Stats 510 Midterm

19342932 Shu Zhou

1. p landing H. until an HH or TT is obtained

So if the coin start with H, it would go like, $HTHTHT\cdots \overset{k-1}{\underset{2}{}} HH$. or TT

Also, if the coin start with T, it would go like $THTHT\cdots\cdots TT$ or HH.

Then if k is odd $\begin{cases} \text{Start with } H: \text{ end with } TT \\ \text{Start with } T: \text{ end with } HH \end{cases}$ $H\boxed{TH}\boxed{TT}$

k is even $\begin{cases} \text{Start with } H: \text{ end with } HH \\ \text{Start with } T: \text{ end with } TT \end{cases}$ $HTHH$

So. $P(X=k) = P(\text{Start with } H) + P(\text{Start with } T)$

$$= \begin{cases} [p(1-p)]^{\frac{k-1}{2}}(1-p) + [p(1-p)]^{\frac{k-1}{2}}p = [p(1-p)]^{\frac{k-1}{2}} & (k \text{ is odd.}) \\ [p(1-p)]^{\frac{k-2}{2}} \times p^2 + [p(1-p)]^{\frac{k-2}{2}}(1-p)^2 = [p(1-p)]^{\frac{k-2}{2}}[2p^2-2p+1] & (k \text{ is even}) \end{cases}$$

Hence $p(X=k) = \begin{cases} [p(1-p)]^{\frac{k-1}{2}} & (k \text{ is odd}) \\ [p(1-p)]^{\frac{k-2}{2}}[2p^2-2p+1] & (k \text{ is even}). \end{cases}$

2. $g(Y) = Y^2 - Y$

i) So if Y is uniformly distributed on $[0,1]$, Let $X = Y^2 - Y$

Then $f_Y(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & y \notin (0,1) \end{cases}$

Then $g(Y)$ is monotoneously increasing on $[\frac{1}{2}, 1]$
decreasing on $[0, \frac{1}{2}]$.

$F_X(x) = P(X \le x) = P(Y^2 - Y \le x) = P(-\sqrt{x+\frac{1}{4}}+\frac{1}{2} \le Y \le \sqrt{x+\frac{1}{4}}+\frac{1}{2}) = F_Y(\sqrt{x+\frac{1}{4}}+\frac{1}{2}) + (1 - F_Y(-\sqrt{x+\frac{1}{4}}+\frac{1}{2}))$

$$= \int_{-\sqrt{x+\frac{1}{4}}+\frac{1}{2}}^{\sqrt{x+\frac{1}{4}}+\frac{1}{2}} 1\, dt$$

$$= \sqrt{x+\tfrac{1}{4}} + \tfrac{1}{2} + \sqrt{x+\tfrac{1}{4}} - \tfrac{1}{2}$$

$$= 2\sqrt{x+\tfrac{1}{4}} \qquad x \in [-\tfrac{1}{4}, 0]$$

Hence $f_X(x) = (x+\frac{1}{4})^{-\frac{1}{2}}$, which holds for $x \in [-\frac{1}{4}, 0]$

ii) If Y follows exponential 1

Then $f_Y(y) = e^{-y}$, $0 < y < \infty$, Then $g(Y)$ is monotonoously increasing on $[\frac{1}{2}, \infty)$, decreasing on $[0, \frac{1}{2}]$

$$F_X(x) = \int_{-\sqrt{x+\frac{1}{4}}+\frac{1}{2}}^{\sqrt{x+\frac{1}{4}}+\frac{1}{2}} e^{-t}dt = -e^{-t}\Big|_{-\sqrt{x+\frac{1}{4}}+\frac{1}{2}}^{\sqrt{x+\frac{1}{4}}+\frac{1}{2}} = e^{\sqrt{x+\frac{1}{4}}-\frac{1}{2}} - e^{-\sqrt{x+\frac{1}{4}}-\frac{1}{2}} \qquad x \in [-\tfrac{1}{4}, \infty)$$

$$f_X(x) = \frac{d}{dx}(F_X(x)) = \frac{1}{2\sqrt{x+\frac{1}{4}}}e^{\sqrt{x+\frac{1}{4}}-\frac{1}{2}} + \frac{1}{2\sqrt{x+\frac{1}{4}}}e^{-\sqrt{x+\frac{1}{4}}-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x+\frac{1}{4}}}\left(e^{\sqrt{x+\frac{1}{4}}-\frac{1}{2}} + e^{-\sqrt{x+\frac{1}{4}}-\frac{1}{2}}\right) \qquad x \in [-\tfrac{1}{4}, \infty)$$

Problem 3.

(a)

$$P(C(x)=m,\ C(x)-X\le t)=P(m-t\le X\le m)=\int_{m-t}^{m} f_X(x)\,dx$$

$$P(C(X)-X\le t)=\sum_{m=1}^{\infty} P(C(x)=m,\ C(x)-X\le t)=\sum_{m=1}^{\infty}\int_{m-t}^{m} f_X(x)\,dx$$

(b) $X=\lambda e^{-\lambda x}$

$$P(F(x)=m,\ C(X)-X\le a,\ X-F(X)\le b)=P(m+1-a\le X\le m+b)=\int_0^{m+b} f_X(x)\,dx-\int_0^{m+1-a} f_X(x)\,dx$$

$$=\int_{m+1-a}^{m+b} f_X(x)\,dx$$

$$=-e^{-\lambda t}\Big|_{m+1-a}^{m+b}$$

$$=-e^{-\lambda(m+b)}+e^{-\lambda(m+1-a)}$$

$$=e^{-\lambda(m+1-a)}-e^{-\lambda(m+b)}$$

Hence $P(C(X)-X\le a,\ X-F(X)\le b)=\sum_{m=0}^{\infty} e^{-\lambda(m+1-a)}-e^{-\lambda(m+b)}$

$$=\sum_{m=0}^{\infty} e^{-\lambda(1-a)}\times e^{-\lambda m}-e^{-\lambda b}\times e^{-\lambda m}$$

$$=\sum_{m=0}^{\infty} e^{-\lambda m}[e^{-\lambda(1-a)}-e^{-\lambda b}]$$

$$=\frac{1}{1-e^{-\lambda}}[e^{-\lambda(1-a)}-e^{-\lambda b}]$$ (according to Geometric Series)

Problem 4.

N_D Democrats; N_R Republicans.

Stop as soon as one Republican in the sample

The probability that it would take at least m draws, with the first m results are all Democrats

$$P(X=m)=\frac{\binom{N_D}{m-1}}{\binom{N}{m-1}}$$