Computer Modeling and Simulation

Lectures 15-17

Constrained Growth Model

- Endemic populations increase rapidly at first, but they eventually encounter resistance from the environment— competitors, predators, limited resources, and disease.
- Thus, the environment tends to limit the growth of populations, so that they usually increase only to a certain level and then do not increase or decrease drastically unless a change in the environment occurs.
- This maximum population size that the environment can support indefinitely is termed the **carrying capacity**.

Carrying Capacity

- In unconstrained growth model, we have

$$\frac{dP}{dt} = rP$$

- When the population is very small, the number of deaths be almost zero, indicating that few individuals are dying.
- Near the carrying capacity, the number of deaths should be almost equal to the number of births, so that the population remains roughly constant.
- For populations larger than the carrying capacity, the fraction should be even larger so that the population decreases in size through deaths.
- To accomplish this dampening of growth, we could compute the number of deaths as a changing fraction of the number of births.
- This fraction is **P/M** where P is the population at any instant and M is carrying capacity.

Logistic Equations

- Thus, we can model the instantaneous rate of change of the number of deaths (*D*) as the fraction *P*/*M* times the instantaneous rate of change of the number of births (*r*), as the following differential equation indicates:

$$\frac{dD}{dt} = \left(r\frac{P}{M}\right)P$$

$$\frac{dP}{dt} = r\left(1 - \frac{P}{M}\right)P$$

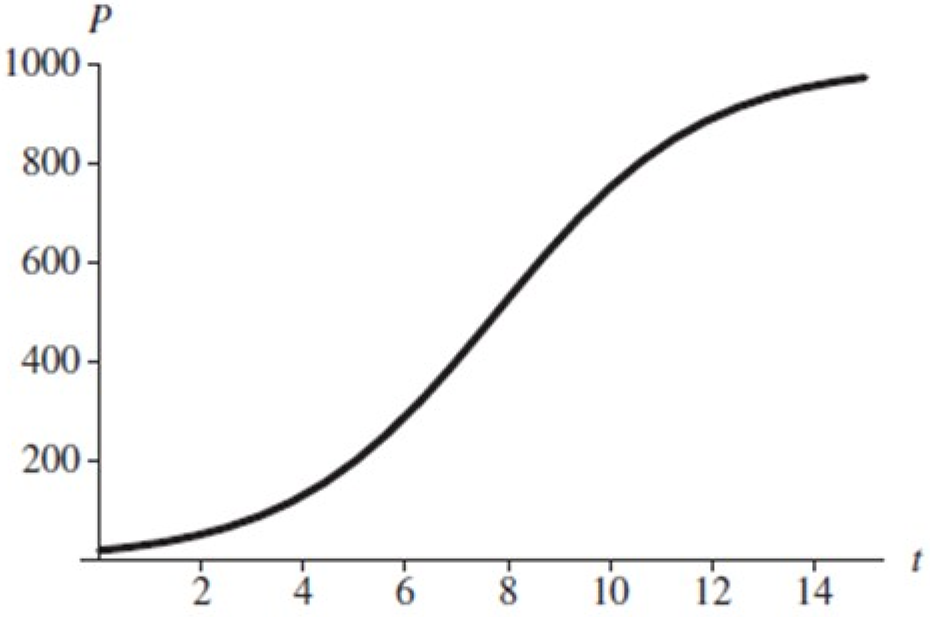
1

$$\Delta P = k\left(1 - \frac{P(t-\Delta t)}{M}\right)P(t-\Delta t), \text{ where } k = r\Delta t$$

