g: one-to-one from $(a, b) \to (c, d)$

$\uparrow$ X $\qquad$ $\uparrow$ $Y \doteq g(X)$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| I(y \in (c, d))$$

COVT as discussed is for 1-1 functions

$\boxed{Y = X^2}$ and x can take both +ve and -ve values $\}$ $g(x) = x^2$ not 1-1 on $(-\infty, \infty)$

Theorem 2.5 Let $\underline{Y} = g(X)$

$P(\underline{X} \in B) = 1$ for some open set B.

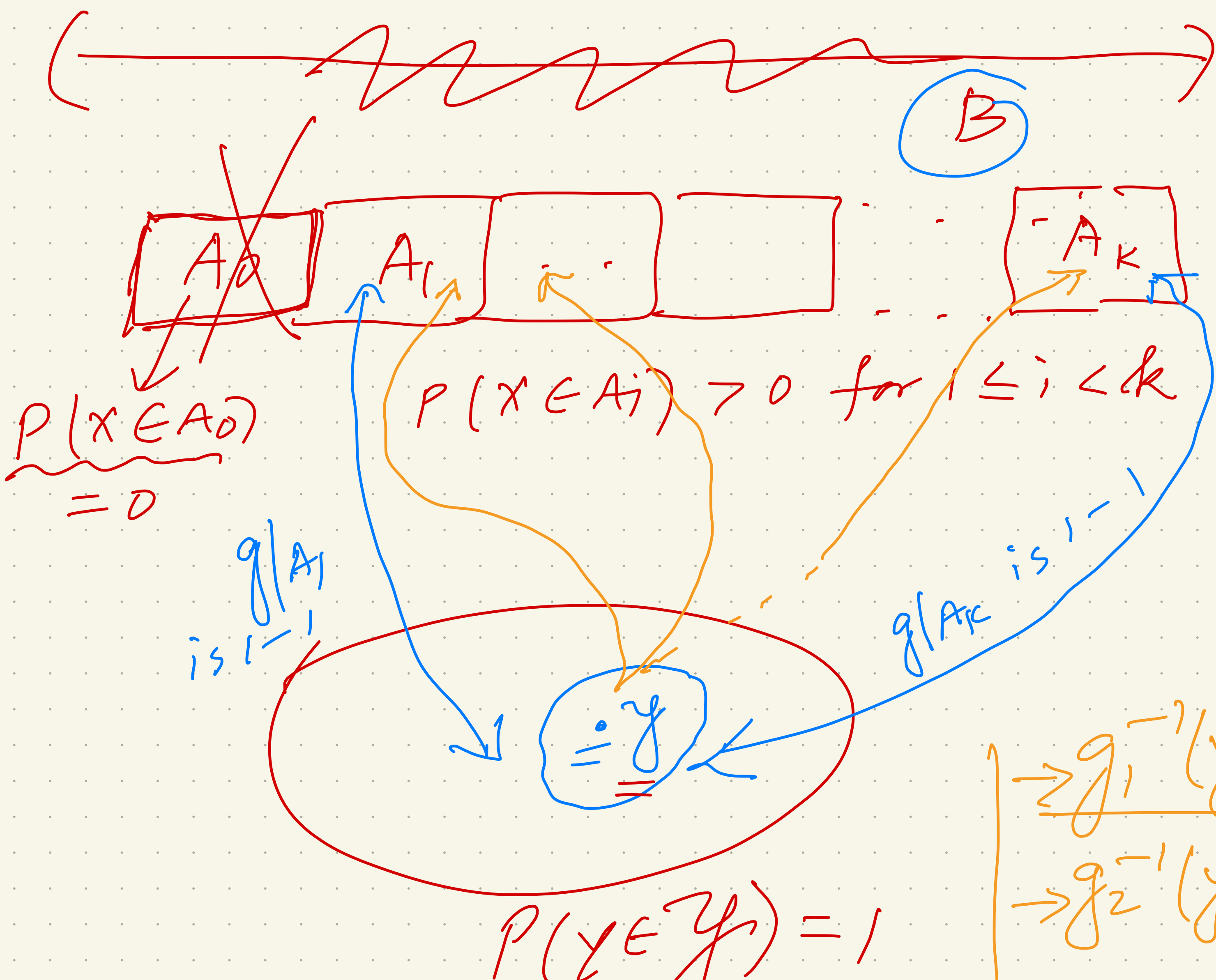
Let $\mathcal{Y}$ be such ~~that~~ $P(Y \in \mathcal{Y}) = 1$

Suppose ~~there~~ is a partition

$(A_0, A_1, \ldots, A_k)$ of B with

$P(X \in A_0) = 0$,

and such that $g|_{A_i}$ is a 1−1 continuously differentiable function between A_i and $\mathcal{Y}$ with non-vanishing derivative.



