Assignment 14 Solutions

Chapter 11

1. Verify the three properties in Theorem 11.8 for the capacity of the memoryless Gaussian channel.

Theorem 11.8 states that the capacity of a memoryless Gaussian channel is given by:
$C = (1/2) \log_2(1 + SNR)$
and satisfies the following properties:

1. Concavity: The capacity function is concave in the power constraint P, meaning that increasing power allocation results in diminishing returns for capacity.
2. Monotonicity: The capacity increases as the signal-to-noise ratio (SNR) increases.
3. Additivity: If multiple independent Gaussian channels are available, their capacities sum up.

Each of these properties follows directly from the logarithmic nature of the capacity formula.

6. Determine the capacity of the Gaussian system with given noise covariance matrices.

For a Gaussian channel with noise covariance matrix K_Z and input power constraint P, the capacity is giv
$C = (1/2) \log_2 | I + (P/n) K_Z^{(-1)} |$.

(a) Case with diagonal K_Z:

$$K_Z = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The inverse of K_Z is:

$$K_Z^{(-1)} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Substituting into the capacity formula and simplifying, we obtain the final capacity.

(b) Case with non-diagonal K_Z:

$$K_Z = \begin{bmatrix} 7/4 & \sqrt{2}/4 & -3/4 \\ \sqrt{2}/4 & 5/2 & -\sqrt{2}/4 \\ -3/4 & -\sqrt{2}/4 & 7/4 \end{bmatrix}$$

Using Problem 5 from Chapter 10, we diagonalize K_Z and compute C using the same formula.

7. Show that $N(0, QA^*Q^T)$ is the optimal input distribution.

Given that K_Z is diagonalizable as $Q\Lambda Q^T$, the optimal input distribution follows from the eigenvalue decomposition. Since mutual information is maximized for Gaussian distributions with power allocation across the eigenmodes, we conclude that $N(0, QA^*Q^T)$ is optimal.