MAST20018

Discrete Maths and Operations Research: Linear Programming

Semester 2, 2018

Lecture Outline

Introduction to Linear Programming

The Geometry of Linear Programming

Geometry of LP in Higher Dimensions

Basic Feasible Solutions

Fundamental Theorem of Linear Programming

Preview of the Simplex Method

The Simplex Method

Solution Possibilities

Non-Standard Formulations

Duality Theory

Sensitivity Analysis

§1 – Introduction to Linear Programming

Synopsis

- What is mathematical optimisation?
- What is linear programming?
- What do we mean by linear constraints and a linear objective function?

§1.1 – Applications of Mathematical Optimisation

Examples of optimisation problems:

- Airline scheduling;
- Shipping;
- Public transportation;
- Supply chain and logistics;
- Defence;
- Mining;
- Electricity and water resources.
- Emergency and disaster management;

§1.2 – What is Mathematical Optimisation?

There are two aspects to optimisation: modelling and solving.

An optimisation model contains:

- **Decision variables**
- **Objective function**
- **Constraints**

§1.3 – Types of Optimisation Formulations

Optimisation models can be placed into distinct classes depending on the best way to express the objective function or the constraints.

For *deterministic* problems, we are most interested in determining whether the objective function or constraints can be expressed linearly or nonlinearly, and whether the variables themselves can be continuous or integral.

These factors determine the class of algorithms available to solve the problem.

Types of Optimisation Formulations

NLP: Nonlinear program	**LP: Linear Program**	**ILP: Integer LP**
max $f(x)$ or **min** $f(x)$	**min** $c^T x$	**min** $c^T x$
$g_i(x) = 0 \quad i = 1, \ldots, k$	$Ax = a$	$Ax = a$
$h_j(x) \leq 0 \quad j = 1, \ldots, m$	$Bx \leq b$	$Bx \leq b$
	$x \geq 0$	$x \geq 0$
$x \in \mathbb{R}^n$		$x \in \mathbb{Z}^n$

In LP and ILP, $c \in \mathbb{R}^n$, and A and B are real matrices with n columns.

§1.4 – Solving Linear Programs

There are many linear programming solvers available both commercially and free for academic use.

Examples of commercial solvers (also with free academic licences) are

- CPLEX,
- Gurobi, and
- FICO Xpress

Common mathematical software, such as MATLAB, have packages for solving linear programs.

Microsoft Excel Solver also solves linear (integer) and nonlinear optimization problems.

Computational speed-up

Bixby[1] tested the various versions of CPLEX on the same computer. One of the data sets was a linear program with 410,640 constraints and 1,584,000 variables.

1988	CPLEX 1.0	29.8 days
1997	CPLEX 5.0	1.5 hours
2002	CPLEX 8.0	86.7 seconds
2003	CPLEX 9.0	59.1 seconds

According to Bixby, the algorithmic improvement (machine independent) is 3,300 times and machine improvement is 1,600 times, giving a total improvement of 5.2 million times in a period of 16 years (1988 to 2004)!!

[1] Bob Bixby, Operations Research, Jan 2002, pp. 1-13, updated in 2004.

§1.5 – Linear Programming

These are optimisation problems whose objective functions and constraints are linear.

Definition 1.1

A real-valued function f, defined on $\Omega \subseteq R^n$ is a *linear function* if and only if there exists a $c \in R^n$ such that

$$f(x) = c_1x_1 + c_2x_2 + ... + c_nx_n.$$

Definition 1.2

A linear equation is an expression of the form

$$c_1x_1 + c_2x_2 + ... + c_nx_n = b$$

A linear inequality is an expression of the form

$$c_1x_1 + c_2x_2 + ... + c_nx_n \leq b$$

or

$$c_1x_1 + c_2x_2 + ... + c_nx_n \geq b.$$

An example of a linear optimisation problem with a linear objective function and linear constraints is:

Example 1.1

$$z^* := \max_x 4x_1 + 3x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 30 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

§1.6 – The Diet Problem

A student wants to plan a nutritious diet, but they are on a limited budget: they want to spend as little money as possible.

Their minimum nutritional daily requirements are as follows: 2000 calories, 55g protein, 800mg calcium. The student is considering the following foods[2]:

Food	Serving size	Energy (Cal)	Protein (g)	Calcium (mg)	Price per serving ($)
Oatmeal	28g	110	4	2	0.30
Chicken	100g	205	32	12	2.40
Eggs	2 large	160	13	54	1.30
Whole milk	250ml	160	8	285	0.90
Cherry pie	170g	420	4	22	0.20
Pork and beans	260g	260	14	80	1.90

[2] Taken from Bob Bixby's notes from the Combinatorial Optimisation at Work meeting
`http://co-at-work.zib.de/berlin2009/downloads/2009-09-24/`

The Diet Problem

We can represent the number of servings of each type of food in the diet by the variables:

x_1	the daily number of servings of Oatmeal
x_2	the daily number of servings of Chicken
x_3	the daily number of servings of Eggs
x_4	the daily number of servings of Milk
x_5	the daily number of servings of Cherry Pie
x_6	the daily number of servings of Pork and Beans

The Diet Problem

and can express the problem as a linear program as follows:

$$\text{Minimise } 0.3x_1 + 2.40x_2 + 1.3x_3 + 0.9x_4 + 2.0x_5 + 1.9x_6,$$

subject to

$$\begin{aligned} 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2000 \\ 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\geq 55 \\ 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\geq 800 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

Here the first three inequalities are functional constraints.

The constraints $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ are not functional constraints.

§1.7 – Formulation of a Linear Program

We are moving towards a convenient form for expressing linear programming problems. Let

- m denote the number of functional constraints
- n denote the number of decision variables
- b_i denote the available level of resource i, $i = 1, 2, 3, ..., m$
- x_j denote the level of activity j, $j = 1, 2, 3, ..., n$
- z denote the value of the objective function
- c_j denote the return/cost per unit of activity j
- a_{ij} denote the amount of resource i consumed /produced by each unit of activity j.

Assumptions of LP Problems

- Activity levesl are continuous (**divisibility**). For activity j, the activity level x_j is a continuous variable.
  (However, quite often a problem requires that the activity levels be integers. In this case, a continuous formulation of such a problem is only an approximation.)
- The contribution of the objective function (and the LHS of each constraint) from decision variable x_j is **proportional** to the level of activity j.
- Independence of the effects of activities (**additivity**):

$$\begin{aligned} f(x_1, ..., x_n) &= c_1 x_1 + ... + c_j x_j + ... + c_n x_n, \\ & \quad a_{i,1} x_1 + ... + a_{i,j} x_j + ... + a_{i,n} x_n \leq b_i, \\ & \text{or} \quad a_{i,1} x_1 + ... + a_{i,j} x_j + ... + a_{i,n} x_n \geq b_i. \end{aligned}$$

- The input parameters are known with **certainty**.

Steps to Take

The steps that need to be taken are:

1. **Choose decision variables**
   These are the items about which we need to make decisions. For linear programs they are continuous. For example, we may need to determine the *start times* for several activities. We could then let our decision variables be: $\mathbf{x}_i :=$ the start time for activity i. You should be remember to include units in your definition of the decision variables.
2. **Write down the objective function**
   This is a linear expression that we wish to either maximise or minimise. For example, we wish to minimise the makespan of a project, or we wish to maximise profit.
3. **Write down the constraints** (and do not forget to write that the variables are non-negative).

Steps to Take

Example 1.2

A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes slabs of steel and can produce either bands (at 200 tonnes/hour) or coils (at 140 tonnes/hour).

Bands and coils can be sold for \$25/tonne and \$30/tonne respectively.

Based on currently booked orders, the company must produce at least 6000 tonnes of bands and 4000 tonnes of coils.

Given that there are 40 hours of production time this week, how many tonnes of bands and coils should be produced to yield the greatest revenue?

Steps to Take

- **Choose the decision variables:**
  - Let x_1 be the amount of time (in hours) devoted to making bands
  - Let x_2 be the amount of time (in hours) devoted to making coils.
- **Write down the objective function:**
  - This is $\max(25 \times 200)x_1 + (30 \times 140)x_2$
- **Write down the constraints:**
  - We need at least 6000 tonnes of bands, so $200x_1 \geq 6000$ and 4000 tonnes of coils, so $140x_2 \geq 4000$. We are limited to 40 hours of production time, so $x_1 + x_2 \leq 40$. The decision variables must be nonnegative, i.e. $x_1, x_2 \geq 0$.

Question: Is this LP feasible?!

Example 1.3 (The (balanced) transportation problem)

This is a classical Operations Research problem with

- a supply of some commodity;
- destinations where the commodity is needed.

The total available supply equals total demand.

The formulation can be adapted for general allocation and scheduling problems.

(The facts in this example have been amended to simplify the problem.)

Adelaide has two water catchment storage facilities:

- Storage 1 can store up to 400 megalitres per day
- Storage 2 can store up to 500 megalitres per day

We assume that the storage facilities are topped up by a third dam with infinite capacity – the storage facilities can therefore supply any amount up to their maximum on demand.

Three secondary dams are supplied from these two facilities:

- Barossa (Dam 1) needs at least 300 megalitres per day
- Happy Valley (Dam 2) needs at least 200 megalitres per day
- Kangaroo Creek (Dam 3) needs at least 400 megalitres per day

The distances between storage facilities and the secondary dams are (in kilometres).

	Dam 1	Dam 2	Dam 3
Storage Facility 1	75	40	85
Storage Facility 2	20	55	75

Task 1

Formulate a linear program that minimises the total transportation distances to meet the demands of the secondary dams, such that capacities of the storage facilities are not violated.

Analysis

- Decision Variables:

  Let x_{ij} be the quantity of water delivered from catchment i to location j, $i = 1, 2$; $j = 1, 2, 3$.

- Objective function:

$$\begin{aligned} f(x) \quad := \quad & 75x_{11} + 40x_{12} + 85x_{13} \\ & +20x_{21} + 55x_{22} + 75x_{23}. \end{aligned}$$

- Constraints:
  Production capacities:

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 400 \text{ (storage 1)} \\ x_{21} + x_{22} + x_{23} &\leq 500 \text{ (storage 2)}. \end{aligned}$$

- Location requirements:

$$x_{11} + x_{21} \geq 300 \text{ (location 1)}$$
$$x_{12} + x_{22} \geq 200 \text{ (location 2)}$$
$$x_{13} + x_{23} \geq 400 \text{ (location 3)}.$$

- Non-negativity:

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0.$$

Complete Formulation

$$
\begin{aligned}
z^* &:= \min z \\
&= 75x_{11} + 40x_{12} + 85x_{13} + 20x_{21} + 55x_{22} + 75x_{23},
\end{aligned}
$$

such that

$$
\begin{array}{lcl}
x_{11} + x_{12} + x_{13} & \leq & 400 \\
x_{21} + x_{22} + x_{23} & \leq & 500 \\
x_{11} + x_{21} & \geq & 300 \\
x_{12} + x_{22} & \geq & 200 \\
x_{13} + x_{23} & \geq & 400 \\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} & \geq & 0.
\end{array}
$$

§2 – The Geometry of Linear Programming

Synopsis

- What is a feasible region?
- How do we obtain the optimal solution via the graphical method?
- Are there pathological cases?

§2.1 – The Feasible Region

There are strong relationships between the geometric and algebraic features of LP problems.

We shall first examine this aspect in two dimensions $(n = 2)$ and then consider higher dimensions. This latter step will involve an understanding of the relationship between geometric and algebraic aspects of linear programming problems.

Example 2.1

Consider the problem

$$z^* := \max z = 4x_1 + 3x_2$$

subject to

$$\begin{array}{rcll} 2x_1 + x_2 & \leq & 40 & \text{(production machine time)} \\ x_1 + x_2 & \leq & 30 & \text{(production packaging time)} \\ x_1 & \leq & 15 & \text{(market demand)} \\ x_1 & \geq & 0 & \\ x_2 & \geq & 0. & \end{array}$$

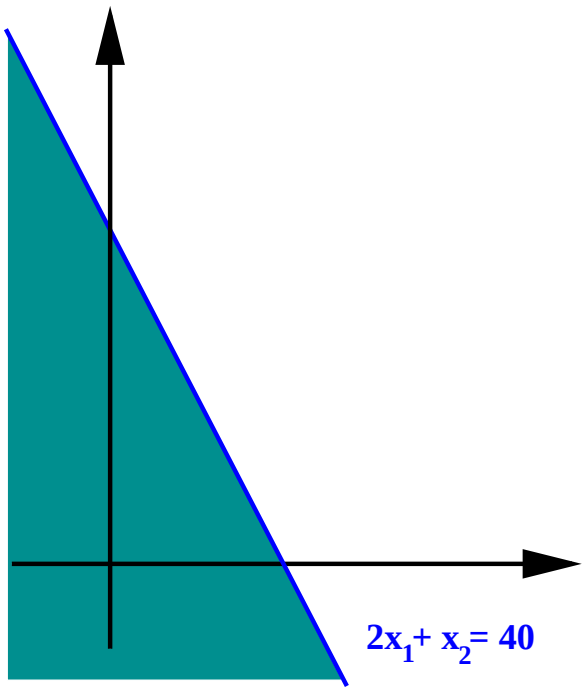
- First constraint:

$$2x_1 + x_2 \leq 40$$

- Corresponding hyperplane:

$$2x_1 + x_2 = 40$$

- Corresponding halfspace:



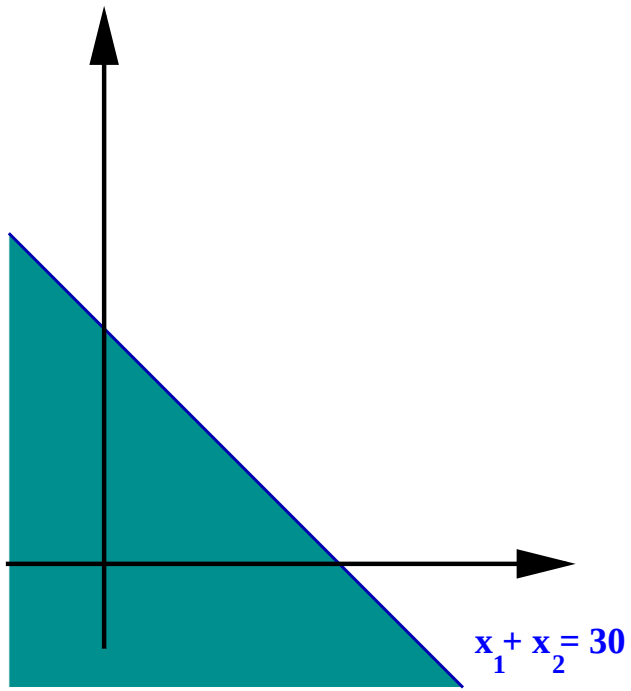
- Second constraint:

$$x_1 + x_2 \leq 30$$

- Corresponding hyperplane:

$$x_1 + x_2 = 30$$

- Corresponding halfspace:



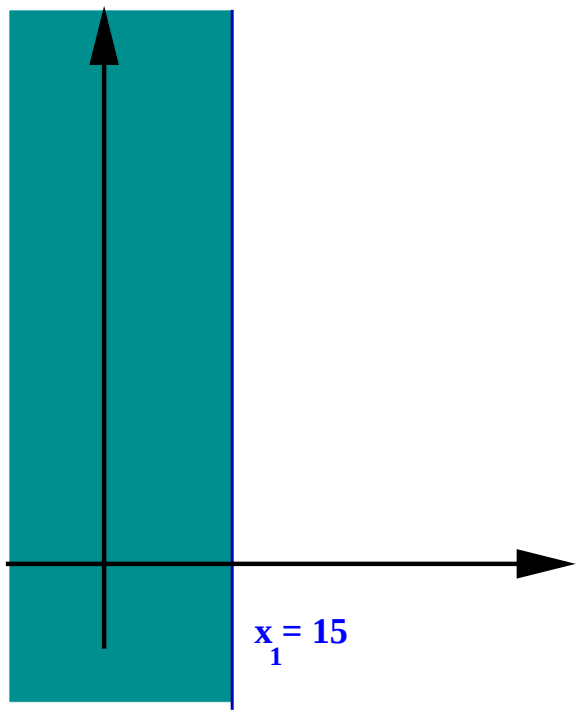
- Third constraint:

$$x_1 \leq 15$$

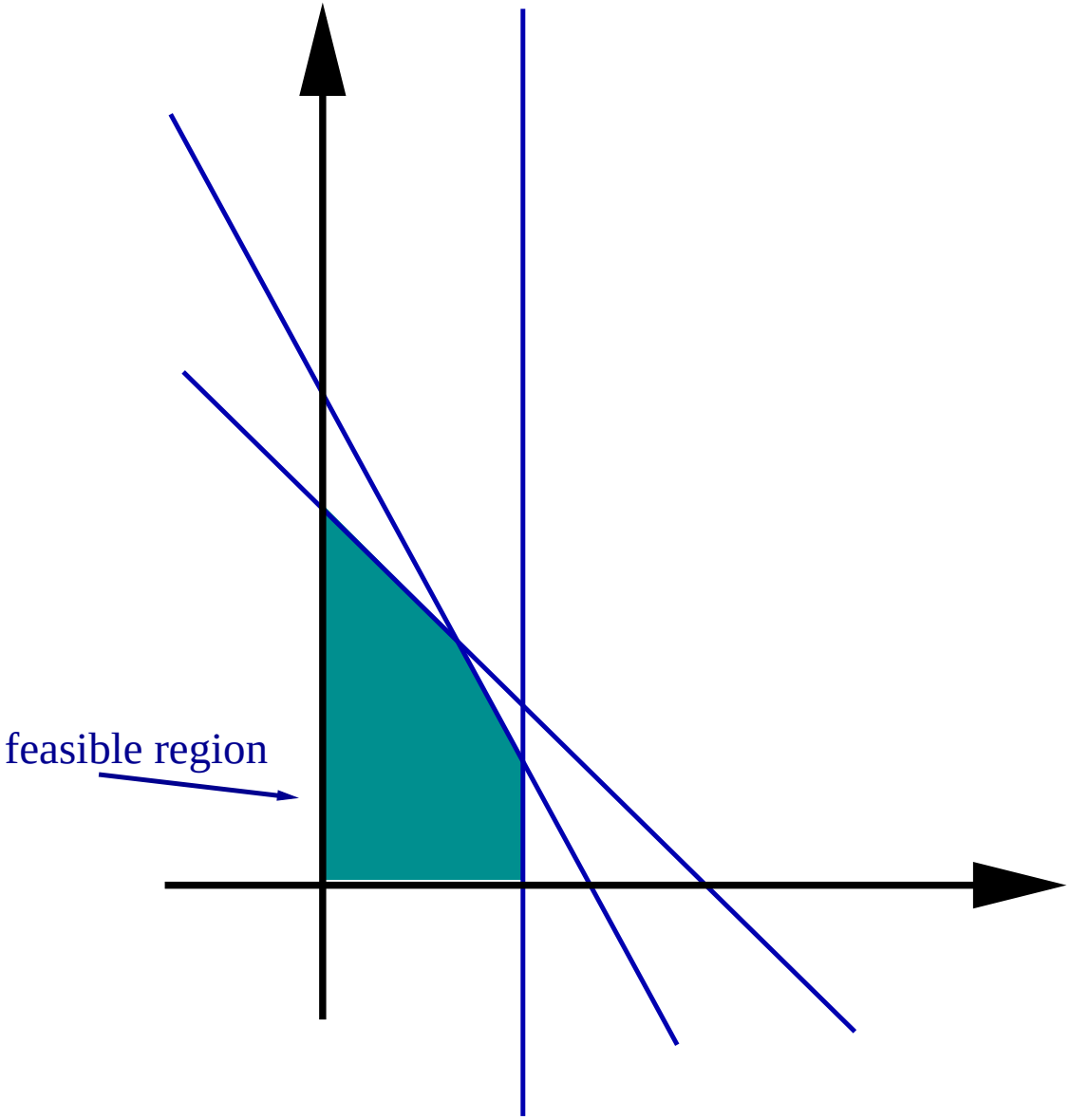
- Corresponding hyperplane:

$$x_1 = 15$$

- Corresponding halfspace:



Taking note of the nonnegativity constraints, the overall feasible region is therefore:



Definition 2.1

For an LP problem with n variables, a vector $x \in \mathbb{R}^n$ is called a feasible solution if it satisfies all constraints of the problem.

Definition 2.2

The set of all feasible solutions of an LP problem is called the feasible region.

§2.2 – The Graphical Method

We can use a graphical method to solve LP problems in two dimensions.

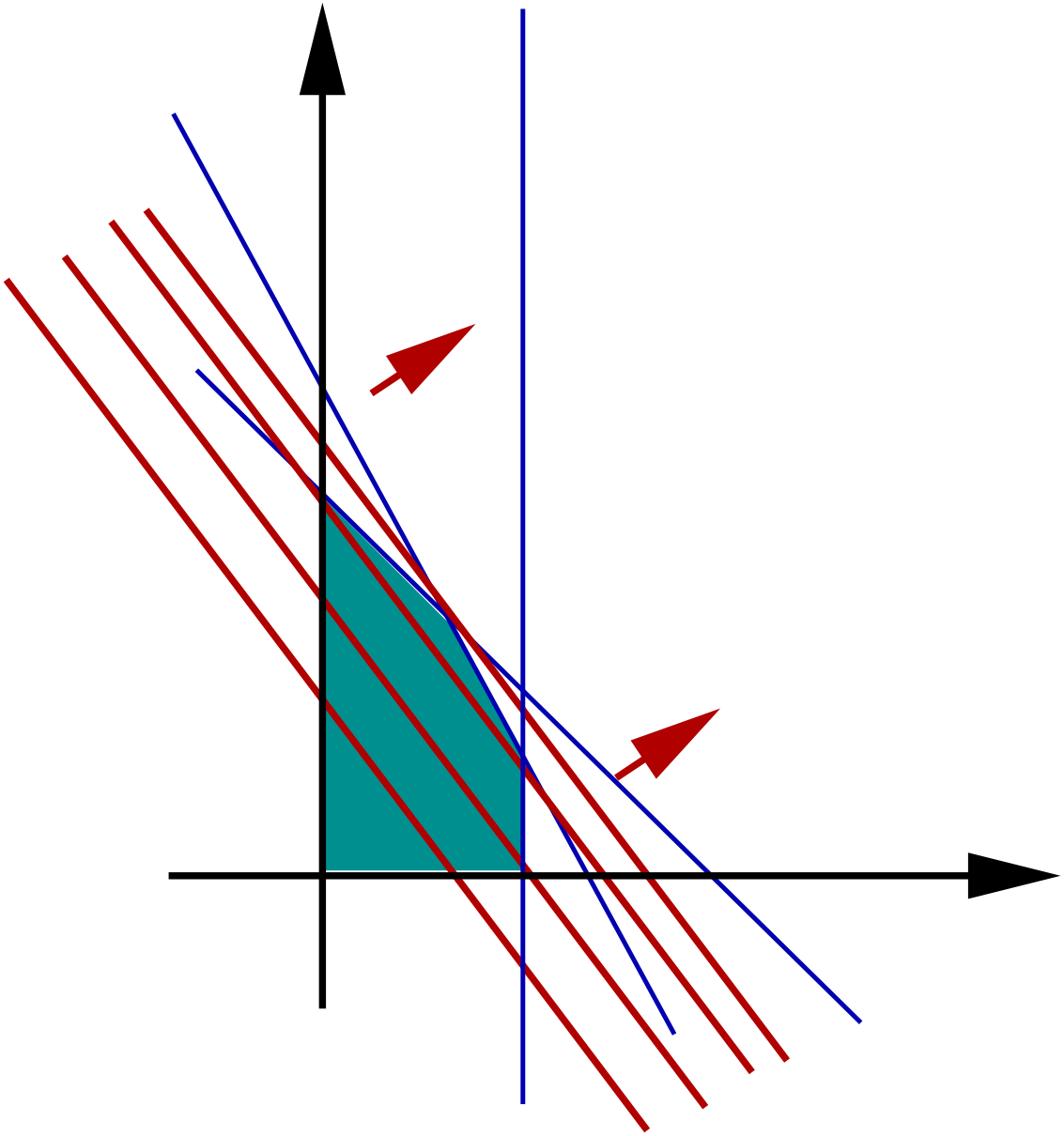
- The objective function is

$$f(x) = z = 4x_1 + 3x_2$$

  Hence

$$x_2 = (z - 4x_1)/3$$

  so that, for a given value of z, the level curve is a straight line with slope $-4/3$. We can plot it for various values of z. We can identify the (x_1, x_2) pair that yields the largest feasible value for z.



Important Observations

- The graphical method can be used to identify the hyperplanes specifying the optimal solution.
- The optimal solution itself is determined by solving the respective equations.
- Don't be tempted to "read" the optimal solution directly from the graph.
- The optimal solution in this example is an extreme point of the feasible region.

