课题组组会-练习 5

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一　练习及结果

1. 在 $x \in [0,1]$ 的均匀网格上尝试使用 Hybrid Least Squares Reconstruction(HLSr) 对 $f(x)$ 及 $g(x)$ 进行 Hyperbolic rDG 的 DG(P0P3)+rDG(P1P2) 重构，其中 $f(x) = 1 + x + x^2 + x^3, g(x) = sin(\pi x)$.

a) 写出对于第 i 个单元的重构超定方程组，并测试重构精度。

b) 如果网格为不均匀网格呢？

2. 在 $x \in [0,1]$ 的均匀网格上尝试使用 Variational Reconstruction(VR) 对 $f(x), g(x)$ 进行 P0P1 重构，其中 $f(x) = 1 + x, g(x) = sin(\pi x)$

a) 尝试调整不同阶次的权重系数及边界面权重系数，测试重构精度。

b) 如果为不均匀网格呢？

解：1.

本题均考虑非均匀网格，均匀网格视为非均匀网格的特例。

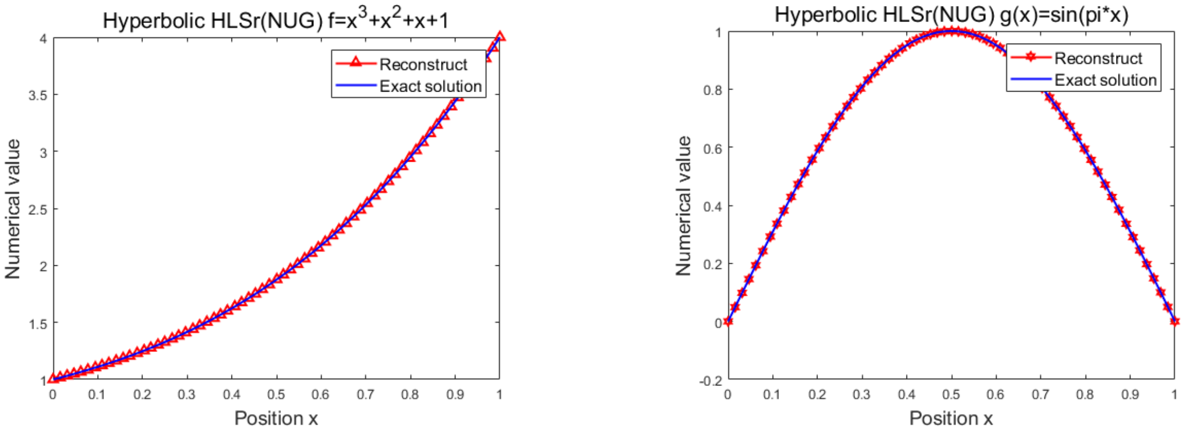
第 i 个单元的重构超定方程组为：

$$\begin{cases} \displaystyle\int_{\Omega_{i+1}} \varphi_i^R d\Omega = \int_{\Omega_{i+1}} \varphi_{i+1} d\Omega \\ \displaystyle\int_{\Omega_{i+1}} v_i^R d\Omega = \int_{\Omega_{i+1}} v_{i+1} d\Omega \\ \displaystyle\frac{\partial v_i^R}{\partial x}|_{x_c^{i+1}} = \frac{\partial v_{i+1}}{\partial x}|_{x_c^{i+1}} \end{cases}$$

