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The Determination of Cloud Masses and Dust Characteristics from Submillimetre Thermal Emission

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SUMMARY

The purpose of this paper is to review the principles by which the dust masses and total masses of interstellar clouds and certain characteristics of interstellar dust grains can be derived from observations of far-infrared and submillimetre thermal emission. To the extent possible, the discussion will be independent of particular grain models.

1 INTRODUCTION

The construction of large infrared telescopes, such as the NASA Kuiper Airborne Observatory, the NASA Infrared Telescope Facility, and the United Kingdom Infrared Telescope, and the development of sensitive far-infrared and submillimetre photometers have made it possible to measure thermal spectra even for very cool clouds ($T < 20$ K).

In nearly all cases, the optical depths, τ, determined from the spectra at wavelengths longer than $\sim 200\ \mu$m are found to be very much less than one. Hence the observed flux densities sample with equal efficiency the emission from all depths in a cloud. It is especially that characteristic of submillimetre flux measurements which one seeks to exploit in determining the characteristics of clouds and grains.

The fundamental difficulty in determining the mass of any cool cloud which is opaque to starlight is that the principal ingredient, molecular hydrogen, is invisible; the kinetic energies of the molecules are far below the minimum energy required to produce observable hydrogen transitions. The methods commonly used to overcome that difficulty are:

(a) *Kinematics*. When gradients in Doppler shifts can reasonably be attributed to rotation of gravitationally bound clouds one may estimate the mass necessary to explain the observed radial velocities. Often, however, the observed gradients can be produced by ejection of unbound material from compact regions within the clouds.

(b) *Extinction*. When the opacity of a cloud is low enough to permit observation of background stars, one may estimate the column density by assuming that the relationship between extinction and hydrogen column density measured in the intercloud medium can be applied with suitable precautions to the cloud. This method can be extended to greater column densities by measuring extinction in the near-infrared (e.g. K band; 2·2 μm) and applying measured relationships between near-infrared and visible extinction.

(c) *Molecular line intensities.* When the column density, $N(x)$, of some molecule, x, is obtained by observing a particular emission line, one may estimate the hydrogen column density, $N(H_2)$, by assuming a ratio $N(H_2)/N(x)$ equal to that measured in a thinner cloud for which $N(H_2)$ has been found by the extinction method. This method is valid only when the excitation temperatures are known and when the optical depth for the observed transition is $\lesssim 1$. For cold clouds, these conditions are often not satisfied for easily observable molecules such as $^{13}C^{16}O$. For example, large variations from cloud to cloud in the apparent ratio of $N(^{13}C^{16}O)$ to $N(^{12}C^{18}O)$ can be produced by differences in line widths and corresponding differences in optical depths for the observed $^{13}C^{16}O$ transitions. The relationship between carbon monoxide abundance and visual extinction has been studied in detail by Frerking, Langer & Wilson (1982).

Cloud masses may also be obtained from estimates of number densities using intensities of collisionally excited lines such as the 2 mm and 2 cm lines of H_2CO, with kinetic temperatures derived from CO line intensities, and velocity gradients determined from emission line widths. The cloud volume is estimated from the size projected on the sky. This technique requires extensive cloud modelling.

(d) *Thermal emission.* Thermal emission can be used as a measure of *dust volume*, or, if a grain density is assumed, as a measure of *dust mass*. The computation of the volume from the flux density depends on the determination of grain emissivity. Even without knowledge of the grain emissivity, one may determine *cloud masses* from thermal emission if the extinction method is used to estimate hydrogen column densities in calibration objects.

The thermal emission method also has limitations, as will be discussed, but it is the only known method for determining the dust mass of a visually opaque cloud without assuming a ratio for the masses of dust and gas. As a method for determining total mass it is independent of the others except in so far as extinction may be used to calibrate both this and the molecular line methods.

The difficulty in determining the composition of interstellar dust is the absence of positive identifying features in either the extinction curve or the thermal emission spectrum. The absorption peaks are broad and are influenced not only by chemical composition but also by crystalline structure, grain mantles, impurities, and grain size. Great effort has been directed toward the discovery of plausible grain models which can account for the extinction curve and for the observed scattering and polarization of starlight. In judging whether a grain model is plausible, one favours constituent elements which are depleted in the gas phase below assumed cosmic abundances. One would prefer to avoid such judgements and instead to improve the knowledge of cosmic abundances by direct analysis of interstellar dust, the most abundant form of observable solid material.

Although no sure analysis has been devised, the range of acceptable grain models can be reduced by measurements of thermal radiation. Emission spectra can provide mean values of emissivities, and when combined with extinction curves, mean values of dielectric constants. Moreover, the polarization spectrum of emitted radiation may provide tests for the dielectric and magnetic properties of individual grain species.