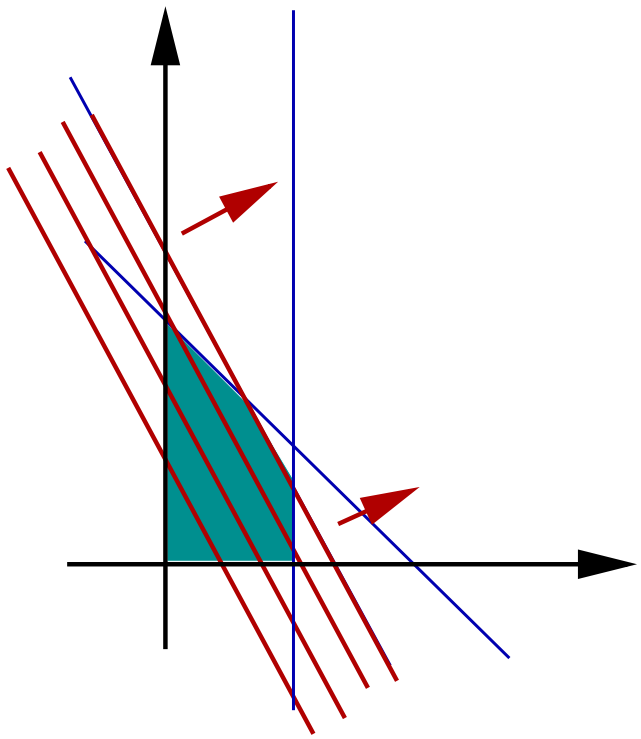
Questions

- What guarantee is there that an optimal solution exists?
- In fact, is there any a priori guarantee that a feasible solution exists?
- Could there be more than one optimal solution?

§2.3 – 'Pathological' Cases (infinitely many solutions)

For two-variable problems the above method works well. However 'pathological' cases still exist.

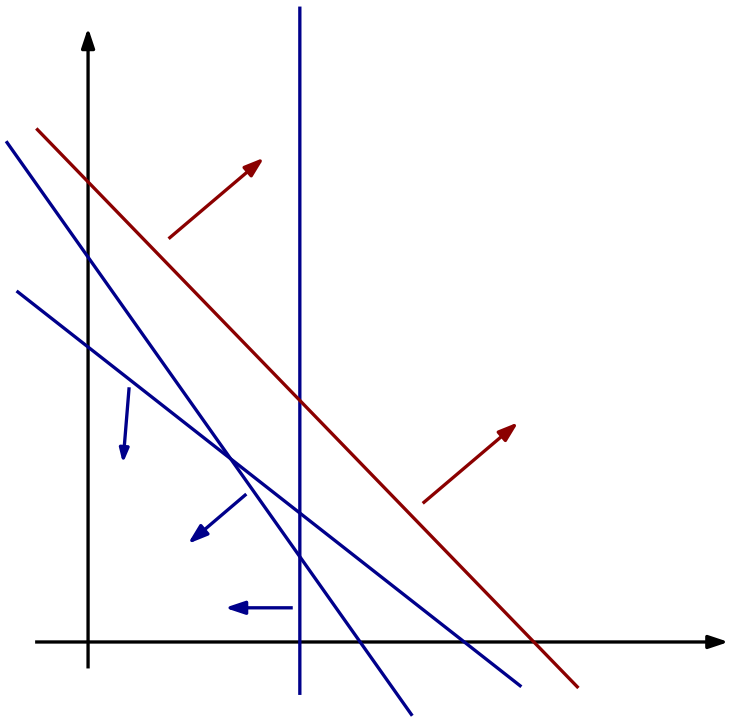
What if $z = 4x_1 + 2x_2$ in the previous example?



There are infinitely many points in the feasible region where $z = 80$.

§2.3 – "Pathological" Cases (infeasibility)

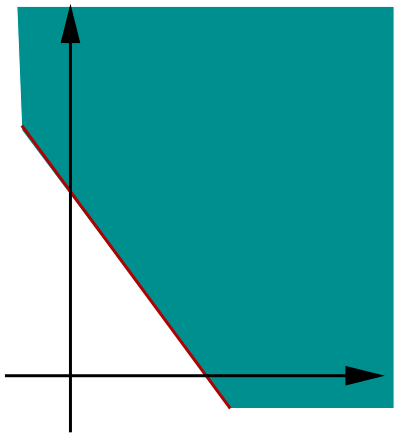
What if we had another constraint $4x_1 + 3x_2 \geq 150$?



In this case the feasible region is empty. Since there is no feasible solution, there is definitely no optimal solution.

§2.3 – "Pathological" Cases (LP is unbounded)

What if we only had the constraint $4x_1 + 3x_2 \geq 101$?

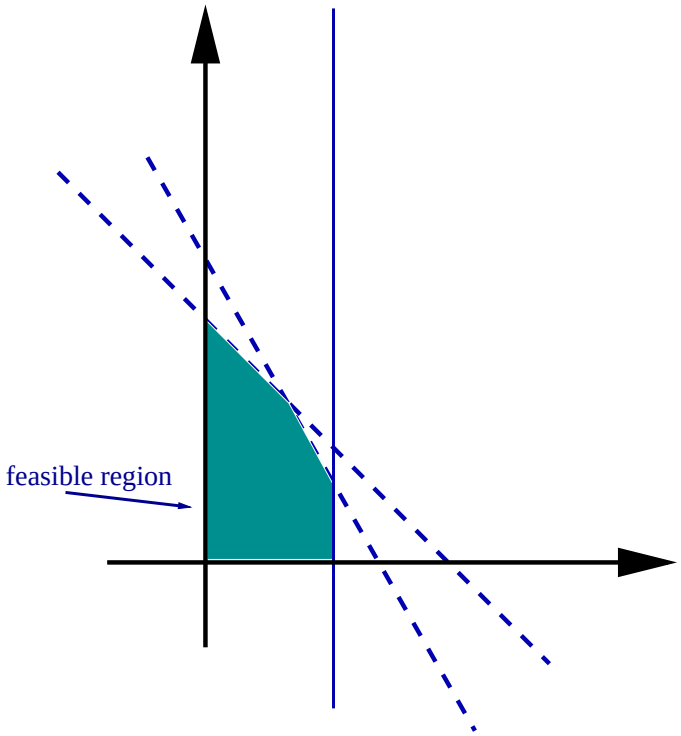


In this case the feasible region is unbounded. For certain optimization problems (for example, if we are minimizing $z = 4x_1 + 3x_2$) there still can be an optimal solution. However, if the objective function increases as we 'head further into the unbounded region', no optimal solution exists.

§2.3 – "Pathological" Cases (feasible region is not closed)

Sometimes the feasible region is not closed. For example,

$$\begin{aligned} 2x_1 + x_2 &< 40 \\ x_1 + x_2 &< 30 \\ x_1 &\leq 15 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



The 'corner point' at $(10, 20)$ is not feasible.
BUT, we will only ever consider closed regions in practice.

§3 – The Geometry of LP in Higher Dimensions

Synopsis

- What are convex sets?
- What is a hyperplane?
- What is a halfspace?
- What is linear independence?
- What is a polytope?
- How do we get from standard form to canonical form?
- What are slack variables?

§3.1 – The Geometry in Higher Dimensions

Let $x^{(1)}, ..., x^{(k)}$ be a collection of points in $\mathbb{R}^n$.

Definition 3.1

A point w such that

$$w = \alpha_1 x^{(1)} + \alpha_2 x^{(2)} + ... + \alpha_k x^{(k)},$$

is a *linear combination* of $x^{(1)}, ..., x^{(k)}$.

Definition 3.2

A point w such that

$$w = \alpha_1 x^{(1)} + \alpha_2 x^{(2)} + ... + \alpha_k x^{(k)},$$

where $0 \leq \alpha_i \leq 1$ and $\sum_i \alpha_i = 1$, is a *convex combination* of $x^{(1)}, ..., x^{(k)}$.

Example 3.1

The point $x = (3, 2, 1)$ is a linear combination of

$$\begin{aligned} x^{(1)} &= (1, 0, 0) \\ x^{(2)} &= (0, 1, 0) \\ x^{(3)} &= (0, 0, 1), \end{aligned}$$

using the coefficients $\alpha_1 = 3$, $\alpha_2 = 2$ and $\alpha_3 = 1$.
The point $y = (9, 4, 1)$ is not a linear combination of

$$\begin{aligned} x^{(1)} &= (3, 1, 0) \\ x^{(2)} &= (2, 4, 0) \\ x^{(3)} &= (4, 3, 0). \end{aligned}$$

Why?

Example 3.2

The point $x = (1/2, 1/3, 1/6)$ is a convex combination of

$$\begin{aligned} x^{(1)} &= (1,0,0) \\ x^{(2)} &= (0,1,0) \\ x^{(3)} &= (0,0,1), \end{aligned}$$

using the coefficients $\alpha_1 = 1/2$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.
The point $y = (3,3,2)$ is a convex combination of

$$\begin{aligned} x^{(1)} &= (3,1,0) \\ x^{(2)} &= (2,4,0) \\ x^{(3)} &= (4,4,6). \end{aligned}$$

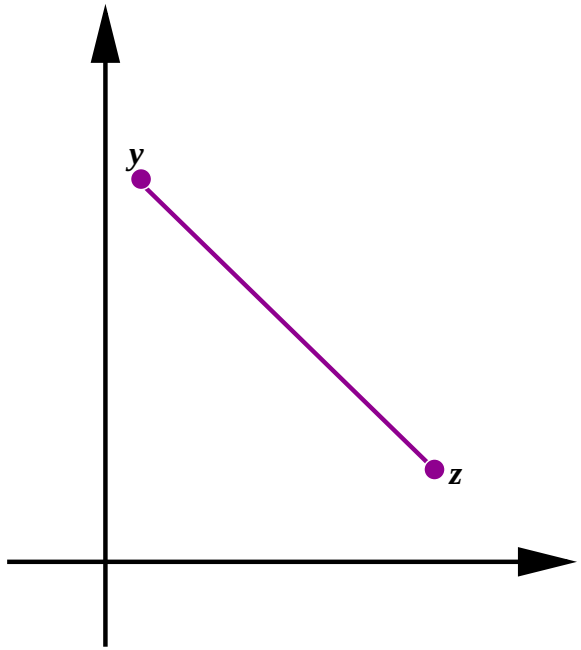
Why?

Definition 3.3

The *line segment* joining two points y, z in $\mathbb{R}^n$ is the collection of all points x such that

$$x = \lambda y + (1 - \lambda) z$$

for some $0 \leq \lambda \leq 1$.



Definition 3.4

A subset $\mathbb{C}$ of $\mathbb{R}^n$ is *convex* if for any two points $y, z \in \mathbb{C}$ and any $0 \leq \lambda \leq 1$, the point

$$\lambda y + (1 - \lambda) z$$

is also in $\mathbb{C}$.

Geometrically this means that, for any pair of points $y, z \in \mathbb{C}$, the line segment connecting these points is in $\mathbb{C}$.

Theorem 3.1

Let $\mathbb{C}_1, \ldots, \mathbb{C}_n$ *be a collection of convex sets. Then*

$$\bigcap_{i=1}^{n} \mathbb{C}_i$$

is a convex set.

That is, the intersection of any finite number of convex sets is a convex set.

Proof.

?? $\square$

Definition 3.5

A set of points $\mathbb{H} \subseteq \mathbb{R}^n$ satisfying a linear equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b,$$

for $(a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0)$, is a *hyperplane*.

Hyperplanes are of dimension $n - 1$. (Why?)

Definition 3.6

The two *closed half-spaces* defined by the hyperplane

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

are the set defined by

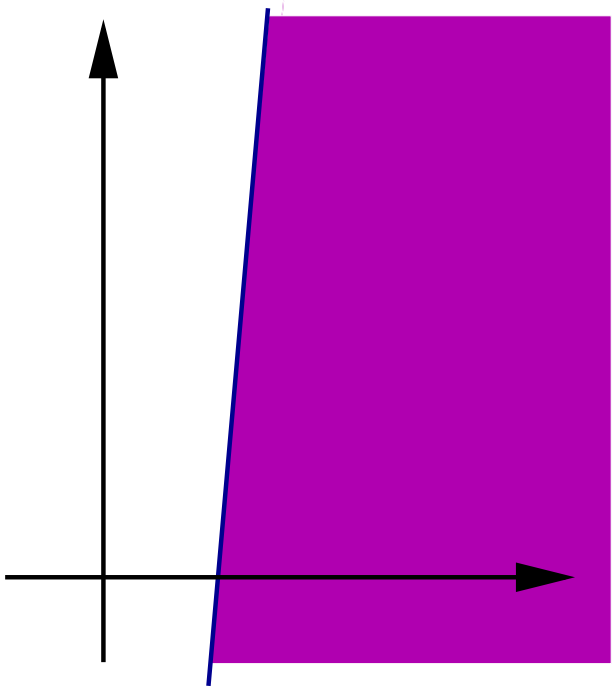
$$a_1x_1 + a_2x_2 + ... + a_nx_n \geq b \quad \text{(positive half space)}$$

and

$$a_1x_1 + a_2x_2 + ... + a_nx_n \leq b \quad \text{(negative half space).}$$

Theorem 3.2

Hyperplanes and their half-spaces are convex sets.



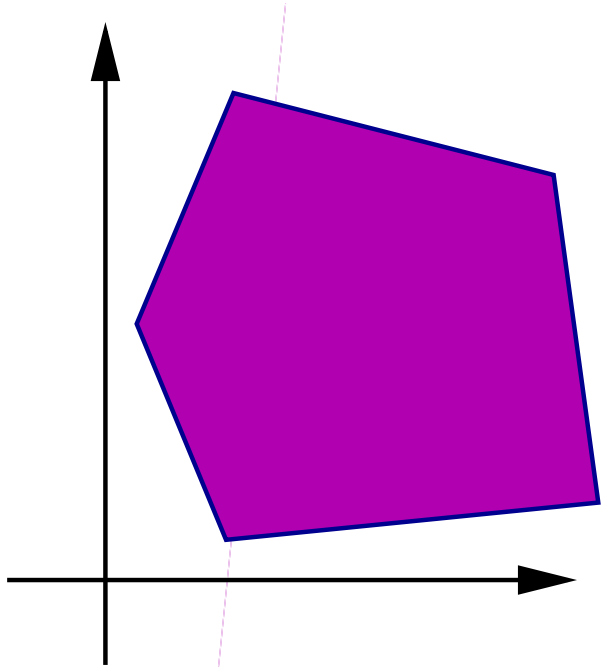
Proof.

??

□

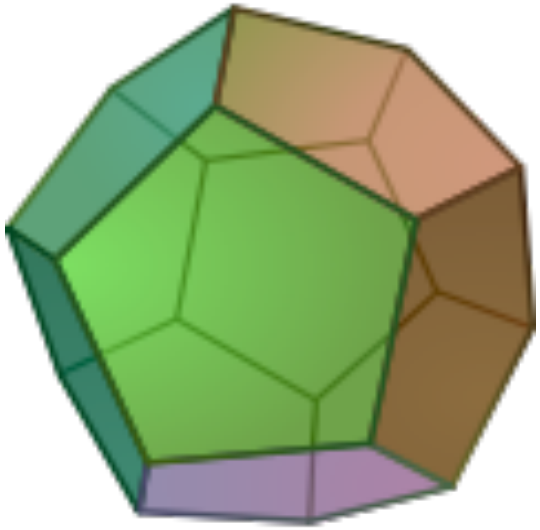
Definition 3.7

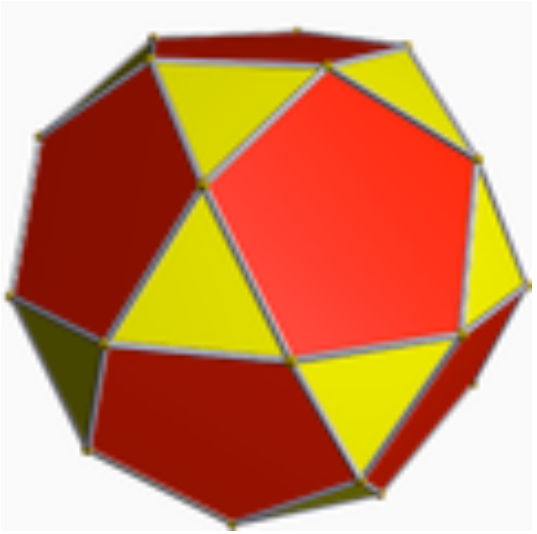
A *polytope* is a set that can be expressed as the intersection of a finite number of closed half-spaces.



Definition 3.8

A *polyhedron* is a non-empty bounded polytope.





Definition 3.9

A point x of a convex set $\mathbb{C}$ is said to be an *extreme point* or vertex of $\mathbb{C}$ if whenever we write

$$x = \lambda y + (1 - \lambda)z,$$

with $y, z \in \mathbb{C}$ and $0 < \lambda < 1$, then $y = z = x$.

Geometrically, this means that there are no points y and z in $\mathbb{C}$ (different from x) with x lying on the line segment connecting y and z.

§3.2 – Linear Independence

Definition 3.10

A collection of vectors $x^{(1)}, ..., x^{(s)}$ in $\mathbb{R}^n$ is said to be *linearly independent* if

$$\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + ... + \alpha_k x^{(k)} = 0,$$

implies that

$$\alpha_1 = \alpha_2 = ... = \alpha_k = 0.$$

Geometrically this means that no vector in the collection can be expressed as a linear combination of the other vectors in the collection.

Example 3.3

- The vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are linearly independent.
- The vectors $(2, 4, 3)$, $(1, 2, 3)$, $(1, 2, 0)$ are not linearly independent.

Theorem 3.3

The set of feasible solutions of the standard LP problem defined by

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & \leq & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & \leq & b_m \end{array}$$

$$x_j \geq 0, j = 1, ..., n$$

is a convex polytope.

Proof.

By definition, a polytope is the intersection of finitely many half-spaces. We have seen

- a half-space is a convex set,
- the intersection of any finite number of convex sets is a convex set.

The fact that the polytope is convex follows. □

Theorem 3.4

If a linear programming problem has exactly one optimal solution, then this solution must be an extreme point of the feasible region.

Proof.

We shall prove this theorem by contradiction. □

Assume, to the contrary, that the problem has exactly one optimal solution, call it x, that is not an extreme point of the feasible region.

Since x is not an extreme point, there are two distinct feasible solutions, say y and z, which are not equal to x and a scalar λ, $0 < \lambda < 1$, such that

$$x = \lambda y + (1 - \lambda)z.$$

If we rewrite the objective function in terms of y and z rather than x, we obtain

$$f(x) = f(\lambda y + (1-\lambda)z).$$

Hence

$$\begin{aligned} f(x) &= \sum_{j=1}^{n} c_j x_j \\ &= \sum_{j=1}^{n} c_j \left[\lambda y_j + (1-\lambda) z_j\right] \\ &= \lambda \sum_{j=1}^{n} c_j y_j + (1-\lambda) \sum_{j=1}^{n} c_j z_j \\ &= \lambda f(y) + (1-\lambda) f(z). \end{aligned}$$

Now, because $0 < \lambda < 1$, there are only three possibilities

$$
\begin{aligned}
&1. \quad f(y) < f(x) < f(z) \\
&2. \quad f(z) < f(x) < f(y) \\
&3. \quad f(y) = f(x) = f(z)
\end{aligned}
$$

Since x is an optimal solution, the first two possibilities cannot occur (why?). Thus, the third case must hold.

But this contradicts the assertion that x is the only optimal solution. Our initial assumption that x is not an extreme point must be wrong.

As an exercise, prove the following:

Theorem 3.5

If a linear programming problem has more than one optimal solution, it must have infinitely many optimal solutions. Furthermore, the set of optimal solutions is convex.

§3.3 – Standard Form

We would like to write our linear programs in a fixed format so that it is easier to apply algorithms, such as the Simplex and Interior Point methods. One such format is the standard form, which means that

- It must be a maximisation problem;
- the constraints must be $\leq$;
- $b_i \geq 0$, for all i, the RHS must be positive;
- $x_i \geq 0$ for all i, all variables are non-negative.

In some cases, we are not be able to achieve $b_i \geq 0$ and $\leq$ simultaneously, and therefore the LP cannot be written in standard form. Later we will introduce the so-called canonical form, and show that any LP can be written in this form.

For the standard form conversion you should familiarise yourself with converting from:

- a minimum objective function to a maximum;
- the case where some RHS constants are negative;
- some constraints are $\geq$;
- some variables are unrestricted.

Definition 3.11

An LP problem is in standard form if it is of the form:

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\leq b_2 \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\leq b_m \\ x_j \geq 0, j = 1, ..., n & \end{aligned}$$

where $b_i \geq 0$ for all i.

With

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

an LP in standard form can be written as:

$$\begin{aligned} \max_x z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

with $b \geq 0$.

The matrix A is called the coefficient matrix of the linear program. Often we write $A = (a_{ij})$.

§3.4 – Canonical Form

If we have an LP in standard form, we can turn it into an LP in canonical form by introducing new variables $x_j \geq 0$, $j = n+1, ..., n+m$ that turn the inequalities into equations:

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \qquad + x_{n+2} & = & b_2 \\
\vdots & \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \qquad\qquad + x_{n+m} & = & b_m
\end{array}$$

$$x_j \geq 0, \quad j = 1, ..., n+m$$

As in the standard form, $b_i \geq 0$ for all i.

The variables $x_j \geq 0$, $j = n+1, ..., n+m$ are called *slack variables*.

Physically, they denote the difference between the right hand side and the left hand side of the inequalities in the standard form.

When a slack variable takes the value zero, the corresponding inequality is *satisfied with equality.*

Lemma 3.1

When $b_i \geq 0$ for all i, the canonical form

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \qquad + x_{n+2} & = & b_2 \\
\vdots & \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \qquad + x_{n+m} & = & b_m \\
x_j \geq 0, \quad j = 1, ..., n+m & &
\end{array}$$

has at least one feasible solution, $x = (0, 0, 0, ..., 0, b_1, b_2, ..., b_m)$

This solution is obtained by:

- *Setting all the original variables to zero*
- *Setting the slack variables to the respective right-hand side values.*

Definition 3.12

In general, an LP problem

$$\max z = \sum_{j=1}^{n+m} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n + a_{1,n+1}x_{n+1} + ... + a_{1,n+m}x_{n+m} & = & b_1 \\ a_{21}x_1 + ... + a_{2n}x_n + a_{2,n+1}x_{n+1} + ... + a_{2,n+m}x_{n+m} & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + a_{m,n+1}x_{n+1} + ... + a_{m,n+m}x_{n+m} & = & b_m \\ x_j \geq 0, \quad j = 1, ..., n+m & & \end{array}$$

is in canonical form if

- there exist m columns of $A = (a_{ij})$ which are $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

  respectively.
- $b_i \geq 0$ for all i.

Example 3.4

The LP problem

$$\max z = 2x_1 - 3x_2 + 4x_3 + x_5$$

$$\begin{aligned} 2x_1 + 0x_2 + 1x_3 + 3x_4 + 0x_5 &= 6 \\ -x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 &= 2 \\ 2x_1 + 0x_2 + 0x_3 + 6x_4 + 1x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 & \end{aligned}$$

is in canonical form because the 3rd, 2nd and 5th columns of the coefficient matrix are, respectively,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that $(0, 2, 6, 0, 2)$ is a feasible solution to this LP problem.

Lemma 3.2

Suppose

$$\max z = \sum_{j=1}^{n+m} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n + a_{1,n+1}x_{n+1} + ... + a_{1,n+m}x_{n+m} & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + a_{m,n+1}x_{n+1} + ... + a_{m,n+m}x_{n+m} & = & b_m \\ x_j \geq 0, \quad j = 1, ..., n+m & & \end{array}$$

is in canonical form and columns $j_1, \ldots, j_m$ *are* $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. *Then the LP has at least one feasible solution, which is given by:*

- *Setting* $x_{j_1} = b_1, \ldots, x_{j_m} = b_m$
- *Setting all other variables to zero.*

§4 – Basic Feasible Solutions

Synopsis

- What is a basic feasible solution (*bfs*);
- Extreme points correspond to basic feasible solutions, and vice versa.

§4.1 – Basic Feasible Solutions

Consider the system of m linear equations with k variables, where $k \geq m$,

$$\begin{array}{rcl} a_{11}x_1 + \ldots + a_{1k}x_k & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + \ldots + a_{mk}x_k & = & b_m \end{array}$$

and assume that ($A = (a_{ij})$ has rank m).

Definition 4.1

A *basic solution* to the system is constructed by

- Selecting m columns whose coefficients are linearly independent
- Setting the $k - m$ variables that correspond to the other columns to zero.
- Solving the $m \times m$ system of equations that remain to assign values to the variables that correspond to the selected columns;

The variables corresponding to the selected m columns are basic variables, and other variables are nonbasic variables.

Definition 4.2

A *basic feasible solution* of an LP problem

$$\begin{aligned} \max / \min \ z &= cx \\ Ax &= b \\ x &\geq 0, \end{aligned}$$

where rank(A) is equal to the number of rows of A, is a basic solution of the linear equation system $Ax = b$ that also satisfies the non-negativity constraints $x_i \geq 0$ for all i.

- Any basic feasible solution is a feasible solution, but the converse is not true.

Example 4.1

Standard Form

The LP

$$\max_x 3x_1 + 4x_2$$

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0, \end{aligned}$$

is in standard form.

Canonical Form

The corresponding canonical form is

$$\max_x 3x_1 + 4x_2$$

$$\begin{aligned} 2x_1 + x_2 + x_3 \qquad &= 4 \\ x_1 + 2x_2 \qquad + x_4 &= 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The 'trivial basic feasible solution', derived by selecting variables x_3 and x_4 to be basic is $x = (0, 0, 4, 3)$.

What are the other basic feasible solutions?

