For simple linear reg $\underset{\in\mathbb{R}}{y_i} = \beta_0 + \beta_1 \underset{\in\mathbb{R}}{x_i} + e_i$. $i=1,\cdots,n$

show that the F-test-statistic for $H_0: E[Y|X] = \beta_0$ v.s $H_A: E[Y|X] = \beta_0 + \beta_1 x$
is $(t\text{-statistic})^2$.

Recall that: $$t\text{-statistic} = \frac{\hat\beta_1}{s.e.(\hat\beta_1)}$$

$$F\text{-statistic} = \frac{(RSS_{H_0} - RSS_{H_A})/(df_{H_0} - df_{H_A})}{RSS_{H_A}/df_{H_A}}$$ $\rightarrow$ $\hat\sigma^2$ estimated from $Y \sim X$.

Specifically, for the simple linear reg. model,

we have $$RSS_{H_0} \overset{\text{practice in Lecture 7}}{=} TSS = \sum_{i=1}^n (y_i - \bar y)^2$$

$$RSS_{H_A} = \sum_{i=1}^n (y_i - \hat y_i)^2$$ with $\hat y_i$ the fitted value from $Y \sim X$.

$$\Rightarrow RSS_{H_0} - RSS_{H_A} = \underbrace{\sum_{i=1}^n (y_i - \bar y)^2}_{TSS} - \underbrace{\sum_{i=1}^n (y_i - \hat y_i)^2}_{RSS \text{ of } Y\sim X}$$

$$\overset{\text{from Lecture 7}}{=} SS_{Reg} = \sum_{i=1}^n (\hat y_i - \bar y)^2 .$$

In addition, $df_{H_0} = n-1$ and $df_{H_A} = n-2$

$$df_{H_0} - df_{H_A} = 1.$$

Overall, we have

$$F\text{-statistic} = \frac{\sum_{i=1}^n (\hat y_i - \bar y)^2}{\hat\sigma^2}$$ $\leftarrow$ estimated σ^2 from $Y \sim X$.

Note that $\sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = \sum_{i=1}^{n} [\underbrace{\hat{\beta}_0 + \hat{\beta}_1 x_i}_{\hat{Y}_i} - \underbrace{(\hat{\beta}_0 + \hat{\beta}_1 \bar{x})}_{\bar{Y}}]^2$

$$= \sum_{i=1}^{n} (\hat{\beta}_1 (x_i - \bar{x}))^2$$

$$= \hat{\beta}_1^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$\Rightarrow$ F-statistic $= \dfrac{\hat{\beta}_1^2 \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}{\hat{\sigma}^2}$

For the $(t\text{-statistic})^2 = \dfrac{\hat{\beta}_1^2}{[s.e(\hat{\beta}_1)]^2} = \dfrac{\hat{\beta}_1^2}{\widehat{Var}(\hat{\beta}_1 | x)}$. $\quad \sigma^2 = Var(e_i|x)$

therefore, we only need to show $\widehat{Var}(\hat{\beta}_1 | x) = \dfrac{\hat{\sigma}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$.

from lecture 5. we know

$$\widehat{Var}\left[\underbrace{\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}}_{2\times 1} \Big| X\right] = \underbrace{\hat{\sigma}^2 (X^T X)^{-1}}_{2\times 2} \quad \text{where } \underset{n\times 2}{X} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

Since $X^T X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$,

$\begin{pmatrix} a & c \\ c & b \end{pmatrix}^{-1}_{2\times 2} = \dfrac{1}{ab - c^2} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}_{2\times 2}$

we have $(X^T X)^{-1} = \dfrac{1}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{bmatrix}_{2\times 2}$.

This implies

$$\widehat{Var}(\hat{\beta}_1 | X) = \hat{\sigma}^2 \frac{1}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} (\underbrace{\sum_{i=1}^{n} x_i}_{n\bar{x}})^2} = \frac{\hat{\sigma}^2}{\sum_{i=1}^{n} x_i^2 - n \bar{x}^2}.$$

Since $\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} (x_i^2 - 2 x_i \bar{x} + \bar{x}^2)$

$$= \sum_{i=1}^{n} x_i^2 - 2 (\underbrace{\sum_{i=1}^{n} x_i}_{n\bar{x}}) \bar{x} + n \bar{x}^2$$

$= \sum_{i=1}^{n} x_i^2 - n \bar{x}^2$, we have $\widehat{Var}(\hat{\beta}_1 | X) = \hat{\sigma}^2 / \sum_{i=1}^{n} (x_i - \bar{x})^2$.

This finishes the proof.