Homework 2 solution

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Problem 1 Let θ be the angle between the line and the x axis. Then we have

$$\theta \sim \text{Uniform}[-\pi/2, 3\pi/2), \qquad Y = \tan(\theta), \tag{1}$$

and we need to find the distribution of Y. Apart from a zero-measure set, we can restrict tan on each interval $(-\pi/2, \pi/2), (\pi/2, 3\pi/2)$ to make it a 1-1 function (say, g_1, g_2). We can calculate the image distribution on each interval and apply Theorem 2.5. (in the note) to get the final result.

It is observed that for $g_1 : (-\pi/2, \pi/2) \to \mathbb{R}, \theta \mapsto \tan(\theta)$, we have

$$\frac{d}{dy} g_1^{-1}(y) = \frac{d}{dy} \arctan(y) = \frac{1}{y^2 + 1}, \tag{2}$$

and for $g_2 : (\pi/2, 3\pi/2) \to \mathbb{R}, \theta \mapsto \tan(\theta)$, we have

$$\frac{d}{dy} g_2^{-1}(y) = \frac{d}{dy} (\arctan(y) - \pi) = \frac{1}{y^2 + 1}. \tag{3}$$

As the pdf of uniform distribution on $(-\pi/2, 3\pi/2)$ is $1/2\pi$, theorem 2.5. tells us that

$$f_Y(y) = \frac{1}{2\pi} \frac{1}{y^2 + 1} + \frac{1}{2\pi} \frac{1}{y^2 + 1} = \frac{1}{\pi(y^2 + 1)}, \quad \forall\, y \in \mathbb{R}. \tag{4}$$

which is pdf of Cauchy distribution.

Note: Although they are the same, I choose to present θ as Uniform distribution on $[-\pi/2, 3\pi/2)$ instead of $[0, 2\pi)$ because arctan is only defined on $(-\pi/2, \pi/2)$. This can makes the solution concise. But you can do it either way. (The other case requires taking care in dividing domains, taking the inverse functions and derivative,...) $\square$

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Problem 2 Because of definition of conditional probability and the fact that $W > i + j$ implies $W > i$, we can write

$$P(W > i + j | W > i) = \frac{P(W > i + j, W > i)}{P(W > i)} = \frac{P(W > i + j)}{P(W > i)}. \tag{5}$$

This together with the memoryless property give

$$P(W > i + j) = P(W > i)P(W > j) \quad \forall\, i, j = 1, 2, \ldots \tag{6}$$

Let $i = 1$, we have

$$P(W > j + 1) = P(W > j)P(W > 1) \quad \forall\, j = 1, 2, \ldots \tag{7}$$

Substitute $j = 1$, then 2, we have

$$P(W > 2) = (P(W > 1))^2, \tag{8}$$

$$P(W > 3) = P(W > 2)P(W > 1) = (P(W > 1))^3. \tag{9}$$

From that and (7), we can easily use induction to show

$$P(W > j) = (P(W > 1))^j = q^j, \quad \forall\, j = 1, 2, \ldots \tag{10}$$

where we denote $q = P(W > 1), p = 1 - q = P(W = 1)$. Hence

$$P(W = j) = P(W > j - 1) - P(W > j) = q^{j-1} - q^j = pq^{j-1}, \quad \forall\, j = 1, 2, \ldots \tag{11}$$

which is exactly what we need. $\square$

Problem 3 Because the event ($[X]=m$ and $X-[X]\le t$) is equivalent to $m\le X\le m+t$, we have

$$P([X]=m, X-[X]\le t)=P(m\le X\le m+t)=\int_m^{m+t}\lambda e^{-\lambda t}=e^{-\lambda m}(1-e^{-\lambda t}), \forall\, m\in\mathbb{N}, t\in[0,1). \tag{12}$$

From this, we have

$$P([X]=m)=\lim_{t\to 1}P([X]=m, X-[X]\le t)=e^{-\lambda m}(1-e^{-\lambda}), \tag{13}$$

and thanks to geometric series,

$$P(X-[X]\le t)=\sum_{m=0}^{\infty}P([X]=m, X-[X]\le t)=\frac{1}{1-e^{-\lambda}}(1-e^{-\lambda t}). \tag{14}$$

Taking the derivative, we have the pdf for $X-[X]$

$$f_{X-[X]}(t)=\frac{\lambda}{1-e^{-\lambda}}e^{-\lambda t}, \quad \forall\, t\in(0,1), \tag{15}$$

and 0 otherwise.

