Computer Modeling and Simulation

Lecture 14

Dynamic Systems

- Dynamic vs Static Systems
- Dynamic Systems
  - Continuous Systems
    - Systems with rate proportional to the amount
      - Population growth models
        - Unconstrained
        - Constrained
      - Drug Dosage Model
    - Force and Motion
    - Random Walk
  - Discrete Systems
    - Discrete Event Systems

Dynamic Systems

- Dynamic systems are those that change with time.
- Examples:
  - population of humans, deer or bacteria etc. changing with time
  - Motion of vehicles (position and speed changing with time)
  - Number of people standing in a queue in front of a service counter changes with time
- Types of Dynamic Systems
  - Continuous Systems
  - Discrete Event Systems

Continuous Systems

- A *continuous system* is one in which the state variable(s) change continuously over time. E.g. the amount of water flow over a dam.
- Continuous systems are represented by Differential Equations.

Differential Equations

- A **differential equation** is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

dy/dx = f(x)

- Here “x” is an independent variable and “y” is a dependent variable
- For example, dy/dx = 5x
- The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity.
- The derivate can be either

Ordinary Differential Equation

Rate of change

- How much a system variable changes with change in time
  - Average rate of change
  - Instantaneous rate of change
- Average rate of change

Average rate of change

Definition Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. Then the **change in time**, Δt, is $\Delta t = b - a$; and the **change in position**, Δs, is $\Delta s = s(b) - s(a)$. Moreover, the **average velocity**, or the **average rate of change of s with respect to t**, of the object from time $a = b - \Delta t$ to time b is

$$\textbf{average velocity} = \frac{\textbf{change in position}}{\textbf{change in time}} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(b) - s(a)}{b - a} = \frac{s(b) - s(b - \Delta t)}{\Delta t}$$

Instantaneous rate of change

Definition The **instantaneous velocity**, or the **instantaneous rate of change of s with respect to t**, at $t = b$ is the number the average velocity, $\dfrac{s(b) - s(b - \Delta t)}{\Delta t}$, approaches as Δt comes closer and closer to 0 (provided the ratio approaches a number). In this case, the **derivative of $y = s(t)$ with respect to t** at $t = b$, written $s'(b)$ or $\left.\dfrac{dy}{dt}\right|_{t=b}$, is the instantaneous velocity at $t = b$. In general, the **derivative of $y = s(t)$ with respect to t** is written as $s'(t)$, or $\dfrac{dy}{dt}$, or dy/dt.

Continuous Systems

- Systems where the change in the system variable is proportional to the quantity of that system variable
  - Population growth models
  - Profits on income
  - Drug dosage model

Population Growth Model

- **Carrying Capacity:**
  - The **maximum** population size of a biological species that can be sustained by that specific environment given the food, habitat, water, and other resources available.
- Unconstrained Growth Model
  - The carrying capacity is infinite
- Constrained Growth Model
  -

Unconstrained Growth Model (Malthusian Model)

- Thomas Malthus gave the Malthusian Model in “An Essay on the Principle of Population (1798)”
- Popular definition of ”Malthusian”: population growth exponentially and food grows linearly
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Malthusian Models

- No constraints, such as competition for food or a predator, exist on growth of the population
- The model where the rate of change of the population is **directly proportional** (∝) to the number of individuals in the population.

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = rP$$

- The constant *r* is the **growth rate**, or **instantaneous growth rate**, or **continuous growth rate**, while *dP/dt* is the **rate of change of the population**.

Differential Equation

- Consider a bacterial population of size 100, an instantaneous growth rate of 10% = 0.10, and time measured in hours. Thus, we have

