MODELING THE SWING OF DOUBLE PENDULUM

Presented by:

Nabeel Ghazali 40195041

Wasay Ejaz 40185576

Hasan mahmud 40166145

Sulaiman Conain 40170570

- A mass suspended freely by a string or a rod
- Two force:
  - Gravitational force (Downwards)
  - Tension in the string (Upwards)
- The tangential component is used to derive the equation of motion

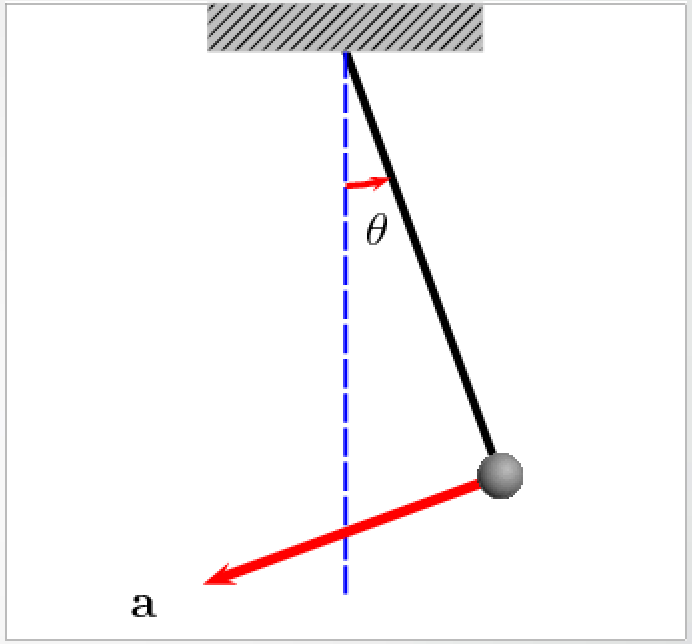
$$\tau = I\alpha \qquad \Rightarrow \qquad -mg\sin\theta\, L = mL^2\frac{d^2\theta}{dt^2}$$

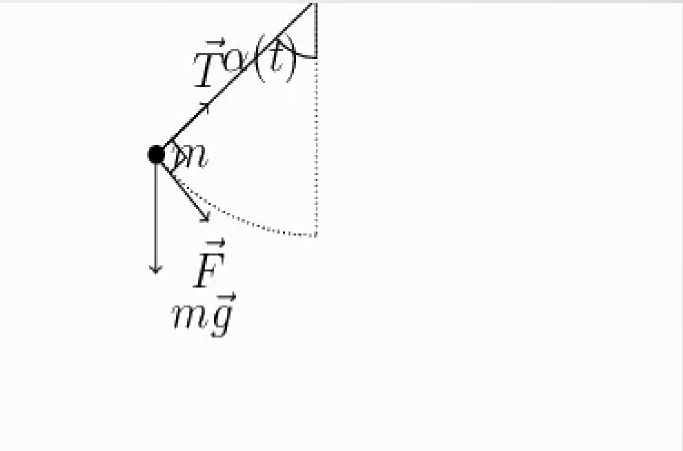
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

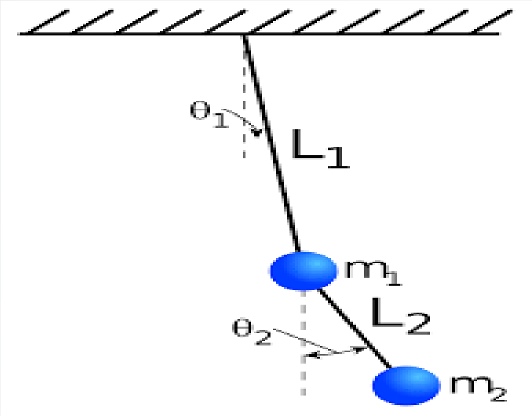
- Rearranging the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

Non-Homogeneous Equation



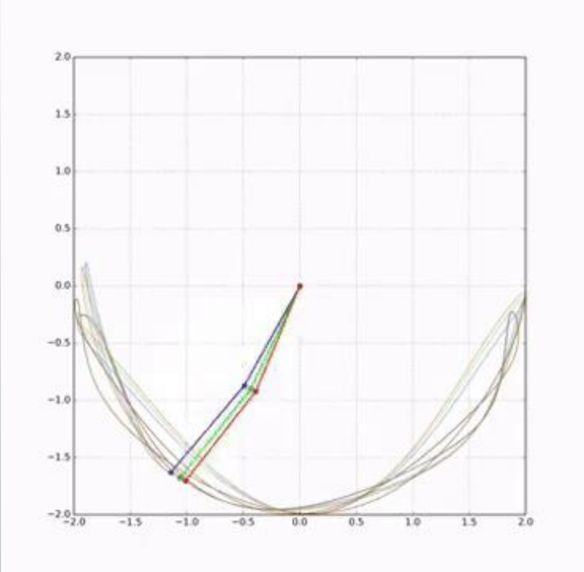




- Difference between a single and double pendulum is masses
- Single pendulum → One mass
- Double pendulum → two masses

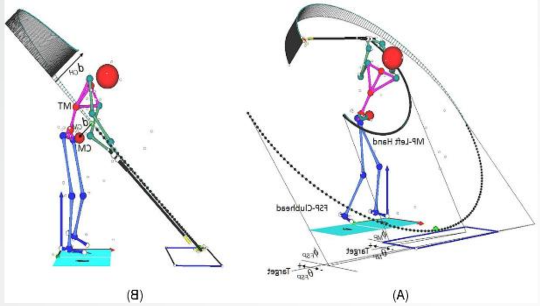
However, this is an oversimplification.

