Stat 510 : Homework 4.

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1 (i) Let (U, V) be distributed jointly with a spherically symmetric density. In other words, let their joint density $f(u, v) = Cg(u^2 + v^2)$, for some non-negative function g. Show that $(\epsilon_1 U, \epsilon_2 V)$ has the same distribution as (U, V), where ϵ_1 and ϵ_2 are either 1 or -1. Deduce that U and V are uncorrelated.

(ii) Now consider a pair of random variables (W_1, W_2) that are independent of (U, V) in (a), where each W_i assumes the value 1 or -1. Let $Z_1 = W_1 U$ and $Z_2 = W_2 V$. Show that (Z_1, Z_2) again has the same distribution as (U, V).

2 . (i) Let Y have a density which is symmetric about 0 and let $X = SY$ where S is independent of Y and assumes values 1 and -1 with probability 1/2. Show that $\mathrm{Cov}(X, Y) = 0$ but that X and Y are *not* independent. (This shows that uncorrelatedness does not necesarily imply independence.)

(ii) A random rectangle is generated in the following way: The base X is chosen to be a U(0,1) random variable and after having generated the base, the height of the rectangle is chosen to be uniform on $(0, X)$. Find the expected circumference and area of the rectangle.

(iii)Two streams cross at a point at right angles. (You can think of the X stream running along the X axis and the Y stream running along the Y axis, intersecting at the origin.) At the intersection, the flow of each stream is regulated via its own lock, say the X lock (for X stream) and the Y lock (for Y stream). At the stroke of midnight, one of the locks is randomly opened and left that way for an hour. The chance that the X-lock is obtained is p_x and the chance of the Y-lock being opened $p_y = 1 - p_x$. Let W be the number of the fish that pass through the intersection in that hour. If the X-lock is opened, this is a Poisson(λ_x) random variable; otherwise a Poisson(λ_y). Find EW and $\mathrm{Var}(W)$.

- 3. Let $(N_1, N_2, \ldots, N_m)$ be distributed as Multinomial$(n, p_1, p_2, \ldots, p_m)$. Calculate: (i) the covariance between N_i and N_j, (iv) the conditional covariance between N_i and N_j for $i, j \le r$ given $(N_{r+1}, \ldots, N_m)$, (iii) an appropriate approximation to the probability that N_1 is larger than 65 when $N = 100$ and $p_1 = 0.25$. (30 points)
  Hint: It might be useful to think of the multinomial vector as $\sum_{i=1}^{n} V_i$ where $V_i = (V_{i,1}, \ldots, V_{i,m})$'s are i.i.d Multinomial $(1, p_1, p_2, \ldots, p_m)$.

- 4. Let (X, Y) be jointly distributed inside the ellipse given by $\{(x, y) : \; a\,x^2 + b\,y^2 = 1\}$. Find the marginal densities of X and Y and the conditional densities of Y given X and X given Y. Are X and Y correlated? Are they independent?

- 5. Before we start this problem, some preliminary facts from calculus. For $\alpha, \beta > 0$,

$$\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} := B(\alpha, \beta)\,,$$

where B is called the beta function. Here Γ is the usual gamma function from calculus.

Suppose Θ follows the Uniform$(0, 1)$ distribution and $(X_1, X_2, \ldots, X_n)|\Theta = \theta$ are i.i.d. Bernoulli(θ). Let $S_n = \sum_{i=1}^n X_i$.

(i) Write down $f(m, \theta)$ the 'mixed' joint density of (S_n, Θ) for (m, θ) in an appropriate set that you should identify. Calculate $p_n(k) := P(S_n = k)$, the pm.f of S_n and find the conditional density of Θ given $S_n = k$, as well as $E(\Theta|S_n)$. Argue that the form of the conditional density makes sense intuitively.
(ii) Write down the 'mixed' joint density of $(X_1, X_2, \ldots, X_n, \Theta)$, say $f(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \theta)$ and use this to calculate the joint pm.f. of $(X_1, X_2, \ldots, X_n)$, i.e. $P(X_1 = \epsilon_1, X_2 = \epsilon_2, \ldots, X_n = \epsilon)$ where the vector $(\epsilon_1, \ldots, \epsilon)$ is a generic point in $\{0, 1\}^n$.
(iii) Argue that the distribution of $(X_1, X_2, \ldots, X_n)$ is *exchangeable*, i.e. $(X_1, X_2, \ldots, X_n)$ has the same distribution as $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ where π is any permutation of the integers 1 through n.
(iv) Compute the conditional distribution of $(X_1, X_2, \ldots, X_n)$ given S_n and the conditional distribution of X_1 given S_n. What is $E(X_1 \mid S_n)$?