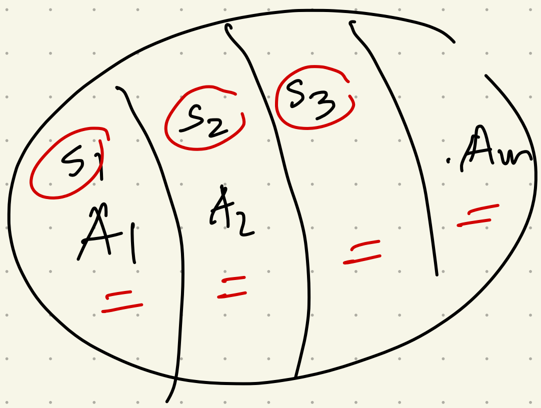
Simple random variables

S : simple random variable ≥ 0

if $S(\omega) = \sum_{i=1}^{m} s_i \, 1(\omega \in A_i)$

$A_1 \cup A_2 \cdots \cup A_m = \Omega$



$(\Omega, \mathcal{A}, P)$

we assume each $P(A_i) > 0$

$$ES = \int S(\omega)\, dP(\omega)$$

$$= \sum_{i=1}^{m} s_i \, P(A_i)$$

For a general non-negative random variable X,

$$EX := \sup\{ES : 0 \le S \le X, \ S \text{ is a simple function}\}$$

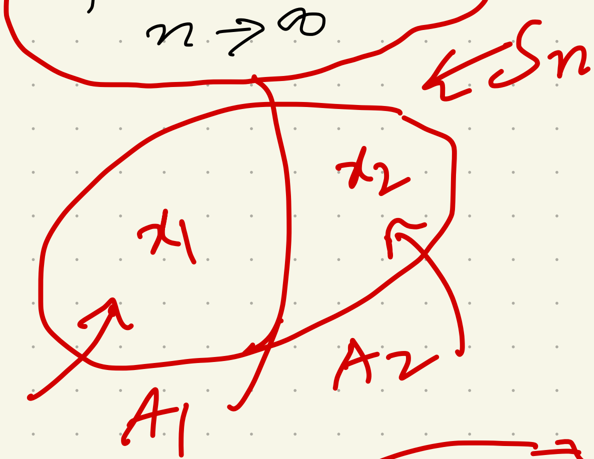
So we allow $EX = \infty$

an alternative route to defining EX is to take $S_1 \le S_2 \le S_3 \le \cdots \uparrow X$

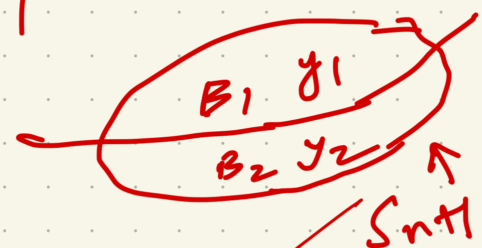
and define $EX = \uparrow \lim_{n \to \infty} ES_n$

$ES_n \le ES_{n+1}$

$\boxed{S_n \le S_{n+1}}$



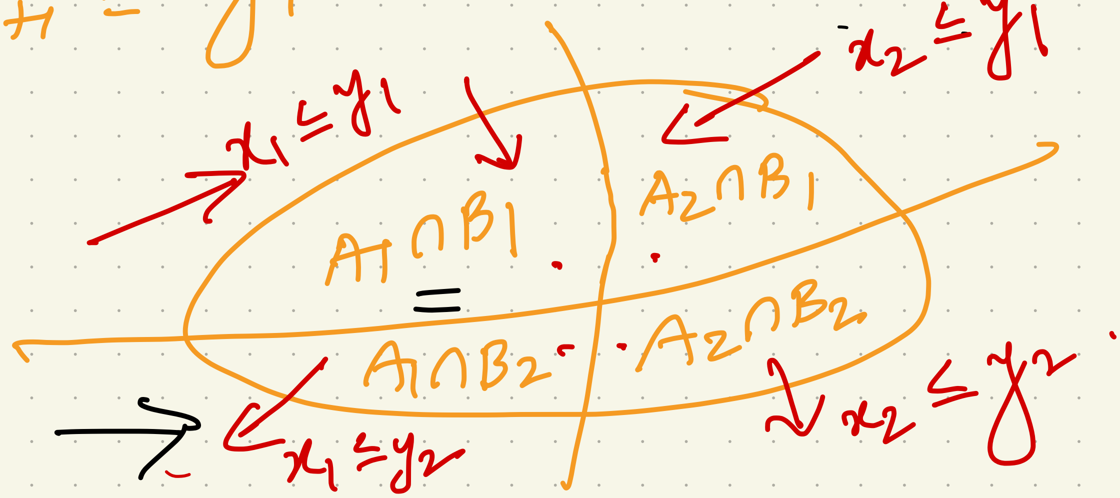
$S_n = x_1 \mathbb{1}_{A_1} + x_2 \mathbb{1}_{A_2}$



