1, For $Y \sim$ Negative Binomial $(r, p)$,

$$P(Y=y) = \binom{r+y-1}{y} p^r (1-p)^y$$

$$= \frac{(r+y-1)(r+y-2)\cdots r}{y!} p^r (1-p)^y$$

$$= \frac{1}{y!} \times \frac{(r+y-1)\cdots r}{r^y} \times p^r \times (r(1-p))^y .$$

When $r(1-p) \to \lambda$, $r \to \infty$, $p \to 1$ we have

- $(r(1-p))^y \to \lambda^y$
- $p^r = \left(1 - \frac{1}{r}(r(1-p))\right)^r$ $\to e^{-\lambda}$
- $\frac{(r+y-1)\cdots r}{r^y} \to 1$ (as $r \to \infty$ and $y$ is fixed)

Hence

$$P(Y=y) \to e^{-\lambda} \frac{\lambda^y}{y!} \quad \forall\, y = 0, 1, 2, \ldots$$

2, For all $t > 0$, because $f(x,y) = f(-x,-y)$

$$P(X/Y > t) = P(X > Yt,\ Y > 0) + P(X < Yt,\ Y < 0)$$

$$= \int_{y=0}^{\infty}\int_{x=ty}^{\infty} f(x,y)\,dx\,dy + \int_{y=-\infty}^{0}\int_{x=-\infty}^{ty} f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{\infty}\int_{x=ty}^{\infty} f(x,y)\,dx\,dy + \int_{y=-\infty}^{0}\int_{x=-\infty}^{ty} f(-x,y)\,dx\,dy$$

$$= \int_{y=0}^{\infty}\int_{x=ty}^{\infty} f(x,y)\,dx\,dy + \int_{y=-\infty}^{0}\int_{x=-yt}^{\infty} f(x,y)\,dx\,dy$$

$$= P(X > |Y|t,\ Y > 0) + P(X > |Y|t,\ Y < 0)$$

$$= P(X/|Y| > t)$$

Similar we also have the same result for $t < 0$. Hence $X/Y$ and $X/|Y|$ have the same law.

[Alternative] $N \sim Poi(\lambda)$,

let $X_1, X_2, \dots, X_N$ be those eggs and

$$X_i = \begin{cases} 1 & \text{if } i\text{-th egg hatches} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow X_1, \dots, X_N \mid N \sim Ber(p)$.

We need to calculate distribution of $X = \sum_{i=1}^{N} X_i$.

For all $x$ being a natural number

$$P(X = x) = \sum_{n=0}^{\infty} P(N = n)\, P(X = x \mid N = n)$$

$$= \sum_{n=x}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{n}{x} p^x (1-p)^{n-x}$$

(because we need $x \geq n$)

$$= e^{-\lambda} \frac{\lambda^x}{x!} p^x \sum_{n=x}^{\infty} \frac{1}{(n-x)!} \lambda^{n-x} (1-p)^{n-x}$$

$$= e^{-\lambda} \frac{\lambda^x}{x!} p^x \cdot e^{\lambda(1-p)}$$

$$= e^{-\lambda p} \frac{(\lambda p)^x}{x!}$$

Hence, $X \sim Poi(\lambda p)$.

3, (i)

We have (by change of variable thm)

$$f_{W_2|W_1}(\omega_2|\omega_1) = \frac{1}{1-\omega_1} \frac{\Gamma(\alpha_2+\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \left(\frac{\omega_2}{1-\omega_1}\right)^{\alpha_2} \left(1-\frac{\omega_2}{1-\omega_1}\right)^{\alpha_3}$$

$\forall\, 0<\omega_2<1-\omega_1.$

Thus

$$f(\omega_1, \omega_2) = f(\omega_1)\, f(\omega_2|\omega_1)$$

$=$ what you need to prove

$\forall\ \omega_1, \omega_2 > 0,\ \omega_1+\omega_2 < 1.$ (S)

This is called the Dirichlet distribution with parameter (Dir) $(\alpha_1, \alpha_2, \alpha_3)$.

(ii) General case:

$$(\Theta_1, \Theta_2) \sim Dir(\alpha_1, \alpha_2, \alpha_3)$$ ((1, 1, 1) in the problem)

$$(N_1, N_2, N_3) | (\Theta_1 = \theta_1, \Theta_2 = \theta_2) \sim Multinom(n, (\theta_1, \theta_2, \theta_3))$$

$(\theta_3 = 1-\theta_1-\theta_2)$

By Bayes' rule

$$f(\theta_1, \theta_2 | N_1, N_2, N_3) = \frac{p(N_1, N_2, N_3 | \theta_1, \theta_2)\, f(\theta_1, \theta_2)}{f(\theta_1, \theta_2)}$$

$$\propto \theta_1^{N_1}\, \theta_2^{N_2}\, \theta_3^{N_3}\ \ \theta_1^{\alpha_1-1}\, \theta_2^{\alpha_2-1}\, \theta_3^{\alpha_3-1}$$

$$\sim Dir(N_1+\alpha_1, N_2+\alpha_2, N_3+\alpha_3).$$

Then

$$E(\Theta_1 | N_1, N_2, N_3) = \frac{N_1+\alpha_1}{n+\alpha_1+\alpha_2+\alpha_3}$$ ($\uparrow$ if $N_1 \uparrow$)

$$E(\Theta_2 | N_1, N_2, N_3) = \frac{N_2+\alpha_2}{n+\alpha_1+\alpha_2+\alpha_3}$$

[Alternative]

$$M_X(t) = M_X(-t) \quad \forall\ t \in [-h, h]$$

but

$$M_X(-t) = E e^{-Xt} = M_{-X}(t).$$

Hence,

$$M_X(t) = M_{-X}(t) \quad \forall\ t \in [-h, h],$$

which implies $X$ and $-X$ have the same distribution.

$\Rightarrow \quad \forall\ g$ odd

$$\begin{aligned} E g(X) &= \int_{\mathbb{R}} g(x)\, dF_X(x) \\ &= -\int_{\mathbb{R}} g(-x)\, dF_X(-x) && (\text{change } x \to -x) \\ &= -\int_{\mathbb{R}} g(x)\, F_{-X}(x) && (g \text{ odd}) \\ &= -E g(X) && (\text{as } X \sim -X) \end{aligned}$$

Hence, $Eg(X) = 0$.