Stat 510: Exam 3

Moulinath Banerjee

December 11, 2017

Announcement: The exam carries 55 points but the maximum possible score is 45 points.

Problem 1: Let $\underline{X} = (X_1, X_2, \ldots, X_p)$ be a random vector and assume that $M_{\underline{X}}(\mathbf{t}) = E(e^{\mathbf{t}^T \underline{X}})$ exists for all sufficiently small $\mathbf{t}$, i.e. $\|\mathbf{t}\| < h_0$, for some $h_0 > 0$. Show that $\underline{X}$ is *exchangeable*, i.e. any permutation of the co-ordinates of $\underline{X}$ has the same distribution as $\underline{X}$, if and only if $M_{\underline{X}}(\mathbf{t})$ is symmetric in its arguments, i.e. $M_{\underline{X}}(\mathbf{t}) = M_{\underline{X}}(\tilde{\mathbf{t}})$ for any $\mathbf{t}$, where $\tilde{\mathbf{t}}$ is formed by an arbitrary permutation of the co-ordinates of $\mathbf{t}$. [10]
Hint: To get a feel for the problem, it might help to consider $p = 2$ first.

.

Problem 2: Consider a random vector (X, Y) that has a joint distribution of the form:

$$f(x,y) = C_{a,b}\, g\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) 1((x,y) \in \mathbb{R}^2)\,,$$

where $g : [0, \infty) \to [0, \infty)$ satisfies $\int_0^\infty g(u)du = 1$, and a, b are positive constants. Define random variables (R, Θ) by $X = aR\cos(\Theta)$ and $Y = bR\sin(\Theta)$, where $0 < R < \infty$ and $0 < \Theta < 2\pi$.

(i) What is the geometric interpretation of (R, Θ)?

(ii) Find the joint distribution of (R, Θ) and show that R and Θ are independent. Compute their marginal densities.

(iii) Calculate explicitly the constant $C_{a,b}$ in terms of a and b. [3 + 12 + 5 = 20]

Problem 3: Let $X \sim \text{Exp}(\lambda)$ [failure time] and $T \sim \text{Exp}(\mu)$ [observation time] and suppose that X and T are independent. We observe the pair (Δ, T) where $\Delta = 1(X \leq T)$. This type of observed data is called 'current-status' data. [**Note:** The symbol := should be interpreted as 'is defined as'.]

(a) Calculate $f_{\Delta|T}(\delta|T = t) := P(\Delta = \delta|T = t)$ for $\delta = 0, 1$ (this is the conditional p.m.f. of Δ given $T = t$, where $t > 0$) and hence deduce that the 'mixed' joint density of (Δ, T), say $f_{T,\Delta}(\delta, t)$, is given by the expression:

$$f_{T,\Delta}(\delta, t) := f_{\Delta|T}(\delta|T = t)\, g_T(t) = (1 - e^{-\lambda t})^{\delta}\, e^{-\lambda t(1-\delta)}\, \mu e^{-\mu t}\,.$$

(b) Calculate $f_{\Delta}(\delta) := P(\Delta = \delta)$, the marginal p.m.f of Δ and $f_{T|\Delta}(t|\Delta = \delta)$, i.e. the conditional p.d.f. of T given $\Delta = \delta$ for the two possible values of δ. Show that

$$E(T|\Delta = 1) = \frac{1}{\lambda}\left[\frac{\lambda + \mu}{\mu} - \frac{\mu}{\mu + \lambda}\right] \quad \text{and} \quad E(T|\Delta = 0) = \frac{1}{\lambda + \mu}\,.$$

(c) Find the bigger of the two. Does this conform to intuition? [8 + 12 + 5 = 25] .