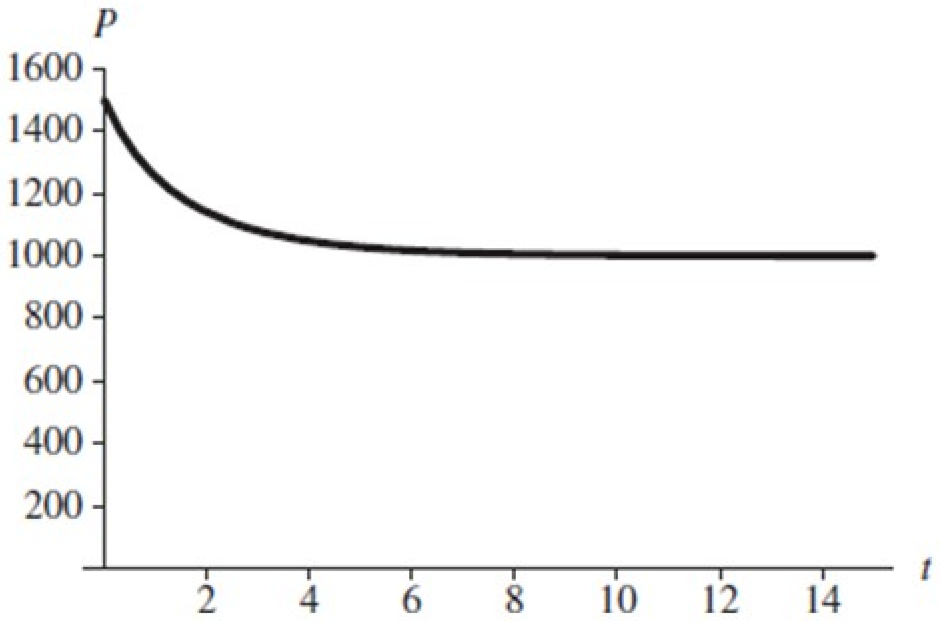
2

Carrying Capacity

- Differential equation (1) and difference equation (2) are called **logistic equations**.
- An **equilibrium solution** for a differential equation is a solution where the derivative is always zero. An **equilibrium solution** for a difference equation is a solution where the change is always zero.



Graph of logistic equation where initial population is 20, carrying capacity is 1000 and instantaneous rate of births is 50% with time in years



Graph of logistic equation where initial population is 1500, carrying capacity is 1000 and instantaneous rate of births is 50% with time in years

SYSTEM DYNAMICS MODELS WITH INTERACTIONS

- **Competition** is the struggle between individuals of a population or between species for the same limiting resource.
- If one individual (species) reduces the availability of the resource to the other, we term that type of competition **exploitative,** or **resource depletion**. This interaction is indirect and may involve removal of the resource or denial of living space.
- If there is direct interaction between individuals (species), where one interferes with or denies access to a resource, we term that competition **interference**. In this form, there may be physical contests for

Modeling Competition

- Sometimes two species are not eating each other but are competing for the same limited food source.
- For example, whitetip sharks (WTS) and blacktip sharks (BTS) in an area might feed on the same kinds of fish in a year when the fish supply is low.
- A large increase in one species, such as BTS, might have a detrimental effect on the ability of the other species, such as WTS, to obtain an adequate amount of food and, therefore, to thrive.
- Also, we expect that superior hunting skills of one species would diminish the food supply for the other species.
- As one species grows, the other shrinks, and vice versa

Modeling Competition

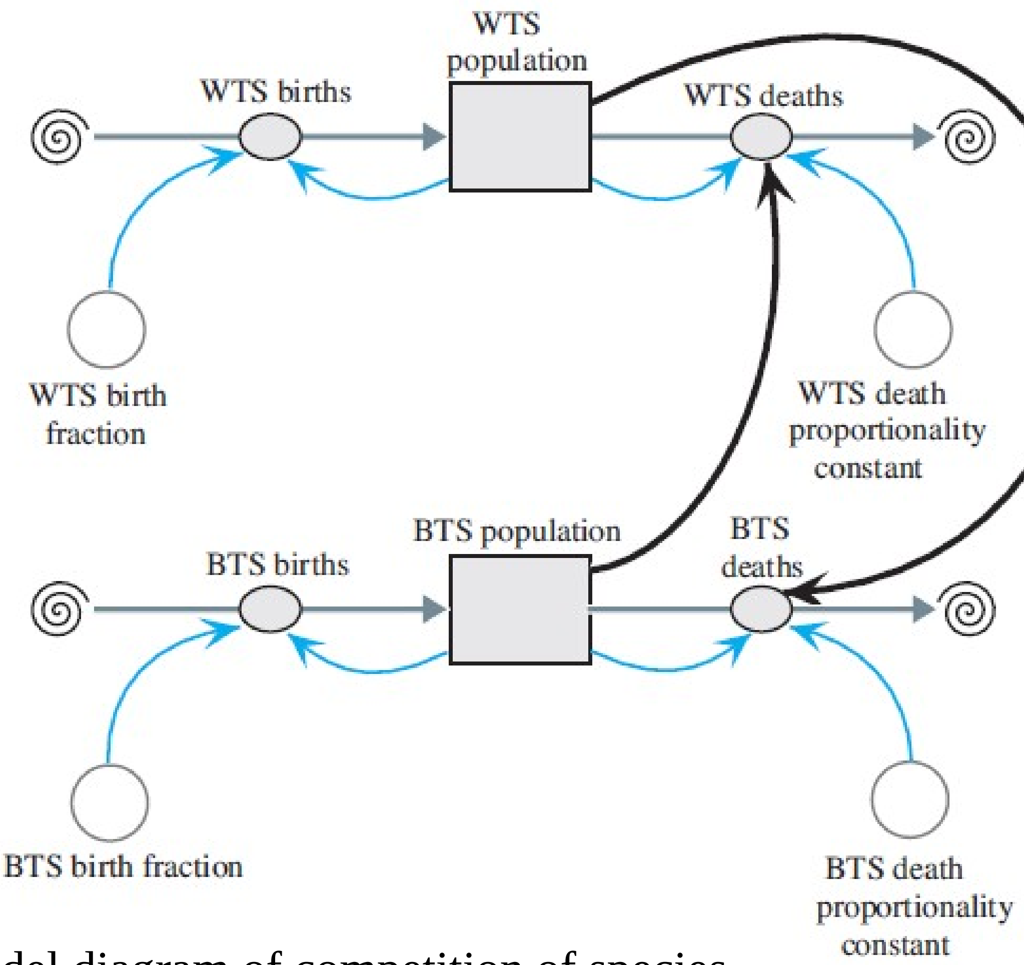
- In an unconstrained growth model, which ignores competition and limiting factors, we consider a population's (P) births to be proportional to the number of individuals in the population (r_1P) and its deaths to follow a similar proportionality (r_2P).
- Thus, in this model, the rate of change of the population is