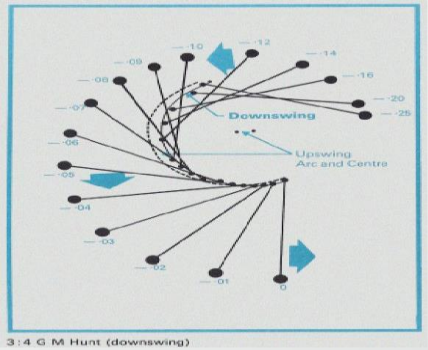
- The behavior by this change of masses has a huge impact
- Double pendulum is an accurate example of chaos theory also know as the butterfly effect
- Small variation such as 1/100th change in the initial angle changes the motion of the two same pendulums



The example of the double pendulums can also be seen in real life

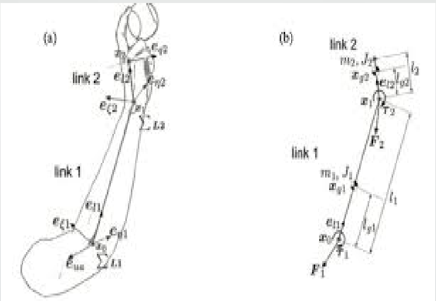
- Golf swing

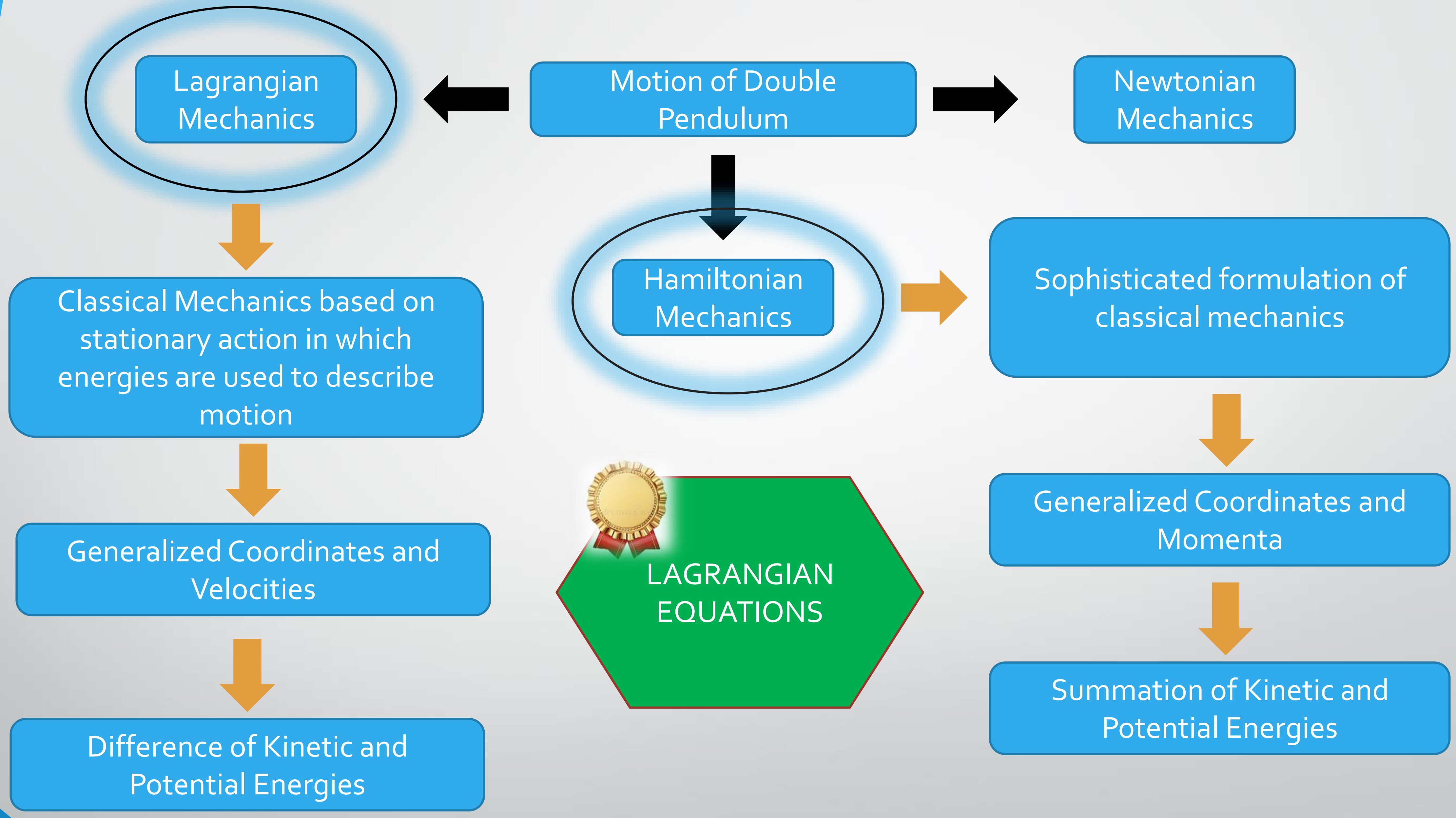




- Tennis serve

- Baseball pitching




Lagrangian Mechanics
Motion of Double Pendulum
Newtonian Mechanics
Hamiltonian Mechanics
Classical Mechanics based on stationary action in which energies are used to describe motion
Sophisticated formulation of classical mechanics
Generalized Coordinates and Velocities
LAGRANGIAN EQUATIONS
Generalized Coordinates and Momenta
Difference of Kinetic and Potential Energies
Summation of Kinetic and Potential Energies

Model Development

- Best explained with the help of two coupled ordinary second order differential equations.
- Study carried out by two different methods

