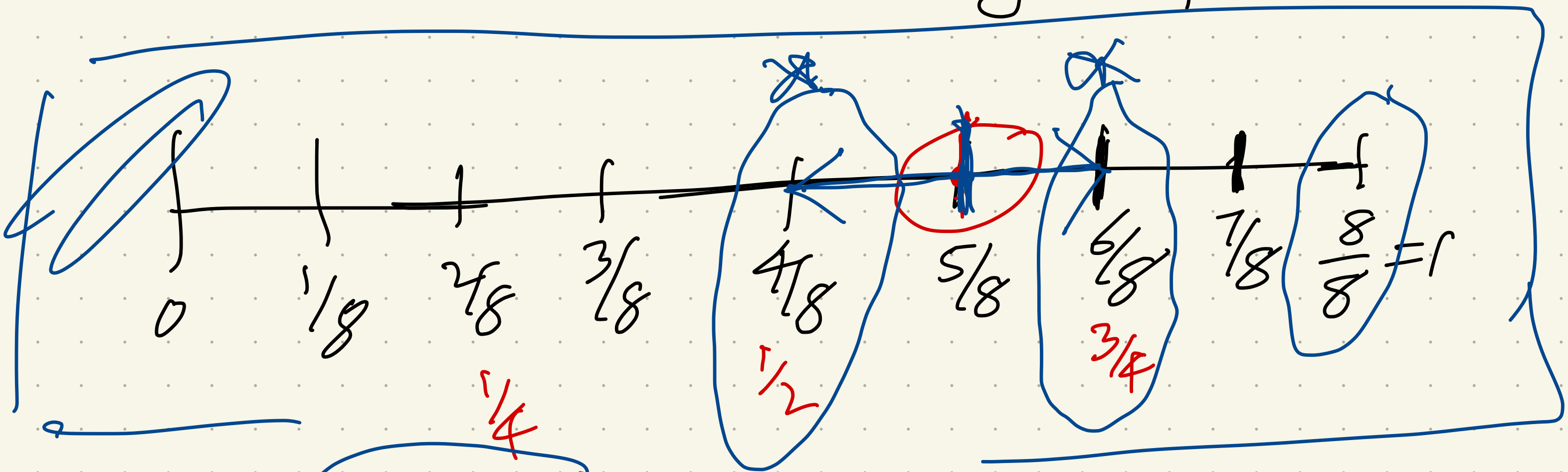
Infinite Coin Tossing Experiment.



$\frac{5}{8} = (1\ 0\ 1)$ $\quad \frac{1}{4}$

e_1^x, $e_2^{(x)} = 0$, $e_3^x = 1$

$\frac{5}{2^3}$

$\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

$\frac{e_1^x}{2} + \frac{e_2^x}{4} + \frac{e_3^x}{8} = \frac{5}{8}$

$x = \frac{5}{8}$

is $x < 3/4$ or $\geq 3/4$?

$x \geq 5/8$, $x < 5/8$

Binary representations of numbers $\leq \frac{5}{8}$

$(0, \cancel{*}, \cancel{*}, \cancel{*}, \ldots)$ $\quad (1, 0, 1)$ $\quad (e_1\, e_2\, e_3 \ldots)$

(a) Anything with 0 at first position!

$\frac{1}{2}$ $\quad \{\omega : e_1(\omega) = 0\}$ $\quad 1, 0, 0$ $\quad k/2^m$

(b) { Anything with $(1, 0, 0, * * * *)$

$P(\{\ \}) = \frac{1}{8}$ $\quad$ So $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

$(1, 0, 1, 0, 0, \ldots 0)$

$$S_3 = \{ e_\ell = 0,\ 1 \le \ell \le i_1 - 1,\ e_{i_1} = 1,$$
$$e_\ell = 0 \text{ for } i_1 + 1 \le \ell \le i_2 - 1,\ e_{i_2} = 1,$$
$$e_{i_2+1} = e_{i_2+2} = \cdots = e_{i_3} = 0 \}$$

