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1. $P(X=x) = -\dfrac{(1-p)^x}{x\log p}$ ; $x = 1,2,3\ldots\ldots$

(a). $\displaystyle\sum_{x=1}^{\infty} P(X=x) = \frac{1}{\log p}\sum_{x=1}^{\infty} -\frac{(1-p)^x}{x}$

$\displaystyle = \frac{1}{\log p}\sum_{x=1}^{\infty}\frac{(-1)^{x+1}}{x}(p-1)^x$

Since we know that. $\displaystyle\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ for $|x|\le 1$;

Hence $\displaystyle\sum_{x=1}^{\infty} P(X=x) = \frac{1}{\log p}\times\log(1+p-1)) = \frac{1}{\log p}\times\log p = 1$; Hence Q.E.D.

(b) The most likely no of sights is obtained by $P(X=1)$

$P(x=1) = \dfrac{p-1}{\log p}$.

(c).. We first calculate $E(x^{(k)})$ to get $E(x)$ and $Var(x)$

$\displaystyle E[x^{(k)}] = \sum_{x=1}^{\infty} x^k\frac{p^x}{-\log(1-p)x} = \frac{p^k}{-\log(1-p)}\sum_{x=k}^{\infty}\frac{x^k p^{(x-k)}}{x}$

$= \dfrac{p^k}{-\log(1-p)}(k-1)!\,(1-p)^{-k}$.

Hence $E(x) = -\dfrac{1}{\log(1-p)}\cdot\dfrac{p}{1-p} = -\dfrac{p}{\log(1-p)(1-p)}$ or equivalently. $-\dfrac{1-p}{p\log p}$.

$Var(x) = E(x^2) - E(x)^2 = \dfrac{1}{-\log p}\cdot\dfrac{(1-p)}{p^2}\left[1 - \dfrac{1-p}{-\ln p}\right]$

2. $M(t_1,\ldots t_n) = E[\exp(t_1x_1 + \ldots\ldots + t_nx_n)]$.

(a). $M(t_1,\ldots t_p) = E[\exp(t_1x_1 + t_2x_2 + \ldots\ldots t_px_p)]$.

(b). If $x_i$ are i.i.d. $N(0,1)$

Then. $\displaystyle m_{x_i}(t) = \int_{-\infty}^{\infty} e^{tx_i}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_i^2}dx_i = e^{\frac{1}{2}t^2}$.

Hence. $M_x(t_1,\ldots t_p) = e^{\frac{1}{2}\sum_{i=1}^{p}t_i^2}$

(c) $Y = \mu + BX$ $X$ is i.i.d $N(0,1)$

Since $X$ is still i.i.d $N_n(0,1)$: $Y$ is just a linear transformation of $X$.

$M_x(t_1,\ldots t_p)$ doesn't depend on any other parameters except for $(t_1,\ldots t_p)$.

Hence. the m.g.f of $Y$ exists; Since the m.g.f of $X$ exists.

$\displaystyle m_{y_i}t = \int_{-\infty}^{\infty} e^{ty_i}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\mu+Bx_i)^2}dx_i \Rightarrow M_Y(t) = \exp\{\mu't + \tfrac{1}{2}t'\Sigma t\}$

$\Rightarrow M_Y(t)$ depends only on $(\mu,\Sigma)$

(d) Since $M_Y(t) = \exp(\mu't + \frac{1}{2}t'\Sigma t)$

If $(\mu_1,\Sigma_1) \neq (\mu_2,\Sigma_2)$.

Then: $M_{\mu_1,\Sigma_1} - M_{\mu_2\Sigma_2} = e^{\mu_1't + \frac{1}{2}t'\Sigma_1 t} - e^{\mu_2't + \frac{1}{2}t'\Sigma_2 t} = 0$

iff $\mu_1't + \frac{1}{2}t'\Sigma_1 t = \mu_2't + \frac{1}{2}t'\Sigma_2 t \Rightarrow \mu_1' = \mu_2'$ ; $\Sigma_1 = \Sigma_2$; which contradicts our assumption

Hence, Q.E.D.

