$\underline{(\Omega, \mathcal{A}, P)}$

$\underline{X} : \Omega \longrightarrow \mathbb{R}$

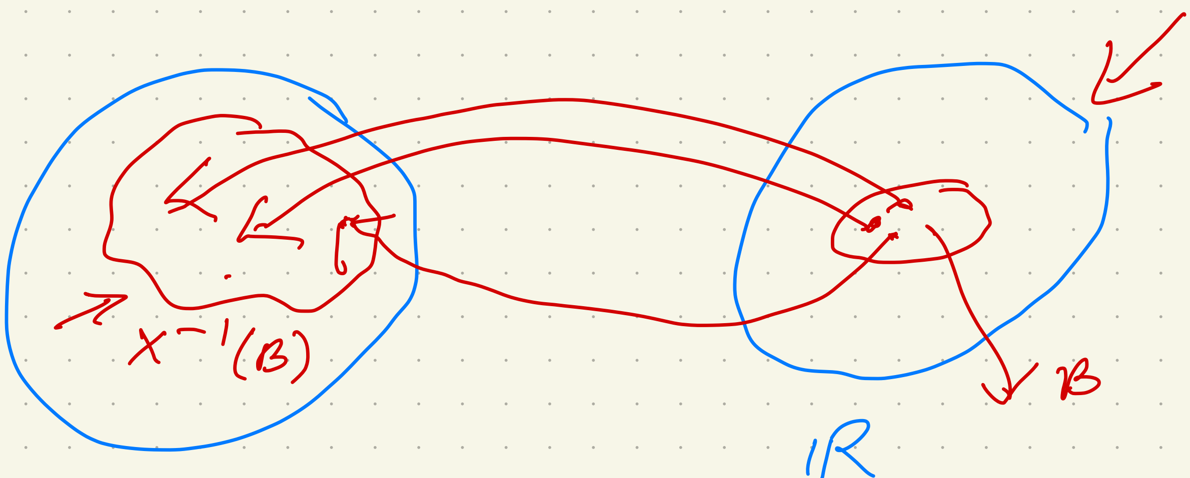
X is r.v if $\underline{X^{-1}(B)} \in \underline{\mathcal{A}}$ for all $B \in \mathcal{B}_{\mathbb{R}} \longrightarrow$ Borel σ field

Induced probability on $\mathcal{B}_{\mathbb{R}}$

Define P_X as:

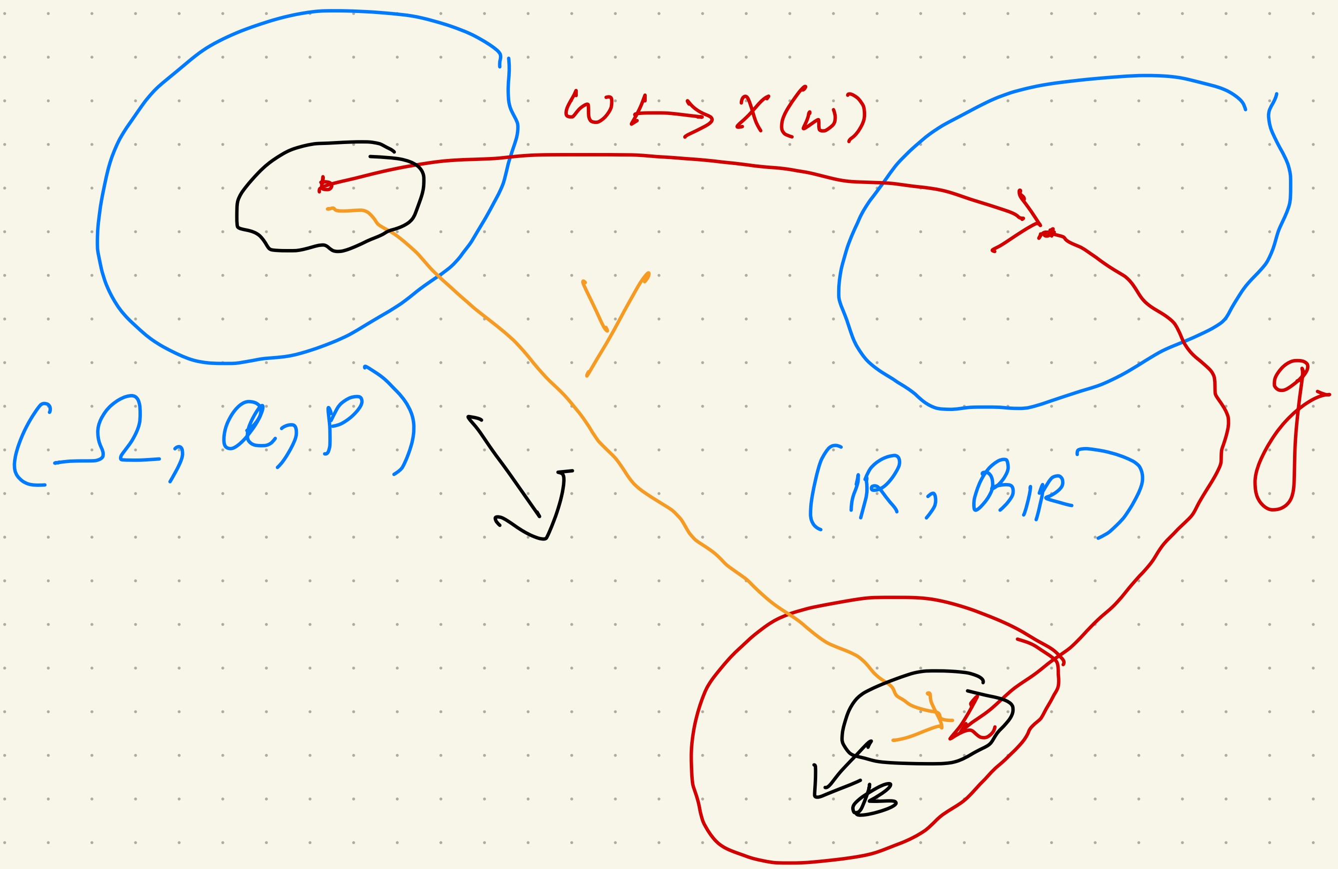
$$\underline{P_X(B) = P[X^{-1}(B)]}$$

P_X : distribution of X



$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$

$P_X(B) = P(X^{-1}(B))$



$(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

$\underline{X : \Omega} \to \mathbb{R}$, $X^{-1}(B) \in \mathcal{A} \ \forall \ B \in \mathcal{B}_{\mathbb{R}}$

$g : \mathbb{R} \to \mathbb{R}$, g measurable

i.e. $g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for all $B \in \mathcal{B}_{\mathbb{R}}$

$$\underline{Y = g(X) \equiv g \circ X}$$

$Y = \Omega \to \mathbb{R}$, Y is also a ~~proper~~ random variable

$\boxed{P_X \text{ and } g}$

Consider P_Y: the distribution of Y.

Let $B \in \mathcal{B}_{\mathbb{R}}$:

$$
\begin{aligned}
P_Y(B) &= P(Y^{-1}(B)) \\
&= P\left[(g(X))^{-1}(B)\right] \\
&= P\left[X^{-1}(g^{-1}(B))\right] \\
&= P_X\left[g^{-1}(B)\right]
\end{aligned}
$$

We can also write:

$$
\begin{aligned}
P_Y(B) &= P\{\omega : Y(\omega) \in B\} \\
&= P_X\left[g^{-1}(B)\right] \\
&= P\left[\omega : X(\omega) \in g^{-1}(B)\right]
\end{aligned}
$$

Random variables X and X' have the same distribution if $P_X = P_{X'}$

i.e $P_X(B) = P_{X'}(B)$

$P(X \in B) \quad P(X' \in B)$

P_X and $P_{X'}$ are identical iff F_X and $F_{X'}$ are identical:

i.e $F_X(y) = F_{X'}(y)$ for every y

$P(X \leq y) \quad P(X' \leq y)$.