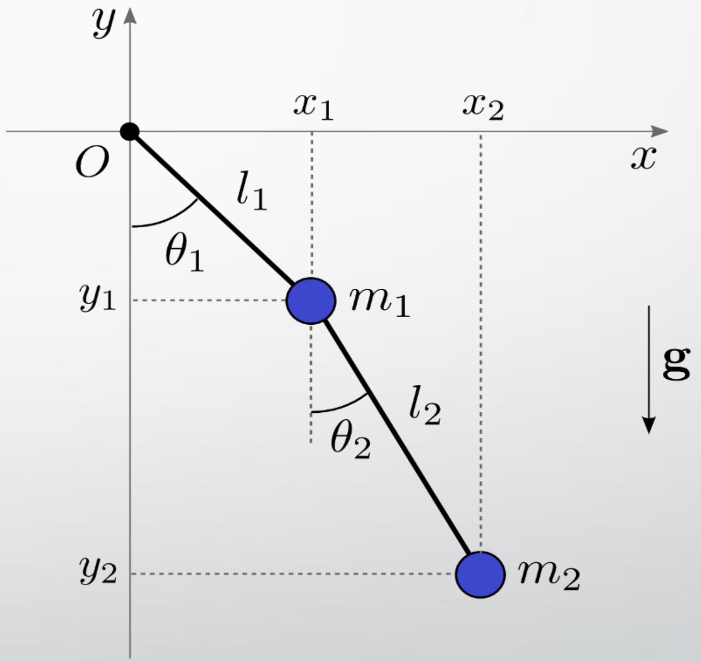
Lagrangian Mechanics

Hamiltonian Mechanics

Model Development - Lagrangian

- Lagrangian mechanics is a formulation of classical mechanics that is based on the principle of stationary action.
- It is described in terms of the generalized coordinates (θ_1 and θ_2) generalized velocities (θ'_1 and θ'_2).
- Lagrangian = T - V.
- Kinetic Energy => $T = \frac{1}{2}m1(\dot{x}1^2+\dot{y}1^2) + \frac{1}{2}m2(\dot{x}2^2+\dot{y}2^2)$
- Potential Energy => $V = \text{m1gy1} + \text{m2gy2}$
- Results from Euler - Lagrangian Equation gives => $\frac{d}{dt}(\frac{\partial L}{\partial \grave{q} i}) - \frac{\partial L}{\partial qi} = 0$

where, qi is nothing but θi ; i =1,2



Model Development - Hamiltonian

- It is a mathematically sophisticated formulation of classical mechanics.
- The system is determined by the generalized coordinates (θ1, θ2) and generalized momenta (p1, p2)
- It is the sum of Kinetic Energy and Potential Energy ; H = T + V

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

Model Development - Hamiltonian

- Hamiltonian of the system is given by :

$$H = \sum_{i=1}^{2} \dot{\theta}_i p_{\theta_i} - L$$

$$H = \frac{m_2 l_2^2 p_{\theta_1}^2 + (m_1 + m_2) l_1^2 p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2 m_2 l_1^2 l_2^2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]} - (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

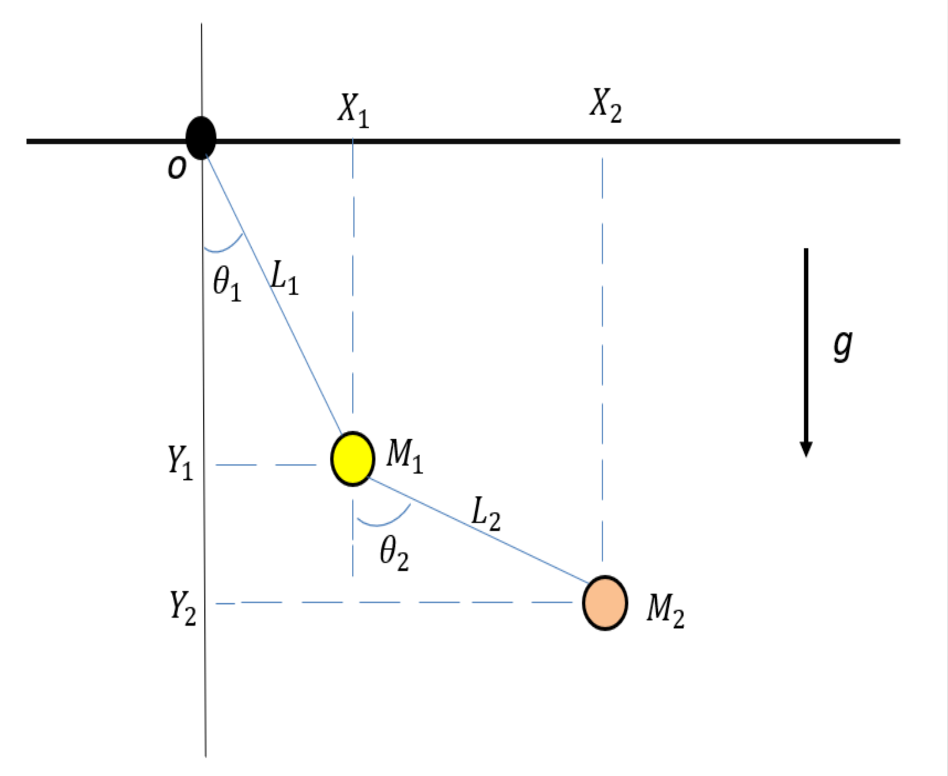
- From Hamiltonian, we can obtain equation of motion of system which is equivalent to Euler Lagrange equation.

- Hamiltonian equation gives first order differential equation.

- We are focusing on Lagrangian equation for analysis as the Hamiltonian gives more complicated expressions that are much complex to analyze.

Euler Lagrange Equations Derivations
&
Simplification of Solution

- formed by attaching a pendulum directly to another one.
- comprises a bob and a massless rigid rod that can only travel vertically.
- The pivot of the first pendulum is fixed to a point O
- Consider a system in which the massless strings L_1 and L_2 connected two pendulums of mass M_1 and M_2.
- θ_1 and θ_2 as the angles the first and second rods make with the vertical direction, respectively.



From the figure , the positions of the rods are given by:

- $X_1 = L_1 \sin\theta_1$
- $Y_1 = - L_1 \cos\theta_1$
- $X_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$
- $Y_2 = - L_1 \cos\theta_1 - L_2 \cos\theta_2$

Differentiating w.r.t time

we obtain the velocities

- $X_1{}' = L_1\theta_1{}' \cos\theta_1$
- $Y_1{}' = L_1\,\theta_1{}' \sin\theta_1$
- $X_2{}' = L_1\theta_1{}' \cos\theta_1 + L_2\theta_2{}' \cos\theta_2$
- $Y_2{}' = L_1\,\theta_1{}' \sin\theta_1 + L_2\,\theta_2{}' \sin\theta_2$

For a double pendulum, the Lagrangian is given by L=T - P, where T and P are kinetic energy and potential energy, respectively.