We begin with a discussion of dust and cloud masses (Sections 2 and 3) followed by a discussion of grain properties (dielectric constants, emissivities, and composition; Sections 4, 5 and 6), and finally by a table of numerical values of submillimetre emission parameters (Section 7).

2 DUST MASS

To the precision now available, one finds empirically that the submillimetre spectrum of a cool cloud can be fitted by the product of the black-body intensity, $B(\nu,T)$ and an emissivity $Q(\nu)$ which varies as ν^{β}, where β increases from ~ 1 at frequency $\nu \gtrsim 50\ \mathrm{cm}^{-1}$ ($\lambda \lesssim 200\ \mu\mathrm{m}$) to $\gtrsim 2$ by $\nu \approx 10\ \mathrm{cm}^{-1}$ ($\sim 1000\ \mu\mathrm{m}$) (e.g. Erickson *et al.* 1981; Schwartz 1982). The spectrum thus gives a measure of a nominal temperature, T, falling within the range of temperatures of the individual dust grains. The determination of the dust mass of a cloud at a known distance is based on measurements of the flux density and T, and of $Q(\nu)$ and the size of the grains.

We consider first an idealized cloud composed of spherical grains of uniform size, composition, and temperature. The cloud is optically thick to starlight, is optically thin to submillimetre radiation, and is heated by internal processes. We shall later consider the effects of changes in these grain and cloud characteristics.

2.1 *Idealized cloud.* The flux density, $F(\nu)$, from a cloud at a distance D containing N spherical dust grains each of cross-section σ, temperature T, and emissivity $Q(\nu)$, is given by

$$F(\nu) = N[\sigma/D^2]\, Q(\nu)\, B(\nu,T). \tag{1}$$

The volume of dust in the cloud is given by

$$V = Nv \tag{2}$$

where v is the volume of an individual grain. Eliminating N from these equations, one obtains

$$V = [F(\nu)D^2/B(\nu,T)][v/\sigma]/Q(\nu). \tag{3}$$

If one assumes a grain density, ρ, one obtains an expression for the dust mass $M_d = V\rho$, or

$$M_d = [F(\nu)D^2/B(\nu,T)][(4/3)a/Q(\nu)]\rho \tag{4}$$

where a is the grain radius (Hildebrand *et al.* 1977).

It may appear that this expression is applicable only for clouds with uniform spherical grains and that the radius, a, must be known for the grains in each cloud under consideration. We show next that such is not the case. (We continue, until Section 2·5, to assume uniform composition and temperature.)

2.2 *Grain size.* The extinction curve for starlight passing through interstellar material is not consistent with attenuation by grains of uniform size (e.g.

Mathis & Wallenhorst 1981). Clearly the quantities a and $Q = Q(a,\nu)$ in equation (4) must be suitably chosen averages.

The problem would appear to be complicated by the increase in a, and hence a change in $Q(a,\nu)$, in dense clouds as indicated by peculiar extinction measurements (Carrasco, Strom & Strom 1973; Coyne, Gehrels & Serkowski 1974; Serkowski, Mathewson & Ford 1975). Fortunately, however, for $\lambda \gg a$, a simplification is imposed by the Kramers-Kronig relations which can be used to connect Q with a and with the complex dielectric constant, $\epsilon(\nu)$, of the grains at a given frequency ν (Jackson 1961; Andriesse 1977). For spherical grains, one may write the relationship in the form

$$Q(\nu) = 8\pi a(\nu/c)\,\mathrm{Im}\,\{-[\epsilon(\nu)-1]/[\epsilon(\nu)+2]\}. \qquad (5)$$

The quantity $a/Q(\nu)$ is thus independent of a for a given value of $\epsilon(\nu)$. Hence, if values a and Q are correctly determined in some medium, the corresponding ratio $a/Q(\nu)$ can be used with equation (4) to obtain the dust mass of any cloud with the same value of $\epsilon(\nu)$. Moreover, if $a/Q(\nu)$ is independent of a, the flux density from a grain must be proportional to σa (*cf.* equation 1); that is, to the grain *volume*, v.

Although these conclusions drawn from equation (5) are strictly true only for spherical grains of uniform composition, they will be important in our consideration of non-spherical grains and grains in which ice mantles surround refractory cores.

When an 'average' $Q(\nu)$ is measured as described in Section 5, the value for each grain is weighted by the contribution of the grain to the flux density and hence by the grain volume, v (or a^3). The corresponding value of a is weighted in the same manner; for a size distribution $n(a)$, the weighted average is $\int an(a)a^3da/\int n(a)a^3da$ where the integrals extend over all values of a.