$S_{n+1} = y_1 \mathbb{1}_{B_1} + y_2 \mathbb{1}_{B_2}$

$ES_n = \boxed{x_1 P(A_1) + x_2 P(A_2)}$

$ES_{n+1} = y_1 P(B_1) + y_2 P(B_2)$



$$x_1 P(A_1) + x_2 P(A_2)$$

$$= x_1 P(A_1 \cap B_1) + x_1 P(A_1 \cap B_2)$$

$$+ x_2 P(A_2 \cap B_1) + x_2 P(A_2 \cap B_2)$$

$$\le \underline{y_1 P(A_1 \cap B_1)} + y_2 P(A_1 \cap B_2)$$

$$\underline{+ y_1 P(A_2 \cap B_1)} + y_2 P(A_2 \cap B_2)$$

$$= y_1 P(B_1) + y_2 P(B_2) \quad !$$

Definition of EX is consistent in the sense ~~that~~ for any $S_1 \le S_2 \le \dots$ such ~~that~~ $S_n \uparrow X$, $\uparrow \lim E(S_n)$ is the same.

For a general random variable X

write $X = X^+ - X^-$.

$X^+ = X \vee 0$, $X^- = (-X) \vee 0$

$X = X^+ - X^-$ $|X| = X^+ + X^-$.

If $EX^+ < \infty$ and $EX^- < \infty$,

then $\boxed{EX = EX^+ - EX^-}$

We rely on p.m.f of X (discrete)
or the p.d.f of X (for continuous)
to write down EX in practice.

Discrete random variable.

X assumes values $x_1, x_2, \cdots$

$p(x_j) = P(X = x_j)$.

$x_1, x_2, \ldots \in \mathbb{R}^+$.

Then $EX = \sum x_j p(x_j)$.

X is a general random variable

i.e X can assume both +ve and −ve values,

then look at $E(|X|) = \sum_{j=1}^{\infty} |x_j| p(x_j)$

where $x_1, x_2, \ldots$

if $E(|X|) < \infty$

then we write $EX = \sum_{j=1}^{\infty} x_j p(x_j)$

For a general fn $g(x)$,

$$E[g(x)] = \sum_j g(x_j) p(x_j)$$

provided $E[|g(x)|] < \infty$

$$= \sum_{j=1}^{\infty} |g(x_j)| p(x_j)$$

X cts. r.v. with p.d.f $f(x)$.

$$E[g(X)] = \int g(x) f(x)\, dx$$

provided $E[|g(X)|] = \int |g(x)| f(x)\, dx < \infty$.

$X \sim N(0,1)$

$$EX = \int_{-\infty}^{\infty} x \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}_{\phi(x)} dx$$

$= 0$

$\phi(x)$ is an even fn.
$x\phi(x)$ " " odd fn

$$E(|X|) = 2\int_0^\infty |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-\frac{1}{2}x^2} dx$$

$$= \sqrt{2/\pi} \left(-e^{-\frac{1}{2}x^2} \Big|_0^\infty \right)$$

$< \infty$

$X \sim$ Cauchy

$$f(x) = \frac{1}{\pi(1+x^2)} \mathbb{1}(x \in \mathbb{R})$$

$$E[|X|] = 2 \int_0^\infty \frac{1}{\pi} \cdot \frac{x}{1+x^2} \, dx = \infty$$

$$\sim \frac{1}{x}.$$

$= \infty$

Check ~~that~~ this is true!