The kinetic energy T is given by:

$$T = \frac{1}{2}\ M_1 {V_1}^2 + \frac{1}{2}\ M_2 {V_2}^2$$

$$= \frac{1}{2}\ M_1\ ({X_1}'^{\,2} + {Y_1}'^{\,2}) + \frac{1}{2}\ M_2\ ({X_2}'^{\,2} + {Y_2}'^{\,2})$$

$$= \frac{1}{2}\ M_1 {L_1}^2\, {\theta_1}'^{\,2} + \frac{1}{2} M_2\ [{L_1}^2\, {\theta_1}'^{\,2} + \ {L_2}^2\, {\theta_2}'^{\,2} + 2 L_1 L_2\ \theta_1'\theta_2'\ \cos(\theta_1 - \ \theta_2\ )]$$

The potential energy of the system is given by,

$$P = M_1 g Y_1 + M_2 g Y_2$$

$$= -\ (M_1 + \ M_2)\mathrm{g} L_1 \cos\theta_1 - M_2 \mathrm{g} L_2 \cos\theta_2$$

The Lagrangian function is defined as,

$$\mathrm{L} = T - P$$

$$= \frac{1}{2}\ (M_1 + M_2) {L_1}^2\, {\theta_1}'^{\,2} + \frac{1}{2}\ M_2 {L_2}^2\, {\theta_2}'^{\,2} + M_2 L_1 L_2\ \theta_1'\theta_2'\ \cos(\theta_1 - \ \theta_2\ )] + (M_1 + \ M_2)\mathrm{g} L_1 \cos\theta_1\ + M_2 \mathrm{g} L_2 \cos\theta_2$$

The Euler-Lagrange equation in terms of generalized coordinates can be expressed as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta'_i}\right) - \frac{\partial L}{\partial \theta'_i} = 0$$

Lagrange's differential equation for θ_1 is,

$$[(M_1 + M_2)L_1 \frac{d^2\theta_1}{dt^2} + M_2 L_2 \frac{d^2\theta_2}{dt^2} \cos(\theta_1 - \theta_2)] + M_2 L_2 (\frac{d\theta_2}{dt})^2 \sin(\theta_1 - \theta_2) + g(M_1 + M_2 \sin\theta_1] = 0$$

Similarly, Lagrange's differential equation for θ_2 is,

$$[M_2 \frac{d^2\theta_2}{dt^2} + M_2 L_1 \frac{d^2\theta_1}{dt^2} \cos(\theta_1 - \theta_2)] - M_2 L_1 (\frac{d\theta_1}{dt})^2 \sin(\theta_1 - \theta_2) + M_2 g \sin\theta_2] = 0$$

Simplification of solution

- To simplify the equations to run in the, MATLAB we can use these below variables,

- $u_1 = \theta_1$(t)
- $u_2 = \theta_1{}'$(t)
- $v_1 = \theta_2$(t)
- $v_2 = \theta_2{}'$(t)
- $\frac{du_1}{dt}$ =u_2(t)
- $\frac{du_1}{dt}$ =v_2(t)

$$[(M_1 + M_2)L_1 \frac{du_2}{dt} + M_2 L_2 \frac{dv_2}{dt} \cos(u_1 - v_1)] + M_2 L_2 (v_2)^2 \sin(u_1 - v_1) + g(M_1 + M_2 \sin u_1] = 0$$

And $[M_2 \frac{dv_2}{dt} + M_2 L_1 \frac{du_2}{dt} \cos(u_1 - v_1)] - M_2 L_1 (u_2)^2 \sin(u_1 - v_1) + M_2 g \sin v_2] = 0$

Using the substitution:

- a= $(M_1 + M_2)L_1$
- b= $M_2 L_2 \cos(u_1 - v_1)$
- c= $M_2 L_1 \cos(u_1 - v_1)$
- d= $M_2 L_2$
- e= - $M_2 L_2 (v_2)^2 \sin(u_1 - v_1)$ - $g(M_1 + M_2) \sin u_1$
- f= $M_2 L_1 (u_2)^2 \sin(u_1 - v_1)$ - $M_2 g \sin v_2$

We can write the system as,

- $a\frac{du_2}{dt} + b\frac{dv_2}{dt} = e$
- $c\frac{du_2}{dt} + d\frac{dv_2}{dt} = f$

Above two Equations can be solved for $\frac{du_2}{dt}$ and $\frac{dv_2}{dt}$ by,

$\frac{du_2}{dt} = \frac{ed - bf}{ab - cd}$ and, $\frac{dv_2}{dt} = \frac{af - ce}{ab - cb}$

Now, these equations are in the form that MATLAB can use.

Implementation

We have used MATLAB 2020a for implementing our project.

