Computer Modeling and Simulation

Lectures 7&8

Key Equations in Queuing Model

Name	Equation	Description
Inter-arrival time	$a_i = ar_i - ar_{i-1}$	Interval between two consecutive arrivals
Mean inter-arrival time	$\bar{a} = \dfrac{\sum a_i}{n}$	Average inter-arrival time
Arrival rate	$\lambda = \dfrac{n}{T}$ $\lambda = \dfrac{1}{\bar{a}}$ (in the long run)	The number of arrivals at unit time
Mean service time	$\bar{s} = \dfrac{\sum s_i}{n}$	Average time for each customer to be served
Service rate	$\mu = \dfrac{1}{\bar{s}}$	Capability of server at unit time

Key Equations in Queuing Model

Mean delay time	$\overline{de} = \frac{\sum (ss_i - ar_i)}{n}$	Average time for each customer to spend in a queue
Mean residence time	$\bar{w} = \frac{\sum (d_i - ar_i)}{n}$	Average time each customer stays in the system
System busy time	$B = \sum s_i$	Total service time of server
System idle time	$I = T - B$	Total idle time of server
System utilization	$\rho = \frac{B}{T}$	The proportion of the time in which the server is busy

Customer-Centric Simulation

Index state variables by customer i instead of event i

t_{enter}^{i}	time customer i enters the system
L_q^i	length of the queue when customer i enters
t_{served}^{i}	time customer i service starts
W_q^i	waiting time in the queue for customer i
t_{exit}^{i}	time customer i exists the system
W_i	total waiting time for customer i

Customer-Centric Simulation

x: Inter-arrival Period
y: Service Time
t_{enter}: Entry Time
L_q: Queue Length
t_{served}: Time Served
W_q: Queue Wait Time
t_{exit}: Exit Time
W: Total Wait Time

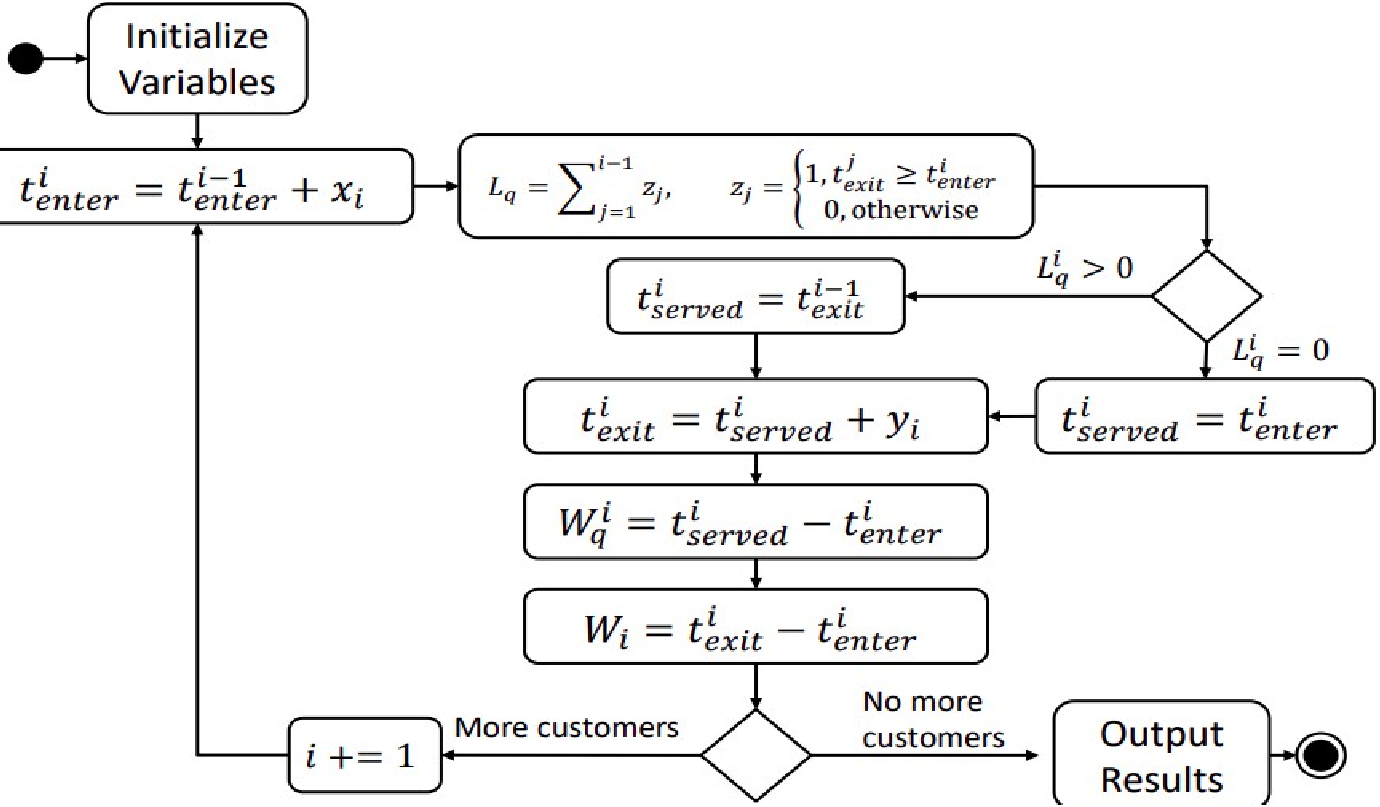
i	x	t_{enter}	L_q	t_{served}	W_q	y	t_{exit}	W
1	**0.43**					**0.90**		
2	**4.49**					**1.66**		
3	**0.95**					**0.02**		
4	**0.03**					**0.46**		
5	**0.28**					**0.51**		

Customer-based Simulation

$\boldsymbol{x}$: Inter-arrival Period
$\boldsymbol{y}$: Service Time
$\boldsymbol{t_{enter}}$: Entry Time
$\boldsymbol{L_q}$: Queue Length
$\boldsymbol{t_{served}}$: Time Served
$\boldsymbol{W_q}$: Queue Wait Time
$\boldsymbol{t_{exit}}$: Exit Time
$\boldsymbol{W}$: Total Wait Time

i	x	t_{enter}	L_q	t_{served}	W_q	y	t_{exit}	W
1	**0.43**	0.00+0.43= 0.43	0	0.43	0.43-0.43= 0	**0.90**	0.43+0.90= 1.33	1.33-0.43= 0.90
2	**4.49**	0.43+4.49= 4.92	0	4.92	4.92-4.92= 0	**1.66**	4.92+1.66= 6.58	6.58-4.92= 1.66
3	**0.95**	4.92+0.95= 5.87	1	6.58	6.58-5.87= 0.71	**0.02**	6.58+0.02= 6.60	6.60-5.87= 0.73
4	**0.03**	5.87+0.03= 5.90	2	6.60	6.60-5.90= 0.70	**0.46**	6.60+0.46= 7.06	7.06-5.90= 1.16
5	**0.28**	5.90+0.28= 6.18	3	7.06	7.06-6.18= 0.88	**0.51**	7.06+0.51= 7.57	7.57-6.18= 1.39

Customer-based Simulation



Customer-based Simulation

$\lambda = 2/3 \qquad \mu = 4/3$

1000-customer Sim.:

- $\overline{W} = 1.49$ min.
- $\overline{W}_q = 0.74$ min.
- $\overline{L}_q{}^* = 1.05$
- $\overline{L}^* = \overline{L}_q{}^* + 1 = 2.05$

Queuing Theory:

- $\overline{W} = \frac{1}{\mu-\lambda} = 1.50$ min.
- $\overline{W}_q = \frac{\lambda}{\mu(\mu-\lambda)} = 0.75$ min.
- $\overline{L}_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = 0.50$
- $\overline{L} = \frac{\lambda}{\mu-\lambda} = 1.00$

