9 ☐ DIFFERENTIAL EQUATIONS

9.1 Modeling with Differential Equations

1. $y = \frac{2}{3}e^x + e^{-2x} \quad \Rightarrow \quad y' = \frac{2}{3}e^x - 2e^{-2x}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\begin{aligned} \text{LHS} = y' + 2y &= \tfrac{2}{3}e^x - 2e^{-2x} + 2\left(\tfrac{2}{3}e^x + e^{-2x}\right) = \tfrac{2}{3}e^x - 2e^{-2x} + \tfrac{4}{3}e^x + 2e^{-2x} \\ &= \tfrac{6}{3}e^x = 2e^x = \text{RHS} \end{aligned}$$

2. $y = -t\cos t - t \quad \Rightarrow \quad dy/dt = -t(-\sin t) + \cos t(-1) - 1 = t\sin t - \cos t - 1$.

$$\begin{aligned} \text{LHS} = t\,\frac{dy}{dt} &= t(t\sin t - \cos t - 1) = t^2\sin t - t\cos t - t \\ &= t^2 \sin t + y = \text{RHS}, \end{aligned}$$

so y is a solution of the differential equation. Also $y(\pi) = -\pi\cos\pi - \pi = -\pi(-1) - \pi = \pi - \pi = 0$, so the initial condition is satisfied.

3. (a) $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2e^{rx}$. Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get $2r^2e^{rx} + re^{rx} - e^{rx} = 0 \quad \Rightarrow \quad (2r^2 + r - 1)e^{rx} = 0 \quad \Rightarrow$ $(2r - 1)(r + 1) = 0 \quad$ [since e^{rx} is never zero] $\quad \Rightarrow \quad r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = ae^{x/2} + be^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$y = ae^{x/2} + be^{-x} \quad \Rightarrow \quad y' = \frac{1}{2}ae^{x/2} - be^{-x} \quad \Rightarrow \quad y'' = \frac{1}{4}ae^{x/2} + be^{-x}$.

$$\begin{aligned} \text{LHS} = 2y'' + y' - y &= 2\left(\tfrac{1}{4}ae^{x/2} + be^{-x}\right) + \left(\tfrac{1}{2}ae^{x/2} - be^{-x}\right) - \left(ae^{x/2} + be^{-x}\right) \\ &= \tfrac{1}{2}ae^{x/2} + 2be^{-x} + \tfrac{1}{2}ae^{x/2} - be^{-x} - ae^{x/2} - be^{-x} \\ &= \left(\tfrac{1}{2}a + \tfrac{1}{2}a - a\right)e^{x/2} + (2b - b - b)e^{-x} \\ &= 0 = \text{RHS} \end{aligned}$$

4. (a) $y = \cos kt \quad \Rightarrow \quad y' = -k\sin kt \quad \Rightarrow \quad y'' = -k^2\cos kt$. Substituting these expressions into the differential equation $4y'' = -25y$, we get $4(-k^2\cos kt) = -25(\cos kt) \quad \Rightarrow \quad (25 - 4k^2)\cos kt = 0 \quad$ [for all t] $\quad \Rightarrow \quad 25 - 4k^2 = 0 \quad \Rightarrow$ $k^2 = \frac{25}{4} \quad \Rightarrow \quad k = \pm\frac{5}{2}$.

(b) $y = A\sin kt + B\cos kt \quad \Rightarrow \quad y' = Ak\cos kt - Bk\sin kt \quad \Rightarrow \quad y'' = -Ak^2\sin kt - Bk^2\cos kt$.

The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\begin{aligned} \text{LHS} = 4y'' + 25y &= 4(-Ak^2\sin kt - Bk^2\cos kt) + 25(A\sin kt + B\cos kt) \\ &= -4Ak^2\sin kt - 4Bk^2\cos kt + 25A\sin kt + 25B\cos kt \\ &= (25 - 4k^2)A\sin kt + (25 - 4k^2)B\cos kt \\ &= 0 \quad \text{since } k^2 = \tfrac{25}{4}. \end{aligned}$$

5. (a) $y = \sin x \quad \Rightarrow \quad y' = \cos x \quad \Rightarrow \quad y'' = -\sin x.$

LHS $= y'' + y = -\sin x + \sin x = 0 \neq \sin x$, so $y = \sin x$ **is not** a solution of the differential equation.

(b) $y = \cos x \quad \Rightarrow \quad y' = -\sin x \quad \Rightarrow \quad y'' = -\cos x.$

LHS $= y'' + y = -\cos x + \cos x = 0 \neq \sin x$, so $y = \cos x$ **is not** a solution of the differential equation.

(c) $y = \frac{1}{2}x\sin x \quad \Rightarrow \quad y' = \frac{1}{2}(x\cos x + \sin x) \quad \Rightarrow \quad y'' = \frac{1}{2}(-x\sin x + \cos x + \cos x).$

LHS $= y'' + y = \frac{1}{2}(-x\sin x + 2\cos x) + \frac{1}{2}x\sin x = \cos x \neq \sin x$, so $y = \frac{1}{2}x\sin x$ **is not** a solution of the differential equation.

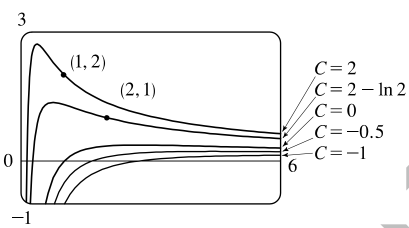
(d) $y = -\frac{1}{2}x\cos x \quad \Rightarrow \quad y' = -\frac{1}{2}(-x\sin x + \cos x) \quad \Rightarrow \quad y'' = -\frac{1}{2}(-x\cos x - \sin x - \sin x).$

LHS $= y'' + y = -\frac{1}{2}(-x\cos x - 2\sin x) + \left(-\frac{1}{2}x\cos x\right) = \sin x =$ RHS, so $y = -\frac{1}{2}x\cos x$ **is** a solution of the differential equation.

6. (a) $y = \dfrac{\ln x + C}{x} \quad \Rightarrow \quad y' = \dfrac{x \cdot (1/x) - (\ln x + C)}{x^2} = \dfrac{1 - \ln x - C}{x^2}.$

$$\begin{aligned}\text{LHS} &= x^2y' + xy = x^2 \cdot \frac{1 - \ln x - C}{x^2} + x \cdot \frac{\ln x + C}{x} \\ &= 1 - \ln x - C + \ln x + C = 1 = \text{RHS},\end{aligned}$$ so y is a solution of the differential equation.

(b)



A few notes about the graph of $y = (\ln x + C)/x$:

(1) There is a vertical asymptote of $x = 0$.

(2) There is a horizontal asymptote of $y = 0$.

(3) $y = 0 \quad \Rightarrow \quad \ln x + C = 0 \quad \Rightarrow \quad x = e^{-C}$, so there is an x-intercept at e^{-C}.

(4) $y' = 0 \quad \Rightarrow \quad \ln x = 1 - C \quad \Rightarrow \quad x = e^{1-C}$, so there is a local maximum at $x = e^{1-C}$.

(c) $y(1) = 2 \quad \Rightarrow \quad 2 = \dfrac{\ln 1 + C}{1} \quad \Rightarrow \quad 2 = C$, so the solution is $y = \dfrac{\ln x + 2}{x}$ [shown in part (b)].

(d) $y(2) = 1 \quad \Rightarrow \quad 1 = \dfrac{\ln 2 + C}{2} \quad \Rightarrow \quad 2 + \ln 2 + C \quad \Rightarrow \quad C = 2 - \ln 2$, so the solution is $y = \dfrac{\ln x + 2 - \ln 2}{x}$ [shown in part (b)].

7. (a) Since the derivative $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

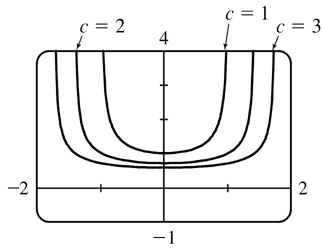
(b) $y = \dfrac{1}{x + C} \quad \Rightarrow \quad y' = -\dfrac{1}{(x + C)^2}.\quad$ LHS $= y' = -\dfrac{1}{(x + C)^2} = -\left(\dfrac{1}{x + C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

(d) If $y(x) = \dfrac{1}{x + C}$, then $y(0) = \dfrac{1}{0 + C} = \dfrac{1}{C}$. Since $y(0) = 0.5$, $\dfrac{1}{C} = \dfrac{1}{2} \quad \Rightarrow \quad C = 2$, so $y = \dfrac{1}{x + 2}$.

8. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y.)

(b) $y = (c - x^2)^{-1/2} \quad \Rightarrow \quad y' = x(c - x^2)^{-3/2}$. RHS $= xy^3 = x[(c - x^2)^{-1/2}]^3 = x(c - x^2)^{-3/2} = y' =$ LHS

(c) 

When x is close to 0, y' is also close to 0.

As x gets larger, so does $|y'|$.

(d) $y(0) = (c - 0)^{-1/2} = 1/\sqrt{c}$ and $y(0) = 2 \quad \Rightarrow \quad \sqrt{c} = \frac{1}{2} \quad \Rightarrow \quad c = \frac{1}{4}$, so $y = \left(\frac{1}{4} - x^2\right)^{-1/2}$.

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \quad \Rightarrow \quad 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \quad \dfrac{P}{4200} < 1 \quad \Rightarrow$ $P < 4200 \quad \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \quad \Rightarrow \quad P > 4200$

(c) $\dfrac{dP}{dt} = 0 \quad \Rightarrow \quad P = 4200$ or $P = 0$

10. (a) $\dfrac{dv}{dt} = -v[v^2 - (1 + a)v + a] = -v(v - a)(v - 1)$, so $\dfrac{dv}{dt} = 0 \quad \Leftrightarrow \quad v = 0, a,$ or 1.

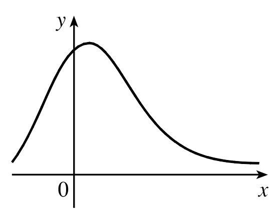
(b) With $0 < a < 1$, $dv/dt = -v(v - a)(v - 1) > 0 \quad \Leftrightarrow \quad v < 0$ or $a < v < 1$, so v is increasing on $(-\infty, 0)$ and $(a, 1)$.

(c) With $0 < a < 1$, $dv/dt = -v(v - a)(v - 1) < 0 \quad \Leftrightarrow \quad 0 < v < a$ or $v > 1$, so v is decreasing on $(0, a)$ and $(1, \infty)$.

11. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y - 1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.



A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can see that $y' < 0$ for some points in the first quadrant.

B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$. Thus, equations A and B are incorrect, so the correct equation is C.

C. $y' = 1 - 2xy$ seems reasonable since:

(1) When $x = 0$, y' could be 1.

(2) When $x < 0$, y' could be greater than 1.

(3) Solving $y' = 1 - 2xy$ for y gives us $y = \dfrac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \to \infty$, $y \to 0^+$, as shown in the figure.

13. (a) $y' = 1 + x^2 + y^2 \geq 1$ and $y' \to \infty$ as $x \to \infty$. The only curve satisfying these conditions is labeled III.

(b) $y' = xe^{-x^2-y^2} > 0$ if $x > 0$ and $y' < 0$ if $x < 0$. The only curve with negative tangent slopes when $x < 0$ and positive tangent slopes when $x > 0$ is labeled I.

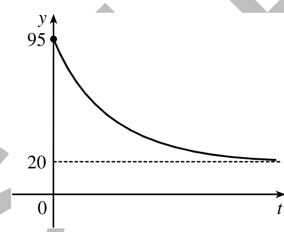
(c) $y' = \dfrac{1}{1 + e^{x^2+y^2}} > 0$ and $y' \to 0$ as $x \to \infty$. The only curve satisfying these conditions is labeled IV.

(d) $y' = \sin(xy)\,\cos(xy) = 0$ if $y = 0$, which is the solution graph labeled II.

14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\dfrac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95^\circ\text{C}$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

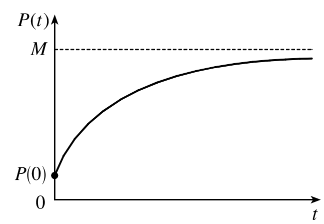
(c)



15. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

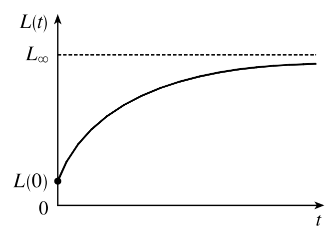
(b) $\dfrac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(c)



16. (a) $\dfrac{dL}{dt} = k(L_\infty - L)$. Assuming $L_\infty > L$, we have $k > 0$ and $dL/dt > 0$ for all t.

(b)



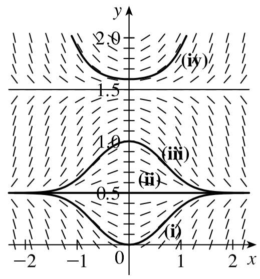
17. If $c(t) = c_s\left(1 - e^{-\alpha t^{1-b}}\right) = c_s - c_s e^{-\alpha t^{1-b}}$ for $t > 0$, where $k > 0$, $c_s > 0$, $0 < b < 1$, and $\alpha = k/(1-b)$, then

$$\frac{dc}{dt} = c_s\left[0 - e^{-\alpha t^{1-b}} \cdot \frac{d}{dt}\left(-\alpha t^{1-b}\right)\right] = -c_s e^{-\alpha t^{1-b}} \cdot (-\alpha)(1-b)t^{-b} = \frac{\alpha(1-b)}{t^b}\,c_s\,e^{-\alpha t^{1-b}} = \frac{k}{t^b}(c_s - c).$$

The equation for c indicates that as t increases, c approaches c_s. The differential equation indicates that as t increases, the rate of increase of c decreases steadily and approaches 0 as c approaches c_s.