Several distributions have been proposed. Mathis, Rumpl & Nordsieck (1977) obtain a fit to the extinction curve in the visible portion of the spectrum using a distribution $n(a) \propto a^{-3.5}$ in the range $0.01\ \mu\mathrm{m} \leq a \leq 0.25\ \mu\mathrm{m}$. Hong & Greenberg (1980) propose a core–mantle grain model with a distribution favouring somewhat larger radii. From ultraviolet scattering data, Witt (1979) infers a distribution including smaller radii. Using the Mathis *et al.* model, we obtain a weighted average radius $a = 0.1\ \mu\mathrm{m}$.

Ideally one should obtain a and $Q(\nu)$ in the same cloud. For the present we must be content with an a from the intercloud medium and a $Q(\nu)$ from a reflection nebula (Section 5).

From the arguments we have advanced, we conclude that in otherwise ideal clouds, variations in a from grain to grain and from cloud to cloud do not influence dust mass determinations if corresponding values of a and $Q(\nu)$ have once been established.

2.3 *Grain shape*. The polarization of starlight indicates the presence of aligned non-spherical grains. One must therefore consider the extent to which the observed flux density from a fixed volume of dust depends on the shape of the grains and the extent to which the substitution of $(4/3)a$ (equation (4)) for v/σ (equation (3)) depends on shape where a is determined from the extinction curve *assuming* spherical grains.

The first question is already answered by equation (5) and the consequent dependence of $F(\nu)$ on grain volume if one adds that the equation is very nearly true even for greatly elongated or flattened shapes (Purcell 1969). Unless one suspects grains in which the ratio of the axes is greater than about 5 one may safely ignore the effect of shape on flux density until Q can be measured with 10 times better precision than has yet been achieved.

The second question, the use of an a from the extinction curve to find v/σ for non-spherical grains, has been answered by Greenberg & Hong (1973) who note

> 'that if one were to take a set of spherical particles and elongate them, the shape of the wavelength dependence of extinction by the randomly oriented elongated particles would be qualitatively like that of smaller spheres. Thus, in order to obtain the same wavelength dependence of extinction, we should have started with larger spheres (but smaller numbers) in order to maintain the same total extinction. Thus, the total volume of material [derived for transparent clouds] turns out to be essentially shape-invariant because it is constrained not only by the amount of visual extinction but also by the wavelength dependence.'

Aside from consideration of techniques for estimating grain size if one assumes or somehow measures a mean projected cross-section, σ, of a grain, one may still ask how the ratio v/σ in equation (3) depends on shape. The answer to this purely geometrical question is readily found for any assumed convex shape using the relationship between mean projected area and total surface area derived by Schwartzschild (see Appendix). The ratio v/σ remains within a factor of 2 of its maximum value, $(4/3)a$, for all regular polyhedrons and all circular cylinders of length between $\sim 1/5$ and ~ 7 times the diameter (here, we define a to be $[\sigma/\pi]^{\frac{1}{2}}$).

We conclude that unless the bulk of dust is made up of very thin needles or flakes, the shape of the grains will have little influence on the determination of dust volume.

2.4 *Grain mantles.* In tenuous clouds exposed to starlight one expects grains to be composed entirely of refractory materials. In cool dense clouds, an absorption feature at 3·1 μm gives evidence for the formation of ice mantles where the mass of ice is usually found to be small compared to the mass of the refractory cores (Merrill, Russell & Soifer 1976). The effects of changes in surface material have been examined in detail by Aannestad (1975). The effect on extinction in the near and mid-infrared can be very great, but the effect on thermal emission at submillimetre wavelengths is relatively small: as we have seen (Section 2.2), the entire volume of a grain contributes to the emitted flux. The dust mass derived from equation (4) should not be significantly changed by mantles unless the mantles make up a major portion of the volume.

2.5 *Grain mixtures.* The temperature of a grain may depend significantly on its composition. Clouds made up of several grain species will exhibit composite thermal spectra with nominal temperatures, T, depending on abundances and temperatures for the individual species. For grains of widely differing emissivities, the temperatures may differ by almost a factor of two (Mezger, Mathis & Panagia 1982).

If the grains of different species also differ in size, then the value of a for the mixture may not be appropriate for grains emitting, say, on the low-frequency side of the spectrum. This does not mean that one must know a frequency-dependent value of a: the empirical relationship $Q(\nu) \propto \nu^{\beta}$ allows for the frequency dependence of any parameter affecting Q. What it does mean is that the 'average' value of Q should properly be determined at a frequency, ν_0, such that the various species contribute to the emitted flux in proportion to their volumes; that is, such that the product $B(\nu,T)Q(\nu)/a$ is the same for each species. Such a frequency will exist for two-component mixtures. The best one can do without assuming a model for grain composition is to determine Q somewhere near the centre of the emission spectrum.

