1) Suppose $\underline{X} = (X_1, \ldots, X_d)$ is exchangeable

Then $\underline{X} =_d (X_{\pi(1)}, \ldots, X_{\pi(d)})$ for any permutation π, where $=_d$ means equality in distribution. (*)

Let $\tilde{t} = (t_{\tau(1)}, \ldots, t_{\tau(d)})$ be a permutation of the vector $t = (t_1, \ldots, t_d)$, where τ is a permutation of $\{1, 2, \ldots, d\}$

Then $M_{\underline{X}}(\tilde{t}) = \mathbb{E}[e^{\tilde{t}^T \underline{X}}]$

$$= \mathbb{E}[\exp(t_{\tau(1)} X_1 + \cdots + t_{\tau(d)} X_d)]$$

$$= \mathbb{E}[\exp(t_1 X_{\tau^{-1}(1)} + \cdots + t_d X_{\tau^{-1}(d)})]$$

where τ^{-1} is the inverse permutation of τ.

Letting $\tilde{X} = (X_{\tau^{-1}(1)}, \ldots, X_{\tau^{-1}(d)})$

we have:

$M_{\underline{X}}(\tilde{t}) = M_{\tilde{X}}(t)$

But $M_{\tilde{X}}(t) = M_{\underline{X}}(t)$ because $\underline{X}$ is exchangeable. Refer back to (*).

Hence, $M_{\underline{X}}(\tilde{t}) = M_{\underline{X}}(t)$ i.e $M_{\underline{X}}$ is symmetric in its arguments.

The other part follows by starting from the symmetry of $M_{\underline{X}}(t)$ and using it to show ~~that~~ $M_{\underline{X}}(t)$ is identical to $M_{\tilde{X}}(t)$ for any $\tilde{X}$ formed by permuting the co-ordinates of $\underline{X}$, from which we deduce $\underline{X} \overset{d}{=} \tilde{X}$ i.e $\underline{X}$ is exchangeable.

2. (a) (R, θ) are the polar co-ordinates of $\left(\frac{x}{a}, \frac{y}{b}\right)$.

(b) $x = aR\cos\theta$

$y = bR\sin\theta$.

Jacobian of (x, y) w.r.t (R, θ):

$$J = \text{abs det} \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial y}{\partial R} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \text{abs det} \begin{bmatrix} a\cos\theta & b\sin\theta \\ -aR\sin\theta & bR\cos\theta \end{bmatrix}$$

$$= abR.$$

$$f(x,y) = C_{a,b}\, g\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) 1\left((x,y)\in \mathbb{R}^2\right)$$

So:

$$f(r,\theta) = C_{a\cdot b}\, g\left(\frac{a^2 r^2 \cos^2\theta}{a^2} + \frac{b^2 r^2 \sin^2\theta}{b^2}\right)$$
$$1\left((r,\theta)\in(0,\infty)\times(0,2\pi)\right)$$
$$\times\ abr$$

$$= C_{a\cdot b}\, g(r^2)\, a\cdot b r$$
$$\times\ 1\left(r\in(0,\infty)\right) 1\left(\theta\in(0,2\pi)\right)$$

$$= 2\pi C_{a,b}\, g(r^2)\, abr\, 1\left(r\in(0,\infty)\right)$$
$$\times\ \frac{1}{2\pi}\, 1\left(\theta\in(0,2\pi)\right)$$

The factorization shows the independence of R and Θ. Also gives marginals

$\Theta \sim \text{Unif}(0, 2\pi)$

(c) $$\int_0^\infty ab\,C_{a,b}\, 2\pi\, r\, g(r^2)\, r\, dr = 1$$

i.e. $$ab\pi\, C_{a,b} \int_0^\infty 2r\, g(r^2)\, dr = 1$$

i.e. $$ab\,\pi\, C_{a,b} \int_0^\infty g(x)\, dx = 1$$

i.e. $$C_{a,b} = \frac{1}{\pi ab \int_0^\infty g(x)\, dx}$$

3. $f_{\Delta|T}(\delta | T = t)$ to be computed.

$$= P(\Delta = \delta | T = t)$$

Take $\delta = 1$.

