Stats 510 Assignment 4.

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(i) $f(u,v) = Cg(u^2+v^2)$

Let $X = \varepsilon_1 U$, $Y = \varepsilon_2 V$; Now, we can deduce the density of X, Y by

$$h(x,y) = f(u(x,y), v(x,y))\left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right|$$

$$= Cg(\varepsilon_1^2 x^2 + \varepsilon_1^2 v^2)\,|\varepsilon_1\varepsilon_2|$$

$= Cg(x^2+y^2)$; Hence, (X,Y) have the same distribution as (U,V)

$Cov(U,V) = E(UV) - E(U)E(V)$.

Since we have known that (X,Y) have the same distribution as (U,V)

Hence $(-U, V)$ have the same distribution as (U,V).

Therefore $E(UV) = E(-UV) = -E(UV) = 0$

Also, $E(U) = E(V) = 0$

Hence $Cov(U,V) = 0$

(ii) $Z_1 = W_1 U$, $Z_2 = W_2 V$; note that $U = W_1 Z_1$, $V = W_2 Z_2$

Then $h(z_1, z_2) = f(U(z_1,z_2), V(z_1,z_2))\left|\frac{\partial u}{\partial z_1}\frac{\partial v}{\partial z_2} - \frac{\partial v}{\partial z_1}\frac{\partial u}{\partial z_2}\right|$

$$= Cg(w_1^2 z_1^2 + w_2^2 z_2^2)\,|w_1 w_2|$$

$= Cg(z_1^2+z_2^2)$; Hence Q.E.D

2. i) $X = SY$; $E(XY) = E(SY^2) = E(S)\cdot E(Y^2)$ (Since S is independent)

$E(S) = 0$; $\Rightarrow E(XY) = 0$

Also, $E(Y) = 0$, Since Y have a density symmetric about 0.

Hence, $Cov(XY) = E(XY) - E(X)E(Y) = 0$

ii) We assume the distribution of Height is Y

Then the circumstance $E(2(X+Y)) = 2EX + 2EY = 1 + 2\times E(E(Y|X)) = 1 + 2\times \underbrace{E(X/2)}_{1/4} = \frac{3}{2}$

area: $E(XY) = E(XE(Y|X)) = E(X^2/2) = \frac{1}{6}$

iii) $p_y = 1 - p_x \Rightarrow W = p_x W_x + (1-p_x)W_y$; where $W_x \sim Poisson(\lambda_x)$; $W_y \sim Poisson(\lambda_y)$

Hence $E(W) = p_x E(W_x) + (1-p_x)E(W_y)$

$= p_x\lambda_x + (1-p_x)\lambda_y$

$E(W^2) = p_x E(W_x^2) + (1-p_x)E(W_y^2)$

$= p_x(\lambda_x + \lambda_x^2) + (1-p_x)(\lambda_y + \lambda_y^2)$

Hence $Var(W) = E(W^2) - E(W)^2 = p_x(\lambda_x+\lambda_x^2) + (1-p_x)(\lambda_y+\lambda_y^2) - (p_x^2\lambda_x^2 + 2p_x(1-p_x)\lambda_x\lambda_y + p_y^2\lambda_y^2)$

$= p_x(\lambda_x+\lambda_x^2) + p_y(\lambda_y+\lambda_y^2) - p_x^2\lambda_x^2 - 2p_x p_y\lambda_x\lambda_y - p_y^2\lambda_y^2.$

3 i) $Cov(N_i,N_j) = E[N_iN_j] - E[N_i]E(N_j)$

$$= \sum_{k\neq l} p_ip_j - \sum_{k=1}^{n} E(I_k^{(i)}) \sum_{l=1}^{n} E(I_l^{(j)})$$

$$= (n^2-n)p_ip_j - np_i \times np_j$$

$$= -np_ip_j$$

ii) $Cov(N_i,N_j|Z) = E(N_iN_j|Z) - E(N_i|Z)E(N_j|Z)$

$$= \sum_{k=l}^{n} \underbrace{E(I_k^{(i)}I_l^{(j)}|Z)}_{0} + \sum_{k\neq l}^{n} E(I_k^{(i)}I_l^{(j)}|Z) - E(N_i|Z)E(N_j|Z)$$

$$= \sum_{k\neq l} E(I_k^{(i)}|Z)E(I_l^{(j)}|Z) - E(N_i|Z)E(N_j|Z)$$

$E(I_k^{(i)}|Z) = (n-\sum_{r=1}^{m} N_i)p_i \Rightarrow Cov(N_i,N_j|Z) = -(n-\sum_{r=1}^{m} N_i)p_ip_j$

iii) $E(N_1) = np_1 = 25$

$Var(N_1) = np_1(1-p_1) = 100\times0.25\times0.75 = 18.75$

Hence. $P(N_1>65) = P\left(\frac{N_1-E(N_1)}{\sqrt{Var(N_1)}} > \frac{65-25}{\sqrt{18.75}}\right) = P(Z>9.238) \approx 0$.