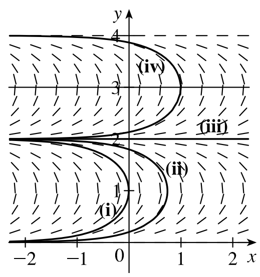
9.2 Direction Fields and Euler's Method

1. (a)



(b) It appears that the constant functions $y = 0.5$ and $y = 1.5$ are equilibrium solutions. Note that these two values of y satisfy the given differential equation $y' = x \cos \pi y$.

2. (a)



(b) It appears that the constant functions $y = 0$, $y = 2$, and $y = 4$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = \tan\left(\frac{1}{2}\pi y\right)$.

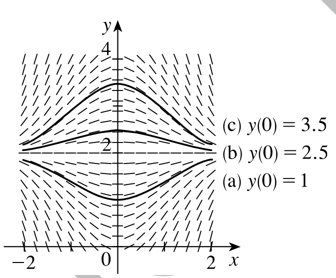
3. $y' = 2 - y$. The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, III is the direction field for this equation. Note that for $y = 2$, $y' = 0$.

4. $y' = x(2 - y) = 0$ on the lines $x = 0$ and $y = 2$. Direction field I satisfies these conditions.

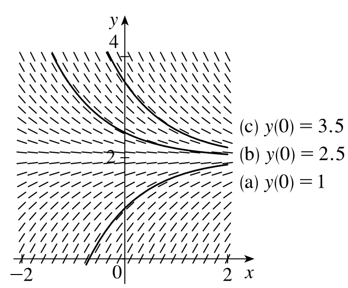
5. $y' = x + y - 1 = 0$ on the line $y = -x + 1$. Direction field IV satisfies this condition. Notice also that on the line $y = -x$ we have $y' = -1$, which is true in IV.

6. $y' = \sin x \sin y = 0$ on the lines $x = 0$ and $y = 0$, and $y' > 0$ for $0 < x < \pi$, $0 < y < \pi$. Direction field II satisfies these conditions.

7.



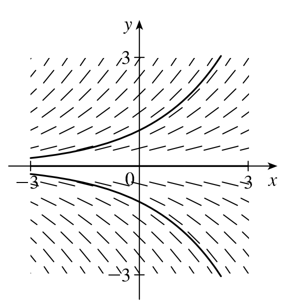
8.



9.

x	y	$y' = \frac{1}{2}y$
0	0	0
0	1	0.5
0	2	1
0	−3	−1.5
0	−2	−1

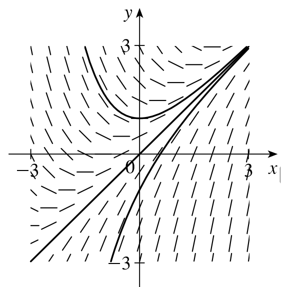
Note that for $y = 0$, $y' = 0$. The three solution curves sketched go through $(0, 0)$, $(0, 1)$, and $(0, -1)$.



10.

x	y	$y' = x - y + 1$
-1	0	0
-1	-1	1
0	0	1
0	1	0
0	2	-1
0	-1	2
0	-2	3
1	0	2
1	1	1

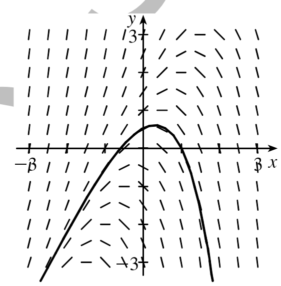
Note that $y' = 0$ for $y = x + 1$ and that $y' = 1$ for $y = x$. For any constant value of x, y' decreases as y increases and y' increases as y decreases. The three solution curves sketched go through $(0, 0)$, $(0, 1)$, and $(0, -1)$.



11.

x	y	$y' = y - 2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

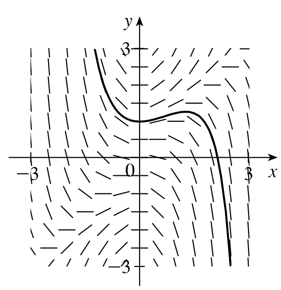
Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.



12.

x	y	$y' = xy - x^2$
2	3	2
-2	-3	2
± 2	0	-4
0	0	0
2	2	0

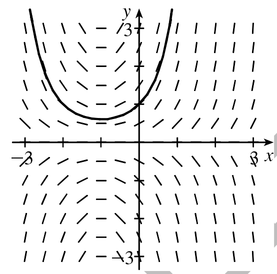
$y' = xy - x^2 = x(y - x)$, so $y' = 0$ for $x = 0$ and $y = x$. The slopes are positive only in the regions in quadrants I and III that are bounded by $x = 0$ and $y = x$. The solution curve in the graph passes through $(0, 1)$.



13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

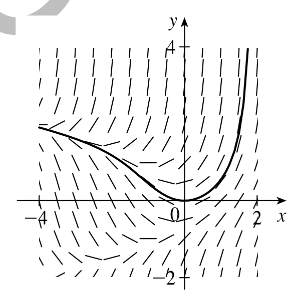
Note that $y' = y(x+1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x+1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



14.

x	y	$y' = x + y^2$
-2	± 1	-1
-2	± 2	2
2	± 1	3
0	± 2	4
0	0	0

Note that $y' = x + y^2 = 0$ only on the parabola $x = -y^2$. The slopes are positive "outside" $x = -y^2$ and negative "inside" $x = -y^2$. The solution curve in the graph passes through $(0, 0)$.



15. $y' = x^2 y - \frac{1}{2}y^2$ and $y(0) = 1$.

In Maple, use the following commands to obtain a similar figure.

```
with(DETools):
ODE:=diff(y(x),x)=x^2*y(x)-(1/2)*y(x)^2;
ivs:=[y(0)=1];
DEplot({ODE},y(x),x=-3..2,y=0..4,ivs,linecolor=black);
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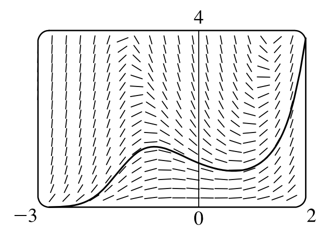
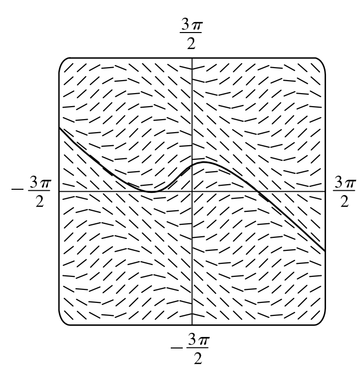


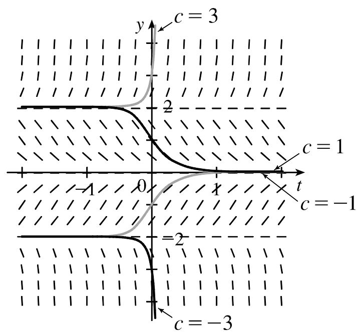
16. $y' = \cos(x + y)$ and $y(0) = 1$.

In Maple, use the following commands to obtain a similar figure.

