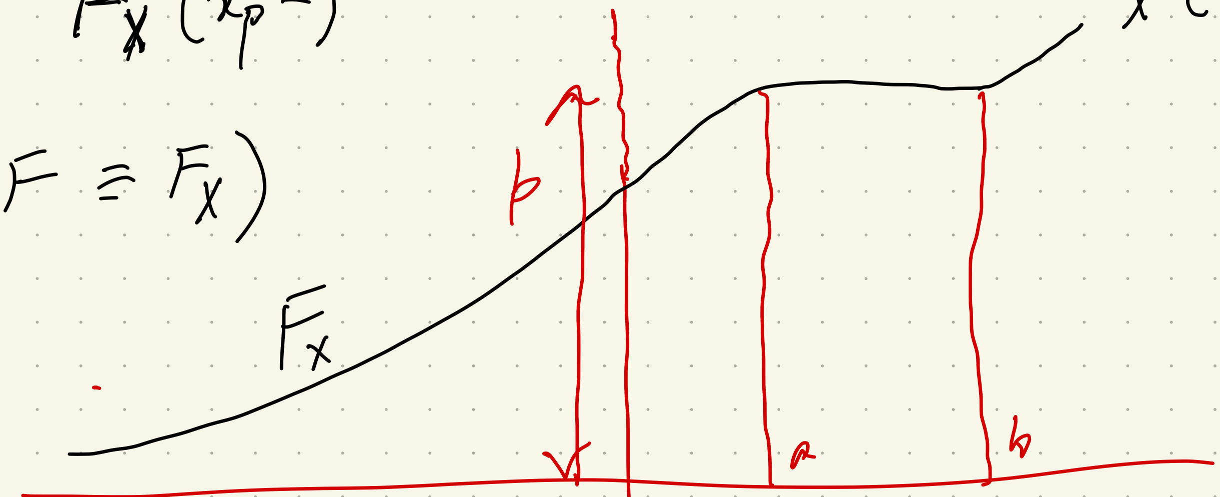
any p'th quantile x_p satisfies

$$\underset{\displaystyle F_X(x_p-)}{\overset{\shortparallel}{P(X < x_p)}} \le p \le \underset{\displaystyle F_X(x_p)}{\overset{\shortparallel}{P(X \le x_p)}}$$

$(F \equiv F_X)$



Suppose $F_X(t) = p$ for all $a \le t \le b$ and $F_X(t) < p$ for $t < a$ and $F_X(t) > p$ for $x > b$, then any number between a and b is a valid candidate for x_p

i.e. $p \le F(u-) \le F(u) \le p \quad \forall u \in [a, b]$

We'll isolate a specific quantile, essentially the smallest one.

Done by definining F_X^{-1} or F^{-1}:
a generalized notion of inverse.
For any $0 < t < 1$,
[set $F^{-1}(t)$ = smallest x such that
$F(x) \geq t$.]

More kosher defn:
$F^{-1}(t) = \inf\{x : F(x) \geq t\}$.

is bounded below.}

$\{x : F(x) \geq t\} = [F^{-1}(t), \infty)$

Note: $\boxed{F(F^{-1}(t)) \geq t.}$

This is by right continuity. Take $\{x_n\}$
of numbers such that $x_n \downarrow F^{-1}(t)$.
So, $F(x_n) \downarrow F(F^{-1}(t))$.
as $x_n \geq F^{-1}(t)$, $F(x_n) \geq F(F^{-1}(t))$
but of course $F(x_n) \geq t$. Shows that
$F(F^{-1}(t)) \geq t$.

$F(F^{-1}(t)-) \leq t$, since

$x < F^{-1}(t)$ means $F(x) < t$.

Important Relation:

$$\boxed{F^{-1}(t) \leq x \iff F(x) \geq t}. \quad (*)$$

equivalently $F^{-1}(t) > x \iff F(x) < t$.

Inverse Distribution Function Technique

Thm 2.1 (Notes):

let X be some r.v with dist. function F

i.e $F(x) = P(X \leq x)$

Define $Y = F^{-1}(U)$, where $U \sim$ Uniform$(0,1)$

r.v.

Then Y has the same distribution as X i.e $P(Y \leq y) = F(y)$.

Proof:

$P(F^{-1}(u) \leq y) = P(F(y) \geq u)$

from $(*)$.