将其转化成 $Ax = b$ 的形式，

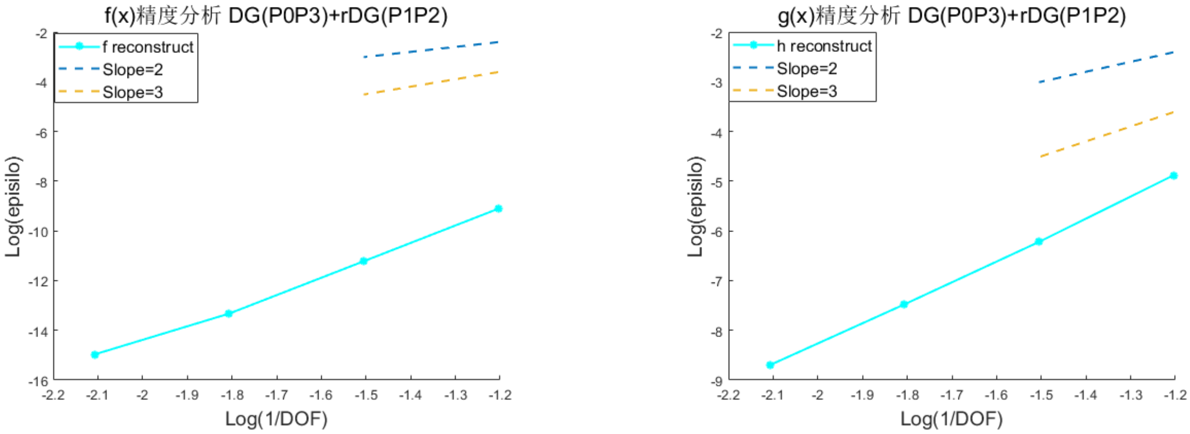
$$\begin{pmatrix} \frac{\overline{B_4^i}}{\Delta x_{i+1}} \\ \\ \frac{\overline{B_3^i}}{\Delta x_{i+1} \Delta x_i} \\ \\ \frac{\left(x_c^{i+1} - x_c^i\right)^2 \Delta x_{i+1}^2}{\Delta x_i^3} \end{pmatrix} \begin{pmatrix} \varphi_{xxx}^{c,R} \Delta x_i^3 \\ \\ \varphi_{xxx}^{c,R} \Delta x_i^3 \\ \\ \varphi_{xxx}^{c,R} \Delta x_i^3 \end{pmatrix} = \begin{pmatrix} \overline{\varphi_{i+1}} - \overline{\varphi_i} - \overline{\varphi_x^i} \Delta x_i \frac{x_c^{i+1} - x_c^i}{\Delta x_i} - \varphi_{xx}^{c,i} \Delta x_i^2 \frac{\overline{B_3^i}}{\Delta x_{i+1}} \\ \\ \overline{\varphi_x^{i+1}} - \overline{\varphi_x^i} \Delta x_i \Delta x_i^{-1} - \varphi_{xx}^c \Delta x_i^2 \frac{x_c^{i+1} - x_c^i}{\Delta x_i^2} \\ \\ \varphi_{xx}^{c,i+1} \Delta x_{i+1}^2 - \varphi_{xx}^c \Delta x_i^2 \frac{\Delta x_{i+1}^2}{\Delta x_i^2} \end{pmatrix}$$

最终得到 $f(x)$ 与 $g(x)$ 的重构比较图以及精度分析图：



(a) f 重构 **(b)** g 重构

图 1: Hyperbolic HLSr DG(P0P3)+rDG(P1P2)



(a) f 精度分析 **(b)** g 精度分析

图 2: 精度分析-单元格数为 8,16,32,64

注: 若对 f 考虑单元格数为 64,128,256,512 的精度分析，则可达到机器误差

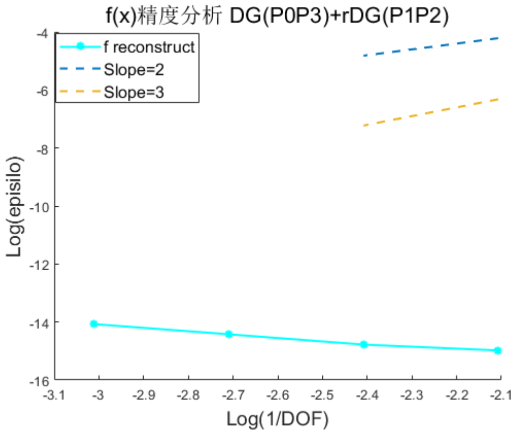


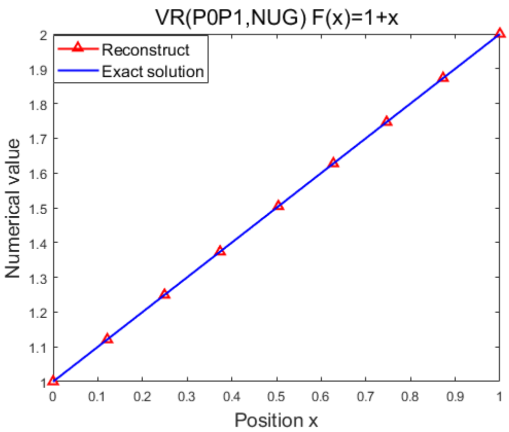
图 3: 精度分析-单元格数为 64,128,256,512

解：2.