```
with(DETools):
ODE:=diff(y(x),x)=cos(x+y(x));
ivs:=[y(0)=1];
DEplot({ODE},y(x),x=-1.5*Pi..1.5*Pi,y=-1.5*Pi..1.5*Pi,
ivs,linecolor=black);
```



17.



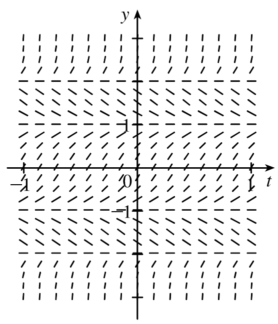
The direction field is for the differential equation $y' = y^3 - 4y$.

$L = \lim_{t\to\infty} y(t)$ exists for $-2 \le c \le 2$;

$L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$.

For other values of c, L does not exist.

18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t\to\infty} y = \infty$ and $\lim_{t\to-\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -1$.
- If $y < -2$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -\infty$.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad \Rightarrow \quad x_0 = 0,\ y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad \Rightarrow \quad y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$, so $y_1 = y\,(0.4) = 1.4$.

(ii) $h = 0.2 \quad \Rightarrow \quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

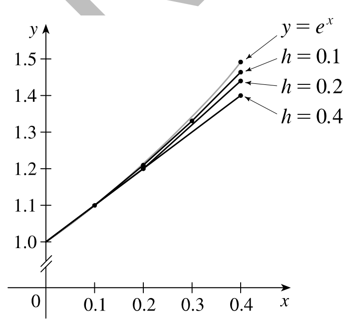
(iii) $h = 0.1 \quad \Rightarrow \quad x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)



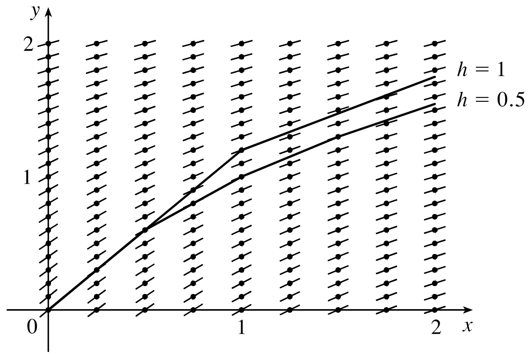
We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

(c) (i) For $h = 0.4$: (exact value) $-$ (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) $-$ (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) $-$ (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20. 

As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$

$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$

$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$

$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$

22. $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x^2y - \frac{1}{2}y^2$. Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, and $x_5 = 1$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2F(0, 1) = 1 + 0.2\left[0^2(1) - \frac{1}{2}(1)^2\right] = 1 + 0.2\left(-\frac{1}{2}\right) = 0.9.$

$y_2 = y_1 + hF(x_1, y_1) = 0.9 + 0.2F(0.2, 0.9) = 0.9 + 0.2\left[(0.2)^2(0.9) - \frac{1}{2}(0.9)^2\right] = 0.8262.$

$y_3 = y_2 + hF(x_2, y_2) = 0.8262 + 0.2F(0.4, 0.8262) = 0.8262 + 0.2\left[(0.4)^2(0.8262) - \frac{1}{2}(0.8262)^2\right] = 0.784377756.$

$y_4 = y_3 + hF(x_3, y_3) = 0.784377756 + 0.2F(0.6, 0.784377756) \approx 0.779328108.$

$y_5 = y_4 + hF(x_4, y_4) \approx 0.779328108 + 0.2F(0.8, 0.779328108) \approx 0.818346876.$

Thus, $y(1) \approx 0.8183$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$

$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$

$$\begin{aligned} y_4 = y_3 + hF(x_3, y_3) &= 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)] \\ &= 1.5452976. \end{aligned}$$

$$\begin{aligned} y_5 = y_4 + hF(x_4, y_4) &= 1.5452976 + 0.1F(0.4, 1.5452976) \\ &= 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264. \end{aligned}$$

Thus, $y(0.5) \approx 1.7616$.

24. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 0$, and $F(x, y) = \cos(x + y)$. Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, and $x_3 = 0.6$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(0, 0) = 0.2\cos(0 + 0) = 0.2(1) = 0.2.$

$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(0.2, 0.2) = 0.2 + 0.2\cos(0.4) \approx 0.3842121988.$

$y_3 = y_2 + hF(x_2, y_2) \approx 0.3842 + 0.2F(0.4, 0.3842) \approx 0.5258011763.$

Thus, $y(0.6) \approx 0.5258$.

(b) Now use $h = 0.1$. For $1 \leq n \leq 6$, $x_n = 0.n$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.1\cos(0 + 0) = 0.1(1) = 0.1.$

$y_2 = y_1 + hF(x_1, y_1) = 0.1 + 0.1\cos(0.2) \approx 0.1980.$

$y_3 = y_2 + hF(x_2, y_2) \approx 0.1980 + 0.1\cos(0.3980) \approx 0.2902.$

$y_4 = y_3 + hF(x_3, y_3) \approx 0.2902 + 0.1\cos(0.5902) \approx 0.3733.$

$y_5 = y_4 + hF(x_4, y_4) \approx 0.3733 + 0.1\cos(0.7733) \approx 0.4448.$

$y_6 = y_5 + hF(x_5, y_5) \approx 0.4448 + 0.1\cos(0.9448) \approx 0.5034.$

Thus, $y(0.6) \approx 0.5034$.

25. (a) $dy/dx + 3x^2y = 6x^2 \quad \Rightarrow \quad y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $\text{H} = h = 1$ and $\text{N} = 1$ for part (i), $\text{H} = 0.1$ and $\text{N} = 10$ for part (ii), and so forth.

$h \to$ H: $0 \to$ X: $3 \to$ Y:

For(I, 1, N): Y + H × Y_1 → Y: X + H → X:

End(loop):

Display Y. [To see all iterations, include this statement in the loop.]

(i) $\text{H} = 1, \text{N} = 1 \quad \Rightarrow \quad y(1) = 3$

(ii) $\text{H} = 0.1, \text{N} = 10 \quad \Rightarrow \quad y(1) \approx 2.3928$

(iii) $\text{H} = 0.01, \text{N} = 100 \quad \Rightarrow \quad y(1) \approx 2.3701$

(iv) $\text{H} = 0.001, \text{N} = 1000 \quad \Rightarrow \quad y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \quad \Rightarrow \quad y' = -3x^2e^{-x^3}$

$$\text{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \text{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.

(i) For $h = 1$: (exact value) − (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$

(ii) For $h = 0.1$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$

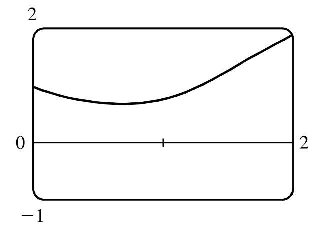
(iii) For $h = 0.01$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

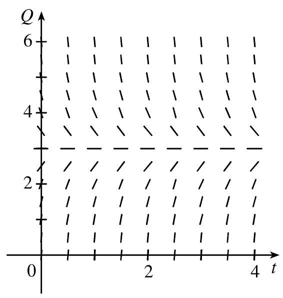
26. (a) We use the program from the solution to Exercise 25 with $\mathrm{Y}_1 = x^3 - y^3$, $\mathrm{H} = 0.01$, and $\mathrm{N} = \frac{2-0}{0.01} = 200$. With $(x_0, y_0) = (0, 1)$, we get $y(2) \approx 1.9000$.

(b)



Notice from the graph that $y(2) \approx 1.9$, which serves as a check on our calculation in part (a).

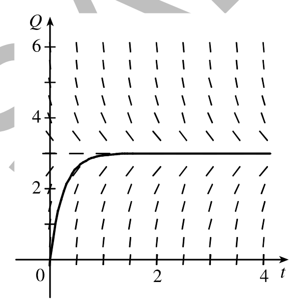
27. (a) $R\,\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ becomes $5Q' + \dfrac{1}{0.05}Q = 60$ or $Q' + 4Q = 12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \;\Rightarrow\; Q = 3$ is an equilibrium solution.

(d)



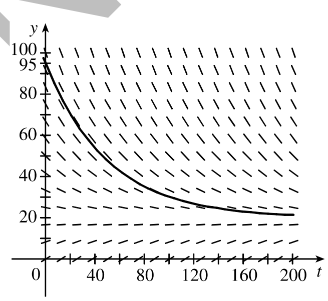
(e) $Q' + 4Q = 12 \;\Rightarrow\; Q' = 12 - 4Q$. Now $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$
$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$
$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$
$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$
$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

28. (a) From Exercise 9.1.14, we have $dy/dt = k(y - R)$. We are given that $R = 20°\mathrm{C}$ and $dy/dt = -1°\mathrm{C/min}$ when $y = 70°\mathrm{C}$. Thus, $-1 = k(70 - 20) \;\Rightarrow\; k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is $20°\mathrm{C}$; that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y-20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$\begin{aligned} y_1 &= y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95-20)\right] = 92 \\ y_2 &= y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92-20)\right] = 89.12 \\ y_3 &= y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12-20)\right] = 86.3552 \\ y_4 &= y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552-20)\right] = 83.700992 \\ y_5 &= y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992-20)\right] = 81.15295232 \end{aligned}$$

Thus, $y(10) \approx 81.15°\text{C}$.

9.3 Separable Equations

1. $\dfrac{dy}{dx} = 3x^2y^2 \;\Rightarrow\; \dfrac{dy}{y^2} = 3x^2\,dx \quad [y \neq 0] \;\Rightarrow\; \int y^{-2}\,dy = \int 3x^2\,dx \;\Rightarrow\; -y^{-1} = x^3 + C \;\Rightarrow$

$\dfrac{-1}{y} = x^3 + C \;\Rightarrow\; y = \dfrac{-1}{x^3 + C}$. $y = 0$ is also a solution.

2. $\dfrac{dy}{dx} = x\sqrt{y} \;\Rightarrow\; \dfrac{dy}{\sqrt{y}} = x\,dx \quad [y \neq 0] \;\Rightarrow\; \int y^{-1/2}\,dy = \int x\,dx \;\Rightarrow\; 2y^{1/2} = \frac{1}{2}x^2 + K \;\Rightarrow$

$\sqrt{y} = \frac{1}{4}x^2 + \frac{1}{2}K \;\Rightarrow\; y = \left(\frac{1}{4}x^2 + C\right)^2$, where $C = \frac{1}{2}K$. $y = 0$ is also a solution.

3. $xyy' = x^2 + 1 \;\Rightarrow\; xy\dfrac{dy}{dx} = x^2 + 1 \;\Rightarrow\; y\,dy = \dfrac{x^2+1}{x}\,dx \quad [x \neq 0] \;\Rightarrow\; \displaystyle\int y\,dy = \int\left(x + \frac{1}{x}\right)dx \;\Rightarrow$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \ln|x| + K \;\Rightarrow\; y^2 = x^2 + 2\ln|x| + 2K \;\Rightarrow\; y = \pm\sqrt{x^2 + 2\ln|x| + C}$, where $C = 2K$.

4. $y' + xe^y = 0 \;\Rightarrow\; \dfrac{dy}{dx} = -xe^y \;\Rightarrow\; e^{-y}\,dy = -x\,dx \;\Rightarrow\; \int e^{-y}\,dy = \int -x\,dx \;\Rightarrow\; -e^{-y} = -\frac{1}{2}x^2 + C \;\Rightarrow$

$e^{-y} = \frac{1}{2}x^2 - C \;\Rightarrow\; -y = \ln\left(\frac{1}{2}x^2 - C\right) \;\Rightarrow\; y = -\ln\left(\frac{1}{2}x^2 - C\right)$

5. $(e^y - 1)y' = 2 + \cos x \;\Rightarrow\; (e^y - 1)\dfrac{dy}{dx} = 2 + \cos x \;\Rightarrow\; (e^y - 1)\,dy = (2 + \cos x)\,dx \;\Rightarrow$

$\int(e^y - 1)\,dy = \int(2 + \cos x)\,dx \;\Rightarrow\; e^y - y = 2x + \sin x + C$. We cannot solve explicitly for y.

6. $\dfrac{du}{dt} = \dfrac{1 + t^4}{ut^2 + u^4t^2} \;\Rightarrow\; \dfrac{du}{dt} = \dfrac{1 + t^4}{t^2(u + u^4)} \;\Rightarrow\; (u + u^4)\,du = \dfrac{1 + t^4}{t^2}\,dt \;\Rightarrow\; \int(u + u^4)\,du = \int(t^{-2} + t^2)\,dt \;\Rightarrow$

$\dfrac{1}{2}u^2 + \dfrac{1}{5}u^5 = -\dfrac{1}{t} + \dfrac{1}{3}t^3 + C$. We cannot solve explicitly for u.

7. $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}} \;\Rightarrow\; \theta\cos\theta\,d\theta = te^{-t^2}\,dt \;\Rightarrow\; \int\theta\cos\theta\,d\theta = \int te^{-t^2}\,dt \;\Rightarrow$

$\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C$ [by parts]. We cannot solve explicitly for θ.

8. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1 + R^2}}{\ln H} \;\Rightarrow\; \dfrac{\ln H}{H^2}\,dH = R\sqrt{1 + R^2}\,dR \;\Rightarrow\; \displaystyle\int\frac{\ln H}{H^2}\,dH = \int R(1 + R^2)^{1/2}\,dR \;\Rightarrow$

$-\dfrac{\ln H}{H} - \dfrac{1}{H} = \dfrac{1}{3}(1 + R^2)^{3/2} + C$ [by parts]. We cannot solve explicitly for H.

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1 = p(t^2-1) + 1(t^2-1) = (p+1)(t^2-1) \quad\Rightarrow\quad \dfrac{1}{p+1}\,dp = (t^2-1)\,dt \quad\Rightarrow$

