X : random variable.

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Consider r.v's such ~~that~~ $M_X(t) < \infty$ for $t \in$ some open interval around 0.

If mgf exists:

$$\mathbb{E}[e^{tX}] = \sum_{j=0}^{\infty} \frac{t^j E(X^j)}{j!}$$

$$\frac{d^n}{dt^n} M_X(t) = E[X^n e^{tX}]$$

$$\mathbb{E}(e^{tX})$$

Moment generating functions when they exist characterize distributional convergence. $\longleftrightarrow$

MGF calculations:

$$\mathbb{E}[e^{tY}] = \exp\left[t\mu + \frac{t^2\sigma^2}{2}\right]$$

standard calculation.

where $Y \sim N(\mu, \sigma^2)$.

Suppose $X \sim \Gamma(\alpha, \lambda)$

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}(x > 0)$$

$(\alpha, \lambda > 0)$.

When $\alpha = 1$, $\Gamma(\alpha, \lambda)$ is precisely the exponential(λ) density.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx < \infty \quad \forall\, \alpha > 0.$$

$\longleftrightarrow$

Exercise: $\displaystyle\int_0^\infty f_X(x)\, dx = 1$.

χ^2 family of distributions is an important subclass. $\chi^2_n \equiv \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

$$\mathbb{E}[e^{tX}]$$

$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{tx} e^{-\lambda x} x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \boxed{\int_0^\infty e^{-(\lambda-t)x} x^{\alpha-1}\, dx}$$

$$\int_0^\infty e^{cx} x^{\alpha-1}\, dx = \infty$$ c is a constant

for any $c > 0$

For convergence, need $t < \lambda$.

Hence, conclude that $\mathbb{E}[e^{tX}] < \infty$ $\forall\, t < \lambda$ and certainly therefore $\forall\, t \in (-\lambda, \lambda)$, and $\forall$

$$\int_0^\infty e^{-(\lambda-t)x} x^{\alpha-1}\, dx = \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}$$

Change of variable argument will show this.

or use the fact that $\Gamma(\lambda - t, \alpha)$ density integrates to 1!

$$\text{So } E[e^{tX}] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\lambda - t)^\alpha}$$

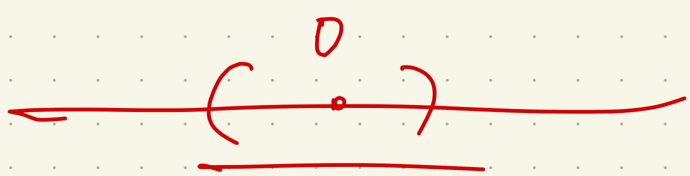
$$= \left(\frac{\lambda}{\lambda - t}\right)^\alpha.$$

Theorem 2.3.11 (CB).

Theorem 4.4 (Notes).

Suppose X and Y are two random variables with finite m.g.f's M_X and M_Y in a common nbhd. of 0.

Then $F_X = F_Y$ iff $M_X(t) = M_Y(t)$ for all t in the nbhd.



the mgf uniquely pins down the distribution.

Corollary 4.5: X and Y bounded random variables, i.e $\exists\, M > 0$ s.t. $|X| \leq M$ w.p. 1 and $|Y| \leq M$ w.p. 1. Then X and Y have the same distribution iff all their moments coincide.

$\Longrightarrow$

$\Longleftarrow$

$M_X(t)$, $M_Y(t)$ are finite for all t

$$E(X^k) = E[Y^k]$$

$$\downarrow$$

$$\int x^k \, dF(x)$$

$\Leftarrow$ the mgfs of both X and Y can be expanded in an infinite series.

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbb{E}(X^j)$$

$$M_Y(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbb{E}(Y^j)$$

since $\mathbb{E}(X^j) = \mathbb{E}(Y^j)$, $M_X(t) = M_Y(t)$

What we've shown here is that for a bounded random variable its moments completely determine its distribution function.

But in general, can find distinct distributions ~~that~~ are different BUT possessing ~~the~~ same set of moments.

