Homework 1 Solution

1.19

(a)For every possible partial derivative of the form $\frac{\partial^4 f(x,y)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$, we only need the following two conditions hold

$$k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$$

$$k_1 + k_2 + k_3 = 4.$$

This is equivalent to place 4 unlabeled balls into 3 labeled boxes and there are $\binom{6}{4} = 15$ ways to do that.

(b)Similarly, it is equivalent to place r unlabeled balls into n labeled bins, which has $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ different methods to do that.

1.21

There are $\binom{2n}{2r}$ ways of choosing $2r$ shoes from a total of $2n$ shoes. Thus there are $\binom{2n}{2r}$ equally likely sample points. There are $\binom{n}{2r}$ ways of choosing $2r$ different shoes styles. There are two ways of choosing within a given shoe style (left shoe or right shoe), which gives 2^{2r} ways of arranging each one of the $\binom{n}{2r}$ arrays. The product of those is the numerator $\binom{n}{2r}2^{2r}$. Thus the probability is $\frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}$.

1.24

Let X be the number of flips to obtain the first head. So, A wins if $X = 1, 3, 5, \ldots$ and B wins if $X = 2, 4, 6, \ldots$. Since before the first head appears, all flips are tails, we have $\mathbb{P}(X = k) = pq^{k-1}$, where $q = 1 - p$ and p is the probability that a head appears in a trial.

(a)$\mathbb{P}(Awins) = \mathbb{P}(X = 1, 3, 5, \ldots) = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \frac{2}{3}$

(b)$\mathbb{P}(Bwins) = \mathbb{P}(X = 2, 4, 6, \ldots) = \sum_{k=1}^{\infty} pq^{2k-1} = \frac{p}{1-q^2} = \frac{1}{2-p}$

(c)$\frac{p}{1-q^2} = \frac{p}{2p-p^2} > \frac{p}{2p} = \frac{1}{2}$

1.31

(a) The number of ordered samples drawn with replacement from the set $\{x_1, \ldots, x_n\}$ is n^n. The number of ordered samples that make up the unordered sample $\{x_1, \ldots, x_n\}$ is $n!$. Thus, the outcome with average $\frac{\sum_{i=1}^n x_i}{n}$ that is obtained by the unordered sample $\{x_1, \ldots, x_n\}$ has probability $\frac{n!}{n^n}$. Any other unordered outcome from $\{x_1, \ldots, x_n\}$, distinct from the unordered sample $\{x_1, \ldots, x_n\}$, will contain m different numbers repeated $k_1, \ldots, k_m$ times where $\sum_{i=1}^m k_i = n$ with at least one of the k_i's satisfying $2 \le k_i \le n$. The probability of obtaining the corresponding average of such outcome is

$\frac{n!}{\prod_{i=1}^m k_i! n^n} < \frac{n!}{n^n}$, since $\prod_{i=1}^m k_i! > 1$.

Thus, the outcome with average $\frac{\sum_{i=1}^n x_i}{n}$ is the most likely.

(c)Since we are drawing with replacement from the set $\{x_1, \ldots, x_n\}$, the probability of choosing any x_i is $\frac{1}{n}$. Therefore the probability of obtaining an ordered sample of size n without x_i is $(1-\frac{1}{n})^n$. To prove that $\lim_{n\to\infty}(1-\frac{1}{n})^n = e^{-1}$, calculate the limit of the log. That is

$$\lim_{n\to\infty} n\log(1-\frac{1}{n}) = \lim_{n\to\infty} \frac{\log(1-\frac{1}{n})}{1/n} = \lim_{x\to 0^+} \frac{\log(1-x)}{x}.$$

L'Hôpital's rule shows that the limit is -1, establishing the result.

