Homework 4 solution

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Problem 1 (1) Using the change of variable formula, we have

$$\begin{aligned}
f_{\epsilon_1 U, \epsilon_2 V}(u,v) &= f_{U,V}(\frac{u}{\epsilon_1}, \frac{v}{\epsilon_2}) \left| \det \begin{pmatrix} \frac{1}{\epsilon_1} & 0 \\ 0 & \frac{1}{\epsilon_2} \end{pmatrix} \right| & (1) \\
&= Cg((\frac{u}{\epsilon_1})^2, (\frac{v}{\epsilon_2})^2) & (2) \\
&= Cg(u^2+v^2) = f_{U,V}(u,v) & (3)
\end{aligned}$$

Thus, $\epsilon_1 U, \epsilon_2 V$ and U, V have the same distribution. This yields the fact that (U,V) and $(-U,V)$ have the same distribution, which implies that U and $-U$ have the same distribution. From the previous homework, we know that $EU = 0$.

Similarly, the fact that (U,V) and $(U,-V)$ have the same distribution yields $EV = 0$. It also implies UV and $-UV$ have the same distribution, which implies $E(UV) = 0$. Combining all those results, we have

$$\mathrm{Cov}(U,V) = E(UV) - EU - EV = 0 - 0 - 0 = 0. \tag{4}$$

(2)

$$\begin{aligned}
P(Z_1 \le z_1, Z_2 \le z_2) &= \sum_{\epsilon_1,\epsilon_2=\pm 1} P(Z_1 \le z_1, Z_2 \le z_2, W_1 = \epsilon_1, W_2 = \epsilon_2) \\
&= \sum_{\epsilon_1,\epsilon_2=\pm 1} P(\epsilon_1 U \le z_1, \epsilon_2 V \le z_2) P(W_1 = \epsilon_1, W_2 = \epsilon_2) \quad \text{(independent)} \\
&= \sum_{\epsilon_1,\epsilon_2=\pm 1} P(U \le z_1, V \le z_2) P(W_1 = \epsilon_1, W_2 = \epsilon_2) \quad \text{(from last part)} \\
&= P(U \le z_1, V \le z_2).
\end{aligned}$$

Here, we are using the fact that

$$\mathbb{P}(W_1 = 1, W_2 = 1) + \mathbb{P}(W_1 = -1, W_2 = 1) + \mathbb{P}(W_1 = 1, W_2 = -1) + \mathbb{P}(W_1 = -1, W_2 = -1) = 1$$

Hence (Z_1, Z_2) and (U,V) have the same distribution. □

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Problem 2 (1) Because Y has a density symmetric about 0, we will have

$$\begin{aligned}\mathbb{E}[Y] &= \int_{\mathbb{R}} y f_Y(y) dy && (5)\\ &= \int_0^{\infty} y f_Y(y) dy - \int_{\infty}^{0} y f_Y(-y) d(y) && (6)\\ &= 0. && (7)\end{aligned}$$

Therefore, $\mathbb{E}[Y] = 0$.

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[\mathbb{E}[XY|Y]] && (8)\\ &= \mathbb{E}[\mathbb{E}[SY^2|Y]] && (9)\\ &= \mathbb{E}[Y^2\mathbb{E}[S|Y]] && (10)\\ &= \mathbb{E}[Y^2\mathbb{E}[S]] = 0, \text{because } \mathbb{E}[S] = 0. && (11)\end{aligned}$$

(You can also show that XY has the same distribution as Y, or symmetric around 0, which implies that its expectation is 0.) Hence,

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0 = 0 \qquad (12)$$

However, they are not independent.