Moment-generating functions and convergence in distribution:

Thm 4.6 $\{X_n\}$: sequence of random variables

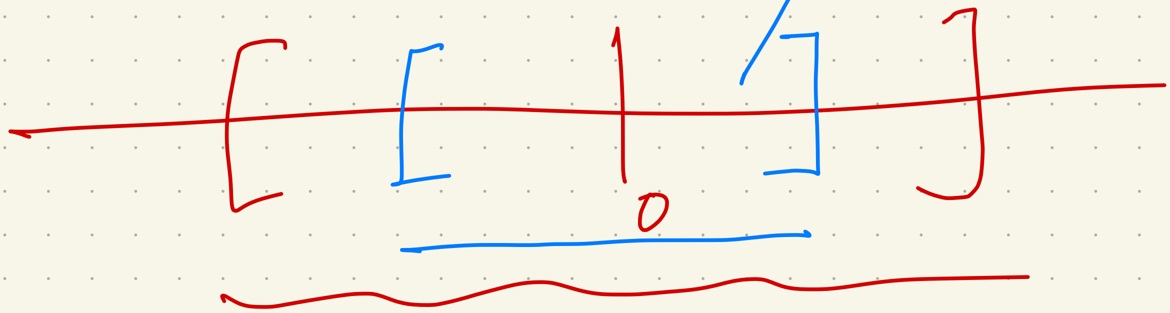
X : some fixed r.v.

Assume: the mgf's of all these random variables exist finitely in some neighborhood of 0.

Suppose ~~that~~ $M_n(t) = E[e^{tX_n}]$

$\longrightarrow M_X(t) = E[e^{tX}]$ for all

t in some neighborhood of 0.



Then $X_n \xrightarrow[d]{} X$ i.e.

$F_{X_n}(x) \longrightarrow F_X(x)$ for all x

such ~~that~~ $F_X(x) = F_X(x-)$.

F_X is continuous at x.

Going to use it later to prove a somewhat general version of the CLT.

Now, special case: convergence of Binomial to Normal.

DeMoivre - Laplace

$S_n \sim Bin(n, p)$.

Normalized S_n:

$$\tilde{S}_n = \frac{S_n - ES_n}{s.d(S_n)} = \frac{S_n - np}{\sqrt{np(1-p)}}$$

We'd like to show $\tilde{S}_n \xrightarrow[d]{} Z$

where $Z \sim N(0,1)$.

Proof:

$$M_n(t) = \mathbb{E}\left[e^{t\tilde{S}_n}\right]$$

$$= \mathbb{E}\left[\exp\left[t\left(\frac{S_n - np}{\sqrt{np(1-p)}}\right)\right]\right]$$

$$= \mathbb{E}\left[\exp\left(\frac{tS_n}{\sqrt{np(1-p)}} - \frac{npt}{\sqrt{np(1-p)}}\right)\right]$$

$$= \left(e^{-tp/\sqrt{np(1-p)}} \right)^n$$

$$\times \; \mathbb{E}\left[\exp\left(\frac{tS_n}{\sqrt{np(1-p)}} \right) \right]$$

$$\Downarrow$$

$$\underline{M_{S_n}}\left(\frac{t}{\sqrt{np(1-p)}} \right)$$

$$\underline{M_{S_n}(t)} = \left[\underline{(1-p) + pe^t} \right]^n.$$

where $S_n \sim \text{Bin}(n, p)$

$$\sum_{j=0}^{n} e^{tj} \binom{n}{j} p^j q^{n-j}$$

$$= \left(e^{-\frac{tp}{\sqrt{np(1-p)}}} \right)^n \times \left[(1-p) + pe^{\frac{t}{\sqrt{np(1-p)}}} \right]^n$$

Binomial mgf multiply through

$$= \left[p\, \underline{e^{\frac{t}{\sqrt{n}}\sqrt{\frac{1-p}{p}}}} + (1-p)\, \underline{e^{-\frac{t}{\sqrt{n}}\sqrt{\frac{p}{1-p}}}} \right]^n =$$