$$dP/dt = r_1P - r_2P = (r_1 - r_2)P$$

- The solution is an exponential function, $P = P_0e^{(r1 - r2)t}$.
- However, with competition, a competing species has a negative impact on the rate of change of a population.
- In this situation, we can model the number of deaths of each species as being proportional to its population size and the population size of the other species.
- Thus, for B being the population of BTS and W the population of WTS, the number of deaths of each species is proportional to the product BW.
- Moreover, the constant of proportionality associated with this proportionality for one species reflects competitive skills of the other species.

Modeling Competition

- In this situation, we can model the number of deaths of each species as being proportional to its population size and the population size of the other species.
  - Δ(deaths of WTS) = *wBW*, where *w* is a WTS death proportionality constant
  - Δ(deaths of BTS) = *bWB* where *b* is a BTS death proportionality constant

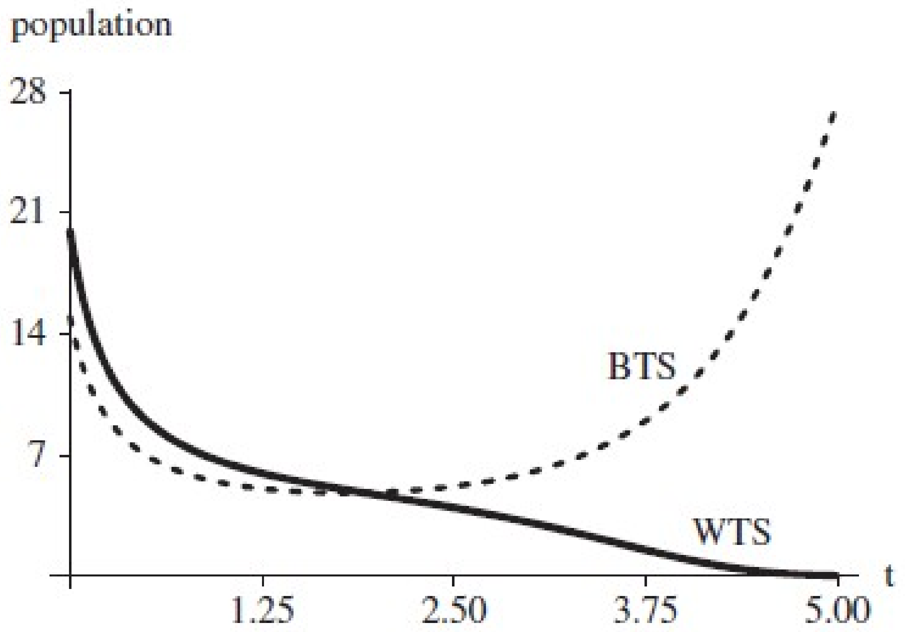


Model diagram of competition of species

Equation Set

Rule of Thumb: A constant of proportionality for a product of populations, such as *BW*, is frequently at least an order of magnitude (decimal point moved one place to the left) less than a constant of proportionality for one population, such as *B* or *W*.

- *BTS_population*(0) = 15
- *BTS_birth_fraction* = 1
- *BTS_births* = *BTS_birth_fraction* * *BTS_population*
- *BTS_death_proportionality_constant* = 0.20
- *BTS_deaths* = (*BTS_death_proportionality_constant* * *WTS_population*) * *BTS_population*
- *WTS_population*(0) = 20
- *WTS_birth_fraction* = 1
- *WTS_births* = *WTS_population* * *WTS_birth_fraction*
- *WTS_death_proportionality_constant* = 0.27
- *WTS_deaths* = (*WTS_death_proportionality_constant* * *BTS_population*) * *WTS_population*



Graph of population change of BTS and WTS over time

Predator-Prey Model

- When one species (**predator**) consumes another species (**prey**) while the latter is still living, the action is **predation**.
- Predator adaptations usually involve better prey detection and capture, whereas prey adaptations normally involve improved abilities to escape and avoid detection.
- In the 1920s, mathematicians Vito Volterra and Alfred Lotka independently proposed a model for populations of a predator species and its prey, such as hawk and squirrel populations in a certain area.
- This model is called **Lotka-Volterra** Model.

Historical Note During the Cultural Revolution in China (1958–1960), Chairman Mao Zedong decreed that all sparrows be killed because they ate too much of the crops and they seemed to be only for pleasure anyway. With reduction in its main predator, the insect population increased dramatically. The insects destroyed much more of the crops than the birds ever did. Consequently, the Chinese reversed the decision that caused the imbalance (PBS 2002).

