UNIVERSITY OF MICHIGAN

PROBABILITY

STATS510

ASSIGNMENT 1

Author: *Shu* ZHOU

ID: 19342932

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Problem 1.36

The probability of hitting less than twice

$$P(H < 2) = P(H = 0) + P(H = 1) = (\frac{4}{5})^{10} + 10 * (\frac{1}{5})(\frac{4}{5})^{9} = 0.3758 \tag{1}$$

Hence

$$P(H \geq 2) = 1 - P(H < 2) = 0.6242 \tag{2}$$

Given that the hit is at least once

$$P(H < 2|H \geq 1) = P(H = 1|H \geq 1) = (\frac{4}{5})^{9} = 0.1342 \tag{3}$$

Hence

$$P(H \geq 2|H \geq 1) = 1 - P(H < 2|H \geq 1) = 0.8658 \tag{4}$$

Problem 1.39

- If A,B are mutually exclusive, we have $A \cap B = \phi$ and $P(A \cap B) = 0$. Since $P(A)$ and $P(B)$ are independent, $P(A \cap B) = P(A)P(B) = 0$. However, $P(A), P(B) > 0$, we have $P(A)P(B) > 0$. Hence, if A,B are mutually exclusive, they are not independent.
- If A, B are independent, we have $P(A \cap B) = P(A)P(B) > 0$.However, when A,B are mutually exclusive, we have $P(A \cap B) = 0$.Hence, if A,B are independent, they are not mutually exclusive.

Problem 1.46

$X_i =$ the number of cells containing exactly i balls
Since we need to calculate X_3, there are three possible values of $X_3 = 0, 1, 2$

- If $X_3 = 2$, then the distribution of balls is {3,3,1}, hence the total number is

$$\binom{7}{2}\binom{5}{1}\binom{7}{3}\binom{4}{3} = 14700 \tag{5}$$

- If $X_3 = 1$, then the possible distributions of balls are {3,4},{3,2,2},{3,2,1,1},{3,1,1,1,1};

$$\binom{7}{1}\binom{6}{1}\binom{7}{3} = 1470 \tag{6}$$

$$\binom{7}{1}\binom{6}{2}\binom{7}{3}\binom{7}{3}\binom{4}{2} = 22050 \tag{7}$$

$$\binom{7}{1}\binom{6}{1}\binom{5}{2}\binom{7}{3}\binom{4}{2}\binom{2}{1} = 176400 \tag{8}$$

$$\binom{7}{3}\binom{6}{4}\binom{7}{3}4! = 88200 \tag{9}$$

Hence, the total number is 288120.

The total number of samples is $7^7 = 823543$, then we have $P(X_3 = 2) = 14700/823543 = 0.018$,$P(X_3 = 1) = 288120/823543 = 0.350$. Hence $P(X_3 = 0) = 1 - 0.018 - 0.350 = 0.632$

