STATS 510 Final Exam

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1. $X, Y \sim N(0,1)$; $U = X$, $V = X/Y$

Hence; $\begin{cases} u = x \\ y = u/v \end{cases}$

For the Jacobian; we have $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \left| -\frac{u}{v^2} \right| = \frac{u}{v^2}$

The Joint density is

$$f_{U,V}(u,v) = f_{X,Y}(u, u/v) \times \frac{u}{v^2} \times I(0<u<\infty, 0<v<\infty)$$

$$= \frac{1}{2\pi} e^{-\frac{u^2}{2}} \times e^{-\frac{u^2/v^2}{2}} \times \frac{u}{v^2} \times I(0<u<\infty, 0<v<\infty)$$

$$= \frac{1}{2\pi} e^{-\frac{u^2(1+\frac{1}{v^2})}{2}} \times \frac{u}{v^2} \times I(0<u<\infty, 0<v<\infty)$$

Hence; The marginal density of v is

$$f_V(v) = \int f_{U,V}(u,v)\,du$$

$$= \int_0^\infty \frac{1}{2\pi} e^{-\frac{u^2(1+\frac{1}{v^2})}{2}} \times \frac{u}{v^2}\,du$$

$$= \frac{1}{2\pi v^2}\int_0^\infty u \times e^{-\frac{u^2(1+\frac{1}{v^2})}{2}}\,du$$

$$= \frac{1}{2\pi v^2} \times \left(-\frac{1}{1+v^2}\right) e^{-\frac{1}{2}u^2(1+v^2)} \Bigg|_0^{+\infty}$$

$$= \frac{1}{2\pi v^2(1+v^2)} \qquad v \in (-\infty, \infty)$$

2. $X \sim f_X(x)$

$P(Y \mid X=x) \sim N(1/x, x^4) \Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}x^2} e^{-\frac{(y-\frac{1}{x})^2}{2x^4}}$

find the p.d.f of $\frac{Y-1/x}{x^2}$;

Hence $f(x,y) = f \cdot \frac{1}{\sqrt{2\pi}x^2} e^{-\frac{(y-\frac{1}{x})^2}{2x^4}}$; we Let $U = X$, $V = \frac{Y-1/x}{x^2}$.

Hence; $\begin{cases} x = u \\ y = v \cdot u^2 + \frac{1}{u} \end{cases}$

For the Jacobian, we have $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2uv - \frac{1}{u^2} & u^2 \end{vmatrix} = |u^2|$

Hence

$$f_{U,V}(u,v) = f_u(u) \cdot \frac{1}{\sqrt{2\pi}\,u^2} e^{-\frac{(v \cdot u^2 + \frac{1}{u} - \frac{1}{u})^2}{2 \times u^4}} \times u^2$$

$$= f_u(u) \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2}, \text{ According to Lemma 4.2.7.}$$

We have $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$

Hence $f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2}$ (which is the p.d.f of $V = \frac{Y - 1/x}{x^2}$

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3. N_1, N_2, N_3. sum of multinomials.

$\{(\Delta i1, \Delta i2, \Delta i3)\}_{i=1}^{n}$ be n. i.i.d Multinomial $(1, \theta_1, \theta_2, \theta_3)$ random vector (only one of $\Delta i1; \Delta i2; \Delta i3$ can be 1)

Hence $f(\Delta i1, \Delta i2, \Delta i3) = \frac{1}{\Delta i1! \cdot \Delta i2! \cdot \Delta i3!} p_1^{\Delta i1} \times p_2^{\Delta i2} \times p_3^{\Delta i3}$

$= p_1^{\Delta i1} \times p_2^{\Delta i2} \times p_3^{\Delta i3} = \theta_1^{\Delta i1} \theta_2^{\Delta i2} \theta_3^{\Delta i3}$

Then $P(N_1 = n_1, N_2 = n_2, N_3 = n_3) = \frac{n!}{n_1! n_2! n_3!} \theta_1^{n_1} \theta_2^{n_2} \theta_3^{n_3}$; $n_3 = n - n_1 - n_2$

$P(N_1 = n_1, N_2 = n_2, N_3 = n_3 | \Delta i1, \Delta i2, \Delta i3) = P(N_1 = \sum_{i=1}^{n} \Delta i1, N_2 = \sum_{i=1}^{n} \Delta i2, N_3 = \sum_{i=1}^{n} \Delta i3)$