Because Y has a density function, there exists a $\epsilon > 0$ s.t.

$$1 > \mathbb{P}(|Y| > \epsilon) > 0$$

It is observed that

$$\mathbb{P}(|X| > \epsilon) = \mathbb{P}(|S||Y| > \epsilon) = \mathbb{P}(|Y| > \epsilon)$$

Then we will have

$$\mathbb{P}(|X| > \epsilon, |Y| > \epsilon) = \mathbb{P}(|Y| > \epsilon) > \mathbb{P}(|Y| > \epsilon)^2 = \mathbb{P}(|X| > \epsilon)\mathbb{P}(|Y| > \epsilon) \qquad (13)$$

□

Thus, they are not independent.

(2) Let Y be the height of the rectangular.

$$\begin{aligned}\mathbb{E}[circumference] &= \mathbb{E}[2X + 2Y]\\ &= 2\mathbb{E}[X] + 2\mathbb{E}[Y]\\ &= 1 + 2\mathbb{E}[\mathbb{E}[Y|X]]\\ &= 1 + 2\mathbb{E}[\frac{X}{2}]\\ &= 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[area] &= \mathbb{E}[XY]\\ &= \mathbb{E}[\mathbb{E}[XY|X]]\\ &= \mathbb{E}[X\mathbb{E}[Y|X]]\\ &= \mathbb{E}[\frac{X^2}{2}]\\ &= \frac{1}{6}\end{aligned}$$

(3) Let $A = 1$ if X-lock is obtained and $A = 0$ if Y-lock is obtained. Then we have

$$\begin{aligned}
\mathbb{E}[W] &= \mathbb{E}[\mathbb{E}[W|A]] \\
&= \mathbb{E}[W|A=1]P(A=1) + \mathbb{E}[W|A=0]P(A=0) \\
&= \lambda_x p_x + \lambda_y p_y
\end{aligned}$$

$$Var(W) = Var(\mathbb{E}[W|A]) + \mathbb{E}[Var(W|A)],$$

where

$$Var(\mathbb{E}[W|A]) = Var(A\lambda_x + (1-A)\lambda_y) = p_x\lambda_x^2 + p_y\lambda_y^2 - (p_x\lambda_x + p_y\lambda_y)^2 = p_x p_y(\lambda_x - \lambda_y)^2$$

$$\mathbb{E}[Var(W|A)] = p_x\lambda_x + p_y\lambda_y$$

Thus,

$$\begin{aligned}
Var(W) &= Var(\mathbb{E}[W|A]) + \mathbb{E}[Var(W|A)] \\
&= p_x p_y(\lambda_x - \lambda_y)^2 + p_x\lambda_x + p_y\lambda_y
\end{aligned}$$

Problem 3 (1) Using the hint, we can write

$$(N_1, \dots, N_m) = \sum_{i=1}^{m} V_i$$

, where $V_i = (V_{i,1}, \dots, V_{i,m})$'s are i.i.d Multinomial $(1, p_1, \dots, p_m)$.

$$\begin{aligned}
\mathbb{E}[N_i N_j] &= \mathbb{E}[(\sum_{k=1}^{n} V_{k,i})(\sum_{k=1}^{n} V_{k,j})] \\
&= \mathbb{E}[\sum_{k_1 \neq k_2} V_{k_1,i} V_{k_2,j}] + \mathbb{E}[\sum_{k=1}^{n} V_{k,i} V_{k,j}] \\
&= n(n-1) p_i p_j + \begin{cases} 0 & i \neq j \\ \mathbb{E}[\sum_{k=1}^{n} V_{k,i}] = np_i & i = j \end{cases}
\end{aligned}$$

because $V_{k,i}$ and $V_{k,j}$ cannot be non-negative at the same time for $i \neq j$, and if $i = j$, as $V_{k,i} = 0$ or 1, $V_{k,i}^2 = V_{k,i}$.

$$\mathbb{E}[N_i] = \mathbb{E}[\sum_{k=1}^{n} V_{k,i}] = np_i$$

$$\mathbb{E}[N_j] = \mathbb{E}[\sum_{k=1}^{n} V_{k,j}] = np_j$$

Thus,

$$\begin{aligned}
\text{Cov}[N_i, N_j] &= \mathbb{E}[N_i N_j] - \mathbb{E}[N_i]\mathbb{E}[N_j] \\
&= -np_i p_j + \begin{cases} 0 & i \neq j \\ np_i & i = j \end{cases} \\
&= \begin{cases} -np_i p_j & i \neq j \\ np_i(1 - p_i) & i = j \end{cases}
\end{aligned}$$