$$dP / dt = 0.10P$$

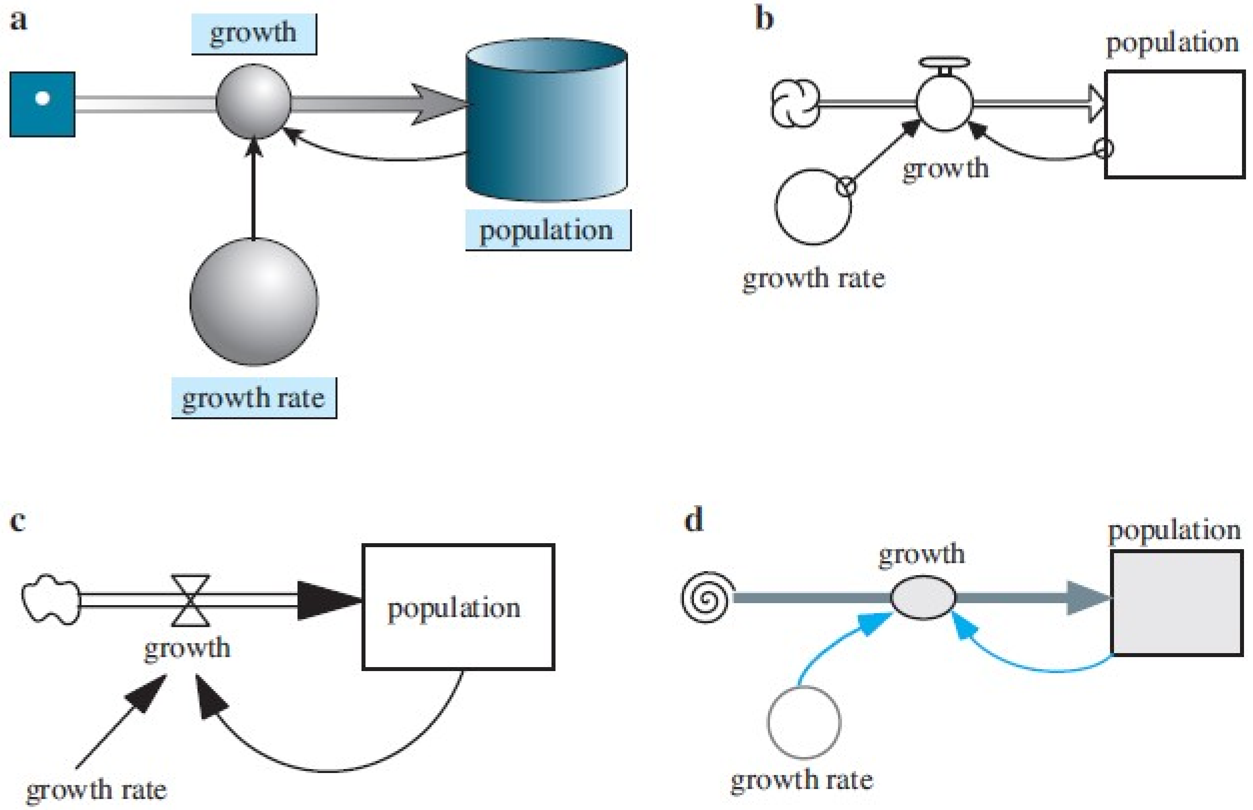
with $P_0 = 100$.

- A **solution** to this differential equation is a function, $P(t)$, whose derivative is $0.10P(t)$, with P

Variables in the Growth Population Model

- A **stock** (**box variable**, or **reservoir**), such as *population*, accumulates with time.
- By contrast, a **converter** (**variable-auxiliary/constant**, or **formula**), such as *growth_rate*, does not accumulate but stores an equation or a constant.
- The growth is the additional number of organisms that join the population. Thus, a **flow** (**rate**), such as *growth*, is an activity that changes the magnitude of a stock and represents a derivative.
- Because both population and growth rate are necessary to determine the growth, we have **arrows** (**connectors**, or **arcs**) from these quantities to the flow indicator.

Graphical representation of model



Diagrams of population models where growth rate is proportional to population:
(a) *Berkeley Madonna®* (b) *STELLA®* (c) *Vensim PLE®* (d) Text's format

Differential Equation

Definitions A **differential equation** is an equation that contains one or more derivatives. An **initial condition** is the value of the dependent variable when the independent variable is zero. A **solution** to a differential equation is a function that satisfies the equation and initial condition(s).

Solution to Differential Equations

- Analytical Solution using Integration

- Numerical Methods

Analytical Solution Using Integrals

- We can solve the differential equation *dP/dt* = = 0.10*P* using a technique called **separation of variables**.

$$\frac{1}{P}dP = 0.10\,dt$$

- Then, we integrate both sides of the equation, as follows:

$$\int \frac{1}{P}dP = \int 0.10\,dt$$

$\ln|P| = 0.10t + C$ for an arbitrary constant C

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In general, the solution to

$$\frac{dP}{dt} = rP \text{ with initial population } P_0$$

is

$$P = P_0 e^{rt}$$

Numerical Methods

- Numerical methods are used when the analytical solution is not possible.
    - Euler Method
    - Runge Kutta Method
- In unconstrained growth model, we have the analytical solution but we will use the Euler method just to demonstrate the use of numerical methods in complex situations where the analytical solution doesn't exist.