$\displaystyle\int \frac{1}{p+1}\,dp = \int (t^2-1)\,dt \quad\Rightarrow\quad \ln|p+1| = \tfrac{1}{3}t^3 - t + C \quad\Rightarrow\quad |p+1| = e^{t^3/3-t+C} \quad\Rightarrow\quad p+1 = \pm e^C e^{t^3/3-t} \quad\Rightarrow$

$p = Ke^{t^3/3-t} - 1$, where $K = \pm e^C$. Since $p = -1$ is also a solution, K can equal 0, and hence, K can be any real number.

10. $\dfrac{dz}{dt} + e^{t+z} = 0 \quad\Rightarrow\quad \dfrac{dz}{dt} = -e^t e^z \quad\Rightarrow\quad \int e^{-z}\,dz = -\int e^t\,dt \quad\Rightarrow\quad -e^{-z} = -e^t + C \quad\Rightarrow\quad e^{-z} = e^t - C \quad\Rightarrow$

$\dfrac{1}{e^z} = e^t - C \quad\Rightarrow\quad e^z = \dfrac{1}{e^t - C} \quad\Rightarrow\quad z = \ln\!\left(\dfrac{1}{e^t - C}\right) \quad\Rightarrow\quad z = -\ln\!\left(e^t - C\right)$

11. $\dfrac{dy}{dx} = xe^y \quad\Rightarrow\quad e^{-y}\,dy = x\,dx \quad\Rightarrow\quad \int e^{-y}\,dy = \int x\,dx \quad\Rightarrow\quad -e^{-y} = \tfrac{1}{2}x^2 + C.$

$y(0) = 0 \quad\Rightarrow\quad -e^{-0} = \tfrac{1}{2}(0)^2 + C \quad\Rightarrow\quad C = -1$, so $-e^{-y} = \tfrac{1}{2}x^2 - 1 \quad\Rightarrow\quad e^{-y} = -\tfrac{1}{2}x^2 + 1 \quad\Rightarrow$

$-y = \ln\!\left(1 - \tfrac{1}{2}x^2\right) \quad\Rightarrow\quad y = -\ln\!\left(1 - \tfrac{1}{2}x^2\right).$

12. $\dfrac{dy}{dx} = \dfrac{x\sin x}{y} \quad\Rightarrow\quad y\,dy = x\sin x\,dx \quad\Rightarrow\quad \int y\,dy = \int x\sin x\,dx \quad\Rightarrow\quad \tfrac{1}{2}y^2 = -x\cos x + \sin x + C$ [by parts].

$y(0) = -1 \quad\Rightarrow\quad \tfrac{1}{2}(-1)^2 = -0\cos 0 + \sin 0 + C \quad\Rightarrow\quad C = \tfrac{1}{2}$, so $\tfrac{1}{2}y^2 = -x\cos x + \sin x + \tfrac{1}{2} \quad\Rightarrow$

$y^2 = -2x\cos x + 2\sin x + 1 \quad\Rightarrow\quad y = -\sqrt{-2x\cos x + 2\sin x + 1}$ since $y(0) = -1 < 0$.

13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u},\ u(0) = -5.\quad \int 2u\,du = \int\left(2t + \sec^2 t\right)dt \quad\Rightarrow\quad u^2 = t^2 + \tan t + C,$

where $[u(0)]^2 = 0^2 + \tan 0 + C \quad\Rightarrow\quad C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5 < 0$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

14. $x + 3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0 \quad\Rightarrow\quad 3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = -x \quad\Rightarrow\quad 3y^2\,dy = \dfrac{-x}{\sqrt{x^2+1}}\,dx \quad\Rightarrow$

$\int 3y^2\,dy = \int -x(x^2+1)^{-1/2}\,dx \quad\Rightarrow\quad y^3 = -(x^2+1)^{1/2} + C.\quad y(0) = 1 \quad\Rightarrow\quad 1^3 = -(0^2+1)^{1/2} + C \quad\Rightarrow$

$C = 2$, so $y^3 = -(x^2+1)^{1/2} + 2 \quad\Rightarrow\quad y = (2 - \sqrt{x^2+1})^{1/3}.$

15. $x\ln x = y\left(1 + \sqrt{3+y^2}\right)y',\ y(1) = 1.\ \int x\ln x\,dx = \int\left(y + y\sqrt{3+y^2}\right)dy \quad\Rightarrow\quad \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,dx$

[use parts with $u = \ln x$, $dv = x dx$] $= \tfrac{1}{2}y^2 + \tfrac{1}{3}(3+y^2)^{3/2} \quad\Rightarrow\quad \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C = \tfrac{1}{2}y^2 + \tfrac{1}{3}(3+y^2)^{3/2}.$

Now $y(1) = 1 \quad\Rightarrow\quad 0 - \tfrac{1}{4} + C = \tfrac{1}{2} + \tfrac{1}{3}(4)^{3/2} \quad\Rightarrow\quad C = \tfrac{1}{2} + \tfrac{8}{3} + \tfrac{1}{4} = \tfrac{41}{12}$, so

$\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + \tfrac{41}{12} = \tfrac{1}{2}y^2 + \tfrac{1}{3}(3+y^2)^{3/2}$. We do not solve explicitly for y.

16. $\dfrac{dP}{dt} = \sqrt{Pt} \quad\Rightarrow\quad dP/\sqrt{P} = \sqrt{t}\,dt \quad\Rightarrow\quad \int P^{-1/2}\,dP = \int t^{1/2}\,dt \quad\Rightarrow\quad 2P^{1/2} = \tfrac{2}{3}t^{3/2} + C.$

$P(1) = 2 \quad\Rightarrow\quad 2\sqrt{2} = \tfrac{2}{3} + C \quad\Rightarrow\quad C = 2\sqrt{2} - \tfrac{2}{3}$, so $2P^{1/2} = \tfrac{2}{3}t^{3/2} + 2\sqrt{2} - \tfrac{2}{3} \quad\Rightarrow\quad \sqrt{P} = \tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3} \quad\Rightarrow$

$P = \left(\tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3}\right)^2.$

17. $y' \tan x = a + y,\ 0 < x < \pi/2 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \quad \Rightarrow \quad \dfrac{dy}{a+y} = \cot x\, dx \quad [a + y \neq 0] \quad \Rightarrow$

$\displaystyle\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\, dx \quad \Rightarrow \quad \ln|a+y| = \ln|\sin x| + C \quad \Rightarrow \quad |a+y| = e^{\ln|\sin x| + C} = e^{\ln|\sin x|} \cdot e^C = e^C\, |\sin x| \quad \Rightarrow$

$a + y = K \sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case

$y = -a$ by allowing K to be zero.) $\quad y(\pi/3) = a \quad \Rightarrow \quad a + a = K \sin\left(\dfrac{\pi}{3}\right) \quad \Rightarrow \quad 2a = K\,\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad K = \dfrac{4a}{\sqrt{3}}.$

Thus, $a + y = \dfrac{4a}{\sqrt{3}} \sin x$ and so $y = \dfrac{4a}{\sqrt{3}} \sin x - a$.

18. $\dfrac{dL}{dt} = kL^2 \ln t \quad \Rightarrow \quad \dfrac{dL}{L^2} = k \ln t\, dt \quad \Rightarrow \quad \displaystyle\int \frac{dL}{L^2} = \int k \ln t\, dt \quad \Rightarrow \quad -\frac{1}{L} = kt \ln t - \int k\, dt$

[by parts with $u = \ln t$, $dv = k\, dt$] $\Rightarrow \quad -\dfrac{1}{L} = kt \ln t - kt + C \quad \Rightarrow \quad L = \dfrac{1}{kt - kt\ln t - C}.$

$L(1) = -1 \quad \Rightarrow \quad -1 = \dfrac{1}{k - k \ln 1 - C} \quad \Rightarrow \quad C - k = 1 \quad \Rightarrow \quad C = k + 1$. Thus, $L = \dfrac{1}{kt - kt \ln t - k - 1}.$

19. $\dfrac{dy}{dx} = \dfrac{x}{y} \quad \Rightarrow \quad y\, dy = x\, dx \quad \Rightarrow \quad \int y\, dy = \int x\, dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C. \quad y(0) = 2 \quad \Rightarrow \quad \frac{1}{2}(2)^2 = \frac{1}{2}(0)^2 + C \quad \Rightarrow$

$C = 2$, so $\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2 \quad \Rightarrow \quad y^2 = x^2 + 4 \quad \Rightarrow \quad y = \sqrt{x^2 + 4}$ since $y(0) = 2 > 0$.

20. $f'(x) = x\, f(x) - x \quad \Rightarrow \quad \dfrac{dy}{dx} = xy - x \quad \Rightarrow \quad \dfrac{dy}{dx} = x(y-1) \quad \Rightarrow \quad \dfrac{dy}{y-1} = x\, dx \quad [y \neq 1] \quad \Rightarrow$

$\displaystyle\int \frac{dy}{y-1} = \int x\, dx \quad \Rightarrow \quad \ln|y-1| = \tfrac{1}{2}x^2 + C. \quad f(0) = 2 \quad \Rightarrow \quad \ln|2-1| = \tfrac{1}{2}(0)^2 + C \quad \Rightarrow \quad C = 0$, so

$\ln|y-1| = \frac{1}{2}x^2 \quad \Rightarrow \quad |y-1| = e^{x^2/2} \quad \Rightarrow \quad y - 1 = e^{x^2/2}$ [since $f(0) = 2$] $\Rightarrow \quad y = e^{x^2/2} + 1.$

21. $u = x + y \quad \Rightarrow \quad \dfrac{d}{dx}(u) = \dfrac{d}{dx}(x+y) \quad \Rightarrow \quad \dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$, but $\dfrac{dy}{dx} = x + y = u$, so $\dfrac{du}{dx} = 1 + u \quad \Rightarrow$

$\dfrac{du}{1+u} = dx \quad [u \neq -1] \quad \Rightarrow \quad \displaystyle\int \frac{du}{1+u} = \int dx \quad \Rightarrow \quad \ln|1+u| = x + C \quad \Rightarrow \quad |1+u| = e^{x+C} \quad \Rightarrow$

$1 + u = \pm e^C e^x \quad \Rightarrow \quad u = \pm e^C e^x - 1 \quad \Rightarrow \quad x + y = \pm e^C e^x - 1 \quad \Rightarrow \quad y = Ke^x - x - 1,$ where $K = \pm e^C \neq 0$.

If $u = -1$, then $-1 = x + y \quad \Rightarrow \quad y = -x - 1$, which is just $y = Ke^x - x - 1$ with $K = 0$. Thus, the general solution is $y = Ke^x - x - 1$, where $K \in \mathbb{R}$.

22. $xy' = y + xe^{y/x} \quad \Rightarrow \quad y' = y/x + e^{y/x} \quad \Rightarrow \quad \dfrac{dy}{dx} = v + e^v$. Also, $v = y/x \quad \Rightarrow \quad xv = y \quad \Rightarrow \quad \dfrac{dy}{dx} = x\,\dfrac{dv}{dx} + v,$

so $v + e^v = x\,\dfrac{dv}{dx} + v \quad \Rightarrow \quad \dfrac{dv}{e^v} = \dfrac{dx}{x} \quad [x \neq 0] \quad \Rightarrow \quad \displaystyle\int \frac{dv}{e^v} = \int \frac{dx}{x} \quad \Rightarrow \quad -e^{-v} = \ln|x| + C \quad \Rightarrow$

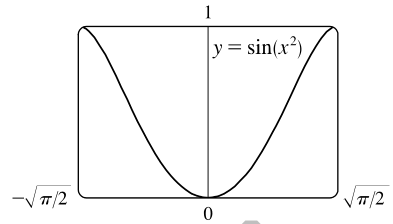
$e^{-v} = -\ln|x| - C \quad \Rightarrow \quad -v = \ln(-\ln|x| - C) \quad \Rightarrow \quad y/x = -\ln(-\ln|x| - C) \quad \Rightarrow \quad y = -x\ln(-\ln|x| - C).$

23. (a) $y' = 2x\sqrt{1-y^2} \;\Rightarrow\; \dfrac{dy}{dx} = 2x\sqrt{1-y^2} \;\Rightarrow\; \dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \;\Rightarrow$

$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \le x^2 + C \le \frac{\pi}{2}$.

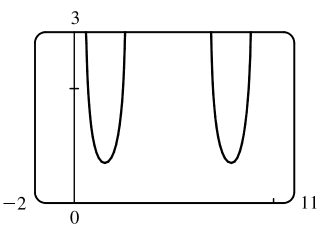
(b) $y(0) = 0 \;\Rightarrow\; \sin^{-1} 0 = 0^2 + C \;\Rightarrow\; C = 0$,

so $\sin^{-1} y = x^2$ and $y = \sin(x^2)$ for $-\sqrt{\pi/2} \le x \le \sqrt{\pi/2}$.

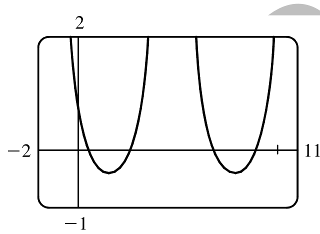


(c) For $\sqrt{1-y^2}$ to be a real number, we must have $-1 \le y \le 1$; that is, $-1 \le y(0) \le 1$. Thus, the initial-value problem $y' = 2x\sqrt{1-y^2}$, $y(0) = 2$ does *not* have a solution.

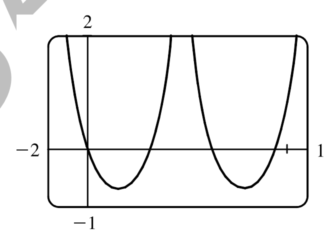
24. $e^{-y}y' + \cos x = 0 \;\Leftrightarrow\; \int e^{-y}\,dy = -\int \cos x\,dx \;\Leftrightarrow\; -e^{-y} = -\sin x + C_1 \;\Leftrightarrow\; y = -\ln(\sin x + C)$. The solution is periodic, with period 2π. Note that for $C > 1$, the domain of the solution is $\mathbb{R}$, but for $-1 < C \le 1$ it is only defined on the intervals where $\sin x + C > 0$, and it is meaningless for $C \le -1$, since then $\sin x + C \le 0$, and the logarithm is undefined.



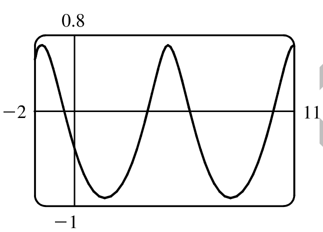
$C = -0.5$



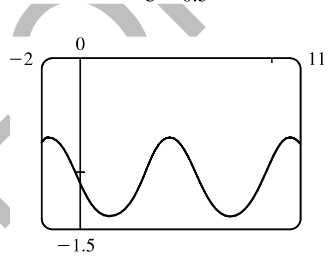
$C = 0.5$



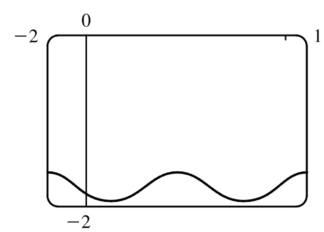
$C = 1$



$C = 1.5$



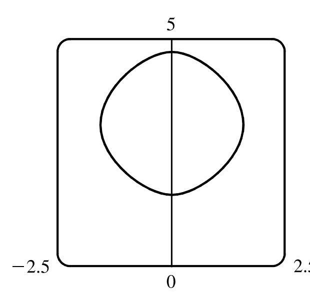
$C = 3$



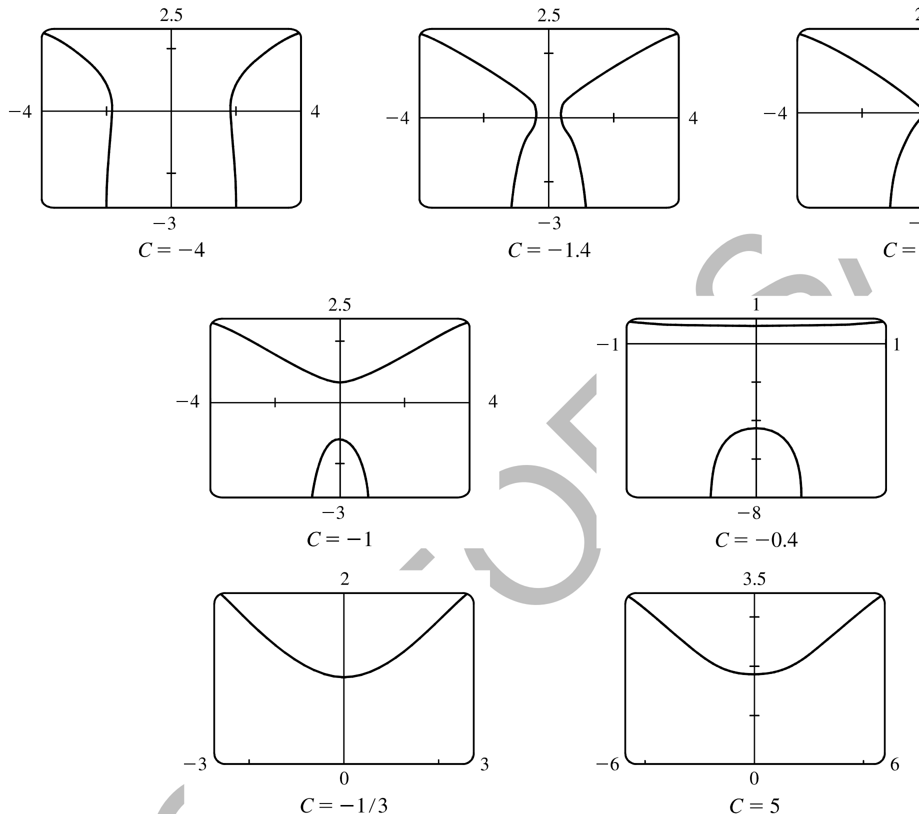
$C = 6$

For $-1 < C < 1$, the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where $\sin x + C \le 0$). For $C = 1$, the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where $\sin x + C = 0 \;\Leftrightarrow\; \sin x = -1$. For $C > 1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

25. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sin y}$, $y(0) = \dfrac{\pi}{2}$. So $\int \sin y\,dy = \int \sin x\,dx \;\Leftrightarrow\; -\cos y = -\cos x + C \;\Leftrightarrow\; \cos y = \cos x - C$. From the initial condition, we need $\cos\frac{\pi}{2} = \cos 0 - C \;\Rightarrow\; 0 = 1 - C \;\Rightarrow\; C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \le \cos x - 1 \;\Leftrightarrow\; 0 \le \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[···]]`.

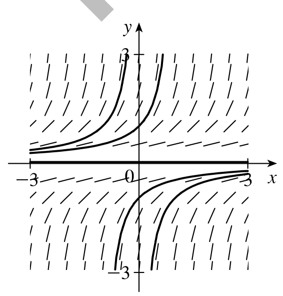


26. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+1}}{ye^y} \quad\Leftrightarrow\quad \int ye^y\,dy = \int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u = y$, $dv = e^y\,dy$, and on the RHS we use the substitution $z = x^2 + 1$, so $dz = 2x\,dx$. The equation becomes $ye^y - \int e^y\,dy = \frac{1}{2}\int \sqrt{z}\,dz \quad\Leftrightarrow$ $e^y(y-1) = \frac{1}{3}(x^2+1)^{3/2} + C$, so we see that the curves are symmetric about the y-axis. Every point (x, y) in the plane lies on one of the curves, namely the one for which $C = (y-1)e^y - \frac{1}{3}(x^2+1)^{3/2}$. For example, along the y-axis, $C = (y-1)e^y - \frac{1}{3}$, so the origin lies on the curve with $C = -\frac{4}{3}$. We use Maple's `plots[implicitplot]` command or `Plot[Evaluate[···]]` in Mathematica to plot the solution curves for various values of C.



It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C < -\frac{4}{3}$, the graph consists of left and right branches. At $C = -\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C = -\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

27. (a), (c)

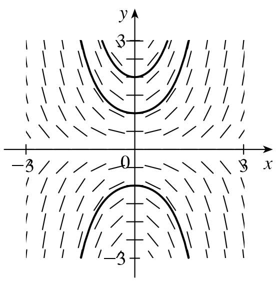


(b) $y' = y^2 \quad\Rightarrow\quad \dfrac{dy}{dx} = y^2 \quad\Rightarrow\quad \displaystyle\int y^{-2}\,dy = \int dx \quad\Rightarrow$

$-y^{-1} = x + C \quad\Rightarrow\quad \dfrac{1}{y} = -x - C \quad\Rightarrow$

$y = \dfrac{1}{K - x}$, where $K = -C$. $y = 0$ is also a solution.

28. (a), (c)



(b) $y' = xy \quad \Rightarrow \quad \dfrac{dy}{dx} = xy \quad \Rightarrow \quad \displaystyle\int \frac{dy}{y} = \int x\,dx \quad \Rightarrow$

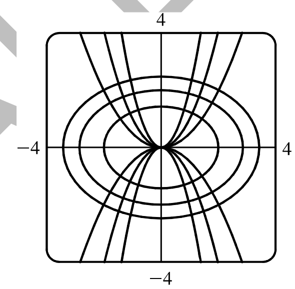
$\ln|y| = \frac{1}{2}x^2 + C \quad \Rightarrow \quad |y| = e^{x^2/2+C} = e^{x^2/2}e^C \quad \Rightarrow$

$y = Ke^{x^2/2}$, where $K = \pm e^C$. Taking $K = 0$ gives us the solution $y = 0$.

29. The curves $x^2 + 2y^2 = k^2$ form a family of ellipses with major axis on the x-axis. Differentiating gives $\dfrac{d}{dx}(x^2 + 2y^2) = \dfrac{d}{dx}(k^2) \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad 4yy' = -2x \quad \Rightarrow \quad y' = \dfrac{-x}{2y}$. Thus, the slope of the tangent line at any point (x, y) on one of the ellipses is $y' = \dfrac{-x}{2y}$, so the orthogonal trajectories must satisfy $y' = \dfrac{2y}{x} \quad \Leftrightarrow \quad \dfrac{dy}{dx} = \dfrac{2y}{x} \quad \Leftrightarrow \quad \dfrac{dy}{y} = 2 = \dfrac{dx}{x} \quad \Leftrightarrow$

$\displaystyle\int \frac{dy}{y} = 2\int \frac{dx}{x} \quad \Leftrightarrow \quad \ln|y| = 2\ln|x| + C_1 \quad \Leftrightarrow \quad \ln|y| = \ln|x|^2 + C_1 \quad \Leftrightarrow$

$|y| = e^{\ln x^2 + C_1} \quad \Leftrightarrow \quad y = \pm x^2 \cdot e^{C_1} = Cx^2$. This is a family of parabolas.

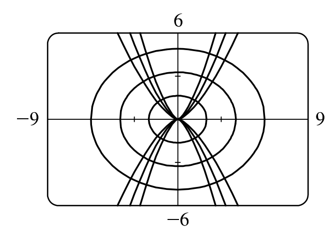


30. The curves $y^2 = kx^3$ form a family of power functions. Differentiating gives $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(kx^3) \quad \Rightarrow \quad 2yy' = 3kx^2 \quad \Rightarrow$

$y' = \dfrac{3kx^2}{2y} = \dfrac{3(y^2/x^3)x^2}{2y} = \dfrac{3y}{2x}$, the slope of the tangent line at (x, y) on one of the curves. Thus, the orthogonal trajectories must satisfy $y' = -\dfrac{2x}{3y} \quad \Leftrightarrow \quad \dfrac{dy}{dx} = -\dfrac{2x}{3y} \quad \Leftrightarrow$

$3y\,dy = -2x\,dx \quad \Leftrightarrow \quad \int 3y\,dy = \int -2x\,dx \quad \Leftrightarrow \quad \frac{3}{2}y^2 = -x^2 + C_1 \quad \Leftrightarrow$

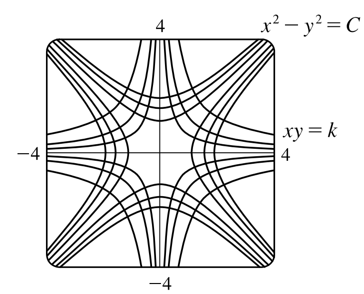
$3y^2 = -2x^2 + C_2 \quad \Leftrightarrow \quad 2x^2 + 3y^2 = C$. This is a family of ellipses.



31. The curves $y = k/x$ form a family of hyperbolas with asymptotes $x = 0$ and $y = 0$. Differentiating gives $\dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(\dfrac{k}{x}\right) \quad \Rightarrow \quad y' = -\dfrac{k}{x^2} \quad \Rightarrow \quad y' = -\dfrac{xy}{x^2}$ [since $y = k/x \quad \Rightarrow \quad xy = k$] $\Rightarrow \quad y' = -\dfrac{y}{x}$. Thus, the slope of the tangent line at any point (x, y) on one of the hyperbolas is $y' = -y/x$, so the orthogonal trajectories must satisfy $y' = x/y \quad \Leftrightarrow \quad \dfrac{dy}{dx} = \dfrac{x}{y} \quad \Leftrightarrow$

$y\,dy = x\,dx \quad \Leftrightarrow \quad \int y\,dy = \int x\,dx \quad \Leftrightarrow \quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \quad \Leftrightarrow$

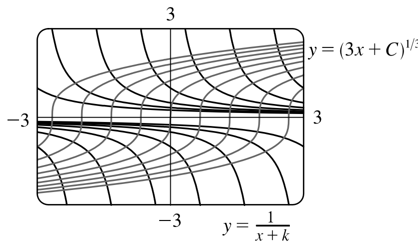
$y^2 = x^2 + C_2 \quad \Leftrightarrow \quad x^2 - y^2 = C$. This is a family of hyperbolas with asymptotes $y = \pm x$.



32. The curves $y = 1/(x + k)$ form a family of hyperbolas with asymptotes $x = -k$ and $y = 0$. Differentiating gives $\dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(\dfrac{1}{x+k}\right) \quad \Rightarrow \quad y' = -\dfrac{1}{(x+k)^2} \quad \Rightarrow \quad y' = -y^2$ [since $y = 1/(x+k)$]. Thus, the slope of the tangent line at any point (x, y) on one of the hyperbolas is $y' = -y^2$, so the orthogonal trajectories must satisfy $y' = 1/y^2 \quad \Leftrightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{y^2} \quad \Leftrightarrow$

$y^2\,dy = dx \quad \Leftrightarrow \quad \int y^2\,dy = \int dx \quad \Leftrightarrow \quad \frac{1}{3}y^3 = x + C_1 \quad \Leftrightarrow$

$y^3 = 3x + C \quad \Leftrightarrow \quad y = (3x + C)^{1/3}$. This is a family of cube root functions with vertical tangents on the x-axis $\quad [y = 0]$.



33. $y(x) = 2 + \displaystyle\int_2^x [t - ty(t)]\,dt \quad \Rightarrow \quad y'(x) = x - xy(x)$ [by FTC 1] $\quad \Rightarrow \quad \dfrac{dy}{dx} = x(1 - y) \quad \Rightarrow$

$\displaystyle\int \frac{dy}{1 - y} = \int x\,dx \quad \Rightarrow \quad -\ln|1 - y| = \frac{1}{2}x^2 + C$. Letting $x = 2$ in the original integral equation gives us $y(2) = 2 + 0 = 2$. Thus, $-\ln|1 - 2| = \frac{1}{2}(2)^2 + C \quad \Rightarrow \quad 0 = 2 + C \quad \Rightarrow \quad C = -2$.

Thus, $-\ln|1 - y| = \frac{1}{2}x^2 - 2 \quad \Rightarrow \quad \ln|1 - y| = 2 - \frac{1}{2}x^2 \quad \Rightarrow \quad |1 - y| = e^{2 - x^2/2} \quad \Rightarrow$

$1 - y = \pm e^{2 - x^2/2} \quad \Rightarrow \quad y = 1 + e^{2 - x^2/2}$ $[y(2) = 2]$.

34. $y(x) = 2 + \displaystyle\int_1^x \frac{dt}{ty(t)}, x > 0 \quad \Rightarrow \quad y'(x) = \frac{1}{xy(x)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{xy} \quad \Rightarrow \quad \int y\,dy = \int \frac{1}{x}\,dx \quad \Rightarrow$

$\frac{1}{2}y^2 = \ln x + C$ $[x > 0]$. Letting $x = 1$ in the original integral equation gives us $y(1) = 2 + 0 = 2$.

Thus, $\frac{1}{2}(2)^2 = \ln 1 + C \quad \Rightarrow \quad C = 2$. $\frac{1}{2}y^2 = \ln x + 2 \quad \Rightarrow \quad y^2 = 2\ln x + 4$ $[> 0]$ $\quad \Rightarrow \quad y = \sqrt{2\ln x + 4}$.

35. $y(x) = 4 + \displaystyle\int_0^x 2t\sqrt{y(t)}\,dt \quad \Rightarrow \quad y'(x) = 2x\sqrt{y(x)} \quad \Rightarrow \quad \frac{dy}{dx} = 2x\sqrt{y} \quad \Rightarrow \quad \int \frac{dy}{\sqrt{y}} = \int 2x\,dx \quad \Rightarrow$

$2\sqrt{y} = x^2 + C$. Letting $x = 0$ in the original integral equation gives us $y(0) = 4 + 0 = 4$.

Thus, $2\sqrt{4} = 0^2 + C \quad \Rightarrow \quad C = 4$. $2\sqrt{y} = x^2 + 4 \quad \Rightarrow \quad \sqrt{y} = \frac{1}{2}x^2 + 2 \quad \Rightarrow \quad y = \left(\frac{1}{2}x^2 + 2\right)^2$.

36. $(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0 \quad \Rightarrow \quad (t^2 + 1)\dfrac{dy}{dt} + y^2 + 1 = 0 \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{-y^2 - 1}{t^2 + 1} \quad \Rightarrow$

$\displaystyle\int \frac{dy}{y^2 + 1} = -\int \frac{dt}{t^2 + 1} \quad \Rightarrow \quad \arctan y = -\arctan t + C \quad \Rightarrow \quad \arctan t + \arctan y = C \quad \Rightarrow$

$\tan(\arctan t + \arctan y) = \tan C \quad \Rightarrow \quad \dfrac{\tan(\arctan t) + \tan(\arctan y)}{1 - \tan(\arctan t)\,\tan(\arctan y)} = \tan C \quad \Rightarrow \quad \dfrac{t + y}{1 - ty} = \tan C = k \quad \Rightarrow$

$t + y = k - kty \quad \Rightarrow \quad y + kty = k - t \quad \Rightarrow \quad y(1 + kt) = k - t \quad \Rightarrow \quad f(t) = y = \dfrac{k - t}{1 + kt}$.

Since $f(3) = 2 = \dfrac{k - 3}{1 + 3k} \quad \Rightarrow \quad 2 + 6k = k - 3 \quad \Rightarrow \quad 5k = -5 \quad \Rightarrow \quad k = -1$, we have $y = \dfrac{-1 - t}{1 + (-1)t} = \dfrac{t + 1}{t - 1}$.

37. From Exercise 9.2.27, $\dfrac{dQ}{dt} = 12 - 4Q \quad \Leftrightarrow \quad \displaystyle\int \frac{dQ}{12 - 4Q} = \int dt \quad \Leftrightarrow \quad -\frac{1}{4}\ln|12 - 4Q| = t + C \quad \Leftrightarrow$

$\ln|12 - 4Q| = -4t - 4C \quad \Leftrightarrow \quad |12 - 4Q| = e^{-4t - 4C} \quad \Leftrightarrow \quad 12 - 4Q = Ke^{-4t}$ $\quad [K = \pm e^{-4C}] \quad \Leftrightarrow$

$4Q = 12 - Ke^{-4t} \quad \Leftrightarrow \quad Q = 3 - Ae^{-4t}$ $\quad [A = K/4]$. $\quad Q(0) = 0 \quad \Leftrightarrow \quad 0 = 3 - A \quad \Leftrightarrow \quad A = 3 \quad \Leftrightarrow$

$Q(t) = 3 - 3e^{-4t}$. As $t \to \infty$, $Q(t) \to 3 - 0 = 3$ (the limiting value).

38. From Exercise 9.2.28, $\dfrac{dy}{dt} = -\frac{1}{50}(y-20) \quad\Leftrightarrow\quad \displaystyle\int \frac{dy}{y-20} = \int\left(-\tfrac{1}{50}\right)dt \quad\Leftrightarrow\quad \ln|y-20| = -\tfrac{1}{50}t + C \quad\Leftrightarrow$

$y - 20 = Ke^{-t/50} \quad\Leftrightarrow\quad y(t) = Ke^{-t/50} + 20.\;\; y(0) = 95 \quad\Leftrightarrow\quad 95 = K + 20 \quad\Leftrightarrow\quad K = 75 \quad\Leftrightarrow$

$y(t) = 75e^{-t/50} + 20.$

39. $\dfrac{dP}{dt} = k(M-P) \quad\Leftrightarrow\quad \displaystyle\int\frac{dP}{P-M} = \int(-k)\,dt \quad\Leftrightarrow\quad \ln|P-M| = -kt + C \quad\Leftrightarrow\quad |P-M| = e^{-kt+C} \quad\Leftrightarrow$