A further complication for mixtures of grains has to do with exposure to ultraviolet radiation. Since the ratio of ultraviolet absorptivity to infrared emissivity may differ from one species to the next, the distribution of temperatures among the various species may be different in clouds having the same mixture of grains and the same mean temperature but different exposure to ultraviolet sources. Hence the mean value of a/Q determined from submillimetre emission by grains in an optical reflection nebula may not be accurate for a dark cloud of the same composition where the value $(a/Q)_i$ for the ith species is weighted differently by the corresponding values of $B(\nu,T_i)$. Ultraviolet exposure may also increase the weighting of very small grains ($a \lesssim 0{\cdot}01$ mm), which may undergo large temperature fluctuations when absorbing single energetic photons (Purcell 1976; Aannestad & Kenyon 1979).

The factors which alter grain temperatures including fluctuations of individual grains, differences in the emissivities of the various grain species, differences in ultraviolet exposure, and large-scale gradients due to differences in exposure to heat sources, all tend to broaden the observed thermal peaks. With measurements of the shapes of the peaks for selected objects one might place upper limits on some of these factors.

Even with such limits, the problems related to mixtures of grains with different emissivities are likely to remain more troublesome in principle than those related to variations only in size or shape or to the growth of mantles; but since the value of Q is still known at best to a factor of 3 for any cloud in any environment, it is likely that mass determinations can be improved by better measurements of Q whether or not the measured Q nicely corresponds to the average of the species according to their volumes.

2.6 *Externally heated clouds.* In the special case of clouds heated only from the outside by short-wavelength radiation for which $\tau \gg 1$, the far-infrared brightness is determined by the incident radiation and the surface properties of the cloud; not by the column density of the dust. For example, Keene (1981) finds approximately the same surface brightness for many globules despite variations in size. We shall leave the analysis of externally heated clouds to papers dealing especially with that problem and with energy transport in clouds (e.g. Leung 1975; Mezger *et al.* 1982).

2.7 *Conclusion to discussion of dust mass.* In clouds which are opaque to starlight, the only known method for determining dust mass without assuming a ratio of gas to dust is to measure thermal emission. The method depends on an assumed grain density and on an estimate of the distribution in grain size (in a reference object) based on the extinction curve, but is otherwise insensitive to grain models. Changes in the size and shape of the grains and the growth of mantles are unimportant except in extreme cases. The accuracy of the dust mass determinations is now limited primarily by the uncertainties in grain size and emissivity. In particular cases, the accuracy may be limited by large temperature variations along the line of sight: the possibility of underestimating the masses of cool components tends to increase with increases in the nominal temperature derived from the peak of the spectrum.

As measurements of grain emissivity improve, it will become necessary to assume specific models in order to estimate how different grain species are weighted in measurements of flux density and emissivity. The problem should be readily manageable for two-component models.

As we will show in later Sections (4, 5, 6), thermal emission measurements may themselves provide tests of alternative models. In particular, it may prove feasible to exploit the separation in temperature according to emissivity, the phenomenon at the crux of the complications for mixtures, to test the relationship of dielectric and magnetic properties of the principal species.

3 CLOUD MASS

In determining a cloud mass from thermal emission one need not know or assume values of either a or Q. One measures the visual extinction, A_v, and the submillimetre optical depth, $\tau(\nu)$, in a calibration object with visible background stars. One assumes that the ratio of A_v to hydrogen column density is the same as that for the intercloud medium and that that ratio and the ratio $A_v/\tau(\nu)$ for the calibration object can be applied to other (denser) clouds.

We shall write explicitly the relationship between dust mass and cloud mass; then the relationship of hydrogen column density to $\tau(\nu)$; then, qualitatively, the conditions favouring accurate results.

3.1 *Relationship of cloud mass to dust mass.* If one introduces a ratio, M_g/M_d, of gas mass to dust mass, $[M_g \approx M(\text{cloud})]$ then one may rewrite equation (4) to give the mass of a cloud:

$$M_c = [F(\nu)D^2/B(\nu,T)]\,[(4/3)\,a/Q(\nu)]\,\rho\,[M_g/M_d]. \qquad (6)$$

Purcell (1969) has shown that if one assumes an extreme value $\epsilon(0) = \infty$, for the static dielectric constant of grain material, one can use the integrated absorption to place a limit on the fraction of the total volume that is solid material. When this limit is combined with measurements of hydrogen column density, one may derive a lower limit ($\sim$ 60) for M_g/M_d.

In principle, one could use measurements of dust volume and hydrogen column density to derive a value for M_g/M_d. What we shall pursue here is the use of flux density and temperature measurements combined with extinction measurements to estimate the entire factor

$$C = [(4/3)\,a/Q(\nu)]\,\rho\,[M_g/M_d] \qquad (7)$$

appearing in equation (6).