(e). We know that for $x = \{x_1 \cdots x_n\}$

$$f_x(x) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{1}{2}x'x\right)$$

The density function of $Y$ is.

$$f_Y(y) = f_x(x(y))\,|J|$$

Where. $J$ is the Jacobian.

$$J = \left|\frac{\partial x_i}{\partial y_j}\right| = |B|^{-1}; \quad \text{As } X = B^{-1}(Y-\mu).$$

There-fore $f_Y(y) = f_x(B^{-1}(y-\mu))|B|^{-1}$

$$= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[B^{-1}(y-\mu)]'[B^{-1}(y-\mu)]\right\}$$

$$= (2\pi)^{-n/2} \cdot |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right\}.$$

Exercise 4.33.

$H|N = x_1 + \cdots + x_n$; $P(x_i = t) = \dfrac{-1}{\log(p)} \dfrac{(1-p)^t}{t}$

As already down in Exercise 1.

$$Ee^{x_i t} = \frac{-1}{\log p}\sum_{x_i=1}^{\infty} \frac{(e^t(1-p))^{x_i}}{x_i} = \frac{-1}{\log p}(-\log\{1-e^t(1-p)\}) = \frac{\log\{1-e^t(1-p)\}}{\log p} = K$$

Hence.

$$E(K)^N = \sum_{n=0}^{\infty}\left(\frac{\log\{1-e^t(1-p)\}}{\log p}\right)^n \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda} \times e^{\frac{\lambda\log(1-e^t(1-p))}{\log p}} \underbrace{\left|\sum_{n=0}^{\infty}\left(\frac{e^{-\frac{\lambda\log(1-e^t(1-p))}{\log p}}\left(\frac{\lambda\log(1-e^t(1-p))}{\log p}\right)^n}{n!}\right)\right|}_{=1}$$

$= 1$; Since it is a sum of Poisson $(J)$, where $J = \dfrac{\lambda\log(1-e^t(1-p))}{\log p}$.

$$= e^{-\lambda} e^{\frac{\lambda\log(1-e^t(1-p))}{\log p}} = E(e^{Ht}).$$

Hence. $E(e^{Ht}) = e^{-\lambda} e^{\frac{\lambda\log(1-e^t(1-p))}{\log p}} = \left(\dfrac{p}{1-e^t(1-p)}\right)^{-\lambda/\log p}$

$E(e^{Ht})$ is the m.g.f of a negative binomial$(r, p)$; with $r = -\lambda/\log p$

Hence, Q.E.D.

Exercise 4.34.

(a) $P(X=x) = \int_0^1 P(X=x|p) f_P(p)\,dp.$

$$= \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{1}{B(a,\beta)} p^{a-1}(1-p)^{\beta-1}\,dp.$$

$$= \binom{n}{x}\frac{\Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta)} \int_0^1 p^{x+a-1}(1-p)^{n-x+\beta-1}\,dp$$

$$= \binom{n}{x}\frac{\Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta)} \frac{\Gamma(x+a)\Gamma(n+\beta-x)}{\Gamma(a+\beta+n)}$$ ; Hence; Q.E.D.

(b). $X|p \sim$ negative binomial $(r, p)$; $P \sim$ beta$(a, \beta)$.

$P(X=x) = \int_0^1 P(X=x|p) f_P(p)\,dp$

$$= \int_0^1 \binom{r+x-1}{x} p^r (1-p)^x \frac{\Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta)} p^{a-1}(1-p)^{\beta-1}\,dp$$

$$= \binom{r+x-1}{x}\frac{\Gamma(a+\beta)\,\Gamma(r+a)\,\Gamma(x+\beta)}{\Gamma(a)\Gamma(\beta)\,\Gamma(x+a+\beta+r)}$$

$EX = E(E(X|p)) = E\left(\frac{r(1-p)}{p}\right)$

$E\left[\frac{1-p}{p}\right] = \int_0^1 \left(\frac{1-p}{p}\right) f_P(p)\,dp = \frac{\beta}{a-1}$

Hence. $EX = \dfrac{r\beta}{a-1}$

$Var(X) = E(Var(X|p)) + Var(E(X|p))$

$\quad = E\left(\frac{r(1-p)}{p^2}\right) + Var\left(\frac{r(1-p)}{p}\right)$

$Var\left(\frac{1-p}{p}\right) = E\left[\left(\frac{1-p}{p}\right)^2\right] - \left[E\left(\frac{1-p}{p}\right)\right]^2$

$\quad = \dfrac{(a+\beta-1)\beta}{(a-1)^2(a-2)}$

$E\left(\frac{r(1-p)}{p^2}\right) = \dfrac{(\beta+1)(a+\beta)}{a(a-1)}$

Hence. $Var(X) = \dfrac{r(\beta+1)(a+\beta)}{a(a-1)} + \dfrac{r^2(a+\beta-1)\beta}{(a-1)^2(a-2)}$

$x_i \mid p_i \sim \text{Bernoulli}(p_i)$; $i = 1, \ldots, n$.