$P - M = Ae^{-kt} \;\; [A = \pm e^C] \quad\Leftrightarrow\quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then

$P(0) = 0 \quad\Leftrightarrow\quad 0 = M + A \quad\Leftrightarrow\quad A = -M \quad\Leftrightarrow\quad P(t) = M - Me^{-kt}.\quad \displaystyle\lim_{t\to\infty} P(t) = M - M\cdot 0 = M.$

40. (a) $\dfrac{dx}{dt} = k(a-x)(b-x),\; a \neq b.$ Using partial fractions, $\dfrac{1}{(a-x)(b-x)} = \dfrac{1/(b-a)}{a-x} - \dfrac{1/(b-a)}{b-x}$, so

$$\int\frac{dx}{(a-x)(b-x)} = \int k\,dt \quad\Rightarrow\quad \frac{1}{b-a}\left(-\ln|a-x| + \ln|b-x|\right) = kt + C \quad\Rightarrow\quad \ln\left|\frac{b-x}{a-x}\right| = (b-a)(kt+C).$$

The concentrations [A] $= a - x$ and [B] $= b - x$ cannot be negative, so $\dfrac{b-x}{a-x} \geq 0$ and $\left|\dfrac{b-x}{a-x}\right| = \dfrac{b-x}{a-x}$.

We now have $\ln\left(\dfrac{b-x}{a-x}\right) = (b-a)(kt+C)$. Since $x(0) = 0$, we get $\ln\left(\dfrac{b}{a}\right) = (b-a)C$. Hence,

$$\ln\left(\frac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\frac{b}{a}\right) \quad\Rightarrow\quad \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt} \quad\Rightarrow\quad x = \frac{b[e^{(b-a)kt}-1]}{be^{(b-a)kt}/a - 1} = \frac{ab[e^{(b-a)kt}-1]}{be^{(b-a)kt}-a}\,\frac{\text{moles}}{\text{L}}.$$

(b) If $b = a$, then $\dfrac{dx}{dt} = k(a-x)^2$, so $\displaystyle\int\frac{dx}{(a-x)^2} = \int k\,dt$ and $\dfrac{1}{a-x} = kt + C$. Since $x(0) = 0$, we get $C = \dfrac{1}{a}$.

Thus, $a - x = \dfrac{1}{kt + 1/a}$ and $x = a - \dfrac{a}{akt+1} = \dfrac{a^2kt}{akt+1}\,\dfrac{\text{moles}}{\text{L}}$. Suppose $x =$ [C] $= a/2$ when $t = 20$. Then

$x(20) = a/2 \quad\Rightarrow\quad \dfrac{a}{2} = \dfrac{20a^2k}{20ak+1} \quad\Rightarrow\quad 40a^2k = 20a^2k + a \quad\Rightarrow\quad 20a^2k = a \quad\Rightarrow\quad k = \dfrac{1}{20a}$, so

$$x = \frac{a^2t/(20a)}{1 + at/(20a)} = \frac{at/20}{1+t/20} = \frac{at}{t+20}\,\frac{\text{moles}}{\text{L}}.$$

41. (a) If $a = b$, then $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2}$ becomes $\dfrac{dx}{dt} = k(a-x)^{3/2} \quad\Rightarrow\quad (a-x)^{-3/2}\,dx = k\,dt \quad\Rightarrow$

$\int(a-x)^{-3/2}\,dx = \int k\,dt \quad\Rightarrow\quad 2(a-x)^{-1/2} = kt + C \quad$ [by substitution] $\quad\Rightarrow\quad \dfrac{2}{kt+C} = \sqrt{a-x} \quad\Rightarrow$

$\left(\dfrac{2}{kt+C}\right)^2 = a - x \quad\Rightarrow\quad x(t) = a - \dfrac{4}{(kt+C)^2}$. The initial concentration of HBr is 0, so $x(0) = 0 \quad\Rightarrow$

$0 = a - \dfrac{4}{C^2} \quad\Rightarrow\quad \dfrac{4}{C^2} = a \quad\Rightarrow\quad C^2 = \dfrac{4}{a} \quad\Rightarrow\quad C = 2/\sqrt{a}\quad$ [C is positive since $kt + C = 2(a-x)^{-1/2} > 0$].

Thus, $x(t) = a - \dfrac{4}{\left(kt + 2/\sqrt{a}\,\right)^2}$.

(b) $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2} \quad\Rightarrow\quad \dfrac{dx}{(a-x)\sqrt{b-x}} = k\,dt \quad\Rightarrow\quad \displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int k\,dt \quad (\star).$

From the hint, $u = \sqrt{b-x} \quad\Rightarrow\quad u^2 = b - x \quad\Rightarrow\quad 2u\,du = -dx$, so

$$\int \frac{dx}{(a-x)\sqrt{b-x}} = \int \frac{-2u\,du}{[a-(b-u^2)]u} = -2\int \frac{du}{a-b+u^2} = -2\int \frac{du}{\left(\sqrt{a-b}\right)^2 + u^2}$$

$$\overset{17}{=} -2\left(\frac{1}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}}\right)$$

So $(\star)$ becomes $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt + C$. Now $x(0) = 0 \quad\Rightarrow\quad C = \dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$ and we have

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}} \quad\Rightarrow\quad \dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right) = kt \quad\Rightarrow$

$t(x) = \dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right).$

42. If $S = \dfrac{dT}{dr}$, then $\dfrac{dS}{dr} = \dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\dfrac{dT}{dr} = 0$ can be written as $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$. Thus,

$\dfrac{dS}{dr} = \dfrac{-2S}{r} \quad\Rightarrow\quad \dfrac{dS}{S} = -\dfrac{2}{r}\,dr \quad\Rightarrow\quad \displaystyle\int \frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \quad\Rightarrow\quad \ln|S| = -2\ln|r| + C$. Assuming $S = dT/dr > 0$

and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k \quad [k = e^C] \quad\Rightarrow\quad S = \dfrac{1}{r^2}k \quad\Rightarrow\quad \dfrac{dT}{dr} = \dfrac{1}{r^2}k \quad\Rightarrow$

$dT = \dfrac{1}{r^2}k\,dr \quad\Rightarrow\quad \displaystyle\int dT = \int \frac{1}{r^2}k\,dr \quad\Rightarrow\quad T(r) = -\frac{k}{r} + A.$

$T(1) = 15 \quad\Rightarrow\quad 15 = -k + A$ **(1)** and $T(2) = 25 \quad\Rightarrow\quad 25 = -\frac{1}{2}k + A$ **(2)**.

Now solve for k and A: -2**(2)** $+$ **(1)** $\quad\Rightarrow\quad -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

43. (a) $\dfrac{dC}{dt} = r - kC \quad\Rightarrow\quad \dfrac{dC}{dt} = -(kC - r) \quad\Rightarrow\quad \displaystyle\int \frac{dC}{kC - r} = \int -dt \quad\Rightarrow\quad (1/k)\ln|kC - r| = -t + M_1 \quad\Rightarrow$

$\ln|kC - r| = -kt + M_2 \quad\Rightarrow\quad |kC - r| = e^{-kt + M_2} \quad\Rightarrow\quad kC - r = M_3e^{-kt} \quad\Rightarrow\quad kC = M_3e^{-kt} + r \quad\Rightarrow$

$C(t) = M_4e^{-kt} + r/k.\ C(0) = C_0 \quad\Rightarrow\quad C_0 = M_4 + r/k \quad\Rightarrow\quad M_4 = C_0 - r/k \quad\Rightarrow$

$C(t) = (C_0 - r/k)e^{-kt} + r/k.$

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim\limits_{t\to\infty} C(t) = r/k$.

As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

44. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10}\,0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10 - x}{10}\cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10-x} = 0.005\,dt \quad\Rightarrow\quad \dfrac{-dx}{10-x} = -0.005\,dt \quad\Rightarrow\quad \ln(10-x) = -0.005t + c \quad\Rightarrow\quad 10 - x = Ce^{-0.005t}$,

where $C = e^c \quad\Rightarrow\quad x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10(1 - e^{-0.005t}) \quad\Rightarrow\quad 0.9 = 1 - e^{-0.005t} \quad\Rightarrow\quad e^{-0.005t} = 0.1 \quad\Rightarrow\quad -0.005t = -\ln 10 \quad\Rightarrow$

$t = 200 \ln 10 \approx 460.517$ days ≈ 1.26 years.

45. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\,\dfrac{\text{kg}}{\text{L}}\right]\left(10\,\dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\,\dfrac{\text{kg}}{\text{min}}$.

$\displaystyle\int \frac{dy}{y} = -\frac{1}{100}\int dt \quad\Rightarrow\quad \ln y = -\frac{t}{100} + C$, and $y(0) = 15 \quad\Rightarrow\quad \ln 15 = C$, so $\ln y = \ln 15 - \dfrac{t}{100}$.

It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

46. Let $y(t)$ be the amount of carbon dioxide in the room after t minutes. Then $y(0) = 0.0015(180) = 0.27\text{ m}^3$. The amount of air in the room is 180 m^3 at all times, so the percentage at time t (in mimutes) is $y(t)/180 \times 100$, and the change in the amount of carbon dioxide with respect to time is

$$\frac{dy}{dt} = (0.0005)\left(2\,\frac{\text{m}^3}{\text{min}}\right) - \frac{y(t)}{180}\left(2\,\frac{\text{m}^3}{\text{min}}\right) = 0.001 - \frac{y}{90} = \frac{9 - 100y}{9000}\,\frac{\text{m}^3}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{9 - 100y} = \int \frac{dt}{9000}$ and $-\dfrac{1}{100}\ln|9 - 100y| = \dfrac{1}{9000}t + C$. Because $y(0) = 0.27$, we have $-\frac{1}{100}\ln 18 = C$, so $-\frac{1}{100}\ln|9 - 100y| = \frac{1}{9000}t - \frac{1}{100}\ln 18 \quad\Rightarrow\quad \ln|9 - 100y| = -\frac{1}{90}t + \ln 18 \quad\Rightarrow$ $\ln|9 - 100y| = \ln e^{-t/90} + \ln 18 \quad\Rightarrow\quad \ln|9 - 100y| = \ln(18e^{-t/90})$, and $|9 - 100y| = 18e^{-t/90}$. Since y is continuous, $y(0) = 0.27$, and the right-hand side is never zero, we deduce that $9 - 100y$ is always negative. Thus, $|9 - 100y| = 100y - 9$ and we have $100y - 9 = 18e^{-t/90} \quad\Rightarrow\quad 100y = 9 + 18e^{-t/90} \quad\Rightarrow\quad y = 0.09 + 0.18e^{-t/90}$. The percentage of carbon dioxide in the room is

$$p(t) = \frac{y}{180} \times 100 = \frac{0.09 + 0.18e^{-t/90}}{180} \times 100 = (0.0005 + 0.001e^{-t/90}) \times 100 = 0.05 + 0.1e^{-t/90}$$

In the long run, we have $\lim\limits_{t\to\infty} p(t) = 0.05 + 0.1(0) = 0.05$; that is, the amount of carbon dioxide approaches 0.05% as time goes on.

47. Let $y(t)$ be the amount of alcohol in the vat after t minutes. Then $y(0) = 0.04(500) = 20$ gal. The amount of beer in the vat is 500 gallons at all times, so the percentage at time t (in minutes) is $y(t)/500 \times 100$, and the change in the amount of alcohol with respect to time t is $\dfrac{dy}{dt} = \text{rate in} - \text{rate out} = 0.06\left(5\,\dfrac{\text{gal}}{\text{min}}\right) - \dfrac{y(t)}{500}\left(5\,\dfrac{\text{gal}}{\text{min}}\right) = 0.3 - \dfrac{y}{100} = \dfrac{30 - y}{100}\,\dfrac{\text{gal}}{\text{min}}$.

Hence, $\displaystyle\int \frac{dy}{30 - y} = \int \frac{dt}{100}$ and $-\ln|30 - y| = \frac{1}{100}t + C$. Because $y(0) = 20$, we have $-\ln 10 = C$, so

$-\ln|30-y| = \frac{1}{100}t - \ln 10 \quad\Rightarrow\quad \ln|30-y| = -t/100 + \ln 10 \quad\Rightarrow\quad \ln|30-y| = \ln e^{-t/100} + \ln 10 \quad\Rightarrow$ $\ln|30-y| = \ln(10e^{-t/100}) \quad\Rightarrow\quad |30-y| = 10e^{-t/100}$. Since y is continuous, $y(0) = 20$, and the right-hand side is never zero, we deduce that $30 - y$ is always positive. Thus, $30 - y = 10e^{-t/100} \quad\Rightarrow\quad y = 30 - 10e^{-t/100}$. The percentage of alcohol is $p(t) = y(t)/500 \times 100 = y(t)/5 = 6 - 2e^{-t/100}$. The percentage of alcohol after one hour is $p(60) = 6 - 2e^{-60/100} \approx 4.9$.

48. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\begin{aligned}\frac{dy}{dt} &= \left(0.05\ \frac{\text{kg}}{\text{L}}\right)\left(5\ \frac{\text{L}}{\text{min}}\right) + \left(0.04\ \frac{\text{kg}}{\text{L}}\right)\left(10\ \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\ \frac{\text{kg}}{\text{L}}\right)\left(15\ \frac{\text{L}}{\text{min}}\right)\\ &= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130-3y}{200}\ \frac{\text{kg}}{\text{min}}\end{aligned}$$

