Continuous Random Vectors.

$(X, Y) \in \mathbb{R}^2$.
with density $f(x,y)$.

Necessary and sufficient conditions for independence of X and Y:

(i) $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

(ii) $\boxed{f_{X,Y}(x,y) = g(x) h(y)}$ for some non-negative functions g, h

(iii) $f_{Y|x}(y) = f_Y(y)$ or

$f_{X|y}(x) = f_X(x)$

$f_{Y|X=x}(y)$

$f_{X|Y=y}(x)$

Same considerations for independence hold for discrete random vector (X,Y) with pdf replaced by pmf.

Lemma 4.27. of CB: ✓

If $f(x,y) = g(x)h(y)$ for non-negative functions g and h, then X is indep. of Y. Converse is also true.

X ind. of $Y \Rightarrow f(x,y) = f_X(x) \cdot f_Y(y)$

and representation as $g(x) \cdot h(y)$ is immediate

$$f(x,y) = \underbrace{c \cdot f_X(x)}_{= g(x)} \cdot \underbrace{\frac{1}{c} \cdot f_Y(y)}_{= h(y)} \qquad c > 0$$

look up proof in CB.

Expectations for a function $g(X,Y)$.

$\mathbb{E}\, g(X)$

$$\mathbb{E}[g(X)] = \int g(x) f(x)\, dx$$

X continuous.

$$= \sum_{x \in \mathcal{X}} g(x) f(x)$$

if X discrete, $f(x)$ is a p.m.f

What about $\mathbb{E}[g(X,Y)]$?

First suppose ~~that~~ g is non-negative

$g : \mathbb{R}^2 \to [0, \infty)$.

Then $\mathbb{E}[g(X,Y)] = \int g(x,y) f(x,y)\, dx\, dy$

For a general function g:

Provided $\mathbb{E}[|g(X,Y)|] < \infty$,

$$\mathbb{E}[g(X,Y)] = \int g(x,y) f(x,y)\, dx\, dy$$

Note ~~that~~ $g(X,Y)$ could just be a function of X or Y.

Suppose $g(X,Y) = \underline{t}(X)$

From previous classes:

$$\underline{\mathbb{E}[\underset{=}{t}(X)]} = \underline{\int t(x) f_X(x)\, dx}$$

By recent definition:

$$\mathbb{E}[t(X)] = \int_{\mathbb{R}^2} t(x) f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} \Big[\underbrace{\int_{\mathbb{R}} f(x,y)\, dy}_{\substack{\| \\ f_X(x)}} \Big] t(x)\, dx$$

$$= \int_{\mathbb{R}} t(x) f_X(x)\, dx.$$

$$\operatorname{Var} g(X,Y) = \underline{\mathbb{E}\left[g(X,Y)^2\right]} - \left(\mathbb{E}\, g(X,Y)\right)^2$$

$$\int g^2(x,y) f_{X,Y}(x,y)\, dx\, dy \text{ etc.}$$

$$\mathrm{Var}[g(X,Y)] = \mathbb{E}\left[(g(X,Y) - \mathbb{E}g(X,Y))^2\right]$$

a quick generalization to d-dimensional random vectors.

$(X_1, \ldots, X_d)$: continuous r.v with density $f(x_1, \ldots, x_d)$

Then:

$$P[X_1 \in A_1, X_2 \in A_2, \ldots, X_d \in A_d]$$

$$= \int_{A_1 \times A_2 \times \cdots \times A_d} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d$$

(for all Borel sets $A_1, \ldots, A_d$)

$\Updownarrow$ equivalent

to the display being true for A_i's intervals.

$$F_{X_1, \ldots, X_d}(x_1, \ldots, x_d) = \int_{(-\infty, x_1] \times \cdots \times (-\infty, x_d]} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d$$

$$f_{X_1,\ldots,X_k}(x_1,\ldots,x_d) \qquad k<d$$

$$= \int_{\mathbb{R}^{d-k}} f(x_1,\ldots,x_d)\, dx_{k+1}\cdots dx_d$$

$$f_{X_{k+1},\ldots,X_d \mid X_1=x_1,\ldots,X_k=x_k}(x_{k+1},\ldots,x_d)$$

$$= \frac{f(x_1,x_2,\ldots,x_d)}{f_{X_{k+1},\ldots,X_d}(x_{k+1},\ldots,x_d)}$$

$(X_1,\ldots,X_k)$ is indep. of $(X_{k+1},\ldots,X_d)$

iff.

$$f(x_1,\ldots,x_d) = g(x_1,\ldots,x_k)\, h(x_{k+1},\ldots,x_d)$$

$$(X_1,\ldots,X_d)$$

$$= (X_1,\ldots,X_{i_1}),(X_{i_1+1},\ldots,X_{i_2})\cdots(X_{i_{m-1}+1},\ldots,X_{i_m})$$

where $i_m = d$

Then these groups of random variables are mutually independent if and only if:

$$f(x_1, x_2, \dots, x_d) = \prod_{l=1}^{m} g_l(x_{i_{l-1}+1}, \dots, x_{i_l})$$

$(i_m = d,\ i_0 = 0)$

$$\mathbb{E}[h(x_1, \dots, x_d)] = \begin{cases} \int h(x_1, \dots, x_d) f(x_1, \dots, x_d)\, dx_1 \dots dx_d & \text{if } (x_1, \dots, x_d) \text{ is cts.} \\ \sum h(x_1, \dots, x_d) f(x_1, \dots, x_d) & \text{if } (x_1, \dots, x_d) \text{ is discrete} \end{cases}$$

p.m.f.