$$S_p = \{ \cdots,\ e_{i_p} = 0 \}$$

$$S_{p+1} = \{ x \}$$

$$P(B_j | A) = \frac{P(B_j \cap A)}{P(A)}$$

$$P(A) = P\left(A \cap \bigcup_{k=1}^{\infty} B_k\right)$$

$$= P\left(\bigcup_{k=1}^{\infty} (A \cap B_k)\right)$$

$$= \sum_{k=1}^{\infty} P(A \cap B_k)$$

$$P(A \cap B_k) = P(B_k) P(A | B_k)$$

$$P(B_j | A) = \frac{P(B_j) P(A | B_j)}{\sum P(A | B_k) P(B_k)}$$

$\{\omega\} = \{ \underline{e_1} \, \underline{e_2} \, \underline{e_3} \cdots \}$

e_i is 0 or 1

Fix an $m \geq 1$

$P(\{\omega\}) \leq P\{ \underline{e_1 e_2 \cdots e_m} \cdots \}$

$\boxed{\underbrace{1, 0, 0, 1, \cdots 0}_{\text{first } m}}$

set of all sequences whose first m outcomes tally with the first m outcomes of $\{\omega\}$

$= \frac{1}{2^m} \longrightarrow 0$ as $m \to \infty$

$P(\{\omega\}) = 0$

m.

$$\mathcal{D}_m = \left\{ \frac{k}{2^m} : 0 \le k \le 2^m \right\}$$

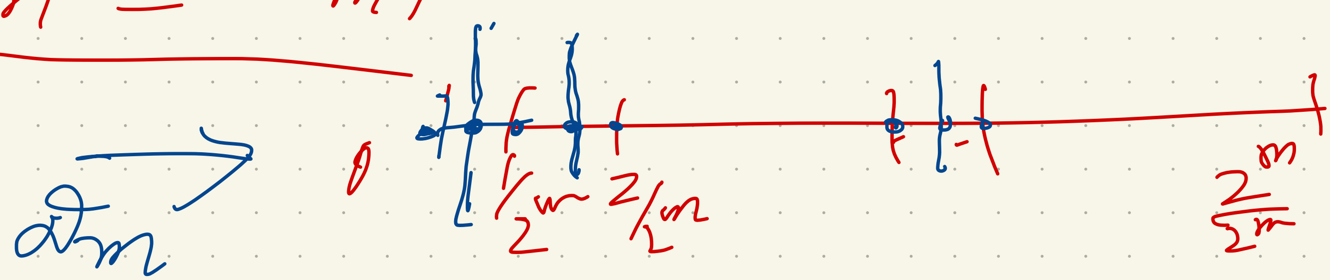
$m = 1$, $\mathcal{D}_1 = \left\{ 0, \frac{1}{2}, 1 \right\}$

$\mathcal{D}_2 = \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \right\}$

$\mathcal{D}_3 = \left\{ 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \dots \right\}$

$\mathcal{D}_4 = \{ \quad \}$

$\mathcal{D}_m \subseteq \mathcal{D}_{m+1}$



$$\Omega = \{\omega_1, \omega_2, \cdots\}$$

$$X : \Omega \longrightarrow \mathbb{R}$$

$$\text{Range of } X = \{X(\omega_1), X(\omega_2), \cdots\}$$

$$= (x_1, x_2, x_3, \cdots)$$

may not be 1-1

Binomial expt.

Coin tossed n times in succession

record $\underbrace{HTTH \cdots T}$ t

$$\# \Omega = 2^n$$

$$\underset{\sim}{w} \in \Omega$$

$$X(w) = \# \text{ of } H\text{'s in } w.$$

$$\mathcal{X} = \{0, 1, \ldots, n\} \quad 0 \le k \le n$$

$$P(X = k) = P(\{w : w \text{ has } k \ H\})$$

$$\text{"} p_X(k)$$

consider any $\underline{\omega \text{ with } k}$ H,

$$\underline{P(\{\omega\})} = p^k q^{n-k}$$

p : prob of H on any single toss

$$p_X(k) = \#\{\omega : \omega \text{ has } k \text{ H}\} \times p^k q^{n-k}$$

$$= \binom{n}{k} p^k q^{n-k}$$

$X \sim \text{Bin}(n, p)$

X follows Bin(n,p)