$$P[\Delta = 1 | T = t]$$

$$= P[X \leq T | T = t]$$

$$= P[X \leq t | T = t]$$

$$= P[X \leq t] = 1 - e^{-\lambda t}.$$

similarly $P[\Delta = 0 | T = t] = e^{-\lambda t}$

so:

$$f_{\Delta|T}(\delta | T = t) = (1 - e^{-\lambda t})^{\delta} (e^{-\lambda t})^{1-\delta}$$

$\delta \in \{0, 1\}$

So:

$$f_{\Delta, T}(\delta, t) = f_{\Delta|T}(\delta|t)\, f_T(t)$$

$$= (1-e^{-\lambda t})^{\delta}\, e^{-\lambda t(1-\delta)}\, \mu e^{-\mu t}.$$

(b) Calculate $f_\Delta(\delta) = P(\Delta = \delta)$

$$P(\Delta = 1) = P(X \le T)$$

$$= \int_0^\infty P(X \le T \mid T = t)\, f_T(t)\, dt$$

$$= \int_0^\infty P(X \le t)\, \mu e^{-\mu t}\, dt$$

$$= \int_0^\infty (1 - e^{-\lambda t})\, \mu e^{-\mu t}\, dt$$

$$= 1 - \int_0^\infty \mu e^{-(\lambda+\mu)t}\, dt$$

$$= 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}.$$

$$P[\Delta = 0] = \frac{\mu}{\lambda+\mu}$$

Find: $f_{T|\Delta}(t|\delta)$

$$= \frac{f_{T,\Delta}(t,\delta)}{f_\Delta(\delta)}$$

$$= \frac{(1-e^{-\lambda t})^\delta e^{-\lambda t(1-\delta)} \mu e^{-\mu t}}{\left(\frac{\lambda}{\lambda+\mu}\right)^\delta \left(\frac{\mu}{\lambda+\mu}\right)^{1-\delta}}$$

so $f_{T|\Delta}(t|1)$

$$= \frac{(1-e^{-\lambda t})\mu e^{-\mu t}}{\lambda/_{\lambda+\mu}}.$$

$$f_{T|\Delta}(t|0) = \frac{e^{-\lambda t}\mu e^{-\mu t}}{\mu/_{\lambda+\mu}}.$$

$$\mathbb{E}[T \mid \Delta = 1]$$

$$= \frac{\lambda+\mu}{\lambda}\left[\int_0^\infty t\mu e^{-\mu t} - \int_0^\infty t\mu e^{-(\lambda+\mu)t}\right]$$

$$= \frac{\lambda+\mu}{\lambda}\left[\frac{1}{\mu} - \frac{\mu}{(\lambda+\mu)^2}\right]$$

$$= \frac{1}{\lambda}\left[\frac{\lambda+\mu}{\mu} - \frac{\mu}{\lambda+\mu}\right]$$

$$\mathbb{E}[T \mid \Delta = 0] = \int_0^\infty t(\lambda+\mu)e^{-(\lambda+\mu)t}\,dt$$

$$= \frac{1}{\lambda+\mu}$$

<u>To compare:</u>

$\mathbb{E}[T \mid \Delta = 1]$ to $\mathbb{E}[T \mid \Delta = 0]$

$$\frac{1}{\lambda}\left[\frac{\lambda+\mu}{\mu} - \frac{\mu}{\lambda+\mu}\right] - \frac{1}{\lambda}\cdot\frac{\lambda}{\mu+\lambda}$$

$$= \frac{1}{\lambda}\left[\frac{\lambda+\mu}{\mu} - \frac{\mu+\lambda}{\lambda+\mu}\right]$$

$$= \frac{1}{\lambda}\left[\frac{\lambda}{\mu} + \cancel{1} - \cancel{1}\right] = \frac{1}{\mu} > 0$$

$$\mathbb{E}[T \mid \Delta = 1] > \mathbb{E}[T \mid \Delta = 0]$$

conforms to intuition

because $\mathbb{E}[T \mid \Delta = 1] = \mathbb{E}[T \mid X \le T]$

and $\mathbb{E}[T \mid \Delta = 0] = \mathbb{E}[T \mid X > T]$

In the first case you are conditioning on a set where T tends to be large relative to X and in the second where T tends to be small relative to X