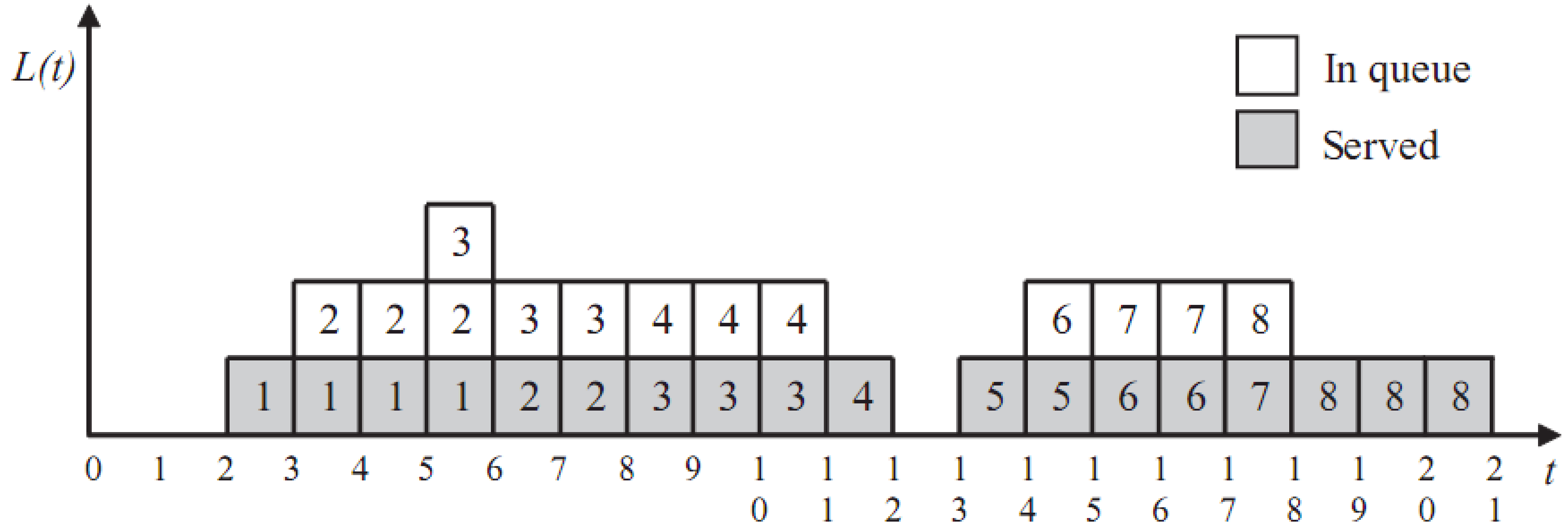
Another Example of Customer based Simulation

- Consider the following example of a grocery store with only one checkout counter.
- The customer waits for the cashier in a line.
- The inter - arrival time, arrival time, service time and departure time for each customer are given in the following table.

Another Example of Customer based Simulation

Customer	Inter-arrival time	Arrival time	Service time	Departure time
1	—	2	4	6
2	1	3	2	8
3	2	5	3	11
4	3	8	1	12
5	5	13	2	15
6	1	14	2	17
7	1	15	1	18
8	2	17	3	21

Simulation of Queuing Model



Little’s Law

- The average number of customers (L) is equal to the arrival rate (λ) multiplied by the average time (w) the customer spends in the system:
- $L = \lambda w$.
- Little ’ s law is meaningful in that the law holds regardless of any kind of the arrival and service distribution.
- Thus, Little ’ s law does not require restricted assumptions for the types of arrival and service patterns.

Little’s Law

- The average customer (L) at time t can be calculated from the figure as follows.
- $L = (1 + 2 + 2 + 3 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 2 + 2 + 2 + 2 + 1 + 1 + 1)/21 = 1.48.$
- The arrival rate (λ) is $8/21 = 0.38$.
- The mean residence time (w) is $(4 + 5 + 6 + 4 + 2 + 3 + 3 + 4)/8 = 3.88$.
- The average number of customers in the system can be calculated from Little’s law: $L = \lambda w = 0.38 \times 3.88 = 1.48$.
- The result of Little’s law is the same as that of simulation.