$p_i \sim \text{beta}(\alpha, \beta)$

(a) $E(Y) = \sum_{i=1}^{n} E(x_i)$

$E(x_i) = E(E(x_i \mid p_i)) = E\left(\sum_{x=0}^{1} x\, p(x_i = x \mid p_i = p_i)\right)$

$= E\left(\sum_{x=0}^{1} x p_i^{x}(1-p_i)^{1-x}\right) = E(p_i)$

Hence $E(x_i) = E(p_i) = \int_0^1 p_i f_p(p_i)\,dp_i = \frac{1}{B(\alpha,\beta)} \int_0^1 p_i^{\alpha}(1-p_i)^{\beta-1}\,dp_i$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$

$= \frac{\alpha}{\alpha+\beta}$.

Therefore. $E(Y) = \frac{n\alpha}{\alpha+\beta}$ ; Q.E.D.

(b). $Var(Y) = \sum_{i=1}^{n} Var(X_i)$.

$Var(X_i) = E[Var(x_i \mid p_i)] + Var(\underbrace{E(x_i \mid p_i)}_{p_i})$.

Hence. ① $Var(E(x_i \mid p_i)) = Var(p_i) = E(p_i^2) - E(p_i)^2 = \int_0^1 p_i^2 f_p(p_i)\,dp_i - \left(\frac{\alpha}{\alpha+\beta}\right)^2$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$

$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$

$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

② $E[Var(x_i \mid p_i)] = E\left(E(x_i^2/p_i) - [E(x_i - p_i)]^2\right)$

$= E(p_i(1-p_i)) = E(p_i) - E(p_i^2)$

$= \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$

Hence $Var(x_i) = \frac{\alpha\beta}{(\alpha+\beta)^2}$.

$Var(Y) = \frac{n\alpha\beta}{(\alpha+\beta)^2}$ ; The random variable $Y$ is binomial$\left(n, \frac{\alpha}{\alpha+\beta}\right)$

(c). $x_i \mid p_i \sim \text{binomial}(n_i, p_i)$, $p_i \sim \text{beta}(\alpha, \beta)$

① $E(x_i) = E(E(x_i \mid p_i))$;

$E(x_i \mid p_i) = \sum_{x=0}^{n_i} x \binom{n_i}{x} p_i^{x}(1-p_i)^{n_i - x} = \boxed{\sum_{x=0}^{\infty} x \binom{n_i - 1}{x-1} p_i^{x-1}(1-p_i)^{n_i - x}}^{=1} \times n_i p_i$

$= n_i p_i$

Hence $E(n_i p_i) = n_i E(p_i) = \frac{n_i \alpha}{\alpha+\beta}$.

Hence $E(Y) = \sum_{i=1}^{k} \frac{n_i \alpha}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} n_i$

$\Rightarrow Var(x_i) = \frac{n_i \alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta+1)(\alpha+\beta)^2}$

$Var(Y) = \sum_{i=1}^{k} \frac{n_i \alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta+1)(\alpha+\beta)^2}$

②. For $Var(Y)$:

$Var(E(x_i \mid p_i)) = Var(n_i p_i) = n_i^2 Var(p_i) = n_i^2(E(p_i^2) - [E(p_i)]^2) = \frac{n_i^2 \alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$

$E(Var(x_i \mid p_i)) = E(n_i p_i (1-p_i)) = n_i [E(p_i) - E(p_i)^2] = \frac{n_i \alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)}$

4.40.