we have used the following pendulum parameters,

$L_1 = 2$, $L_2 = 4$

$m_1 = 4$, $m_2 = 2$

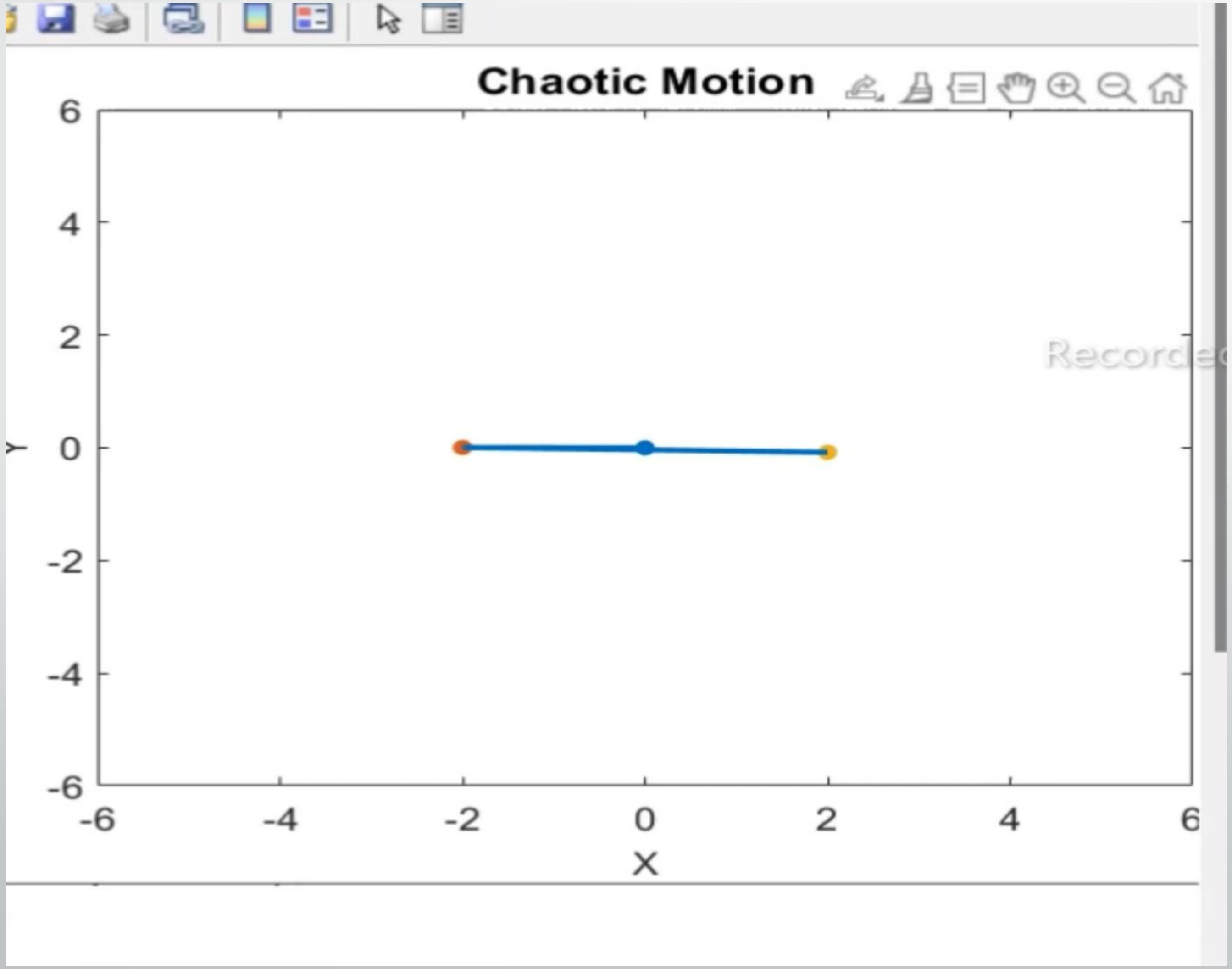
$g = 9.8$

The initial conditions is as follows,

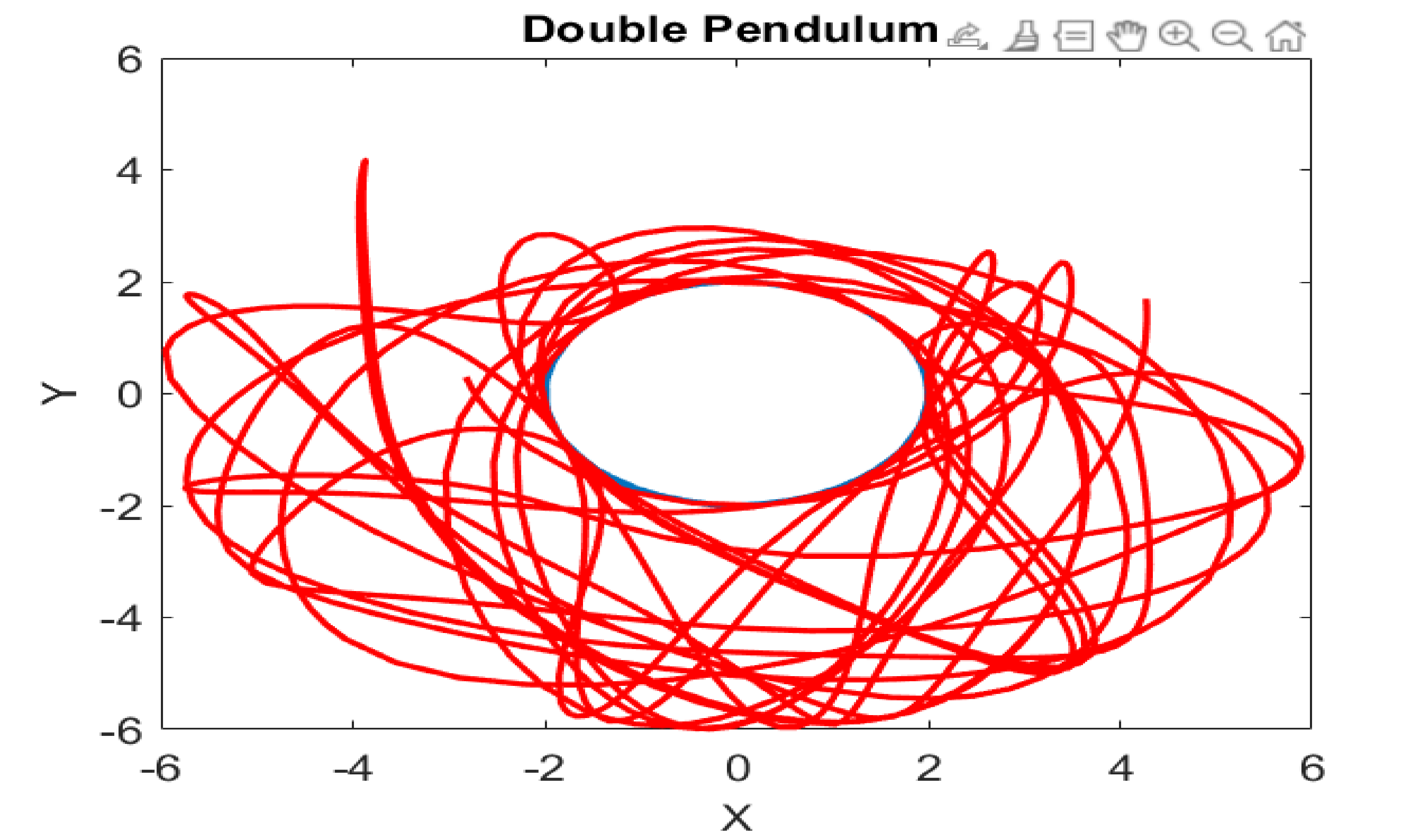
$u_1 = 1.5$ $u_2(0) = 0.0 v_1(0) = 3.0$ $v_2(0) = 0.0$

```
function yprime = SimplifiedEquations(t, y)
l1=2;
l2=4;
m1=4;
m2=2;
g=9.8;
y_prime=zeros (4,1);
a = (m1+m2)*l1 ;
b = m2*l2*cos(y(1)-y(3));
c = m2*l1*cos(y(1)-y(3));
d = m2*l2 ;
e = -m2*l2*y(4)* y(4)*sin(y(1)-y(3))-g*(m1+m2)*sin(y(1)) ;
f =m2*l1*y(2)*y(2)*sin(y(1)-y(3))-m2*g*sin(y(3)) ;
yprime(1) = y(2);
yprime(3)= y(4) ;
%Final Equations
yprime(2)= (e*d-b*f)/(a*d-c*b) ;
yprime(4)= (a*f-c*e)/(a*d-c*b) ;
yprime=yprime';
end
```