$= \frac{n!}{\sum \Delta i1! \sum \Delta i2! \sum \Delta i3!} \theta_1^{\sum \Delta i1} \theta_2^{\sum \Delta i2} \theta_3^{\sum \Delta i3}$

Hence, According to Bayes' Rule $P(\Delta i1, \Delta i2, \Delta i3 | N_1 = n_1, N_2 = n_2, N_3 = n_3) = \prod_{j=1}^{3} \frac{\theta_j^{\Delta ij + \sum \Delta ij - n_j} \times n_j!}{\sum \Delta ij!}$

(b). $Cov(\Delta 1k, \Delta 2L)$ for $k \neq l$ and $1 \leq k, l \leq 3$. conditional on $(N_1, N_2, N_3) = (n_1, n_2, n_3)$

$= Cov(\Delta 1k, \Delta 2L | N_1 = n_1, N_2 = n_2, N_3 = n_3)$

$= E(\Delta 1k \times \Delta 2L | N_1 = n_1, N_2 = n_2, N_3 = n_3) - \mu_{\Delta 1k} \times \mu_{\Delta 2L}$; $\mu_{\Delta 1k} = \frac{|\Delta 11 + \Delta 12 + \Delta 13|}{3} = \frac{1}{3} = \mu_{\Delta 2L}$ ← not related to n_1, n_2, n_3

$= E(\Delta 1k \times \Delta 2L | N_1 = n_1, N_2 = n_2, N_3 = n_3) - \frac{1}{9}$

$= E(\Delta 1k | N_1 = n_1, N_2 = n_2, N_3 = n_3) \times E(\Delta 2L | N_1 = n_1, N_2 = n_2, N_3 = n_3) - \frac{1}{9}$

$= 0$;

4. $X \sim exp(\lambda)$; $Y \sim Exp(\mu)$ $\Delta = I(X \leq Y)$; M be the minimum of X and Y

Hence $P(M > m | \Delta = 1) = P(M > m | X \leq Y)$; under this condition

$= P(X > m | X \leq Y)$

$= P(X \in (m, n], Y \in [n, +\infty))$

$= P(X \in (m, n]) P(Y \in [n, +\infty))$ (Since X and Y are independent).

$= \int_m^n \lambda e^{-\lambda x} dx + \int_n^{+\infty} \mu e^{-\mu y} dy$

$\equiv \int_m^{+\infty} \lambda e^{-\lambda x} dx$

$= 1 - e^{-\lambda m} = P(M > m)$ which is not relavant to Δ.

Hence; Δ and M are independent.

5. (i). $X_1, X_2 \dots X_n$ be a sequence of identically distributed mean-0 random variables

$S_n \equiv \frac{X_1 X_2 + X_2 X_3 + \dots + X_n X_{n+1}}{n} \to_p 0$

$S_n := \sum_{\substack{i=1 \\ j=i+1}}^{n} x_i x_j$

$E(S_n) = E(\sum_{\substack{i=1 \\ j=1}}^{n} x_i x_j)$ → follows a

$= E(E(\sum_{\substack{i=1 \\ j=i+1}}^{n} x_i x_j | x_i))$

$= 0$.

$Var(S_n) = E(S_n^2) - (E(S_n))^2$

$= E(S_n^2)$

$= 0$.

Hence $S_n \equiv \frac{x_1 x_2 + x_2 + \dots + x_n x_{n+1}}{n} \to 0$,

Q.E.D.

(ii) $U_1, U_2 \dots U_n$ be i.i.d uniform (0,1), we first calculate the problem with $n=2$

Let $Z = U_1 U_2$; Hence.

$F_Z(z) = P(Z \leq z) = \int_{u=0}^{1} P(U_2 \leq z/u) f_{U_1}(u) du = \int_{u=0}^{z} du + \int_{u=z}^{1} \frac{z}{u} du = z - z \log z$.

Hence $f_Z(z) = -\log z$

We take into account the third variable. U_3. Hence $V = U_1 U_2 U_3$

$F_V(v) = P(V \leq v) = \int_{u=0}^{1} P(U_3 \leq v/u) f_Z(u) du = -\int_{u=0}^{v} \log u du - \int_{u=v}^{1} \frac{v}{u} \log u du \Rightarrow$

$f_V(v) = \frac{1}{2} (\log v)^2$ (see next page)

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Hence, according to mathematical induction, we can deduce that

$$f_{\prod_{i=1}^{n} U_n}(c) = \frac{(-\log c)^{n-1}}{(n-1)!} I(0<c\leq 1);$$ which is the p.d.f of c.

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