Denote $g|_{A_i}$ by g_i

$\to g_1^{-1}(y)$

$\to g_2^{-1}(y)$

$\vdots$

$\to g_k^{-1}(y)$

Then

$$f_Y(y) = \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right|$$

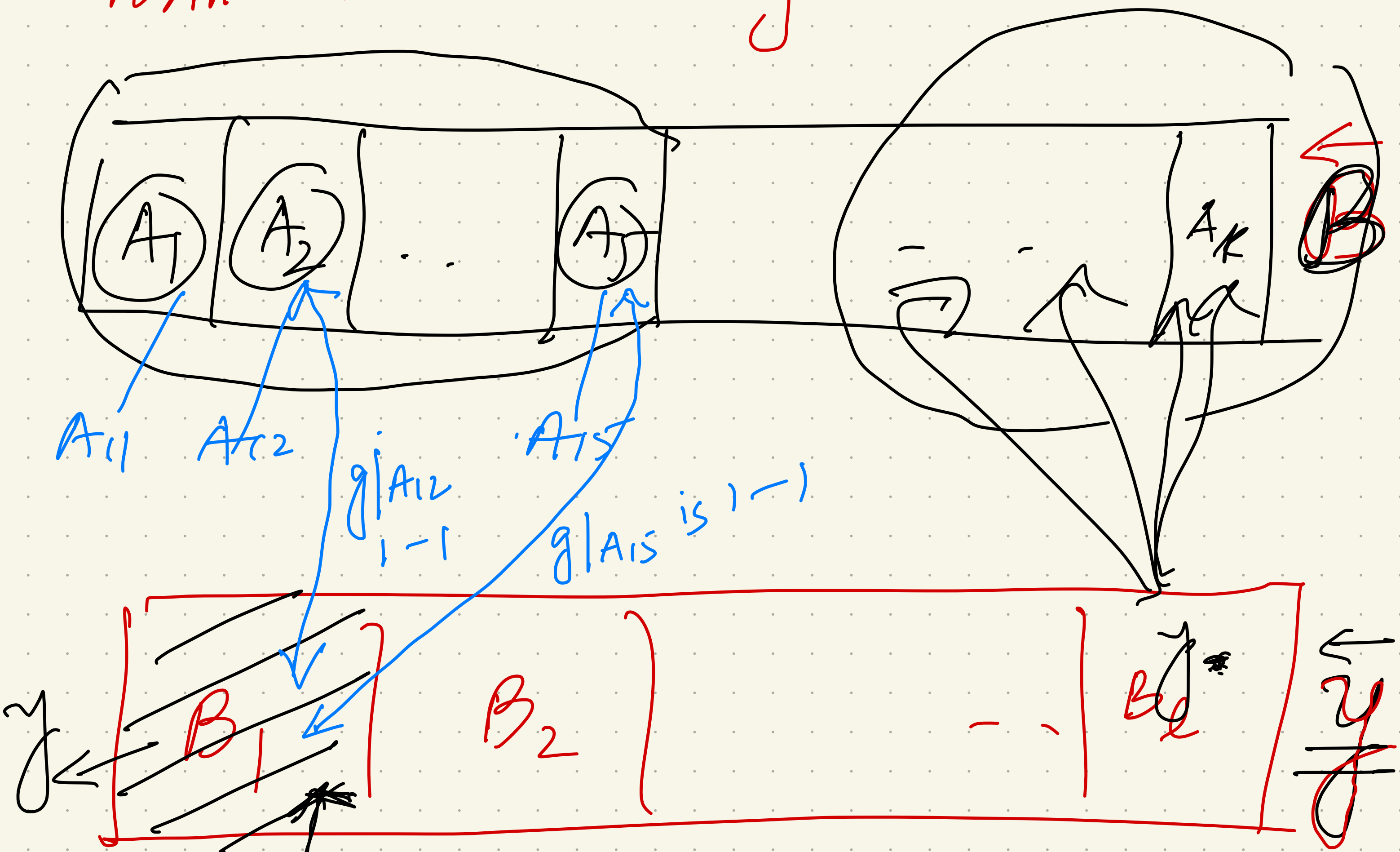
Another extension.

When $\mathcal{Y}$ itself can be partitioned.