$$x \le y \Rightarrow F(x) \le F(y)$$

$$\underline{F(x)} = P(X \le x).$$

$$\{X \le x\} \subseteq \{X \le y\}$$

$$P(\{X \le x\}) \le P(\{X \le y\}) = F(y),$$

$$\left.\lim_{x \to \infty} F(x)\right\}$$

$\underbrace{x_n \uparrow \infty}$ look at $\underline{F(x_n)}$

$$F(x_n) = P(\underbrace{X \le x_n}_{A_n}) = P(A_n)$$

$$A_1, A_2, A_3, \ldots$$

$$x_n < x_{n+1}$$

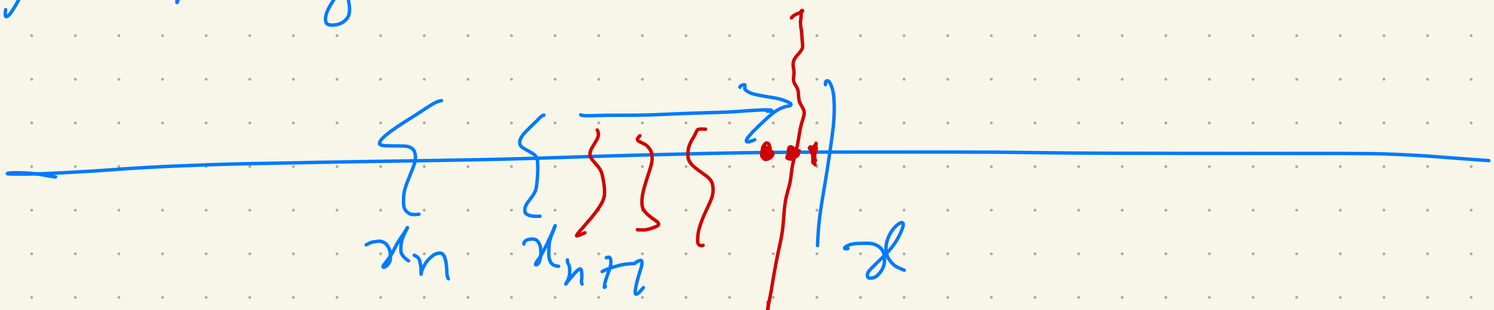
$$\{X \le x_n\} \subseteq \{X \le x_{n+1}\}$$

$$A_1 \subseteq A_2 \subseteq \cdots$$

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \{X \le x_j\} = \{X \in \mathbb{R}\} = \Omega$$

$$\underbrace{A_j \nearrow \Omega}_{P(A_j) = F(x_j)}, \quad P(A_j) \nearrow P(\Omega) = 1$$

Want to show F has **LL property**



$x_n \uparrow x$

$x_n < x$

$F \uparrow$

$A_n = \{X \le x_n\}$

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

$$\bigcup_{n=1}^{\infty} A_n = \{X < x\}.$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P(X < x)$$

"

$$\lim_{n} P(A_n) = \lim_{n\to\infty} F(x_n)$$

$F(x_n) \longrightarrow P(X < x)$

$F(x-)$

$$\boxed{F(x-) \le F(x).}$$

"$\lim_{y \to x-} F(y)$"

