课题组组会-练习 11

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一　练习及结果

在不考虑源项的情况下，Navier-Stokes 方程可写作

$$\frac{\partial U}{\partial t}+\frac{\partial F_j}{\partial x_j}=\frac{\partial G_j}{\partial x_j}$$

其中

$$U=\begin{pmatrix}\rho\\ \rho v_i\\ \rho E\end{pmatrix},F_j=\begin{pmatrix}\rho v_j\\ \rho v_i v_j+p\delta_{ij}\\ v_j\left(\rho E+p\right)\end{pmatrix},G_j=\begin{pmatrix}0\\ \tau_{ij}\\ v_l\tau_{lj}+q_j\end{pmatrix}.$$

请利用以下参考量，推倒无量纲形式下的 Navier-Stokes 方程组。

长度	l_{ref}	L	$x_i^*=\frac{x_i}{l_{ref}}$
温度	T_{ref}	T_∞	$T^*=\frac{T}{T_\infty}$
压强	p_{ref}	p_∞	$p^*=\frac{p}{p_\infty}$
密度	ρ_{ref}	ρ_∞	$\rho^*=\frac{\rho}{\rho_\infty}$
速度	V_{ref}	$\sqrt{\frac{p_{ref}}{\rho_{ref}}}$	$v_i^*=\frac{v_i}{V_{ref}}$
能量	E_{ref}	V_∞^2	$E^*=\frac{E}{E_{ref}}$
时间	t_{ref}	$\frac{l_{ref}}{V_{ref}}$	$t^*=\frac{t}{t_{ref}}$
黏性系数	μ_{ref}	μ_∞	$\mu^*=\frac{\mu}{\mu_\infty}$
导热系数	k_{ref}	k_∞	$k^*=\frac{k}{k_\infty}$
雷诺数	Re	$\frac{\rho_{ref}V_{ref}l_{ref}}{\mu_{ref}}$	
普朗特数	Pr	$\frac{\mu_{ref}\gamma R}{k_{ref}(\gamma-1)}$	

解:

注意：$\frac{\partial mx}{\partial nt}=\frac{m}{n}\frac{\partial x}{\partial t}$

(1) 考虑连续性方程

$$\frac{\partial\left(\rho^*\rho_\infty\right)}{\partial\left(t^*t_{ref}\right)}+\frac{\partial\left(\rho^*\rho_\infty v_j^*V_{ref}\right)}{\partial\left(x_j^*l_{ref}\right)}=0$$

$$\Rightarrow\frac{\partial\rho^*}{\partial t^*}+\frac{\partial\rho^*v_j^*}{\partial x_j^*}=0$$

(2) 考虑动量方程

$$\frac{\partial\left(\rho^*\rho_\infty v_i^*V_{ref}\right)}{\partial\left(t^*t_{ref}\right)}+\frac{\partial\left(\rho^*\rho_\infty v_i^*v_j^*V_{ref}^2+p^*p_\infty\delta_{ij}\right)}{\partial\left(x_j^*l_{ref}\right)}$$

$$=\frac{\partial}{\partial\left(x_j^*l_{ref}\right)}\left(\mu^*\mu_\infty\left(\frac{\partial\left(v_i^*V_{ref}\right)}{\partial\left(x_j^*\_ref\right)}+\frac{\partial\left(v_j^*V_{ref}\right)}{\partial\left(x_i^*l_{ref}\right)}\right)-\frac{2}{3}\mu^*\mu_\infty\frac{\partial\left(v_k^*V_{ref}\right)}{\partial\left(x_k^*l_{ref}\right)}\delta_{ij}\right)$$

$$\Rightarrow\frac{\partial\rho^*v_i^*}{\partial t^*}+\frac{\partial\rho^*v_i^*v_j^*+p^*\delta_{ij}}{\partial x_j^*}=\frac{\partial\tau_{ij}^*/Re}{\partial x_j^*}$$

其中 $\tau_{ji}^*=\mu^*\left(\frac{\partial v_i^*}{\partial x_j^*}+\frac{\partial v_j^*}{\partial x_i^*}\right)-\frac{2}{3}\mu^*\frac{\partial v_k^*}{\partial x_k^*}\delta_{ij}$

(3) 考虑能量方程

$$\frac{\rho_\infty E_{ref}}{t_{ref}}\frac{\partial\rho^*E^*}{\partial t^*}+\frac{V_{ref}}{l_{ref}}\left(\frac{\partial}{\partial x_j^*}\left(v_j^*\rho^*E^*\right)\rho_\infty E_{ref}+\frac{\partial p^*v_j^*}{\partial x_j^*}p_\infty\right)$$

$$=\frac{1}{l_{ref}}\left(\frac{V_{ref}^2\mu_\infty}{l_{ref}}\frac{\partial v_i^*\tau_{lj}^*}{\partial x_j^*}+\frac{T_\infty k_\infty}{l_{ref}}\frac{\partial q_j^*}{\partial x_j^*}\right)$$

$$\Rightarrow\frac{\partial\rho^*E^*}{\partial t^*}+\frac{v_j^*\rho^*E^*+\frac{p_\infty}{\rho_\infty V_\infty^2}v_j^*p^*}{\partial x_j^*}=\frac{\partial\frac{p_{ref}}{ReV_\infty^2\rho_{ref}}v_l^*\tau_{lj}^*+\frac{k_\infty T_\infty}{Re\mu_\infty V_\infty^2}q_j^*}{\partial x_j^*}$$

其中 $q_j^*=k^*\frac{\partial T^*}{\partial x_j^*}$.

综上，用无 * 的符号表示无量纲参数，得到新的 Navier-Stokes 方程:

$$\frac{\partial U}{\partial t}+\frac{\partial F_j}{\partial x_j}=\frac{\partial G_j}{\partial x_j}$$

其中

$$U = \begin{pmatrix} \rho \\ \rho v_i \\ \rho E \end{pmatrix}, F_j = \begin{pmatrix} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j \rho E + \frac{p_\infty}{\rho_\infty V_\infty^2} p v_j \end{pmatrix}, G_j = \begin{pmatrix} 0 \\ \frac{\tau_{ji}}{Re} \\ \frac{p_\infty}{\rho_\infty Re V_\infty^2} v_l \tau_{lj} + \frac{k_\infty T_\infty}{\mu_\infty Re V_\infty^2} q_j \end{pmatrix}.$$