Hence, $\displaystyle\int \frac{dy}{130-3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t + C$. Because $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \quad\Rightarrow\quad \ln|130-3y| = -\frac{3}{200}t + \ln 130 = \ln(130e^{-3t/200})$, and $|130-3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}(1 - e^{-3t/200})$ kg.

(b) After one hour, $y = \frac{130}{3}(1 - e^{-3\cdot 60/200}) = \frac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

49. Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \quad\Rightarrow\quad mv' + vm' = gm \quad\Rightarrow$ $mv' + v(km) = gm \quad\Rightarrow\quad v' + vk = g \quad\Rightarrow\quad \dfrac{dv}{dt} = g - kv \quad\Rightarrow\quad \displaystyle\int \frac{dv}{g-kv} = \int dt \quad\Rightarrow$ $-(1/k)\ln|g-kv| = t + C \quad\Rightarrow\quad \ln|g-kv| = -kt - kC \quad\Rightarrow\quad g - kv = Ae^{-kt}$. $v(0) = 0 \quad\Rightarrow\quad A = g$. So $kv = g - ge^{-kt} \quad\Rightarrow\quad v = (g/k)(1 - e^{-kt})$. Since $k > 0$, as $t \to \infty$, $e^{-kt} \to 0$ and therefore, $\lim\limits_{t\to\infty} v(t) = g/k$.

50. (a) $m\dfrac{dv}{dt} = -kv \quad\Rightarrow\quad \dfrac{dv}{v} = -\dfrac{k}{m}\,dt \quad\Rightarrow\quad \ln|v| = -\dfrac{k}{m}t + C$. Since $v(0) = v_0$, $\ln|v_0| = C$. Therefore, $\ln\left|\dfrac{v}{v_0}\right| = -\dfrac{k}{m}t \quad\Rightarrow\quad \left|\dfrac{v}{v_0}\right| = e^{-kt/m} \quad\Rightarrow\quad v(t) = \pm v_0 e^{-kt/m}$. The sign is $+$ when $t = 0$, and we assume v is continuous, so that the sign is $+$ for all t. Thus, $v(t) = v_0 e^{-kt/m}$. $ds/dt = v_0 e^{-kt/m} \quad\Rightarrow$ $s(t) = -\dfrac{mv_0}{k}e^{-kt/m} + C'$.

From $s(0) = s_0$, we get $s_0 = -\dfrac{mv_0}{k} + C'$, so $C' = s_0 + \dfrac{mv_0}{k}$ and $s(t) = s_0 + \dfrac{mv_0}{k}(1 - e^{-kt/m})$.

The distance traveled from time 0 to time t is $s(t) - s_0$, so the total distance traveled is $\lim\limits_{t\to\infty}[s(t) - s_0] = \dfrac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k > 0$ to conclude that $\lim\limits_{t\to\infty} e^{-kt/m} = 0$.

(b) $m\,\dfrac{dv}{dt} = -kv^2 \;\Rightarrow\; \dfrac{dv}{v^2} = -\dfrac{k}{m}\,dt \;\Rightarrow\; \dfrac{-1}{v} = -\dfrac{kt}{m} + C \;\Rightarrow\; \dfrac{1}{v} = \dfrac{kt}{m} - C$. Since $v(0) = v_0$,

$C = -\dfrac{1}{v_0}$ and $\dfrac{1}{v} = \dfrac{kt}{m} + \dfrac{1}{v_0}$. Therefore, $v(t) = \dfrac{1}{kt/m + 1/v_0} = \dfrac{mv_0}{kv_0t + m}$. $\dfrac{ds}{dt} = \dfrac{mv_0}{kv_0t + m} \;\Rightarrow$

$s(t) = \dfrac{m}{k}\displaystyle\int \dfrac{kv_0\,dt}{kv_0t + m} = \dfrac{m}{k}\ln|kv_0t + m| + C'$. Since $s(0) = s_0$, we get $s_0 = \dfrac{m}{k}\ln m + C' \;\Rightarrow$

$C' = s_0 - \dfrac{m}{k}\ln m \;\Rightarrow\; s(t) = s_0 + \dfrac{m}{k}(\ln|kv_0t + m| - \ln m) = s_0 + \dfrac{m}{k}\ln\left|\dfrac{kv_0t + m}{m}\right|$.

We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t) = \dfrac{v_0}{1 + (kv_0/m)t}$ and $s(t) = s_0 + \dfrac{m}{k}\ln\left|1 + \dfrac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which $v_0 > 0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k, v_0, and m are all positive), and $\lim\limits_{t\to\infty} s(t) = \infty$. In other words, the object travels infinitely far. However, $\lim\limits_{t\to\infty} v(t) = 0$. When $v_0 < 0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object approaches $-\infty$ as t approaches $m/k(-v_0)$: $\lim\limits_{t\to -m/(kv_0)} s(t) = -\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim\limits_{t\to -m/(kv_0)} v(t) = -\infty$ when $v_0 < 0$, showing that the speed of the object increases without limit.

51. (a) $\dfrac{1}{L_1}\dfrac{dL_1}{dt} = k\dfrac{1}{L_2}\dfrac{dL_2}{dt} \;\Rightarrow\; \dfrac{d}{dt}(\ln L_1) = \dfrac{d}{dt}(k\ln L_2) \;\Rightarrow\; \displaystyle\int \dfrac{d}{dt}(\ln L_1)\,dt = \int \dfrac{d}{dt}(\ln L_2^k)\,dt \;\Rightarrow$

$\ln L_1 = \ln L_2^k + C \;\Rightarrow\; L_1 = e^{\ln L_2^k + C} = e^{\ln L_2^k}e^C \;\Rightarrow\; L_1 = KL_2^k$, where $K = e^C$.

(b) From part (a) with $L_1 = B$, $L_2 = V$, and $k = 0.0794$, we have $B = KV^{0.0794}$.

52. (a) $\dfrac{dV}{dt} = a(\ln b - \ln V)V \;\Rightarrow\; \dfrac{dV}{dt} = -aV(\ln V - \ln b) \;\Rightarrow\; \dfrac{dV}{V\ln(V/b)} = -a\,dt \;\Rightarrow$

$\displaystyle\int \dfrac{dV}{V\ln(V/b)} = \int -a\,dt \;\Rightarrow\; \int \dfrac{1}{u}\,du = \int -a\,dt \quad \begin{bmatrix} u = \ln(V/b), \\ du = (1/V)\,dV \end{bmatrix} \;\Rightarrow\; \ln|u| = -at + k \;\Rightarrow$

$|u| = e^{-at}e^k \;\Rightarrow\; u = Ce^{-at}$ [where $C = \pm e^k$] $\Rightarrow\; \ln(V/b) = Ce^{-at} \;\Rightarrow\; \dfrac{V}{b} = e^{Ce^{-at}} \;\Rightarrow$

$V = be^{Ce^{-at}}$ with $C \neq 0$.

(b) $V(0) = 1 \;\Rightarrow\; 1 = be^{Ce^{-a(0)}} \;\Rightarrow\; 1 = be^C \;\Rightarrow\; b = e^{-C}$, so $V = e^{-C}e^{Ce^{-at}} = e^{Ce^{-at} - C} = e^{C(e^{-at} - 1)}$.

53. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k, $dA/dt = k\sqrt{A}\,(M - A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for

dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\sqrt{A}\,(-1)\frac{dA}{dt} + (M-A)\cdot \tfrac{1}{2}A^{-1/2}\frac{dA}{dt}\right] = \tfrac{1}{2}kA^{-1/2}\frac{dA}{dt}\,[-2A+(M-A)]$$

$$= \tfrac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M-A)\right][M-3A] = \tfrac{1}{2}k^2(M-A)(M-3A)$$

This is 0 when $M-A=0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y=M$, as in the logistic model] and when $M-3A=0 \quad\Leftrightarrow\quad A(t)=M/3$. This represents a maximum by the First Derivative Test, since $\dfrac{d}{dt}\left(\dfrac{dA}{dt}\right)$ goes from positive to negative when $A(t)=M/3$.

(b) From the CAS, we get $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt}-1}{Ce^{\sqrt{M}kt}+1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M, we substitute $t=0$ and $A=A_0=A(0)$: $A_0 = M\left(\dfrac{C-1}{C+1}\right)^2 \quad\Leftrightarrow\quad (C+1)\sqrt{A_0} = (C-1)\sqrt{M} \quad\Leftrightarrow$

$C\sqrt{A_0}+\sqrt{A_0} = C\sqrt{M}-\sqrt{M} \quad\Leftrightarrow\quad \sqrt{M}+\sqrt{A_0} = C\sqrt{M}-C\sqrt{A_0} \quad\Leftrightarrow$

$\sqrt{M}+\sqrt{A_0} = C\left(\sqrt{M}-\sqrt{A_0}\right) \quad\Leftrightarrow\quad C = \dfrac{\sqrt{M}+\sqrt{A_0}}{\sqrt{M}-\sqrt{A_0}}$. [Notice that if $A_0=0$, then $C=1$.]

54. (a) According to the hint we use the Chain Rule: $m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = mv\dfrac{dv}{dx} = -\dfrac{mgR^2}{(x+R)^2} \quad\Rightarrow$

$\displaystyle\int v\,dv = \int \frac{-gR^2\,dx}{(x+R)^2} \quad\Rightarrow\quad \frac{v^2}{2} = \frac{gR^2}{x+R}+C$. When $x=0$, $v=v_0$, so $\dfrac{v_0^2}{2} = \dfrac{gR^2}{0+R}+C \quad\Rightarrow$

$C = \tfrac{1}{2}v_0^2 - gR \quad\Rightarrow\quad \tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \dfrac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0, and its height will be $x=h$. Solving for v_0: $-\tfrac{1}{2}v_0^2 = \dfrac{gR^2}{h+R} - gR \quad\Rightarrow\quad \dfrac{v_0^2}{2} = g\left[-\dfrac{R^2}{R+h}+\dfrac{R(R+h)}{R+h}\right] = \dfrac{gRh}{R+h} \quad\Rightarrow$

$v_0 = \sqrt{\dfrac{2gRh}{R+h}}$.

(b) $v_e = \displaystyle\lim_{h\to\infty} v_0 = \lim_{h\to\infty}\sqrt{\frac{2gRh}{R+h}} = \lim_{h\to\infty}\sqrt{\frac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2\cdot 32\text{ ft/s}^2\cdot 3960\text{ mi}\cdot 5280\text{ ft/mi}} \approx 36{,}581\text{ ft/s} \approx 6.93\text{ mi/s}$

APPLIED PROJECT How Fast Does a Tank Drain?

1. (a) $V = \pi r^2 h \quad\Rightarrow\quad \dfrac{dV}{dt} = \pi r^2\dfrac{dh}{dt}$ [implicit differentiation] $\Rightarrow$

$$\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{\pi r^2}\left(-a\sqrt{2gh}\right) = \frac{1}{\pi 2^2}\left[-\pi\left(\tfrac{1}{12}\right)^2\sqrt{2\cdot 32}\,\sqrt{h}\right] = -\tfrac{1}{72}\sqrt{h}$$

(b) $\dfrac{dh}{dt} = -\tfrac{1}{72}\sqrt{h} \quad\Rightarrow\quad h^{-1/2}\,dh = -\tfrac{1}{72}\,dt \quad\Rightarrow\quad 2\sqrt{h} = -\tfrac{1}{72}t + C.$

$h(0)=6 \quad\Rightarrow\quad 2\sqrt{6} = 0+C \quad\Rightarrow\quad C = 2\sqrt{6} \quad\Rightarrow\quad h(t) = \left(-\tfrac{1}{144}t+\sqrt{6}\right)^2.$

(c) We want to find t when $h = 0$, so we set $h = 0 = \left(-\frac{1}{144}t + \sqrt{6}\right)^2 \Rightarrow t = 144\sqrt{6} \approx 5$ min 53 s.

2. (a) $\dfrac{dh}{dt} = k\sqrt{h} \Rightarrow h^{-1/2}\,dh = k\,dt \quad [h \neq 0] \Rightarrow 2\sqrt{h} = kt + C \Rightarrow$

$h(t) = \frac{1}{4}(kt + C)^2$. Since $h(0) = 10$ cm, the relation $2\sqrt{h(t)} = kt + C$ gives us $2\sqrt{10} = C$. Also, $h(68) = 3$ cm, so $2\sqrt{3} = 68k + 2\sqrt{10}$ and

$k = -\dfrac{\sqrt{10} - \sqrt{3}}{34}$. Thus,

$$h(t) = \frac{1}{4}\left(2\sqrt{10} - \frac{\sqrt{10} - \sqrt{3}}{34}\,t\right)^2 \approx 10 - 0.133t + 0.00044t^2.$$

Here is a table of values of $h(t)$ correct to one decimal place.

t (in s)	$h(t)$ (in cm)
10	8.7
20	7.5
30	6.4
40	5.4
50	4.5
60	3.6

(b) The answers to this part are to be obtained experimentally. See the article by Tom Farmer and Fred Gass, *Physical Demonstrations in the Calculus Classroom,* College Mathematics Journal 1992, pp. 146–148.

3. $V(t) = \pi r^2 h(t) = 100\pi h(t) \Rightarrow \dfrac{dV}{dh} = 100\pi$ and $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = 100\pi\,\dfrac{dh}{dt}$.

Diameter $= 2.5$ inches $\Rightarrow$ radius $= 1.25$ inches $= \frac{5}{4} \cdot \frac{1}{12}$ foot $= \frac{5}{48}$ foot. Thus, $\dfrac{dV}{dt} = -a\sqrt{2gh} \Rightarrow$

$100\pi\,\dfrac{dh}{dt} = -\pi\left(\frac{5}{48}\right)^2\sqrt{2 \cdot 32h} = -\dfrac{25\pi}{288}\sqrt{h} \Rightarrow \dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{1152} \Rightarrow \int h^{-1/2}\,dh = \int -\frac{1}{1152}\,dt \Rightarrow$