$(*)$: $F^{-1}(t) \leq x \iff F(x) \geq t$

use this with $t \equiv u$ and $x \equiv y$

$= P(U \leq F(y))$

$\boxed{0 \leq F(y) \leq 1}$

$= F(y)$

$\left(P(U \leq f) = f \right.$
where $\left. 0 < f < 1 \right)$

$\boxed{\text{Theorem 2.2}}$: If X is a continuous random variable, then $F(X)$ has the uniform distribution.

$\left. Y = \underline{F}(\underline{X}) \right\}$

Proof: X is continuous.

$$Y = F(X), \quad 0 < t < 1$$

$$P(Y \le t) = P(F(X) \le t)$$

$$= \underbrace{\boxed{P(F(X) < t)}}_{T_1} + \underbrace{P(F(X) = t)}_{T_2}$$

Go to (*):

$$F^{-1}(t) \le x \iff F(x) \ge t.$$

equivalent: $F^{-1}(t) > x \iff F(x) < t$

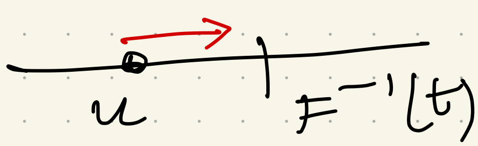
$$T_1 = P(F^{-1}(t) > X) \qquad P(X = w) = 0$$

$$= P(X < F^{-1}(t))$$

$$= P(X \le F^{-1}(t))$$

$$= \boxed{F(F^{-1}(t)) \ge t.}$$

Since F is continuous, $\lim_{u \to F^{-1}(t)-} F(u) = F(F^{-1}(t))$



But for $u < F^{-1}(t)$, $F(u) < t$!

So $F(F^{-1}(t)) \le t$.]

We conclude $F(F^{-1}(t)) = t$

So $T_1 = t$.

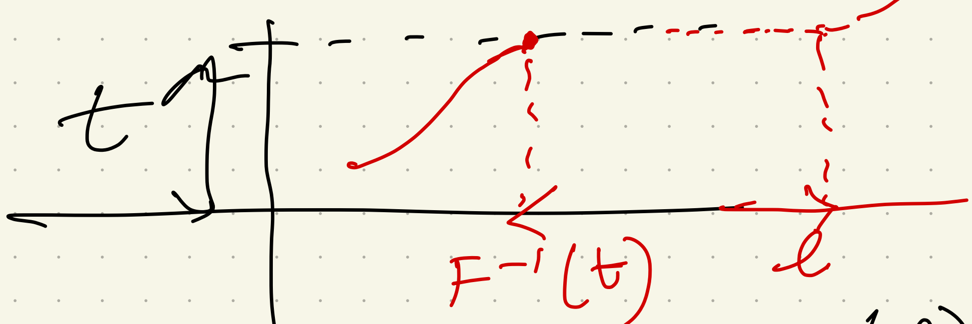
$P(F(X) = t) = 0$

Define $\ell = \sup\{x : F(x) = t\}$

$F^{-1}(t)$ belongs to this set.

$\{F(X) = t\}$

$= \{X \in [F^{-1}(t), \ell]\}$.