Lotka-Volterra Model

- Let s be the number of squirrels in the area and h be the number of hawks. If no hawks are present, the change in s from time $t - \Delta t$ to time t is

$$\Delta s = s(t) - s(t - \Delta t)$$

$$= (\text{squirrel growth at time } t - \Delta t) * \Delta t$$

$$= k_s * s(t - \Delta t) * \Delta t \text{ for constant } k_s$$

- However, this prey's population is reduced by an amount proportional to the product of the number of hawks and the number of squirrels, $h(t - \Delta t) * s(t - \Delta t)$. Thus, with a proportionality constant k_{hs} for this reduction, the change in the number of squirrels from time $t - \Delta t$ to time t is as follows:

$$\Delta s = s(t) - s(t - \Delta t)$$

$$= (\text{squirrel growth at time } t - \Delta t) * \Delta t$$

$$= (k_s * s(t - \Delta t) - k_{hs} * h(t - \Delta t) * s(t - \Delta t)) * \Delta t$$

Lotka-Volterra Model

$$\Delta h = h(t) - h(t - \Delta t)$$
$$= (\text{hawk growth at time } t - \Delta t) * \Delta t$$
$$= (k_{sh} * s(t - \Delta t) * h(t - \Delta t) - k_h * h(t - \Delta t)) * \Delta t$$

- Although the deaths of the squirrels and the births of the hawks are both proportional to the product of the number of possible interactions of the two populations, their constants of proportionality, k_{hs} and k_{sh}, respectively, are probably different. For instance, 2% of the possible interactions might result in the death of a squirrel, while only 1% of the possible interactions might contribute to the birth of a hawk.

Lotka-Volterra Model

- We can express the predator-prey model, known as the **Lotka-Volterra model**,
- as the following pair of difference equations for the change in prey (here, change in
- the squirrel population, Δs) and change in predator (here, change in the hawk population, Δh) from time $t - \Delta t$ to time t:

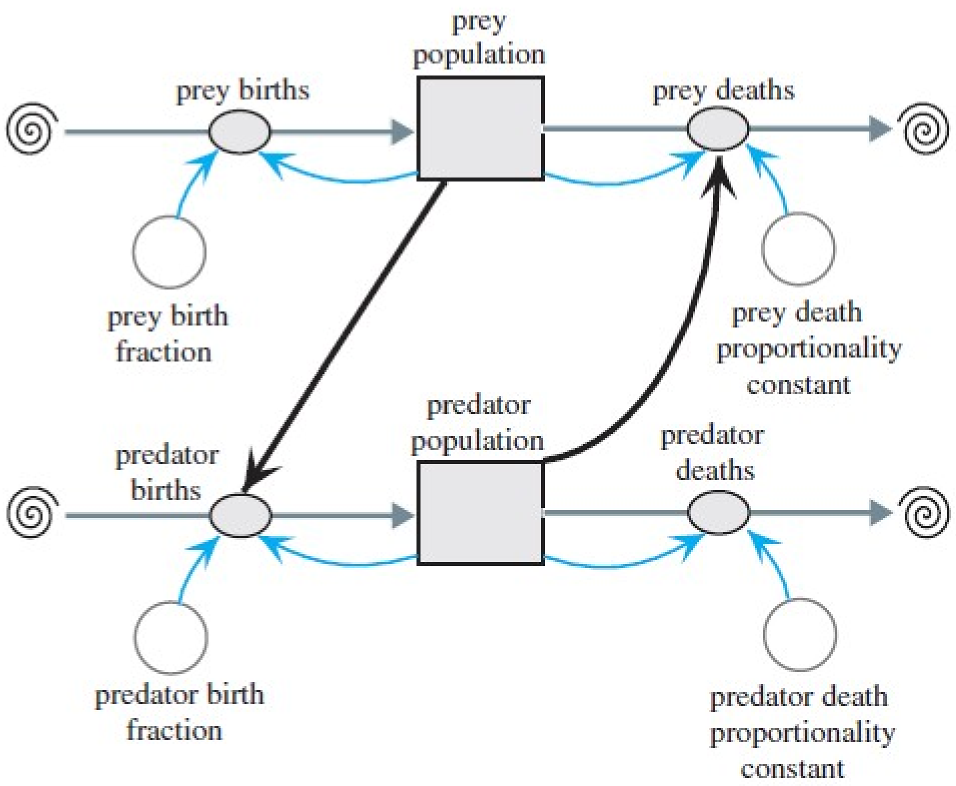
$$\Delta s = (k_s * s(t - \Delta t) - k_{hs} * h(t - \Delta t) * s(t - \Delta t)) * \Delta t$$

$$\Delta h = (k_{sh} * s(t - \Delta t) * h(t - \Delta t) - k_h * h(t - \Delta t)) * \Delta t$$