3.2 *The Relationship of cloud mass to optical depth.* By using the ratio $R = A_v/E(B-V)$ of visual extinction, A_v, to colour excess, $E(B-V)$ (the excess over unreddened values of the difference between blue and visual magnitudes) and the ratio $N(H+H_2)/E(B-V)$ of hydrogen column density to colour excess as measured in the intercloud medium (Bohlin, Savage & Drake 1978), and assuming that the same numbers apply to a thin cloud, one obtains the column density per unit extinction, $N(H+H_2)/A_v$.

The submillimetre optical depth is given by

$$\tau(\nu) = F(\nu)[\pi\theta_{\frac{1}{2}}^2 B(\nu,T)]^{-1}, (\tau \ll 1) \tag{8}$$

where $F(\nu)$ is the flux density within a beam of angular radius $\theta_{\frac{1}{2}}$. In a cloud thick enough to allow a measurement of $\tau(\nu)$ but thin enough to allow a measurement of A_v one may obtain a ratio $A_v/\tau(\nu)$ from which to derive column densities in any cloud for which $\tau(\nu)$ is known (assuming $A_v/\tau(\nu) \approx$ constant and $N(H+H_2)/A_v \approx$ constant). Thus

$$N(H+H_2) = \tau(\nu)\,[A_v/\tau(\nu)]\,[N(H+H_2)/E(B-V)]\,[1/R]. \tag{9}$$

The mass of the cloud within the area $\pi\theta_{\frac{1}{2}}^2 D^2$ covered by the beam is

$$M_c = [\pi\theta_{\frac{1}{2}}^2 D^2]\,[N(H+H_2)m_H\mu]$$

where m_H is the mass of a hydrogen atom and μ is the ratio of total gas mass to hydrogen mass ($\approx$ 1·36). With equations (8) and (9), this becomes

$$M_c = [F(\nu)D^2/B(\nu,T)]C \tag{10a}$$

where $$C = [A_v/\tau(\nu)]\,[N(H+H_2)/E(B-V)]\,[1/R]\,m_H\mu \tag{10b}$$

or $$C = [N(H+H_2)/\tau(\nu)]m_H\mu. \tag{10c}$$

This analysis has been presented by Keene, Hildebrand & Whitcomb (1982) using the data of Whitcomb *et al.* (1981) for the reflection nebula NGC 7023. The numerical values are given in Table I.

3.3 *Conclusion to discussion of cloud mass.* When applying the factor C, evaluated by means of equation (10), to the measurement of a dense cloud, one need not be concerned, as in the case of dust mass, with choosing a frequency at which each grain species emits in proportion to its volume. On the other hand, one does not entirely skirt the problem of differences in temperature among different grain species (Section 2.5). It is still necessary that the various species emit in the same proportions as in the calibration object. To satisfy that condition, the temperature and ultraviolet exposure of the calibration object should be as near as possible to those of the clouds to be investigated. It would be desirable to extend the calibration to visually opaque clouds by measuring the extinction at, say, 2 μm and applying the known relationship of 2 μm to visual extinction.

In choosing a freqeuney at which to measure and apply the calibration one seeks to minimize the dependence of flux density on temperature by minimizing $h\nu/kT$. To good approximation maps made at $\lambda \gtrsim 350\ \mu$m follow the distribution in column density. Hot regions in large molecular clouds generally have relatively low mass and have little influence on submillimetre emission. For example, a map of the SgrA molecular cloud at 540 μm shows no maximum at the position of the Galactic Centre (Hildebrand *et al.* 1978).

The column density mapped by the submillimetre flux density is, of course, the *dust* column density. Whether the ratio of gas to dust is in fact constant, as implied by the calibration procedure, is a matter to be explored by very careful comparisons from cloud to cloud and point to point of thermal emission and molecular line emission taking into account all the factors other than gas column density which may influence the intensity of the observed lines (Frerking *et al.* 1982).

4 DIELECTRIC CONSTANTS

It is evident (equation (5)) that a determination of a and Q imposes a constraint on the mean value of the imaginary part of the expression $[\epsilon(\nu)-1]/[\epsilon(\nu)+2]$ where $\epsilon(\nu)$ is the complex dielectric constant.

If the extinction curve is known over a sufficient range of wavelengths, one may also determine the value of $[\epsilon(0)-1]/[\epsilon(0)+2]$ where $\epsilon(0)$ is the static dielectric constant. Purcell (1969) has shown that the static electric susceptibility $\chi'(0)$ of a cloud thinly populated by grains is given by

$$\chi'(0) = (2/\pi)\int_0^\infty [\chi''(\nu)/\nu]\, d\nu \tag{11}$$

where $\chi''(\nu)$, the imaginary part of the susceptibility, is related to the total cross-section by

$$4\pi\chi'' = n\lambda\sigma_t = n(c/2\pi\nu)\pi a^2\, Q_{\mathrm{ext}}(\nu).$$

Here, n = number of grains per unit volume, σ_t = total cross-section, and Q_{ext} = extinction efficiency (= emissivity, Q, at long wavelengths). For spherical grains

$$\chi'(0) = na^3\,[\epsilon(0)-1]/[\epsilon(0)+2]. \tag{12}$$

Combining these expressions we have

$$(4\pi^2 a/c)\{[\epsilon(0)-1]/[\epsilon(0)+2]\} = \int_0^\infty \nu^{-2} Q_{\mathrm{ext}}(\nu) d\nu \tag{13}$$

(see Aannestad & Purcell 1973).