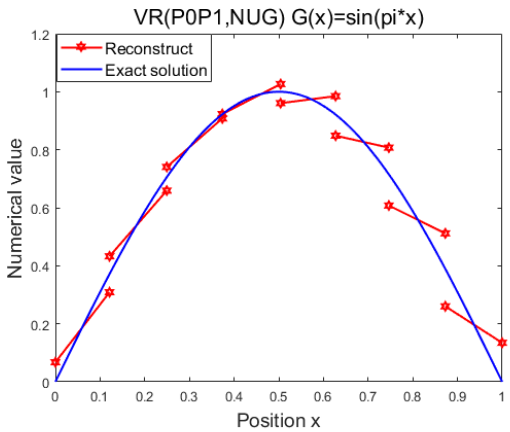
本题均考虑非均匀网格，均匀网格视为非均匀网格的特例。

设置权重系数 $\omega_0 = 1, \omega_1 = 0.5, \omega_b = 1$, 对 $f(x) = 1 + x$ 和 $g(x) = sin(\pi x)$ 进行 VR 重构，以下给出 Nelem=8 时候的重构图：

并对该情况下的 f, g 进行精度分析:

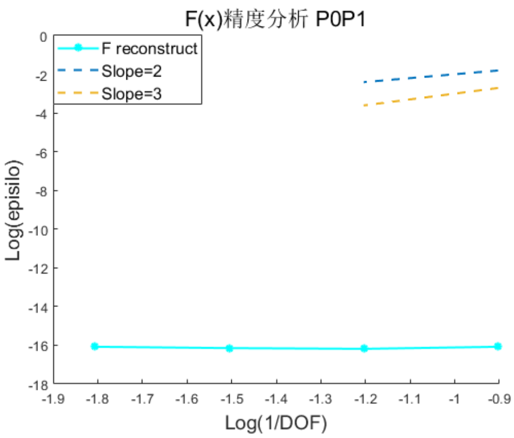


(a) f 重构

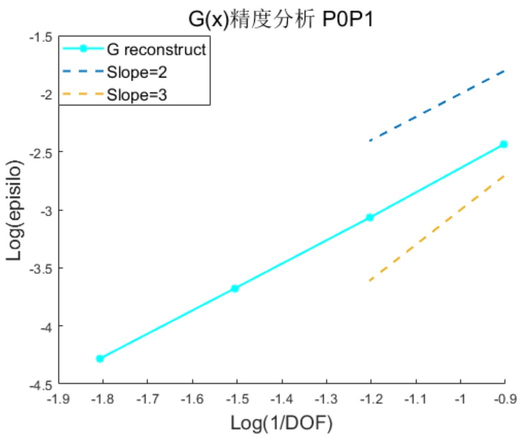


(b) g 重构

图 4: VR 重构 (Nelem=8)



(a) f 精度分析



(b) g 精度分析

图 5: VR 精度分析

调整权重系数以及边界面权重系数，测试重构精度，得到以下结论：

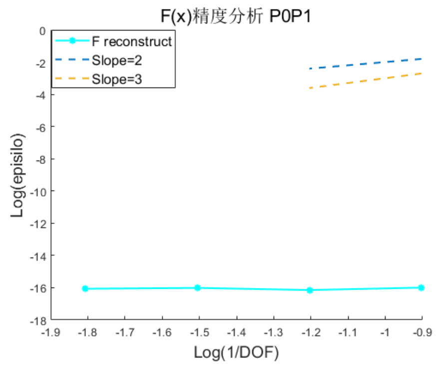
表 1: $G(x)$ 精度与 ω_0,ω_1,ω_b 的关系

ω_0	ω_1	ω_b	精度表现
0	—	—	差
—	0	—	差
—	—	$\uparrow 1$	$\uparrow$
—	$\downarrow 0.25$	—	$\uparrow$
$\uparrow 1$	—	—	$\uparrow$