$$ds/dt = k_s s - k_{hs} hs$$

$$dh/dt = k_{sh} sh - k_h h$$

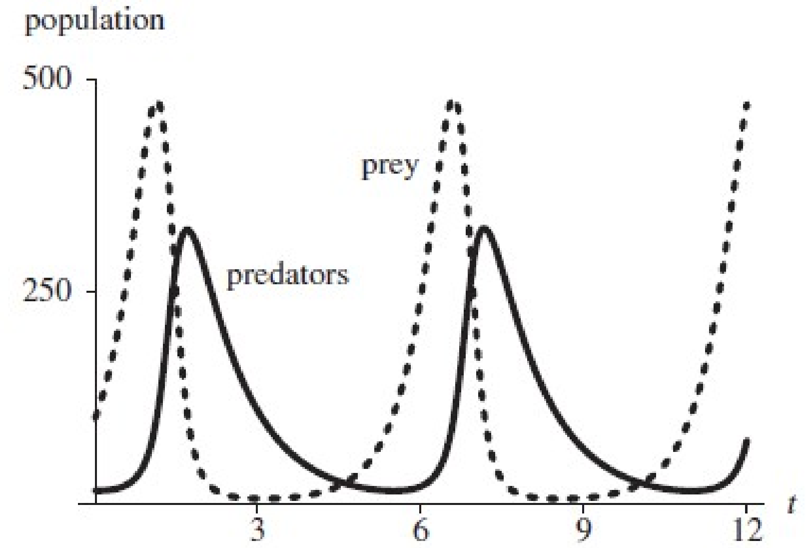
Lotka-Volterra Model



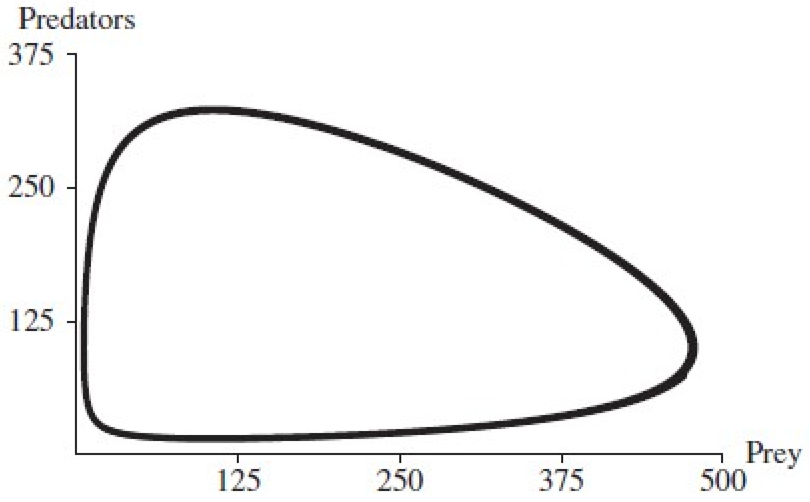
Predator-prey diagram

Equation set

- *predator_population*(0) = 15
- *predator_birth_fraction* = 0.01
- *predator_births* = (*predator_birth_fraction* * *prey_population*) * *predator_population*
- *predator_death_proportionality_constant* = 1.06
- *predator_deaths* = *predator_death_proportionality_constant* * *predator_population*
- *prey_population*(0) = 100
- *prey_birth_fraction* = 2
- *prey_births* = *prey_birth_fraction* * *prey_population*
- *prey_death_proportionality_constant* = 0.02
- *prey_deaths* = *(prey_death_proportionality_constant * predator_population) **
- *prey_population*



Graph of populations versus time in months



Graph of predator population versus prey population

Force and Motion

- We want to model skydiving now.
- For that we model velocity and acceleration first.
- The instantaneous rate of change, or derivative, of position (*s*) with respect to time (*t*) is **velocity** (*v*).
- Moreover, the instantaneous rate of change of velocity with respect to time is **acceleration** (*a*). In derivative notation, we have the following:
- We use these derivatives in modeling the motion of a ball when, on a windless day, someone standing on a bridge holds a ball over the side and tosses the ball straight up into the air.