Now assume ~~that~~ apart from a set which has probability 0 under the distribution of Y,

$\mathcal{Y} = B_1 \cup B_2 \ldots \cup B_\ell$ where this is a partition, and suppose ~~that~~ for each B_j, there's a subcollection $\{A_{ij}\}$ of $\{A_1, A_2, \ldots, A_k\}$ with $g|_{A_{ij}}$ being 1-1 from A_{ij} to B_j

and nicely continuously differentiable with non vanishing derivative.



$$f_Y(\underline{y}) = \sum_{\underline{A_{ij}}} \frac{f_X(g_{ij}^{-1}(y))}{\left|\frac{d}{dy} g_{ij}^{-1}(y)\right|}$$

if $\underline{y \in B_i}$

Exercise:

2.6 (i) $Y = X^2$, $X \sim N(0,1)$

Find f_Y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right], x \in \mathbb{R}$$

$g(x) = x^2$ is NOT 1-1 !

Apply 2.5, with $\mathcal{Y} = (0,\infty)$

$(P(Y \in \mathcal{Y}) = 1)$ and with $A_0 = \{0\}$

$A_1 = (0,\infty)$, $A_2 = (-\infty, 0)$.

$P(X \in A_0) = 0$

$g|_{A_1}$ is 1-1 from $\boxed{(0,\infty) \text{ to } (0,\infty)}$

$g|_{A_2}$ is 1-1 from $\boxed{(-\infty,0) \text{ to } (0,\infty)}$.

Consider $y \in \mathcal{Y}$, $y > 0$.

$$f_Y(y) = \overbrace{f_X(g_1^{-1}(y)) \left| \frac{d}{dy} g_1^{-1}(y) \right|}^{\text{1'st term}}$$

$$+ \overbrace{f_X(g_2^{-1}(y)) \left| \frac{d}{dy} g_2^{-1}(y) \right|}^{\text{2'nd term.}}$$

$g_1 : \underline{(0, \infty)} \rightarrow \underline{(0, \infty)}$ with $g_1(x) = x^2$

$g_2 : \underline{(-\infty, 0)} \rightarrow \underline{(0, \infty)}$ with $g_2(x) = x^2$

2'nd term: $g_2^{-1}(y) = -\sqrt{y}$

$$\frac{d}{dy} g_2^{-1}(y) = -\underline{\frac{1}{2\sqrt{y}}}$$

so, 2'nd term:

$$\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(-\sqrt{y})^2\right] \frac{1}{2\sqrt{y}}$$

$$\left. = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} \right\} \text{equals 1'st term}$$

So, finally:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{y}} \exp\left(-\frac{y}{2}\right) \mathbb{1}(y>0)$$

