Section 4 of Notes:

Distributional Convergence:

Defn 4.1: $\{X_n\}$ sequence of r.v's.

We say that $X_n \xrightarrow{d} X$ if

for every x such that $F_X(x-) = F_X(x)$

we have $\lim_{n\to\infty} F_n(x) = F_X(x)$,

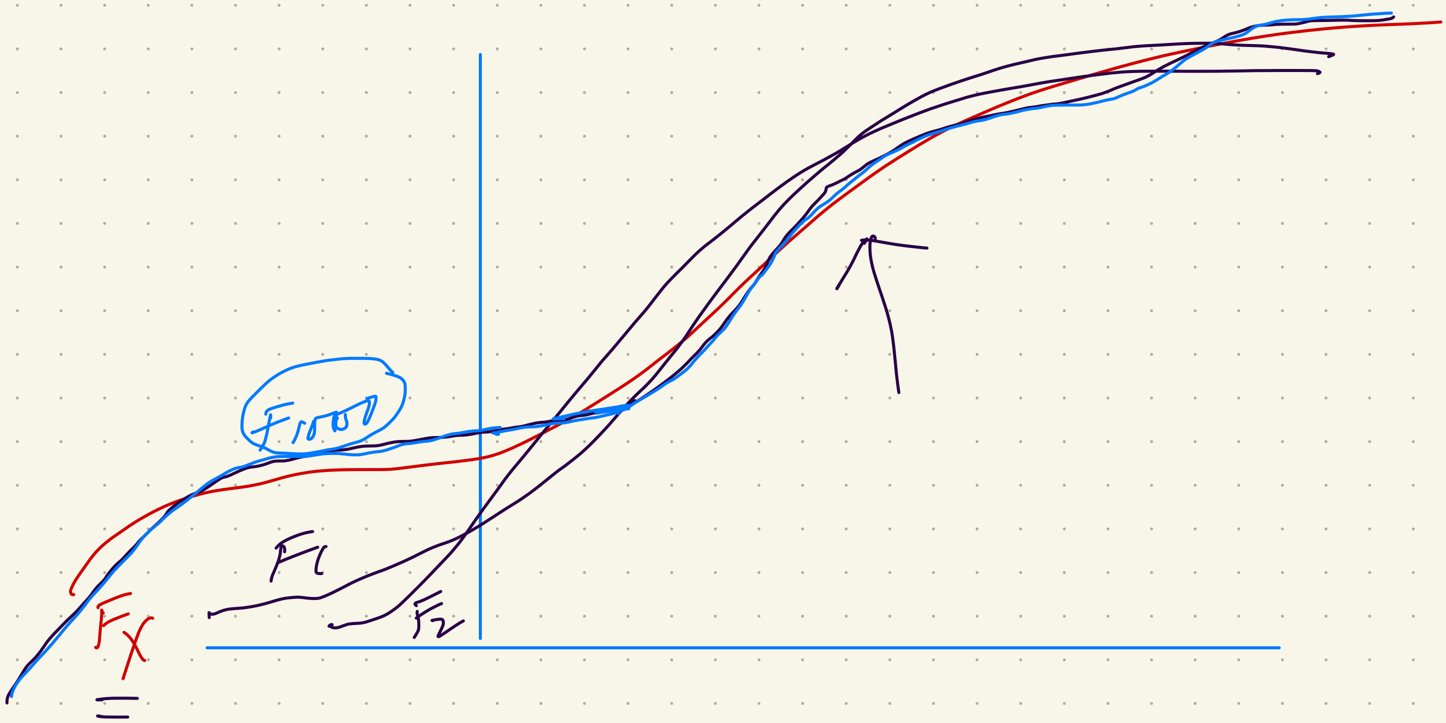
where F_n is the dist. func. of X_n and

F_X that of F

$$\underline{P(X_n \le x)} \to P(X \le x)$$

$$\left.\begin{matrix} F_X(x-) = F_X(x) \\ \shortparallel \qquad\quad \shortparallel \\ P(X<x) \quad P(X \le x) \end{matrix}\right\} \iff \underline{P(X=x)=0}$$

If F_X is continuous, then of course reduces to $F_n(x) \xrightarrow[n\to\infty]{} F(x)$ for ALL x.