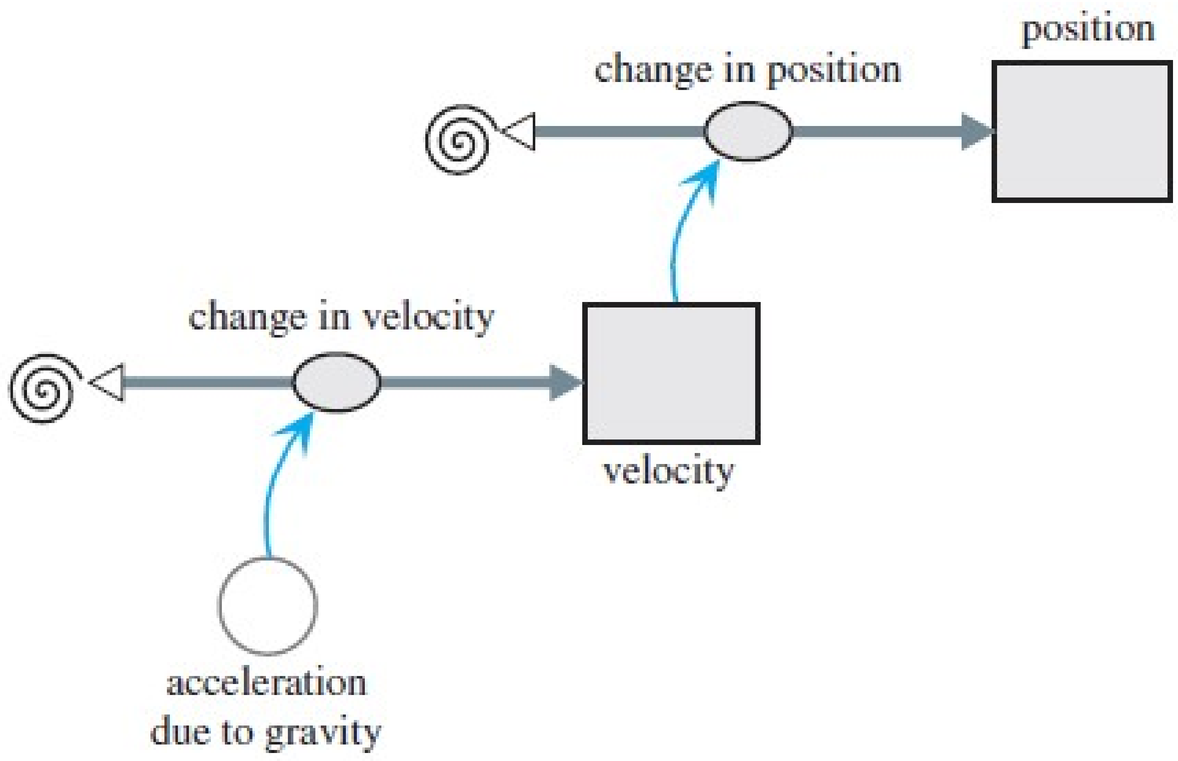
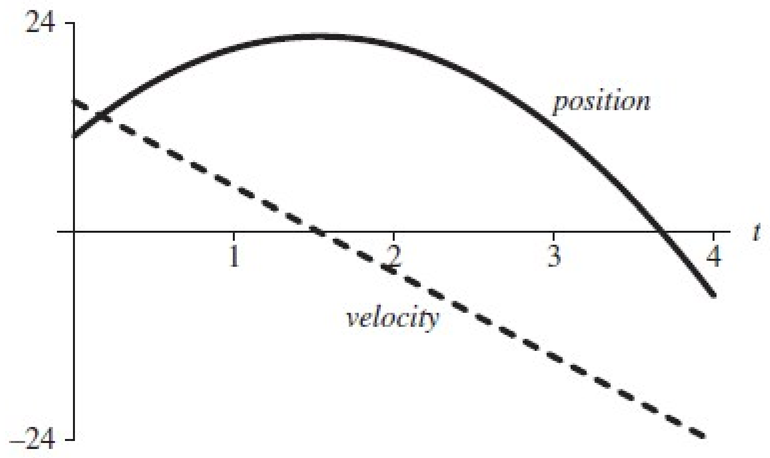
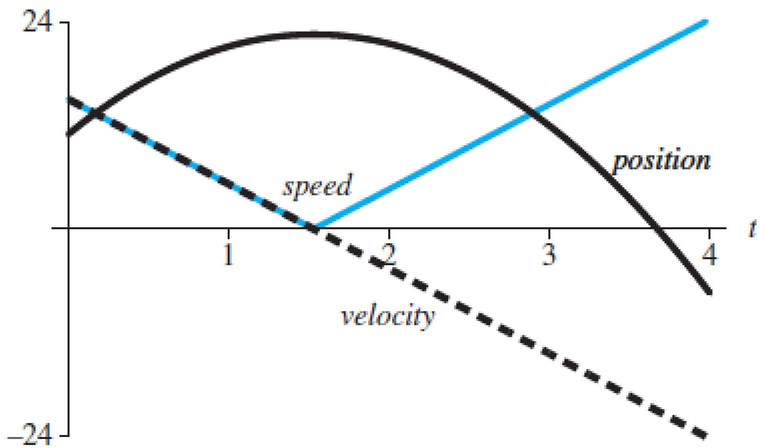


Diagram of motion of ball thrown straight up



Graph of velocity (m/s) and position (m) of ball versus time (s)



Graph of velocity (m/s), position (m), and speed (m/s) of ball versus time (s)