Finite Difference Equation

- Euler method uses finite difference equations

(new population) = (old population) + (change in population)

or

population(*t*) = *population*(*t* – Δ*t*) + Δ*population*

Δ*population* = *population*(*t*) - *population*(*t* – Δ*t*)

- The above equation is called a **finite difference equation.**

growth (dP/dt) = Δ*population /* Δtime

growth = *(population*(*t*) - *population*(*t* – Δ*t*))/ Δtime

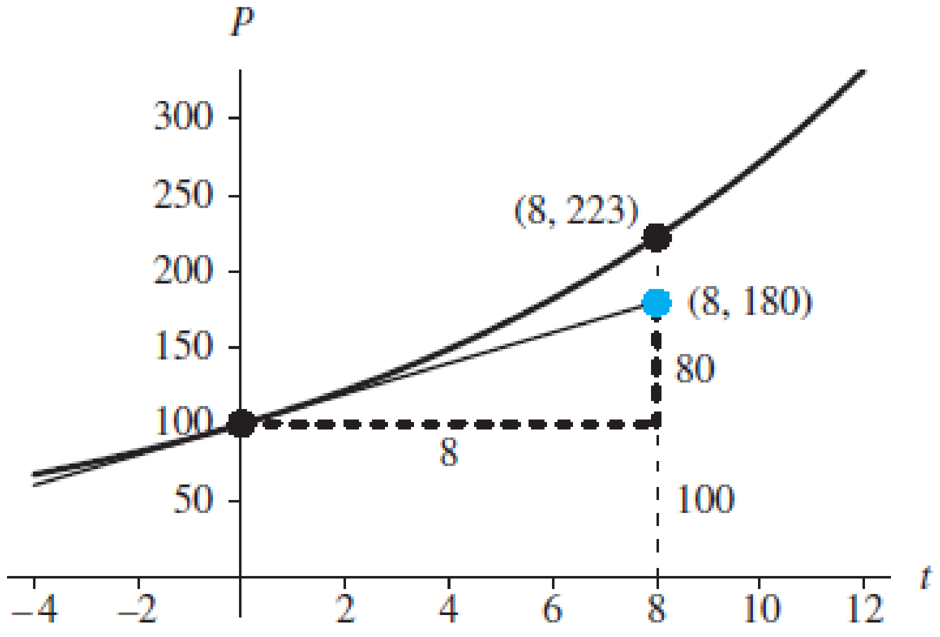
- Computer programs and system dynamics tools employ such finite difference equations to solve differential equations.

Euler’s method

- growth_rate = 0.10
- population(0) = 100
- growth(t) = growth_rate * population(t - Δt)
- population(t) = population(t - Δt) + growth(t) * Δt

- Starting with $P_0 = P(0) = 100$ and using $\Delta t = 8$. In this situation, $t = 8$, $t - \Delta t = 0$, growth(t) is the derivative at that time is $P'(0) = 0.1(100) = 10$, which is the slope of the tangent line to the curve $P(t)$ at (0,100).
- We multiply Δt, 8, by this derivative at the previous time step, 10, to obtain the estimated change in P, 80.
- Consequently, the estimate for P_1 is as follows:

$$
\begin{aligned}
\text{estimate for } P_1 &= \text{previous value of } P + \text{estimated change in } P \\
&= P_0 + P'(0)\Delta t \\
&= 100 + 10(8) \\
&= 180
\end{aligned}
$$



Actual point, (8, 223), and point obtained by Euler’s method, (8, 180)

Simulation Program

Algorithm 1 Algorithm for simulation of unconstrained growth

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize length of time step Δt
numIterations $\leftarrow$ *simulationLength*/Δt
for i going from 1 through *numIterations* do the following:
    growth $\leftarrow$ *growthRate* * *population*
    population $\leftarrow$ *population* + *growth* * Δt
    $t \leftarrow i * \Delta t$
    display t, *growth*, and *population*

Simulation Program

Algorithm 2 Alternative algorithm to Algorithm 1 for simulation of unconstrained growth that does not display *growth*

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize Δt
growthRatePerStep **←** ***growthRate*** ***** $\boldsymbol{\Delta t}$
numIterations ← *simulationLength*/Δt
for *i* going from 1 through *numIterations* do the following:
    population **←** ***population*** **+** ***growthRatePerStep*** ***** ***population***
    $t \leftarrow i * \Delta t$
    display *t* and *population*