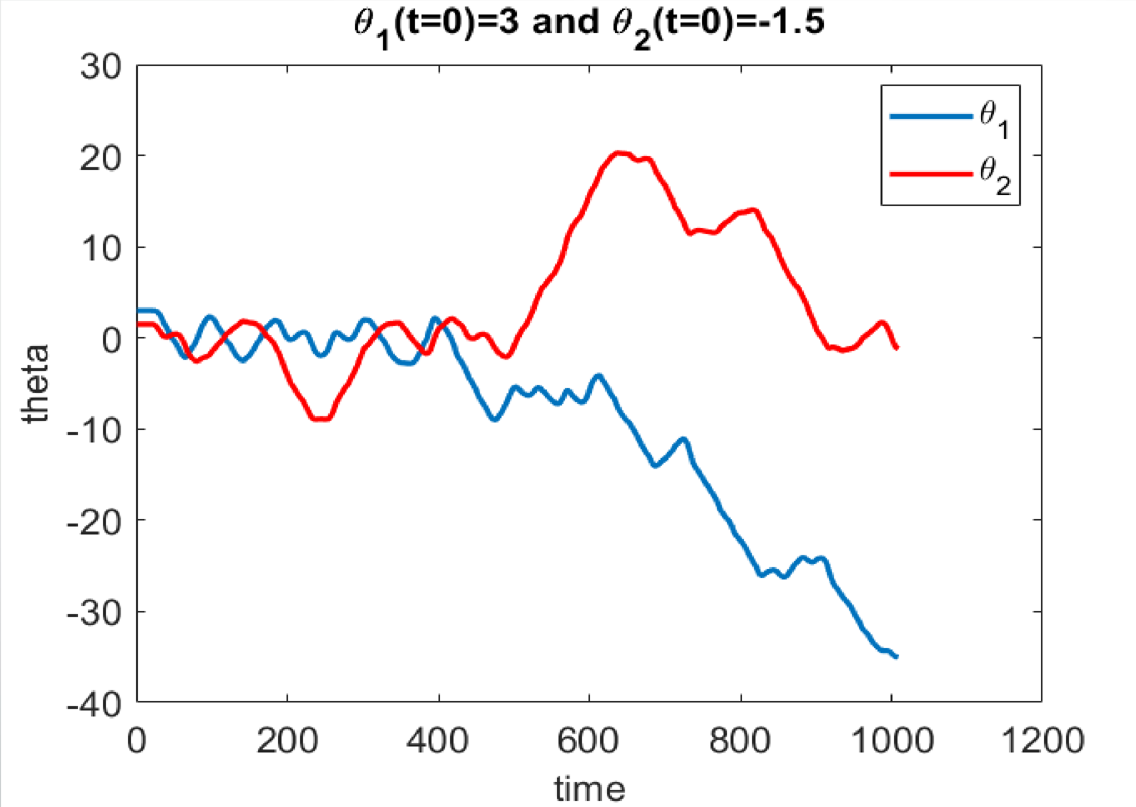
- When have also implemented the animation, chaotic behaviour of the double pendulum and also the plot of theta1 vs theta2 at initial values 3 and 1.5 respectively.



- chaotic behavior of the double pendulum.

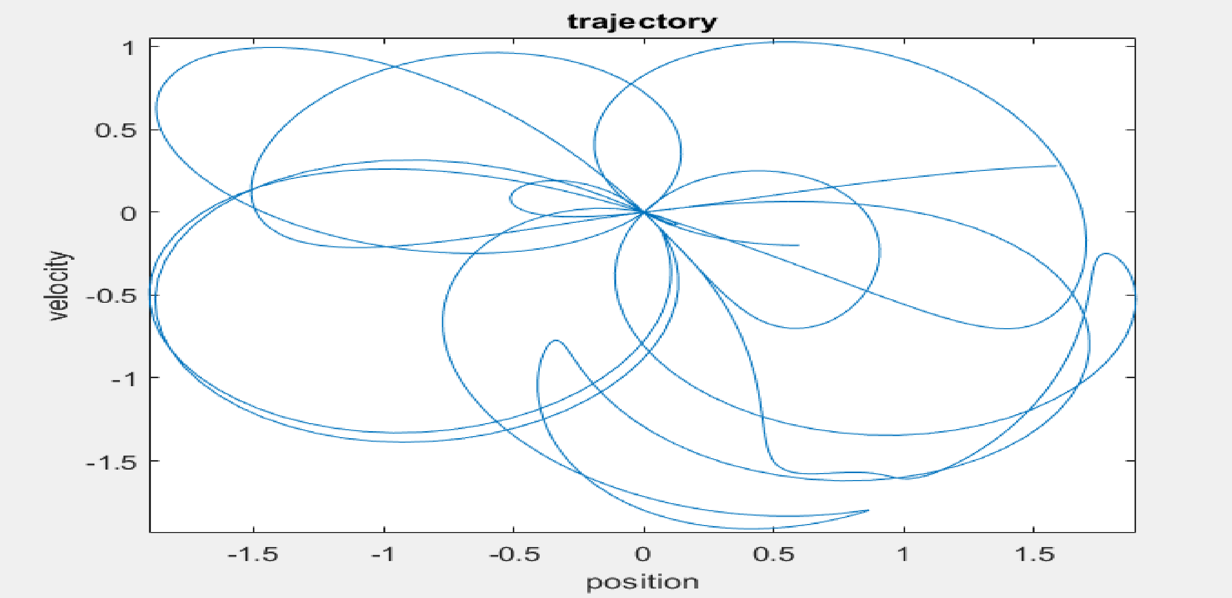


- Initial angles are theta1 = 3.0 and theta2 = 1.5
- For the initial few seconds the behaviour looks the same.
- After some time passes we can see the drastic changes.