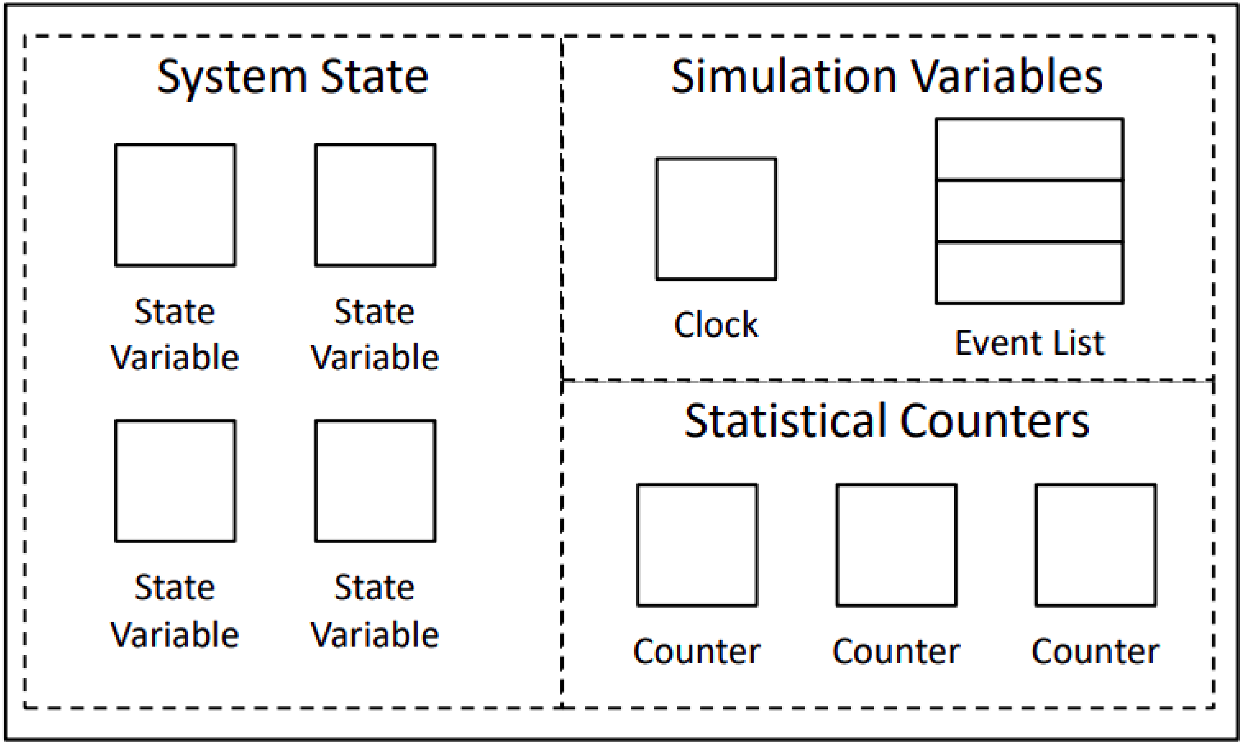
Limitation of Customer-based Simulation

- Need to maintain history of all customers
- Huge space complexity

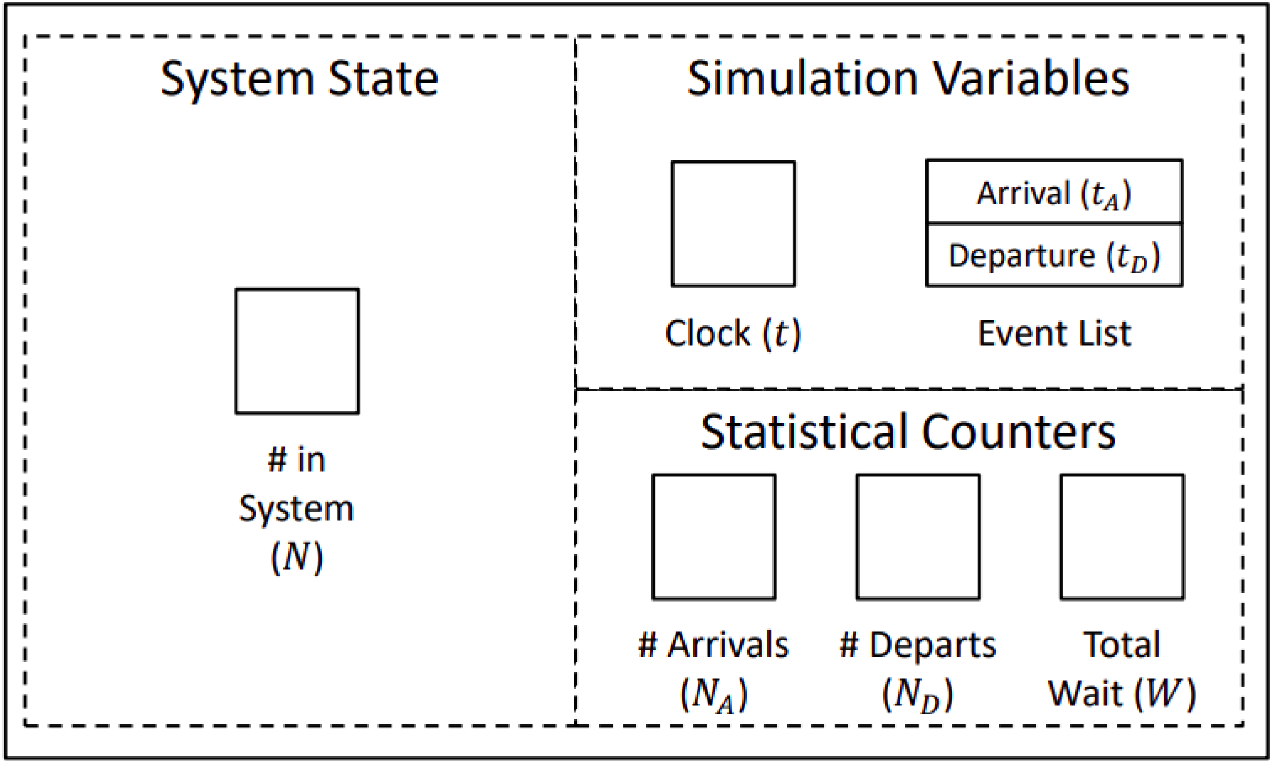
Discrete Event Simulation-Model Constructs

- Time variable: Store the current simulation time
- Event list/stack: List of future events with associated execution times
- System state variables: Store state variables which persist across time
- Counter variables: Store derived state variables that record useful observations for analysis

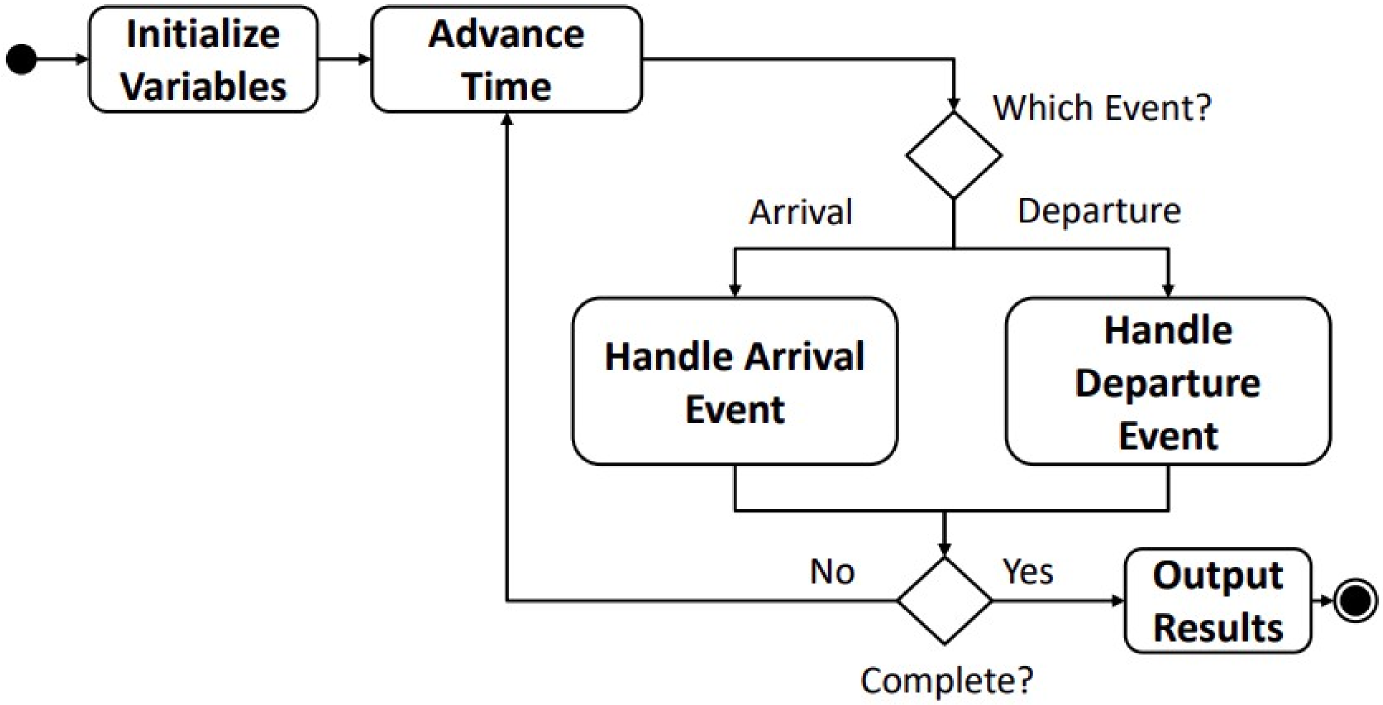
Discrete Event Simulation-Model Constructs



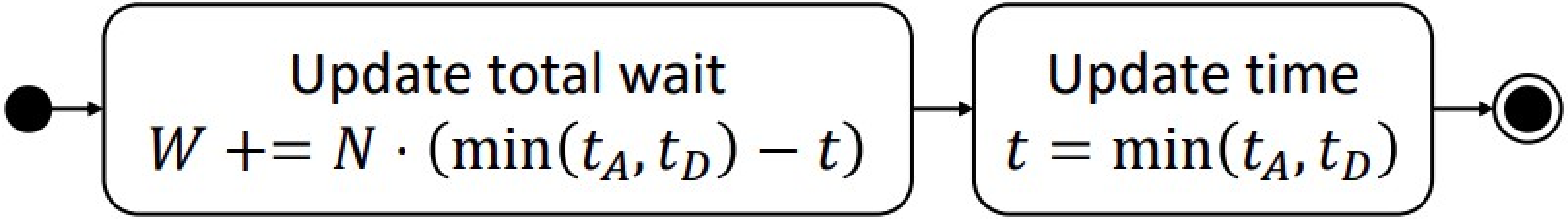
Discrete Event Simulation-Model Constructs