One important case:

$h(x_1, \dots, x_n) = a + b_1 X_1 + \dots + b_n X_n$.

$\mathbb{E}[a + b_1 X_1 + \dots + b_n X_n]$

$= a + b_1 \mathbb{E} X_1 + \dots + b_n \mathbb{E} X_n$

almost immediately from the characterization of $\mathbb{E}$ as an integral or a sum.

$Var[a + b_1 X_1 + \dots + b_n X_n]$

ubiquitous in statistics.

$= Var[b_1 X_1 + \dots + b_n X_n]$.

Covariance: A measure of a linear association between X and Y.

i.e it captures how effectively Y can be approximated by a line of the form $a + bX$, or vice-versa.

$$Cov(X_1, X_2) = \mathbb{E}\left[(X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2)\right]$$

positive values of Cov indicate a monotone $\uparrow$ relationship between X_1 and X_2.

negative values of Cov indicate a monotone $\downarrow$ relationship.

Covariance $\longrightarrow$ Correlation

$$Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_{X_1} \cdot \sigma_{X_2}}$$

$$= \mathbb{E}\left[\left(\frac{X_1 - \mathbb{E}X_1}{\sigma_{X_1}}\right) \times \left(\frac{X_2 - \mathbb{E}X_2}{\sigma_{X_2}}\right)\right]$$

σ_{X_1} : s.d. of X_1

σ_{X_2} : s.d. of X_2

$$\tilde{X}_1 = \frac{X_1 - \mathbb{E}X_1}{\sigma_{X_1}}$$

$$\tilde{X}_2 = \frac{X_2 - \mathbb{E}X_2}{\sigma_{X_2}}$$

$\mathbb{E}\tilde{X}_1 = \mathbb{E}\tilde{X}_2 = 0, \ \sigma_{\tilde{X}_1} = \sigma_{\tilde{X}_2} = 1$

Fact: $-1 \leq \rho_{X_1, X_2} \leq 1$.

$\rho_{X_1, X_2} = \pm 1 \Rightarrow X_2 = a + bX_1$ for some $a, b \Longleftrightarrow X_1 = \tilde{a} + \tilde{b} X_2$ for some $\tilde{a}, \tilde{b}$.

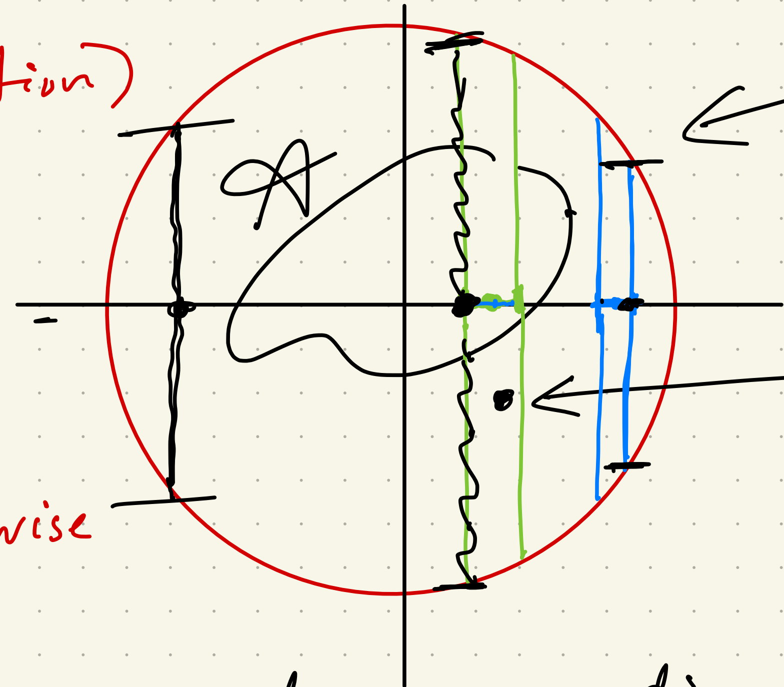
If $\rho_{X_1, X_2} = 0$, this indicates no linear association between X_1 and X_2 but does not imply ~~that~~ X_1 and X_2 are independent.