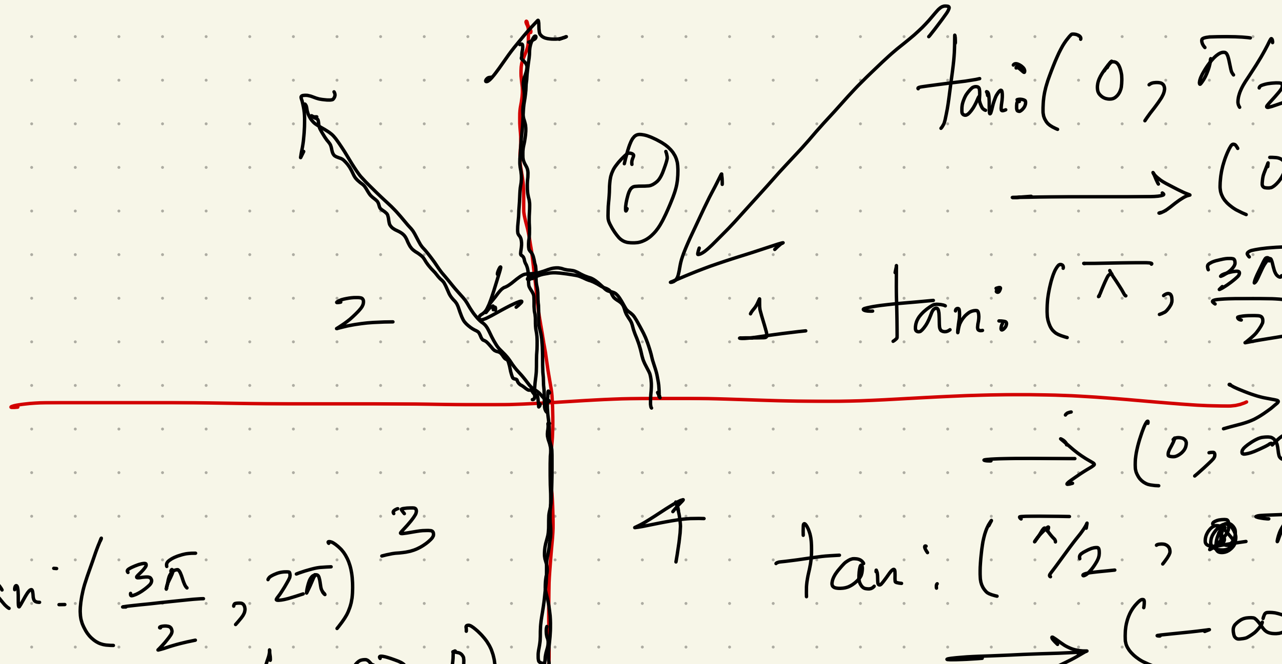
- χ_1^2 density or a $\Gamma(1/2, 1/2)$

application of the further extension:

$\Theta \sim$ Uniform on $(0, 2\pi)$.

$Y = \underline{\tan(\Theta)}$. Find dist. of Y.

$$f_\Theta(\theta) = \frac{1}{2\pi} \mathbb{1}(0 < \theta < 2\pi)$$



tan: $(0, \pi/2) \longrightarrow (0, \infty)$

tan: $(\pi, \frac{3\pi}{2}) \longrightarrow (0, \infty)$

tan: $(\pi/2, \pi) \longrightarrow (-\infty, 0)$

tan: $(\frac{3\pi}{2}, 2\pi) \longrightarrow (-\infty, 0)$

A's

$$\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$$

$$\left(0, \frac{\pi}{2}\right) \xrightarrow[1-1]{\tan^{(1)}} (0, \infty) \qquad \left(\pi, \frac{3\pi}{2}\right) \xrightarrow[1-1]{\tan^{(2)}} (0, \infty)$$

$$\left(\frac{\pi}{2}, \pi\right) \xrightarrow{1-1} (-\infty, 0) \qquad \left(\frac{3\pi}{2}, 2\pi\right) \xrightarrow{1-1} (-\infty, 0)$$

B's: $(0, \infty) \cup (-\infty, 0)$

y

Take $y > 0$

$$f_Y(y) = \frac{f_\Theta\left(\tan^{(1)^{-1}}(y)\right)}{\left|\frac{d}{dy}\tan^{(1)^{-1}}(y)\right|} + \frac{f_\Theta\left(\tan^{(2)^{-1}}(y)\right)}{\left|\frac{d}{dy}\tan^{(2)^{-1}}(y)\right|}$$

Check same expression holds for $y<0$

$$= \frac{1}{2\pi}\frac{1}{1+y^2} + \frac{1}{2\pi}\frac{1}{1+y^2} = \frac{1}{\pi(1+y^2)}$$

Conclude:

$$f_Y(y) = \frac{1}{\pi} \frac{1}{1+y^2} \mathbb{1}(y \in \mathbb{R})$$

Cauchy density or t_1 density.

Section 3

X → typically a low dimensional function (functional) is of interest.
r.v

A measure of centrality.

central tendency.

Average | Expectation.

$x_1, x_2, \ldots, x_n$ } Avg(x_i's) $= \dfrac{x_1 + \ldots + x_n}{n}$

X : assumes values

$\{ x_1, x_2, \ldots \}$.

Let $p(x_j) = P(X = x_j)$

$$E[g(X)] = \sum_{j=1}^{\infty} g(x_j)\, p(x_j)$$

provided $E[|g(X)|] < \infty$

$*$ $E[|g(X)|] = \sum |g(x_j)|\, p(x_j)$

$*$

X is continuous with p.d.f $f(x)$,

then

$$E[g(X)] = \int_{\mathbb{R}} g(x) f(x)\, dx$$

provided $\int_{\mathbb{R}} |g(x)| f(x)\, dx < \infty$.

$$\parallel$$

$$E[|g(X)|]$$

To define expectations:

First define $E(X)$ for any non-negative random variable.

For any general Y, write

$$Y = Y^+ - Y^-$$

$$\left.\begin{aligned} Y^+ &= Y \mathbb{1}(Y \geq 0) \\ Y^- &= -Y \mathbb{1}(Y < 0) \end{aligned}\right\}$$