Suppose we select x_2 and x_3 to be basic. Then, the reduced system is

$$\begin{aligned} x_2 + x_3 &= 4 \\ 2x_2 &= 3. \end{aligned}$$

This yields the basic feasible solution

$$x = (0, 3/2, 5/2, 0).$$

If we select x_1 and x_2 to be the basic variables, the reduced system is

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_1 + 2x_2 &= 3. \end{aligned}$$

This yields the basic feasible solution

$$x = (5/3, 2/3, 0, 0).$$

If we select x_1 and x_3 as basic variables, the reduced system is

$$\begin{aligned} 2x_1 + x_3 &= 4 \\ x_1 &= 3 \end{aligned}$$

This yields the basic solution

$$x = (3, 0, -2, 0)$$

which is not feasible.

The two other basic solutions are $(2, 0, 0, 1)$, which is basic feasible, and $(0, 4, 0, -5)$, which is not basic feasible.

Lemma 4.1

(Lemma 3.2 restated) If

$$\max z = \sum_{j=1}^{n+m} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n + a_{1,n+1}x_{n+1} + ... + a_{1,n+m}x_{n+m} & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + a_{m,n+1}x_{n+1} + ... + a_{m,n+m}x_{n+m} & = & b_m \\ x_j \geq 0, \quad j = 1, ..., n+m & & \end{array}$$

is in canonical form and columns $j_1, j_2, \ldots, j_m$ *are* $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

Then $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ *defined by*

$$x_{j_1} = b_1, \ldots, x_{j_m} = b_m, \quad \textit{all other } x_j = 0$$

is a basic feasible solution.

Example 4.2

The LP problem

$$\max z = 2x_1 - 3x_2 + 4x_3 + x_5$$

$$\begin{aligned} 2x_1 + 0x_2 + 1x_3 + 3x_4 + 0x_5 &= 6 \\ -x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 &= 2 \\ 2x_1 + 0x_2 + 0x_3 + 6x_4 + 1x_5 &= 2 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

is in canonical form.

So $x = (0, 2, 6, 0, 2)$ is a basic feasible solution, with x_2, x_3 and x_5 basic variables.

§4.2 – Extreme Points and Basic Feasible Solutions

Why are we interested in basic feasible solutions?

It is because they correspond to extreme points.

Geometry		Algebra
Extreme Point	$\Longleftrightarrow$	Basic Feasible Solution

Consider the standard form linear programming problem:

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

such that

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + ... + a_{2n}x_n & \leq & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n & \leq & b_m \end{array}$$

$$x_j \geq 0, j = 1, ..., n.$$

The feasible region is a convex polyhedron S, which is a subset of $\mathbb{R}^n$.

The canonical form of this linear program is

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

such that

$$\begin{aligned} a_{11}x_1 + ... + a_{1n}x_n + ... + a_{1,n+m}x_{n+m} &= b_1 \\ a_{21}x_1 + ... + a_{2n}x_n + ... + a_{2,n+m}x_{n+m} &= b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + ... + a_{m,n+m}x_{n+m} &= b_m \end{aligned}$$

$$x_j \geq 0, j = 1, ..., n + m.$$

where (a_{ij}) for $i = 1, \ldots m$ and $j = n + 1, \ldots n + m$ forms the $m \times m$ identity matrix.

The coefficient matrix (a_{ij}) for $i = 1, \ldots m$ and $j = 1, \ldots n + m$ clearly has rank m (since the final m columns are linearly independent).

By the definition of standard form, we also know that $b_i \geq 0$, for all i.

Theorem 4.1

The vector $\tilde{x} \equiv (x_1, \ldots, x_n)$ *in* $\mathbb{R}^n$ *is an extreme point of* S *if and only if* x *in* $\mathbb{R}^{n+m}$ *is a basic feasible solution of the set of equations*

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n + ... + a_{1,n+m}x_{n+m} & = & b_1 \\ a_{21}x_1 + ... + a_{2n}x_n + ... + a_{2,n+m}x_{n+m} & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + ... + a_{m,n+m}x_{n+m} & = & b_m \end{array}$$

$$x_j \geq 0, j = 1, ..., n + m.$$

Proof:

This is an 'if and only if' theorem. Frequently we have to prove theorems like this by dealing with the 'if' and 'only if' parts separately. We do this here.

'if' $\Leftarrow$

In this part we have to show that if $x = (x_1, \ldots, x_{n+m})$ is a basic feasible solution of the equations then $\tilde{x}$ is an extreme point of S.

We use ideas that we shall return to later in the subject. Corresponding to any basic feasible solution $x \in \mathbb{R}^{n+m}$, we can partition the set of indices into those corresponding to *basic variables*, which correspond to the m linearly independent columns of the coefficient matrix, and those corresponding to *non-basic variables* which are set to be zero. Thus we can write

$$x = [x_B, 0].$$

A slight subtlety is that some of the components in x_B can also be zero.

We can also partition A according to basic and non-basic variables, so that

$$A = [A_B, A_{NB}].$$

Then

$$Ax = b$$

is the same as

$$[A_B, A_{NB}] \begin{bmatrix} x_B \\ 0 \end{bmatrix} = b,$$

which implies that

$$A_B x_B = b.$$

By definition, the columns of A_B are linearly independent and so A_B is nonsingular. Thus

$$x_B = A_B^{-1} b.$$

Let $y \equiv (y_1, \ldots, y_{n+m})$ be a feasible solution to the set of equations. Because the final m columns of the coefficient matrix correspond to the identity matrix, we know that, for any $i = 1, \ldots, m$,

$$y_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} y_j.$$

This means the value of $\tilde{y} \equiv (y_1, \ldots, y_n)$ determines the value of y, and vice versa.

With this machinery, we can now prove the 'if' part. We use proof by contradiction.

Let $x \equiv (x_1, \ldots, x_{n+m})$ be a basic feasible solution to the set of equations, and assume that $\tilde{x} \equiv (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is not an extreme point of S.

Then there exist feasible $\tilde{y} = (y_1, \ldots, y_n)$ and $\tilde{z} = (z_1, \ldots, z_n)$, not equal to $\tilde{x}$, and $\lambda \in (0, 1)$ such that

$$\tilde{x} = \lambda \tilde{y} + (1 - \lambda)\tilde{z}.$$

Since $\tilde{y}$ and $\tilde{z}$ are feasible, we can add nonnegative slack variables $(y_{n+1}, \ldots, y_{n+m})$ and $(z_{n+1}, \ldots, z_{n+m})$, as on the previous slide, such that

$$\begin{aligned} Ay &= b \\ Az &= b. \end{aligned}$$

where $y = (y_1, \ldots, y_{n+m})$ and $z = (z_1, \ldots, z_{n+m})$.

Moreover, since, for any $i = 1, \ldots, m$,

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j,$$

we know that

$$x_{n+i} = \lambda y_{n+i} + (1 - \lambda) z_{n+i}$$

and so we can write

$$x = \lambda y + (1 - \lambda) z.$$

Partition y and z according to the basic/non-basic partition of x so that $y = [y_B, y_{NB}]$ and $z = [z_B, z_{NB}]$.

Looking first at the non-basic variables, we observe that

$$0 = \lambda y_{NB} + (1 - \lambda) z_{NB}.$$

Furthermore y_{NB} and z_{NB} are nonnegative. This implies that $y_{NB} = z_{NB} = 0$.

It follows that

$$[A_B, A_{NB}] \begin{bmatrix} y_B \\ 0 \end{bmatrix} = b$$

and

$$[A_B, A_{NB}] \begin{bmatrix} z_B \\ 0 \end{bmatrix} = b.$$

which, because A_B is nonsingular, implies that

$$y_B = z_B = [A_B]^{-1} b.$$

Thus x, y and z are all the same, which implies that $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ are all the same, and this contradicts our assumption that $\tilde{y}$ and $\tilde{z}$ are not equal to $\tilde{x}$.

We conclude that $\tilde{x} = (x_1, \ldots, x_n)$ cannot be written as a convex combination of two different feasible points. It must therefore be an extreme point.

'only if' $\Rightarrow$

In this part we have to show that if $\tilde{x} = (x_1, \ldots, x_n)$ is an extreme point of S, then $x = (x_1, \ldots, x_{n+m})$, where

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j,$$

is a basic feasible solution of the constraint equations in the canonical form.

We start by partitioning x in a different manner from the 'if' part of the proof.

Here we partition x into $[x^+, 0]$, where the components in x^+ are strictly positive.

We also partition A in the same way so that $A = [A^+, A^0]$ and

$$[A^+, A^0] \begin{bmatrix} x^+ \\ 0 \end{bmatrix} = b.$$

So we have,

$$A^+ x^+ = b.$$

We want to show that the columns of A^+ are linearly independent. Again, we do this using a proof by contradiction.

Assume that the columns of A^+ are not linearly independent.

Then there is a non-zero w^+ (not necessarily nonnegative) such that

$$A^+ w^+ = 0.$$

Since $x^+ > 0$, there exists ϵ such that $x^+ > \epsilon w^+$ and $x^+ > -\epsilon w^+$. (These inequalities should be read coordinate-wise.)

We have

$$[A^+, A^0] \begin{bmatrix} x^+ + \epsilon w^+ \\ 0 \end{bmatrix} = A^+(x^+ + \epsilon w^+) = A^+ x^+ + \epsilon A^+ w^+ = b.$$

Since $x^+ + \epsilon w^+ > 0$, $(x^+ + \epsilon w^+, 0)$ is a feasible solution to the constraint equations, and its first n components correspond to a point in the feasible region S.

Similarly, since $x^+ - \epsilon w^+ > 0$, the first n components of $(x^+ - \epsilon w^+, 0)$ correspond to a point in the feasible region S.

Now

$$x^+ = \frac{1}{2}(x^+ + \epsilon w^+) + \frac{1}{2}(x^+ - \epsilon w^+),$$

so that,

$$[x^+, 0] = \frac{1}{2}[x^+ + \epsilon w^+, 0] + \frac{1}{2}[x^+ - \epsilon w^+, 0]$$

and x can be written as a convex combination of two distinct feasible points.

Furthermore, the points $[x^+, 0]$, $[x^+ + \epsilon w^+, 0]$ and $[x^+ - \epsilon w^+, 0]$ must differ in at least one of their first n components.

So $\tilde{x}$ can be written as a convex combination of two other feasible points in S, which contradicts the assumption that it is an extreme point of S.

So the columns of A^+ must be linearly independent. Since there cannot be any more than m linearly independent vectors in $\mathbb{R}^m$, this means that there cannot be any more than m positive components in x.

We'd like to be able to say immediately that this means that x is a basic feasible solution, but we can't quite say this because it is possible that there are $s < m$ columns in A^+, when there are less than m positive components in x.

However, if this is the case, there must $m - s$ columns of A^0 that we can add to those of A^+ to form a linearly-independent set of m columns, because we know that there are m linearly independent columns of A overall, and we see that x is a basic feasible solution with some of its basic variables equal to zero.

This proves the 'only if' part.

$\square$

§5 – Fundamental Theorem of Linear Programming

Synopsis

- Fundamental theorem of Linear Programming

Theorem 5.1 (Fundamental Theorem of Linear Programming)

Consider the linear programming problem in canonical form

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1(n+m)}x_{n+m} & = & b_1 \\ a_{21}x_1 + ... + a_{2(n+m)}x_{n+m} & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{m(n+m)}x_{n+m} & = & b_m \\ x_j \geq 0,\ j = 1, \ldots, n+m & & \end{array}$$

(a) *If this problem has a feasible solution, then it must have a basic feasible solution.*

(b) *If this problem has an optimal solution, then it must have an optimal basic feasible solution.*

Proof

Part (a)

Let x be a feasible solution. We need to prove that the problem has a basic feasible solution.

As in the proof of the 'only if' part of the previous theorem, we partition x into a strictly positive part x^+, with s entries, and a zero part. Without loss of generality we may assume that this partition is $x = [x^+, 0]$.

We also partition $A = [A^+, A^0]$ in the same way. Then $Ax = b$ becomes

$$\begin{bmatrix} A^+, & A^0 \end{bmatrix} \begin{bmatrix} x^+ \\ 0 \end{bmatrix} = b,$$

that is,

$$A^+ x^+ = b.$$

Case (1): The columns of A^+ are linearly independent.

In this case, $s \leq m$, and either

- $s = m$, in which case x is a basic feasible solution by definition, or
- $s < m$, in which case we can add $m - s$ further columns to make a linearly independent set as we did in the 'only if' part of the previous theorem.

In either case we obtain a basic feasible solution.

Case (2): The columns of A^+ are linearly dependent.

In this case we will iteratively construct a new feasible solution such that the corresponding A^+ (for the new feasible solution) has independent columns. In this way we reduce to Case (1).

Since the columns of A^+ are linearly dependent, there is a non-zero w^+ such that

$$A^+ w^+ = 0.$$

Without loss of generality, we can assume $w_j > 0$ for at least one j.

As before, we have

$$[A^+, A^0] \begin{bmatrix} x^+ - \epsilon w^+ \\ 0 \end{bmatrix} = A^+(x^+ - \epsilon w^+) = b$$

for any value of ϵ.

In addition, as long as $\epsilon \geq 0$ and $x^+ \geq \epsilon w^+$ (that is, $x_j \geq \epsilon w_j$ for each j with $w_j > 0$), $(x^+ - \epsilon w^+, 0)$ is nonnegative.

Now choose

$$\epsilon^* = \min \left\{ \frac{x_j}{w_j} : w_j > 0 \right\} \ (> 0).$$

Then $(x^+ - \epsilon^* w^+, 0)$ is a feasible solution. Moreover, one component of $x^+ - \epsilon^* w^+$ is zero, with the rest nonnegative.

Thus we have constructed a new feasible solution $(x^+ - \epsilon w^+, 0)$ whose number of positive components is reduced by at least one from that of x.

If the columns of the new A^+ with respect to this new feasible solution are linearly independent, the LP problem has a feasible solution by Case (1).

Otherwise, we replace x by $(x^+ - \epsilon w^+, 0)$ and repeat the argument above. We then get a third feasible solution whose number of positive components is reduced by at least one from that of $(x^+ - \epsilon w^+, 0)$.

Continue like this, all the time reducing the number of positive components by at least one.

This process must terminate in a finite number of iterations (why?) with a feasible solution such that the columns of the corresponding A^+ are linearly independent.

We then conclude by Case (1) that the LP problem has at least one basic feasible solution.

Part (b)

Starting with an optimal feasible solution x, we can construct an optimal basic feasible solution in exactly the same way as in Part (a).

Case (1): The columns of A^+ are linearly independent.

From the proof of Part (a) we see that x is an optimal basic feasible solution.

Case (2): The columns of A^+ are linearly dependent.

Using the notation from Part (a),

$$\begin{aligned} c^T \begin{bmatrix} x^+ - \epsilon w^+ \\ 0 \end{bmatrix} &= \sum_{j=1}^{s} c_j (x_j - \epsilon w_j) \\ &= \sum_{j=1}^{s} c_j x_j - \epsilon \sum_{j=1}^{s} c_j w_j \\ &= c^T x - \epsilon \sum_{j=1}^{s} c_j w_j. \end{aligned}$$

Since x is an optimal solution (that is, $c^T x$ achieves the maximum) and $(x^+ - \epsilon w^+, 0)$ is a feasible solution, we have

$$\sum_{j=1}^{s} c_j w_j \geq 0.$$

Take a δ such that

$$0 < \delta \leq \min\left\{\frac{-x_j}{w_j} : w_j < 0\right\}.$$

(The right-hand side is defined as ∞ if there are no w_j that are negative.)

Following the logic of Case (2) of Part (a), we can show that $(x^+ + \delta w^+, 0)$ is a feasible solution whose objective value is

$$\begin{aligned} c^T \begin{bmatrix} x^+ + \delta w^+ \\ 0 \end{bmatrix} &= \sum_{j=1}^{s} c_j(x_j + \delta w_j) \\ &= c^T x + \delta \sum_{j=1}^{s} c_j w_j \end{aligned}$$

Since x is optimal and $\delta > 0$, we must have

$$\sum_{j=1}^{s} c_j w_j \leq 0.$$

Combining the two inequalities above, we have

$$\sum_{j=1}^{s} c_j w_j = 0,$$

which implies

$$c^T \begin{bmatrix} x^+ - \epsilon w^+ \\ 0 \end{bmatrix} = c^T x.$$

In other words, the value of the objective function is the same at $(x^+ - \epsilon w^+, 0)$ as at $(x^+, 0)$.

Following the same recursive procedure as in Case (2) of Part (a), we can construct a basic feasible solution with the same objective value as x (and so is optimal) such that the columns of the corresponding A^+ are linearly independent, reducing to Case (1). $\square$

Corollary 5.1

If the feasible region of the linear programming problem is not empty then it must have at least one extreme point.

Corollary 5.2

The feasible region possesses at most a finite number of extreme points (Can you suggest an upper bound?)

Corollary 5.3

If the linear programming problem possesses an optimal solution, then there is an optimal solution which is an extreme point of the feasible region.

§6 – Preview of the Simplex Method

Synopsis

- Methods for solving LPs
- Idea of the Simplex Method
- An example
- The Simplex Method

§6.1 – Methods for Solving LPs

- Given a linear program with an optimal solution, at least one of the optimal solutions is an extreme point of the feasible region.
- So how about solving the problem by enumerating all the extreme points of the feasible region?

Since extreme points of the feasible region correspond to basic feasible solutions, for an LP in canonical form with $n+m$ variables and m constraints, there are at most

$$\binom{n+m}{m} = \frac{(n+m)!}{m!n!}$$

extreme points.

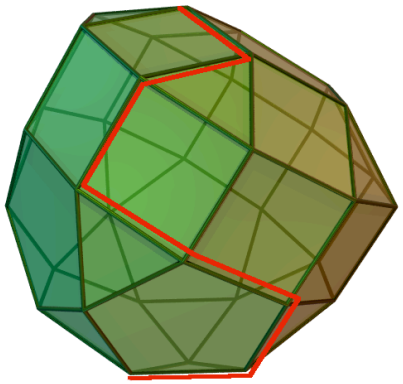
For large n and m, this yields a very large number of possible extreme points. This is an example of the Curse of Dimensionality.

For example for $n = m = 50$, the maximum number of extreme points is 10^{29}.

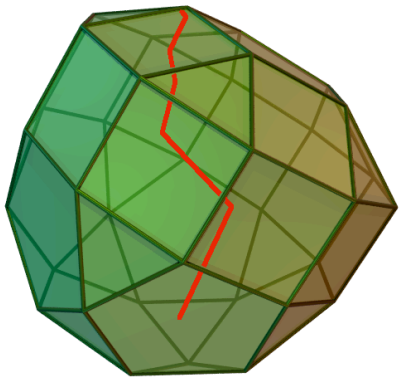
It is impractical to find an optimal solution by enumerating all extreme points.

Methods for Solving LPs

- Simplex, [Dantzig, 1947]
  - Starts at an extreme point and moves to another extreme point with improved objective value.

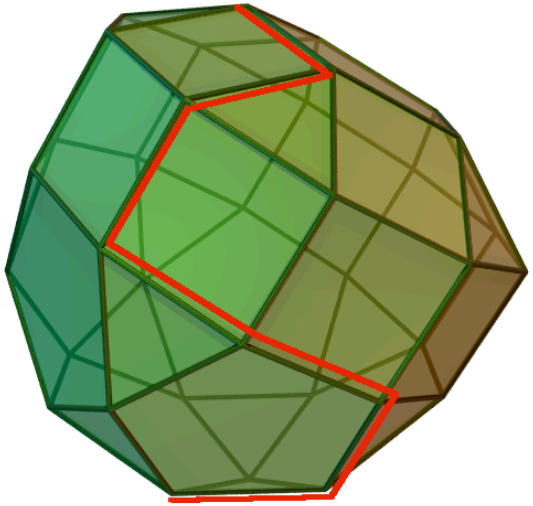


- Interior Point [Karmarkar, 1984]
  - Moves from the (relative) interior of the region or faces, towards the optimal solution.



§6.2 – Idea of the Simplex Method

The Simplex Method is an algorithm that 'jumps' from one extreme point of the feasible region to another in such a way that the value of the objective function is improved (or at least does not become worse) at each stage.



Idea of the Simplex Method

We initialise with a particular basic feasible point and have a criterion to check whether it is optimal. If it is, then we stop. Otherwise we perform an iteration to generate a new basic feasible point and check again. We keep on iterating until the algorithm terminates.

Input: A *Basic Feasible Solution*

if Optimality criterion is satisfied **then**

Exit. We are done.

else

Generate a new *Basic Feasible Solution* that is at least as good as the old one.

end if

The iterative step involves moving along an edge of the feasible region to get to a neighbouring extreme point.

§6.3 – An Example

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{aligned} 2x_1 - x_2 &\leq 1 \\ -3x_1 + 4x_2 &\leq 13 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

This LP problem is in standard form.

Convert it to canonical form by introducing slack variables.

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

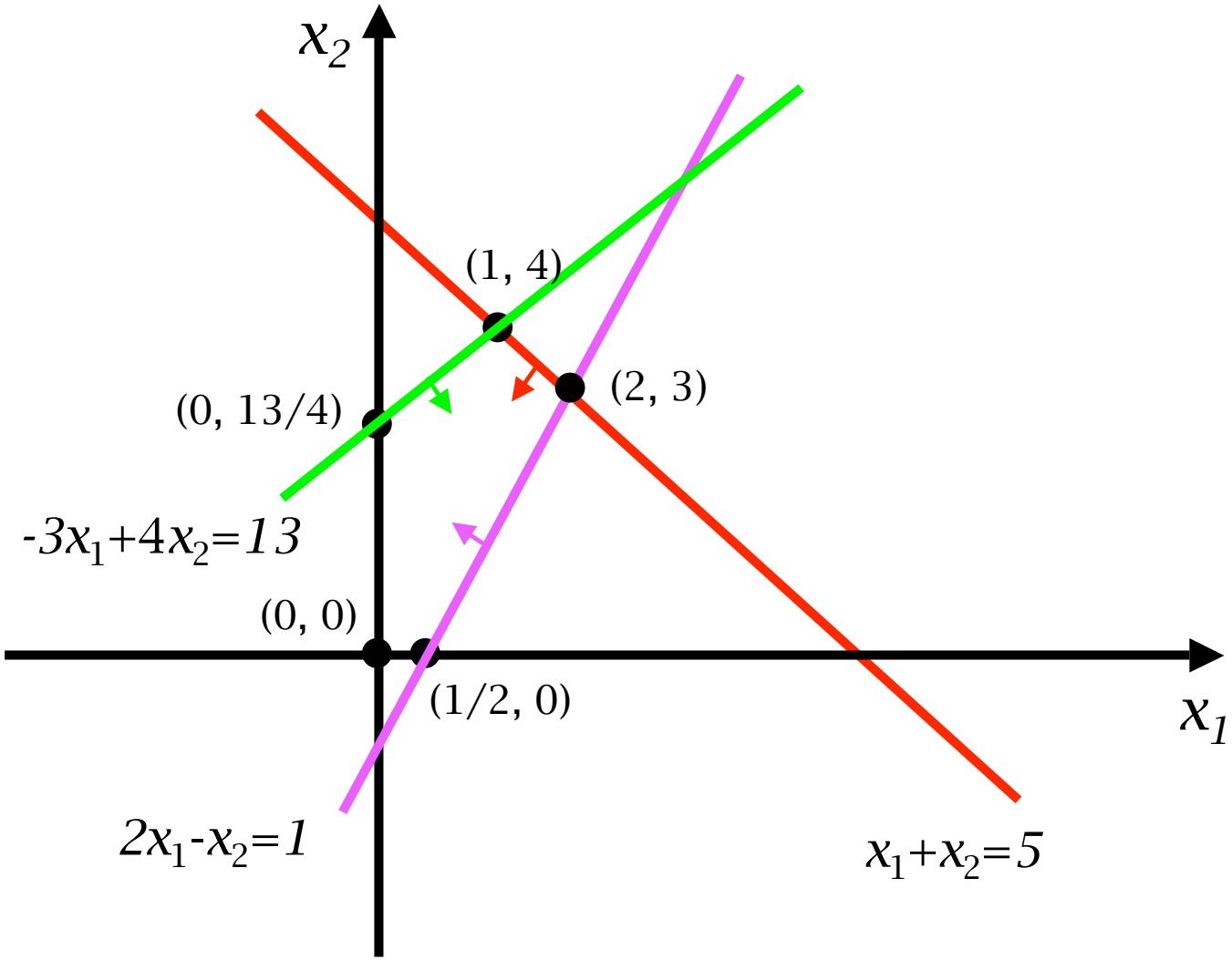
Initial basic variables: x_3, x_4, x_5 (which are the slack variables)

Initial bfs: $x = (0, 0, 1, 13, 5)$.

Initial value of objective function: $z = 3 \cdot 0 + 2 \cdot 0 = 0$

If either x_1 or x_2 were to become a basic variable and take on a positive value, the value of z would increase. Why?

How about if the cost coefficient of x_1 in z is negative?



The initial bfs $x = (0, 0, 1, 13, 5)$ corresponds to the extreme point $(0, 0)$ in $\mathbb{R}^2$.

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Which one of x_1 and x_2 should we choose?

Close to the point $(0, 0)$, there are 3 units of increase in z per unit increase of x_1, and 2 units of increase in z per unit increase of x_2.

Either x_1 or x_2 can be the entering basic variable.