The modification of (12) for non-spherical grains (Purcell 1969) is significant only if $\epsilon(0)$ is found to be much larger than the values ($\lesssim 4$) for typical insulating materials *and* if the grains are very far from spherical (ratio of axes $\gtrsim 4$ or $\lesssim 1/4$).

The evaluation of the integral is now limited primarily by the uncertainty in $Q_{\mathrm{ext}}(\nu)$ at infrared and submillimetre wavelengths.

5 EMISSIVITY

In the preceding discussion we have emphasized the importance of the emissivity, $Q(\nu)$, in estimating dust masses of clouds and dielectric characteristics of dust grains. We now review the principles involved in measuring $Q(\nu)$ at long wavelengths.

For (UV) wavelengths comparable to grain radii, the absorption cross-section, σ_{abs}, of a grain is nearly equal to the geometrical cross-section, $\sigma_{\mathrm{geom}} = \pi a^2$, and, by Babinet's principle, the scattering cross-section, σ_{scat}, must also be nearly equal to σ_{geom}. Hence for these wavelengths the total cross-section,

$$\sigma_{\mathrm{tot}} = \sigma_{\mathrm{abs}} + \sigma_{\mathrm{scat}}, \tag{14}$$

TABLE I

Submillimetre emission factors

Factor		Section or (eqn) in text	1983 Chicago assumptions	Range of values; comments	Ref.
β	Index giving frequency dependence of Q $[F(\nu) \propto \nu^{\beta} B(\nu,T)]$	2	1 for 50 μm $\leq \lambda \leq$ 250 μm 2 for $\lambda >$ 250 μm	$\sim$1 at 100 μm to $\geq$2 at 1000 μm	a
ρ	Density of grain material		3 g cm^{-3}	Amorphous C: 1·9 Graphite 2·25 Enstatite 3·1 SiC 3·2 Olivine 3·2–3·4 Magnetite 5·0	
a	Average grain radius weighted by $n(a)a^3$	2·2	0·1 μm	0·25–0·01 μm +smaller sizes	b c
Q(UV)	Grain emissivity at 0·15–0·30 μm	5	3	2·5–3·5	d
$\frac{Q(\mathrm{UV})}{Q_{125}}$	Ratio of Q(UV) to grain emissivity at 125 μm	5	4000	>700 1250 700–2500 if $\beta = 2$* 2000–5000 if $\beta = 1$* *At $\lambda >$ 50 μm	e f g g
Q_{250}	Grain emissivity at 250 μm (assuming $\beta = 1$)		$(125/250)^1(3/4000) = 1/2666$		
$\frac{N(\mathrm{H}+\mathrm{H}_2)}{\tau_\lambda}$	Ratio of hydrogen column density to optical depth at wavelength λ	(9)	$1{\cdot}2 \times 10^{25}$ atom cm^{-2} at 400 μm $(1{\cdot}2 \times 10^{25})(250/400)^2$ atom cm^{-2} at 250 μm	$7–26 \times 10^{24}$ atom cm^{-2} at 400 μm; depends on β	h

TABLE I—*concluded*

Submillimetre emission factors

Factor		Section or (eqn) in text	1983 Chicago assumptions	Range of values; comments	Ref.
C_{250}	Coeff. for estimates of cloud mass (250 μm) $[C = C_{250}(\lambda/250)^{\beta}]$	(10)	10 g cm^{-2}	Value probably good within a factor of 2	i
$\frac{(4/3)a\rho}{Q_{250}}$	Coeff. for estimates of dust mass (250 μm) $[Q = Q_{250}(250/\lambda)^{\beta}]$	(4)	0·1 g cm^{-2}	Value probably good within a factor of 3 or 4 for measured cloud (NGC 7023)	j

[a] Schwartz (1982); Erickson *et al.* (1981).
[b] Mathis *et al.* (1977); Mathis & Wallenhorst (1981).
[c] Lillie & Witt (1976).
[d] Aannestad (1975).
[e] Keene (1981).
[f] Harvey, Thronson & Gatley (1980): result converted to correspond to the 125 μm figure of Whitcomb *et al.* (1981).
[g] Whitcomb *et al.* (1981).
[h] Keene *et al.* (1982).
[i] Derived from expression for $N(H+H_2)_{250}$; *cf.* equations (10) and (11).
[j] From assumed values of a, ρ and Q_{250}.