here $B = (-\infty, y]$

relate p.m.f of Y to that of X in discrete case

relate p.d.f of Y to that of X in the continuous case

Start with X discrete

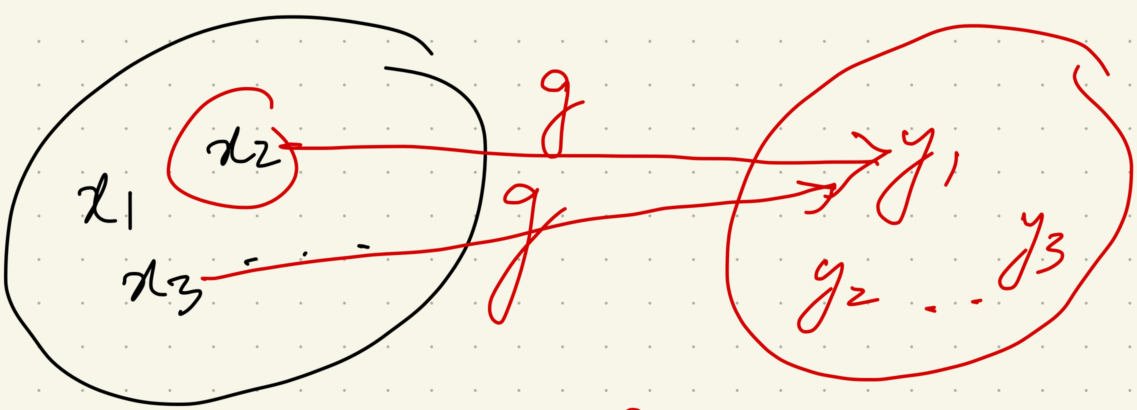
Theorem 2.3

X : discrete random variable assuming values $\{x_1, x_2, \ldots\}$

associated p.m.f $\{p_1, p_2, p_3, \ldots\}$

$p_j = P(X = x_j)$. $\quad p_j > 0$

$Y = g(X)$. suppose Y assumes values $\{y_1, y_2, y_3, \ldots\}$



p_Y : want to find

what is $p_Y(y_\ell) = P(Y = y_\ell)$.

$P(Y = y_\ell)$.

$= P\left[X \in \{x_j : g(x_j) = y_\ell\}\right]$

$$= \sum_{j : g(x_j) = y_\ell} \boxed{P(X = x_j)} \quad p_X(x_j)$$

$$= \sum_{g^{-1}(\{y_\ell\})} p_X(x_j)$$

Example 2.4: X discrete random variable.

$X \in \{\pm m, \ m = 0, 1, 2, 3, \dots\}$

$p_X(0) = \frac{1}{2}$, $\quad p_X(m) = P(X = m) = \frac{2}{3} \cdot \frac{1}{2^{m+1}}$

$p_X(-m) = \frac{1}{3} \cdot \frac{1}{2^{m+1}}$

What's an intuitive way of generating an X like this?

Find p.m.f of $Y = |X| \equiv g(X)$.

$P(Y=0) = P(X=0) = \frac{1}{2}$

$P(Y=m) = P(X \in g^{-1}(m))$, $\underline{m>0}$

$= P(X \in \{-m, m\})$

$= p_X(m) + p_X(-m) = \frac{1}{2^{m+1}}$

Continuous random variables with p.m.f's.

$\left.\underline{\text{Change of Variable Theorem}}\right\}$:

$\boxed{\text{Theorem 2.4}}$

X : continuous random variable

$P(X \in I) = 1$. $I = (\underset{=}{a}, \underset{=}{b})$

X has a p.d.f on I denoted f_X.

So for any $a < c < d < b$

$P(X \in (c,d)) = \int_c^d f_X(t)\,dt$

define $Y = g(X)$

find p.d.f of Y.

Not all g's produce a continuous Y.