However, it seems 'reasonable' to believe that bringing in x_1 will lead to a greater increase in the objective function z.

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Greedy Rule

If z is written in terms of the current nonbasic variables, then the largest positive cost coefficient determines the entering (basic) variable.

So in this example we choose x_1 to be the entering basic variable.

If we write the objective function $z = 3x_1 + 2x_2$ as

$$z - 3x_1 - 2x_2 = 0,$$

this means we choose the most negative coefficient in this equation.

How about if all cost coefficients are negative or zero? For example,

$$\max \ z = -2x_1 - 4x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

The current value of z is maximum because allowing any nonbasic variable to become positive would not increase the value of z.

Optimality Criterion

If z is written in terms of the current non-basic variables and if all cost coefficients are negative or zero, then the current value of z is maximum and the current bfs is optimal.

Returning to our example ...

$$\max \; z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

We want to increase the entering variable x_1 as much as possible.

But we do not want to leave the feasible region.

What is the maximum increase of x_1 without causing any basic variable to become negative?

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Since x_2 remains non-basic, its value is equal to zero. So we have

$$x_3 = 1 - 2x_1 \geq 0 \Rightarrow 2x_1 \leq 1 \Rightarrow x_1 \leq \frac{1}{2}$$

$$x_4 = 13 + 3x_1 \geq 0 \Rightarrow -3x_1 \leq 13, \text{ satisfied by any } x_1 \geq 0$$

$$x_5 = 5 - x_1 \geq 0 \Rightarrow x_1 \leq 5 \Rightarrow x_1 \leq \frac{5}{1}$$

The largest allowable positive value for x_1 is $1/2$.

If $x_1 = 1/2$, then $x_3 = 0$ and so x_3 becomes the leaving (basic) variable.

Ratio Test

Assume that the entering basic variable has been chosen. For every equation for which the coefficient of the entering variable is positive, form the ratio of the resource value to that coefficient.

Select the equation that produces the smallest of these ratios.

The basic variable for the selected equation is the leaving (basic) variable.

Restoring the Canonical Form

Restore the canonical form with basic variables x_1, x_4 and x_5 by pivoting on $2x_1$ in the first equation.

$$\max \ z = 3x_1 + 2x_2$$

$$\begin{array}{rcl} 2x_1 - x_2 + x_3 & = & 1 \\ -3x_1 + 4x_2 \qquad + x_4 & = & 13 \\ x_1 + x_2 \qquad\qquad + x_5 & = & 5 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

$$\begin{array}{rcl} x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 & = & \frac{1}{2} \\ \frac{5}{2}x_2 + \frac{3}{2}x_3 + x_4 & = & \frac{29}{2} \\ \frac{3}{2}x_2 - \frac{1}{2}x_3 \qquad + x_5 & = & \frac{9}{2} \end{array}$$

So the bfs is now $\left(\frac{1}{2}, 0, 0, \frac{29}{2}, \frac{9}{2}\right)$ and the objective function value is $z = 3(\frac{1}{2}) + 2(0) = \frac{3}{2}$.

$$\begin{array}{rcl} x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 & = & \frac{1}{2} \\ \frac{5}{2}x_2 + \frac{3}{2}x_3 + x_4 & = & \frac{29}{2} \\ \frac{3}{2}x_2 - \frac{1}{2}x_3 \qquad + x_5 & = & \frac{9}{2} \end{array}$$

Express z in terms of the non-basic variables x_2 and x_3

The first constraint leads to $x_1 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3$ and so

$$z = 3x_1 + 2x_2 = 3\left(\frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3\right) + 2x_2 = \frac{3}{2} + \frac{7}{2}x_2 - \frac{3}{2}x_3$$

that is

$$z - \frac{3}{2} - \frac{7}{2}x_2 + \frac{3}{2}x_3 = 0$$

Exercise: verify that this can be obtained from $z - 3x_1 - 2x_2 = 0$ by pivoting on $2x_1$.

Now we have a new canonical system:

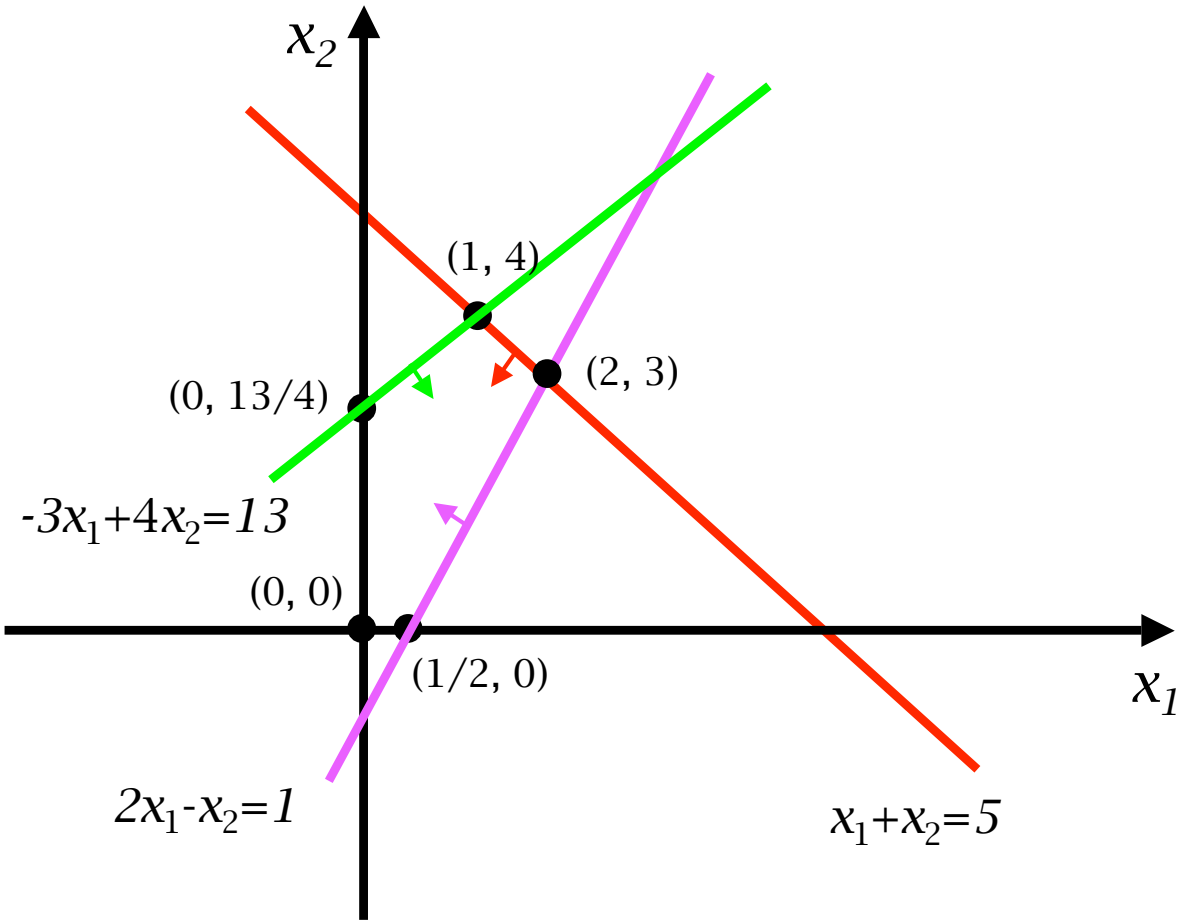
$$\max \ z = \frac{3}{2} + \frac{7}{2}x_2 - \frac{3}{2}x_3$$

$$\begin{array}{rcl} x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 & = & \frac{1}{2} \\ \frac{5}{2}x_2 + \frac{3}{2}x_3 + x_4 & = & \frac{29}{2} \\ \frac{3}{2}x_2 - \frac{1}{2}x_3 \qquad + x_5 & = & \frac{9}{2} \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Current basic variables: x_1, x_4, x_5.

Current bfs: $\left(\frac{1}{2}, 0, 0, \frac{29}{2}, \frac{9}{2}\right)$

Current value $z = 3/2$



The current bfs $\left(\frac{1}{2}, 0, 0, \frac{29}{2}, \frac{9}{2}\right)$ corresponds to the extreme point $\left(\frac{1}{2}, 0\right)$ in $\mathbb{R}^2$.

What should we do with this new canonical form? Repeat the procedure!

§7 – The Simplex Method

Input: A max Linear Program in *standard form*
Output: An optimal solution

```
Construct canonical form and obtain a basic feasible solution
while There are negative reduced costs do
    Select entering variable with most negative reduced cost.
    Select leaving variable using the ratio test.
    Pivot the tableau.
end while
```

Here the reduced costs are the coefficients of the variables when we write the objective function as an equation such that all terms are on the same side and the coefficient of z is equal to 1.

More Details

To come...

- Initialisation
- Streamlining the process
- Optimality test
- Iteration
- LP with no feasible solution
- LP with no optimal solution
- LP with multiple optimal solutions

§7.1 – Initialisation

We start with an LP in standard form:

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & \leq & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & \leq & b_m \end{array}$$

$$x_j \geq 0, \quad j = 1, ..., n$$

To initialise the Simplex Algorithm, we transform the LP into canonical form by introducing slack variables.

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} & = & b_2 \\
\vdots & \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \qquad\qquad + x_{n+m} & = & b_m
\end{array}$$

$$x_j \geq 0, \quad j = 1, \ldots, n+m$$

If we use the slack variables $x_{n+1} \ldots, x_{n+m}$ as basic variables, we obtain the initial basic feasible solution, namely

$$(0, \ldots, 0, b_1, \ldots, b_m).$$

Example 7.1

$$\max z = 4x_1 + 3x_2$$

$$\begin{aligned} 2x_1 + x_2 &\leq 40 \\ x_1 + x_2 &\leq 30 \\ x_1 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

can be rewritten as

$$\max z = 4x_1 + 3x_2$$

$$\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 40 \\ x_1 + x_2 \qquad + x_4 & = & 30 \\ x_1 \qquad\qquad\qquad + x_5 & = & 15 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

where x_3, x_4, x_5 are slack variables.

The initial basic feasible solution is $x = (0, 0, 40, 30, 15)$.

Summary of the Initialisation Step

- Convert the standard LP problem to canonical form
- Select the slack variables as basic variables
- Obtain the initial basic feasible solution

Comments:

- This is simple.
- This initial basic feasible solution is not necessarily a good choice: it can be (very) far from an optimal solution.

§7.2 – Iteration

- We are at an extreme point of the feasible region.
- We want to move to an adjacent extreme point.
- We want to move to a better extreme point.

Observation:

- Basic feasible solutions which differ only in that one basic variable is interchanged with one non-basic variable correspond to adjacent feasible extreme points.

Moving to an adjacent extreme point

- Step 1: Select which non-basic variable becomes basic
- Step 2: Determine which basic variable becomes non-basic
- Step 3: Reconstruct a new canonical form reflecting this change

§7.3 – The Simplex Tableau

- It is convenient to describe how the Simplex Method works using a table (= tableau).
- There are a number of different layouts for these tables.
- In this subject, all of us should use the layout specified in these lecture slides.

- It is convenient to incorporate the objective function into the formulation as a functional constraint.
- We can do this by viewing z, the value of the objective function, as a decision variable, and introduce the additional constraint

$$z = \sum_{j=1}^{n} c_j x_j$$

or equivalently

$$z - c_1 x_1 - c_2 x_2 - \ldots - c_n x_n = 0.$$

The Initial Tableau

BV	x_1	...	x_n	x_{n+1}	...	x_{n+m}	RHS
x_{n+1}	a_{11}	...	a_{1n}	1	...	0	b_1
x_{n+2}	a_{21}	...	a_{2n}	0	...	0	b_2
$\cdots$							
x_{n+m}	a_{m1}	...	a_{mn}	0	...	1	b_m
z	$-c_1$	...	$-c_n$	0	...	0	0

- We refer to the last row as the z-row.
- We omitted the z-column since it is $(0, 0, \ldots, 1)^T$, and it stays like this along the way.
- The reduced costs of $x_1, \ldots, x_n$ are $-c_1, \ldots, -c_n$, respectively, and the reduced costs of the basic variables $x_{n+1}, \ldots, x_{n+m}$ are zero. (See Section 6.4.)
- The tableau above is in canonical form.
- In general, a tableau is said to be in canonical form if
  - all $b_i \geq 0$,
  - m of the columns form the $m \times m$ identity matrix (not necessarily in order),
  - the reduced costs for basic variables are all equal to zero.

Example 7.2

$$\max z = 4x_1 + 3x_2$$

such that

$$\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 40 \\ x_1 + x_2 \qquad + x_4 & = & 30 \\ x_1 \qquad\qquad\qquad + x_5 & = & 15 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

can be transformed to

$$\max z$$

such that

$$\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 40 \\ x_1 + x_2 \qquad + x_4 & = & 30 \\ x_1 \qquad\qquad\qquad + x_5 & = & 15 \\ z - 4x_1 - 3x_2 & = & 0 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

The Initial Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	2	1	1	0	0	40
x_4	1	1	0	1	0	30
x_5	1	0	0	0	1	15
z	-4	-3	0	0	0	0

§7.4 – Selecting a New Basic Variable (Entering Variable)

Question: Which one of the current non-basic variables should we change so that it becomes basic?
Answer: The z-row

$$z - c_1x_1 - c_2x_2 - ... - c_nx_n = 0$$

tells us how the value of the objective function z changes initially, as we change the decision variables.

Initially all $x_1, \ldots, x_n$ are non-basic variables. So if we decide to add x_j to the basis, then it will take on some nonnegative value.

The objective function will not improve unless $c_j > 0$ (that is, the reduced cost $-c_j < 0$). Thus we want to select a non-basic variable with a negative (or at least non-positive) reduced cost.

Moreover, since we are trying to maximize the objective function, we select a non-basic variable with the largest value of c_j, that is the most negative reduced cost.

Greedy Rule (Restated)

In a maximisation problem, we choose a currently non-basic variable with the most negative reduced cost to be the new basic variable (often called the entering variable).

That is, we bring a non-basic variable with the most negative reduced cost to the basis.

If there are two or more such non-basic variables, choose any one of them.

However, to prevent cycling we usually choose the one with smallest subscript. This issue will be discussed later.

Example (continued)

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	2	1	1	0	0	40
x_4	1	1	0	1	0	30
x_5	1	0	0	0	1	15
z	-4	-3	0	0	0	0

The most negative reduced cost in the z-row is -4, corresponding to $j = 1$.

Thus, we select x_1 as the new basic variable.

§7.5 – Determining the New Non-basic Variable (Leaving Variable)

Suppose we decided to select x_j as a new basic variable.

Since the number of basic variables is fixed at m, we have to take one variable out of the basis.

Which one?

Observation

- As we increase x_j from zero, we can expect that, sooner or later, one or more of the basic variables will become negative.
- We take the first such variable out of the basis and set it to zero.

Example (continued)

$$\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 40 \\ x_1 + x_2 \qquad + x_4 & = & 30 \\ x_1 \qquad\qquad\qquad + x_5 & = & 15 \end{array}$$

- Suppose we select x_1 as the new basic variable.
- Since x_2 is a non-basic variable and is staying non-basic, its value is zero. Thus the above system can be simplified.

Each equation involves only two variables:

- The new basic variable x_1
- The old basic variable associated with the respective constraint.

$$\begin{array}{rcl} 2x_1 + x_3 & = & 40 \\ x_1 \qquad + x_4 & = & 30 \\ x_1 \qquad\qquad + x_5 & = & 15. \end{array}$$

i.e.

$$\begin{array}{rcl} x_3 & = & 40 - 2x_1 \\ x_4 & = & 30 - x_1 \\ x_5 & = & 15 - x_1. \end{array}$$

Since we need $x_3, x_4, x_5 \geq 0$, we need

$$\begin{aligned} 2x_1 &\leq 40 \\ x_1 &\leq 30 \\ x_1 &\leq 15 \end{aligned}$$

That is,

$$\begin{aligned} x_1 &\leq \frac{40}{2} \\ x_1 &\leq \frac{30}{1} \\ x_1 &\leq \frac{15}{1} \end{aligned}$$

If the last inequality holds, then the other two inequalities hold. So we can increase x_1 up to 15. And if $x_1 = 15$, then $x_5 = 0$ and so we should take x_5 out of the basis.

More generally ....

If we have selected x_j as the new basic variable, then the ith functional constraint becomes

$$a_{ij}x_j + x_i = b_i$$

where x_i is an old basic variable.

We need

$$x_i = b_i - a_{ij}x_j \geq 0.$$

If $a_{ij} \leq 0$, then this is always satisfied since $x_j \geq 0$ and $b_i \geq 0$.

If $a_{ij} > 0$, then the above is equivalent to $x_j \leq \frac{b_i}{a_{ij}}$. So we need

$$x_j \leq \min_i \left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0 \right\} \text{ (maximum allowable increase of } x_j\text{)}$$

If this minimum is $\frac{b_{i*}}{a_{i*j}}$ and if we set $x_j = \frac{b_{i*}}{a_{i*j}}$, then $x_{i*} = 0$ and so x_{i*} comes out of the basis.

Ratio Test (Restated)

Given that we have selected the new basic variable x_j, we take out of the basis the old basic variable corresponding to row i such that

$$a_{ij} > 0$$

and the ratio

$$Ratio_i := \frac{b_i}{a_{ij}}$$

attains its smallest value.

Such a variable to be taken out of the basis is often called the leaving variable.

Example (continued)

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	2	1	1	0	0	40
x_4	1	1	0	1	0	30
x_5	1	0	0	0	1	15
z	-4	-3	0	0	0	0

$$\begin{aligned} Ratio_1 &= \frac{40}{2} = 20 \\ Ratio_2 &= \frac{30}{1} = 30 \\ Ratio_3 &= \frac{15}{1} = 15. \end{aligned}$$

Thus we take x_5 out of the basis.

§7.6 – Restoring the Canonical Form – Pivot Operation

- We interchanged a basic variable with a non-basic variable.
- So we have a new basis.
- We have to construct the simplex tableau for the new set-up.
- This is done by performing one pivot operation.
- We choose the entry in the column of the entering variable and the row of the leaving variable as the pivot entry.
- We make this pivot entry 1 and all other entries in the same column 0 by elementary row operations (just as in linear algebra).

Example (continued)

Old Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	2	1	1	0	0	40
x_4	1	1	0	1	0	30
x_5	**1**	0	0	0	1	15
z	-4	-3	0	0	0	0

The pivot entry is **1** in the x_1-column and x_5-row.

Pivoting on this entry we obtain .....

Example (continued)

New Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	1	1	0	-2	10
x_4	0	1	0	1	-1	15
x_1	1	0	0	0	1	15
z	0	-3	0	0	4	60

Note that this new tableau is in canonical form. Good! We will keep canonical form along the way!

Note the change of bases.

Current basis: x_3, x_4, x_1

Previous basis: x_3, x_4, x_5

How do we "read" a simplex tableau?

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	1	1	0	-2	10
x_4	0	1	0	1	-1	15
x_1	1	0	0	0	1	15
z	0	-3	0	0	4	60

New basis:

$$x_3, x_4, x_1.$$

New basic feasible solution:

$$x = (15, 0, 10, 15, 0).$$

New value of the objective function:

$$z = 60.$$

§7.7 – Optimality Criterion (Restated)

If there are non-basic variables with negative reduced costs, then we have a chance to improve the objective function by adding one of these variables to the basis.

On the other hand, if all the non-basic variables have nonnegative coefficients in the z-row of the simplex tableau, then we cannot improve the objective function and we stop.

If all the reduced costs are nonnegative, the current solution is optimal.

Example (continued)

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	1	1	0	-2	10
x_4	0	1	0	1	-1	15
x_1	1	0	0	0	1	15
z	0	-3	0	0	4	60

- Optimality Test: There is at least one negative reduced cost. Hence we have to continue with the algorithm.

Steps in each iteration

In each iteration we have three steps:

- Variable in: Greedy Rule
- Variable Out: Ratio Test
- Tableau update: Pivot Operation

Example (continued)

- Variable in: There is only one variable with negative reduced costs, that is x_2. So we set $j = 2$ and put x_2 in the basis (i.e. x_2 is the entering variable).
- Variable out: We conduct the ratio test using the right-hand side column (RHS) and the column of the new basic variable x_2.

x_2	...	RHS	Ratio
1	...	10	10
1	...	15	15
0	...	15	–

The minimum ratio is attained at the first row. We therefore take out of the basis the basic variable associated with row 1, that is, x_3 is the leaving variable. We set $i = 1$.

Updating tableau: We have to conduct the pivot operation on $(i = 1, j = 2)$.

Old Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	**1**	1	0	-2	10
x_4	0	1	0	1	-1	15
x_1	1	0	0	0	1	15
z	0	-3	0	0	4	60

The pivot entry is the one in the x_2-column and x_3-row.

After pivoting on $(i = 1, j = 2)$, we obtain:

New Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	1	0	-2	10
x_4	0	0	-1	1	1	5
x_1	1	0	0	0	1	15
z	0	0	3	0	-2	90

- The current basis consists of x_2, x_4 and x_1.
- The current basic feasible solution is $x = (15, 10, 0, 5, 0)$.
- The value of the objective function at this point is equal to $z = 90$.
- Optimality Test: There is at least one non-basic variable with negative reduced cost so we continue.

Example (continued)

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	1	0	-2	10
x_4	0	0	-1	1	**1**	5
x_1	1	0	0	0	1	15
z	0	0	3	0	-2	90

- Variable in: The variable with the most negative reduced cost is x_5, so we place x_5 in the basis and set $j = 5$.
- Variable out: We conduct the ratio test on the column of x_5.

x_5	...	RHS	Ratio
-2	...	10	–
1	...	5	5
1	...	15	15

The minimum ratio is attained at $i = 2$. Thus we take out of the basis the basic variable associated with row 2, that is x_4. We conduct the pivot operation on $(i = 2, j = 5)$.

After the pivot operation on $(i = 2, j = 5)$, we obtain:

New Tableau

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	5
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

- The new basis consists of x_2, x_5 and x_1.
- The new basic feasible solution is $x = (10, 20, 0, 0, 5)$.
- The value of the objective function at this point is equal to $z = 100$.
- Optimality Test: All the reduced costs are nonnegative, hence we stop. The current solution is optimal. Stop!

An Important Note

When we stop, we have to do two things:

- Write down the optimal solution and optimal value of the objective function, namely x^* and z^*.
- Check these values.
  - Are the constraints satisfied?
  - Is the value of z^* consistent with the value of x^*?

Example (continued)

From the final tableau we obtain the optimal solution

$$x^* = (10, 20, 0, 0, 5)$$

and the optimal value of the objective function

$$z^* = 100.$$

An optimal solution to the original problem (before introducing the slack variables) is

$$(x_1^*, x_2^*) = (10, 20).$$

Check objective value:

$$z^* = 4x_1^* + 3x_2^* = 4 \times 10 + 3 \times 20 = 100.$$

Check constraints:

$$2x_1^* + x_2^* \leq 40 \qquad (2 \times 10 + 1 \times 20 = 40, \text{ OK})$$

$$x_1^* + x_2^* \leq 30 \qquad (1 \times 10 + 1 \times 20 = 30, \text{ OK})$$

§7.8 – Complexity of the Simplex Algorithm

In theory the Simplex Algorithm is an *exponential time* algorithm. This means that in the worst case the time taken to solve a problem of size n can grow exponentially with n.

However, in almost all practical cases, the Simplex Algorithm does not seem to perform anywhere nearly as badly as this worst case bound. For example, the Simplex Algorithm has been used to solve practical problems with tens of thousands of constraints and hundreds of thousands of variables.

As mentioned earlier, there are polynomial time algorithms for solving LP problems.

§7.9 – Example

Example 7.3

A water desalination plant bases its production on profit alone. This plant can produce drinking grade ($60 per megalitre), irrigation grade ($35 per megalitre) and industry grade ($20 per megalitre) water for sale.

There are four filtration processes: electro-dialysis with capacity (48 units per week), freezing (60 units per week), reverse osmosis (16 units per week) and carbon (5 units per week).

A megalitre of drinking water requires 8, 12 and 4 units of the first three and no carbon filtering. A megalitre of irrigation water requires 6, 7, 3 and 1 unit respectively. A megalitre of industry water requires only 1, 4 and 1 unit of the first three and no carbon filtering.

Example (continued)

Let x_1 be the number of megalitres of drinking water produced per week,

Let x_2 be the number of megalitres of irrigation water produced per week

Let x_3 be the number of megalitres of industry water produced per week.

To maximise our weekly profit, we need to solve

$$\max_x z = 60x_1 + 35x_2 + 20x_3$$

$$\begin{aligned} 8x_1 + 6x_2 + x_3 &\leq 48 \\ 12x_1 + 7x_2 + 4x_3 &\leq 60 \\ 4x_1 + 3x_2 + x_3 &\leq 16 \\ x_2 &\leq 5 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

Example (continued)

- Step 1: We add slack variables and construct the canonical form. This yields the first *basic feasible solution*.

$$\max_x z = 60x_1 + 35x_2 + 20x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7$$

$$\begin{array}{rcl}
8x_1 + 6x_2 + x_3 + x_4 & = & 48 \\
12x_1 + 7x_2 + 4x_3 + x_5 & = & 60 \\
4x_1 + 3x_2 + x_3 + x_6 & = & 16 \\
x_2 + x_7 & = & 5
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Example (continued)

- We rewrite this formulation as a Simplex Tableau.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_4	8	6	1	1	0	0	0	48
x_5	12	7	4	0	1	0	0	60
x_6	4	3	1	0	0	1	0	16
x_7	0	1	0	0	0	0	1	5
z	-60	-35	-20	0	0	0	0	0

Initial basic feasible solution: $(0, 0, 0, 48, 60, 16, 5)$.