4. $ax^2+by^2=1$

$f(x,y) = \frac{ab}{\pi}$; $f_X(x) = \int_y f(x,y)\,dy = \frac{2ab}{\pi}\sqrt{\frac{1-ax^2}{b}}$

Similarly $f_Y(y) = \frac{2ab}{\pi}\sqrt{\frac{1-by^2}{a}}$

Then $f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{ab}{\pi}}{\frac{2ab}{\pi}\sqrt{\frac{1-ax^2}{b}}} = \frac{1}{2}\sqrt{\frac{b}{1-ax^2}}$

Similarly ; $f(x|y) = \frac{2ab}{\pi}\sqrt{\frac{a}{1-by^2}}$

They are not independent, but they are uncorrelated

5. (i) $f(m,\theta)$ $\Theta \sim U(0,1)$ $\Rightarrow g(\theta)=1$ for $0\le\theta\le1$

$(X_1,X_2,\dots X_n)|\Theta=\theta$; i.i.d Bernoulli θ. $h(S_n) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$.

$$P(S_n=k) = \int_0^1 f(k,\theta)\,d\theta = \int_0^1 \binom{n}{k}\theta^k(1-\theta)^{n-k}\,d\theta$$

$$= \binom{n}{k}B(k+1, n-k+1).$$

To get the conditional expectation, we first need to calculate the conditional density

$$f(\theta|S_n) = \frac{f(S_n,\theta)}{f(S_n)} = \frac{\cancel{\binom{n}{m}}\theta^{S_n}(1-\theta)^{n-S_n}}{\cancel{\binom{n}{m}}B(S_n+1,n-S_n+1)}$$

Hence $E(\theta|S_n) = \int_0^1 \theta\cdot\frac{\theta^{S_n}(1-\theta)^{n-S_n}}{B(S_n+1,n-S_n+1)}\,d\theta = \frac{B(S_n+2,n-S_n+1)}{B(S_n+1,n-S_n+1)}$

ii) $P(x_1,\dots x_n|\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$; Since $P(\theta)=1$.

Hence $P(x_1,\dots x_n;\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$

$$\Rightarrow P(X_1=\varepsilon_1, X_2=\varepsilon_2,\dots X_n=\varepsilon_n) = \int_0^1 \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\,d\theta = \int_0^1 \theta^{\sum\varepsilon_i}(1-\theta)^{n-\sum\varepsilon_i}\,d\theta$$

$$= B(\sum\varepsilon_i+1, n-\sum\varepsilon_i+1)$$

iii) $(X_{\pi(1)}, \cdots, X_{\pi(n)})$

; Since $\{\varepsilon_1, \dots \varepsilon_n\}$ is a generic point in $\{0,1\}^n$, according to the de Finetti Theorem

$$P(X_{\pi(1)}=\varepsilon_1, X_{\pi(2)}=\varepsilon_2, \cdots, X_{\pi(n)}=\varepsilon_n) = \int_0^1 \theta^{\sum x_{\pi(i)}}(1-\theta)^{n-\sum x_{\pi(i)}} dF(\theta) = P(X_1=\varepsilon_1, X_2=\varepsilon_2, \cdots X_{\pi(n)}=\varepsilon_n)$$

$$= \int_0^1 \theta^{S_n}(1-\theta)^{n-S_n} d\theta$$

Hence, the distribution is exchangeable

iv) $E(X_1|S_n)$

$$f(X_1|S_n) = \frac{f(x_1, S_n)}{f(S_n)} = \frac{n\theta}{\binom{n}{S_n}B(S_n+1, n-S_n+1)}$$

Then $E(x_1|S_n) = \dfrac{\int_0^1 n\theta\, d\theta}{\binom{n}{S_n}B(S_n+1, n-S_n+1)} = \dfrac{n\theta^2}{2\binom{n}{S_n}B(S_n+1, n-S_n+1)}$