An Analytical Solution to the Extreme Skydiver Problem

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On August 16, 1960, at 7:12 a.m., United States Air Force Captain Joseph W. Kittinger Jr. looked out from his balloon gondola at a height of over 19 miles and… jumped.[1] Kittinger's altitude and free-fall speed record lasted until Felix Baumgartner's jump from a height of nearly 24 miles on Oct. 14, 2012.[2] Baumgartner broke several records, including for vertical speed, where he was the first human to break the sound barrier without assisted power.

In most introductory physics courses, skydiving dynamics are briefly discussed, if at all, when velocity-dependent drag forces are introduced. In these cases, jump altitudes are low enough to assume a constant air density, resulting in a simple differential equation solvable by separation of variables.

The extreme skydiver problem presents the additional difficulty of air density changing with height and thus provides a good example of "complexity creep." In the skydiver case, the problem allows students to see how a typical physics model increases in complexity under zero, constant, and variable drag conditions. Of course, each successive model must reduce to the previous version under the appropriate conditions.

With computer modeling and tools becoming ubiquitous, it is common for students and instructors to seek software solutions instead of attempting an analytical one. In the cases of interest here, past high-altitude skydiver studies have mostly relied on numerical methods, including spreadsheet applications.[3-7] I have to admit, when I first read about Kittinger's jump a number of years ago, I went straight to commercial software to model his jump. However, analytical solutions are like pictures, that is, an equation is worth 1000 lines of code and much more appealing to look at. Therefore, after Baumgartner's well-documented jump, I decided to reexamine the problem.

In this article, analytical solutions are derived for velocity-dependent drag forces with a variable atmospheric density. The results are compared with Baumgartner's recorded jump data and agreement is quite good. Although some of the mathematical methods used here are second-year college subjects, instructors at all levels can pick and choose how to cover the mathematics or just simply discuss the physics and the results as they see fit.

Newton's second law: Low-altitude skydiver

The typical approach to treating the low-altitude skydiver problem is to add a velocity-dependent drag force to the standard free-fall problem. In this case, Newton's second law gives the resultant acceleration as

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = -g + bv^n, \tag{1}$$

where y is the vertical displacement, v is the velocity, and g = 9.81 m/s^2 is the acceleration due to gravity. In Eq. (1) the drag term is given as bv^n, where b is a constant, and n is typically given a value of 1 or 2. When terminal velocity v_t is reached, $dv/dt = 0$, giving $b = g/v_t^n$.

In this study, we will set $n = 2$ as it is the accepted value more applicable to fast-moving larger objects. The case for n = 1 and other related topics are discussed in the appendix.[8]

When $n = 2$, the drag force is customarily given the specific form

$$F_\mathrm{d} = \frac{1}{2}\rho_0 ADv^2, \tag{2}$$

where ρ_0 = 1.23 kg/m^3 is the air density at sea level, A the cross-sectional area of the falling object, and D the dimensionless drag coefficient whose value is determined by the shape of the object and the air flow around it. At terminal velocity, $F_\mathrm{d} = mg$, giving the terminal velocity from Eq. (2) as

$$v_\mathrm{t} = \sqrt{\frac{2mg}{D\rho_0 A}}. \tag{3}$$

Using the definitions of Eqs. (2) and (3) in Eq. (1) gives the solutions

$$v = -v_\mathrm{t}\tanh(gt/v_\mathrm{t} + c_1) \tag{4}$$

$$y = -(v_\mathrm{t}^2/g)\ln[\cosh(gt/v_\mathrm{t} + c_1)] + c_2, \tag{5}$$

where c_1 and c_2 are the constants of integration chosen to give the correct initial velocity and altitude, respectively. In the limit $b \rightarrow 0$, Eqs. (4) and (5) revert to the simple free-fall solutions

$$v = -gt + v_0 \tag{6}$$

$$y = -gt^2/2 + v_0 t + y_0, \tag{7}$$

where v_0 and y_0 are the initial velocity and altitude.

Newton's second law: High-altitude skydiver

In the extreme skydiver problem, air density variation with height can be modeled by placing an altitude-dependent term in the drag force. One of the simplest expressions is given by the barometric formula derived from kinetic theory and the ideal gas law,[9]

$$\rho = \rho_0 e^{-mgy/kT}, \tag{8}$$

where m = 28.95 g/mol is the mean molecular mass of the atmosphere, $k = 1.38\times10^{-23}$ J/K is Boltzmann's constant, and T is the assumed atmospheric mean temperature. Since $\lambda = mg/kT$ depends on temperature and temperature varies with altitude, no single value of λ accurately captures the entire range of interest, a point we will come back to later. Replacing ρ_0 in Eq. (2) with ρ of Eq. (8) gives the acceleration as

$$\frac{dv}{dt} = -g + be^{-\lambda y}v^2. \tag{9}$$

Using the identity, $dv/dt = (dy/dt)(dv/dy) = v\,(dv/dy)$, and setting $u = v^2$ in Eq. (9) results in the expression

$$\frac{du}{dy} - 2be^{-\lambda y}u = -2g. \tag{10}$$

Equation (10) has the form

$$\frac{du}{dy} + P(y)u = Q(y) \tag{11}$$

with the solution, obtained by the integration factor method,

$$u = \frac{\int Q(y)e^{\int P(y)dy}dy}{e^{\int P(y)dy}}. \tag{12}$$

Using the specific forms of $P(y)$ and $Q(y)$ from Eq. (10) in Eq. (12) gives

$$u = (-2g)\exp[(-2b/\lambda)\exp(-\lambda y)]\int\exp[(2b/\lambda)\exp(-\lambda y)]dy. \tag{13}$$