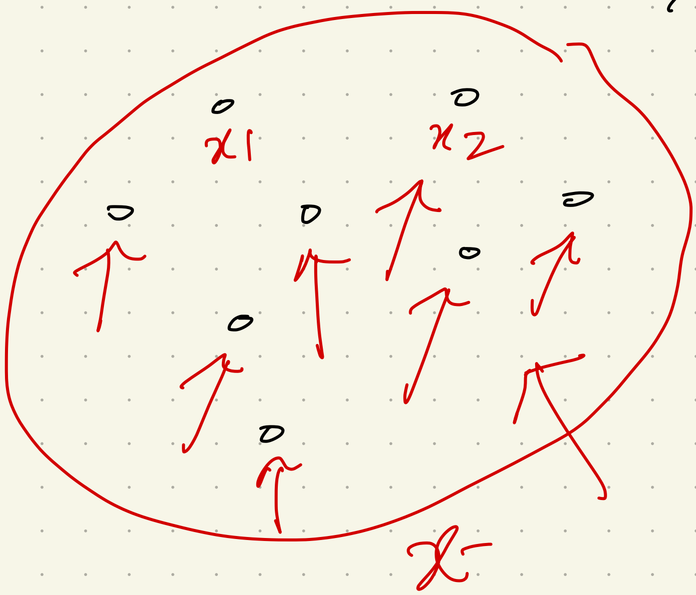
Mixed random variable:

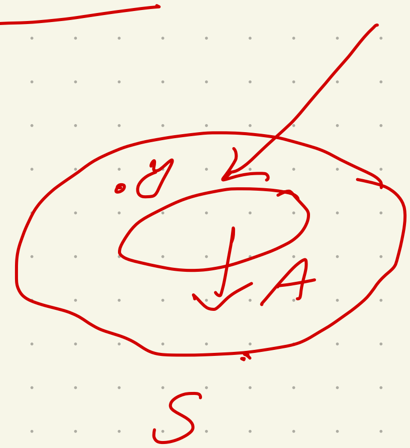
X is called 'mixed' if there is a set $\mathcal{X} = \{x_1, x_2, \ldots\}$, a Borel set $S \subseteq \mathbb{R}$, a function $p : \mathcal{X} \to [0,1]$ and a function $f : S \to [0, \infty)$ such that:

$$P(X = x_j) = p(x_j)$$

and for any measurable $A \subseteq S$,

$$P(X \in A) = \int_A f(x)\,dx.$$





$$P(X = y) = 0$$

For such a 'mixed' X,

$$E[g(X)] = \sum_{x_j} g(x_j)\,p(x_j) + \int_S g(x)\,f(x)\,dx$$

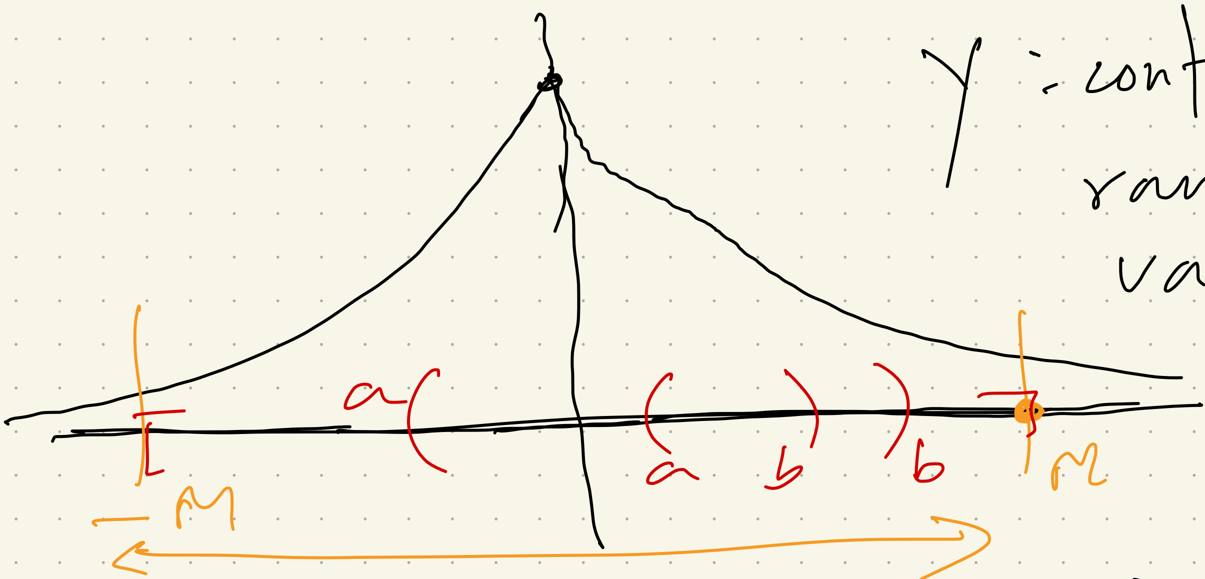
again if $E[|g(X)|] < \infty$.