(2) Let $\tilde{N} = N_1 + \cdots + N_r$. We know that $\tilde{N}, N_{r+1}, \dots, N_m \sim \text{Multinomial}(n, \sum_{i=1}^{r} p_i, p_{r+1}, \dots, p_m)$. Thus,

$$\begin{aligned}
\mathbb{P}(N_{r+1} = n_{r+1}, \dots, N_m = n_m) &= \mathbb{P}(\tilde{N} = n - \sum_{i=1}^{r} n_i, N_{r+1} = n_{r+1}, \dots, N_m = n_m) \\
&= \frac{n!}{n_{r+1}! \dots n_m! (n - \sum_{i=r+1}^{m} n_i)!} \left(\prod_{i=r+1}^{m} p_i^{n_i} \right) (\sum_{i=1}^{r} p_i)^{n - \sum_{i=r+1}^{m} n_i},
\end{aligned}$$

where $\sum_{i=r+1}^m n_i \leq n, n_i \in \mathbb{N}$ Therefore,

$$\begin{aligned}
& \mathbb{P}(N_1 = n_1, \ldots, N_r = n_r | N_{r+1} = n_{r+1}, \ldots, N_m = n_m) \\
= & \frac{\mathbb{P}(N_1 = n_1, \ldots, N_r = n_r, N_{r+1} = n_{r+1}, \ldots, N_m = n_m)}{\mathbb{P}(N_{r+1} = n_{r+1}, \ldots, N_m = n_m)} \\
= & \frac{\frac{n!}{n_1! \ldots n_m!} \prod_{i=1}^m p_i^{n_i}}{\frac{n!}{n_{r+1}! \ldots n_m! (n - \sum_{i=r+1}^m n_i)!} \left(\prod_{i=r+1}^m p_i^{n_i}\right) \left(\sum_{i=1}^r p_i\right)^{n - \sum_{i=r+1}^m n_i}} \\
= & \frac{(\sum_{i=1}^r n_i)! \prod_{i=1}^r p_i^{n_i}}{n_1! \ldots n_r! (\sum_{i=1}^r p_i)^{\sum_{i=1}^r n_i}} \\
= & \frac{(\sum_{i=1}^r n_i)!}{n_1! \ldots n_r!} \prod_{i=1}^r \left(\frac{p_i}{\sum_{j=1}^r p_j}\right)^{n_i},
\end{aligned}$$

where $\sum_{i=1}^m n_i = n$ and $n_i \in \mathbb{N}$, which is Multinomial$(n - \sum_{i=r+1}^m n_i, \frac{p_1}{\sum_{i=1}^r p_i}, \ldots, \frac{p_r}{\sum_{i=1}^r p_i})$. Then use the conclusion in (i) we could get the conditional covariance is

$$\begin{cases} -(n - \sum_{i=r+1}^m n_i) \frac{p_i}{\sum_{k=1}^r p_k} \frac{p_j}{\sum_{k=1}^r p_k} & i \neq j \\ (n - \sum_{i=r+1}^m n_i) \frac{p_i}{\sum_{k=1}^r p_k} (1 - \frac{p_i}{\sum_{k=1}^r p_k}) & i = j \end{cases}$$

(3) We have $N_1 \sim Bin(100, 0.25)$. Because $n = 100$ is large, use Central Limit Theorem to approximate this Binomial distribution is reasonable. We have

$$P(X > 65) = P\left(\frac{X - 100 \times 0.25}{10 \times \sqrt{0.25 \times 0.75}} > \frac{65 - 100 \times 0.25}{10 \times \sqrt{0.25 \times 0.75}}\right) \approx P(Z > 9.24), \tag{14}$$

where Z is the standard normal distribution. □

Problem 4 The area of the ellipse is $\pi/\sqrt{ab}$.