Problem 1.51

$$P(X=0) = \binom{25}{4} / \binom{30}{4} = 0.462 \tag{10}$$

$$P(X=1) = \binom{5}{1}\binom{25}{3} / \binom{30}{4} = 0.420 \tag{11}$$

$$P(X=2) = \binom{5}{2}\binom{25}{2} / \binom{30}{4} = 0.109 \tag{12}$$

$$P(X=3) = \binom{5}{3}\binom{25}{1} / \binom{30}{4} = 0.0091 \tag{13}$$

$$P(X=4) = 1/\binom{30}{4} = 0.0002 \tag{14}$$

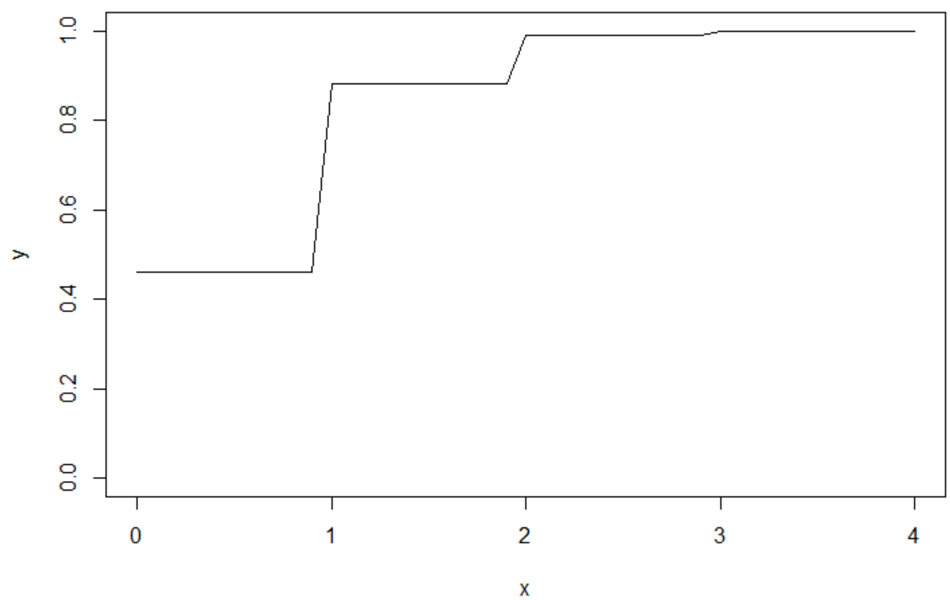

Figure 1: Cdf

Problem 1.19

(a). The number of partial derivatives is $\binom{6}{4} = 15$

(b). There are total n slots for total r index numbers, each slot is labelled with a variable. Hence we need to consider the total number of different arrangements of index numbers, which is equal to the arrangement of $n-1$ walls + r balls, Hence, we have

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} \tag{15}$$

total arrangements,Q.E.D.

Problem 1.21

The total number of possible picks is $\binom{2n}{2r}$
The number of unmatched picks is $\binom{n}{2r}$ (type of shoe) $\times$ 2^{2r}(left or right)
Hence

$$P = \frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}} \tag{16}$$

Problem 1.24

(a).

$$P(A) = \sum_{n=0}^{\infty}(\frac{1}{2})^{2n+1} = \frac{1}{2}\frac{1}{1-\frac{1}{4}} = 2/3 \tag{17}$$

(b).

$$P(A) = \sum_{n=0}^{\infty} p(1-p)^{2n} = \frac{p}{1-(1-p)^2} = \frac{p}{(1-1+p)(1+1-p)} = \frac{1}{2-p} \tag{18}$$

(c). Since $P(A) = \frac{1}{2-p}$, it is obvious that $P(A) > 0.5$ for any $(0 < P < 1)$

Problem 1.31

(a). If we pick n numbers from the set $\{x_1, x_2, ...x_n\}$ with replacement, the total number of picks is n^n. If we pick exactly $\{x_1, x_2, ...x_n\}$, then the situation is equal to pick n numbers from the set $\{x_1, x_2, ...x_n\}$ without turning them back, the total number of picks is $nPn = n!$. Hence the probability is $\frac{n!}{n^n}$

If we do not pick exactly $\{x_1, x_2, ...x_n\}$, assume we pick total m different numbers. Then this situation is equivalent to put n balls into m slots, the total number of picks is $\binom{n}{m}$(ways of picks) $\times\binom{m+n-1}{n}$(ways of arrangments) $< n!$, Therefore the outcome with $\frac{\{x_1,x_2,....x_n\}}{n}$ is the most likely

For each number x_i, the probability to pick it is $\frac{1}{n}$, then for n picks, the probability not to pick it is $(1-\frac{1}{n})^n$

$$\lim_{n->\infty}(1-\frac{1}{n})^n = \lim_{n->\infty}\frac{1}{(1+\frac{1}{n-1})^n} = \lim_{n->\infty}\frac{1}{(1+\frac{1}{n-1})^{n-1}(1+\frac{1}{n-1})} = \frac{1}{e\times 1} = \frac{1}{e} \tag{19}$$

2

The probability of getting exactly x success in N trials, with the probability of success on a single trial being p is:

$$P(X = x) = \binom{n}{x}p^x(1-p)^{n-x} \tag{20}$$

Hence, the probability that we need N trials of the experiment for this to happen is

$$P(total) = \binom{N}{m_1}p^{m1} \times \binom{N-m_1}{m_2}p^{m_2}..... \times \binom{N-\sum_{i=1}^{k-1} m_i}{m_k}p^{m_k} \times (1-\sum m_i p)^{(N-\sum m_i)} \tag{21}$$

Assume $m_0 = 0$, Hence we can summarize the equation into

$$P(total) = (1-\sum_{i=1}^{k} m_i p)^{(N-\sum_{i=1}^{k} m_i)}\prod_{i=1}^{k}\binom{N-\sum_{j=0}^{i-1} m_j}{m_i}p^{m_i} \tag{22}$$

3

The Accept rate of Type-1 student is

$$P(Type-1) = \sum_{i=1}^{k} p_i a_{i,1} \tag{23}$$

Similarly, the accept rate of Type-2 student is

$$P(Type-2) = \sum_{i=1}^{k} q_i a_{i,2} \tag{24}$$

Since we have $a_{k,1} < a_{k,2}$ for each k, assume k = 2, we can induce that

$$q * a_{1,2} + (1-q) * a_{2,2} - p * a_{1,1} - (1-p) * a_{1,2} \leq (q-p)(a_{1,2} - a_{2,2}) \tag{25}$$

$Eq.(25) < 0$ when

- $q - p < 0$ and $a_{1,2} - a_{2,2} > 0$
- $q - p > 0$ and $a_{1,2} - a_{2,2} < 0$

Hence, the admissions data for the whole university taken together may demonstrate a huge bias towards Type 1 students.