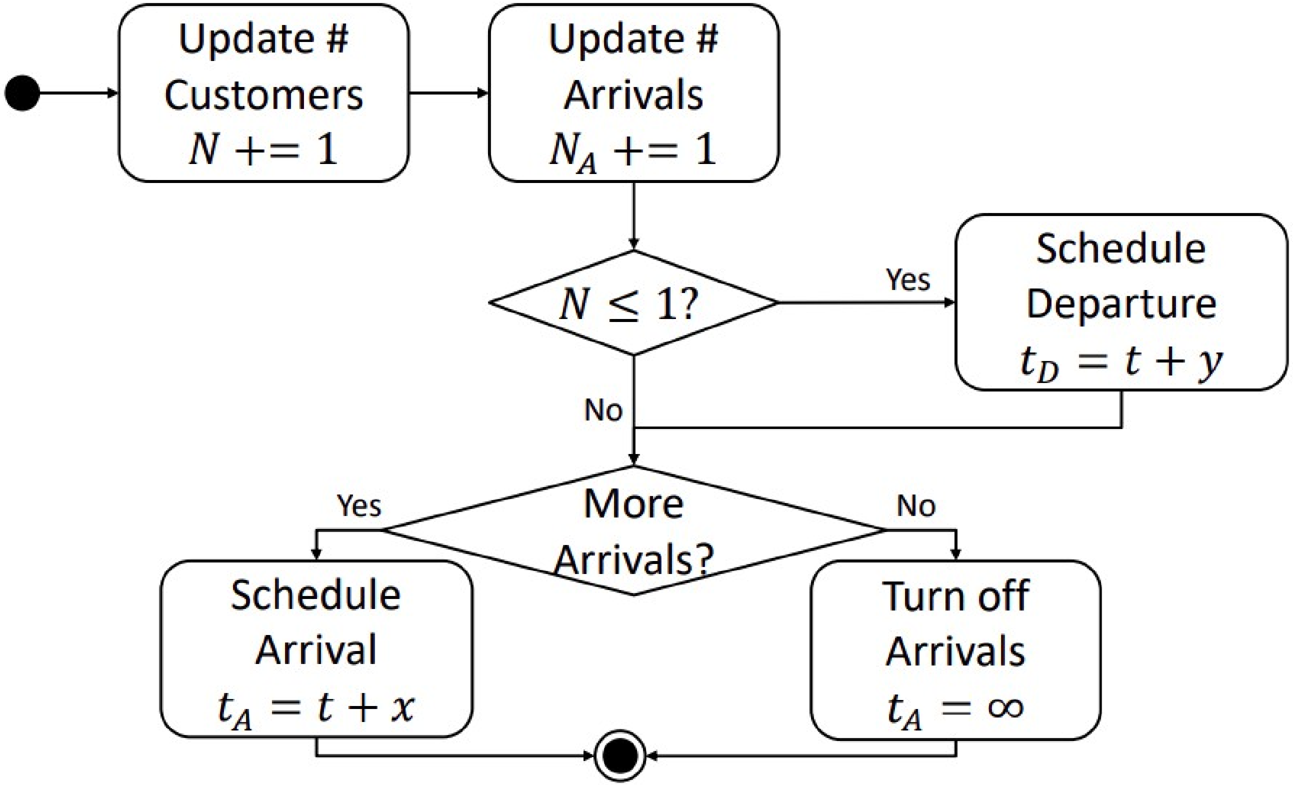
Queuing Activity Diagram



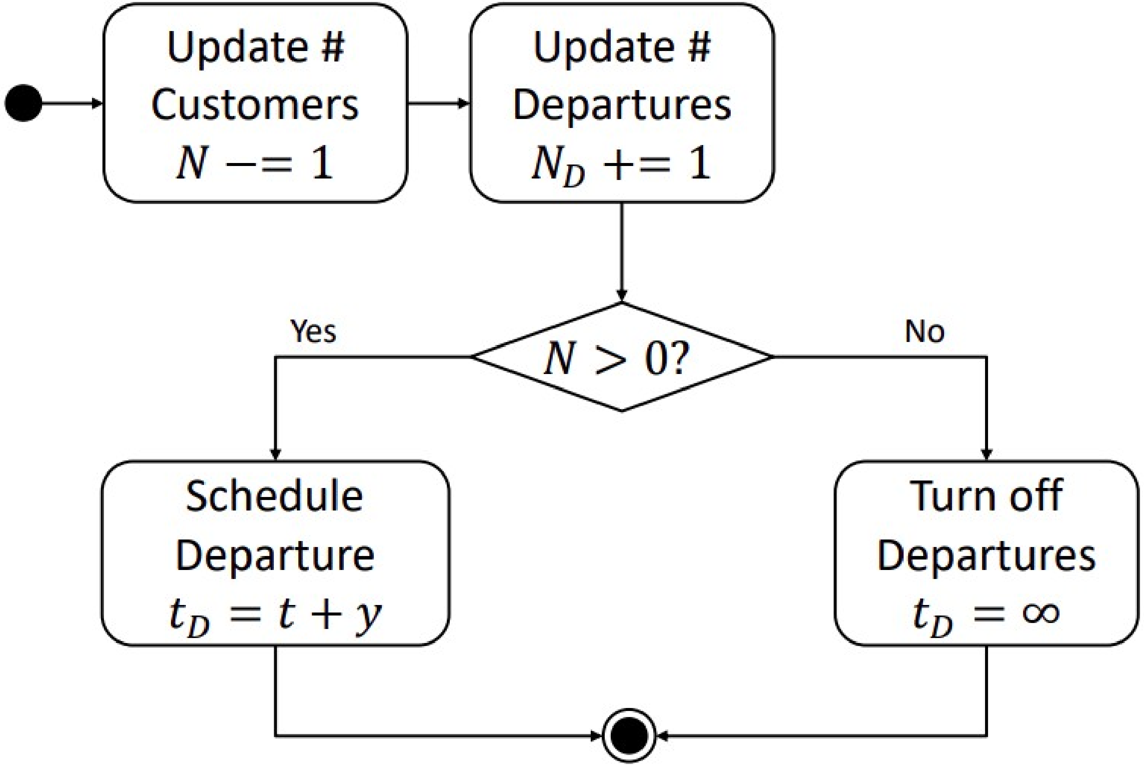
Advance Time



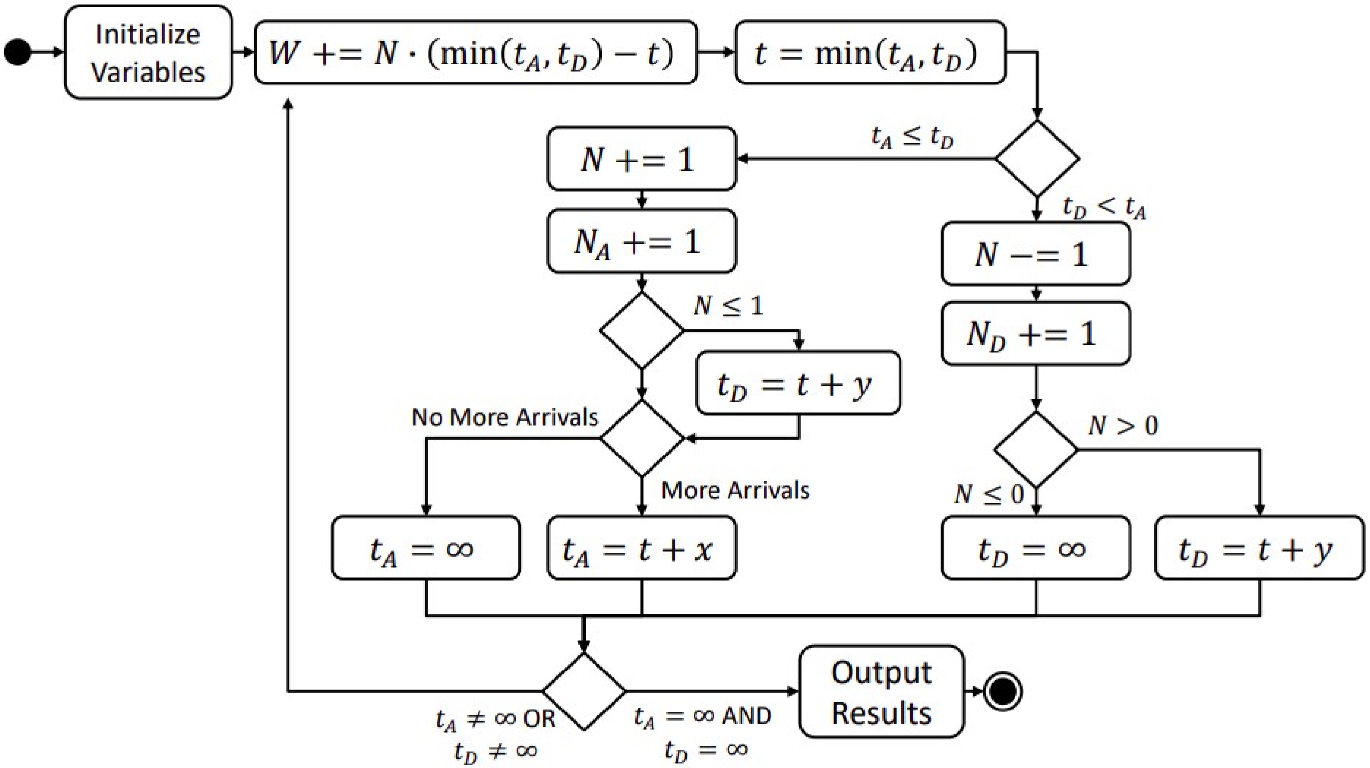
Handle Arrival Event



Handle Departure Event

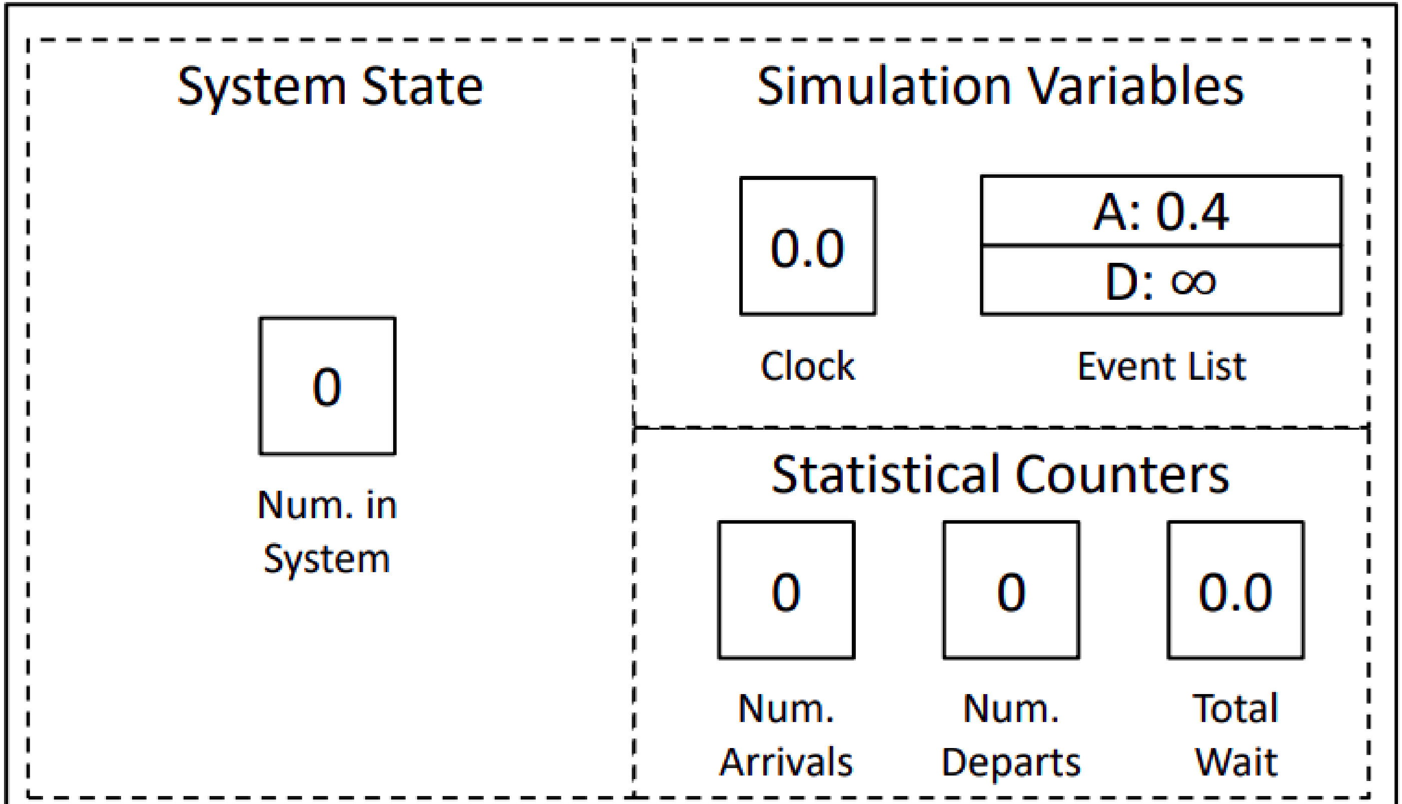