$$\boxed{F(x) - F(x-)}$$

$= P(X \le x) - P(X < x)$

$= P(X = x)$

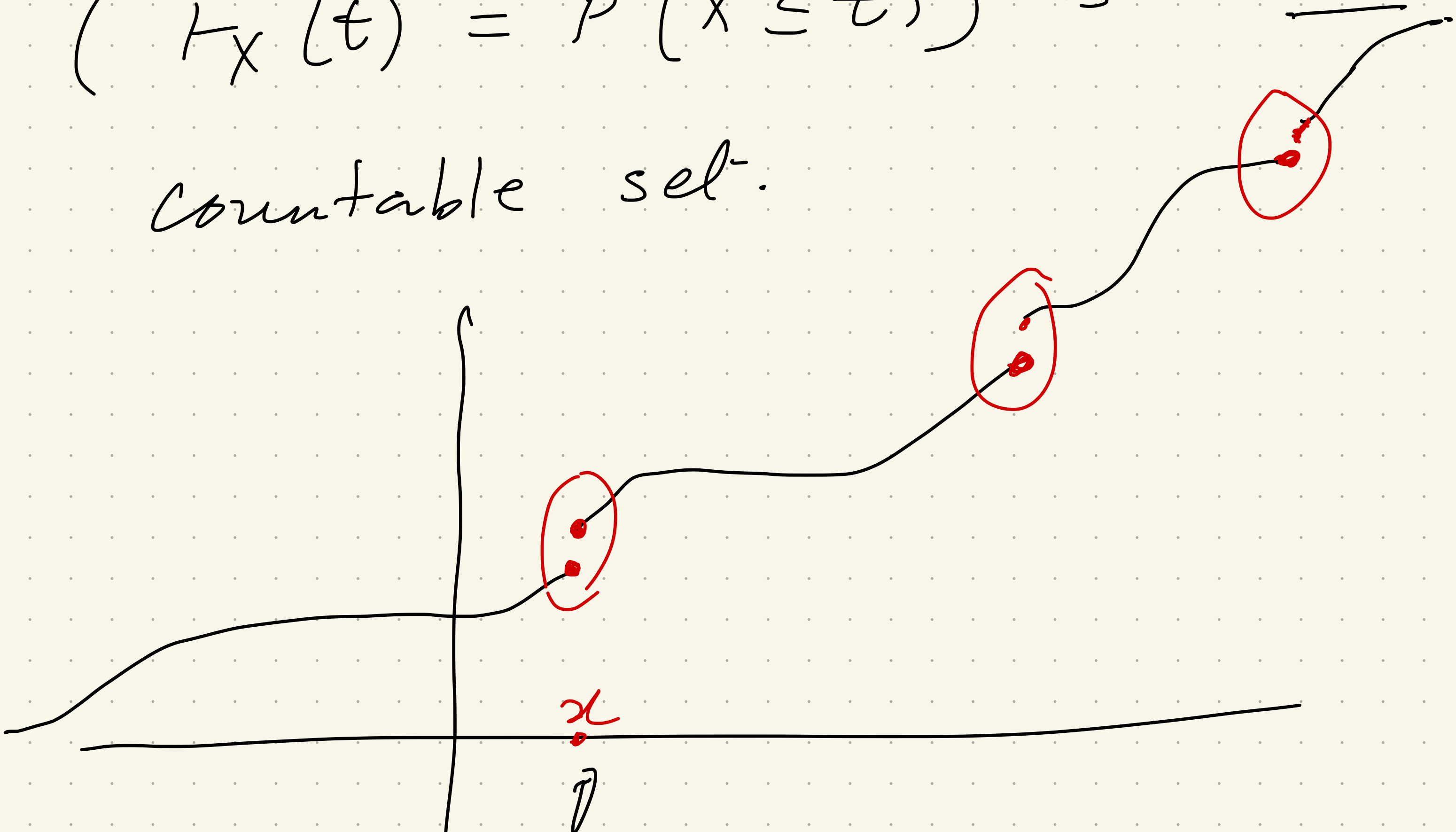
because $\{X \le x\} = \{X < x\} \cup \{X = x\}$

Continuous r.v's

are precisely those whose dist. fns

are continuous.

Point to note: The number of discontinuities of any F_X ($F_X(t) = P(X \le t)$) is a countable set.

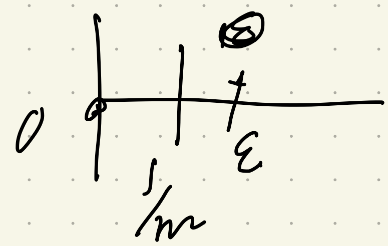


Jump discontinuities

$F_X(x-)$ $\xrightarrow{\text{jumps}}$ to $F_X(x)$

How many jumps of height

$> \frac{1}{m}$ can you have?



$m \in \mathbb{N}$

$\boxed{\text{at most } m} < \infty$

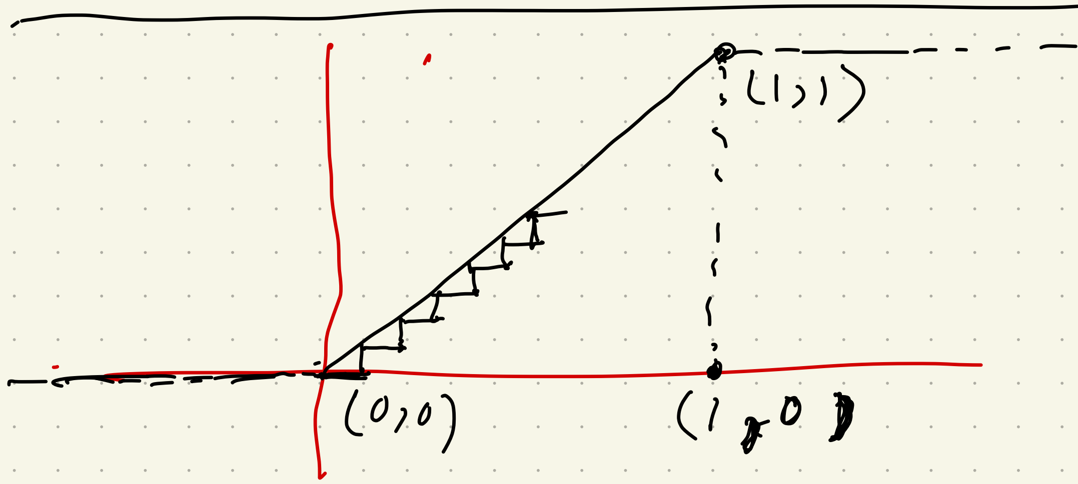
Set of jumps $= \bigcup_{m=1}^{\infty} \left\{ \text{Jumps of height} > \frac{1}{m} \right\}$