$$|y| = y^+ + y^-$$
$$y = y^+ - y^-$$

We want to write $\boxed{Ey = Ey^+ - Ey^-}$

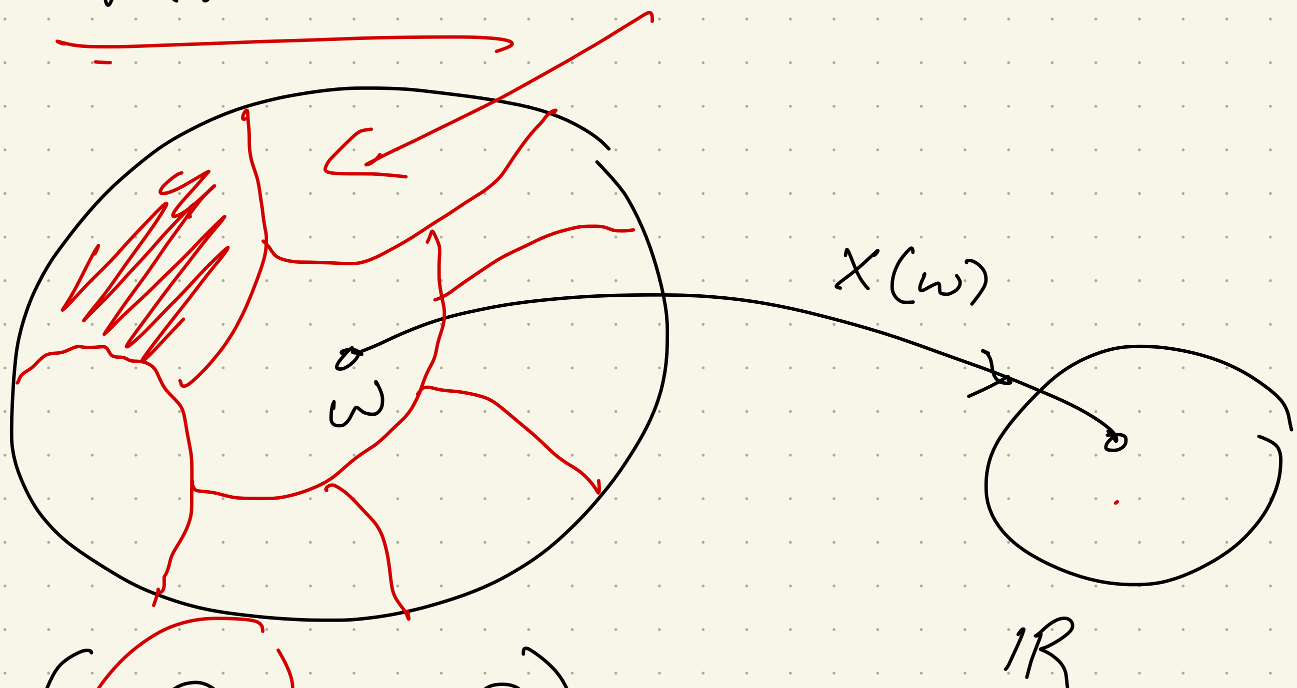
Basically makes sense when both are finite

will require Ey^+ and Ey^- to be finite. That is essentially equivalent to $E(|y|) < \infty$

Non-negative random variables always have an expectation → could be infinite.

But a general random variable is only assigned an expectation if its absolute magnitude has finite expectation.

X : some non-negative random variable



$(\Omega, \mathcal{A}, P)$

consider a simple random variable first. Call it S

$$S(\omega) = \sum_{j=1}^{\ell} x_j \, \mathbb{1}(\omega \in A_j)$$

Where $A_1, A_2, \ldots, A_\ell$ form a partition of Ω

Let $\tilde{S}$ be a non-negative simple random variable.

$$\tilde{S}(\omega) = \sum \underline{\tilde{x}_j} \, 1(\omega \in \tilde{A}_j)$$

$$\tilde{A}_1 \cup \ldots \cup \tilde{A}_\ell = \Omega$$

$$\underline{\tilde{x}_j \geq 0}$$

$E[\tilde{S}]$ is defined intuitively

$$\infty \sum_{j=1}^{\ell} \tilde{x}_j P(\tilde{A}_j)$$

If $X \geq 0$, take a sequence of non-negative simple random variables $\tilde{S}_1 \leq \tilde{S}_2 \leq \tilde{S}_3 \leq \dots$

. . . such ~~that~~

for each $\omega \in \Omega$, $X(\omega) = \uparrow \lim_{j \uparrow \infty} \tilde{S}_j(\omega)$

$$\left.\begin{aligned} &\underline{\text{Define } E[\bar{X}] = \uparrow \lim_{j\uparrow\infty} E[\tilde{S}_j]} \\ &\tilde{S} \text{ and } \tilde{S}' \text{ are non-negative simple random variables} \end{aligned}\right\}$$

and $\tilde{\mathfrak{s}} \leq \tilde{\mathfrak{s}}'$,

then $E\tilde{\mathfrak{s}} \leq E(\tilde{\mathfrak{s}}')$!

<u>Prove...</u>