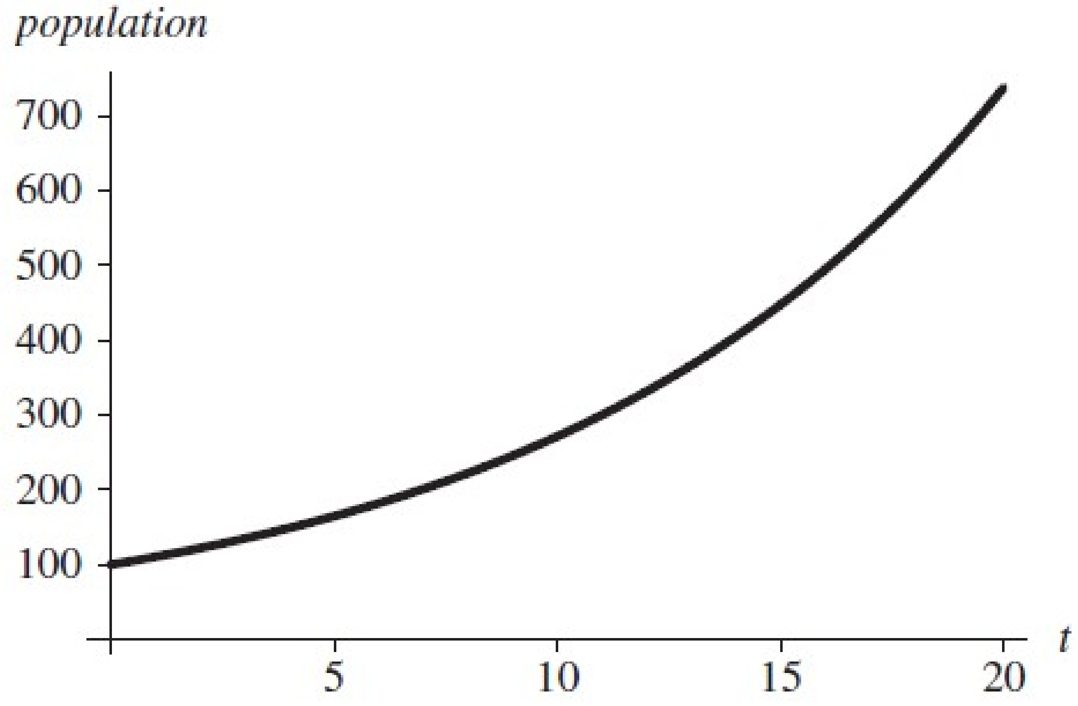
Numerical Solution by Euler's Method

t	$population(t)$	=	$population(t - \Delta t)$	+	$(growth)$	*	Δt
0.000	100.000000						
0.005	100.050000	=	100.000000	+	10.000000	*	0.005
0.010	100.100025	=	100.050000	+	10.005000	*	0.005
0.015	100.150075	=	100.100025	+	10.010003	*	0.005
0.020	100.200150	=	100.150075	+	10.015008	*	0.005
0.025	100.250250	=	100.200150	+	10.020015	*	0.005
0.030	100.300375	=	100.250250	+	10.025025	*	0.005
0.035	100.350525	=	100.300375	+	10.030038	*	0.005
0.040	100.400701	=	100.350525	+	10.035053	*	0.005

Table of Estimated Populations, Where the Initial Population is 100, the Continuous Growth Rate is 10% per Hour, and the Time Step is 0.005 h

t (h)	*growth*	*population*
0.000	10.00	100.00
1.000	11.05	110.51
2.000	12.21	122.13
3.000	13.50	134.98
4.000	14.92	149.17
5.000	16.49	164.85
6.000	18.22	182.18
7.000	20.13	201.34
8.000	22.25	222.51
9.000	24.59	245.90
10.000	27.18	271.76
11.000	30.03	300.33
12.000	33.19	331.91
13.000	36.68	366.81
14.000	40.54	405.38
15.000	44.80	448.00
16.000	49.51	495.11
17.000	54.72	547.16
18.000	60.47	604.69
19.000	66.83	668.27
20.000		738.54

Table of Estimated Growths and Populations, Reported on the Hour, Where the Initial Population is 100, the Growth Rate is 10%, and the Time Step is 0.005 h



Graph Population vs time