Expand as a series expression inside exponent (✱).

$$p\left(1 + \frac{t}{\sqrt{n}}\sqrt{\frac{1-p}{p}} + \frac{t^2}{2n}\frac{1-p}{p} + \frac{t^3}{3!\, n^{3/2}}\left(\frac{1-p}{p}\right)^{3/2} + \dots\right)$$

$$+ (1-p)\left[1 - \frac{t}{\sqrt{n}}\sqrt{\frac{p}{1-p}} + \frac{t^2}{2n}\frac{p}{1-p} - \frac{t^3}{3!\, n^{3/2}}\left(\frac{p}{1-p}\right)^{3/2} + \dots\right]$$

$$= \left\{ p + \cancel{\frac{t}{\sqrt{n}}\sqrt{p(1-p)}} + \frac{t^2}{2n}(1-p) + (1-p) \cancel{- \frac{t}{\sqrt{n}}\sqrt{p(1-p)}} + \frac{t^2}{2n}p + r_n \right\}$$

Check that $r_n = O\left(\frac{1}{n^{3/2}}\right)$

So $(*)$ reduces to:

$$\left(1 + \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$

$$= \left(1 + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \to e^{t^2/2}$$

as $n \to \infty$, by a standard result from calculus. $\left\{ \left(1 + \frac{y_n}{n}\right)^n \to e^y \text{ when } y_n \to y \right\}$

$M_X(t) = e^{t^2/2}$ when $X \sim N(0,1)$!

So: $M_n(t) \longrightarrow M_X(t)$

Therefore $\tilde{S}_n \xrightarrow{\;\;}_{d} N(0,1)$

by Theorem 4.6.

Normal approximation to Binomial has been discussed.

Discussed in the context of p, the probability of H, remaining invariant to n.

But in many applications $p \equiv p_n$

$S_n \sim \mathrm{Bin}(n, p_n)$

Ex 4.5 Suppose $S_n \sim \mathrm{Bin}(n, p_n)$ such that $\boxed{n p_n (1-p_n)} = \mathrm{Var}(S_n)$

$\longrightarrow \infty$.

Then
$$\frac{S_n - n p_n}{\sqrt{n p_n (1-p_n)}} \longrightarrow N(0,1)$$

Essentially same techniques as in p fixed case but more finesse.

Nature of convergence changes when the variance stabilizes in the limit way.

$$Var(S_n) = \underline{np_n(1-p_n)} \longrightarrow \lambda$$

consider $\underline{S_n} \sim Bin(n, p_n)$.

$n \to \infty$, $p_n \to 0$, $\underline{np_n} \longrightarrow \underline{\lambda} > 0$

$np_n(1-p_n) \longrightarrow \lambda$

Then $S_n \sim$ Poisson(λ) in distribution as $n \to \infty$.

<u>Way 1</u>: $P(S_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$

do this!

$$P(Poi(\lambda) = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

$S_n \sim Bin(n, p_n)$. $np_n \longrightarrow \lambda$.

$$M_{S_n}(t) = \mathbb{E}\left[e^{tS_n}\right]$$

$$= \left[(1-p_n) + p_n e^t\right]^n$$

$$= \left[1 + \frac{x_n}{n}\right]^n$$

where $x_n = np_n(e^t - 1)$

$$x_n \longrightarrow \lambda(e^t - 1).$$

Therefore:

$M_{S_n}(t) \longrightarrow e^{\lambda(e^t-1)}$ as $n \to \infty$

$$\lim_{n\to\infty}\left(1 + \frac{x_n}{n}\right)^n = e^{\lim_{n\to\infty} x_n}$$

Exercise: If $X \sim Poi(\lambda)$,

$M_X(\lambda) = e^{\lambda(e^t-1)}$

Conclude ~~that~~ $S_n \longrightarrow Poi(\lambda)$.

$X \sim Poi(\lambda)$.

$$p(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$

$$= \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \boxed{\sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}} = e^{\lambda(e^{t}-1)} !$$