becomes $$\sigma_{tot}(\text{UV}) \approx 2\sigma_{geom}, \ (\lambda \approx a). \tag{15}$$
(A resonance near 0·2 μm increases σ_{tot}(UV) to $\sim 3\,\sigma_{geom}$.) For wavelengths much greater than grain radii, $\sigma_{scat} \to 0$ and
$$\sigma_{tot}(\text{IR}) = \sigma_{abs}(\text{IR}), \ (\lambda \gg a). \tag{16}$$

If one divides these expressions through by σ_{geom}, one obtains the corresponding expressions for the extinction, scattering, and absorption efficiencies:
$$Q_{ext} = Q_{scat} + Q_{abs}, \tag{17}$$
$$Q_{ext}(\text{UV}) \approx 2 \text{ (or somewhat more)}, \ (\lambda \approx a) \tag{18}$$
and $$Q_{ext}(\text{IR}) = Q_{abs}(\text{IR}), \ (\lambda \gg a) \tag{19}$$
where, by Kirchhoff's law, Q_{abs}(IR) = emissivity, $Q(\nu)$.

The determination of $Q(\nu)$ at $\lambda \gg a$ consists in measuring the ratio $Q(\nu)/Q_{ext}$(UV), and multiplying by the known value of Q_{ext}(UV). The measurement of this ratio, as discussed in detail by Whitcomb *et al.* (1981), can be carried out by two methods. Briefly they are as follows: in the first, one finds $Q(\nu)/Q_{ext}$(UV) from the equivalent ratio of optical depths τ(IR)/τ(UV). The long-wavelength optical depth, τ(IR), is the ratio ($\ll 1$) of the observed brightness of a (thin) cloud to the brightness of a black body of the same temperature (equation (8)). The UV optical depth is obtained by measuring the visual or near IR extinction of background stars. In the second method, one carries out an energy balance for grains in a reflection nebula heated by a UV-emitting star of known spectral type. The energy absorbed from the UV flux of the star is set equal to the observed thermal emission. If one assumes that the star-to-grain distance is equal to the projected distance and neglects UV absorption between the star and the grain one obtains an upper limit to the energy absorbed and hence a lower limit to $Q(\nu)$.

In carrying out the energy balance one must use a value for the fraction of the incident UV energy absorbed by the grain. We have implicitly taken that fraction, Q_{abs}(UV)/Q_{ext}(UV) to be 1/2 in equation (18). Whitcomb has noticed that the UV and IR observations of a reflection nebula can be combined to give a measure of the albedo, $\gamma = Q_{scat}/Q_{ext}$, and hence of $(1-\gamma) = Q_{abs}/Q_{ext}$. The probability, P, that a UV photon from an imbedded star will escape from a cloud is just the ratio of the UV luminosity to the total luminosity,
$$P = L(\text{UV})/[L(\text{UV})+L(\text{IR})]. \tag{20}$$
But P is also given by
$$P = \gamma^{\langle m \rangle} \tag{21}$$
where $\langle m \rangle$ is the mean number of times a photon is scattered before it escapes. If the mean scattering angle is small, as indicated by the observations of Lillie & Witt (1976), then $\langle m \rangle \approx \tau$. Using the measured values of P (equation (20)) and τ (equation (8)), one obtains a value of γ (0·36) in good agreement with that obtained from observations of the diffuse galactic light (0·40; Lillie & Witt 1976).

In measuring an 'average' $Q(\nu)$, values for individual grains are weighted according to size as discussed in Section 2.2. The problem of determining $Q(\nu)$ in mixtures of grains is discussed in Section 2.5.

6 GRAIN COMPOSITION: POLARIZATION OF EMITTED RADIATION

We have thus far considered only average properties of grains. We have discussed individual grain species only to the extent of arguing that the mix of species is not important with regard to determinations of dust volume. With regard to the evolution of the interstellar medium, however, the question is fundamental.

Models of dust grain composition are based largely on examination of the extinction curve and on indirect evidence from studies of the chemical composition of interstellar gas as discussed, for example, by Field (1973). In the model of Mathis *et al.* (1977), graphite grains, suggested by an absorption feature at 0·22 μm (Hoyle & Wickramasinghe 1962), and silicate grains, suggested by a feature at 10 μm (Gillett, Low & Stein 1968; Woolf & Ney 1969; Gilman 1969), are present in approximately equal numbers and account for essentially the entire grain population. If that model is correct, then the graphite grains will be at a higher temperature than the silicate grains since graphite is a better ultraviolet and visible absorber and a poorer far-IR and submillimetre emitter (Mezger *et al.* 1982). Consequently the flux on the short-wavelength side of the composite emission peak should be mostly from graphite and the flux on the long-wavelength side should be mostly from silicates.