The integral in Eq. (13) can't be put in terms of elementary functions; however, by making the substitution $z = (2b/\lambda)\exp(-\lambda y)$, the integral can be transformed to the exponential integral,

$$Ei(z) = \int_{-\infty}^{z} \frac{\exp(x)}{x}dx, \tag{14}$$

where x is a dummy variable. Although $Ei(z)$ is somewhat unwieldy to work with, it has been well studied and documented.[10] There are also online calculators available as well as analytical approximations.[11] The solution for velocity vs. altitude in terms of z is thus given by

$$v[z(y)] = \sqrt{u} = \left\{\frac{2g}{\lambda}e^{-z}[Ei(z)+c]\right\}^{1/2}, \tag{15}$$

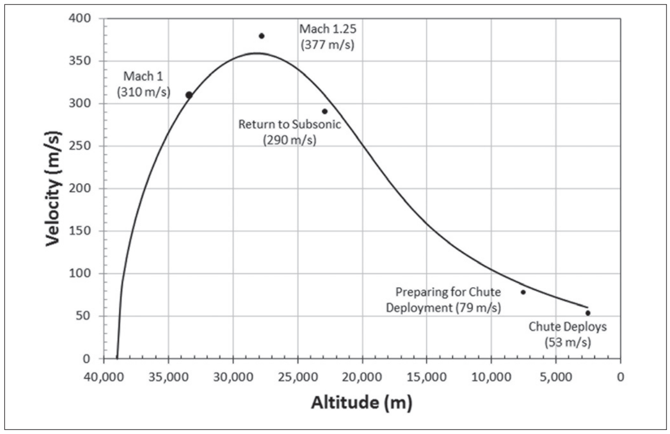

Fig. 1. Calculated velocity vs. altitude curve compared to reported data points given in Table I. The curve was calculated using Eq. (15).

where c is a constant of integration determined by setting $v = v_0$ at $y = y_0$. For $v_0 = 0$, $c = -Ei(z_0)$.

Kittinger's data were incomplete and inconsistent, making model verification difficult; therefore, Eq. (15) will be tested against Baumgartner's better documented jump. But before doing so, values for v_t and λ have to be determined. One method is to use Eq. (3) to calculate v_t. Unfortunately, although mass can easily be measured, D and A are somewhat less defined for irregular shaped objects, not only because of equipment profiles—extreme skydivers wear space suits—but also because skydivers usually change their positions during a jump between horizontal and vertical profiles. Fortunately, the choices somewhat limit themselves since terminal velocities for skydivers in the horizontal position are known to typically be in the range of 50 m/s to 60 m/s. In this study, we will use the lower value since it is the value quoted by the United States Parachute Association and possibly more in line with a bulkier suit-protected human form.[12]

Trying to determine λ based on a chosen mean atmospheric temperature turns into a frustrating guessing game over large altitude changes. In this study, most of the interesting high-altitude dynamics occur in the 20,000- to 40,000-m range; therefore, λ was chosen to give the accepted air density $\rho/\rho_0 \approx 0.015$ kg/m^3 at 30,000 m according to the United States Standard Atmosphere model.[13] Using this condition gives $\lambda \approx 1.40\times10^{-4}$/m.

Table I. Comparison of recorded data with calculated values from Eq. (15).

Jump Parameter	Variable (units)	Recorded	Calculated
Maximum Altitude (m)	Jump Point, $t = 0$, $v = 0$	38,969	–
Mach 1.0 (310 m/s)	Altitude (m)	33,446	33,161
Note: Mach number is temperature, therefore altitude, dependent	Time (s)	34	35
Maximum Speed	Speed (m/s)	377.1 Mach 1.25	359 Mach 1.20
	Altitude (m)	27,833	28,144
	Time (s)	50	50
Mach 1.0 (290 m/s)	Altitude (m)	22,961	21,874
	Time (s)	64	69
Total Supersonic Time	Time (s)	30	34
Preparing for Chute Deployment 7619 m	Speed (m/s)	79	88
	Time (s)	180	165
Chute Deploys 2567 m	Speed (m/s)	53	61
	Total Free-Fall Time	260	235

