5.21. For $X_1, \dots, X_n \overset{iid}{\sim}$ with pdf f and median m. Then

$$\begin{aligned} P(\max(X_1, \dots, X_n) > m) &= 1 - P(\max(X_1, \dots, X_n) \le m) \\ &= 1 - P(\text{all } X_i < m) \\ &= 1 - \prod_{i=1}^{n} P(X_i < m) \\ &= 1 - \left(\tfrac{1}{2}\right)^n. \end{aligned}$$

5.29. Let $X_i =$ weight of i-th book

$\Rightarrow$ X_i's are i.i.d with $E X_i = 1$, $\operatorname{var} X_i = 0.05^2$.

$$\begin{aligned} P\left(\sum_{i=1}^{100} X_i > 100.4\right) &\approx P\left(Z > \frac{1.004 - 1}{0.05/10}\right) \\ &= P(Z > .8) \\ &= 0.2119. \end{aligned}$$

5.30. We approximate by CLT $\bar{X}_1 \sim N(\mu, \sigma^2/n)$

$\bar{X}_2 \sim N(\mu, \sigma^2/n)$

and they are independent $\Rightarrow \bar{X}_1 - \bar{X}_2 \sim N(0, 2\sigma^2/n)$

We have

$$\begin{aligned} P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) &= P\left(\frac{-\sigma/5}{\sigma/\sqrt{n/2}} < \frac{\bar{X}_1 - \bar{X}_2}{\sigma/\sqrt{n/2}} < \frac{\sigma/5}{\sigma/\sqrt{n/2}}\right) \\ &\approx P\left(-\frac{1}{5}\sqrt{\frac{n}{2}} < Z < \frac{1}{5}\sqrt{\frac{n}{2}}\right), \end{aligned}$$

which ≈ 0.99. Therefore $\sqrt{n}/5\sqrt{2} = 2.576 \Rightarrow n \approx 332$.

5.31. $\sigma_{\bar{X}}^2 = 9/100$. If we use Chebyshev inequality,

$$P(-3k/10 < \bar{X} - \mu < 3k/10) \geq 1 - 1/k^2,$$

which we want to $> .9$. Therefore,

$$k \geq \sqrt{10} = 3.16, \quad 3k/10 = .9487.$$

Hence, if we use Chebyshev, we get the .9 conficlent interval being $[-.9487, .9487]$.

Using CLT, $\bar{X} - \mu / .3 \approx N(0,1)$. Therefore

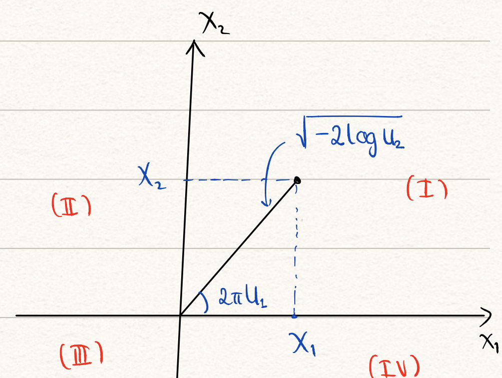
$$.9 = P(-1.645 < Z < 1.645)$$

$$\approx P\left(-1.645 < \frac{\bar{X} - \mu}{.3} < 1.645\right)$$

$$= P(-.4935 < \bar{X} - \mu < .4935),$$

which leads to a narrower confident interval (better!)

5.50.



We have

$$\begin{cases} \tan(2\pi U_1) = X_2/X_1, \\ -2\log U_2 = X_1^2 + X_2^2, \end{cases}$$

which implies

$$U_1 = \begin{cases} \frac{1}{2\pi} \arctan(X_2/X_1) & \text{if } X_1, X_2 > 0. & \text{(I)} \\ -\frac{1}{2\pi} \arctan(X_2/X_1) + \frac{1}{2} & \text{if } X_1 < 0, X_2 > 0 & \text{(II)} \\ \frac{1}{2\pi} \arctan(X_2/X_1) + \frac{1}{2} & \text{if } X_1 < 0, X_2 < 0 & \text{(III)} \\ -\frac{1}{2\pi} \arctan(X_2/X_1) + 1 & \text{if } X_1 > 0, X_2 < 0 & \text{(IV)} \end{cases}$$

and $U_2 = \exp\left(-\frac{X_1^2}{2} - \frac{X_2^2}{2}\right)$ (1-1 map)

By the change of variable formula

$$f_{X_1,X_2}(x_1, x_2) = f_{U_1,U_2}(u_1, u_2) \left| \det \frac{\partial (u_1,u_2)}{\partial (x_1, x_2)} \right|$$

$$= 1 \times \frac{1}{2\pi} \begin{vmatrix} -x_2/x_1^2 \ \frac{1}{1+(x_2/x_1)^2} & \frac{1}{x_1} \frac{1}{1+(x_2/x_1)^2} \\ -x_1 \exp\left(-\frac{x_1^2}{2} + \frac{x_2^2}{2}\right) & -x_2 \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) \end{vmatrix}$$

$$= \frac{1}{2\pi} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2}\right)$$

$$= \mathrm{pdf}_{N(0,1)}(x_1) \times \mathrm{pdf}_{N(0,1)}(x_2).$$

$\square$