Note: So we have the distribution of $[X]$ is geometric. Another way to derive the distributions of $[X]$ and $X-[X]$ is from (12), we can see that the joint distribution is product of a function of m and a function of t. Therefore $[X]$ and $X-[X]$ are independent, and

$$P([X]=m)\propto e^{-\lambda m}, P(X-[X]\le t)\propto 1-e^{-\lambda t}, \tag{16}$$

finding the normalized constant and we get what we need. $\square$

Problem 4 (1) We have

$$\begin{aligned}
EX^r &= \int_0^\infty x^r dF(x) \\
&= \int_0^\infty \left(\int_0^y r y^{r-1} dy \right) dF(x) \\
&\overset{Fubini}{=} \int_0^\infty \left(\int_y^\infty r y^{r-1} dF(x) \right) dy \\
&= \int_0^\infty \left(\int_y^\infty dF(x) \right) r y^{r-1} dy \\
&= \int_0^\infty (1 - F_X(y)) r y^{r-1} dy.
\end{aligned}$$

(2) Because $Y = Y^+ - Y^-$, we have

$$EY = EY^+ - EY^-. \tag{17}$$

As both Y^+ and Y^- are non-negative, we can apply the result in part (1) to get

$$EY = EY^+ - EY^- = \int_0^\infty P(Y > y)dy - \int_0^\infty P(-Y > y)dy = \int_0^\infty (P(Y > y) - P(Y < -y))dy \tag{18}$$

(3) If $P(Y > y) \geq P(Y < -y)$, we have the integrand above being non-negative for all value of $y > 0$, therefore the integral is also non-negative.

(4) By noticing that

$$\int_0^\infty \lambda e^{-\lambda y} dy = 1, \int_0^\infty y \lambda e^{-\lambda y} dy = 1/\lambda, \tag{19}$$

for all $\lambda > 0$, we have $\int f(y)dy = 1/2 + 1/2 = 1$, and because $f(y) \geq 0$ for all y. It is a pdf. Moreover,

$$EY = \int y f(y) dy = \frac{1}{2} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) > 0. \tag{20}$$

To prove this using the result in (3), we observe that for $y \geq 0$

$$P(Y > y) = \frac{1}{2} e^{-\lambda_1 y} > \frac{1}{2} e^{-\lambda_2 y} = P(Y < -y), \tag{21}$$

and the integral of a positive function is positive. Therefore $EY > 0$. □

Problem 5 (1) Because of the continuity of probability measure, we have

$$P(X > 1/n) \xrightarrow{n\to\infty} P(X > 0). \tag{22}$$

Because the limit is positive, there must exist some m such that

$$P(X > 1/m) > 0. \tag{23}$$

Let

$$Y = \begin{cases} 1/m, & \text{if } X > 1/m, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

We have Y is a simple function. Because of its definition and the fact that $X \geq 0$, it is straight forward that $X \geq Y$. Moreover,

$$EY = 1/m \times P(X > 1/m) > 0. \tag{25}$$

Hence

$$EX = \sup_{Z \text{ simple } \leq X} EZ \geq EY > 0. \tag{26}$$

(2) Because $X_1 \leq X_2$, the set of simple variables being less than or equal to X_1 is a subset of the set of simple variables being less than or equal to X_2. Hence

$$EX_1 = \sup_{Z \text{ simple } \leq X_1} EZ \leq \sup_{Z \text{ simple } \leq X_2} EZ = EX_2. \tag{27}$$

$\square$

Problem 6 With the similar reason to Problem 3, we have

$$P(X > x + y) = P(X > x)P(X > y), \quad \forall\, x, y \geq 0. \tag{28}$$

Let $g(x) = \log P(X > x)$, we have $g(x)$ is a continuous function and

$$g(x + y) = g(x) + g(y), \quad \forall\, x, y \geq 0. \tag{29}$$

Because our final goal is to prove $P(X > x) = -\lambda x$ for some $\lambda > 0$, we can prove $g(x) = -\lambda x$ instead. (That is proving g is a linear function!). First we see that for all $n \in \mathbb{N}$, we can use induction (similar to problem 3) to show

$$g(nx) = ng(x), \quad \forall\, n \in \mathbb{N}, x \in \mathbb{R}. \tag{30}$$

For any $y > 0$ and $m \in \mathbb{N}$, we can substitute $x = y/m$ to the equation above to get

$$g\left(\frac{n}{m}y\right) = ng\left(\frac{y}{m}\right) = \frac{n}{m}g(y), \quad \forall\, n, m \in \mathbb{N}, x \in \mathbb{R}, \tag{31}$$

as $g(y) = g(m \times y/m) = mg\left(\frac{y}{m}\right)$. Let $y = 1$ and notice that every rational number q can be written in terms of n/m where $n, m \in \mathbb{N}$,

$$g(q) = qg(1) \quad \forall\, q \in \mathbb{Q}, q > 0. \tag{32}$$

Let $\lambda = -g(1) > 0$ (as $P(X > 1) < 1$), we have

$$g(q) = -\lambda q \quad \forall\, q \in \mathbb{Q}, q > 0. \tag{33}$$

For any real number x, there exists a sequence of rational numbers $q_1, q_2, \ldots$ such that

$$q_n \to x \quad (n \to \infty), \tag{34}$$

therefore, by the continuity of g,

$$g(x) = \lim_{n\to\infty} g(q_n) = \lim_{n\to\infty} -\lambda q_n = -\lambda x. \quad \forall\, x \in \mathbb{R}, x > 0. \tag{35}$$

□