$$\begin{aligned}
\mathbb{E}[X] &= \int_R \int_R \frac{x\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)dxdy \\
&= \int_R \int_{-R} \frac{-x\sqrt{ab}}{\pi} I(a(-x)^2 + by^2 \leq 1)d(-x)dy \\
&= -\int_R \int_R \frac{x\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)dxdy \\
&= -\mathbb{E}[X].
\end{aligned}$$

Therefore $\mathbb{E}[X] = 0$.

$$\begin{aligned}
\mathbb{E}[XY] &= \int_R \int_R \frac{xy\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)dxdy \\
&= \int_R \int_{-R} \frac{-xy\sqrt{ab}}{\pi} I(a(-x)^2 + by^2 \leq 1)d(-x)dy \\
&= -\int_R \int_R \frac{xy\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)dxdy \\
&= -\mathbb{E}[XY].
\end{aligned}$$

Therefore $\mathbb{E}[XY] = 0$.

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

Thus, they are not correlated.

The marginal density for X can be calculated as

$$\begin{aligned}
f_X(x) &= \int_R \frac{\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)dy \\
&= \frac{2\sqrt{a(1 - ax^2)}}{\pi} I(x \in [-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}])
\end{aligned}$$

Thus,

$$\begin{aligned}
f_{Y|X=x}(y) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{\frac{\sqrt{ab}}{\pi} I(ax^2 + by^2 \leq 1)}{\frac{2\sqrt{a(1-ax^2)}}{\pi} I(x \in [-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}])} \\
&= \frac{1}{2}\sqrt{\frac{b}{1 - ax^2}} I(x \in [-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}], y \in [-\sqrt{\frac{1 - ax^2}{b}}, \sqrt{\frac{1 - ax^2}{b}}])
\end{aligned}$$

Similarly, we can get the conditional density for $X|Y$ as

$$f_{X|Y=y}(x) \frac{1}{2}\sqrt{\frac{a}{1 - by^2}} I(y \in [-\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{b}}], x \in [-\sqrt{\frac{1 - by^2}{a}}, \sqrt{\frac{1 - by^2}{a}}])$$

As $f_X(x) \neq f_{X|Y=y}(x)$, X and Y are not independent. $\square$

Problem 5 (1)

$$\begin{aligned} f(m,\theta) &= \mathbb{P}(S_n = m|\Theta = \theta) f_\Theta(\theta) \\ &= \frac{n!}{m!(n-m)!}\theta^m(1-\theta)^{n-m} I(\theta \in [0,1], 0 \le m \le n, m \in \mathbb{N}), \end{aligned}$$

$$\begin{aligned} p_n(m) &= \mathbb{P}(S_n = m) = \int_0^1 \frac{n!}{m!(n-m)!}\theta^m(1-\theta)^{n-m} d\theta \\ &= \frac{n!}{m!(n-m)!}\frac{\Gamma(m+1)\Gamma(n-m+1)}{\Gamma(n+2)} \\ &= \frac{n!}{m!(n-m)!}\frac{m!(n-m)!}{(n+1)!} \\ &= \frac{1}{n+1} \end{aligned}$$

where $m = 0, 1, 2, \ldots, n$.

$$\begin{aligned} f_{\Theta|S_n=k}(\theta) &= \frac{f(k,\theta)}{\mathbb{P}(S_n = k)} \\ &= \frac{\frac{n!}{m!(n-m)!}\theta^m(1-\theta)^{n-m}}{\frac{1}{n+1}} \\ &= \frac{(n+1)!}{m!(n-m)!}\theta^m(1-\theta)^{n-m}, \end{aligned}$$

where $\theta \in [0,1]$ and $m \in [0,n], n, m \in \mathbb{N}$, and 0 for all other situations.

$$\begin{aligned} E[\Theta|S_n = m] &= \int_0^1 \frac{(n+1)!}{m!(n-m)!}\theta^m(1-\theta)^{n-m} d\theta \\ &= \frac{k+1}{n+2}, \end{aligned}$$

where $m \in [0,n], n, m \in \mathbb{N}$. It makes sense since the larger S_n we observe, the larger θ tends to be. And it is between 0 and 1 and not equal to 0 or 1.