PointCare

- PointCare.m gives us insights into the dynamics of our project.
- To construct this two-dimensional PointCare we take theta2 = 0 (whenever the pendulum passes vertical state). so that we get theta1 plot.



Hamilton

- Using a Hamiltonian with low energy conditions we can easily calculate the energy in joules.
- Example: yo[θ_1, θ_2, $P\theta_1$, $P\theta_2$] = [0.2,0.2828,0,0], the energy = 28.620.
- at this low energy we can easily predict the behaviour of the system.

Hamiltaion.m

```
function energy = Hamiltaion(y0)
t1 = y0(1);
t2 = y0(2);
v1 = y0(3);
v2 = y0(4);
g = 9.8;
energy = abs(((3*(v2^2)-2*v1*v2*cos(t1-t2))/(2+2*(sin(t1-t2)^2)))- 2*g*cos(t1)-g*cos(t2));
end
```

Workspace

Name	Value
energy	28.6200
s	*1x1 struct*
t	*1x1001 double*
T	10
x	*1001x4 double*
x1	*1001x1 double*
x2	*1001x1 double*
y0	[0.2000,0.2828,0,0]
y1	*1001x1 double*
y2	*1001x1 double*

EULER'S METHOD

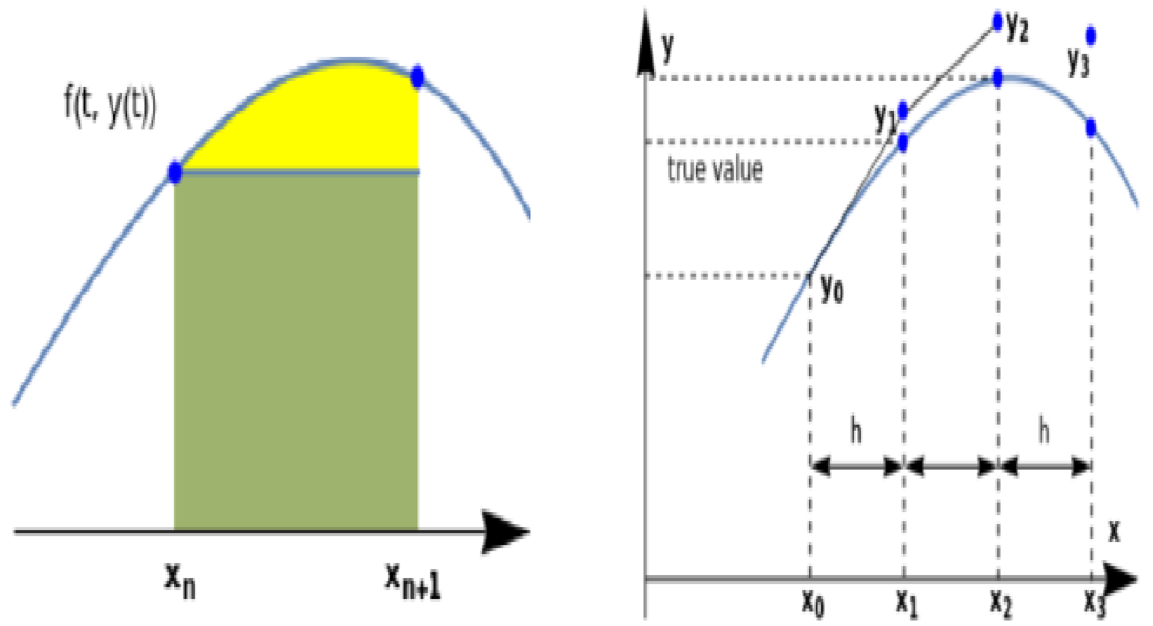
- The slope at the beginning of the interval is taken as an approximation of the average slope over the whole interval, this approximation is called Euler's method.

