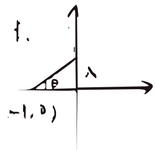
Stats 510 Assignment 2.

1934 2932 Shu Zhou

1.



Let θ denotes the angle that a line with fixed point of rotation makes with the x-axis.

Then $\tan\theta = \lambda \Rightarrow \theta = \arctan\lambda$

$$d\theta = \frac{1}{1+\lambda^2}\, d\lambda$$

So the distribution of the angle θ is given by

$$\frac{d\theta}{\pi} = \frac{1}{\pi}\frac{1}{1+\lambda^2}\, d\lambda$$

Hence, p.d.f of Y is given by $f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2} \quad y \in R.$

2. $P(W=j) > 0$ for all $j \geq 1$

$P(W > i+j \mid W > i) = P(W > j)$

$$P(W > i+j \mid W > i) = \frac{P(W > i+j,\, W > i)}{P(W > i)} = \frac{P(W > i+j)}{P(W > i)} = P(W > j) \Rightarrow P(W > i+j) = P(W > i)P(W > j)$$

$$\sum_{j=1}^{\infty} P(W > i+j) = \sum_{i=1}^{\infty} P(W > i)P(W > j) \quad ①$$

Also $\sum_{j=1}^{\infty} P(W > i+j) = \sum_{j=1}^{\infty} P(W > i)P(W > j) \quad ②$

We let $j = k$ in ① and $i = k$ in ②, hence we have

$\sum_{k=1} P(W > i)P(W > k) = \sum_{j}^{\infty} P(W > k)P(W > j)$; Hence consequently

$\Rightarrow P(W > k)\sum_{j=2}^{\infty} P(W > j) = P(W > k+1)\sum_{j=1}^{\infty} P(W > j)$

Let Q denotes $\sum_{j=2}^{\infty} P(W > j)$; Also we evaluate $k=1$; Assume $P(X=1) = p$

Hence $Q = \frac{1-p}{p}$

Therefore $(1-p)P(X > k) = P(X > k+1)$; Since $P(X \geq 1) = 1$

Hence $P(X > k) = (1-p)^{k-1}$

Hence $P(X = k) = (1-p)^{k-1} - (1-p)^k = p(1-p)^{k-1} = pq^{k-1}$, Thus W is Geometrically distributed

3. x follows the exponential distribution

$f(x) = \lambda e^{-\lambda x}$

Since $[x]$ can only take integer values,

$$P([x] = m,\, X - [x] \leq t) = \int_m^{m+t} \lambda e^{-\lambda x}\, dx = e^{-\lambda m} - e^{-\lambda(m-t)} = e^{-\lambda m}(1 - e^{-\lambda t})$$

Hence, Q.E.D.

4. $P(X \geq 0) = 1$.

① $E(x) = \int_0^{\infty} P(x > t)\, dt$

$E(x^r) = \int_0^{\infty} x^r\, dF_x(x) = -\int_0^{\infty} x^r\, d\bar{F}_X(x)$

Integration by parts gives that

$$E(x^r) = -x^r \bar{F}_X(x)\Big|_0^{\infty} + \int_0^{\infty} r x^{r-1}\, \bar{F}_X(x)\, dx$$

Since $E(x) < \infty$ $\lim_{x\to\infty} x^r \bar{F}_X(x) = 0$

Hence

$$E(x^r) = \int_0^{\infty} r x^{r-1} \bar{F}_X(x)\, dx$$

$$= \int_0^{\infty} r x^{r-1}(1 - F_X(y))\, dy.$$

ii) $Y = Y^+ - Y^-$

We definine $Y^+(\omega) = \max\{Y(\omega), 0\}$. $Y^-(\omega) = -\min\{Y(\omega), 0\}$.