on $[F^{-1}(t), \ell]$ $F(x) = t$

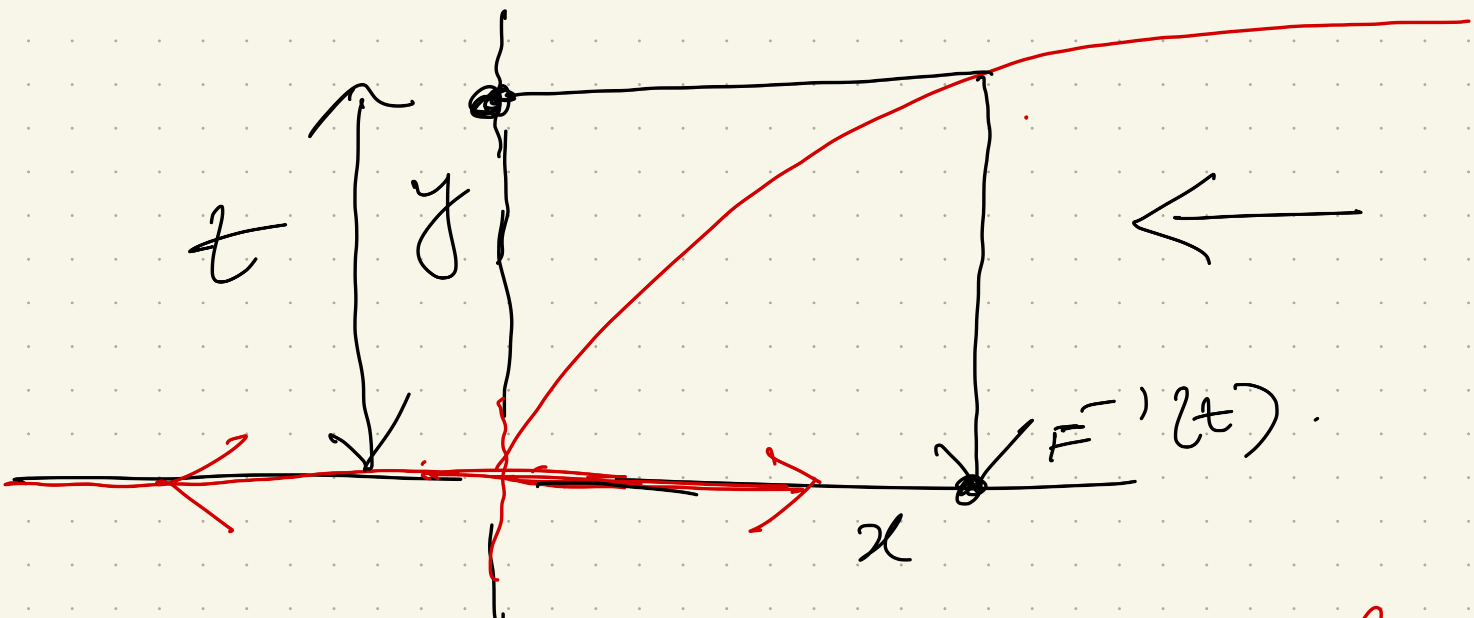
T_2

$$P(\{F(X) = t\}) = \cancel{0}\ \underbrace{F(\ell) - F(F^{-1}(t))}_{=0}$$

We've shown $P(Y \le t) = t$.

<u>Exercise</u>: $\boxed{F(x)} = 1 - e^{-\lambda x}$, if $x \geq 0$
$= 0$ if $x < 0$

Given U, want to manufacture X that has distribution F.



To generate Y which has dist func F, set $Y = F^{-1}(U)$.

Finding the inverse boils down to expressing x in terms of $F(x)$

$$F(x) = 1 - e^{-\lambda x}$$

so $e^{-\lambda x} = 1 - F(x)$

so $x = -\frac{1}{\lambda} \log(1 - \underbrace{F(x)}_{= t})$.

so, $x \to F^{-1}(t)$

so $F^{-1}(t) = -\frac{1}{\lambda} \log(1-t)$ for $0 < t < 1$.

So:

Set $Y = -\frac{1}{\lambda} \log(1-u)$.

Then we know Y has distribution function F.

(b) X takes values in $\{1, 2, 3, \dots\}$

$\left.\begin{array}{l} P(X=j) = p_j \end{array}\right\}$ Think Geometric

Given a uniform random variable U, I'd like to generate an r.v $\tilde{X}$ ~~that~~ has the same distribution as X.