Example (continued)

- Step 2: There are negative reduced costs, so we continue.
- Step 3: We select the non-basic variable with the most negative reduced cost x_1 as the new basic variable.
- Step 4: We conduct the ratio test on the column of the new basic variable. Row 3 yields the minimum ratio so we take out the basic variable x_6 associated with row 3.
- Step 5: We perform the pivot operation on $(i = 3, j = 1)$.

Example (continued)

Old Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_4	8	6	1	1	0	0	0	48
x_5	12	7	4	0	1	0	0	60
x_6	**4**	3	1	0	0	1	0	16
x_7	0	1	0	0	0	0	1	5
z	-60	-35	-20	0	0	0	0	0

Example (continued)

After performing the pivot operation on $(i = 3, j = 1)$, we obtain:

New Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_4	0	0	-1	1	0	-2	0	16
x_5	0	-2	1	0	1	-3	0	12
x_1	1	$3/4$	$1/4$	0	0	$1/4$	0	4
x_7	0	1	0	0	0	0	1	5
z	0	10	-5	0	0	15	0	240

Example (continued)

- Step 2: There are negative reduced costs, so we continue.
- Step 3: We select the non-basic variable with the most negative reduced cost x_3 as the new basic variable.
- Step 4: We conduct the ratio test on the column of the new basic variable. Row 2 yields the minimum ratio so we take out the basic variable x_5 associated with row 2.
- Step 5: We perform the pivot operation on $(i = 2, j = 3)$.

Old Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_4	0	0	-1	1	0	-2	0	16
x_5	0	-2	**1**	0	1	-3	0	12
x_1	1	$3/4$	$1/4$	0	0	$1/4$	0	4
x_7	0	1	0	0	0	0	1	5
z	0	10	-5	0	0	15	0	240

After the pivot operation on $(i = 2, j = 3)$, we obtain:

New Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_4	0	-2	0	1	1	-5	0	28
x_3	0	-2	1	0	1	-3	0	12
x_1	1	$5/4$	0	0	$-1/4$	1	0	1
x_7	0	1	0	0	0	0	1	5
z	0	0	0	0	5	0	0	300

- Step 2: All the reduced costs are nonnegative. Thus, we Stop! The current solution is optimal.
- Report: The optimal solution is

$$x^* = (1, 0, 12, 28, 0, 0, 5)$$

and the optimal value of the objective function is equal to

$$z* = 300$$

- Don't forget to check the results!

§8 – Solution Possibilities

We saw before that linear programming problems can have

- multiple optimal solutions,
- an unbounded objective function, or
- an empty feasible set.

We want to know how to recognise these situations when using the Simplex Algorithm.

§8.1 – Multiple Optimal Solutions

If a non-basic variable x_j in the final simplex tableau has a zero reduced cost, then the corresponding linear programming problem has multiple optimal solutions.

This follows because we can pivot on the column corresponding to x_j, thus bringing it into the basis and removing one of the variables currently in the basis without changing the value of the objective function.

If there are two optimal solutions, then there must be infinitely many optimal solutions. Why?

Example

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	1	0	-2	10
x_4	0	0	-1	1	1	5
x_1	1	0	0	0	1	15
z	0	0	2	0	0	80

The current feasible solution $x = (15, 10, 0, 5, 0)$ is optimal.
The non-basic variable x_5 has a reduced cost equal to zero.
Pivoting on the $(i = 2, j = 5)$-entry, for example, we obtain:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	5
x_1	1	0	1	-1	0	10
z	0	0	2	0	0	80

So $x = (10, 20, 0, 0, 5)$ is another optimal solution.
Any point on the line segment between $(15, 10, 0, 5, 0)$ and $(10, 20, 0, 0, 5)$ is an optimal solution.

§8.2 – Unbounded Problems (No Optimal Solution)

This refers to the situation when the objective function can take arbitrarily large value (for a maximisation problem) or arbitrarily small value (for a minimisation problem) in the feasible region. Therefore no optimal solution exists.

In terms of the simplex tableau, this case occurs when the value of the new basic variable can be increased indefinitely without causing any one of the old basic variables to become negative.

In other words, this occurs when the column of the new basic variable consists of non-positive elements only, that is, when the Ratio Test fails to identify a variable to be taken out of the basis.

Example

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_5	0	-6	0	1	1	25
x_1	1	-2	0	6	0	40
x_3	0	0	1	1	0	10
z	0	-3	0	2	0	80

According to the Greedy Rule, x_2 is the new basic variable.

If we conduct the Ratio Test on the x_2 column we fail to find a variable to be taken out of the basis because there is no positive element in this column.

This means that x_2 can be increased indefinitely. Thus the problem is unbounded: it has no optimal solution.

§8.3 – No Feasible Solutions

Some linear programming problems do not have feasible solutions (that is, the feasible region is empty, or equivalently the problem is infeasible).

How does this show in the simplex tableau?
This is a very important issue. However, problems in standard form always have feasible solutions, so the appropriate place to recognise when there are no feasible solutions is in the conversion of a problem to standard form. We shall discuss this in the next section.

§8.4 – Cycling and Bland's Rule

Question: Is it possible that the simplex procedure we described will never stop?

Answer: Yes!

Reason: If there is a change in the basis but not in the value of the objective function, that is, a basic variable whose value is zero leaves the basis, we can cycle indefinitely between the two solutions.

Thus cycling is caused by degenerate basic feasible solutions. They are basic feasible solutions in which at least one basic variable has a value of zero.

Example

Table 1

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	$1/2$	$-11/2$	$-5/2$	9	1	0	0	0
x_6	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
z	-10	57	9	24	0	0	0	0

Example (continued)

Table 2

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_1	1	-11	-5	18	2	0	0	0
x_6	0	4	2	-8	-1	1	0	0
x_7	0	11	5	-18	-2	0	1	1
z	0	-53	-41	204	20	0	0	0

Example (continued)

Table 3

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_1	1	0	$1/2$	-4	$-3/4$	$11/4$	0	0
x_2	0	1	$1/2$	-2	$-1/4$	$1/4$	0	0
x_7	0	0	$-1/2$	4	$3/4$	$-11/4$	1	1
z	0	0	$-29/2$	98	$27/4$	$53/4$	0	0

Example (continued)

Table 4

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	2	0	1	-8	$-3/2$	$11/2$	0	0
x_2	-1	1	0	2	$1/2$	$-5/2$	0	0
x_7	1	0	0	0	0	0	1	1
z	29	0	0	-18	-15	93	0	0

Example (continued)

Table 5

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	-2	4	1	0	$1/2$	$-9/2$	0	0
x_4	$-1/2$	$1/2$	0	1	$1/4$	$-5/4$	0	0
x_7	1	0	0	0	0	0	1	1
z	20	9	0	0	$-21/2$	$141/2$	0	0

Example (continued)

Table 6

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	-4	8	2	0	1	-9	0	0
x_4	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
z	-22	93	21	0	0	-24	0	0

Example (continued)

Table 7 = Table 1

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	$1/2$	$-11/2$	$-5/2$	9	1	0	0	0
x_6	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
z	-10	57	9	24	0	0	0	0

Table 7 = Table 1 $\Rightarrow$ Table 8 = Table 2 $\Rightarrow$ Table 9 = Table 3, ...

In practice, cycling is not a major issue because it occurs rarely. It is, however, of theoretical interest.

The following simple rule can be used to prevent cycling.

Bland's Rule (Smallest Subscript Rule)

- Among all variables with negative reduced costs, choose the variable with the smallest subscript as the entering variable (i.e. choose the leftmost negative reduced cost).
- In using the Ratio Test, if two or more variables compete for leaving the basis (i.e. with the smallest ratio), choose the variable with the smallest subscript as the leaving variable.

Theorem 8.1

When applying Bland's Rule the Simplex Algorithm terminates in a finite number of iterations.

In this course you are not required to follow Bland's rule.

Example (continued, but using Bland's Rule)

Table 6

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	-4	8	2	0	1	-9	0	0
x_4	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
z	-22	93	21	0	0	-24	0	0

Example (continued, but using Bland's Rule)

Table 7'

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	0	-4	-2	8	1	-1	0	0
x_1	1	-3	-1	2	0	2	0	0
x_7	0	3	1	-2	0	-2	1	1
z	0	27	-1	44	0	20	0	0

Example (continued, but using Bland's Rule)

Table 8'

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	0	2	0	4	1	-5	2	2
x_1	1	0	0	0	0	0	1	1
x_3	0	3	1	-2	0	-2	1	1
z	0	30	0	42	0	18	1	1

Success!

§9 – Non-Standard Formulations

Recall that the Simplex Algorithm relies very much on the Canonical Form, which, in turn, requires the correct Standard Form, that is,

- the optimisation criterion is $opt = \max$,
- all RHS coefficients are nonnegative,
- all functional constraints are "$\leq$" type inequalities (which ensures m slack variables in the initial tableau and so the initial bfs), and
- all variables are nonnegative.

Not all programs can be written in Standard Form. So extra steps are needed to convert such programs to the Canonical Form.

Standard Form

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & \leq & b_m \end{array}$$

$$x_j \geq 0, \quad j = 1, \ldots, n$$

where all $b_i \geq 0$.

Non-Standard Formulations

There are 5 scenarios we must deal with:

- we have a minimisation problem;
- a variable is unrestricted in sign (urs);
- some part of $b = (b_1, \ldots, b_m)$ is negative;
- we have a "$\geq$" constraint;
- we have a "$=$" constraint.

An Example

$$\min z = x_1 - x_2$$

$$\begin{array}{rcl} x_1 + x_2 & \leq & -5 \\ x_1 & \geq & -10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Here "$x_2 \in \mathbb{R}$" indicates that x_2 is unrestricted in sign.

§9.1 – Minimisation Problems

There are two ways to deal with minimisation problems:

1. Convert the problem to a maximisation problem by multiplying the objective function by -1.
2. Change the Greedy Rule and the Optimality Criterion a tiny bit.

Minimisation Problems – Method 1

Observe that the problem

$$\min_x f(x)$$

is equivalent to the problem

$$\max_x -f(x)$$

in that both have the same set of optimal solutions.

Also, if the minimum value of $f(x)$ is z_{min} and the maximum value of $-f(x)$ is z_{max}, then $z_{min} = -z_{max}$.

So, an easy way to deal with a minimisation problem, is to multiply the coefficients of the objective function by -1 and maximize the new objective function.

However, you have to remember that the solution you obtain is the negative of the solution you want!!!

Example – Method 1

$$\min z = x_1 - x_2$$

$$\begin{array}{rcl} x_1 + x_2 & \leq & 5 \\ x_1 & \leq & 10 \\ x_2 & \leq & 3 \\ \\ x_1, x_2 & \geq & 0 \end{array}$$

Example – Method 1

For method 1, we multiply the objective function by -1 to obtain a maximisation problem:

$$\max \hat{z} = -x_1 + x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_1 \qquad &\leq 10 \\ x_2 &\leq 3 \\ \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example – Method 1

Initial tableau for the maximisation problem:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	1	1	0	0	5
x_4	1	0	0	1	0	10
x_5	0	1	0	0	1	3
$\hat{z}$	1	-1	0	0	0	0

The entering variable is x_2, and the leaving variable is x_5.

Example – Method 1

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	0	1	0	-1	2
x_4	1	0	0	1	0	10
x_2	0	1	0	0	1	3
$\hat{z}$	1	0	0	0	1	3

After one pivot we see that there are no negative reduced costs. Therefore the optimality criterion is satisfied.

Example – Method 1

The optimal solution is

$$x = (0, 3, 5, 10, 0).$$

The optimal solution for the original problem (ignoring the slack variables) is

$$(x_1, x_2) = (0, 3).$$

The optimal value of the modified objective function is

$$\hat{z} = 3.$$

Thus the optimal value of the original objective function is

$$z = -3.$$

Minimisation Problems – Method 2

Another option is to work on the minimisation problem but change the Simplex Algorithm a bit. This is not hard to do – we just have to change our approach to the reduced costs.
For a minimisation LP problem,

- the Optimality Criterion is that we stop if all the reduced costs are nonpositive,
- the Greedy Rule is that we choose the column with the most positive reduced cost, and
- the Ratio Test remains the same.

Example – Method 2

$$\min z = x_1 - x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_1 \quad &\leq 10 \\ x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

		$\Downarrow$				
BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	1	1	0	0	5
x_4	1	0	0	1	0	10
x_5	0	1	0	0	1	3
z	-1	1	0	0	0	0

We select x_2 to enter the basis and x_5 to leave. Will the choices in pivots always be the same for methods 1 and 2?

Example – Method 2

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	0	1	0	-1	2
x_4	1	0	0	1	0	10
x_2	0	1	0	0	1	3
z	-1	0	0	0	-1	-3

We can see that there are no nonnegative reduced costs, so the optimality criterion is satisfied. The optimal solution is $(0, 3, 2, 10, 0)$, with optimal value $z^* = -3$.

The optimal solution for the problem, ignoring the slack variables, is

$$(x_1, x_2) = (0, 3).$$

§9.2 – Negative RHS

We convert a negative RHS to a non-negative RHS by multiplying the respective constraint by -1.

Remember when doing so we have to reverse the inequality sign. That is, change $\leq$ to $\geq$ and vice versa.

Example (revisited)

$$\min z = x_1 - x_2$$

$$\begin{array}{rcl} x_1 + x_2 & \leq & -5 \\ x_1 & \geq & -10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Example (continued)

Convert to a maximisation problem.

And then make the RHS coefficients nonnegative.

$$\max \hat{z} = -x_1 + x_2$$

$$\begin{array}{rcl} -x_1 - x_2 & \geq & 5 \\ -x_1 & \leq & 10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Observe that in fixing the negative RHS we have created another problem! The first constraint is of type "$\geq$".

§9.3 – Unrestricted Variables

What do we do when some variable x_j is unrestricted in sign? We address this by using the fact that any number (positive or negative) can be expressed as the difference of two positive numbers.

Thus, if x_j is unrestricted in sign, we can introduce two additional variables, say $x_j^{(1)}$ and $x_j^{(2)}$, and set $x_j = x_j^{(1)} - x_j^{(2)}$ with $x_j^{(1)}, x_j^{(2)} \geq 0$.

Clearly, if $x_j^{(1)} > x_j^{(2)}$ then $x_j > 0$; if $x_j^{(1)} < x_j^{(2)}$ then $x_j < 0$; and if $x_j^{(1)} = x_j^{(2)}$ then $x_j = 0$.

Thus, x_j is indeed unrestricted in sign (which we write as $x_j \in \mathbb{R}$).

Example (continued)

Let $x_2 = x_3 - x_4$, where $x_3, x_4 \geq 0$. Then

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl} -x_1 - x_3 + x_4 & \geq & 5 \\ -x_1 & \leq & 10 \\ x_3 - x_4 & = & 3 \\ \\ x_1, x_3, x_4 & \geq & 0 \end{array}$$

§9.4 – $\geq$ Constraints

When we convert from standard form to canonical form we introduce slack variables to get from inequality ($\leq$) to equality ($=$). We use a similar idea for the $\geq$ constraints.
We convert a "$\geq$" constraint to an "$=$" constraint by introducing a *surplus variable*. These are similar to slack variables except they are subtracted at the beginning.

Example (continued)

The constraint

$$-x_1 - x_3 + x_4 \geq 5$$

can be written as

$$-x_1 - x_3 + x_4 - x_5 = 5$$

where

$$x_5 \geq 0$$

is the surplus variable.

Now we obtain:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl} -x_1 - x_3 + x_4 - x_5 & = & 5 \\ -x_1 & \leq & 10 \\ x_3 - x_4 & = & 3 \end{array}$$

§9.5 – Equality Constraints

When we have equality constraints the strategy that we will adopt involves adding yet more variables to our program: artificial variables.

We call the new variables 'artificial' because they are used temporarily, and ultimately will become zero if our original program is feasible.

Example (continued)

$$-x_1 - x_3 + x_4 - x_5 = 5$$

can be rewritten as

$$-x_1 - x_3 + x_4 - x_5 + y_1 = 5$$

where $y_1 \geq 0$ is an artificial variable.

- The second equation is equivalent to the first if and only if y_1 is equal to zero.
- We try to force this to happen by getting the artificial variable out of the basis.

Similarly, by introducing an artificial variable $y_2 \geq 0$, $x_3 - x_4 = 3$ can be written as

$$x_3 - x_4 + y_2 = 3$$

Example (continued)

We now have:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl}
-x_1 - x_3 + x_4 - x_5 + y_1 & = & 5 \\
-x_1 & \leq & 10 \\
x_3 - x_4 + y_2 & = & 3 \\
x_1, x_3, x_4, x_5, y_1, y_2 & \geq & 0
\end{array}$$

Example (continued)

As usual, for a "$\leq$" type constraint, we introduce a slack variable.

Introducing a slack variable x_6 for the second constraint, we obtain:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl}
-x_1 - x_3 + x_4 - x_5 + y_1 & = & 5 \\
-x_1 + x_6 & = & 10 \\
x_3 - x_4 + y_2 & = & 3 \\
x_1, x_3, x_4, x_5, x_6, y_1, y_2 & \geq & 0
\end{array}$$

We are now in canonical form with basic variables y_1, x_6, y_2.

§9.6 – Summary: procedure for transforming any LP problem to canonical form

1. For a minimisation problem, convert it to a maximisation problem by multiplying the objective function by -1
2. For each negative RHS coefficient $b_i < 0$, multiply the corresponding functional constraint by -1 (and change the direction of the inequality)
3. For each variable x_j which is unrestricted in sign, introduce new variables $x_j^{(1)}, x_j^{(2)} \geq 0$ and replace x_j by $x_j^{(1)} - x_j^{(2)}$
4. For each "$\geq$" functional constraint, introduce a surplus variable so as to obtain a "$=$" constraint
5. For each "$=$" functional constraint, introduce an artificial variable
6. For each "$\leq$" functional constraint, introduce a slack variable
7. Now the problem is in canonical form whose basic variables are the slack and artificial variables

Example (top to toe)

Original problem:

$$\min z = x_1 - x_2$$

$$\begin{array}{rcl} x_1 + x_2 & \leq & -5 \\ x_1 & \geq & -10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Example (top to toe)

Convert to a maximisation problem:

$$\max \hat{z} = -x_1 + x_2$$

$$\begin{array}{rcl} x_1 + x_2 & \leq & -5 \\ x_1 & \geq & -10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Example (top to toe)

Make all RHS coefficients nonnegative:

$$\max \hat{z} = -x_1 + x_2$$

$$\begin{array}{rcl} -x_1 - x_2 & \geq & 5 \\ -x_1 & \leq & 10 \\ x_2 & = & 3 \end{array}$$

$$\begin{array}{rcl} x_1 & \geq & 0, \\ x_2 & \in & \mathbb{R} \end{array}$$

Example (top to toe)

Introduce two variables $x_3, x_4 \geq 0$ and replace the urs variable x_2 by $x_3 - x_4$:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl} -x_1 - x_3 + x_4 & \geq & 5 \\ -x_1 & \leq & 10 \\ x_3 - x_4 & = & 3 \\ \\ x_1, x_3, x_4 & \geq & 0 \end{array}$$

Example (top to toe)

Introduce the surplus variable x_5 for the first functional constraint:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl} -x_1 - x_3 + x_4 - x_5 & = & 5 \\ -x_1 & \leq & 10 \\ x_3 - x_4 & = & 3 \\ \\ x_1, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Example (top to toe)

Introduce the artificial variables y_1, y_2 for the two "$=$" constraints:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl}
-x_1 - x_3 + x_4 - x_5 + y_1 & = & 5 \\
-x_1 & \leq & 10 \\
x_3 - x_4 + y_2 & = & 3 \\
x_1, x_3, x_4, x_5, y_1, y_2 & \geq & 0
\end{array}$$

Example (top to toe)

Introduce the slack variable x_6 for the "$\leq$" constraint:

$$\max \hat{z} = -x_1 + x_3 - x_4$$

$$\begin{array}{rcl}
-x_1 - x_3 + x_4 - x_5 \qquad + y_1 & = & 5 \\
-x_1 \qquad\qquad\qquad + x_6 & = & 10 \\
x_3 - x_4 \qquad\qquad\qquad + y_2 & = & 3 \\
\\
x_1, x_3, x_4, x_5, x_6, y_1, y_2 & \geq & 0
\end{array}$$

We are now in canonical form with basic variables y_1, x_6, y_2.

Our (initial) basic feasible solution here is

$$(x_1, x_3, x_4, x_5, x_6, y_1, y_2) = (0, 0, 0, 0, 10, 5, 3)$$

Example (top to toe)

Observe that the initial basis y_1, x_6, y_2 consists of:

- slack variables that arise from 'less than or equal to constraints', and
- artificial variables that arise from 'greater than or equal to constraints' or equality constraints.

Ignoring the values of the slack, surplus and artificial variables in $(0, 0, 0, 0, 10, 5, 3)$, we obtain $(x_1, x_3, x_4) = (0, 0, 0)$.

But is this a basic feasible solution to the original problem? No!

Why?

Initialization Revisited

A solution to the transformed system gives a feasible solution to the original system if and only if all artificial variables are equal to zero.

How can we ensure that the artificial variable stays out of the basis?

Two methods can be used to achieve this:

- The two-phase Simplex Algorithm
- The Big M method (which is essentially equivalent, but not covered in this subject)

§9.7 – The Two-phase Method

The two-phase method consists of

- Phase 1, which drives out the artificial variables by finding a basic feasible solution for which the artificial variables are non-basic and have value zero, and
- Phase 2, which starts from this basic feasible solution (ignoring the artificial variables) and produces an optimal solution.

Phase 1

Let

$$w = \text{sum of the artificial variables}$$

and

$$w^* = \text{minimum value of } w \text{ subject to the constraints.}$$

Because the artificial variables must satisfy the nonnegativity constraint, $w^* = 0$ if and only if all the artificial variables are equal to zero. Thus,

- the goal in Phase 1 is to minimize w.

Example

$$\max z = -3x_1 - 5x_2$$

$$\begin{array}{rcl}
x_1 \qquad\qquad + x_4 \qquad\qquad & = & 4 \\
2x_2 \qquad\qquad + y_1 \qquad & = & 12 \\
3x_1 + 2x_2 - x_3 \qquad\qquad + y_2 & = & 18 \\
x_1, x_2, x_3, x_4, y_1, y_2 & \geq & 0
\end{array}$$

where y_1 and y_2 are artificial variables.

Phase 1

$$\min w = y_1 + y_2$$

$$\begin{array}{rcl}
x_1 + x_4 & = & 4 \\
2x_2 + y_1 & = & 12 \\
3x_1 + 2x_2 - x_3 + y_2 & = & 18 \\
x_1, x_2, x_3, x_4, y_1, y_2 & \geq & 0
\end{array}$$

We use the 'minimise' version of the simplex method to achieve this.

BV	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	1	0	0	1	0	0	4
y_1	0	2	0	0	1	0	12
y_2	3	2	-1	0	0	1	18
w	0	0	0	0	-1	-1	0

Note that this tableau is not in canonical form, because there are non-zero coefficients in the w-row corresponding to the basic variables. To restore the canonical form, we add Row 2 and Row 3 to the w-row.

New Tableau in Canonical Form

BV	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	1	0	0	1	0	0	4
y_1	0	**2**	0	0	1	0	12
y_2	3	2	-1	0	0	1	18
w	3	4	-1	0	0	0	30

Tableau after pivoting on $(i = 2, j = 2)$

BV	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	1	0	0	1	0	0	4
x_2	0	1	0	0	$1/2$	0	6
y_2	**3**	0	-1	0	-1	1	6
w	3	0	-1	0	-2	0	6

Tableau after pivoting on $(i = 3, j = 1)$

BV	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	0	0	$1/3$	1	$1/3$	$-1/3$	2
x_2	0	1	0	0	$1/2$	0	6
x_1	1	0	$-1/3$	0	$-1/3$	$1/3$	2
w	0	0	0	0	-1	-1	0

- All the artificial variables are out of the basis.
- The minimum value of w is $w^* = 0$.
- This is the end of Phase 1.
- Note that we now have a basic feasible solution for the original problem, obtained by ignoring the y_1 and y_2-columns.

Phase 2

We now put back the original objective function

$$z = -3x_1 - 5x_2$$

Tableau for Phase 2.

BV	x_1	x_2	x_3	x_4	RHS
x_4	0	0	$1/3$	1	2
x_2	0	1	0	0	6
x_1	1	0	$-1/3$	0	2
z	3	5	0	0	0

Note that this tableau is not in canonical form.
To restore canonical form we add $(-3)\times$ Row 3 and $(-5)\times$ Row 2 to the z-row.

After these row operations we obtain:

Tableau for Phase 2 in Canonial Form

BV	x_1	x_2	x_3	x_4	RHS
x_4	0	0	$1/3$	1	2
x_2	0	1	0	0	6
x_1	1	0	$-1/3$	0	2
z	0	0	1	0	-36

This tableau satisfies the Optimality Criterion. So the corresponding solution is already optimal.