Complete Activity Diagram



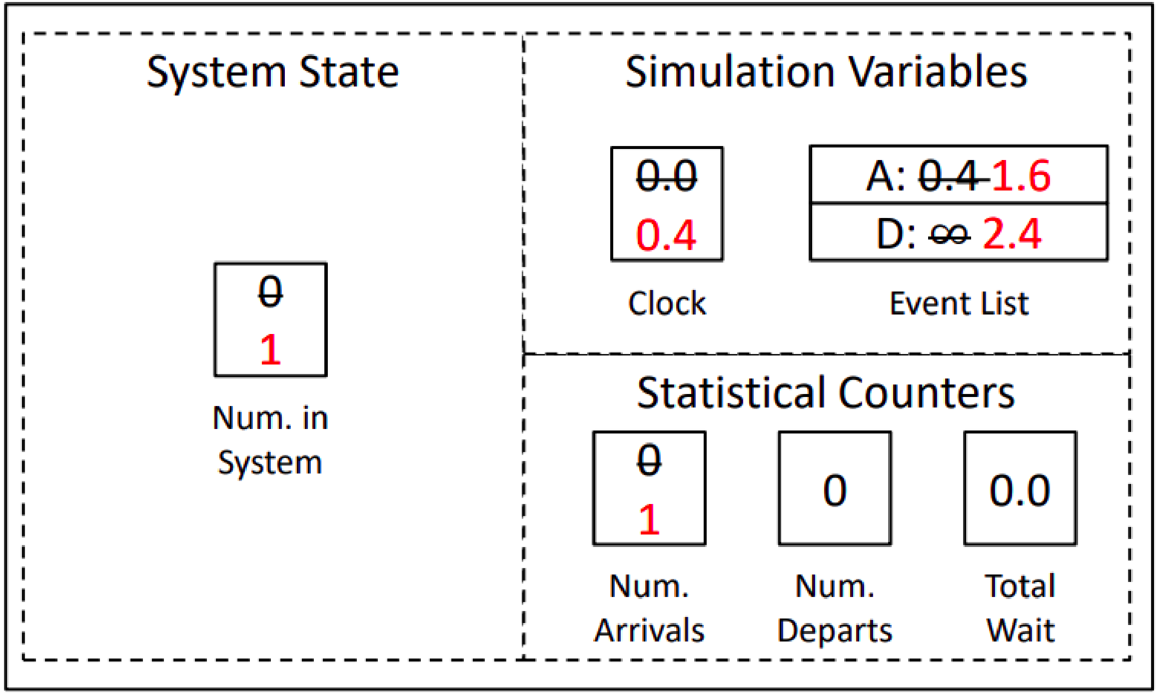
Initialize Simulation

Inter-arrival times: 0.4, 1.2, 0.5, 1.7   Service times: 2.0, 0.7, 0.2, 1.1



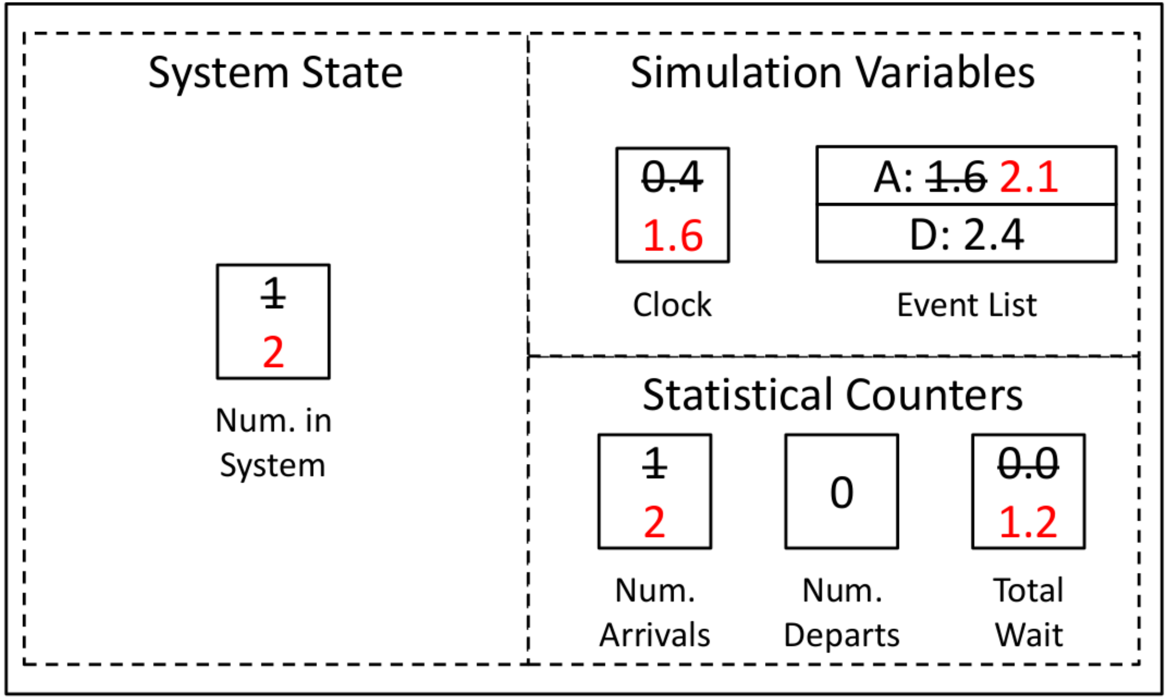
Arrival Event @ $t = 0.4$