Helipad Example:

$\mathbb{1}(\text{condition})$

$= 1$ if condition holds

$= 0$ otherwise



Circular Helipad

chopper lands randomly.

(X, Y): random co-ordinates of chopper

$$f_{X,Y}(x,y) = \boxed{\frac{1}{\pi}\, \mathbb{1}\{x^2+y^2<1\}}$$

$$P((X,Y)\in A) = \frac{\text{area}(A)}{\pi}.$$

area of unit disk.

Let's try to find the behavior of ~~the~~ the marginals and the conditionals.

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$= \int_{\mathbb{R}} \frac{1}{\pi} \mathbb{1}\{x^2+y^2 \le 1\}\,dy$$

$\boxed{X \in (-1,1)}$

$P[(X,Y) \in \text{Unit disk}]$

$= 1.$

So interested only in $-1 < x < 1$.

X is NOT uniformly distributed on $(-1,1)$

$$= \int_{\mathbb{R}} \frac{1}{\pi} \mathbb{1}\{y^2 \le 1-x^2\}\,dy$$

$$= \int_{\mathbb{R}} \frac{1}{\pi} \mathbb{1}\{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2\sqrt{1-x^2}}{\pi}.$$

By symmetry:

$$f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi} \mathbb{1}\{-1 < y < 1\}$$

Intuitively, why is X not uniform on $[-1, 1]$?

density of x is symmetric unimodal about 0

indicates that values closer to 0 are more likely than values far away.

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$x \in (-1, 1)$

$$= \frac{\frac{1}{\cancel{\pi}} \mathbb{1}\{x^2+y^2<1\}}{\frac{2\sqrt{1-x^2}}{\cancel{\pi}}}$$

Uniform on permissible range of $Y|X=x$

$$= \frac{1}{2\sqrt{1-x^2}} \mathbb{1}\{y \in (-\sqrt{1-x^2}, \sqrt{1-x^2})\}$$

$$\mathbb{E}X = \int_{-1}^{1} \boxed{x \cdot \frac{2\sqrt{1-x^2}}{\pi}}\, dx = 0$$

$$\mathbb{E}Y = \text{similar expression}$$

odd function

If f is an even density function i.e $f(-t) = f(t)$ for all t and X has density f, $\Updownarrow$

$\mathbb{E}X = 0$

(basically saying ~~that~~ the integral of an odd function over $\mathbb{R}$ is 0)

<u>Question</u>: Are X and Y dependent?

$=$

Are X and Y uncorrelated i.e is $\rho_{X,Y} = 0$?

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

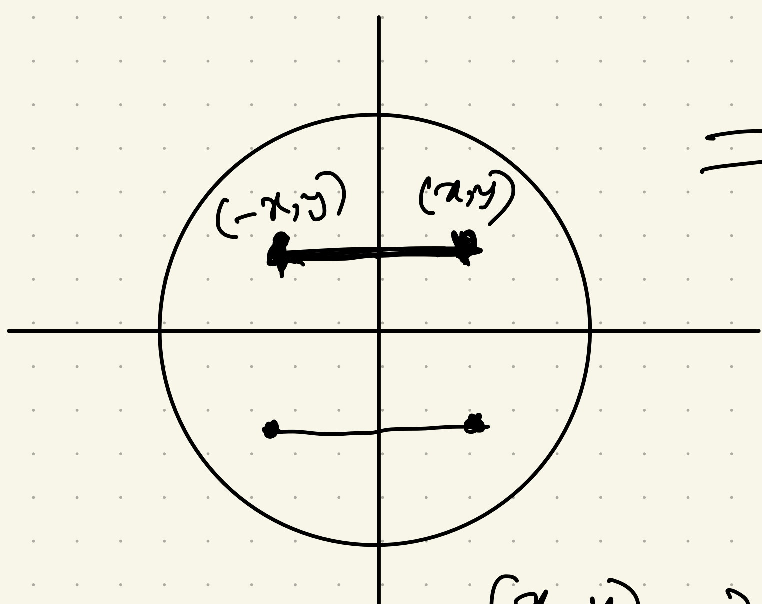
$\downarrow$ simplification

$$= \mathbb{E}(XY) - \mathbb{E}X \cdot \mathbb{E}Y.$$

Here, we have $\mathbb{E}X = \mathbb{E}Y = 0$.

Need to compute $\mathbb{E}(XY)$.

$$\mathbb{E}(XY) = \int_{\mathbb{R}^2} xy \frac{1}{\pi} \mathbb{1}\{x^2+y^2 \le 1\} \, dx \cdot dy.$$



$$= \frac{1}{\pi} \int_{x^2+y^2 \le 1} xy \, dx \cdot dy$$

$$= 0$$

X and Y are uncorrelated but NOT independent

$$(x,y) \to xy.$$

$$x \cdot y = -((-x) \cdot y)$$

$$\Rightarrow \rho_{XY} = 0$$