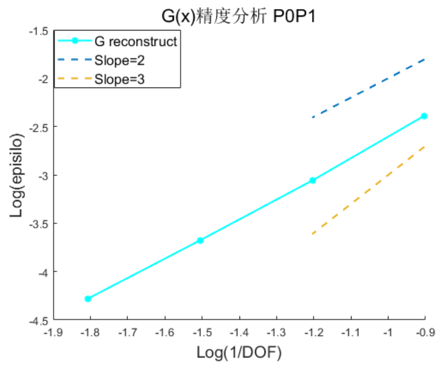
表 2: $F(x)$ 精度与 ω_0,ω_1,ω_b 的关系

ω_0	ω_1	ω_b	精度表现
0	—	—	差
—	0	—	差
—	—	$\uparrow 1$	$\uparrow$
—	$\uparrow 0.5$	—	$\uparrow$
$\downarrow 0.5$	—	—	$\uparrow$

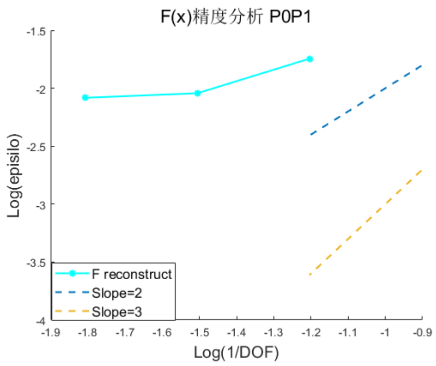
下面给出部分精度分析图：



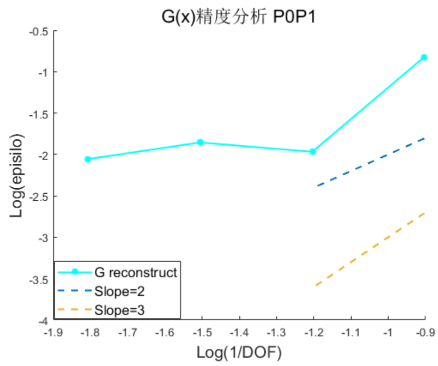
(a) $\omega_0 = 1, \omega_1 = 0.5, \omega_b = 0$



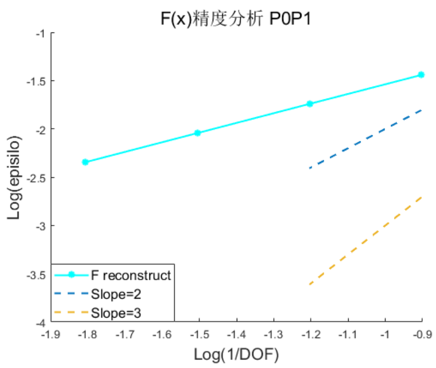
(b) $\omega_0 = 1, \omega_1 = 0.5, \omega_b = 0$



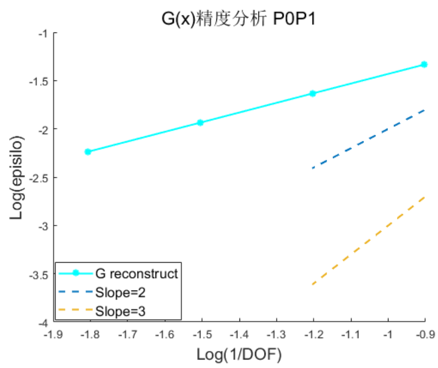
(c) $\omega_0 = 1, \omega_1 = 0, \omega_b = 0$



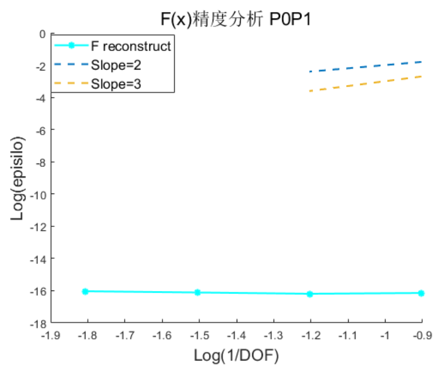
(d) $\omega_0 = 1, \omega_1 = 0, \omega_b = 0$