Inter-arrival times: ~~0.4~~, 1.2, 0.5, 1.7 Service times: 2.0, 0.7, 0.2, 1.1



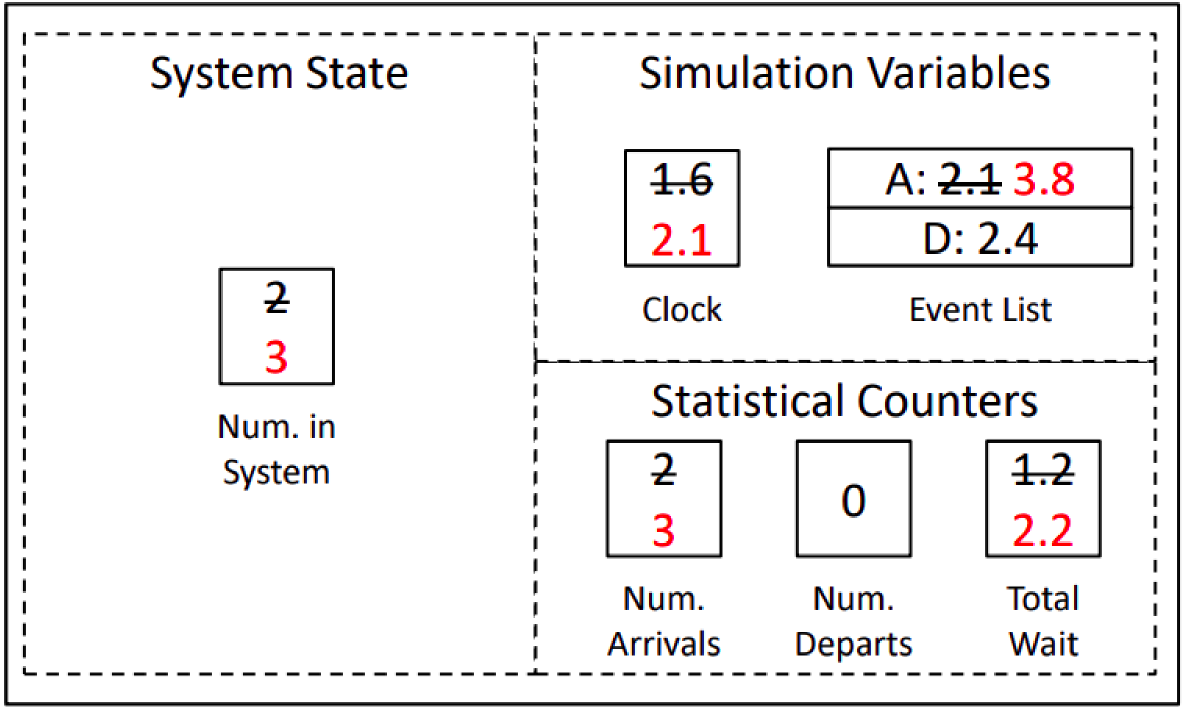
Arrival Event @ t = 1.6

Inter-arrival times: ~~0.4, 1.2~~, 0.5, 1.7 Service times: ~~2.0~~, 0.7, 0.2, 1.1