Example 3.1:

Consider Y such that:

$$f_Y(y) = \frac{1}{2}\lambda e^{-\lambda|y|}, \; y \in \mathbb{R}.$$

double exponential density



Y: continuous random variable.

$$\tilde{X} = M \, \mathbb{1}(Y \geq M) + (-M)\, \mathbb{1}(Y \leq -M)$$

$$+ \; Y \cdot \mathbb{1}(-M < Y < M).$$

Censored version of Y

$P(\cancel{\tilde{X}} \in (a,b))$ where $-M < a < b < M$

$$P(\tilde{X} \in (a,b)) = P(Y \in (a,b)) = \int_{(a,b)} f_Y(y)\, dy$$

For any $S \subseteq (-M, M)$

$$P(Y \in S) = P(\tilde{X} \in S) = \int_S f_Y(y)\,dy$$

$$P(\tilde{X} = M) = P(Y \geq M) = \frac{1}{2} e^{-\lambda M}$$

$$P(\tilde{X} = -M) = P(Y \leq -M) = \frac{1}{2} e^{-\lambda M}$$

$\tilde{X}$: mixed

$$E\tilde{X} = \cancel{M \cdot \frac{1}{2} e^{-\lambda M} + (-M) \frac{1}{2} e^{-\lambda M}}$$

$$+ \underbrace{\int_{(-M,M)} x \frac{1}{2} e^{-\lambda|x|}\,dx}_{=0} \quad \text{odd function}$$

$$= 0$$

Moment of a random variable:

For any integer $k \geq 0$,

k'th moment of X is defined as

$E(X^k)$ (provided $E(|X|^k) < \infty$)

Change of Variable Result for Expectations.

$$X \longmapsto Y$$

via $Y = g(X)$.

suppose X is discrete.

$$E[g(X)] = \sum g(x_j)p(x_j)$$

Notice ~~that~~ Y is also discrete

Y assumes $\{\underline{y_1}, \underline{y_2}, \ldots, \}$

$= g(\mathcal{X})$.

$\mathcal{X}$: totality of values assumed by X.

Y has its own p.m.f.

$\tilde{p}(y_j) = P(Y = y_j)$

$$EY = \sum y_j \tilde{p}(y_j)$$

For consistency, we ask ~~that~~

$$\sum y_j \tilde{p}(y_j) \stackrel{?}{=} \sum g(x_j)\, p(x_j)$$

$$\boxed{\sum_j g(\underline{x_j})\, p(x_j)}$$

$\mathcal{X}$

$\mathcal{X}_1 = \{x_j : g(x_j) = y_1\}$

$\mathcal{X}_2 = \{x_j : g(x_j) = y_2\}$

$\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 \cdots = \mathcal{X}$

$$= \sum_{\ell} \sum_{j \in \mathcal{X}_\ell} \underline{g(x_j)}\, p(x_j)$$

$$= \sum_{\ell} \sum_{j \in \mathcal{X}_\ell} y_\ell\, p(x_j)$$

$$= \sum_{\ell} y_\ell \left(\sum_{j \in \mathcal{X}_\ell} p(x_j) \right)$$

$$= \sum y_\ell P(Y = y_\ell)$$

$$p(Y = y_\ell) = \sum_{x_j : g(x_j) = y_\ell} \underline{p(x_j)}$$