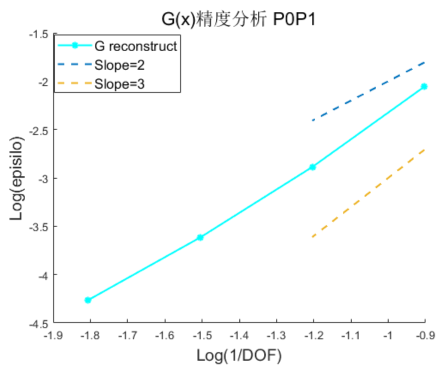
(e) $\omega_0 = 0, \omega_1 = 1, \omega_b = 0$



(f) $\omega_0 = 0, \omega_1 = 1, \omega_b = 0$



(g) $\omega_0 = 0.5, \omega_1 = 0.5, \omega_b = 0$



(h) $\omega_0 = 0.5, \omega_1 = 0.5, \omega_b = 0$

图 6: VR 精度分析 (调整权重系数)

二　附录 (代码, 仅展示部分)

VR

```
clc
clear all
close all
%Unit=8;endx=1;deltax=endx/Unit;numberx=Unit+1;omega0=1;omega1=0.5;omegab=1;
%记录内点位置，上下浮动不超过百分之 5
Grid=zeros(1,numberx);
Deltax=zeros(1,Unit);
for i=2:numberx-1
Grid(1,i)=(i-1)*deltax+(0.1*rand(1)-0.05)*deltax;
end
Grid(1,numberx)=endx;
for i=2:numberx
Deltax(i-1)=Grid(1,i)-Grid(1,i-1);%记录每个单元的区间长度
end
f=@(x)1+x;F=@(x)1;
h=@(x)sin(pi*x);H=@(x)pi*cos(pi*x);
Unumsolution=zeros(1,Unit);
Ureconstruct=zeros(Unit,1);
A=sparse(1:Unit,1:Unit,0,Unit,Unit);
R=zeros(Unit,1);
Unumsolution1=zeros(1,2);
Unumsolution2=zeros(2,numberx-1);
Acc=zeros(3,4);a1=[1/8,1/16,1/32,1/64];a2=[1/8,1/16];

%for i=1:numberx-1
Unumsolution(1,i)=1+0.5*(Grid(i+1)+Grid(i));
end

%构建大型稀疏矩阵
for iface=2:numberx-1
ieL=iface-1;xciL=0.5*(Grid(ieL)+Grid(ieL+1));
ieR=iface;xciR=0.5*(Grid(ieR)+Grid(ieR+1));
dLR=0.5*(Deltax(ieR)+Deltax(ieL));
A(ieL,ieL)=A(ieL,ieL)+2*(omega0^2*((Grid(iface)-xciL)/Deltax(ieL))^2+
    omega1^2*dLR^2/Deltax(ieL)^2)/dLR;
A(ieL,ieL+1)=A(ieL,ieL+1)-2*(omega0^2*((Grid(iface)-xciL)/Deltax(ieL))
    *((Grid(iface)-xciR)/Deltax(ieR))+omega1^2*dLR^2/(Deltax(ieR)*Deltax(
```

```
ieL)))/dLR;
R(ieL)=R(ieL)-2*omega0^2*(Unumsolution(ieL)-Unumsolution(ieR))*((Grid(iface)-xciL)/Deltax(ieL))/dLR;

A(ieR,ieR)=A(ieR,ieR)+2*(omega0^2*((Grid(iface)-xciR)/Deltax(ieR))^2+omega1^2*dLR^2/Deltax(ieR)^2)/dLR;
A(ieR,ieR-1)=A(ieR,ieR-1)-2*(omega0^2*((Grid(iface)-xciL)/Deltax(ieL))*((Grid(iface)-xciR)/Deltax(ieR))+omega1^2*dLR^2/(Deltax(ieR)*Deltax(ieL)))/dLR;
R(ieR)=R(ieR)+2*omega0^2*(Unumsolution(ieL)-Unumsolution(ieR))*((Grid(iface)-xciR)/Deltax(ieR))/dLR;
end

%B.C
%left
iface=1;
ieR=iface;
xciR=0.5*(Grid(ieR)+Grid(ieR+1));
A(ieR,ieR)=A(ieR,ieR)+4*omegab^2*((Grid(iface)-xciR)/Deltax(ieR))^2/Deltax(ieR);
R(ieR)=R(ieR)+4*omegab^2*(1-Unumsolution(ieR))*((Grid(iface)-xciR)/Deltax(ieR))/Deltax(ieR);