Essential to relax the requirement for discontinuity points of F_X

Example: Let $x_n \longrightarrow x$.

Given any $\varepsilon > 0$, $x \in (x_n - \varepsilon, x_n + \varepsilon)$ for all $n > N_\varepsilon$.

$\Updownarrow$

$x_n \in (x - \varepsilon, x + \varepsilon)$

X_n: r. v. degenerate at x_n

$P(X_n = x_n) = 1$

X: r. v degenerate at x. $P(X = x) = 1$

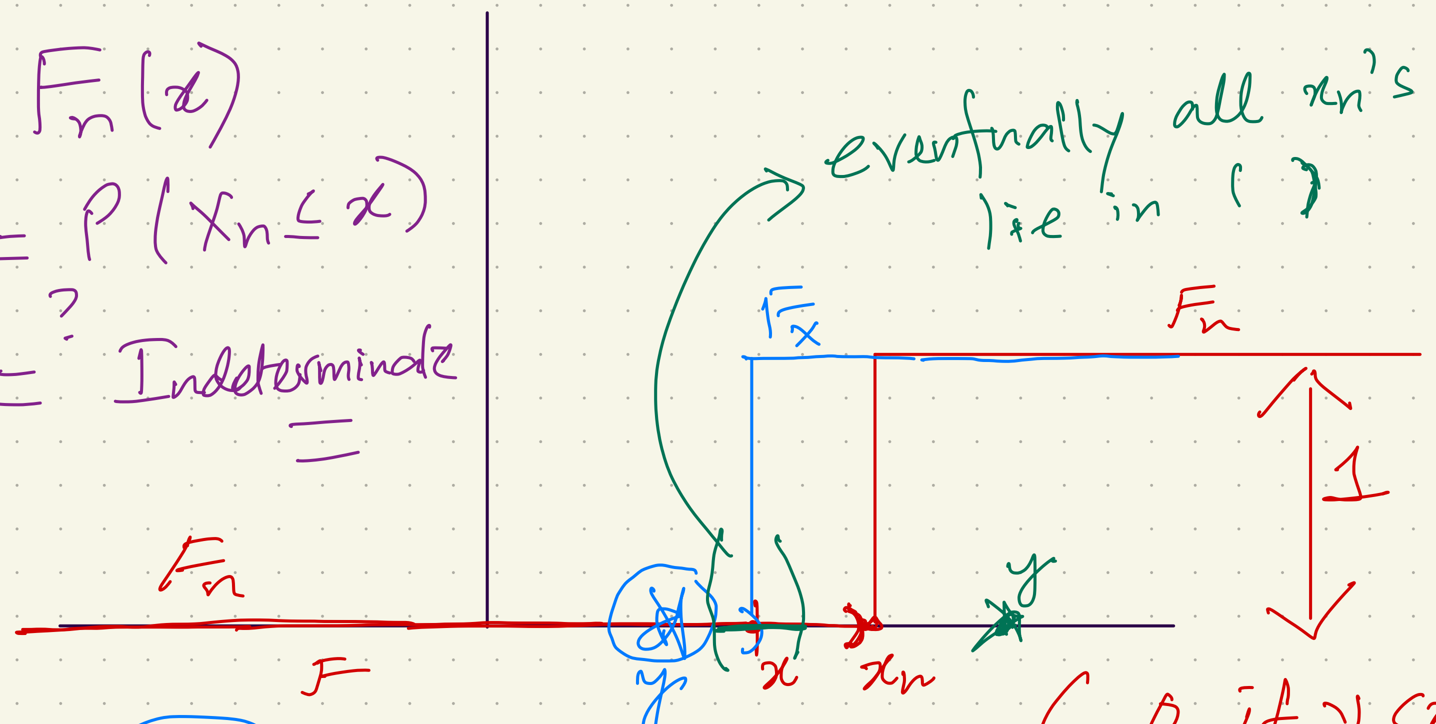
How does F_n look like?