The separation in temperature provided by the difference in the emissivities of the grains provides an opportunity to test the model by examining another distinguishing characteristic, the polarization properties. The 0·22 μm feature (graphite?) shows no enhanced polarization (Gehrels 1974). The vicinity of the 10 μm feature (silicates?) shows strong polarization in some objects, probably due to grain alignment (Dyck & Beichman 1974). One should therefore expect to find stronger polarization on the long-wavelength side of the emission peak than on the short-wavelength side if the graphite-silicate model is correct. The polarization should be normal to that at 10 μm if it is correct that the 10 μm polarization is due to selective absorption by aligned grains.

For other models one would expect qualitatively different results. If, for example, the carbon is amorphous rather than crystalline as proposed by Draine (1981) and by Czyzak, Hirth & Tabak (1982), then the carbon emissivity at $\lambda \gtrsim 100\ \mu$m would be higher than that of silicates. An amorphous carbon-silicate mixture would therefore exhibit stronger polarization on the short-wavelength side of the emission peak.

The results of the Cornell group (Gull *et al.* 1980) and the University College, London group (Cudlip *et al.* 1982) in their infrared polarimetry of Orion are consistent with the interpretation that the short-wavelength side of the spectrum is dominated by emission from grains of low magnetic susceptibility (e.g. graphite). The polarization reported by Cudlip *et al.* (1982) at $\lambda_{\rm eff} = 77\ \mu$m is $2{\cdot}2 \pm 0{\cdot}4$ per cent, nearly orthogonal to the polarization in the near-infrared. The question to be resolved is whether the polarization at $\lambda \gg 100\ \mu$m is larger or smaller than this value.

It has long been recognized that clouds containing aligned grains should emit polarized radiation (e.g. Stein 1966). It is now possible that submillimetre polarimetry could become a practical technique for investigation of grain characteristics.

7 NUMERICAL VALUES OF SUBMILLIMETRE EMISSION PARAMETERS

In Table I we present numerical values for the quantities which enter into estimates of dust mass, cloud mass, and grain characteristics as derived from submillimetre emission. To avoid any implication of finality, our values are labelled '1983 Chicago assumptions'. There is no basis for assigning statistical errors. Instead, we give a range of values consistent with various assumptions and measurements. We do not attempt to evaluate the inaccuracy in the implicit assumption that values for one cloud can be applied to another.

Nowhere have we assumed a value for the ratio, M_g/M_d, of gas mass to dust mass. The value one would infer from the coefficients given in the last two lines of the table ($10/0{\cdot}1 = 100$) is subject to the large errors claimed for those coefficients.

An important goal of this paper is to encourage improved measurements of the quantities presented in the table.

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APPENDIX: AVERAGE PROJECTION OF A CONVEX OBJECT ON TO A PLANE

S.Chandrasekhar and R.G.Sachs have called my attention to a theorem by K.Schwarzchild which states that the average shadow of a convex body is just one-quarter of the total surface area.

The theorem was rediscovered by M.Schwarzschild during World War II while he was working with Chandrasekhar and Sachs at the army ballistics laboratory in Aberdeen, Maryland. Subsequently, Schwarzschild learned from G.Birkhoff of his father's earlier work (M.Schwarzschild, private communication). Sachs recalls that A.Wintner was the first to recognize that the theorem was a rediscovery. I have been unable, even with the help of these distinguished colleagues, to find the original publication. It can doubtless be found somewhere in the Goettingen Archives where Martin Schwarzschild has sent his father's papers.

The theorem is easy to prove when one knows that there is a simple answer. I offer the following:

Divide the convex surface into n elements with $n/2$ exposed, on the average, to the observer. Let $\mathbf{a}_i$ = outward vector representing the ith element, and $\hat{\mathbf{r}}$ = unit vector toward the observer. Then $\mathbf{a}_i \cdot \hat{\mathbf{r}}$ = projection of the ith element on the sky.

The average value of that projection for all directions of $\mathbf{a}_i$ in the hemisphere facing the observer, i.e. for all rotations of the grain for which the ith element is visible, is just $a_i/2$*. Hence the mean projected area, σ, of the (average) $n/2$ visible elements is

$$\sigma = \sum_{i=1}^{n/2} a_i/2 = (1/4) \sum_{i=1}^{n} a_i$$

or

$$\sigma = A/4$$

where A = total surface area.

* $\int_0^{\pi/2} a_i \cos\theta \, (2\pi \sin\theta)\, d\theta \Big/ \int_0^{\pi/2} (2\pi\sin\theta)\, d\theta = a_i/2.$