Stats 510 Assignment 6.

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Exercise 5.21 ① $\frac{3}{4}$

② $P(\max(X_1, \dots X_n) > m) = 1 - P(\forall X_i: X_i \le m)$

$= 1 - (\frac{1}{2})^n$

Exercise 5.29

Let X denote the weight of booklets

Thus X_i is the weight of the ith booklet.

Then $EX_i = 1$; $\mathrm{Var}\, X_i = 0.05^2$

Hence $P(\sum_{i=1}^{100} X_i > 100.4) = P(\frac{1}{100}\sum_{i=1}^{100} X_i > 1.004)$

$= P(\bar{X_i} > 1.004)$.

According to Central Limit Theorem

$$P(\bar{X_i} > 1.004) = P(Z > \frac{0.004}{0.05/\sqrt{100}}) = 0.212.$$

Exercise 5.30

$\bar{X}_1 \sim n(\mu, \sigma^2/n)$; $\bar{X}_2 \sim n(\mu, \sigma^2/n)$.

Hence $\bar{X}_1 - \bar{X}_2 \sim n(0, 2\sigma^2/n)$.

$P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) \approx 0.99$

$$0.99 = \left(\frac{-\sigma/5}{\sigma/\sqrt{n/2}} < \frac{\bar{X}_1 - \bar{X}_2}{\sigma/\sqrt{n/2}} < \frac{\sigma/5}{\sigma/\sqrt{n/2}}\right).$$

Hence $0.99 = P(-\frac{1}{5}\sqrt{\frac{n}{2}} < Z < \frac{1}{5}\sqrt{\frac{n}{2}})$.

$\frac{1}{5}\sqrt{\frac{n}{2}} = 2.576 \Rightarrow n = 50 \cdot (2.576)^2 = 332.$

Exercise 5.31

$\sigma^2 = 9$; $\sigma_{\bar{x}}^2 = 9/100$

① By Chebychev's Inequality, we have.

$$P(-3k/100 < \bar{x} - \mu < 3k/100) \ge 1 - 1/k^2 \ge 0.09 \Rightarrow k \ge \sqrt{10}$$

Then $3k/10 = 0.9487$

$$P(-0.9487 < \bar{x} - \mu < 0.9487) \ge 0.9$$

② By CLT, we have.

$$(\bar{x} - \mu)/\sqrt{0.09} \sim n(0,1).$$

$$0.09 = P(-1.645 < \frac{\bar{x} - \mu}{0.3} < 1.645) = P(-0.4935 < \bar{x} - \mu < 0.4935)$$

Hence, the restriction of chebychev's inequality is not as restricted as the normal approximation by CLT, which yields that chebychev's inequality is a weak restriction.

Exercise 5.50.

$x_1 = \cos(2\pi U_1)\sqrt{-2\log U_2}$

$x_2 = \sin(2\pi U_1)\sqrt{-2\log U_2}$

We need to transform x_1 and x_2.

$\frac{x_1}{x_2} = \tan(2\pi U_1)$

① $U_1 = \arctan\left(\frac{x_1}{x_2}\right) \cdot \frac{1}{2\pi}$

$x_1^2 + x_2^2 = -2\log U_2 \Rightarrow$ ② $U_2 = e^{-(x_1^2+x_2^2/2)}$

③ Jocobian of transformation is

$$J = \begin{vmatrix} \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} \\ \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2\pi x_1^2}\frac{x_2}{\left(1+\frac{x_2^2}{x_1^2}\right)} & \frac{1}{2\pi x_1\left(1+\frac{x_2^2}{x_1^2}\right)} \\ -x_1 e^{-(x_1^2+x_2^2)/2} & -x_2 e^{-(x_1^2+x_2^2)/2} \end{vmatrix}$$

$$= \frac{1}{2\pi\,\cancel{(x_1^2+x_2^2)}} e^{-(x_1^2+x_2^2)/2} \cdot \cancel{(x_1^2+x_2^2)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}.$$

Which is the product of the p.d.f of two normal variables x_1 and x_2.

Hence. Q.E.D