Exam 3

Moulinath Banerjee

University of Michigan

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Announcement: The exam carries 48 points but the maximum possible score is 42 points.

Problem 1: [12 points] Let Y denote the number of failures incurred before the r'th success (r fixed) in a sequence of Bernoulli trials with success probability p. Show by a direct calculation (i.e. not using m.g.f's) that as $r \to \infty$ and $p \to 1$ such that $r(1-p) \to \lambda$ for some $\lambda > 0$,

$$P(Y = y) = e^{-\lambda} \frac{\lambda^y}{y!} \,.$$

Note: Remember that if x_n converges to x, $(1 + x_n/n)^n$ converges to e^x.

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Problem 2: [12 points] Let (X, Y) have p.d.f. $f(x, y) = g(x^2 + y^2)$ for $(x, y) \in \mathbb{R}^2$. Show that X/Y and $X/|Y|$ have the same distribution.
Hint: One approach to this is trying to express $P(X/Y > t)$ and $P(X/|Y| > t)$ in terms of the joint probabilities of (X, Y). The symmetry of f will obviously play an important role.

[OR]

An insect lays N eggs which may be assumed to be distributed as Poisson(λ) for some $\lambda > 0$. Each of these laid eggs hatches independently with probability p. Let X be the number of eggs that hatch. Find the distribution of X.

Problem 3: [12 + 12 points] Let $\alpha_1, \alpha_2, \alpha_3 > 0$.
(i) If $W_1 \sim \text{Beta}(\alpha_1, \alpha_2 + \alpha_3)$ and $W_2|W_1 \sim (1 - W_1)\text{Beta}(\alpha_2, \alpha_3)$ show that the distribution of (W_1, W_2) is given by:

$$f_{\alpha_1,\alpha_2,\alpha_3}(w_1, w_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} w_1^{\alpha_1 - 1} w_2^{\alpha_2 - 1} (1 - w_1 - w_2)^{\alpha_3 - 1} 1((w_1, w_2) \in \mathcal{S}),$$

where $\mathcal{S}$ needs to be specified by you. Explicit calculations are required.

(ii) Let (Θ_1, Θ_2) be a pair of random variables generated from the p.d.f $f_{1,1,1}(\theta_1, \theta_2)$ (as above). Given observed realizations (θ_1, θ_2) of (Θ_1, Θ_2), (N_1, N_2, N_3) are generated from a multinomial distribution with parameters $(n, \theta_1, \theta_2, 1 - \theta_1 - \theta_2)$. Show that the conditional distribution of (Θ_1, Θ_2) belongs to the family $f_{\alpha_1,\alpha_2,\alpha_3}$ for some parameters $\alpha_1, \alpha_2, \alpha_3$ (that you need to identify). How do $E(\Theta_1|(N_1, N_2, N_3))$ and $E(\Theta_2|N_1, N_2, N_3))$ behave in terms of N_1, N_2, N_3?

[OR] to (ii)

Show that if X is a random variable whose m.g.f $M_X(t)$ exists for all $|t| < h$ and is an even function on $(-h, h)$, then $E(g(X)) = 0$ for all *odd* functions g for which $E(g(X))$ is well-defined as a finite quantity. (You may assume that X has a p.d.f f if it helps, but this is not necessary to solve the problem.)