1.36

Let X denotes the number of times that the target is hit. Then we will have

$$P(X=0) = \left(1-\frac{1}{5}\right)^{10} = \left(\frac{4}{5}\right)^{10}$$

$$P(X=1) = \binom{10}{1}\left(1-\frac{1}{5}\right)^{9}\frac{1}{5} = \frac{10\times 4^9}{5^{10}}$$

$$\begin{aligned} P(\text{being hit at least twice}) &= P(X \ge 2) \\ &= 1 - P(X=0) - P(X=1) \\ &= 1 - \left(\frac{4}{5}\right)^{10} - \frac{10\times 4^9}{5^{10}} \\ &= 0.624 \end{aligned}$$

$$\begin{aligned} P(X \ge 2 | X \ge 1) &= \frac{P(X\ge 2, X\ge 1)}{P(X\ge 1)} \\ &= \frac{P(X\ge 2)}{P(X\ge 1)} \\ &= \frac{1-\left(\frac{4}{5}\right)^{10} - \frac{10\times 4^9}{5^{10}}}{1-\left(\frac{4}{5}\right)^{10}} \\ &= 0.699 \end{aligned}$$

1.39

(a) Suppose A and B are mutually exclusive. Then $A \cap B = \emptyset$ and $P(A \cap B) = 0$. If A and B are independent, then $0 = P(A \cap B) = P(A)P(B)$. But this is impossible, since $P(A), P(B) > 0$. Thus A and B cannot be independent.

(b) If A and B ar independent and both have positive probability, then

$$0 < P(A)P(B) = P(A \cap B).$$

This implies $A \cap B \neq \emptyset$, that is, A and B are not mutually exclusive.

1.46

There are 7^7 equally likely sample points. The possible values of X_3 are 0, 1 and 2. Only the pattern 331 (3 balls in one cell, 3 balls in another cell and 1 ball in a third cell) yields $X_3 = 2$. The number of sample points with this pattern is $\binom{7}{2}\binom{7}{3}\binom{4}{3}5 = 14700$. So $P(X_3 = 2) = 14700/7^7 \approx .0178$. There are 4 patterns that yield $X_3 = 1$. The number of sample points that give each of these patterns is given below.

$$\#(\text{pattern is } 34) = 7\binom{7}{3}6 = 1470$$

$$\#(\text{pattern is } 322) = 7\binom{7}{3}\binom{6}{2}\binom{4}{2} = 22050$$

$$\#(\text{pattern is } 3211) = 7\binom{7}{3}6\binom{4}{2}\binom{5}{2}2! = 176400$$

$$\#(\text{pattern is } 31111) = 7\binom{7}{3}\binom{6}{4}4! = 88200$$

The summation of them is 288120 and therefore $P(X_3 = 1) = \frac{288120}{7^7} \approx .3498$.

Thus,

$$P(X_3 = 0) = 1 - P(X_3 = 1) - P(X_3 = 2) \approx .6322$$

Another reasonable solution is to consider balls to be not distinguishable. Under this assumption, we have the total number of equally possible events to be $\binom{n+r-1}{r} = \binom{13}{7}$. Then

$$P(X_3 = 2) = P(X_3 = 2, X_1 = 1) = \frac{\binom{7}{2}\binom{5}{1}}{\binom{13}{7}},$$

$$\begin{aligned} P(X_3 = 1) &= P(X_3 = 1, X_4 = 1) + P(X_3 = 1, X_2 = 2) \\ &\quad + P(X_3 = 1, X_2 = 1, X_1 = 1) + P(X_3 = 1, X_1 = 4) \\ &= \frac{\binom{7}{1}\binom{6}{1} + \binom{7}{1}\binom{6}{2} + \binom{7}{1}\binom{6}{1}\binom{5}{2} + \binom{7}{1}\binom{6}{4}}{\binom{13}{7}}, \end{aligned}$$

and

$$P(X_3 = 0) = 1 - P(X_3 = 1) - P(X_3 = 2).$$

1.51

In the sample space, there are $\binom{30}{4}$ combinations equally possible.