(a) $\int_x \int_y f(x,y)\,dy\,dx = 1.$

$\int_x \int_y f(x,y)\,dy\,dx$

$= \int_0^1 \int_0^{1-x} C x^{a-1} y^{b-1} (1-x-y)^{c-1} dy\,dx$ let $\lambda = \frac{y}{x-1}$ ; $d\lambda = \frac{dy}{x-1}$

$= \int_0^1 \int_0^1 C x^{a-1} [(x-1)\lambda]^{b-1} (1-x)^{c-1} (1-\lambda)^{c-1} (x-1)\,d\lambda\,dx$

$= C \left[\int_0^1 x^{a-1} (1-x)^{b+c-1} dx\right] \left[\int_0^1 \lambda^{b-1} (1-\lambda)^{c-1} d\lambda\right]$

$= C \cdot \frac{\Gamma(a)\cancel{\Gamma(b+c)}}{\Gamma(a+b+c)} \cdot \frac{\Gamma(b)\Gamma(c)}{\cancel{\Gamma(b+c)}}$

Thus ; $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

(b). For $X$.

$f_X(x) = \int_0^{1-x} f(x,y)\,dy = \int_0^1 C x^{a-1} (1-x)^{c-1} [(1-x)t]^{b-1} (1-t)^{c-1} (1-x)\,dt$

$= C x^{a-1} (1-x)^{b+c-1} \int_0^1 \lambda^{b-1} (1-\lambda)^{c-1} dt.$

$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1}$ ; Hence $X \sim \text{beta}(a, b+c)$

Similarly; for $Y$

$f_Y(y) = \int_0^{1-y} f(x,y)\,dy = \frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(a+c)} y^{b-1} (1-y)^{a+c-1}$, Hence $Y \sim \text{beta}(b, a+c)$

(c) $f(y|x) = \frac{f(x,y)}{f_X(x)} = \dfrac{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1}}$

$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \frac{1}{(1-x)} \left(\frac{y}{1-x}\right)^{b-1} \left(1 - \frac{y}{1-x}\right)^{c-1}$

Hence. $Y|X = x \sim \text{beta}(b,c)$

$f_X(x) = \frac{1}{B(a, b+c)} [1-(1-x)]^{a-1} [1-x)^{b+c-1} = f_{1-x}(1-x).$

Hence. $1-x \sim \text{beta}(b+c, a)$ ; we know that $Y \sim \text{beta}(b, a+c)$

Hence. $\frac{Y}{1-x} \sim \text{beta}(b,c).$

(d) $E(XY) = \int_0^1 \int_0^{1-x} xy f(x,y)\,dy\,dx$

$= C \int_0^1 \int_0^{1-x} x^a y^b (1-x)^{c-1} \left(1 - \frac{y}{1-x}\right)^{c-1} dy\,dx$

$= C \left[\int_0^1 x^a (1-x)^{b+c} dx\right] \left[\int_0^1 \lambda^b (1-\lambda)^{c-1} d\lambda\right]$

$= C \cdot B(a+1, b+c+1) . B(b+1, c)$

$= \frac{ab}{(a+b+c+1)(a+b+c)}$

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \int_0^1 x f_X(x)\,dx = \frac{a}{a+b+c}$. Similarly. $E(Y) = \frac{b}{a+b+c}.$

Hence. $\text{cov}(XY) = \frac{-ab}{(a+b+c+1)(a+b+c)^2}$

Exercise 4.47.

(a) $P(Z \le z) = P(X \le z)P(Y<0) + P(X \ge -z)P(Y<0)$

Since $Y \sim N(0,1)$; Hence $E(Y)=0$.

$P(Y<0) = P(Y>0) = \frac{1}{2}$

Hence $P(Z \le z) = P(X \le z)$ ①

Similarly; $P(Z > z) = P(X > z)$; ②

from ①, ②, we can conclude that $P(Z \le z) = P(X \le z)$; i.e. $X$ and $Z$ are identical

Thus, $Z \sim n(0,1)$

(b) $P(Z>0) = P(Z>0, XY>0) + P(Z>0, XY<0)$

$= P(X>0, XY>0) + P(X>0, XY<0)$

$= P(X>0, Y>0) + P(X<0, Y>0)$ $\Rightarrow$ either $X>0$ and $Y>0$ or $X<0$, $Y>0$

$P(Z<0) = P(X<0, Y<0) + P(X>0, Y<0)$. $\Rightarrow$ either $X<0$, $Y<0$ or $X>0$, $Y<0$.

Hence, in both cases, $Z$ and $Y$ have the same sign. their joint distribution is not bivariate normal.