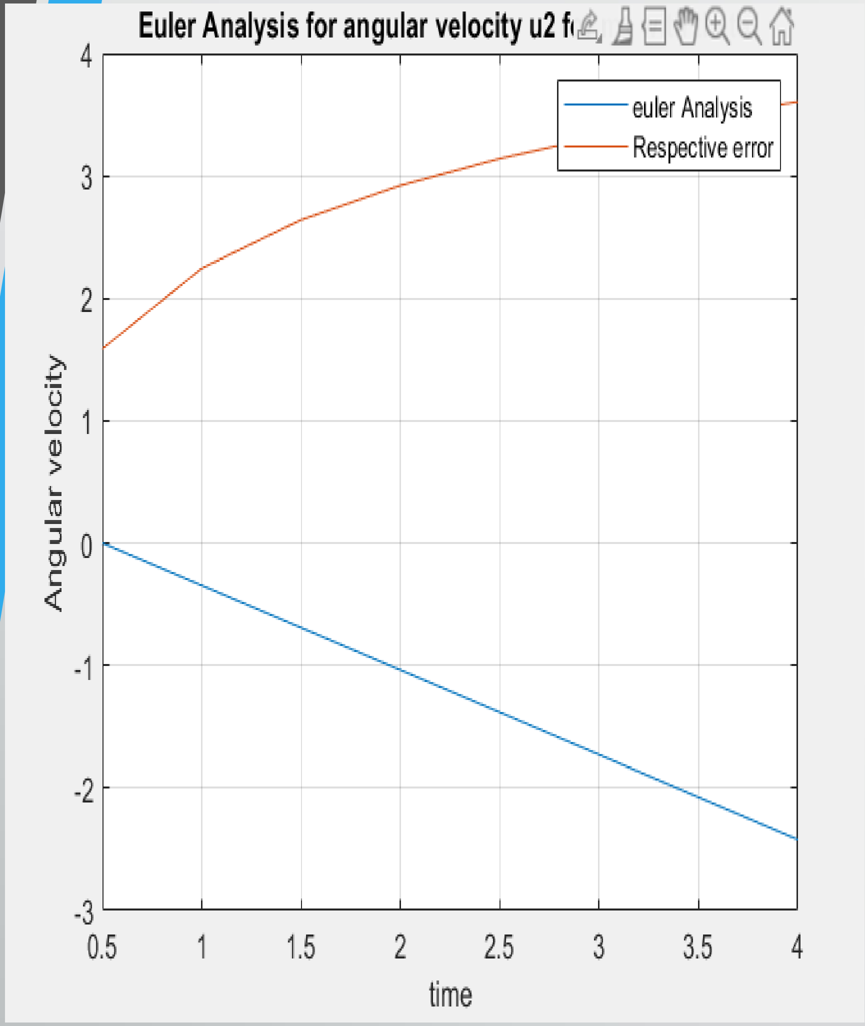
y'=f(x, y) , y(xo)=yo

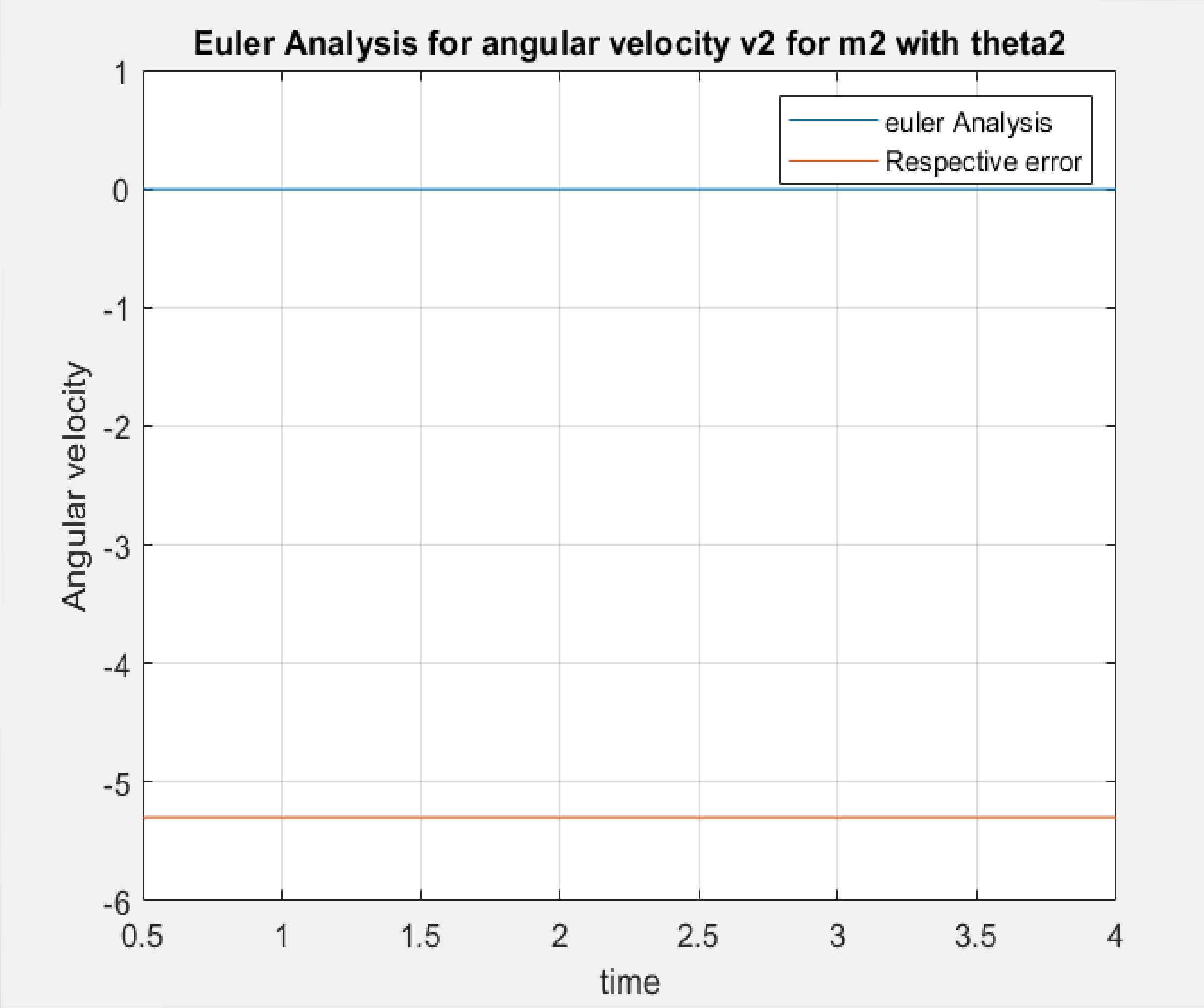
- Euler method or the tangent line method is used for solving numerically the initial value problem.

yn+1 = yn + h f(xn, yn).

$x_n = a + nh$ for $n=0,1,2,\dots,m$; where $h = \frac{b-a}{m}$




Euler Analysis for angular velocity u2 f
euler Analysis
Respective error
Angular velocity
time
4
3
2
1
0
-1
-2
-3
0.5
1
1.5
2
2.5
3
3.5
4


Euler Analysis for angular velocity v2 for m2 with theta2
euler Analysis
Respective error
Angular velocity
time
1
0
-1
-2
-3
-4
-5
-6
0.5
1
1.5
2
2.5
3
3.5
4

- Euler's analysis for angular velocity v2

LE= log(error2)

LH= log(x) x=0.5:h:4

Dr= LH (end) – LH (1)

Nr= LE (:, end) – LE (:,1)

Slope = Nr / Dr

- Euler analysis for angular velocity u2

LE= log(error1)

LH= log(x) x=0.5:h:4

Dr= LH (end) – LH (1)

Nr= LE (:, end) – LE (:,1)

Slope = Nr / Dr

Command Window

```
LE =

    1.5892    2.2465    2.6397    2.9212    3.1406    3.3204    3.4728    3.6049

>> LH= log(x)

LH =

   -0.6931         0    0.4055    0.6931    0.9163    1.0986    1.2528    1.3863

>> Dr= LH(end)-LH(1)

Dr =

    2.0794

>> Nr= LE(:,end)-LE(:,1)

Nr =

    2.0157

>> Slope = Nr / Dr

Slope =

    0.9694
```

Command Window

```
LE =

   -5.2983   -5.2983   -5.2983   -5.2983   -5.2983   -5.2983   -5.2983   -5.2983

>> LH= log(x)

LH =

   -0.6931         0    0.4055    0.6931    0.9163    1.0986    1.2528    1.3863

>> Dr= LH(end)-LH(1)

Dr =

    2.0794

>> Nr= LE(:,end)-LE(:,1)

Nr =

   2.6432e-06

>> Slope = Nr / Dr

Slope =

   1.2711e-06
```