$$F_X(j) = P(X \leq j) = \sum_{l=1}^{j} p_l$$

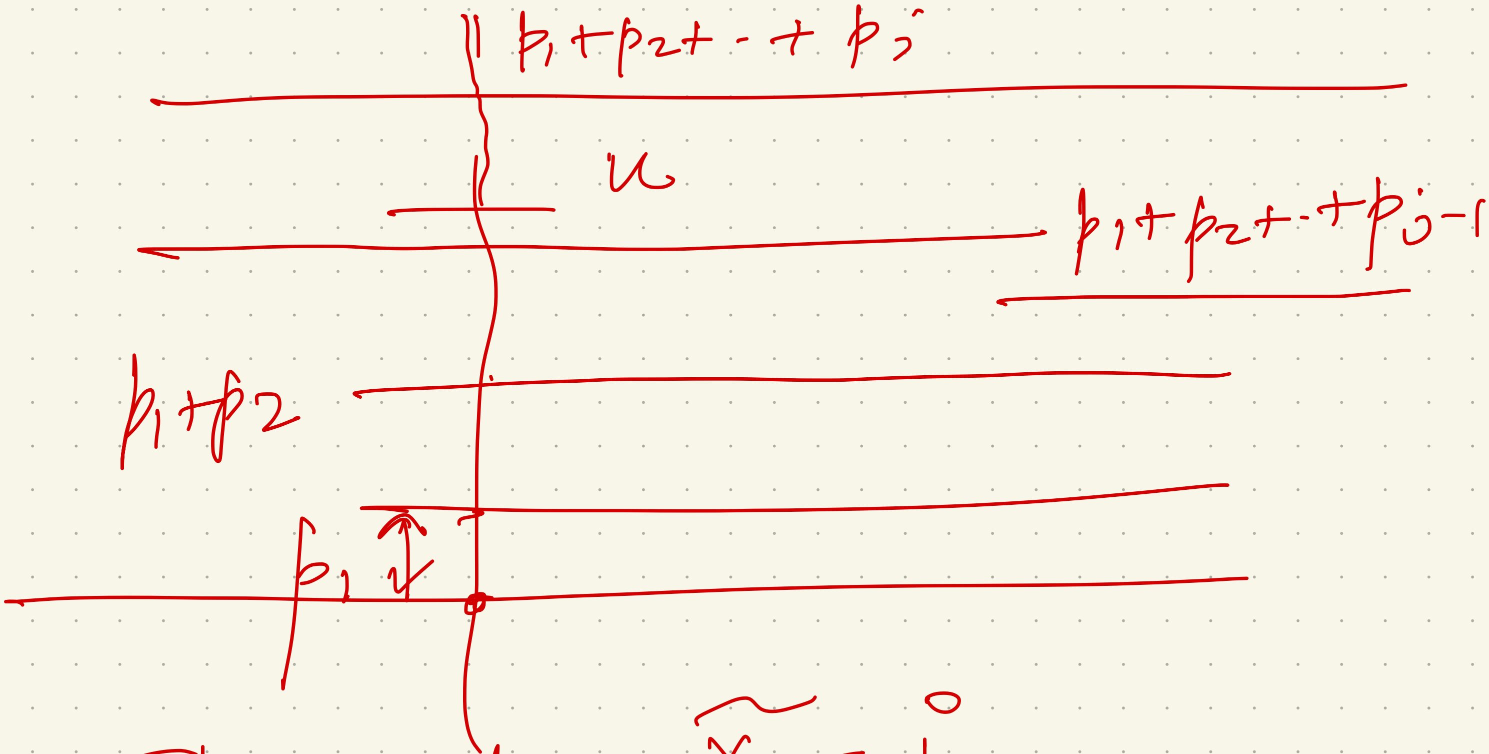
---

and $F_X(t) = \sum_{l=1}^{j} p_l$ for $j \leq t < j+1$

$$\tilde{X} = F_X^{-1}(U)$$

Generate ~~the~~ U.

Find ~~that~~ $\boxed{\text{unique } j_0}$ such that $\sum_{l=1}^{j-1} p_l < U \leq \sum_{l=1}^{j} p_l$



Then define $\tilde{X} = j$

$$P(\tilde{X} = j) = P\left[\sum_{l=1}^{j-1} p_l < u \le \sum_{l=1}^{j} p_l\right]$$

$$= \sum_{l=1}^{j} p_l - \sum_{l=1}^{j-1} p_l$$

$$= p_j$$

Check ~~that~~ this recipe is the inverse distribution function technique

Dealt with p.m.f's before

the equivalent of p.m.f for continuous random variables.

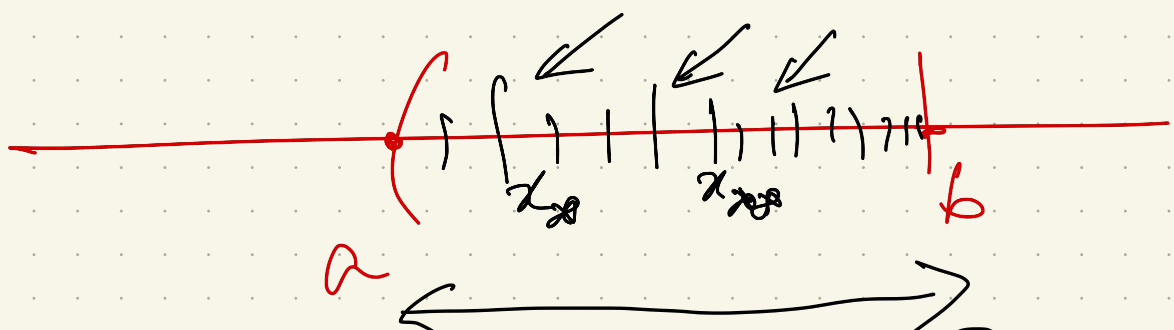
$$\mathcal{X} = \{x_1, x_2, \dots\}$$

$$p(x_j) = P(X = x_j) > 0$$

Discrete.