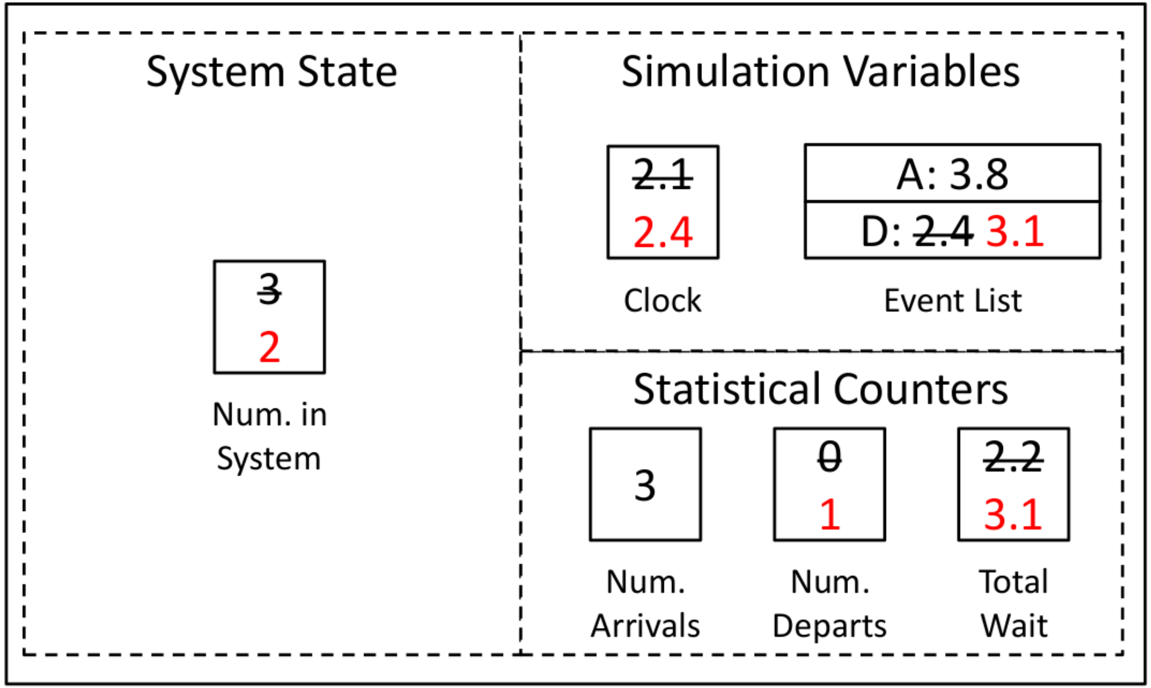
Arrival Event @ t = 2.1

Inter-arrival times: ~~0.4, 1.2, 0.5~~, 1.7 Service times: ~~2.0~~, 0.7, 0.2, 1.1



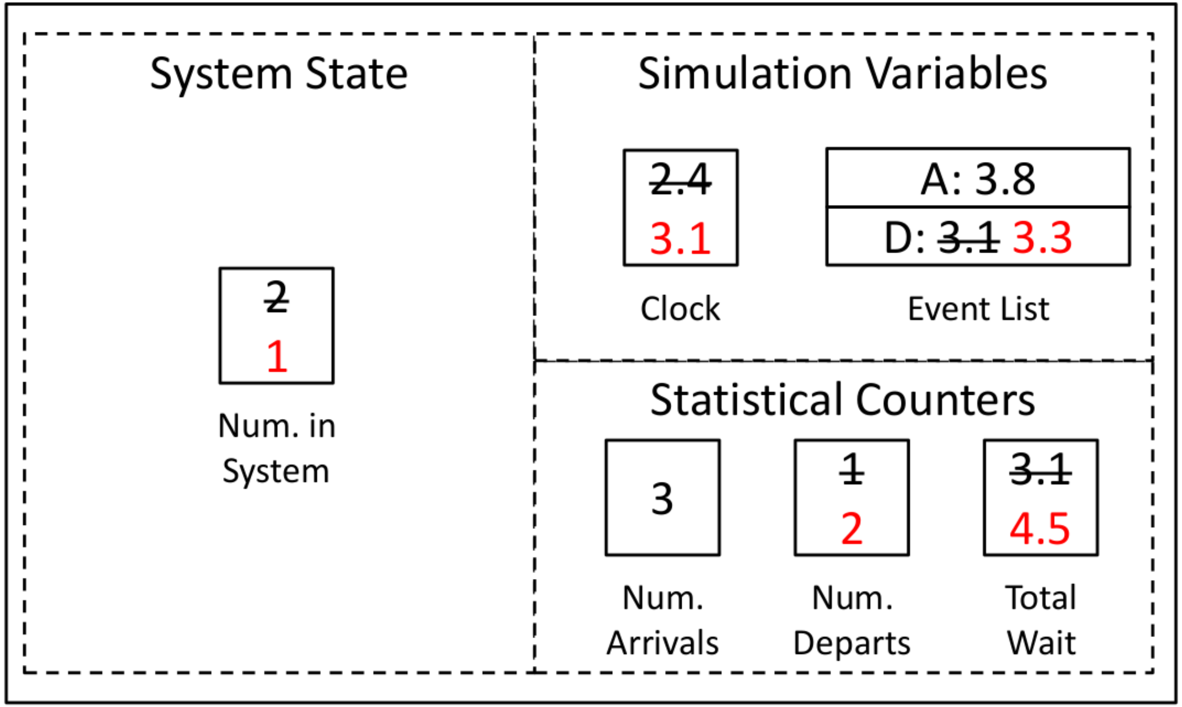
Arrival Event @ t = 2.4

Inter-arrival times: ~~0.4, 1.2, 0.5, 1.7~~ Service times: ~~2.0~~, 0.7, 0.2, 1.1



Departure Event @ t = 3.1

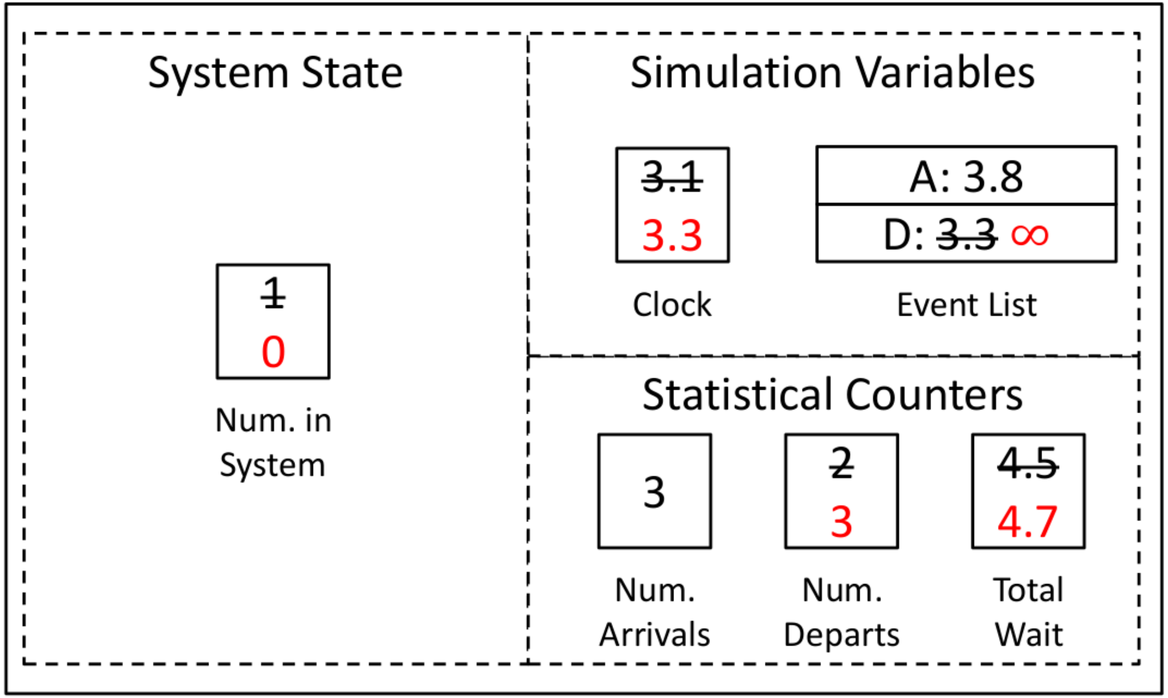
Inter-arrival times: ~~0.4, 1.2, 0.5, 1.7~~ Service times: ~~2.0, 0.7~~, 0.2, 1.1