$F_n(x)$

$= P(X_n \le x)$

$\overset{?}{=}$ Indeterminate

$=$



$$F_n(y) = P(X_n \le y) = \begin{cases} 0 & \text{if } y < x_n \\ 1 & \text{if } y \ge x_n \end{cases}$$

If you take any $y < x$,

$F_X(x) = 0$

What is $F_n(y)$ for all sufficiently large n? $= 0$

Similarly if $y > x$, $F_n(y) = 1$ for all sufficiently large n.

Conclude $F_n(y) \longrightarrow F(y)$ for all $y \neq x$.

Suppose $\underline{x_n} = x - \frac{1}{2m}$ if $\underline{n = 2m}$

$= x + \frac{1}{2m+1}$ if $n = 2m+1$

$x_n \to x$

$\underline{F_n}(x) = 1$, when n is even

$F_n(x) = 0$ when n is odd

$F_X(x) = 1$.

<u>convergence</u> fails.

Distributional convergence would fail even in the simplest case if $\exists$ required convergence of F_n at x.

$\longleftrightarrow$

Convergence in distribution of the discrete uniform on $[0,1]$ to the continuous uniform on $[0,1]$:

X_n is a r.v taking values $\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1\}$ each w.p. $\frac{1}{n}$.

$P\left(X_n = \frac{m}{n}\right) = \frac{1}{n}$ for $1 \le m \le n$

$$\boxed{\begin{array}{l}\text{Generate } u \sim \text{Uniform}(0,1). \\ \text{Set } X_n = \frac{m}{n} \text{ if } \left(\frac{m-1}{n} < u \le \frac{m}{n}\right) \\ \qquad m = 1, 2, \dots, n.\end{array}}$$

$$\underline{\underline{F_{X_n}(x)}} = P(X_n \le x)$$

$$= \frac{\lfloor nx \rfloor}{n}.$$

$\frac{1}{n} \times$ no. of grid points of spacing $\frac{1}{n}$ that lie to the left of x.

Let $X \sim \text{Uniform}(0,1)$

$$F_X(x) = x, \quad 0 < x < 1$$

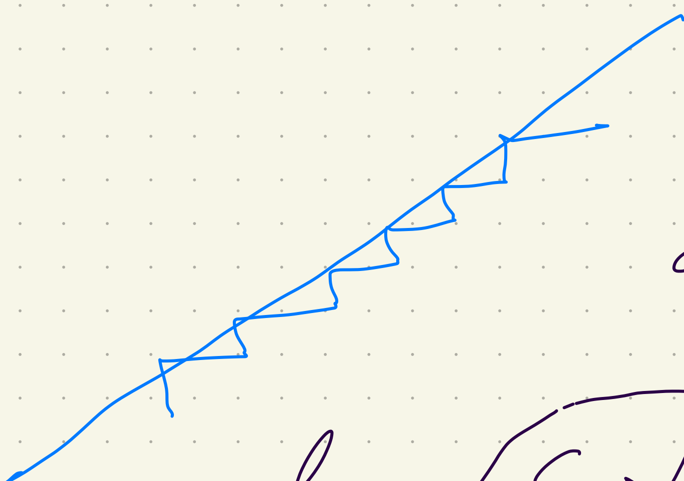
$$|F_{X_n}(x) - F_X(x)|$$

$$= \left| \frac{\lfloor nx \rfloor}{n} - x \right| = \left| \frac{\lfloor nx \rfloor - nx}{n} \right|$$

$$\le \frac{1}{n} \to 0.$$

so $X_n \xrightarrow{d} X$.

Draw pictures representing this convergence



How do we check distributional convergence?

$$\{\text{some normal distribution}\} \xleftarrow{d} \{X_n\} \to \text{sequence of estimators of a parameter of interest}$$

$\boxed{\text{Theorem 4.1:}}$ $\{X_n\}$ a sequence of non-negative random variables and suppose ~~that~~ $X_n \le X_{n+1}$ $\forall n$ and $\underline{X_n} \longrightarrow \underline{X}$ (on a set of probability $\underline{1}$)

$(X_n(\omega) \uparrow x(\omega)$ for $\omega \in \Omega_0$, $P(\Omega_0) = 1)$

Then $EX_n \uparrow EX$. } (MCT)

$\boxed{\text{Theorem 4.2}}$ $\underline{\text{DCT}}$

Let X_n be a sequence of random variables such ~~that~~ $\underline{X_n \longrightarrow X}$ on a set of probability 1. Suppose $|X_n| \le \underline{Z}$ with probability 1 and assume ~~that~~ $\underline{EZ < \infty}$. Then, $|X| \le Z$ with prob. 1.