Force and Motion Model involving friction

- **Kinetic friction,** or **drag**, is a force. This force between objects is in the opposite direction to a moving object and tends to slow motion. Thus, kinetic friction dampens motion of an object.
- When an object moves through a fluid, such as air or water, the fluid friction is a function of the object's velocity. For example, the faster we pedal a bicycle, the harder it is for us to do so. As our velocity increases, so does the friction of the air on our bodies

Different models for fluid friction

Table 3.1.1
Summary of Several Models for Magnitude of Fluid Friction

Name	*Formula*	*Meanings of Symbols*	*When to Use*
Stokes's friction	$F = kv$	k constant v velocity	Very small object moving slowly through fluid
Newtonian friction	$F = 0.5CDAv^2$	C coefficient of drag D density of fluid A object's projected area in direction of movement v velocity	Larger objects moving faster through fluid
Newtonian friction through air	$F = 0.65Av^2$	A object's projected area in direction of movement v velocity	Larger objects with $C = 1$ moving faster through sea-level air

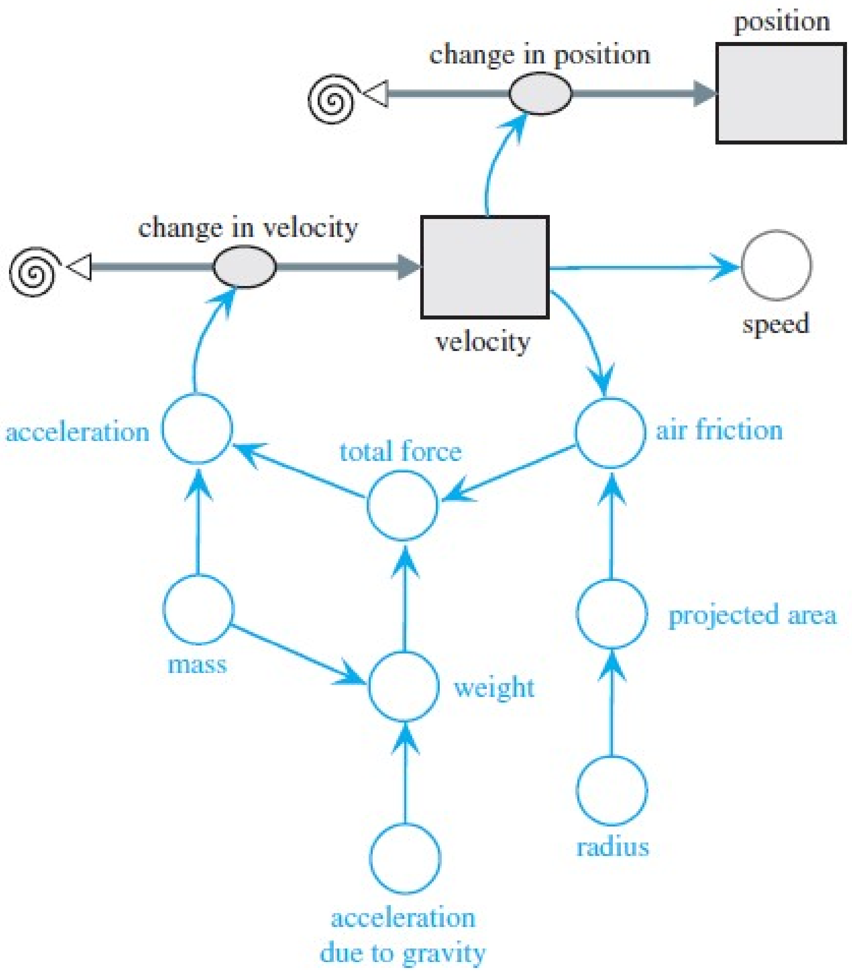
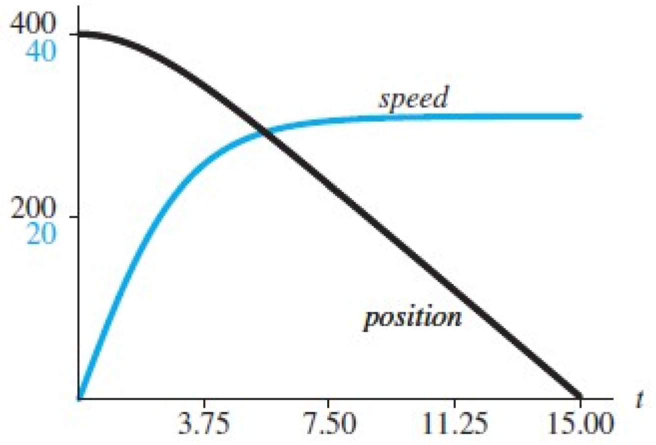


Diagram for motion of ball under influence of air friction

Equation Set for friction during fall model

mass = 0.5 kg
acceleration_due_to_gravity = –9.81 m/s2
radius = 0.05 m
weight = *mass* * *acceleration_due_to_gravity*
projected_area = 3.14159 * *radius*^2
air_friction = –0.65 * *projected_area* * *velocity* * ABS(*velocity*)
total_force = *weight* + *air_friction*
acceleration = *total_force*/*mass*
change_in_velocity = *acceleration*
change_in_position = *velocity*
speed = ABS(*velocity*)
velocity(0) = 0 m/s
velocity(*t*) = *velocity*(*t* – Δ*t*) + (*change_in_velocity*) * Δ*t*
position(0) = 400 m
position(t) = *position(t – Δt)* + *(change_in_position)* * Δ*t*