%Right
iface=numberx;
ieL=iface-1;
xciL=0.5*(Grid(ieL)+Grid(ieL+1));
A(ieL,ieL)=A(ieL,ieL)+4*omegab^2*((Grid(iface)-xciL)/Deltax(ieL))^2/Deltax(ieL);
R(ieL)=R(ieL)+4*omegab^2*(2-Unumsolution(ieL))*((Grid(iface)-xciL)/Deltax(ieL))/Deltax(ieL);


%Thomas 解三对角矩阵
L=zeros(1,Unit);U=zeros(1,Unit);C=zeros(1,Unit);
U(1)=A(1,1);
for i=2:numberx-1
L(i)=A(i,i-1)/U(i-1);
U(i)=A(i,i)-L(i)*A(i-1,i);
end
```

```matlab
Y=zeros(Unit,1);
Y(1)=R(1);
for i=2:numberx-1
Y(i)=R(i)-L(i)*Y(i-1);
end
Ureconstruct(numberx-1)=Y(numberx-1)/U(numberx-1);
for i=numberx-2:-1:1
Ureconstruct(i)=(Y(i)-A(i,i+1)*Ureconstruct(i+1))/U(i);
end





%figure
k=1;
x=Grid(k):1*(Grid(k+1)-Grid(k)):Grid(k+1);
xci=(Grid(k+1)+Grid(k))/2;
p=@(x)Unumsolution(1,k)+Ureconstruct(k,1)*(x-xci)/Deltax(k);
y=p(x);
plot(x,y,'-r^','linewidth',1.5);hold on
H1=plot(x,y,'-r^','linewidth',1.5);hold on

for k=2:numberx-1
x=Grid(k):1*(Grid(k+1)-Grid(k)):Grid(k+1);
xci=(Grid(k+1)+Grid(k))/2;
p=@(x)Unumsolution(1,k)+Ureconstruct(k,1)*(x-xci)/Deltax(k);
y=p(x);
plot(x,y,'-r^','linewidth',1.5);
end

hold on
x=Grid(1):0.01*(Grid(numberx)-Grid(1)):Grid(numberx);
plot(x,f(x),'-b','linewidth',1.5);
H2=plot(x,f(x),'-b','linewidth',1.5);
lgd=legend([H1,H2],'Reconstruct','Exact solution');
lgd.FontSize=12;
xlabel('Position x','fontsize',14)
ylabel('Numerical value','fontsize',14)
title('VR(POP1,NUG) F(x)=1+x','fontsize',16)
```

```
hold off

%计算精度
Acc(1,1)=Accuracy(8);
Acc(1,2)=Accuracy(16);
Acc(1,3)=Accuracy(32);
Acc(1,4)=Accuracy(64);
for k=1:3
accuracyf(k)=(log10(Acc(1,k+1))-log10(Acc(1,k)))./(log10(a1(1,k+1))-
    log10(a1(1,k)));
end

figure
hold on
plot(log10(a1),log10(Acc(1,:)),'-c*','linewidth',1.5)
H1=plot(log10(a1),log10(Acc(1,:)),'-c*','linewidth',1.5);

H2=plot(log10(a2),2*log10(a2),'--','linewidth',1.5);
plot(log10(a2),3*log10(a2),'--','linewidth',1.5)
H3=plot(log10(a2),3*log10(a2),'--','linewidth',1.5);
lgd=legend([H1,H2,H3],'F reconstruct','Slope=2','Slope=3');
lgd.FontSize=12;
xlabel('Log(1/DOF)','fontsize',14)
ylabel('Log(episilo)','fontsize',14)
title('F(x)精度分析 POP1','fontsize',16)
```