The optimal solution is

$$x = (2, 6, 0, 2)$$

and the optimal value of the objective function is

$$z = -36.$$

The example above is atypical in the sense that no iteration is needed in Phase 2.

In general, if the initial tableau (in canonical form) for Phase 2 does not meet the Optimality Criterion, then we need to run the Simplex Algorithm beginning with this tableau until we find an optimal solution for the original problem or discover that the problem is unbounded.

Another example

$$\max z = 80x_1 + 60x_2 + 42x_3$$

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 12 \\ 5x_1 + 6x_2 + 3x_3 &\geq 15 \\ 2x_1 - 3x_2 + x_3 &= 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Another example (continued)

This is equivalent to

$$\max z = 80x_1 + 60x_2 + 42x_3$$

$$\begin{array}{rcl}
2x_1 + 3x_2 + x_3 + x_4 & = & 12 \\
5x_1 + 6x_2 + 3x_3 \quad - x_5 + y_1 & = & 15 \\
2x_1 - 3x_2 + x_3 \quad + y_2 & = & 8 \\
x_1, x_2, x_3, x_4, x_5, y_1, y_2 & \geq & 0
\end{array}$$

where x_4 is a slack variable, x_5 is a surplus variable and y_1 and y_2 are artificial variables.

Another example (continued)

Phase 1

$$\min w = y_1 + y_2$$

$$\begin{array}{rcl}
2x_1 + 3x_2 + x_3 + x_4 & = & 12 \\
5x_1 + 6x_2 + 3x_3 \quad - x_5 + y_1 & = & 15 \\
2x_1 - 3x_2 + x_3 \quad + y_2 & = & 8 \\
x_1, x_2, x_3, x_4, x_5, y_1, y_2 & \geq & 0
\end{array}$$

Another example (continued)

Phase 1

Initial tableau (not in canonical form)

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
x_4	2	3	1	1	0	0	0	12
y_1	5	6	3	0	-1	1	0	15
y_2	2	-3	1	0	0	0	1	8
w	0	0	0	0	0	-1	-1	0

Restore the canonical form by adding Rows 2 and 3 to the w-row.

Another example (continued)

Phase 1

Initial tableau in canonical form

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
x_4	2	3	1	1	0	0	0	12
y_1	5	6	3	0	-1	1	0	15
y_2	2	-3	1	0	0	0	1	8
w	7	3	4	0	-1	0	0	23

Another example (continued)

Phase 1

Tableau after pivoting on $(i = 2, j = 1)$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
x_4	0	$3/5$	$-1/5$	1	$2/5$	$-2/5$	0	6
x_1	1	$6/5$	$3/5$	0	$-1/5$	$1/5$	0	3
y_2	0	$-27/5$	$-1/5$	0	$2/5$	$-2/5$	1	2
w	0	$-27/5$	$-1/5$	0	$2/5$	$-7/5$	0	2

Another example (continued)

Phase 1

Tableau after pivoting on $(i = 3, j = 5)$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
x_4	0	6	0	1	0	0	-1	4
x_1	1	$-3/2$	$1/2$	0	0	0	$1/2$	4
x_5	0	$-27/2$	$-1/2$	0	1	-1	$5/2$	5
w	0	0	0	0	0	-1	-1	0

The optimal value is $w^* = 0$, and the artificial variables y_1 and y_2 have been driven out of the basis.

This is the end of Phase 1.

Another example (continued)

Phase 2

Initial tableau for Phase 2 (not in canonical form)

	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	6	0	1	0	4
x_1	1	$-3/2$	$1/2$	0	0	4
x_5	0	$-27/2$	$-1/2$	0	1	5
z	-80	-60	-42	0	0	0

Restore the canonical form by adding 80 times Row 2 to the z-row.

Another example (continued)

Phase 2

Initial tableau for Phase 2 in canonical form

	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	6	0	1	0	4
x_1	1	$-3/2$	$1/2$	0	0	4
x_5	0	$-27/2$	$-1/2$	0	1	5
z	0	-180	-2	0	0	320

Another example (continued)

Phase 2

Tableau after pivoting on $(i = 1, j = 2)$

	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	$1/6$	0	$2/3$
x_1	1	0	$1/2$	$1/4$	0	5
x_5	0	0	$-1/2$	$9/4$	1	14
z	0	0	-2	30	0	440

Another example (continued)

Phase 2

Tableau after pivoting on $(i = 2, j = 3)$

	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	1/6	0	2/3
x_3	2	0	1	1/2	0	10
x_5	1	0	0	5/2	1	19
z	4	0	0	31	0	460

This is the end of Phase 2.

Report:
The optimal solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 2/3, 10, 0, 19)$
The optimal value of the objective function is $z^* = 460$.

The optimal solution to the original problem is
$(x_1, x_2, x_3) = (0, 2/3, 10)$ with optimal value $z^* = 460$.

Complications of the two-phase method

Let w^* be the optimal value of w obtained in Phase 1.

- Case 1: If $w^* = 0$ and all the artificial variables are non-basic, then we have a basic feasible solution to the original problem. Continue with Phase 2.
- Case 2: If $w^* = 0$ but at least one artificial variable is in the basis, then there are two possible cases
  - Case 2a: we can use pivot operations to take all artificial variables out of the basis and then continue with Phase 2.
  - Case 2b: the constraint corresponding to the zero artificial variable is redundant.
- Case 3: If $w^* > 0$, the problem is infeasible (that is, the feasible region is empty).

Case 2a: An example

Suppose we get the following tableau in Phase 1, where y_1 and y_2 are artificial variables.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	0	11	6	1	0	0	2
y_1	0	$-1/2$	-1	0	1	-1	0
x_1	1	-1	-1	0	0	1	3
w	0	$1/2$	1	0	0	2	0

We have $w^* = 0$ but y_1 remains as a basic variable. Note that there exist nonzero entries in the y_1-row and the columns of non-artificial variables, i.e. $-1/2$ and -1. So we can pivot on one of these entries to drive y_1 out of the basis.

Case 2a: An example (continued)

Pivoting on $-1/2$ yields

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_4	0	0	-16	1	22	-22	2
x_2	0	1	2	0	-2	2	0
x_1	1	0	1	0	-2	3	3
w	0	0	0	0	1	1	0

(You can choose to pivot on -1 in the previous tableau.)

Now we have $w^* = 0$ and all artificial variables are non-basic. So we can proceed to Phase 2 just as in Case 1.

Case 2b: An example

Suppose we get the following tableau in Phase 1, where y_1, y_2 and y_3 are artificial variables.

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	RHS
x_2	0	1	-2	2	3	0	-1	4
y_2	0	0	0	0	1	1	-1	0
x_1	1	0	6	-10	-11	0	5	28
w	0	0	0	0	1	0	3	0

We have $w^* = 0$ but y_2 remains as a basic variable. Unlike the previous example, all entries in the y_2-row and the columns of non-artificial variables are zero. So we cannot pivot on such entries to drive y_2 out of the basis.

The constraint containing y_2 is redundant. So we can ignore the y_2-row and continue with Phase 2.

§9.8 – Summary: procedure for solving LP problems by using the Simplex Algorithm and the Two-phase Method

1. If the given LP problem is not in canonical form, convert it to canonical form by following the "procedure for transforming any LP problem to canonical form" (you may need to introduce slack variables, surplus variables and/or artificial variables).
2. If no artificial variable is introduced in the transformation to canonical form, apply the Simplex Algorithm to the canonical form directly.
3. If artificial variables are introduced, apply the two-phase Method.

§10 – Duality Theory

Duality is a very elegant and important concept within the field of operations research, just as in many other branches of mathematics.

In operations research, the theory of duality was first developed in relation to linear programming, but it has many applications, and perhaps even a more natural and intuitive interpretation, in several related areas such as nonlinear programming, network optimisation and game theory.

Every linear program has associated with it a related linear program called its *dual*. When we are talking about the dual we call the original problem the *primal*.

§10.1 – Motivating Examples

Example 10.1

A small company is being set up to engage in the production of office furniture. The company proposes to manufacture tables, desks and chairs. The production of a table requires 8 kgs of wood and 5 kgs of metal and is sold for \$80, a desk uses 6 kgs of wood and 4 kgs of metal and is sold for \$60, and a chair requires 4 kgs of both metal and wood and is sold for \$50.

The company CEO has engaged you to determine the best production strategy for this company, given that only 100 kgs of wood and 60 kgs of metal are available each week.

Example (continued)

Let x_1 be the number of tables manufactured each week, x_2 be the number of desks manufactured each week and x_3 be the number of chairs manufactured each week. Note that x_1, x_2 and x_3 have to be nonnegative.

Assuming we can immediately sell the furniture we produce, the weekly income in dollars is $80x_1 + 60x_2 + 50x_3$.

The values of x_1, x_2 and x_3 are constrained by the availability of wood and metal.

The number of kgs of wood used is $8x_1 + 6x_2 + 4x_3$. This has to be less than or equal to 100. The number of kgs of metal used is $5x_1 + 4x_2 + 4x_3$. This has to be less than or equal to 60.

Example (continued)

This leads to the linear program

$$\max_x z = 80x_1 + 60x_2 + 50x_3$$

subject to

$$\begin{aligned} 8x_1 + 6x_2 + 4x_3 &\leq 100 \\ 5x_1 + 4x_2 + 4x_3 &\leq 60 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example (continued)

Now let's extend the situation further. Assume that there is a much bigger company which has been the lone producer of this type of furniture for many years. Its management does not appreciate the competition from the new company. This bigger company has decided to attempt to put the new company out of business by buying up the new company's resources of wood and metal.

The problem for the large company is to decide on the prices that it should offer for the wood and metal.

How should the company make this decision?

Example (continued)

Let y_1 be the price, in dollars, offered for a kg of wood and y_2 be the price, in dollars, offered for a kg of metal. Clearly y_1 and y_2 have to be nonnegative.

Then the total expense of the buyout is $100y_1 + 60y_2$. The company obviously wants to minimise this.

What are the constraints on y_1 and y_2?

Example (continued)

It takes 8 kgs of wood and 5 kgs of metal to make a table. It would cost the large company $8y_1 + 5y_2$ to buy this 8 kgs of wood and 5 kgs of metal.

On the other hand, the small company makes $80 from selling the table. If $8y_1 + 5y_2$ is less than this amount, then the small company has the potential to come back to the resource supplier with a better offer, even if it manufactures only tables. Thus we need $8y_1 + 5y_2 \geq 80$.

Similarly we have $6y_1 + 4y_2 \geq 60$ and $4y_1 + 4y_2 \geq 50$.

Example (continued)

This leads to

The Dual Problem

$$\min_y w = 100y_1 + 60y_2$$

subject to

$$\begin{aligned} 8y_1 + 5y_2 &\geq 80 \\ 6y_1 + 4y_2 &\geq 60 \\ 4y_1 + 4y_2 &\geq 50 \\ y_1, y_2 &\geq 0. \end{aligned}$$

Example 10.2

An individual has a choice of two types of food to eat, meat and potatoes, each offering varying degrees of nutritional benefit. He has been warned by his doctor that he must receive at least 400 units of protein, 200 units of carbohydrates and 100 units of fat from his daily diet.

Given that a kg of steak costs \$10 and provides 80 units of protein, 20 units of carbohydrates and 30 units of fat, and that a kg of potatoes costs \$2 and provides 40 units of protein, 50 units of carbohydrates and 20 units of fat, he would like to find the minimum cost diet which satisfies his nutritional requirements.

Example (continued)

Let x_1 be the number of kgs of steak that the man buys and x_2 be the number of kgs of potatoes that he buys. These have to be nonnegative.

The man wants to minimise $10x_1 + 2x_2$, subject to

$$80x_1 + 40x_2 \geq 400$$

$$20x_1 + 50x_2 \geq 200$$

and

$$30x_1 + 20x_2 \geq 100.$$

Example (continued)

The Primal Problem

$$\min_x z = 10x_1 + 2x_2$$

subject to

$$\begin{aligned} 80x_1 + 40x_2 &\geq 400 \\ 20x_1 + 50x_2 &\geq 200 \\ 30x_1 + 20x_2 &\geq 100 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Example (continued)

A chemical company hopes to attract this man away from his present diet by offering him synthetic nutrients in the form of pills. The problem for the company is to determine the prices per unit for their synthetic nutrients.

The company wants to generate the highest possible revenue from the transaction, but needs to keep the prices low enough so that the man can't satisfy his requirements from the natural foods in a cheaper way.

Example (continued)

Let y_1 be the price, in dollars, offered for a unit of protein, y_2 be the price, in dollars, offered for a unit of carbohydrate, and y_3 be the price, in dollars, offered for a unit of fat.

Then the man will have to pay $400y_1 + 200y_2 + 100y_3$ to satisfy his dietary requirements. The company wants to maximise this.

If $80y_1 + 20y_2 + 30y_3 > 10$, then the man could do better by buying just steak, and if $40y_1 + 50y_2 + 20y_3 > 2$, then the man could do better by buying just potatoes. Thus we need

$$80y_1 + 20y_2 + 30y_3 \leq 10$$

and

$$40y_1 + 50y_2 + 20y_3 \leq 2.$$

Example (continued)

This leads to

The Dual Problem

$$\max_{y} w = 400y_1 + 200y_2 + 100y_3$$

subject to

$$\begin{aligned} 80y_1 + 20y_2 + 30y_3 &\leq 10 \\ 40y_1 + 50y_2 + 20y_3 &\leq 2 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

Comments

Each of the two examples describes some kind of competition between two decision makers. The primal/dual relationship is often interpreted in the context of economics and game theory.

§10.2 – The Dual of a Linear Programming Problem in Standard Form

In this section we formalise our notions about the relationship between the primal and dual versions of linear programs.

We start by defining the relationship between a primal linear program in standard form and its dual.

A Primal Problem

$$\max_x z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & \leq & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & \leq & b_m \end{array}$$

$$x_1, x_2, ..., x_n \geq 0.$$

Note: we do not require all b_i to be nonnegative.

The Corresponding Dual Problem

$$\min_{y} w = \sum_{i=1}^{m} b_i y_i$$

$$\begin{array}{rcl} a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m & \geq & c_1 \\ a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m & \geq & c_2 \\ \vdots & \vdots & \vdots \\ a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m & \geq & c_n \end{array}$$

$$y_1, y_2, ..., y_m \geq 0.$$

It is convenient to express a linear program and its dual using matrix notation.

Primal LP	Dual LP
$\max_x z = cx$	$\min_y w = yb$
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

$$A = (a_{ij})_{m \times n}; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}; c = (c_1, \ldots, c_n); x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; y = (y_1, \ldots, y_m)$$

Once again, in this formulation we don't impose the restriction that b has to be nonnegative.

Example 10.3

The Primal Problem

$$\max_x z = 5x_1 + 3x_2 - 8x_3 + 12x_5$$

$$\begin{array}{rcl} 3x_1 - 8x_2 \qquad\quad + 9x_4 - 15x_5 & \leq & 20 \\ 18x_1 + 5x_2 - 8x_3 + 4x_4 + 12x_5 & \leq & 30 \end{array}$$

$$x_1, x_2, \ldots, x_5 \geq 0.$$

Example (continued)

The Dual Problem

$$\min_y w = 20y_1 + 30y_2$$

$$\begin{aligned} 3y_1 + 18y_2 &\geq 5 \\ -8y_1 + 5y_2 &\geq 3 \\ -8y_2 &\geq -8 \\ 9y_1 + 4y_2 &\geq 0 \\ -15y_1 + 12y_2 &\geq 12 \\ y_1, y_2 &\geq 0. \end{aligned}$$

Example 10.4

The Primal Problem

$$\max_x z = 4x_1 + 10x_2 - 9x_3$$

$$\begin{array}{rcl} 5x_1 - 18x_2 + 5x_3 & \leq & 15 \\ -8x_1 + 12x_2 - 8x_3 & \leq & 8 \\ 12x_1 - 4x_2 + 8x_3 & \leq & 10 \\ 2x_1 \qquad\quad - 5x_3 & \leq & 5 \end{array}$$

$$x_1, x_2, x_3 \geq 0.$$

Example (continued)

The Dual Problem

$$\min_x w = 15y_1 + 8y_2 + 10y_3 + 5y_4$$

$$\begin{array}{rcl} 5y_1 - 8y_2 + 12y_3 + 2y_4 & \geq & 4 \\ -18y_1 + 12y_2 - 4y_3 & \geq & 10 \\ 5y_1 - 8y_2 + 8y_3 - 5y_4 & \geq & -9 \end{array}$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

The Dual of the Dual

Let us think about constructing the dual of the dual of an LP problem in standard form. That is, what is the dual of

$$\min_y w = yb$$

subject to

$$yA \geq c$$

$$y \geq 0$$

To find the second dual, we first transform to equivalent non-standard form:

$$\max_y -w = -yb$$

$$-yA \leq -c$$

$$y \geq 0$$

i.e.

$$\max_y -w = -b^T y^T$$

$$-A^T y^T \leq -c^T$$

$$y^T \geq 0$$

Now we take this as the primal problem in standard form.

We can easily apply our rules for constructing the dual to obtain:

$$\min_x -z = -x^T c^T$$

$$-x^T A^T \geq -b^T$$

$$x^T \geq 0$$

$$\min_x -z = -cx$$

$$-Ax \geq -b$$

$$x \geq 0$$

Notice that this is the same as

$$\max_x z = cx$$

$$Ax \leq b$$

$$x \geq 0$$

We can easily apply our rules for constructing the dual to obtain:

$$\min_x -z = -x^T c^T$$

$$-x^T A^T \geq -b^T$$

$$x^T \geq 0$$

$$\min_x -z = -cx$$

$$-Ax \geq -b$$

$$x \geq 0$$

Notice that this is the same as

$$\max_x z = cx$$

$$Ax \leq b$$

$$x \geq 0$$

The dual of the dual is the primal.

§10.3 – The Dual of a Linear Programming Problem in Non-standard Form

What happens when the primal problem is not in standard form? That is, what happens if we have:

- $\geq$ constraints;
- equality constraints;
- a $\min$ objective and/or
- unrestricted variables?

There are some steps we need to take to get to the dual (although when you have practised this a bit you will be able to skim over the steps).

Our objective in this subsection is to 'standardise' the primal so that we can obtain the dual easily.

We are not interested in solving the primal or the dual at this stage.

The approach here is similar to the one that we used when we dealt with non-standard formulations in the context of the simplex method.

However, as we are not trying to solve the problem (at this stage) we are not interested in slack, surplus or artificial variables.

We also allow the RHS to be negative.

$\geq$ constraints

$\geq$ constraints are easy to deal with, as we allow the RHS to be negative (at this stage).
So we can turn these constraints into $\leq$ constraints simply by multiplying through by -1 (which flips the inequality).

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,$$

can be written as

$$-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i.$$

Equality Constraints

Equality constraints require a few extra steps. First we observe that every equality constraint can be expressed as the intersection of two inequality constraints.
So

$$\sum_{j=1}^{n} a_{ij}x_j = b_i$$

is rewritten as

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

and

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i.$$

Notice how we gained a "$\geq$" constraint.
This, in turn, can be rewritten as

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

and

$$-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i.$$

So we end up with two "$\leq$" constraints and we are happy.

Unrestricted Variables and $\min$ Objective

We replace the unrestricted variable by the difference of two non-negative variables.
For example, we replace $x_3 \in \mathbb{R}$ with $x_3 = x_3^{(1)} - x_3^{(2)}$ where $x_3^{(1)}, x_3^{(2)} \geq 0$.
We replace $\min f(x)$ with $\max -f(x)$.

Example 10.5

Primal (non-standard)

$$\max_x z = x_1 + x_2 + x_3$$

$$\begin{aligned} 2x_2 - x_3 &\geq 4 \\ x_1 - 3x_2 + 4x_3 &= 5 \\ x_1 - 2x_2 &\leq 3 \end{aligned}$$

$$x_1, x_2 \geq 0, x_3 \in \mathbb{R}$$

Example (continued)

Primal (equivalent non-standard form)

$$\max_x z = x_1 + x_2 + x_3^{(1)} - x_3^{(2)}$$

$$\begin{aligned}
-2x_2 + x_3^{(1)} - x_3^{(2)} &\leq -4 \\
x_1 - 3x_2 + 4x_3^{(1)} - 4x_3^{(2)} &\leq 5 \\
-x_1 + 3x_2 - 4x_3^{(1)} + 4x_3^{(2)} &\leq -5 \\
x_1 - 2x_2 &\leq 3
\end{aligned}$$

$$x_1, x_2, x_3^{(1)}, x_3^{(2)} \geq 0$$

Example (continued)

From here, to obtain the dual,

- we create m dual variables $y_1 \ldots, y_m$;
- we switch the objective function and RHS coefficients;
- the new objective is labelled w;
- the $\max$ becomes a $\min$;
- we transpose the A coefficient matrix to obtain n constraints;
- all the inequalities flip.

Example (continued)

Dual (equivalent non-standard)

$$\min_y w = -4y_1 + 5y_2^{(1)} - 5y_2^{(2)} + 3y_3$$

$$\begin{aligned} y_2^{(1)} - y_2^{(2)} + y_3 &\geq 1 \\ -2y_1 - 3y_2^{(1)} + 3y_2^{(2)} - 2y_3 &\geq 1 \\ y_1 + 4y_2^{(1)} - 4y_2^{(2)} &\geq 1 \\ -y_1 - 4y_2^{(1)} + 4y_2^{(2)} &\geq -1 \end{aligned}$$

$$y_1, y_2^{(1)}, y_2^{(2)}, y_3 \geq 0$$

Example (continued)

Now we can perform the *reverse* of some of the steps we have taken to get to pseudo-standard form to yield the non-standard dual form.

Dual (non-standard)

$$\min_y w = -4y_1 + 5y_2 + 3y_3$$

$$\begin{aligned} y_2 + y_3 &\geq 1 \\ -2y_1 - 3y_2 - 2y_3 &\geq 1 \\ y_1 + 4y_2 &= 1 \end{aligned}$$

$$y_1, y_3 \geq 0, y_2 \in \mathbb{R}$$

Streamlining the Conversion

- An *equality constraint* in the primal LP generates a dual variable that is *unrestricted in sign*.
- An *unrestricted* variable in the primal LP generates an *equality constraint* in the dual LP.

Primal LP	Dual LP
$\max$	$\min$
Constraint i	Variable i
$\leq$ form	$y_i \geq 0$
$=$ form	$y_i \in \mathbb{R}$
hline Variable j	Constraint j
$x_j \geq 0$	$\geq$ form
$x_j \in \mathbb{R}$	$=$ form
Costs	Resources
Resources	Costs

Primal-dual in Matrix Form

$\mathbf{a}^i$: ith row of $A = (a_{ij})_{m\times n}$, $i = 1, \ldots, m$.

$\mathbf{a}_j$: jth column of $A = (a_{ij})_{m\times n}$, $j = 1, \ldots, n$.

$c : 1 \times n;\ x : n \times 1;\ b : m \times 1;\ y : 1 \times m$

Primal	Dual
$\max_x z = cx$	$\min_y w = yb$
$\mathbf{a}^i x \le b_i,\ \ i \in M$	$y\mathbf{a}_j \ge c_j,\ \ j \in N$
$\mathbf{a}^i x = b_i,\ \ i \in \overline{M}$	$y\mathbf{a}_j = c_j,\ \ j \in \overline{N}$
$x_j \ge 0,\ \ j \in N$	$y_i \ge 0,\ \ i \in M$
$x_j \in \mathbb{R}\ ,\ \ j \in \overline{N}$	$y_i \in \mathbb{R}\ ,\ \ i \in \overline{M}$

$M, \overline{M}$: row-index sets which partition $\{1, \ldots, m\}$

$N, \overline{N}$: column-index sets which partition $\{1, \ldots, n\}$

The Dual of the Dual (General)

Theorem 10.1

Under the setting above, if a linear program (D) is the dual of another linear program (P), then the dual linear program of (D) is (P).

In other words, the dual of the dual is the primal.

Proof: Exercise.

Example 10.6

Please note that this example is an infeasible LP problem – we are using it purely to demonstrate the conversion from primal to dual.

Primal (non-standard)

$$\max_x z = 5x_1 + 4x_2$$

$$\begin{aligned} 3x_1 - 8x_2 &\geq -6 \\ x_1 + 6x_2 &= -5 \\ 8x_1 + 3x_2 &= 10 \end{aligned}$$

$$x_2 \geq 0, x_1 \in \mathbb{R}$$

Example (continued)

We first convert $\geq$ constraints to $\leq$ constraints.

Primal (equivalent non-standard form)

$$\max_x z = 5x_1 + 4x_2$$

$$\begin{aligned} -3x_1 + 8x_2 &\leq 6 \\ x_1 + 6x_2 &= -5 \\ 8x_1 + 3x_2 &= 10 \end{aligned}$$

$$x_2 \geq 0, x_1 \in \mathbb{R} \ .$$

Example (continued)

Dual (non-standard form)

$$\min_y w = 6y_1 - 5y_2 + 10y_3$$

$$\begin{aligned} -3y_1 + y_2 + 8y_3 &= 5 \\ 8y_1 + 6y_2 + 3y_3 &\geq 4 \end{aligned}$$

$$y_1 \geq 0, y_2, y_3 \in \mathbb{R}$$

§10.4 – The Duality Theorems

Next we look at some interesting and important relationships between the primal and the dual.
These relationships can provide us with information about our problem while we are trying to solve it.
They are also important for numerous applications of linear programming in many domains.

