# 课题组组会-练习 10

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## 一　练习及结果

1. 已知在笛卡尔坐标系下，算子 $\nabla$=$(\partial_x, \partial_y, \partial_z)$, 对于标量 $\psi$,$\phi$, 矢量 $A = (A_x, A_y, A_z), B = (B_x, B_y, B_z)$.

(1) 用 $\nabla$ 算子表示 $\psi$ 的梯度 grad $\psi$, A 的散度 div A, A 的旋度 curl A。

(2) 试证明以下微分恒等式

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$$

$$\nabla \cdot (\psi A) = \psi\nabla \cdot A + A \cdot \nabla\psi$$

2. 证明 $\iiint_V \frac{D(\rho dV)}{Dt} = \frac{D}{Dt}\iiint_V \rho dV$.

解:1

(1)

$\text{grad}\psi=\nabla\psi=\left(\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial z}\right)$

div $\text{A}=\nabla\cdot\text{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$

curl A[1]$=\nabla\times\text{A}=\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

(2)

$$
\begin{aligned}
(i)\quad \nabla\left(\psi\phi\right) &= \left(\frac{\partial\psi\phi}{\partial x},\frac{\partial\psi\phi}{\partial y},\frac{\partial\psi\phi}{\partial z}\right) \\
&= \left(\frac{\partial\psi}{\partial x}\phi+\psi\frac{\partial\phi}{\partial x},\frac{\partial\psi}{\partial y}\phi+\psi\frac{\partial\phi}{\partial y},\frac{\partial\psi}{\partial z}+\psi\frac{\partial\phi}{\partial z}\right) \\
&=\phi\left(\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial z}\right)+\psi\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right) \\
&=\phi\nabla\psi+\psi\nabla\phi.
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad \nabla\cdot\left(A+B\right) &=\nabla\cdot\left(A_x+B_x,A_y+B_y,A_z+B_z\right) \\
&=\frac{\partial A_x}{\partial x}+\frac{\partial B_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial B_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial B_z}{\partial z} \\
&=\nabla\cdot A+\nabla\cdot B
\end{aligned}
$$

$$
\begin{aligned}
(iii)\quad \nabla\cdot\left(\psi A\right) &=\nabla\left(\psi A_x,\psi A_y,\psi A_z\right) \\
&=\frac{\partial\left(\psi A_x\right)}{\partial x}+\frac{\partial\left(\psi A_y\right)}{\partial y}+\frac{\partial\left(\psi A_z\right)}{\partial z} \\
&=\frac{\partial\psi}{\partial x}A_x+\psi\frac{\partial A_x}{\partial x}+\frac{\partial\psi}{\partial y}A_y+\psi\frac{\partial A_y}{\partial y}+\frac{\partial\psi}{\partial z}A_z+\psi\frac{\partial A_z}{\partial z} \\
&=\psi\nabla\cdot A+A\cdot\nabla\psi
\end{aligned}
$$

解:2

$$
\begin{aligned}
\frac{D\left(\rho dV\right)}{Dt} &=\frac{\partial\left(\rho dV\right)}{\partial t}+\overrightarrow{v}\cdot\nabla\left(\rho dV\right) \\
\Rightarrow \iiint_V\frac{D\left(\rho dV\right)}{Dt} &=\iiint_V\frac{\partial\left(\rho dV\right)}{\partial t}+\iiint_V\overrightarrow{v}\cdot\nabla\left(\rho dV\right) \\
&=\iiint_V\frac{\partial\left(\rho dV\right)}{\partial t}+\iiint_V u\frac{\partial\left(\rho dV\right)}{\partial x}+v\frac{\partial\left(\rho dV\right)}{\partial y}+w\frac{\partial\left(\rho dV\right)}{\partial z} \\
&=\iiint_V\frac{\partial\left(\rho dV\right)}{\partial t}+u\iiint_V\frac{\partial\left(\rho dV\right)}{\partial x}+v\iiint_V\frac{\partial\left(\rho dV\right)}{\partial y}+w\iiint_V\frac{\partial\left(\rho dV\right)}{\partial z}
\end{aligned}
$$

并且 $\frac{D}{Dt}\iiint_V\rho dV=\frac{\partial\iiint_V\rho dV}{\partial t}+\overrightarrow{v}\cdot\nabla\left(\iiint_V\rho dV\right)$

[1] B 站讲解视频:https://www.bilibili.com/video/BV1a541127cX

$\therefore$ 若 $\iiint_V \frac{\partial(\rho dV)}{\partial t} = \frac{\partial \iiint_V \rho dV}{\partial t}$, 则等式成立。

等价于

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iiint_V \rho \frac{\partial dV}{\partial t} = \frac{\partial \iiint_V \rho dV}{\partial t}$$

考虑右式离散情况 $\sum_{i=1}^{N(t)}$

$$\begin{aligned}
\frac{\partial \sum_i^{N(t)} \rho_i \left(x, y, z, t\right) dV \left(i, t\right)}{\partial t} &= \lim_{t_2 \to t_1} \frac{\sum_{i=1}^{N(t_2)} \rho_i \left(x, y, z, t_2\right) dV \left(i, t_2\right) - \sum_{i=1}^{N(t_1)} \rho_i \left(x, y, z, t_1\right) dV \left(i, t_1\right)}{t_2 - t_1} \\
&= \sum_{i=1}^{N(t_1)} \lim_{t_2 \to t_1} \frac{\rho_i \left(x, y, z, t_2\right) dV \left(i, t_2\right) - \rho_i \left(x, y, z, t_1\right) dV \left(i, t_1\right)}{t_2 - t_1}
\end{aligned}$$

右 $=$ 左 $\Rightarrow$ 离散情况下等式成立 $\Rightarrow$ 连续情况下等式成立 (总可以证明)。