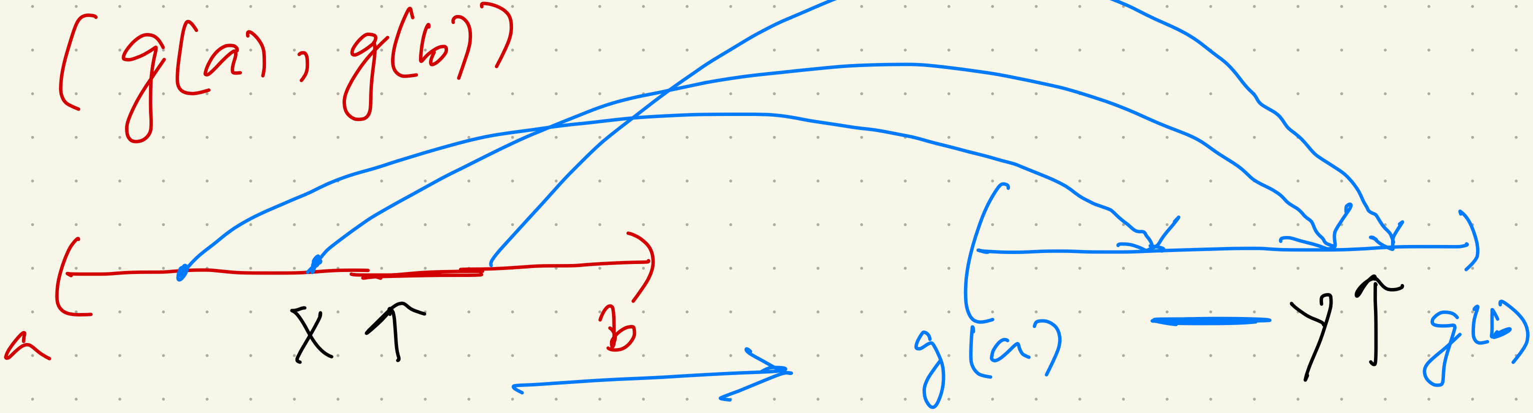
Note that g needs to be sufficiently rich/complex for $g(X)$ to be continuous and have a pdf.

Regularity conditions on g:

Assume g is continuously differentiable and that $g'(x) \neq 0$ for $x \in I$.

$\Rightarrow$ g is either $\uparrow$ on I or $\downarrow$ on I.

and then g is a 1-1 continuously differentiable fn. from (a, b) to $(g(a), g(b))$

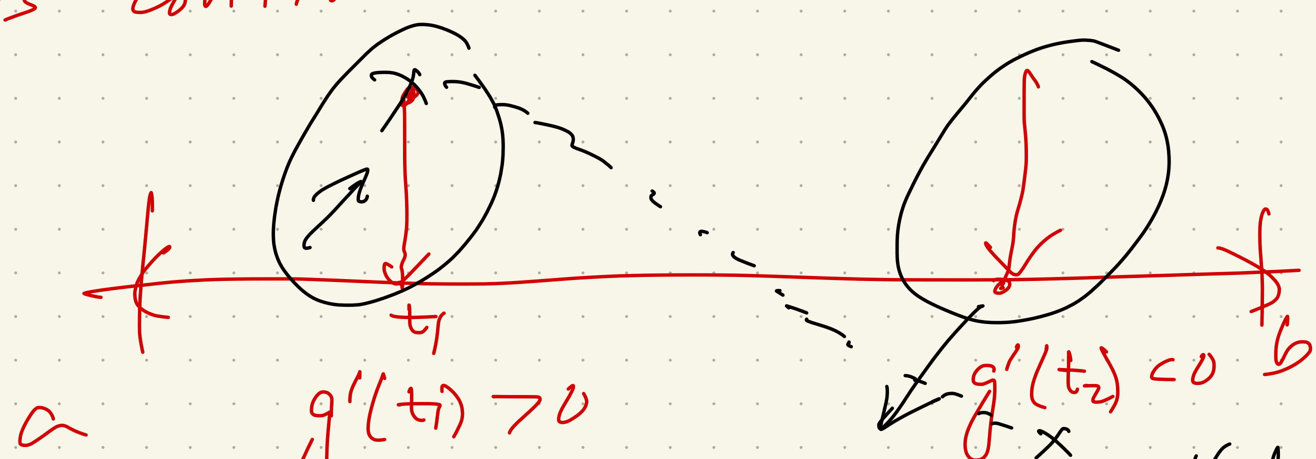


$g'(x) \neq 0 \ \forall x \in I_2$

you know $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$

because were $g(x_1) = g(x_2)$, the mean value theorem gives you a point between x_1, x_2, say x^* s.t. $g'(x^*) = 0$.