Then $\underline{EX_n \longrightarrow EX}$ (and all expectations are finite)

4.2. Moment-generating functions.

X : random variable.

Define $M_X : \mathbb{R} \longrightarrow [0, \infty)$

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$ (Laplace transform)

$M_X(0) = 1$.

We'll say ~~that~~ X has a (finite) mgf if $M_X(t) < \infty$ for all $t \in (-h_0, h_0)$ for some $h_0 > 0$

For M_X to exist finitely in a nbhd of 0, it is essential that ~~the~~ behavior of X in the tails i.e $\{|X| \geq t\}$, is like that of an exponential random variable

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \mathbb{1}(x \in \mathbb{R})$$

X has no mgf. $\mathbb{E}\left[e^{tX}\right] = \infty \quad \forall t \neq 0$.

$E[e^{t|X|}] < \infty$ for $t \in (-h_0, h_0)$

$\Longleftrightarrow$ whenever $E[e^{tX}] < \infty$ for $t \in (-h_0, h_0)$

If $\underline{t \leq 0}$, this is immediate since

$e^{t|X|} \leq 1$.

$t > 0$.

$$e^{t|X|} = e^{tX} \mathbb{1}(X \geq 0) + e^{-tX} \mathbb{1}(X < 0)$$

$$E[e^{t|X|}] = E[e^{tX} \mathbb{1}(X \geq 0)] + E[e^{-tX} \mathbb{1}(X < 0)]$$

$$\leq \mathbb{E}[e^{tX}] + \mathbb{E}[e^{-tX}]$$

$$< \infty.$$

since $t \in (-h_0, h_0)$.

Theorem 4.3 If X has finite mgf $M_X(t)$ on $(-h_0, h_0)$, then

$$\mathbb{E}[e^{tX}] = \sum_{j=0}^{\infty} \frac{t^j \, \mathbb{E}(X^j)}{j!}.$$

$$e^{tx} = \sum_{j=0}^{\infty} \frac{t^j x^j}{j!}$$

$$\mathbb{E}[e^{tx}] = \mathbb{E}\left[\sum_{j=0}^{\infty} \frac{t^j x^j}{j!}\right] \stackrel{?}{=} \sum_{j=0}^{\infty} \frac{t^j \mathbb{E}(x^j)}{j!}$$

viable

in discrete case

$$\sum_{m=0}^{\infty} \underbrace{p_m}_{P(X=m)} \sum_{j=0}^{\infty} \frac{t^j m^j}{j!} \stackrel{\text{if able}}{=} \sum_{j=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{t^j m^j}{j!} p_m \right)$$

can I swap the sums?

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbb{E}(x^j)$$

I'm assuming X ~~a~~ takes values $0, 1, 2, \ldots,$ with probs $p_0, p_1, \ldots$

Proof: $S_n = \sum_{j=0}^{n} \frac{t^j X^j}{j!}$ $\Big\}$ $\circledast$

and $S_\infty = \sum_{j=0}^{\infty} \frac{t^j X^j}{j!}$

Then $S_n \longrightarrow S_\infty$ on a set of prob 1.

look at $|S_n|$

$$\le \sum_{j=0}^{\infty} \frac{|t|^j |X|^j}{j!} = e^{|t||X|}.$$

so $|S_\infty| \le e^{|t||X|}.$

Now $\mathbb{E}\left[e^{|t||X|}\right] < \infty$ $\forall\, t \in (-h_0, h_0)$

(we've shown this)

apply DCT directly,

$$E S_n \longrightarrow E\left(\sum_{j=0}^{\infty} \frac{t^j X^j}{j!}\right)$$

equal to

$$= \sum_{j=0}^{n} \frac{t^j E(X^j)}{j!} \xrightarrow[n\to\infty]{} \sum_{j=0}^{\infty} \frac{t^j E(X^j)}{j!} < \infty$$