Departure Event @ t = 3.3

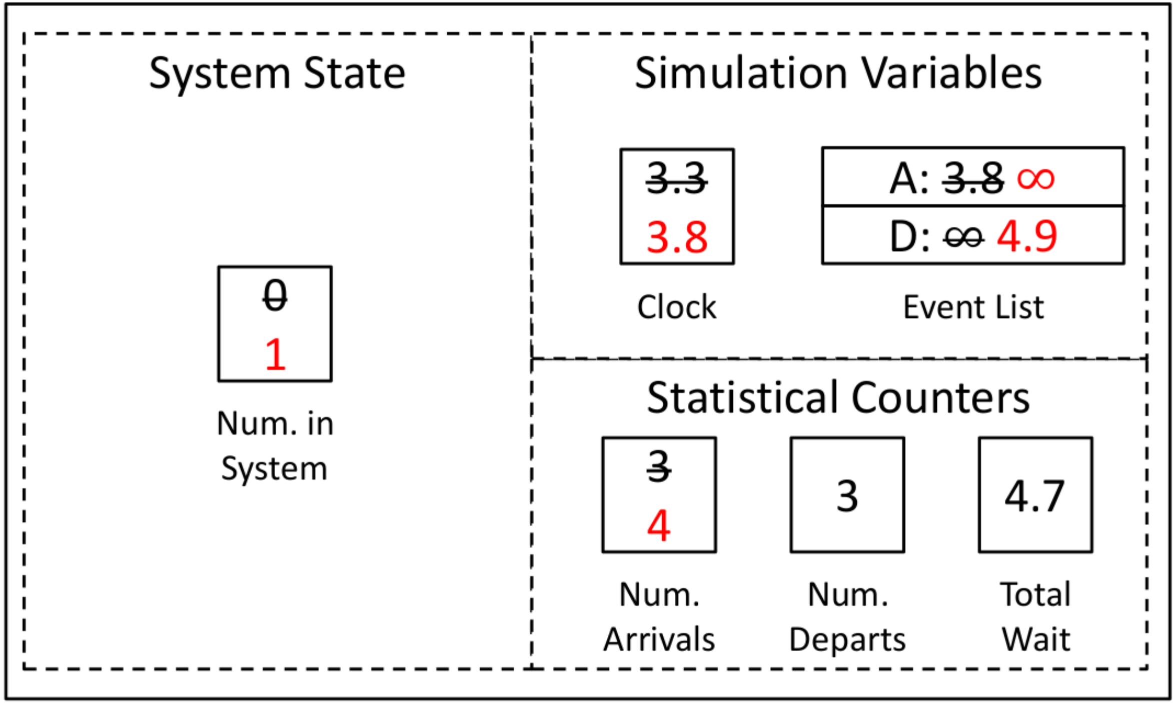
Inter-arrival times: ~~0.4, 1.2, 0.5, 1.7~~

Service times: ~~2.0, 0.7, 0.2~~, 1.1



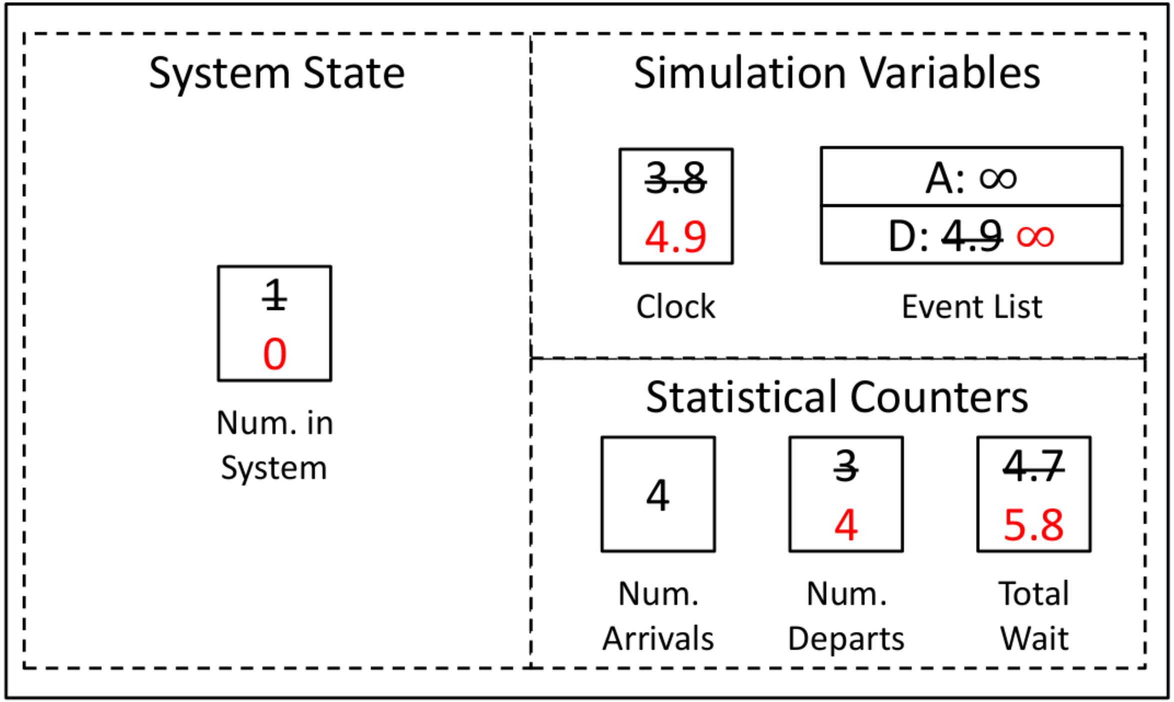
Arrival Event @ t = 3.8

Inter-arrival times: ~~0.4, 1.2, 0.5, 1.7~~ Service times: ~~2.0, 0.7, 0.2, 1.1~~



Departure Event @ t = 4.9

Inter-arrival times: ~~0.4, 1.2, 0.5, 1.7~~ Service times: ~~2.0, 0.7, 0.2, 1.1~~



Event-based Simulation

$\boldsymbol{x}$: 0.4, 1.2, 0.5, 1.7

$\boldsymbol{y}$: 2.0, 0.7, 0.2, 1.1

$\boldsymbol{t}$: Clock Time
$\boldsymbol{t_A}$: Next Arrival Time
$\boldsymbol{t_D}$: Next Departure Time
$\boldsymbol{N}$: Number Customers in System

$\boldsymbol{N_A}$: Number Arrivals (Cumulative)
$\boldsymbol{N_D}$: Number Departures (Cumulative)
$\boldsymbol{W}$: Total Wait Time (Cumulative)

$\boldsymbol{i}$	$\boldsymbol{t}$	$\boldsymbol{t_A}$	$\boldsymbol{t_D}$	$\boldsymbol{N}$	$\boldsymbol{N_A}$	$\boldsymbol{N_D}$	$\boldsymbol{W}$
5	2.4	3.8	2.4+0.7= 3.1	3-1= 2	3	0+1= 1	2.2+3*0.3= 3.1
6	3.1	3.8	3.1+0.2= 3.3	2-1= 1	3	1+1= 2	3.1+2*0.7= 4.5
7	3.3	3.8	∞	1-1= 0	3	2+1= 3	4.5+1*0.2= 4.7
8	3.8	∞	3.8+1.1= 4.9	0+1= 1	3+1= 4	3	4.7
9	4.9	∞	∞	1-1= 0	4	3+1= 4	4.7+1*1.1= 5.8

Customer vs Event based Simulation

i	x	t_{enter}	L_q	t_{served}	W_q	y	t_{exit}	W
1	0.4	0.4	0	0.4	0.0	2.0	2.4	2.0
2	1.2	1.6	1	2.4	0.8	0.7	3.1	1.5
3	0.5	2.1	2	3.1	1.0	0.2	3.3	1.2
4	1.7	3.8	0	3.8	0.0	1.1	4.9	1.1

i	t	t_A	t_D	N	N_A	N_D	W
1	0.0	0.4	∞	0	0	0	0
2	0.4	1.6	2.4	1	1	0	0
3	1.6	2.1	2.4	2	2	0	1.2
4	2.1	3.8	2.4	3	3	0	2.2
5	2.4	3.8	3.1	2	3	1	3.1
6	3.1	3.8	3.3	1	3	2	4.5
7	3.3	3.8	∞	0	3	3	4.7
8	3.8	∞	4.9	1	4	3	4.7
9	4.9	∞	∞	0	4	4	5.8