Countable union of finite sets is countable.

$X \sim \text{Uniform}(0,1)$

cdf of X

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$



Exercise: X_n being a random variable such ~~that~~

$$P\left(X_n = \frac{i}{n}\right) = \frac{1}{n} \text{ for } 1 \le i \le n$$

Uniform random variable on uniform grid.

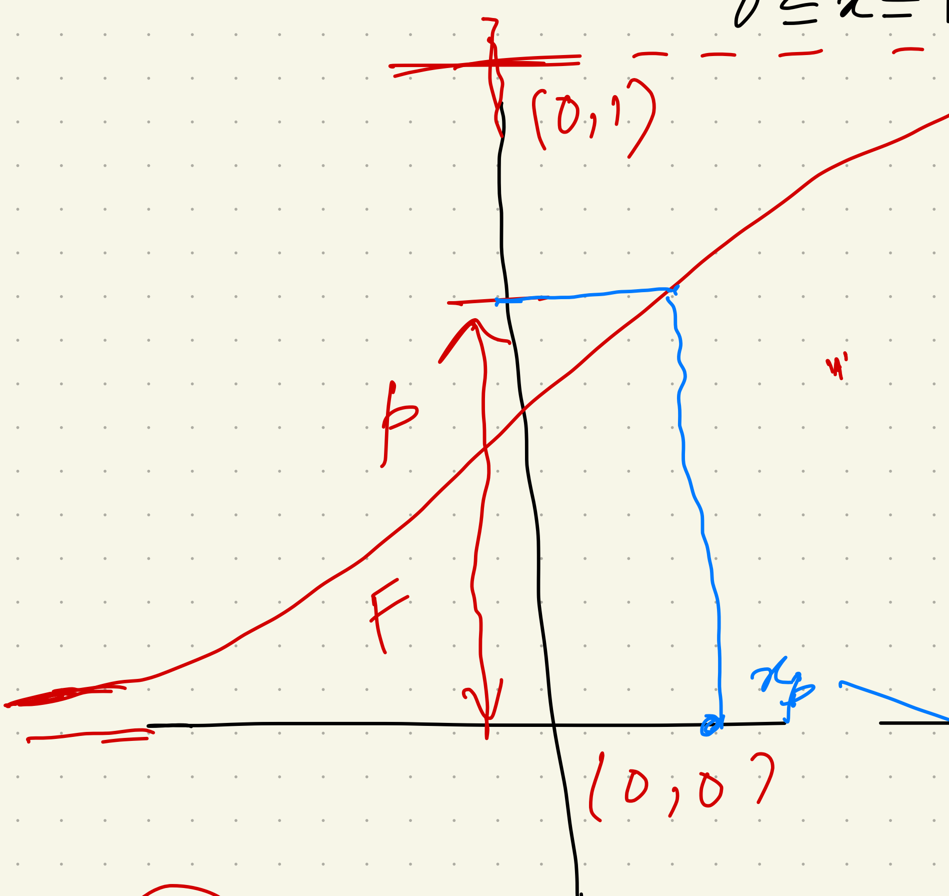
Plot the distribution function of X_n

— call this F_n.