Results

Table I contains published time, speed, and altitude jump data from Ref. 2 for major phases of the jump and compares the data to Eq. (15) calculations. Although, as stated earlier, approximate analytical functions have been developed for $Ei(z)$, an online calculator was used for the calculations that follow.[14] See also related discussion in the appendix.[8]

Figures 1-3 show how Table I data points graphically compare to $v(y)$, $v(t)$, and $y(t)$ curves generated from Eq. (15). Equation (15) is too complex to solve analytically for $v(t)$ or $y(t)$. Those plots were obtained, using a simple spreadsheet tool, by taking small slices of $v(y)$, where the acceleration is assumed to be roughly constant and finding the estimated time

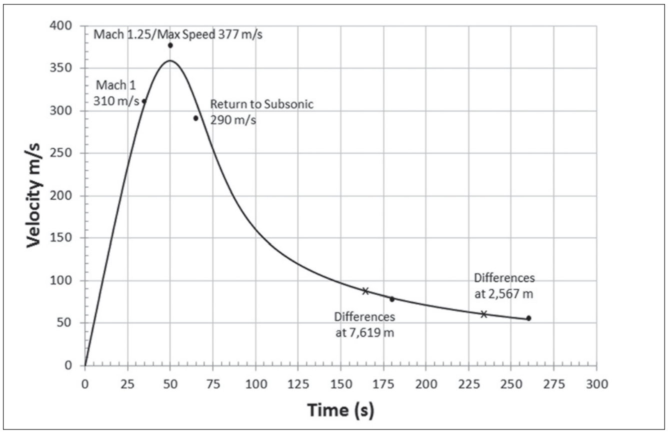

Fig. 2. Calculated velocity-vs.-time curve compared to reported data points given in Table I. The curve was calculated using Eq. (15) and assuming $\Delta t = \Delta y/v_{avg}$, where $v_{avg} = (v_1 + v_2)/2$.

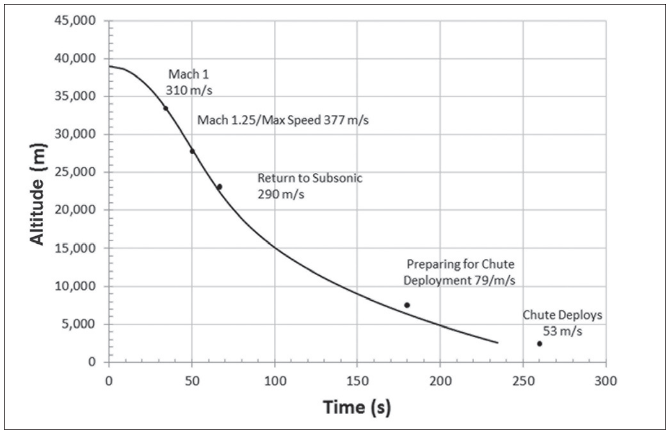

Fig. 3. Calculated altitude-vs.-time curve compared to reported data points given in Table I. The curve was calculated using Eq. (15) and assuming $\Delta t = \Delta y/v_{avg}$, where $v_{avg} = (v_1 + v_2)/2$.

between two altitude points from $\Delta t = \Delta y/v_{avg}$, where $v_{avg} = (v_1 + v_2)/2$.

Although the overall agreements are quite good, there are some noticeable differences. For example, in regards to the maximum speed, the calculated curves are based on a horizontal profile terminal velocity, whereas the skydiver most likely would have gone into a vertical diving profile in order to achieve minimum resistance and record-breaking speeds. Before the curves in Figs. 1 and 2 peak, drag would be negligible and the orientation of the skydiver would have little effect between real and calculated values. As the air density increases, a vertical profile would cause an overshoot seen in the figures. After achieving peak speed, the diver would eventually change back to a horizontal position.

Differences between calculated values and recorded data increase as the skydiver reaches lower altitudes, as can be seen in Table I starting at the point officially reported as "Preparing for Chute Deployment." This is due to the earlier choice of using a value for λ to fit high-altitude air densities. This results in a "thinner" lower atmosphere, producing higher speeds and shorter free-fall times. This is clear in the curve's tail of Fig. 2, where the differences between actual (•) and calculated (x) points at the two respective altitudes are noticeable.

Finally, it should be noted how the early portions of the curves behave as $v(y) \rightarrow (2gy)^{1/2}$, $v(t) \rightarrow gt$, and $y(t) \rightarrow -gt^2/2$, as they should since at these portions of the curves the drag force is negligible (velocity curves were plotted using a positive axis).

Conclusion

Given the complex structure of the atmosphere, the complicated fluid dynamics of the air flow around an irregular human form, and the changing positions of the skydiver, the formulation derived here provides a fairly good description of the dynamics of an extreme skydiver and can be generalized to any high-altitude falling object. Although the use of the single parameter barometric formula to describe atmospheric density allows for analytical solutions, it also has limitations since no single parameter adequately captures air density over the entire range of interest. Better atmospheric models do exist, but their use would be offset by the loss of analytical solutions.

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