(2)

$$\begin{aligned} f(\epsilon_1, \ldots, \epsilon_n, \theta) &= P(X_1 = \epsilon_1, \ldots, X_n = \epsilon_n|\Theta = \theta) f_\Theta(\theta) \\ &= \theta^{\sum_{i=1}^n \epsilon_i}(1-\theta)^{n-\sum_{i=1}^n \epsilon_i} I(\theta \in [0,1]) \end{aligned}$$

where $(\epsilon_1, \ldots, \epsilon_n)$ is a generic point in $\{0,1\}^n$ and the "mixed density" is 0 for all other situations.

$$\begin{aligned} \mathbb{P}(X_1 = \epsilon_1, \ldots, X_n = \epsilon_n) &= \int_0^1 \theta^{\sum_{i=1}^n \epsilon_i}(1-\theta)^{n-\sum_{i=1}^n \epsilon_i} d\theta \\ &= \frac{(\sum_{i=1}^n \epsilon_i)!(n-\sum_{i=1}^n \epsilon_i)!}{(n+1)!} \end{aligned}$$

where $(\epsilon_1, \ldots, \epsilon_n)$ is a generic point in $\{0,1\}^n$ and the probability is 0 for all other situations.

(3) From the result of (2) we can see that the pmf is only a function of $\sum_{i=1}^n X_i$. As

$$\sum_{i=1}^n X_i = \sum_{i=1}^n X_{\pi(i)}$$

for any permutation, the pmf will not change and thus they have the same distribution. Therefore, the distribution is exchangeable.

(4)

$$\begin{aligned}\mathbb{P}(X_1 = \epsilon_1, \dots, X_n = \epsilon_n | S_n = m) &= \frac{\mathbb{P}(X_1 = \epsilon_1, \dots, X_n = \epsilon_n)}{\mathbb{P}(S_n = m)} \\ &= \frac{\frac{(\sum_{i=1}^n \epsilon_i)!(n - \sum_{i=1}^n \epsilon_i)!}{(n+1)!}}{\frac{1}{n+1}} \\ &= \frac{m!(n-m)!}{n!},\end{aligned}$$

where $\sum_{i=1}^n \epsilon_i = m$, $(\epsilon_1, \dots, \epsilon_n)$ is a generic point in $\{0,1\}^n$, $0 \le m \le n$, $m, n \in \mathbb{N}$ and the probability is 0 for all other situations.

$$\begin{aligned}\mathbb{P}(X_1 = \epsilon_1 | S_n = m) &= \sum_{\sum_{i=2}^n \epsilon_i = m - \epsilon_1} \mathbb{P}(X_1 = \epsilon_1, \dots, X_n = \epsilon_n | S_n = m) \\ &= \sum_{\sum_{i=2}^n \epsilon_i = m - \epsilon_1} \frac{m!(n-m)!}{n!} \\ &= \frac{(n-1)!}{(m-\epsilon_1)!(n-1-m+\epsilon_1)!} \frac{m!(n-m)!}{n!} \\ &= \begin{cases} \frac{m}{n} & \epsilon = 1 \\ \frac{n-m}{n} & \epsilon = 0 \end{cases}\end{aligned}$$

For $0 \le \epsilon_1 \le m \le n$, $\epsilon_1 = 0, 1$, $n, m \in \mathbb{N}$ and the probability is 0 for all other situations. Therefore,

$$\mathbb{E}[X_1 | S_n = m] = \frac{m}{n}.$$