Graph of position (m) and speed (m/s) of object versus time (s) under influence of friction

Modeling Skydive

- For simplicity, we consider someone jumping out of a stationary helicopter at 2000 m (about 6562 ft), and we ignore changes in air density.
- The model for a skydive out of a helicopter has two phases.
- For our model, the main difference in these two phases is the projected area in the direction of motion, down.
  - One where the person is in a free fall
    - The crosssectional area of a jumper in the stable arch position with arms arched back and legs bent at the knees is approximately 0.4 m2 (about 4.3 ft2).
  - The other after the parachute opens, when the larger surface area results in more air resistance.
    - Parachutes vary in their designs, but 28 m2 (about 301 ft2) is a reasonable value.

- We trigger the pull of the ripcord by the height (*position*) above the ground, say, 1000 m (about 3281 ft).
- Thus, the diagram contains a converter/variable (*position_open*) for this quantity and nectors/arrows from *position* to *position_open* and from *position_open* to *projected_area*.

Modeling Skydive

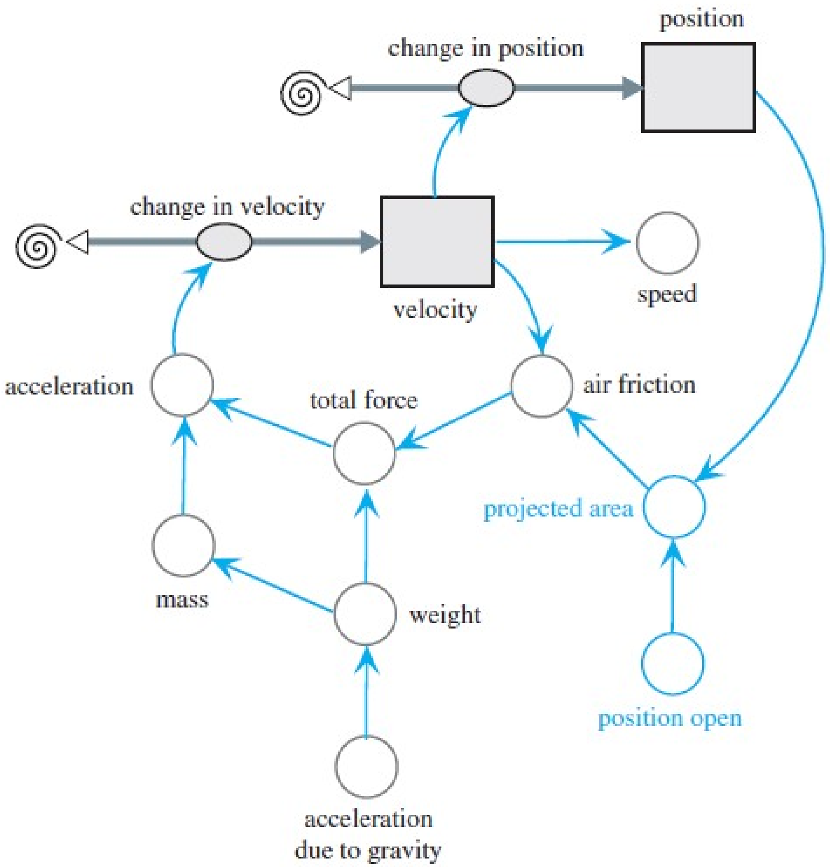
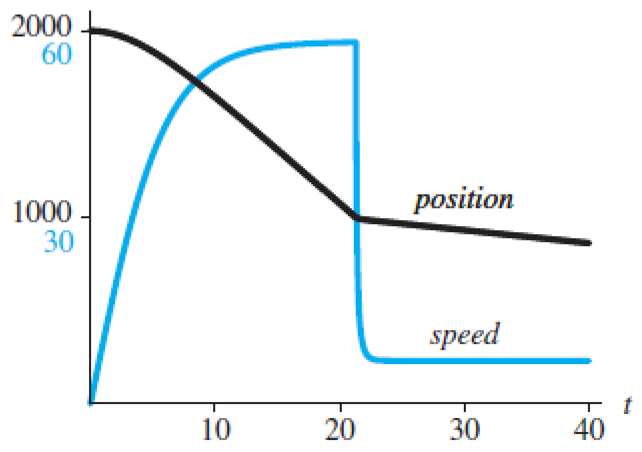


Diagram of skydiver's motion under influence of air friction

if (*position* > *position_open*)
  projected_area ← 0.4
else
  projected_area ← 28



Position (m) and speed (m/s) versus time (s) of skydiver