STA286 Lecture 08

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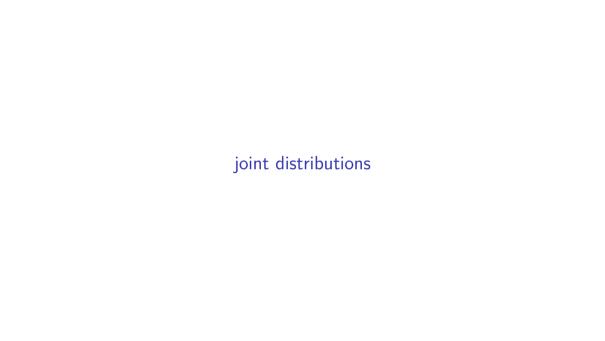
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 X_1 and X_2 are not the same functions. But the have the same p.m.f.:

$$p(x) = \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{1-x}, \ x \in \{0, 1\}.$$

We say X_1 and X_2 are identically distributed.



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Main interest is in statements like:

$$P(X = x, Y = y)$$
 or $P(a < X < b, c < Y < d)$

where the "comma" notation is a compact way of writing, say:

$$P(\{X=x\}\cap\{Y=y\})$$



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They also have cdfs:

$$F(x, y) = P(X \leqslant x, Y \leqslant y)$$

but we won't focus much on these.

a joint pmf

A gas distribution company has pipes with diameters 1, 1.5, and 1.75 inches. The pipes are used at pressures of 2, 1, and 0.5 pounds per square inch.

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X and Y might have the following joint probability mass function:

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e.g. the probability that a randomly selected pipe has diameter X=1.5 and pressure Y=0.5 is:

$$P(X = 1.5, Y = 0.5) = 0.1$$

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A joint density also must be non-negative and integrate to 1 over all \mathbb{R}^2 .

Two electronic components fail at times X and Y according to the following joint density (measured in years):

$$f(x,y) \begin{cases} 2e^{-x-2y} & : x > 0, y > 0 \\ 0 & : \text{ otherwise} \end{cases}$$

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Is this actually a valid joint density? It is non-negative, and:

$$\iint_{\mathbb{R}^{2}} f(x, y) \, dxdy = \int_{0}^{\infty} \int_{0}^{\infty} 2e^{-x-2y} \, dxdy = \int_{0}^{\infty} e^{-x} \, dx \int_{0}^{\infty} 2e^{-2y} \, dy = 1$$

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marginal distributions

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For example, look again at the joint distribution for diameter X and pressure Y of the randomly selected pipe:

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A statement about X alone could be something like P(X=1), which is just 0.075 + 0.11 + 0.16 = 0.345

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Here are the "marginals" for both X and Y in the gas example:

		X		
Y	1	1.5	1.75	Marginal
0.5	0.075	0.100	0.150	0.325
1	0.110	0.080	0.090	0.280
2	0.160	0.140	0.095	0.395
Marginal	0.345	0.320	0.335	1.000

Reconsider the two electronic components example:

$$f(x,y) \begin{cases} 2e^{-x-2y} & : x > 0, y > 0 \\ 0 & : \text{ otherwise} \end{cases}$$

A statement about, say, Y alone might be P(Y > 1), which would be the integral on the entire half plane y > 1:

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But in the end, the calculation only ever involved $2e^{-2y}$ once x was "integrated out."

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This is a valid density. Let X and Y have this joint density.

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conditional pmf and conditional density

Given the joint pmf p(x, y) for X and Y, the conditional pmf for X given Y = y is:

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That's because X and Y have a special relationship...to be revisited.

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Unlike marginal densities, conditional densities can sometimes be visualized. They are just "slices" of the joint density.