Weak Duality

Theorem 10.2 (Weak Duality Theorem)

Consider the following primal-dual pair:

Primal	*Dual*
$\max_x z = cx$	$\min_y w = yb$
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then

$$cx \leq yb.$$

Proof

Suppose x is a feasible solution to the primal problem and y is a feasible solution to the dual problem. Then

$$Ax \leq b, \quad x \geq 0$$

$$yA \geq c, \quad y \geq 0$$

Since $y \geq 0$, multiplying both sides of $Ax \leq b$ by y gives

$$yAx \leq yb.$$

Since $x \geq 0$, multiplying both sides of $yA \geq c$ by x yields

$$yAx \geq cx.$$

Combining these two inequalities, we obtain

$$cx \leq yb$$

as required. $\square$

Theorem 10.2 means that the value of the primal objective function at any primal feasible solution is bounded above by the value of the dual objective function at any dual feasible solution.

In particular,

- the optimal objective function value for the primal LP is bounded above by the objective value of any feasible solution of the dual LP;
- the optimal objective function value for the dual LP is bounded below by the objective value of any feasible solution of the primal LP.

Corollary 10.1

If the objective function of the primal is unbounded on the feasible region, then the dual has no feasible solutions.

Corollary 10.2

If the objective function of the dual is unbounded on the feasible region, then the primal has no feasible solutions.

It is possible that both the primal and the dual problems have no feasible solutions.

Corollary 10.3

Let x^* be a feasible solution to the primal and y^* be a feasible solution to the dual. If

$$cx^* = y^*b,$$

then x^* must be an optimal solution to the primal and y^* must be an optimal solution to the dual.

Proof:
From the Weak Duality Theorem we have, for any feasible solution x of the primal and any feasible solution y of the dual,

$$cx \leq yb.$$

So $cx \leq y^*b$, which means that $cx \leq cx^*$ and so x^* is optimal. Similarly $y^*b \leq yb$, and y^* is an optimal solution to the dual. $\square$

Strong Duality

Theorem 10.3 (Strong Duality Theorem)

Consider the following primal-dual pair:

Primal	*Dual*
$\max_x z = cx$	$\min_y w = yb$
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

If an optimal solution exists for either the primal or its dual, then an optimal solution exists for both and the corresponding optimal objective function values are equal, that is $z^* = w^*$.

The algebra of the Simplex Method

To prove the Strong Duality Theorem, we need to look at the algebra of the Simplex Algorithm in more detail. Consider the LP

$$\begin{aligned} \max z &= cx \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

where $A = (a_{ij})_{m \times n}$ has rank m and

$$c = (c_1, \ldots, c_n), \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

The algebra of the Simplex Method

Recall that a basic solution is obtained by choosing m independent columns of A to correspond to basic variables, setting the non-basic variables to zero, and solving the equation $Ax = b$.
Let A_B be the $m \times m$ submatrix of A consisting of the columns corresponding to basic variables, and A_N be the $m \times (n - m)$ submatrix of A consisting of the columns corresponding to non-basic variables.

The algebra of the Simplex Method

Relabelling the variables when necessary, we write

$$A = (A_N, A_B), \quad x = \begin{bmatrix} x_N \\ x_B \end{bmatrix},$$

where the $(n-m) \times 1$ vector x_N consists of non-basic variables and the $m \times 1$ vector x_B consists of basic variables.

Partition $c = (c_N, c_B)$ in the same way as x.

The tableau is then

BV	x_N	x_B	RHS
?	A_N	A_B	b
z	$-c_N$	$-c_B$	0

The algebra of the Simplex Method

We now use the structure of the tableau to write the basic variables in terms of the non-basic variables.

$Ax = b$ gives $(A_N, A_B) \begin{bmatrix} x_N \\ x_B \end{bmatrix} = b$, so that

$$A_N x_N + A_B x_B = b.$$

Since A_B is invertible (as its columns are linearly independent), we have

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

The algebra of the Simplex Method

For the value of the objective function, we also have

$$\begin{aligned} z &= cx \\ &= (c_N, c_B) \begin{bmatrix} x_N \\ x_B \end{bmatrix} \\ &= c_N x_N + c_B x_B \\ &= c_N x_N + c_B (A_B^{-1} b - A_B^{-1} A_N x_N) \\ &= c_B A_B^{-1} b - (c_B A_B^{-1} A_N - c_N) x_N \end{aligned}$$

and so

$$z + (c_B A_B^{-1} A_N - c_N) x_N = c_B A_B^{-1} b$$

The algebra of the Simplex Method

Now assume that the tableau is in canonical form. Then the reduced costs of the basic variables are equal to zero, and its structure is

BV	x_N	x_B	RHS
x_B	$A_B^{-1}A_N$	I	$A_B^{-1}b$
z	$c_B A_B^{-1}A_N - c_N$	0	$c_B A_B^{-1}b$

The algebra of the Simplex Method

By setting $x_N = 0$, this gives the basic solution

$$x = \begin{bmatrix} 0 \\ A_B^{-1} b \end{bmatrix},$$

which is feasible if and only if $A_B^{-1} b \geq 0$. If x is feasible, the objective value at x is

$$z = c_B A_B^{-1} b.$$

The reduced cost of non-basic x_j is

$$c_B A_B^{-1} \cdot (j\text{th column of } A) - c_j.$$

The 0-matrix under the column x_B in the tableau can be viewed as $c_B A_B^{-1} A_B - c_B$.

Summary: How is the Tableau Updated?

The starting tableau is

BV	x_N	x_B	RHS
?	A_N	A_B	b
z	$-c_N$	$-c_B$	0

or, in compact form

BV	x	RHS
?	A	b
z	$-c$	0

Summary: How is the Tableau Updated?

The canonical tableau is

BV	x_N	x_B	RHS
x_B	$A_B^{-1}A_N$	I	$A_B^{-1}b$
z	$c_B A_B^{-1} A_N - c_N$	0	$c_B A_B^{-1} b$

or, in compact form

BV	x	RHS
x_B	$A_B^{-1}A$	$A_B^{-1}b$
z	$c_B A_B^{-1} A - c$	$c_B A_B^{-1} b$

Observe that we get from the starting tableau to this tableau by multiplying on the left by the matrix

$$\begin{bmatrix} A_B^{-1} & 0 \\ c_B A_B^{-1} & 1 \end{bmatrix}$$

Proof of the Strong Duality Theorem

Recall that we are dealing with

Primal	Dual
$\max_x z = cx$	$\min_y w = yb$
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

$$A : m \times n;\ b : m \times 1;\ c : 1 \times n;\ x : n \times 1;\ y : 1 \times m$$

Introducing slack variables $x_s = \begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$, the primal problem can be converted to:

$$\max z = cx + 0x_s$$

$$(A, I) \begin{bmatrix} x \\ x_s \end{bmatrix} = b$$

$$x, x_s \geq 0$$

where I is the $m \times m$ identity matrix.

Suppose the primal problem has an optimal solution. By the Fundamental Theorem of Linear Programming, the above problem therefore has an optimal basic feasible solution (x^*, x_s^*).

Let A_B be the matrix of the columns of (A, I) corresponding to the basic variables in (x^*, x_s^*).

Now we use the result from the "algebra of the Simplex Method" with A and x replaced by (A, I) and $\begin{bmatrix} x \\ x_s \end{bmatrix}$, respectively.

Initial tableau:

BV	x	x_s	RHS
x_s	A	I	b
z	$-c$	0	0

Tableau for the optimal solution (x^*, x_s^*):

BV	x	x_s	RHS
x_B	$A_B^{-1}A$	$A_B^{-1}I$	$A_B^{-1}b$
z	$c_B A_B^{-1}A - c$	$c_B A_B^{-1}I - 0$	$c_B A_B^{-1}b$

Since this is the tableau for the optimal (x^*, x_s^*), we have

$$c_B A_B^{-1} A - c \geq 0$$

$$c_B A_B^{-1} I - 0 = c_B A_B^{-1} \geq 0$$

and the optimal value is $z^* = cx^* = c_B A_B^{-1} b$.

Let $y^* = c_B A_B^{-1}$. The inequalities above can be written as

$$y^* A \geq c, \quad y^* \geq 0.$$

In other words, y^* is a feasible solution to the dual problem.

Since $cx^* = c_B A_B^{-1} b = y^* b$, by Corollary 10.3, y^* is an optimal solution to the dual problem and moreover $w^* = y^* b = cx^* = z^*$.

We have proved that if the primal has an optimal solution, then so does the dual, and moreover their optimal objective values are equal.

Note that the dual can be expressed as

$$\max -w = -yb, \;\; -yA \leq -c, \;\; y \geq 0$$

and its dual is the primal. From what we proved above it follows that if the dual has an optimal solution, then so does the primal, and their optimal objective values are equal.

This completes the proof of the Strong Duality Theorem. $\square$

Solving the Primal and the Dual at One Time

The proof above implies the following: If

BV	x	x_s	RHS
x_B	$A_B^{-1}A$	$A_B^{-1}I$	$A_B^{-1}b$
z	$c_BA_B^{-1}A - c$	$c_BA_B^{-1}I - 0$	$c_BA_B^{-1}b$

is the optimal tableau for the primal, then $c_BA_B^{-1}$ is an optimal solution to the dual and $c_BA_B^{-1}A - c$ yields the values of the surplus variables in the dual problem.

Theorem 10.4

In an optimal tableau for the primal (in standard form), the reduced costs for the slack variables give rise to an optimal solution to the dual problem, and the reduced costs for the original variables give rise to the values of the surplus variables in the dual problem.

Thus, when solving the primal LP (in standard form) by using the Simplex Algorithm, we also solve the dual LP.

Example 10.7

Initial Tableau for the Primal Problem

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_4	8	6	4	1	0	100
x_5	5	4	4	0	1	60
z	-80	-60	-50	0	0	0

where x_4 and x_5 are slack variables (introduced when converting standard form to canonical form).

Example (continued)

After a few pivot operations .....

Final Tableau for the Primal Problem

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	$-2/5$	$-12/5$	1	$-8/5$	4
x_1	1	$4/5$	$4/5$	0	$1/5$	12
z	0	4	14	0	16	960

From this optimal tableau for the primal we know that

- $(x_1, x_2, x_3) = (12, 0, 0)$ is an optimal solution to the primal problem;
- $(y_1, y_2) = (0, 16)$ is an optimal solution to the dual problem;
- and both problems have the optimal objective value $z^* = w^* = 960$.

Example 10.8

Primal	Dual
$\max z = x_1 + x_2 + x_3$	$\min w = y_1 + y_2 + y_3$
s.t.	s.t.
$8x_1 + 4x_2 + 2x_3 \leq 1$	$8y_1 + 2y_2 + y_3 \geq 1$
$2x_1 + 8x_2 + 4x_3 \leq 1$	$4y_1 + 8y_2 + 2y_3 \geq 1$
$x_1 + 2x_2 + 8x_3 \leq 1$	$2y_1 + 4y_2 + 8y_3 \geq 1$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$

Initial tableau for the primal problem (where x_4, x_5, x_6 are slack variables):

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	8	4	2	1	0	0	1
x_5	2	8	4	0	1	0	1
x_6	1	2	8	0	0	1	1
z	-1	-1	-1	0	0	0	0

Next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	$1/2$	$1/4$	$1/8$	0	0	$1/8$
x_5	0	7	$7/2$	$-1/4$	1	0	$3/4$
x_6	0	$3/2$	$31/4$	$-1/8$	0	1	$7/8$
z	0	$-1/2$	$-3/4$	$1/8$	0	0	$1/8$

This goes on to

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	$14/31$	0	$4/31$	0	$-1/31$	$3/31$
x_5	0	$196/31$	0	$-6/31$	1	$-14/31$	$11/31$
x_3	0	$6/31$	1	$-1/62$	0	$4/31$	$7/62$
z	0	$-11/31$	0	$7/62$	0	$3/31$	$13/62$

and

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	0	0	$1/7$	$-1/14$	0	$1/14$
x_2	0	1	0	$-3/98$	$31/196$	$-1/14$	$11/196$
x_3	0	0	1	$-1/98$	$-3/98$	$1/7$	$5/49$
z	0	0	0	$5/49$	$11/196$	$1/14$	$45/196$

From the final tableau we have

- $(x_1, x_2, x_3) = (1/14, 11/196, 5/49)$ is an optimal solution to the primal problem;
- $(y_1, y_2, y_3) = (5/49, 11/196, 1/14)$ is an optimal solution to the dual problem; and
- the optimal objective values for both problems is $45/196$.

§10.5 – Complementary Slackness Conditions

Again we consider

Primal	Dual
$\max_x z = cx$	$\min_y w = yb$
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

where, as before,

$$A = (a_{ij})_{m\times n}; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}; c = (c_1, \ldots, c_n); x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; y = (y_1, \ldots, y_m)$$

Let $\mathbf{a}^i$ denote the ith row of A, $i = 1, \ldots, m$.

Let $\mathbf{a}_j$ denote the jth column of A, $j = 1, \ldots, n$.

Then the two problems can be written as

Primal	Dual
$\max_x z = cx$	$\min_y w = yb$
$\mathbf{a}^i x \leq b_i, \;\; i = 1, \ldots, m$	$y\mathbf{a}_j \geq c_j, \;\; j = 1, \ldots, n$
$x \geq 0$	$y \geq 0$

Recall that

- the ith dual variable y_i corresponds to the ith functional constraint $\mathbf{a}^i x \leq b_i$ of the primal problem; and
- the jth primal variable x_j corresponds to the jth functional constraint $y\mathbf{a}_j \geq c_j$ of the dual problem.

Theorem 10.5 (The Complementary Slackness Conditions)

Under the setting above, let x be a feasible solution to the primal problem and y a feasible solution to the dual problem.

Then x is optimal to the primal and y is optimal to the dual if and only if

$$\begin{aligned} y_i(b_i - \mathbf{a}^i x) &= 0, \ i = 1, \ldots, m \\ (y\mathbf{a}_j - c_j)x_j &= 0, \ j = 1, \ldots, n, \end{aligned}$$

that is,

$$y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \ \ i = 1, \ldots, m$$

$$\left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) x_j = 0, \ \ j = 1, \ldots, n$$

The Complementary Slackness Conditions state that

- if a primal constraint is non-binding, then the corresponding dual variable is zero (that is, if a dual variable is non-zero, then the corresponding primal constraint is binding), and
- if a dual constraint is non-binding, then the corresponding primal variable is zero (that is, if a primal variable is non-zero, then the corresponding dual constraint is binding).

In other words,

- either a primal variable is zero or the corresponding dual constraint is satisfied with equality, and
- either a primal constraint is satisfied with equality or the corresponding dual variable is zero.

The Complementary Slackness Conditions constitute one of the most important results in linear programming.

Proof: Introduce the slack variables $s_1, s_2, \ldots s_m$ for the primal problem and the surplus variables $t_1, t_2, \ldots t_n$ for the dual problem.

Primal LP

$$\max_{x,s} z = c_1x_1 + ... + c_nx_n + 0s_1 + 0s_2 + ... + 0s_m$$

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + s_2 & = & b_2 \\
 & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + s_m & = & b_m
\end{array}$$

$$x_1, x_2, ..., x_n, s_1, s_2, ..., s_m \geq 0.$$

Note that

$$s_i = b_i - \mathbf{a}^i x, \quad i = 1, \ldots, m.$$

Dual LP

$$\min_{y,t} w = b_1 y_1 + ... + b_m y_m + 0t_1 + ... + 0t_n$$

$$\begin{array}{lcl}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m - t_1 & = & c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \qquad - t_2 & = & c_2 \\
 & & \vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \qquad\qquad - t_n & = & c_n
\end{array}$$

$$y_1, y_2, ..., y_m, t_1, t_2, ..., t_n \geq 0.$$

Note that

$$t_j = y\mathbf{a}_j - c_j, \quad j = 1, \ldots, n.$$

The ith constraint of the primal LP is

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i.$$

Multiplying this by y_i, and summing for $i = 1$ to m, we have

$$\begin{aligned}
\sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij}x_j + s_i \right) &= \sum_{i=1}^{m} b_i y_i \\
\sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} y_i a_{ij} \right) + \sum_{i=1}^{m} y_i s_i &= \sum_{i=1}^{m} b_i y_i
\end{aligned}$$

Now subtract $\sum_{j=1}^{n} c_j x_j$ from both sides to get

$$\sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) + \sum_{i=1}^{m} y_i s_i = \sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j$$

Since $t_j = y\mathbf{a}_j - c_j = \sum_{i=1}^{m} y_i a_{ij} - c_j$, this can be rewritten as

$$\sum_{j=1}^{n} x_j t_j + \sum_{i=1}^{m} y_i s_i = \sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j$$

By the Strong Duality Theorem, if x and y are optimal solutions to the primal and dual respectively, then

$$\sum_{i=1}^{m} b_i y_i = \sum_{j=1}^{n} c_j x_j$$

and so the above equation becomes

$$\sum_{j=1}^{n} x_j t_j + \sum_{i=1}^{m} y_i s_i = 0$$

Since all terms are nonnegative, this implies that each term is zero, that is,

$$y_i(b_i - \mathbf{a}^i x) = y_i s_i = 0, \;\; i = 1, \ldots, m$$

$$(y\mathbf{a}_j - c_j)x_j = x_j t_j = 0, \;\; j = 1, \ldots, n$$

On the other hand, if $x_j t_j = 0$ for all j and $y_i s_i = 0$ for all i, then

$$\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} y_i b_i$$

and so x and y are optimal by Corollary 10.3. $\square$

Example

In Example 10.7 the final tableau for the primal problem is:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	$-2/5$	$-12/5$	1	$-8/5$	4
x_1	1	$4/5$	$4/5$	0	$1/5$	12
z	0	4	14	0	16	960

(Note that x_4, x_5 take the roles of s_1, s_2 respectively.)

So $(x_1, x_2, x_3) = (12, 0, 0)$ and $(y_1, y_2) = (0, 16)$ are optimal solutions to the primal and dual problems respectively.

We have $(s_1, s_2) = (4, 0)$ and $(t_1, t_2, t_3) = (0, 4, 14)$. The values of t_1, t_2, t_3 are exactly the reduced costs for the original variables!

We see that $y_i s_i = 0$ for $i = 1, 2$ and $t_j x_j = 0$ for $j = 1, 2, 3$.

That is, if $y_i \neq 0$ then $s_i = 0$, and if $s_i \neq 0$ then $y_i = 0$. If $x_j \neq 0$ then $t_j = 0$, and if $t_j \neq 0$ then $x_j = 0$.

Applications of the Complementary Slackness Conditions

The Complementary Slackness Theorem has a number of applications. It can be used to

- calculate the solution of the dual (respectively primal) when the solution of the primal (respectively dual) is known,
- verify whether a proposed solution of either the primal or dual is optimal, and
- for certain structured problems it can be used to design an algorithm to solve the problem.

We will give an example for each of the first two applications.

For the third application, if you choose to do the MSc subject "Network Optimisation" (which is offered in odd years), you will see beautiful applications of the complementary slackness conditions in algorithm design.

Example 10.9

The linear program

$$\max_x z = 4x_1 + x_2$$

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ 5x_1 + x_2 &\leq 55 \\ -x_1 + 2x_2 &\leq 3 \end{aligned}$$

$$x_1, x_2 \geq 0,$$

has an optimal solution $x_1^* = 5, x_2^* = 4$. What is the optimal solution of the dual?

Example (continued)

The dual linear program is

$$\min_{y} w = y_1 + 55y_2 + 3y_3$$

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \end{aligned}$$

$$y_1, y_2, y_3 \geq 0.$$

Example (continued)

The complementary slackness conditions are

$$\begin{aligned} y_1(1 - x_1 + x_2) &= 0 \\ y_2(55 - 5x_1 - x_2) &= 0 \\ y_3(3 + x_1 - 2x_2) &= 0 \\ x_1(4 - y_1 - 5y_2 + y_3) &= 0 \\ x_2(1 + y_1 - y_2 - 2y_3) &= 0 \end{aligned}$$

Notice that these comprise five non-linear equations for the five variables x_1, x_2, y_1, y_2, y_3.

Example (continued)

Substituting $x_1^* = 5, x_2^* = 4$ into the complementary slackness conditions, we get

$$\begin{aligned} 0y_1^* &= 0 \\ 26y_2^* &= 0 \\ 0y_3^* &= 0 \\ 5(4 - y_1^* - 5y_2^* + y_3^*) &= 0 \\ 4(1 + y_1^* - y_2^* - 2y_3^*) &= 0 \end{aligned}$$

The second equation gives $y_2^* = 0$ and then the fourth and fifth equations give $y_1^* = 9, y_3^* = 5$. Check that this is feasible for the dual.

The solution to the dual is thus $(y_1^*, y_2^*, y_3^*) = (9, 0, 5)$ at which point $w^* = 24$. Check that this is also equal to z^*.

Example 10.10

For the linear program

$$\max_x z = 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5$$

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 &\leq 1 \\ x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 &\leq 1 \\ 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 &\leq 22 \end{aligned}$$

$$x_1, \ldots, x_5 \geq 0,$$

it has been proposed that the optimal solution is $x_1^* = 0, x_2^* = 2, x_3^* = 0, x_4^* = 7, x_5^* = 0$. Is this correct?

Example (continued)

The dual linear program is

$$\min_y w = y_1 + y_2 + 22y_3$$

$$\begin{aligned} 2y_1 + y_2 + 5y_3 &\geq 8 \\ -3y_1 + 7y_2 + 4y_3 &\geq -9 \\ 4y_1 + 3y_2 - 6y_3 &\geq 12 \\ y_1 - 2y_2 + 2y_3 &\geq 4 \\ 3y_1 + y_2 + 3y_3 &\geq 11 \end{aligned}$$

$$y_1, y_2, y_3 \geq 0.$$

Example (continued)

The primal variables x_2^* and x_4^* are positive. So the complementary slackness conditions give

$$\begin{aligned} -9 + 3y_1^* - 7y_2^* - 4y_3^* &= 0 \\ 4 - y_1^* + 2y_2^* - 2y_3^* &= 0. \end{aligned}$$

Also, since the second primal constraint is non-binding, we have

$$y_2^* = 0.$$

Solving, we see that $y_1^* = 34/10$ and $y_3^* = 3/10$.

Example (continued)

We need to check whether $(y_1^*, y_2^*, y_3^*) = (34/10, 0, 3/10)$ is dual feasible.
Unfortunately

$$4y_1^* + 3y_2^* - 6y_3^* = 118/10 \ngeq 12,$$

and so the solution to the complementary slackness relations is not feasible.

Therefore the postulated solution is not optimal for the primal.

§10.6 – Weak Duality, Strong Duality and Complementary Slackness Conditions for General LP

So far we have developed the Weak Duality, Strong Duality and Complementary Slackness Conditions for primal-dual pairs in standard form.

It is important to understand that all these results are valid for LP problems in non-standard form (i.e. possibly with $=$ constraints and variables which are unrestricted in sign).

The statements of these results under the general setting are exactly the same as what we saw for standard forms.

In the following we just state the Complementary Slackness Conditions under the general setting.

Theorem 10.6 (The Complementary Slackness Conditions)

Consider

Primal	*Dual*
$\max_x z = cx$	$\min_y w = yb$
$\mathbf{a}^i x \leq b_i, \ i \in M$	$y\mathbf{a}_j \geq c_j, \ j \in N$
$\mathbf{a}^i x = b_i, \ i \in \overline{M}$	$y\mathbf{a}_j = c_j, \ j \in \overline{N}$
$x_j \geq 0, \ j \in N$	$y_i \geq 0, \ i \in M$
$x_j \in \mathbb{R}\ , \ j \in \overline{N}$	$y_i \in \mathbb{R}\ , \ i \in \overline{M}$

Let x be a feasible solution to the primal problem and y a feasible solution to the dual problem. Then x is optimal to the primal and y is optimal to the dual if and only if

$$y_i(b_i - \mathbf{a}^i x) = 0, \ \textit{for all } i$$
$$(y\mathbf{a}_j - c_j)x_j = 0, \ \textit{for all } j$$

§11 – Sensitivity Analysis

So far we have assumed that the various parameters that describe a linear programming problem are all given.
However, in real life, this is rarely the case. For example, we might have only an estimate of the profit that we can make on each of our products.
In such situations we want to know how the solution to a linear programming problem will vary if the parameters change.
Sensitivity Analysis is the subject which deals with these issues.
This section is intended to be a very brief introduction to Sensitivity Analysis.

Ingredients of LP Models:

- A linear objective function
- A system of linear constraints
  - RHS values
  - Coefficient matrix (LHS)
  - Signs ($=$, $\leq$, $\geq$)

Problem 11.1

How does the optimal solution change as some of these elements change?

It is instructive to classify the changes into two categories:

- structural changes, and
- parametric changes.

Structural Changes

These are

- addition of a new decision variable,
- addition of a new constraint,
- loss of a decision variable, and/or
- removal of a constraint.

How is the solution affected if the linear program and its dimension grow or shrink? Specific questions that we may be interested in are

- Will adding a new decision variable change the optimal solution?
- How much does the optimal solution change (how much does the basis change)?
- Does the removal of a particular constraint change the solution much?

We will not discuss structural changes in this course.

Parametric Changes

These are

- changes in one or more of the coefficients c_i of the objective function,
- changes in the RHS values b_j, or
- changes in the coefficients of the coefficient matrix $A = (a_{ij})$.

