is even,  $\mathbb{E}(X^r) = 1/3$

**198. Constant Random Variable**

$$M_X(t) = e^{2t}; \mathbb{P}(X = 2) = 1$$

**199. Cumulant generating function for Poisson**

$$K_X(t) = \lambda(e^t - 1); \text{Skew}(X) = \lambda^{-1/2}; \text{Kurt}(X) = 1/\lambda$$

**200. Cumulant generating function for exponential**

$$K_X(t) = -\ln(1 - t/\lambda); \text{Skew}(X) = 2; \text{Kurt}(X) = 6$$

**201. Inversion formula for Laplace transform**

$$L_X(t) = \frac{1}{2(t+1)} + \frac{1}{t+2}; \text{ the cdf of } X \text{ is } F_X(x) = \frac{1}{2}(1 - e^{-x}) + \frac{1}{2}(1 - e^{-2x}), x \geq 0.$$

**202. Recognising Distributions**

(a)  $X \stackrel{d}{=} \text{Bi}(7, \frac{1}{4})$

(b)  $X \stackrel{d}{=} \text{G}(\frac{1}{2})$

(c)  $X \stackrel{d}{=} \gamma(r, 2)$

(d)  $X \stackrel{d}{=} \text{Pn}(3)$

**203. Poisson Probability**

$$\binom{t}{y} \left( \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \right)^y \left( \frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \right)^{t-y}$$

**204. Linear Combination of Normals**

Use the moment generating function for normal random variables.

**205. Independent Normals**

(a)  $X + Y \stackrel{d}{=} N(5, 9); 0.9525$

(b)  $X - Y \stackrel{d}{=} N(-3, 9); 0.9525$

(c)  $3X + 4Y \stackrel{d}{=} N(19, 130); 0.4641$

**206. IQ**

0.0571

**207. Achievement Test**

0.9822

**208. Credit Card**

0.9544

**209. Party Guests**

0.0793

**210. Heads Before Tails**

0.9938

**211. Poisson Identity**

Proof required.

# ***MAST20004 Probability***

***Semester 2, 2020***

## ***Problem Sheet 7 Answers***

### ***Stochastic Processes***

#### **212. Wire**

0.325

#### **213. Even and Odd Numbers of Events**

(a)  $(1 + e^{-2\lambda\alpha}) / 2$

(b)  $(1 - e^{-2\lambda\alpha}) / 2$

#### **214. Trees**

(a)  $e^{-\lambda\pi d^2}$

(b)  $\sum_{i=0}^{n-1} e^{-\lambda\pi d^2} (\lambda\pi d^2)^i / i!$

#### **215. Poisson Process**

Use  $\mathbb{E}(N(t)) = \mathbb{V}(N(t)) = \lambda t$  and apply Chebyshev's inequality.

#### **216. Bank**

Condition on the event  $N(t) = n$  and use the law of total probability.

#### **217. Absorbing Markov Chain**

1/2

#### **218. Die**

Let  $Z_n$  be the outcome of the  $n$ th toss. Then  $X_{n+1} = \max(X_n, Z_{n+1})$ .  $\{X_n : n = 1, 2, \dots\}$  is a Markov chain since  $X_{n+1}$  only depends on  $X_n$ . The state space is  $\{1, 2, 3, 4, 5, 6\}$  and the transition probability matrix is

$$P = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**219. Trout**

$P = \begin{pmatrix} 10/11 & 1/11 \\ 8/9 & 1/9 \end{pmatrix}$  and  $\pi = (88/97 \quad 9/97)$ . So the proportion of trout is approximately 9.3%

**220. Emmett**

(a)  $2/3$

(b)  $1/4$

**221. Product of Transition Matrices**

Consider the  $ij$  entry of  $PQ$  and then show that the sum of the entries in the  $i$ th row sum to one.

**222. Vacation**

0.2358

**223. Another Vacation**

0.5469