You also know that g' cannot be both +ve and −ve. → because g' is continuous.



Intermediate Value Theorem

either $g'(t) > 0$ for all $t \Rightarrow g \uparrow$

or $g'(t) < 0$ for all $t \Rightarrow g \downarrow$

Define $Y = g(X)$.

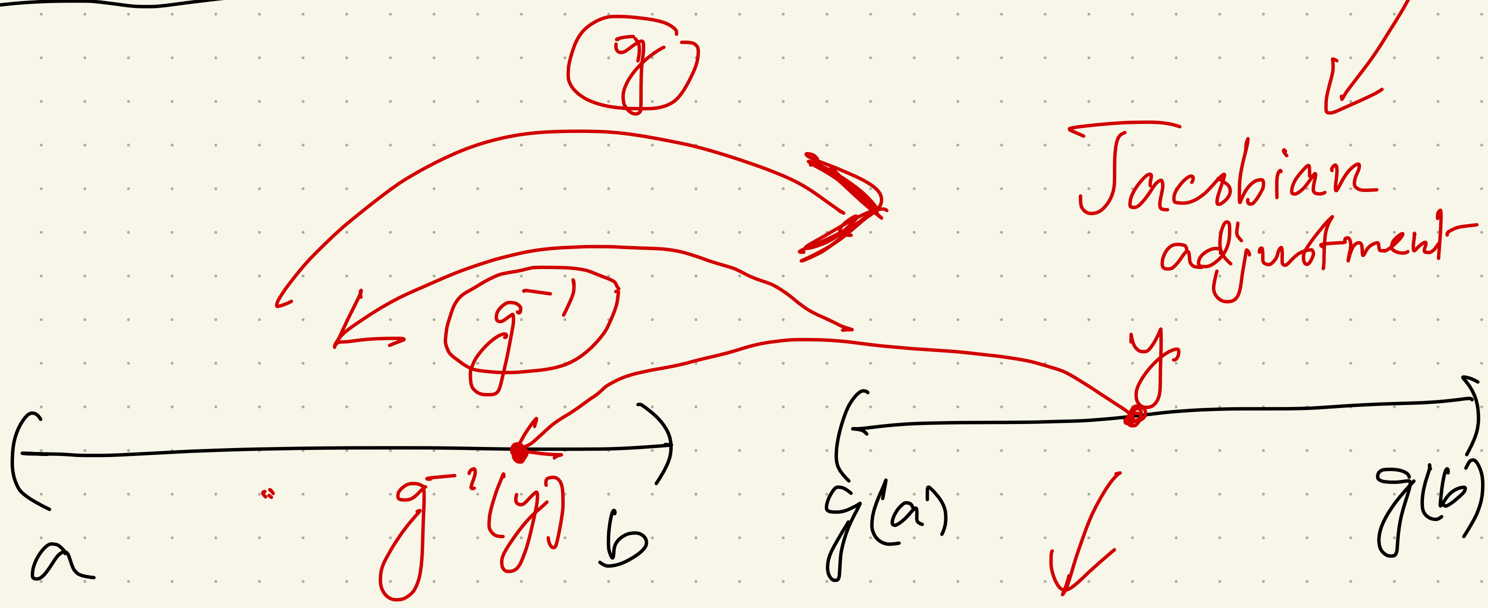
So $P(Y \in (g(a), g(b))) = 1$

(g is being assumed $\uparrow$)

Then the density function of Y is given by:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \mathbb{1}(y \in g(I))}$$

Jacobian adjustment



$f_Y(y)$

$$\boxed{\frac{d}{dy} g^{-1}(y)) = \frac{1}{g'(g^{-1}(y))}}$$

chain rule

Proof:

$I = (a, b)$, $g \uparrow$.