How is the solution affected by such perturbations? For example:

- How much can I change the RHS b_j before the basis changes?
- How much can I change the cost coefficients c_i before the basis changes?
- What is the percentage change in the objective function value when the first two occur?

We will discuss what to do if there are post-hoc changes in b_j or c_i. There are two important aspects of the new solution that we need to check:

- feasibility, and
- optimality.

So first we need to be able to calculate the new solution.

Simplex Algebra provides the required formulas.

Review: Algebra of the Simplex Method

$$\begin{aligned} \max z &= cx \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

where $A = (a_{ij})_{m \times n}$ is assumed to have rank m and

$$c = (c_1, \ldots, c_n), \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Suppose we have an optimal basic feasible solution whose basis matrix is A_B (which must be a non-singular $m \times m$ submatrix of A). Let A_N be the $m \times (n - m)$ submatrix of A consisting of the rest of the columns.

Review: How is the tableau updated?

Tableau (which may be non-canonical):

BV	x_N	x_B	RHS
–	A_N	A_B	b
z	$-c_N$	$-c_B$	0

Optimal tableau:

BV	x_N	x_B	RHS
x_B	$A_B^{-1}A_N$	I	$A_B^{-1}b$
z	$c_B A_B^{-1} A_N - c_N$	0	$c_B A_B^{-1} b$

So the optimal tableau has RHS column

$$\hat{b} = A_B^{-1} b$$

and reduced costs (of the non-basic variables)

$$-\hat{c}_N = c_B A_B^{-1} A_N - c_N.$$

These tell us how the new RHS column $\hat{b}$ and the new cost coefficients $\hat{c}_N$ change as b and/or c change.

§11.1 – Principles

Suppose that we want to know what happens if b_j has changed from our original estimate.
We calculate the new $\hat{b}^{(\text{new})}$ according to the formula

$$\hat{b}^{(\text{new})} = A_B^{-1} b^{(\text{new})}$$

on the previous slide.
The optimal solution for the original problem becomes infeasible for the modified problem if and only if at least one component of $\hat{b}^{(\text{new})}$ is negative. If this happens, we may need to perform a number of actions to find the new solution (if one even exists). This would involve introducing new artificial variables before executing the Two-Phase Simplex method.
If the optimal solution for the original problem is feasible for the new problem, then it is still optimal. How do we know this?

What about changes to c_i?
We calculate the new $-\hat{c}_N^{(\text{new})}$ (reduced costs) using

$$-\hat{c}_N^{(\text{new})} = c_B^{(\text{new})} A_B^{-1} A_N - c_N^{(\text{new})}.$$

Can this change make our current optimal solution infeasible?
It is still optimal if and only if all reduced costs are nonnegative (for a maximisation problem).
If it is no longer optimal, then at least one of the reduced costs will be negative. We therefore continue with the Simplex Algorithm as usual.

Summary of the situation when there are changes to b or c

If all the components of the new RHS are nonnegative, then

- the point with the same basic variables as the old optimal solution is still feasible for the modified problem;
- the new values of the basic variables are simply equal to the new values of the RHS;
- if there are changes to c also, the resulting point may not remain optimal. This depends on what happens to the new reduced costs.

If at least one of the components of the new RHS is negative, then

- the point with the same basic variables as the old optimal solution is not feasible for the modified problem. Corrective action must be taken in order to solve the new system or prove that its infeasible. This will involve setting up a new instance of the Two-Phase Simplex method.

If all the components of $-\hat{c}_N$ satisfy the optimality condition, and all the components of the new RHS are nonnegative, then

- the basic variables in the old optimal solution are still basic in the new optimal solution. The new tableau is still optimal and the solution can be read off.

If at least one of the components of $-\hat{c}_N$ violates the optimality condition,and all the components of the new RHS are nonnegative, then

- the basic variables in the old optimal solution are not basic in the new optimal solution. We need to perform one or more pivot operations to get the new tableau.

Case 1: A negative value occurs in the RHS column after changing b

Consider the following Simplex tableau satisfying the Optimality Criterion.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	5
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

The current optimal solution is $x^* = (10, 20, 0, 0, 5)$ with $z^* = 100$.

Case 1: A negative value occurs in the RHS column after changing b

Suppose that a change to the *original* value of b produces the following new RHS

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	-5
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

Case 1: A negative value occurs in the RHS column after changing b

We can multiple the second row by -1 in order to attempt to restore canonical form

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	1	-1	-1	5
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

Case 1: A negative value occurs in the RHS column after changing b

Note that the system is still NOT in canonical form. It is not obvious how to proceed using pivot operations alone. In fact, the new system may not even be feasible.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	1	-1	-1	5
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

Case 1: A negative value occurs in the RHS column after changing b

We introduce a new artificial variable y_1. We then append a column for y_1 to the Simplex tableau and initialize Phase 1 of the Two-Phase Simplex method (i.e., we maximise $w = -y_1$). Observe that y_1 replaces x_5 as a basic variable. Note that the system is not yet in canonical form.

BV	y_1	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	0	1	-1	2	0	20
y_1	1	0	0	1	-1	-1	5
x_1	0	1	0	1	-1	0	10
w	1	0	0	0	0	0	0

Case 1: A negative value occurs in the RHS column after changing b

We convert to canonical form by subtracting the y_2 row from the w row.

BV	y_1	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	0	1	-1	2	0	20
y_1	1	0	0	1	-1	-1	5
x_1	0	1	0	1	-1	0	10
w	0	0	0	-1	1	1	-5

Case 1: A negative value occurs in the RHS column after changing b

One pivot operation following the Greedy rule and Ratio test gives the final Phase 1 tableau:

BV	y_1	x_1	x_2	x_3	x_4	x_5	RHS
x_2	1	0	1	0	1	-1	25
x_3	1	0	0	1	-1	-1	5
x_1	-1	1	0	0	0	1	5
w	1	0	0	0	0	0	0

Case 1: A negative value occurs in the RHS column after changing b

We initialise Phase 2 with basic variables x_1, x_2, x_3 and with the z-coefficients from the first Simplex tableau.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	1	-1	25
x_3	0	0	1	-1	-1	5
x_1	1	0	0	0	1	5
z	0	0	1	2	0	100

Case 1: A negative value occurs in the RHS column after changing b

Restoring canonical form we get:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	1	-1	25
x_3	0	0	1	-1	-1	5
x_1	1	0	0	0	1	5
z	0	0	0	3	1	95

Which satisfies the Optimality criterion. The new optimal solution is $x^* = (5, 25, 5, 0, 0)$ with $z^* = 95$. Observe that the optimal value of z decreased in total by 5 as a result of the change in b, and that x_3 replaced x_5 as an optimal basic variable.

Case 2: A negative value does NOT occur in the RHS column after changing b

Suppose that a change to the *original* value of b produces the following new RHS for the same example as before

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	1
x_1	1	0	1	-1	0	10
z	0	0	1	2	0	100

In this case the modified problem is still feasible and, because only b has been modified, the z-row remains unchanged and the Optimality criterion is still satisfied. The new optimal solution is $x^* = (10, 20, 0, 0, 1)$ with $z^* = 100$.

Case 3: A change in c produces a z-row that no longer satisfies the Optimality criterion

Suppose that a change to the *original* value of c produces the following new z-row for the same example as before

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	5
x_1	1	0	1	-1	0	10
z	0	0	-1	2	0	100

Case 3: A change in c produces a z-row that no longer satisfies the Optimality criterion

One iteration of the Simplex Algorithm gives the optimal tableau:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	1	1	0	1	0	30
x_5	1	0	0	0	1	15
x_3	1	0	1	-1	0	10
z	1	0	0	1	0	110

Observe that the optimal basic variables have changed and the new solution is $x^* = (0, 30, 10, 0, 15)$ with $z^* = 110$

Case 4: A change in c produces a z-row that still satisfies the Optimality criterion

Suppose that a change to the *original* value of c produces the following new z-row for the same example as before (note that a change in a single component of c can simultaneously change a coefficient in the z-row AND the value of the objective function in the bottom right-hand corner. How?)

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	-1	2	0	20
x_5	0	0	-1	1	1	5
x_1	1	0	1	-1	0	10
z	0	0	2	2	0	80

Observe that the Optimality criterion is still satisfied. The optimal basic variables and their values remain unchanged. The optimal objective value has changed to $z^* = 80$

§11.2 – Change in the RHS

Sometimes we want to know how much we can vary the parameters before the solution is affected. That is, we are interested in the *range of perturbation*.
Suppose that we change one of the elements of b, say b_k, by δ so that the new b is equal to the old one except that the new value of b_k is equal to $b_k + \delta$. That is,

$$b^{(\text{new})} = b + \delta e_k$$

where e_k is the kth column of the identity matrix.

The new final RHS value is given by

$$\hat{b}^{(\text{new})} = A_B^{-1} b^{(\text{new})} = A_B^{-1}(b + \delta e_k)$$

This yields

$$\begin{aligned} \hat{b}^{(\text{new})} &= A_B^{-1} b + \delta A_B^{-1} e_k \\ &= A_B^{-1} b + \delta (A_B^{-1})_{\cdot k} \end{aligned}$$

where $(A_B^{-1})_{\cdot k}$ denotes the kth column of A_B^{-1}.

Since $\hat{b}^{(\text{old})} = A_B^{-1} b$, we have

$$\hat{b}^{(\text{new})} = \hat{b}^{(\text{old})} + \delta (A_B^{-1})_{\cdot k}.$$

The old basis remains optimal after the change if and only if $\hat{b}^{(\text{new})} \geq 0$, which occurs if and only if

$$\delta(A_B^{-1})_{\cdot k} \geq -\hat{b}^{(\text{old})}$$

or equivalently

$$\delta(A_B^{-1})_{i,k} \geq -\hat{b}_i^{(\text{old})}, \;\; i = 1, 2, \ldots, m$$

where $(A_B^{-1})_{i,k}$ denotes the entry of A_B^{-1} in the ith row and kth column, and $\hat{b}_i^{(\text{old})}$ is the ith component of $\hat{b}^{(\text{old})}$.

So the old basis remains optimal if and only if

$$\delta \geq \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,k}}$$

for all i such that $(A_B^{-1})_{i,k} > 0$, and

$$\delta \leq \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,k}}$$

for all i such that $(A_B^{-1})_{i,k} < 0$. We can write this in a more formal way as follows:

Let

$$P_k = \{i \mid i \in \{1, ..., m\}, (A_B^{-1})_{i,k} > 0\}$$

and let

$$N_k = \{i \mid i \in \{1, ..., m\}, (A_B^{-1})_{i,k} < 0\}.$$

Then the old basis remains optimal if and only if

$$\max_{i \in P_k} \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,k}} \leq \delta \leq \min_{i \in N_k} \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,k}}$$

Example 11.1

If

$$b = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

$$A_B^{-1} = \begin{bmatrix} 2 & 4 & -16 \\ 0 & 4 & -8 \\ 0 & -1 & 3 \end{bmatrix},$$

how much can we change the second component of b without changing the optimal solution?

Example (cont'd)

$$\hat{b}^{(\text{old})} = A_B^{-1} b = \begin{bmatrix} 2 & 4 & -16 \\ 0 & 4 & -8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 16 \\ 4 \end{bmatrix}$$

$$(A_B^{-1})_{.2} = \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

Since $(A_B^{-1})_{1,2} = 4 > 0, (A_B^{-1})_{2,2} = 4 > 0, (A_B^{-1})_{3,2} = -1 < 0$,

$$\max_{i \in P_2} \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,2}} = \max_{i=1,2} \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,2}} = \max \left\{ \frac{-48}{4}, \frac{-16}{4} \right\} = -4.$$

$$\min_{i \in N_2} \frac{-\hat{b}_i^{(\text{old})}}{(A_B^{-1})_{i,2}} = \frac{-\hat{b}_3^{(\text{old})}}{(A_B^{-1})_{3,2}} = \frac{-4}{-1} = 4$$

Example (cont'd)

Thus

$$4 \geq \delta \geq -4.$$

Therefore the old basis (whatever it is) will remain optimal if and only if the value of b_2 is in the interval $[20 - 4, 20 + 4] = [16, 24]$.

We also have an alternative method to find the range of δ...

To determine the critical values of δ, we simply compute

$$\hat{b}^{(\text{new})} = A_B^{-1} b^{(\text{new})} \ \ (= \hat{b}^{(\text{old})} + \delta (A_B^{-1})_{\cdot k}).$$

Set all its components to be nonnegative and then solve inequalities.

Equivalently, we can solve

$$\delta (A_B^{-1})_{\cdot k} \geq -\hat{b}^{(\text{old})}$$

that is

$$\delta (A_B^{-1})_{i,k} \geq -\hat{b}_i^{(\text{old})}, \ \ i = 1, 2, \ldots, m.$$

These alternative approaches should produce the same range of δ.

Example (cont'd)

$$b^{(\text{old})} = b = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}, \quad b^{(\text{new})} = \begin{bmatrix} 48 \\ 20+\delta \\ 8 \end{bmatrix}$$

$$\begin{aligned} \hat{b}^{(\text{new})} &= A_B^{-1} b^{(\text{new})} \\ &= \begin{bmatrix} 2 & 4 & -16 \\ 0 & 4 & -8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 48 \\ 20+\delta \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 48+4\delta \\ 16+4\delta \\ 4-\delta \end{bmatrix} \end{aligned}$$

Equivalently,

$$\hat{b}^{(\text{new})} = \hat{b}^{(\text{old})} + \delta (A_B^{-1})_{\cdot 2} = \begin{bmatrix} 48 \\ 16 \\ 4 \end{bmatrix} + \delta \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 48+4\delta \\ 16+4\delta \\ 4-\delta \end{bmatrix}$$

Example (cont'd)

Thus the non-negativity criterion for the basic variables to remain the same is

$$\begin{aligned} 48 + 4\delta \geq 0 \quad &\text{hence} \quad \delta \geq -12 \\ 16 + 4\delta \geq 0 \quad &\text{hence} \quad \delta \geq -4 \\ 4 - \delta \geq 0 \quad &\text{hence} \quad \delta \leq 4 \end{aligned}$$

which is equivalent to

$$-4 \leq \delta \leq 4,$$

and so

$$16 \leq b_2 \leq 24.$$

Thus we obtain the same result that the old basis remains optimal if and only if b_2 is in the interval $[16, 24]$.

§11.3 – Changes in the Cost Vector

Suppose that the value of c_k changes for some k.
How will this affect the optimal solution to the LP problem?
We can distinguish between two cases, when

- x_k is not in the old optimal basis, and when
- x_k is in the old optimal basis.

Our reasoning makes extensive use of our expression for the cost coefficients of the final tableau in terms of the original tableau

$$-\hat{c}_N = c_B A_B^{-1} A_N - c_N.$$

So the kth component of $-\hat{c}_N$ is given by

$$(-\hat{c}_N)_k = (c_B A_B^{-1} A_N)_k - c_k.$$

Change in the cost coefficient of a variable not in the optimal basis

If the original cost coefficient of a variable that is not in the old optimal basis changes from c_k to $c_k + \delta$, then

$$\begin{aligned}(-\hat{c}_N^{(\text{new})})_k &= (c_B A_B^{-1} A_N)_k - (c_k + \delta) \\ &= (-\hat{c}_N^{(\text{old})})_k - \delta.\end{aligned}$$

Thus, for a maximisation problem, the basis (and indeed the optimal solution) remains the same provided that

$$(-\hat{c}_N^{(\text{old})})_k - \delta \geq 0,$$

which is the same as saying that

$$\delta \leq (-\hat{c}_N^{(\text{old})})_k$$

Example 11.2

Suppose that the reduced costs in the final simplex tableau are given by

$$(-\hat{c}_N^{(\text{old})}, -\hat{c}_B^{(\text{old})}) = (2, 3, 4, 0, 0, 0)$$

with x_5, x_6 and x_4 (in order) comprising the final basic variables.

How much can we change the value of c_1 without changing the basis?

Example (cont'd)

First we observe that x_1 is not in the basis and that our problem is a maximisation problem (why?). Note also that $k = 1$.
Here

$$(-\hat{c}_N^{(\text{old})})_1 = 2$$

and so

$$\delta \leq (-\hat{c}_N^{(\text{old})})_1$$

is the same as

$$\delta \leq 2.$$

This means that as long as the new c_1 is less than or equal to the old c_1 plus 2 (i.e. $c_1^{(\text{new})} \leq (-\infty, c_1^{(\text{old})} + 2]$, the optimal solution will not be affected.

Note that we do not need to know the old value of c_1 to reach this conclusion.

Change in the cost coefficient of a variable in the optimal basis

If the original cost coefficient of a variable that is in the old optimal basis changes from c_k to $c_k + \delta$, then

$$\begin{aligned} -\hat{c}_N^{(\text{new})} &= (c_B + \delta e_p) A_B^{-1} A_N - c_N \\ &= (c_B A_B^{-1} A_N - c_N) + \delta e_p A_B^{-1} A_N \\ &= -\hat{c}_N^{(\text{old})} + \delta e_p A_B^{-1} A_N \\ &= -\hat{c}_N^{(\text{old})} + \delta (A_B^{-1} A_N)_{p\cdot} \end{aligned}$$

where x_k corresponds to the pth row in the final tableau, e_p is the pth row of the identity matrix, and $(A_B^{-1} A_N)_{p\cdot}$ is the pth row of $A_B^{-1} A_N$. (This is because c_N is unchanged and c_k is a component of c_B.)

Therefore, if we have a maximisation problem, then the optimal solution remains optimal if and only if

$$-(\hat{c}_N^{(\text{old})})_j + \delta (A_B^{-1} A_N)_{p,j} \geq 0, \text{ for all non-basic } j.$$

Equivalently,

$$\delta \geq \frac{(\hat{c}_N^{(\text{old})})_j}{(A_B^{-1}A_N)_{p,j}}, \text{ for all non-basic } j \text{ with } (A_B^{-1}A_N)_{p,j} > 0$$

and

$$\delta \leq \frac{(\hat{c}_N^{(\text{old})})_j}{(A_B^{-1}A_N)_{p,j}}, \text{ for all non-basic } j \text{ with } (A_B^{-1}A_N)_{p,j} < 0$$

Thus the optimal solution remains optimal if and only if

$$\max_j \frac{(\hat{c}_N^{(\text{old})})_j}{(A_B^{-1}A_N)_{p,j}} \leq \delta \leq \min_j \frac{(\hat{c}_N^{(\text{old})})_j}{(A_B^{-1}A_N)_{p,j}},$$

where $\max_j$ is taken over all j such that $(A_B^{-1}A_N)_{p,j} > 0$ and $\min_j$ is over all j such that $(A_B^{-1}A_N)_{p,j} < 0$.

A direct analysis

The formula above determines the range that δ can fall in without affecting the optimal solution.

Alternatively, as before, we can use our expression for $-\hat{c}_N^{(\text{new})}$ to calculate the new cost coefficients in terms of δ and $-\hat{c}_N^{(\text{old})}$, and then proceed to determine the range of δ by solving a system of inequalities.

Another alternative is to use a direct analysis, which we illustrate in the next example.

Example 11.2 (cont'd)

As before suppose that the reduced costs in the final simplex tableau are given by

$$(-\hat{c}_N^{(\text{old})}, -\hat{c}_B^{(\text{old})}) = (2, 3, 4, 0, 0, 0)$$

with x_5, x_6 and x_4 (in the order they appear in the tableau) comprising the final basic variables.

Suppose that the changes occur in c_4.

Since x_4 corresponds to row 3 in the final tableau, we need to know the 3rd row $(A_B^{-1}A_N)_{3\cdot}$ of $A_B^{-1}A_N$.

Suppose that this row is $(3, -4, 0, 1, 0, 0)$. Then the final tableau is

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_5	–	–	–	0	1	0	–
x_6	–	–	–	0	0	1	–
x_4	3	-4	0	1	0	0	–
z	2	3	4	0	0	0	–

Example 11.2 (cont'd)

If we add δ to the old c_4, we would have instead

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_5	–	–	–	0	1	0	–
x_6	–	–	–	0	0	1	–
x_4	3	-4	0	1	0	0	–
z	2	3	4	$-\delta$	0	0	–

Restoring the canonical form of the x_4 column, we obtain

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_5	–	–	–	0	1	0	–
x_6	–	–	–	0	0	1	–
x_4	3	-4	0	1	0	0	–
z	$2+3\delta$	$3-4\delta$	4	0	0	0	–

Example 11.2 (cont'd)

Thus, for a maximisation problem, to ensure that the current basis remains optimal, we need

$$2 + 3\delta \geq 0$$

and

$$3 - 4\delta \geq 0$$

so that

$$-2/3 \leq \delta \leq 3/4.$$

Thus the old optimal solution remains optimal if we keep the change in c_4 in the interval $[-2/3, 3/4]$, i.e. $c_4^{(\text{new})} \in [c_4 - 2/3, c_4 + 3/4]$ (where c_4 means $c_4^{(\text{old})}$).

From the tableau above we can see that if $\delta < -2/3$ then x_1 will enter the basis, and if $\delta > 3/4$ then x_2 will enter the basis.

Reminder

Ensure that you read questions in this area carefully.

If you are asked about the range of admissible changes in say b_1, then it is not sufficient to report the admissible values of δ. You have to translate the range of admissible changes in δ into the range of admissible changes in b_1.

The same requirements apply to changes in cost coefficients.

For example, if $b_1 = 12$ and you have obtained $-2 \leq \delta \leq 3$, then the admissible range of b_1 is $[12 - 2, 12 + 3] = [10, 15]$.

§11.4 – A Hint of Parametric Programming

So far we have considered only the impact of discrete changes in the cost coefficients or the RHS values. Often one needs to understand the impact of continuous systematic changes in these values to the optimal solution. The study of such problems is usually called parametric (linear) programming.

We will not discuss parametric programming in detail. Instead we will use an example to illustrate the main idea for parametric programs where the cost coefficients contain parameters.

Example 11.3

It is known that

$$\max z = x_1 + 2x_2$$

$$\begin{aligned} x_1 + 3x_2 &\leq 8 \\ x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

has the following optimal tableau (which can be obtained by using the Simplex Method):

BV	x_1	x_2	x_3	x_4	RHS
x_2	0	1	$1/2$	$-1/2$	2
x_1	1	0	$-1/2$	$3/2$	2
z	0	0	$1/2$	$1/2$	6

where x_3 and x_4 are slack variables.

Example 11.3 (cont'd)

Solve the following LP problem, where β is a parameter, $0 \leq \beta < \infty$.

$$\max z(\beta) = (1+\beta)x_1 + (2-\beta)x_2$$

$$\begin{aligned} x_1 + 3x_2 &\leq 8 \\ x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Observe that

- changes occurs in both cost coefficients,
- there is a parameter involved, and
- functional constraints are unchanged.

Example 11.3 (cont'd)

When $\beta = 0$ the parametric problem is the same as the non-parametric problem. Since they have the same functional constraints, the optimal tableau of the latter gives a basic feasible solution to the former. However, we need to update the z-row.

Since for the optimal basis we have

$$z = (1+\beta)x_1 + (2-\beta)x_2$$

we have to add $-\beta$ to the reduced cost of x_1 and β to the reduced cost of x_2. This gives the following tableau.

BV	x_1	x_2	x_3	x_4	RHS
x_2	0	1	$1/2$	$-1/2$	2
x_1	1	0	$-1/2$	$3/2$	2
z	$-\beta$	β	$1/2$	$1/2$	6

Example 11.3 (cont'd)

Restoring canonical form by using the elementary row operations $R_3' = R_3 - \beta R_1 + \beta R_2$, we obtain:

BV	x_1	x_2	x_3	x_4	RHS
x_2	0	1	$1/2$	$-1/2$	2
x_1	1	0	$-1/2$	$3/2$	2
z	0	0	$1/2-\beta$	$1/2+2\beta$	6

This tableau is optimal if and only if

$$1/2 - \beta \geq 0 \text{ and } 1/2 + 2\beta \geq 0.$$

Thus the tableau above is optimal if and only if

$$0 \leq \beta \leq 1/2,$$

and in this case

$$x^*(\beta) = (2, 2), \quad z^*(\beta) = 6.$$

Example 11.3 (cont'd)

Since the upper bound $1/2$ of β corresponds to x_3, x_3 should be the entering variable when $\beta > 1/2$. The ratio test shows that we should pivot on the $(i = 1, j = 3)$-entry and take x_2 out of the basis. Pivoting on this entry we obtain:

BV	x_1	x_2	x_3	x_4	RHS
x_3	0	2	1	-1	4
x_1	1	1	0	1	4
z	0	$-1+2\beta$	0	$1+\beta$	$4+4\beta$

This tableau is optimal if and only if

$$-1 + 2\beta \geq 0 \text{ and } 1 + \beta \geq 0,$$

that is,

$$1/2 \leq \beta.$$

In this case we have

$$x^*(\beta) = (4, 0), \quad z^*(\beta) = 4 + 4\beta.$$

Example 11.3 (cont'd)

Summary:

When $0 \leq \beta \leq 1/2$, the optimal solution is

$$x^*(\beta) = (2, 2)$$

and the optimal value is

$$z^*(\beta) = 6.$$

When $1/2 \leq \beta < \infty$, the optimal solution is

$$x^*(\beta) = (4, 0)$$

and the optimal value is

$$z^*(\beta) = 4 + 4\beta.$$