STA286 Lecture 14

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Path to $E(X)$ is easiest through the mgf:

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = (q + pe^t)^n$$

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Theorem: Let X be the sum of n independent Bernoulli(p) random variables $X_1, \ldots, X_n$. Then $X \sim$ Binomial(n, p).

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Proof:

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So:

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and

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = np(1-p)$$

$$SD(X) = \sqrt{np(1-p)}$$

number of Bernoulli trials until the first 1—Geometric(p)

We've done most of the work on this distribution.

$$p(y) = (1-p)^{y-1}p, \quad y \in \{1, 2, 3, \ldots\}$$

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Variance is inevitably tedious. $E(Y^2) = \frac{d^2}{dt^2} M_Y(t)\Big|_{t=0} = \frac{2-p}{p^2}$, resulting in $\text{Var}(Y) = \frac{q}{p^2}$

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We say $Y \sim$ Geometric(p), due to the geometric rate of decay in the pmf.

cdf and the "reliability function" for a Geometric(p)

The cdf for $Y \sim \text{Geometric}(p)$ comes in handy sometimes.

$$F_Y(y) = \begin{cases} 0 & : y < 1 \\ 1 - (1-p)^k & : k \leqslant y < k+1 \end{cases}$$

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The *reliability* or *survival* function of a random variable is defined as

$$R(x) = P(X > x) = 1 - P(X \leqslant x) = 1 - F(x).$$

For Y in this case:

$$R_Y(y) = \begin{cases} 1 & : y \leqslant 1 \\ (1-p)^k & : k < y \leqslant k+1 \end{cases}$$

the memorylessess of a Geometric(p) distribution

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- independence (of trials)

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The memorylessness of the Bernoulli process can be seen through a property of the Geometric(p) distribution. Suppose $Y \sim$ Geometric(p).

$$
\begin{aligned}
P(Y > j + k | Y > j) &= \frac{P(Y > j + k, Y > j)}{P(Y > j)} \\
&= \frac{P(Y > j + k)}{P(Y > j)} = \frac{(1-p)^{j+k}}{(1-p)^j} = (1-p)^k = P(Y > k)
\end{aligned}
$$

number of Bernoulli trials until the r^{th} 1

Denote by W the number of trials in a Bernoulli(p) process until the r^{th} 1 occurs.

We'll say W has a "Negative Binomial" distribution, or $W \sim \text{NegBin}(r, p)$.

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It will always be 3 1s out of 10 trials, follow by a 1. So the probability is:

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In general: $P(W = k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r, \quad k \in \{r, r+1, r+2, \ldots\}$

NegBin(r, p)

The name comes from this version of the Binomial theorem with negative exponent:

$$(1-a)^{-r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} a^{k-r}$$

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$$\begin{aligned} M_W(t) &= \sum_{k=r}^{\infty} e^{tk} \binom{k-1}{r-1} (1-p)^{k-r} p^r \\ &= (pe^t)^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} (qe^t)^{k-r} \\ &= \left(\frac{pe^t}{1-qe^t} \right)^r \end{aligned}$$

NegBin(r, p)

Theorem: Let W be the sum of r independent random variables $Y_1, \ldots, Y_r$ with Geometric(p) distributions. Then $W \sim$ NegBin(r, p).

Proof: $M_{Y_1+\cdots+Y_r}(t) = M_{Y_1}(t) \cdots M_{Y_r}(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$

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$$\mathrm{Var}(W) = \mathrm{Var}(Y_1) + \cdots + \mathrm{Var}(Y_r) = \frac{rq}{p^2}$$

the hypergeometric distributions

Reminder: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways to choose k items out of n, without replcement.

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Quality control methods often deal with this sort of situation:

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Quality control methods often deal with this sort of situation:

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X could take on integer values between 0 and 5 inclusive, with probabilities:

$$P(X = x) = \frac{\binom{5}{x}\binom{95}{10-x}}{\binom{100}{10}} = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

N is the population size. n is the sample size. k is the number of "defective" (in this example.)

and we say X has a Hypergeometric distribution with parameters N, n, and k.

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Hypergeometric is a "sampling without replacement" version of the Binomial. If N and n are both large, it turns out:

$$\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \approx \binom{n}{x}\left(\frac{k}{N}\right)^x\left(1-\frac{k}{N}\right)^{n-x}$$

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All calculations involving these distributions are tedious. It can be shown (no easy way):

$$E(X) = n\frac{k}{N} \qquad \mathrm{Var}(X) = n\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$