F_Y : distribution function of Y.

$$F_Y(y) = P(Y \le y)$$

$$= \begin{cases} 0 & \text{if } y \le g(a) \\ * & \text{if } y \in (g(a), g(b)) \\ 1 & \text{if } y \ge g(b) \end{cases}$$

$$P(Y \le y) = \underbrace{P(Y \le g(a))}_{=0} + P(g(a) < Y \le y)$$

$$= P(g(a) < Y \le y)$$

$$= P(g(a) < g(X) \le y)$$

applying g^{-1}

$$= P(a < X \le g^{-1}(y))$$

$$P(a < X \le g^{-1}(y))$$

$$= P(X \le g^{-1}(y)), \text{ since } P(X>a) = 1$$

so: $F_Y(y) = P(Y \le y)$

$$= P(X \le g^{-1}(y)).$$

$$= F_X(g^{-1}(y)).$$

so $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \frac{d}{dy}\left(F_X(g^{-1}(y))\right)$$

$\downarrow$ chain rule

Try the same derivation with $g\downarrow$

$$= f_X(g^{-1}(y)) \frac{d}{dy}(g^{-1}(y)).$$

gives me the desired expression.

where did I lose it?

Exercise 2.5 $\quad U \sim \text{Uniform}(0,1)$.

define $W = -\frac{1}{\lambda} \log U$.

$\boxed{\lambda > 0}$.

Find density of W.

Transfer to the notation of our theorem:

$U \equiv X$. $(a,b) = (0,1)$

$$f_U(u) = 1 \cdot \mathbb{1}(0 < u < 1)$$

$$W = g(U) = -\frac{1}{\lambda} \log U.$$

$$(g(a), g(b)) = (0, \infty).$$

$$u \longmapsto -\frac{1}{\lambda} \log(u) \equiv g(u)$$

is a strictly $\downarrow$ function from $(0,1)$ to $(0,\infty)$

so W has a density on $(0,\infty)$

$$g(u) = -\frac{1}{\lambda}\log u$$

$$g'(u) = -\frac{1}{\lambda u} < 0 \ !$$

$$g(u) = -\frac{1}{\lambda}\log u.$$

Finding inverse means express u in terms of $g(u)$.

$$\text{so } +\log u = -\lambda g(u)$$

$$\Rightarrow u = \exp[-\lambda g(u)].$$

So $g^{-1}: (0,\infty)$ to $(0,1)$

with $g^{-1}(w) = \exp[-\lambda w]$

$(w \equiv g(u))$

$$\underline{f_W(w)} = f_U(g^{-1}(w))\left|\frac{d}{dw}g^{-1}(w)\right| \mathbb{1}\{w \in (0,\infty)\}.$$

$$f_U(g^{-1}(w)) = 1.$$

$$\begin{tikzcd} (0, 1) \arrow[r, bend left, "g"] & \lbrack 0, w \ldots \infty) \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

$$\frac{d}{dw} g^{-1}(w)$$

$$= \frac{d}{dw}\left[e^{-\lambda w}\right] = \underline{\underline{-\lambda e^{-\lambda w}}}$$

$$f_W(w) = \lambda e^{-\lambda w} \mathbb{1}(w \in (0, \infty))$$

This is the exponential λ density.

$$\tilde{W} = -\frac{1}{\lambda}\log(1-U).$$

<u>Find density of</u> $\tilde{W}$.

$\tilde{U} = 1 - U$

<u>use</u> COVT! $\underline{\tilde{U}} \sim \mathrm{Unif}(0,1)$

$$\tilde{w} = -\frac{1}{\lambda} \log \tilde{u}$$

$$\tilde{u} \sim \mathrm{Unif}(0,1).$$

so $\tilde{w} \sim \exp(\lambda)$.

$g \uparrow$ or $\downarrow$ and we considered a mapping from an interval to an interval.

Would like generalizations.

What if g is a many to one function?

What if g is a many to many function?

$$\boxed{g(x) = x^2}$$

Next Class.