Homework 2

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Problem 1: A line is drawn through a point (-1,0) in a random direction. Let $(0, Y)$ denote the point at which it cuts the Y-axis. Find the (p.d.f) density of Y.

Problem 2: Let W be a random variable taking values in the set of integers $\{1, 2, 3, \ldots\}$ with $P(W = j) > 0$ for all $j \geq 1$, and having the so-called 'memoryless' property that

$$P(W > i + j \mid W > i) = P(W > j)\,.$$

Show that

$$P(W = j) = q^{j-1}\, p$$

for some $0 < p < 1$. Thus W is Geometrically distributed.

Problem 3: Let X follow the Exponential(λ) distribution and $[X]$ denote the greatest integer not exceeding X. Show that:

$$P([X] = m, X - [X] \leq t) = e^{-\lambda\, m}\, (1 - e^{-\lambda\, t})\,.$$

Hence, calculate the p.m.f of $[X]$ and the p.d.f of $X - [X]$.

Problem 4: Let X be a non-negative random variable. Thus $P(X \geq 0) = 1$.
(i) Let $r \geq 1$. Show that $E(X^r) = \int_0^\infty r\, x^{r-1} (1 - F_X(y))\, dy$ where F_X is the distribution function of X.
(ii) Consider now a random variable Y which can assume both positive and negative values (and assume that $E(|Y|) < \infty$). Let Y^+ denote the maximum of Y and 0 and Y^- the negative of the minimum of Y and 0. Show that $Y = Y^+ - Y^-$. Use this to develop an expression for EY in terms of its distribution function F_Y.
(iii) Suppose Y has heavier tails on the right than on the left: i.e. $P(Y > y) \geq P(Y < -y)$ for all $y > 0$. Argue that $E(Y) \geq 0$.
(iv) Let Y be a random variable with p.d.f

$$f(y) = \frac{1}{2}\lambda_1\, e^{-\lambda_1 |y|}\, 1(y \geq 0) + \frac{1}{2}\lambda_2\, e^{-\lambda_2 |y|}\, 1(y < 0)\,,$$

with $\lambda_1 > \lambda_2$. [Verify that f is indeed a valid probability density.]
Show that $E(Y) > 0$ first by direct calculation, and then by using the result in (iii).

Problem 5: Using the supremum based definition of the expectation of a non-negative random variable show that:
(a) If X is a non-negative random variable and $P(X > 0) > 0$, then $EX > 0$. [Argue first that for some positive integer m, P(X ¿ 1/m) ¿ 0.
$(b)If$X$_1, X_2$ are non-negative random variables and $X_1 \leq X_2$, then $E(X_1) \leq E(X_2)$.

Problem 6 (supplementary): If X is a continuous random variable with $P(X \geq 0) = 1$ and satisfying the memoryless property $P(X > x + y | X > x) = P(X > y)$, show that X has the Exponential(λ) distribution for some $\lambda > 0$.