2016. Exam 1.

Problem 1: The labels on the balls are not important here.

(i) You're doing a random experiment. In each run of the experiment a ball ends up in one of r boxes with equal i.e $\frac{1}{r}$ probability.

There are N runs (N balls).

So $P[N_1 = n_1, N_2 = n_2, \ldots, N_r = n_r]$

where N_i is random # of balls in i'th box is

$$\frac{N!}{\prod_{j=1}^{r} n_j!} \left(\frac{1}{r}\right)^{n_1} \left(\frac{1}{r}\right)^{n_2} \cdots \left(\frac{1}{r}\right)^{n_r}$$

$$= \frac{N!}{\prod_{j=1}^{r} n_j!} \left(\frac{1}{r}\right)^{N}$$

(ii) $(N_i, N_j, N - N_i - N_j)$

$$\sim \text{Mult-inomial}\left(N, \frac{1}{r}, \frac{1}{r}, \frac{r-2}{r}\right)$$

N_i, N_j each follow

$$\text{Binomial}\left(N, \frac{1}{r}\right)$$

From these observations ; the quantities needed in (ii) can be derived.

Problem 3.

Given $i < j < k$,

$$P(X_i = 1, X_j = 0, X_k = 1)$$

$$= \frac{{}^{N_D}P_2 \; {}^{N_R}P_1 \; \cancel{\binom{N-3}{n-3}} \; {}^{N-3}P_{n-3}}{\cancel{\binom{N}{n}} \; {}^{N}P_{n}} = \frac{N_D(N_D-1)N_R}{N(N-1)(N-2)}$$

where $N_D = Np$

$N_R = N(1-p)$

which can be simplified and does not depend on i, j, k or n.

Similar to a HW3 problem

Problem 2.

(i) $P(R=k) = P(R=k, \text{first trial is } H)$

$+ P(R=k, \text{first trial is } T)$

$= p^k q + q^k p$ where $q = 1-p$

for $k \geq 1$.

Now ER can be found easily.

(ii) Let's see how the first two runs come about.

You get m consecutive H's at the start ($m \geq 1$), then ℓ consecutive T's and then an H.

Then sum of runs is $m+\ell$ and the probability is $p^m q^\ell p$.

Similarly, starting with T's, the prob is $q^m p^\ell q$.

$P($ first run has length m, second has length $\ell)$

$= p^m q^\ell \cdot p + q^m p^\ell \cdot q$

$= p^{m+1} q^\ell + q^{m+1} p^\ell$.

$P($ first run + second run $= k)$

$k \geq 2$

$$= \sum_{\substack{(m,\ell) \\ m+\ell=k \\ m\geq 1,\, \ell \geq 1}} \left(p^{m+1} q^\ell + q^{m+1} p^\ell\right)$$

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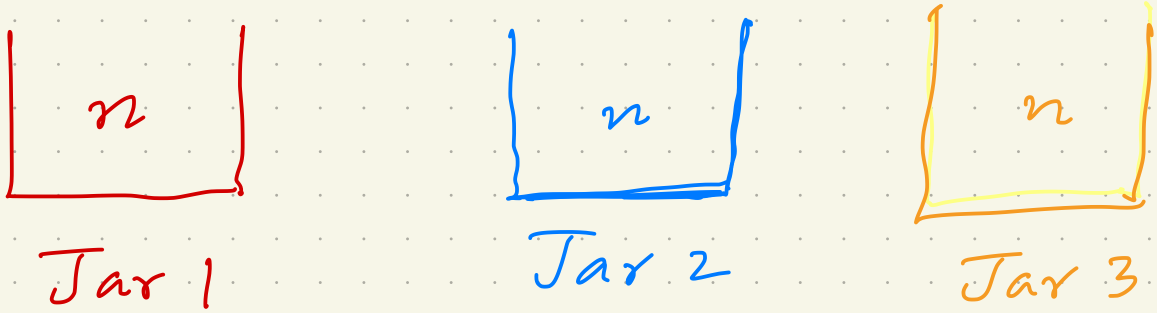

(1) The event can happen in 6 ways.
(We assume $r \neq s$)

p reaches into Jar 1, finds it empty and at this point Jar 2 has r, Jar 3 has s, or Jar 2 has s, Jar 3 has r

or p could reach into Jar 2 and find it empty etc. etc.

Consider the prob. of the underlined event. This means that before the last reach into 1, there were n reaches into 1, $n-s$ reaches into 2, and $n-r$ reaches into 3, a total of $3n-r-s$ reaches

Each jar is equally likely to be picked (w.p. $\frac{1}{3}$)

This probability is then:

$$\underbrace{\frac{(3n-r-s)!}{n!\,(n-r)!\,(n-s)!}}_{\text{the total number of different permutations of } 1,2,3 \text{ into } 3n-r-s \text{ positions}} \left(\frac{1}{3}\right)^{3n-r-s} \underbrace{\left(\frac{1}{3}\right)}_{\text{the last reach into } 1}$$

the total number of different permutations of 1, 2, 3 into $3n-r-s$ positions (i.e all possible sequences in which 1, 2, 3 can be drawn before the last reach into 1).

The rest of the calculation is easy.

Note: When $r=s$, there are only 3 different ways, not 6!

Problem 2

(a) Verification is straightforward and skipped. (Monotone increasing, continuous and $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$).

$$F(x) = 1 - \frac{1}{(1+x)^2}, \quad x > 0$$

$$\Rightarrow \frac{1}{(1+x)^2} = 1 - F(x)$$

$$\Rightarrow (1+x)^2 = \frac{1}{1-F(x)}$$

$$\Rightarrow x = \sqrt{\frac{1}{1-F(x)}} - 1.$$

So, $F^{-1}(p) = \sqrt{\frac{1}{1-p}} - 1$ for $(0 < p < 1)$

So, given a uniform $(0,1)$, say U,

$$F^{-1}(U) = \sqrt{\frac{1}{1-U}} - 1 \sim F$$

$$EX = \int_0^\infty (1 - F(x))\,dx = \int_0^\infty \frac{dx}{(1+x)^2}$$

(b) Suppose $U \sim \text{Uniform}(0,1)$.

Then X has the same distribution as $F^{-1}(U)$.

Then $EX = E(F^{-1}(U)) = \int_0^{\infty} F^{-1}(p)\,dp$

Problem 3.

$$EX(X-1)\dots(X-r+1) = \sum_{x=0}^{+\infty} x(x-1)\dots(x-r+1)\frac{e^{-\lambda}\lambda^x}{x!}$$

(other terms $=0$)
$$= \sum_{x=r}^{+\infty} x(x-1)\dots(x-r+1)\frac{e^{-\lambda}\lambda^x}{x!}$$

(simplify)
$$= \lambda^r \sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}$$

(sum pdf poi=1)
$$= \lambda^r$$

Hence, we have $EX = \lambda$, $EX(X-1) = \lambda^2$, $EX(X-1)(X-2) = \lambda^3$

$\Rightarrow$ $EX = \lambda$, $EX^2 = \lambda^2 + \lambda$, $EX^3 = \lambda^3 + 3(\lambda^2+\lambda) - 2\lambda$
$$= \lambda^3 + 3\lambda^2 + \lambda.$$

$$\Rightarrow \quad E(X-\lambda)^3 = EX^3 - 3\lambda EX^2 + 3\lambda^2 EX - \lambda^3$$
$$= (\lambda^3 + 3\lambda^2 + \lambda) - 3(\lambda^2+\lambda)\lambda + 3\lambda^3 - \lambda^3$$
$$= \lambda.$$