Hence If Y can take both positive and negative values

If Y is positive,

$$Y^+ - Y^- = Y(\omega) - 0 = Y(\omega)$$

If Y is negative

$Y^+ - Y^- = 0 + Y(\omega) = Y(\omega)$. Hence $Y = Y^+ - Y^-$ is verfied

Hence $EY = E[Y^+ - Y^-] = E[Y^+] - E[Y^-]$

iii) $P(Y \geq y) \geq P(Y < -y)$ for all $y > 0$;

$$EY = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^{\infty} y f_Y(y) dy + \int_{-\infty}^{0} y f_Y(y) dy$$

iv) $f(y) = \frac{1}{2}\lambda_1 e^{-\lambda_1|y|} (y \geq 0) + \frac{1}{2}\lambda_2 e^{-\lambda_2|y|} (y < 0)$

It is obvious that $f(y) \geq 0$ for all y. ①

$$\int_{-\infty}^{\infty} f(y) dy = \int_0^{\infty} \frac{1}{2}\lambda_1 e^{-\lambda_1 y} dy + \int_{-\infty}^{0} \frac{1}{2}\lambda_2 e^{-\lambda_2|y|} dy$$

$= \frac{1}{2} + \frac{1}{2} = 1$, ② Hence according to ① and ②, f is a valid probability density

$$EY = \int_{-\infty}^{\infty} y f(y) dy = \int_0^{\infty} \frac{1}{2} y \lambda_1 e^{-\lambda_1|y|} dy + \int_{-\infty}^{0} \frac{1}{2}\lambda_2 e^{\lambda_2 y} dy$$

$$= \frac{1}{2}\lambda_1 \times e^{-\lambda_1 y}(y+1) \times \left(\frac{1}{-\lambda_1}\right)\Big|_0^{\infty} + \frac{1}{2} e^{\lambda_2 y}(y-1)\Big|_{-\infty}^{0}$$

$$= -\frac{1}{2} e^{-\lambda_1 y}(y+1)\Big|_0^{\infty} + \frac{1}{2} e^{\lambda_2 y}(y-1)\Big|_{-\infty}^{0}$$

$= \frac{1}{2} + \frac{1}{2} = 1 > 0$ Hence $EY > 0$

According to the result in (iii) For all $y > 0$; $P(Y \geq y) : P\left(\frac{1}{2}\lambda_1 e^{-\lambda_1|y|} - y \geq 0\right) \geq P\left(\frac{1}{2}\lambda_1 e^{-\lambda_1|y|} + y < 0\right) = P(Y < -y) = 0$

Hence, $EY > 0$.

5. The suprome definition is

a) $E(X) = \sup\{E(Y) : Y \text{ is simple and } 0 \leq Y \leq X\}$

Since $P(X > 0) > 0$; $\exists m$ $\forall x$; such that for all $x > 1/m$ $P(X > 1/m) > 0$.

Hence $E(X) = \sup_{0 \leq Y \leq X} \left\{\int_0^{\infty} Y P(Y)\right\} > 0$ since we can simply take $Y = 1/m > 0$.

b). Given that $X_1 \leq X_2$, since both X_1, X_2 is simple

$E(X_2) = \sup_{X_1 \leq X_2} E(X_1) \geq E(X_1)$; Hence $E(X_2) \geq E(X_1)$; Q.E.D.

6. $P(X > x+y \mid X > x) = P(X > y)$

Hence $\int_y^{+\infty} f(x) dx = \dfrac{\int_{x+y}^{\infty} f(x) dx}{\int_x^{\infty} f(x) dx}$ we assume $y = dx$, hence $\int_{dx}^{+\infty} f(X) dX = \dfrac{\int_{x+dx}^{\infty} f(X) dX}{\int_x^{\infty} f(x) dX}$

Hence $\Rightarrow \int_0^{dx} f(X) dX = f(0) dx = \dfrac{f(x) dx}{1 - \int_0^x f(x) dx}$ differentiate according to x $\Rightarrow$ $-f(x) = \dfrac{f'(x)}{f(0)}$; Hence when $x > 0$, we solve the diff. Eq. $f(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$. hence. Q.E.D