$$P(X = 0) = \frac{\binom{5}{0}\binom{25}{4}}{\binom{30}{4}} \approx 0.4616$$

$$P(X = 1) = \frac{\binom{5}{1}\binom{25}{3}}{\binom{30}{4}} \approx 0.4196$$

$$P(X = 2) = \frac{\binom{5}{2}\binom{25}{2}}{\binom{30}{4}} \approx 0.1095$$

$$P(X = 3) = \frac{\binom{5}{3}\binom{25}{1}}{\binom{30}{4}} \approx 0.0091$$

$$P(X = 4) = \frac{\binom{5}{4}\binom{25}{0}}{\binom{30}{4}} \approx 0.0002$$

For the cumulative distribution function, we have

Table 1: Values of cumulative distribution function

x	$x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$	$3 \leq x < 4$	$4 \leq x$
$F(x)$	=0	≈0.4616	≈0.8812	≈0.9917	≈0.9998	=1

Exercise 2. There are a couple of ways to understand the question. In this solution, we understand the question as follows: Suppose we do N trials, what is the probability that the time N is the first time that we have exactly m_1 outcomes of type 1, m_2 outcomes of type 2, ..., m_k outcomes of type k?

It can be seen that this probability is 0 if $N < m_1 + \dots m_k$. To calculate it when $N \geq m_1 + \dots m_k$, we can see that to make N is first time for that to happens, we need to have one of the followings

(1) at the time $N-1$, we have $m_1 - 1$ outcomes of type 1, m_2 outcomes of type 2, ..., m_k outcomes of type k, and get type 1 in the last trial; or

(2) at the time $N-1$, we have m_1 outcomes of type 1, $m_2 - 1$ outcomes of type 2, ..., m_k outcomes of type k, and get type 2 in the last trial; or

....

(k) at the time $N-1$, we have m_1 outcomes of type 1, m_2 outcomes of type 2, ..., $m_k - 1$ outcomes of type k, get type k in the last trial.

Because those events are disjoint, and thanks to the multinomial distribution, we have the probability that we need to calculate equals to

$$\sum_{i=1}^{k} \binom{N-1}{m_1, \dots, m_i - 1, \dots, m_k, N - \sum_{j=1}^{k} m_j} p_1^{m_1} \dots p_i^{m_i - 1} \dots p_k^{m_k} p_i$$

$$= \sum_{i=1}^{k} \binom{N-1}{m_1, \dots, m_i - 1, \dots, m_k, N - \sum_{j=1}^{k} m_j} p_1^{m_1} \dots p_k^{m_k}.$$

Exercise 3. Using the total probability rule, we have the probability that a Type 1 student is admitted is

$$\sum_{k=1}^{M} P(\text{admitted}|\text{apply to dept. i}) P(\text{apply to dept. i}) = \sum_{k=1}^{M} a_{k,1} p_k. \tag{1}$$

Similarly, the probability that a Type 2 student is admitted is

$$\sum_{k=1}^{M} a_{k,2} q_k. \tag{2}$$

In the next part, we need to show an example such that $a_{k,1}$ is slightly less than $a_{k,2}$ for all k, but there is still a bias towards students of Type 1. To make the example simple, we only consider 2 departments. First we consider that case where $a_{1,1} = a_{1,2} =: a_1$, $a_{2,1} = a_{2,2} =: a_2$. Because $p_1 + p_2 = q_1 + q_2 = 1$, we have

$$\begin{aligned}\sum_{k=1}^{M} a_{k,1}p_k - \sum_{k=1}^{M} a_{k,2}q_k &= a_1(p_1 - q_1) + a_2(q_2 - p_2) \\ &= a_1(p_1 - q_1) - a_2(p_1 - q_1) \\ &= (a_1 - a_2)(p_1 - q_1),\end{aligned}$$

which can be significantly greater than 0 if $a_1 > a_2$ and $p_1 > q_2$. Intuitively, if the department 1 is much easier to students than department 2, and there are many more Type 1 students applying to department 1, then there would be a huge bias towards Type 1 students.

In the case where $a_{k,1}$ is slightly less than $a_{k,2}$ for all k (says, $a_{k,2} = a_{k,2} + \delta$, where δ is small). We can see that the quantity $(a_1 - a_2)(p_1 - q_1)$ above would be slightly smaller (by an amount proportional to δ). So when δ is small, this quantity will still be greater than 0, and we still have a bias towards students of Type 1.