$$\left| \sum_{j=0}^{\infty} \frac{t^j E(X^j)}{j!} \right|$$

$$\leq \sum_{j=0}^{\infty} \frac{|t|^j |E(X^j)|}{j!}$$

$$\leq \sum_{j=0}^{\infty} \frac{|t|^j E(|X|^j)}{j!} = E\left[e^{|t||X|}\right] < \infty$$

$$|E(X^j)| \leq E(|X^j|)$$

$$= E(|X|^j)$$

Basically we're using $E(|Y|) \geq |EY|$

(we assume $E(|Y|) < \infty$)

$\underline{E}(|Y|) < \infty$, $|Y| = Y^+ + Y^-$

$$E(|Y|) = \underbrace{EY^+}_{<\infty} + \underbrace{EY^-}_{<\infty}$$

$$EY = EY^+ - EY^-$$

Certainly, if $a, b \geq 0$, $\underline{a+b} \geq |a-b|$.

2'nd part of Theorem 4.3:

$$E(X^n) = M_X^{(n)}(0)$$

$$\equiv \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}$$

$$\frac{d^n}{dt^n} M_X(t) \Big|_{t=0}$$

$$= \frac{d^n}{dt^n} \left[\sum_{j=0}^{\infty} \frac{t^j E(X^j)}{j!} \right] \Bigg|_{t=0}$$

provided $\exists$ can swap summing and differentiation

$$= \sum_{j=0}^{\infty} \frac{d^n}{dt^n} \left(\frac{t^j E(X^j)}{j!} \right) \Bigg|_{t=0}$$

only the term with $j = n$ contributes and gives $E(X^n)$!

And in general:

$$\left[\frac{d^n}{dt^n} M_X(t) = \frac{d^n}{dt^n} E(e^{tX}) \overset{?}{=} E\left[\frac{d^n}{dt^n} e^{tX} \right] \right.$$

$$= E[x^n e^{tx}].$$

How about?

Consider $n = 1$.

Want to show:

$$\frac{d}{dt} E[e^{tX}] = E[Xe^{tX}].$$

Consider any sequence $h_n \to 0$.

$$\frac{d}{dt} E[e^{tX}]$$

$$= \lim_{n\to\infty} \frac{E[e^{(t+h_n)X}] - E[e^{tX}]}{h_n}$$

$$= \lim_{n\to\infty} E\Big[\underbrace{e^{tX} \frac{e^{h_n X} - 1}{h_n}}_{Y_n}\Big]$$

$$\lim_{n\to\infty} Y_n = Xe^{tX} \text{ (verify)}.$$

so $E(Y_n) \to E[Xe^{tX}]$

provided an appropriate dominating random variable can be obtained.

$$|Y_n| = \left| e^{tX} \frac{e^{h_n X} - 1}{h_n} \right|$$

$$\leq e^{|t||X|} \left| \frac{e^{h_n X} - 1}{h_n} \right|$$

$$= e^{|t||X|} \left| X + \frac{h_n X^2}{2!} + \frac{h_n^2 X^3}{3!} + \cdots \right|$$

$$\leq e^{|t||X|} \left(|X| + \frac{c_0 |X|^2}{2!} + \frac{c_0^2 |X|^3}{3!} + \cdots \right)$$

where $c_0 > 0$ is chosen so ~~that~~ $h_n < c_0$ for all sufficiently large n and such ~~that~~ $c_0 + |t| < \underline{h_0}$.

(remember $-h_0 < t < h_0$)

so $|Y_n| \le e^{|t||x|} \left| \frac{e^{c_0|x|} - 1}{c_0} \right|$

$$\le \frac{e^{(|t|+c_0)|x|} + e^{|t||x|}}{c_0}$$

$$\stackrel{\text{def}}{=} Z.$$

and $\underbrace{EZ < \infty}$, since $|t|, |t|+c_0 < h_0$.

By DCT:

$$E(Y_n) \xrightarrow[n\to\infty]{} E(Xe^{tX})$$

$$= \frac{d}{dt} E[e^{tX}]\ !$$