X assumes values $x_1, x_2, \dots$

$$P(X \in (a,b]) = \sum_{x_j \in (a,b]} p(x_j)$$



What if X is continuous?

$$P(X \in (a,b]) = \int_{(a,b]} f(u)\,du$$

prob. density function.

$$f(u) \geq 0$$

Normal(μ, σ^2) family

$\mu \in \mathbb{R}, \ \sigma > 0$

We say $X \sim N(\mu, \sigma^2)$ distribution

follows

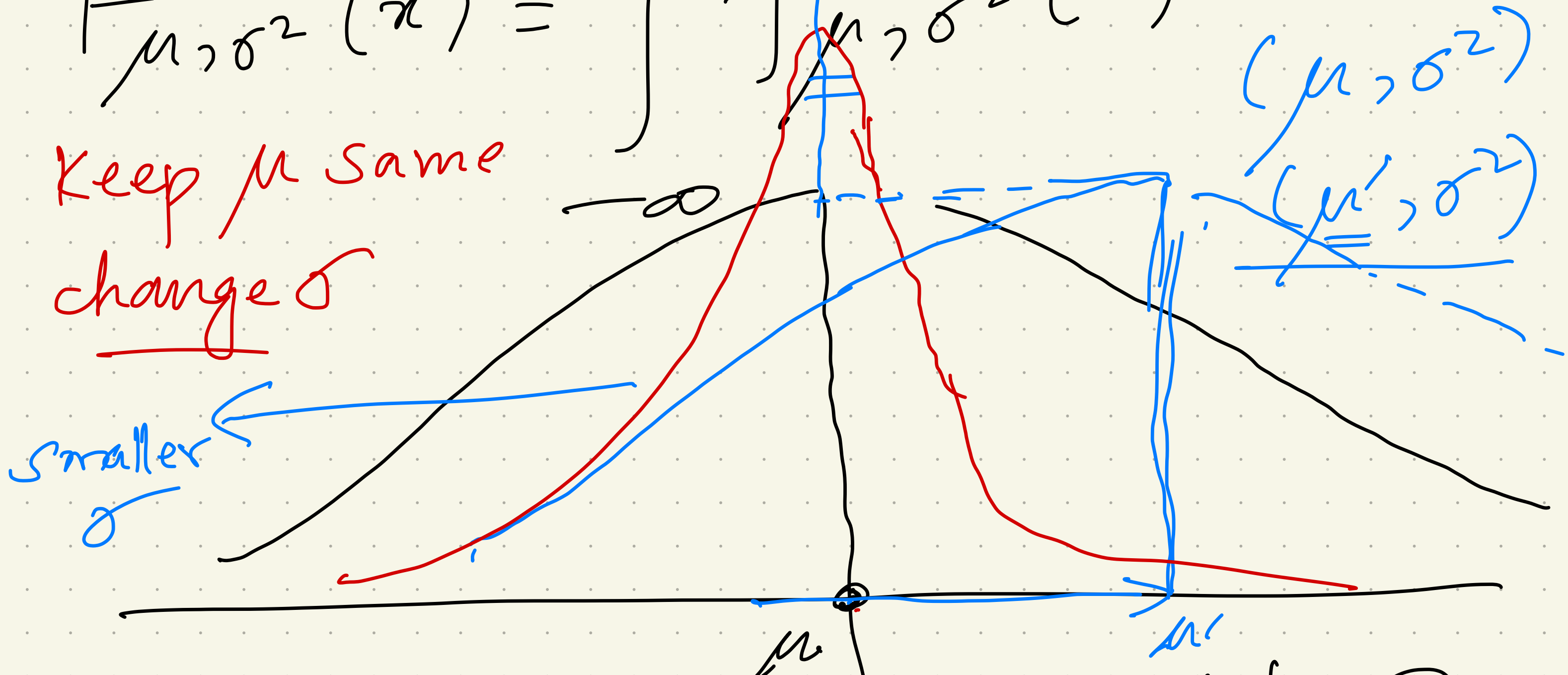
if its density is:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$F_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} f_{\mu,\sigma^2}(t)\, dt$$

Keep μ same

change σ



spread of curve is governed by σ