Let F be the distr. func. of X

~ Uniform $(0,1)$

Show ~~that~~ $\sup_{0 \le x \le 1} | F_n(x) - F(x) | \longrightarrow 0$



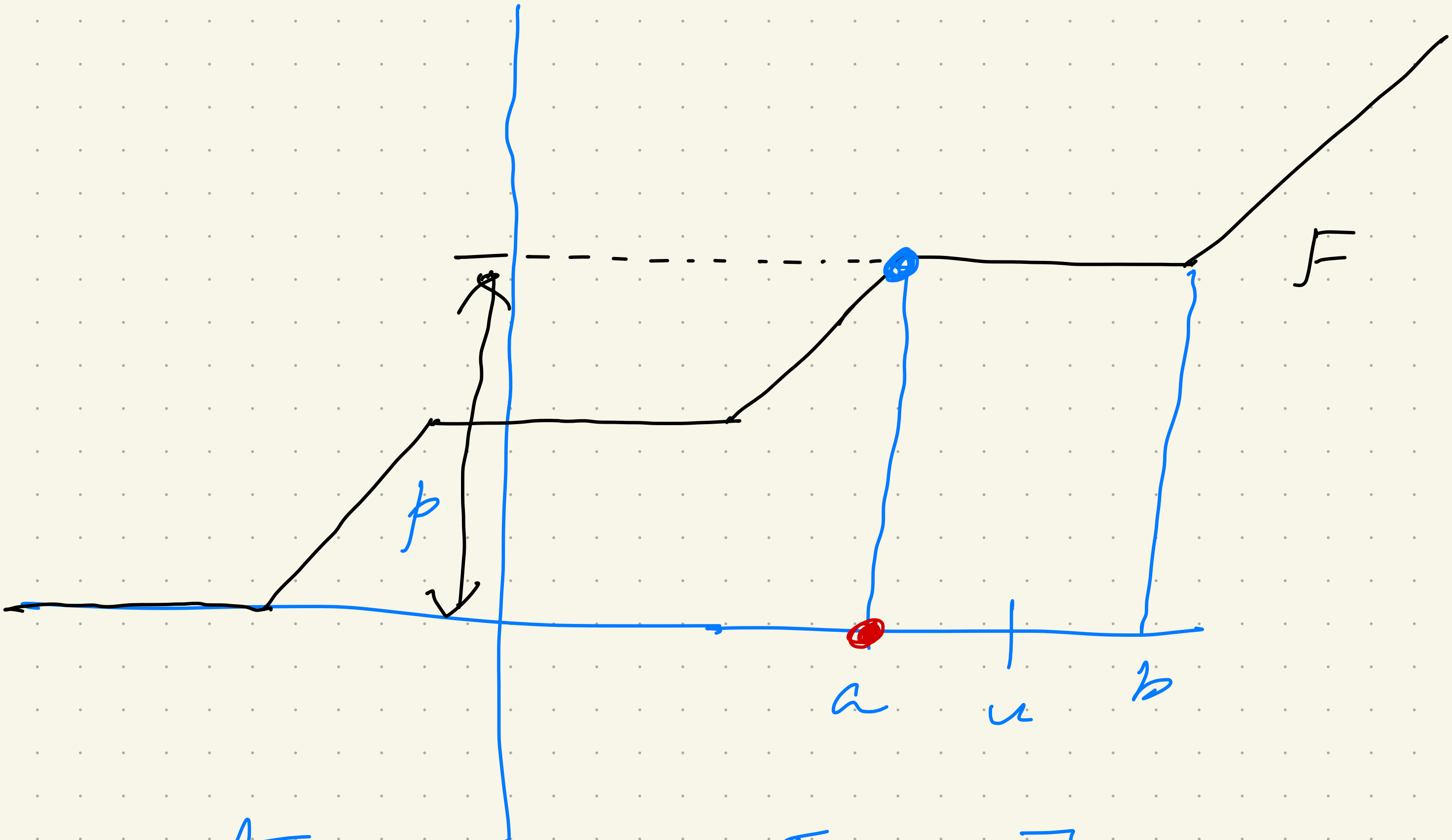
In this case, I have a well defined $F^{-1}(p)$ = unique x s.t. $F(x) = p$

$F(x_p) = p$

$F(x_p-) = p$

x_p is the unique p^{th} quantile.

x_p $\quad F(x_p-) \le p \le F(x_p)$



For any $u \in [a, b]$,

$p = F(u-) \le p \le F(u) = p$

infinitely many quantiles.

Focus on a. $a = \inf\{x : F(x) \geq p\}$ $\overset{"}{=}$ $F^{-1}(p)$.

very important

Point: For any F_X and any $0 < p < 1$

define: $F_X^{-1}(p) = \inf\{x : F(x) \geq p\}$

$=$ smallest x s.t. $F(x) \geq p$.