$2\sqrt{h} = -\frac{1}{1152}t + C \Rightarrow \sqrt{h} = -\frac{1}{2304}t + k \Rightarrow h(t) = \left(-\frac{1}{2304}t + k\right)^2$. The water pressure after t seconds is $62.5h(t)$ lb/ft^2, so the condition that the pressure be at least 2160 lb/ft^2 for 10 minutes (600 seconds) is the condition $62.5 \cdot h(600) \geq 2160$; that is, $\left(k - \frac{600}{2304}\right)^2 \geq \frac{2160}{62.5} \Rightarrow \left|k - \frac{25}{96}\right| \geq \sqrt{34.56} \Rightarrow k \geq \frac{25}{96} + \sqrt{34.56}$. Now $h(0) = k^2$, so the height of the tank should be at least $\left(\frac{25}{96} + \sqrt{34.56}\right)^2 \approx 37.69$ ft.

4. (a) If the radius of the circular cross-section at height h is r, then the Pythagorean Theorem gives $r^2 = 2^2 - (2 - h)^2$ since the radius of the tank is 2 m. So $A(h) = \pi r^2 = \pi[4 - (2 - h)^2] = \pi(4h - h^2)$. Thus, $A(h)\,\dfrac{dh}{dt} = -a\sqrt{2gh} \Rightarrow$

$\pi(4h - h^2)\,\dfrac{dh}{dt} = -\pi(0.01)^2\sqrt{2 \cdot 10h} \Rightarrow (4h - h^2)\,\dfrac{dh}{dt} = -0.0001\sqrt{20h}$.

(b) From part (a) we have $(4h^{1/2} - h^{3/2})\,dh = \left(-0.0001\sqrt{20}\right)dt \Rightarrow \frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = \left(-0.0001\sqrt{20}\right)t + C$.

$h(0) = 2 \Rightarrow \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \Rightarrow C = \left(\frac{16}{3} - \frac{8}{5}\right)\sqrt{2} = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $h = 0$: $0 = \left(-0.0001\sqrt{20}\right)t + C \Rightarrow$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}/15}{0.0001\sqrt{20}} = \frac{11{,}200\sqrt{10}}{3} \approx 11{,}806 \text{ s} \approx 3 \text{ h } 17 \text{ min}$$

APPLIED PROJECT Which Is Faster, Going Up or Coming Down?

1. $mv' = -pv - mg \quad \Rightarrow \quad m\,\dfrac{dv}{dt} = -(pv + mg) \quad \Rightarrow \quad \displaystyle\int \frac{dv}{pv + mg} = \int -\frac{1}{m}\,dt \quad \Rightarrow$

$\dfrac{1}{p}\,\ln(pv + mg) = -\dfrac{1}{m}t + C \quad [pv + mg > 0]$. At $t = 0$, $v = v_0$, so $C = \dfrac{1}{p}\,\ln(pv_0 + mg)$.

Thus, $\dfrac{1}{p}\,\ln(pv + mg) = -\dfrac{1}{m}t + \dfrac{1}{p}\,\ln(pv_0 + mg) \quad \Rightarrow \quad \ln(pv + mg) = -\dfrac{p}{m}t + \ln(pv_0 + mg) \quad \Rightarrow$

$pv + mg = e^{-pt/m}(pv_0 + mg) \quad \Rightarrow \quad pv = (pv_0 + mg)e^{-pt/m} - mg \quad \Rightarrow \quad v(t) = \left(v_0 + \dfrac{mg}{p}\right)e^{-pt/m} - \dfrac{mg}{p}$.

2. $y(t) = \int v(t)\,dt = \displaystyle\int \left[\left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}\right] dt = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m}\left(-\frac{m}{p}\right) - \frac{mg}{p}t + C.$

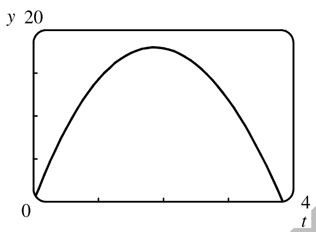
At $t = 0$, $y = 0$, so $C = \left(v_0 + \dfrac{mg}{p}\right)\dfrac{m}{p}$. Thus,

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p} - \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}e^{-pt/m} - \frac{mgt}{p} = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

3. $v(t) = 0 \quad \Rightarrow \quad \dfrac{mg}{p} = \left(v_0 + \dfrac{mg}{p}\right)e^{-pt/m} \quad \Rightarrow \quad e^{pt/m} = \dfrac{pv_0}{mg} + 1 \quad \Rightarrow \quad \dfrac{pt}{m} = \ln\left(\dfrac{pv_0}{mg} + 1\right) \quad \Rightarrow$

$t_1 = \dfrac{m}{p}\ln\left(\dfrac{mg + pv_0}{mg}\right)$. With $m = 1$, $v_0 = 20$, $p = \frac{1}{10}$, and $g = 9.8$, we have $t_1 = 10\ln\left(\frac{11.8}{9.8}\right) \approx 1.86$ s.

4.



The figure shows the graph of $y = 1180(1 - e^{-0.1t}) - 98t$. The zeros are at $t = 0$ and $t_2 \approx 3.84$. Thus, $t_1 - 0 \approx 1.86$ and $t_2 - t_1 \approx 1.98$. So the time it takes to come down is about 0.12 s longer than the time it takes to go up; hence, going up is faster.

5. $y(2t_1) = \left(v_0 + \dfrac{mg}{p}\right)\dfrac{m}{p}(1 - e^{-2pt_1/m}) - \dfrac{mg}{p}\cdot 2t_1$

$\quad = \left(\dfrac{pv_0 + mg}{p}\right)\dfrac{m}{p}\left[1 - (e^{pt_1/m})^{-2}\right] - \dfrac{mg}{p}\cdot 2\dfrac{m}{p}\,\ln\left(\dfrac{pv_0 + mg}{mg}\right)$

Substituting $x = e^{pt_1/m} = \dfrac{pv_0}{mg} + 1 = \dfrac{pv_0 + mg}{mg}$ (from Problem 3), we get

$y\,(2t_1) = \left(x\cdot\dfrac{mg}{p}\right)\dfrac{m}{p}(1 - x^{-2}) - \dfrac{m^2g}{p^2}\cdot 2\ln x = \dfrac{m^2g}{p^2}\left(x - \dfrac{1}{x} - 2\ln x\right)$. Now $p > 0$, $m > 0$, $t_1 > 0 \quad \Rightarrow$

$x = e^{pt_1/m} > e^0 = 1. \quad f(x) = x - \dfrac{1}{x} - 2\ln x \quad \Rightarrow \quad f'(x) = 1 + \dfrac{1}{x^2} - \dfrac{2}{x} = \dfrac{x^2 - 2x + 1}{x^2} = \dfrac{(x-1)^2}{x^2} > 0$

for $x > 1 \quad \Rightarrow \quad f(x)$ is increasing for $x > 1$. Since $f(1) = 0$, it follows that $f(x) > 0$ for every $x > 1$. Therefore, $y(2t_1) = \dfrac{m^2g}{p^2}\,f(x)$ is positive, which means that the ball has not yet reached the ground at time $2t_1$. This tells us that the time spent going up is always less than the time spent coming down, so *ascent is faster*.

9.4 Models for Population Growth

1. (a) Comparing the given equation, $\dfrac{dP}{dt} = 0.04P\left(1 - \dfrac{P}{M}\right)$, to Equation 4, $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$, we see that the carrying capacity is $M = 1200$ and the value of k is 0.04.

(b) By Equation 7, the solution of the equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$. Since $P(0) = P_0 = 60$, we have $A = \dfrac{1200 - 60}{60} = 19$, and hence, $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$.

(c) The population after 10 weeks is $P(10) = \dfrac{1200}{1 + 19e^{-0.04(10)}} \approx 87$.

2. (a) $dP/dt = 0.02P - 0.00004P^2 = 0.02P(1 - 0.002P) = 0.02P(1 - P/500)$. Comparing to Equation 4, $dP/dt = kP(1 - P/M)$, we see that the carrying capacity is $M = 500$ and the value of k is 0.02.

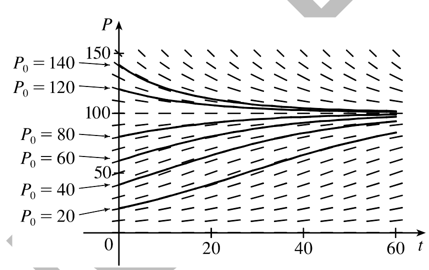
(b) By Equation 7, the solution of the equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$. Since $P(0) = P_0 = 40$, we have $A = \dfrac{500 - 40}{40} = 11.5$, and hence, $P(t) = \dfrac{500}{1 + 11.5e^{-0.02t}}$.

(c) The population after 10 weeks is $P(10) = \dfrac{500}{1 + 11.5e^{-0.02(10)}} \approx 48$.

3. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 4, $dP/dt = kP(1 - P/M)$, we see that the carrying capacity is $M = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)

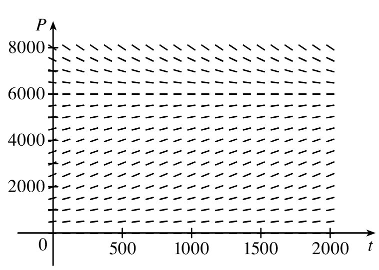


All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

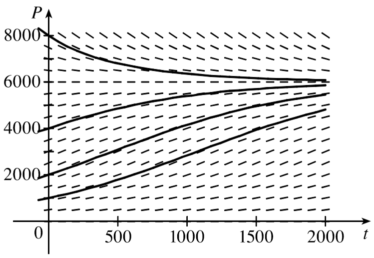
(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

4. (a) $M = 6000$ and $k = 0.0015 \quad \Rightarrow \quad dP/dt = 0.0015P(1 - P/6000)$.

(b)



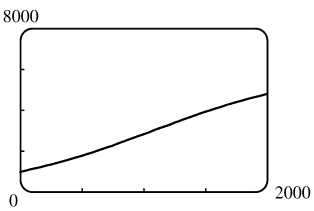
All of the solution curves approach 6000 as $t \to \infty$.

(c) 

The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise 9.2.25 for a possible program to calculate $P(50)$. [In this case, we use $\text{X} = 0$, $\text{H} = 1$, $\text{N} = 50$, $\text{Y}_1 = 0.0015y(1 - y/6000)$, and $\text{Y} = 1000$.] We find that $P(50) \approx 1064$.

(e) Using Equation 7 with $M = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{6000}{1 + Ae^{-0.0015t}}$, where $A = \dfrac{M - P_0}{P_0} = \dfrac{6000 - 1000}{1000} = 5$. Thus, $P\,(50) = \dfrac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1$, which is extremely close to the estimate obtained in part (d).

(f) 

The curves are very similar.

5. (a) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right) \;\Rightarrow\; y(t) = \dfrac{M}{1 + Ae^{-kt}}$ with $A = \dfrac{M - y(0)}{y(0)}$. With $M = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \;\Rightarrow\; \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \;\Rightarrow\; 2 = 1 + 3e^{-0.71t} \;\Rightarrow\; e^{-0.71t} = \frac{1}{3} \;\Rightarrow\; -0.71t = \ln\frac{1}{3} \;\Rightarrow\; t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

6. (a) $\dfrac{dP}{dt} = 0.4P - 0.001P^2 = 0.4P(1 - 0.0025P)$ $\left[\frac{0.001}{0.4} = 0.0025\right]$ $= 0.4P\left(1 - \dfrac{P}{400}\right)$ $\left[0.0025^{-1} = 400\right]$

Thus, by Equation 4, $k = 0.4$ and the carrying capacity is 400.

(b) Using the fact that $P(0) = 50$ and the formula for dP/dt, we get

$$P'(0) = \left.\frac{dP}{dt}\right|_{t=0} = 0.4(50) - 0.001(50)^2 = 20 - 2.5 = 17.5.$$

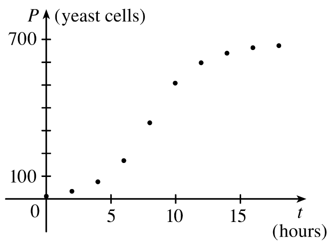
(c) From Equation 7, $A = \dfrac{M - P_0}{P_0} = \dfrac{400 - 50}{50} = 7$, so $P = \dfrac{400}{1 + 7e^{-0.4t}}$. The population reaches 50% of the carrying capacity, 200, when $200 = \dfrac{400}{1 + 7e^{-0.4t}} \;\Rightarrow\; 1 + 7e^{-0.4t} = 2 \;\Rightarrow\; e^{-0.4t} = \frac{1}{7} \;\Rightarrow\; -0.4t = \ln\frac{1}{7} \;\Rightarrow\; t = \left(\ln\frac{1}{7}\right)/(-0.4) \approx 4.86$ years.

7. Using (7), $A = \dfrac{M - P_0}{P_0} = \dfrac{10{,}000 - 1000}{1000} = 9$, so $P(t) = \dfrac{10{,}000}{1 + 9e^{-kt}}$. $P(1) = 2500 \quad \Rightarrow \quad 2500 = \dfrac{10{,}000}{1 + 9e^{-k(1)}} \quad \Rightarrow$

$1 + 9e^{-k} = 4 \quad \Rightarrow \quad 9e^{-k} = 3 \quad \Rightarrow \quad e^{-k} = \frac{1}{3} \quad \Rightarrow \quad -k = \ln\frac{1}{3} \quad \Rightarrow \quad k = \ln 3$. After another three years, $t = 4$,

and $P(4) = \dfrac{10{,}000}{1 + 9e^{-(\ln 3)4}} = \dfrac{10{,}000}{1 + 9\left(e^{\ln 3}\right)^{-4}} = \dfrac{10{,}000}{1 + 9(3)^{-4}} = \dfrac{10{,}000}{1 + \frac{1}{9}} = \dfrac{10{,}000}{\frac{10}{9}} = 9000$.

8. (a)



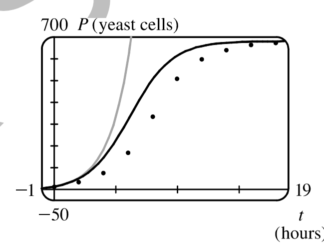
From the graph, we estimate the carrying capacity M for the yeast population to be 680.

(b) An estimate of the initial relative growth rate is $\dfrac{1}{P_0}\dfrac{dP}{dt} = \dfrac{1}{18} \cdot \dfrac{39 - 18}{2 - 0} = \dfrac{7}{12} = 0.58\overline{3}$.

(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680 - 18}{18} = \frac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9}e^{-7(7/12)}} \approx 420$ yeast cells

9. (a) We will assume that the difference in birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 2000. Thus,

$k \approx \dfrac{1}{P}\dfrac{dP}{dt} = \dfrac{1}{6.1 \text{ billion}}\left(\dfrac{20 \text{ million}}{\text{year}}\right) = \dfrac{1}{305}$, and $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) = \dfrac{1}{305}P\left(1 - \dfrac{P}{20}\right)$ with P in billions.

(b) $A = \dfrac{M - P_0}{P_0} = \dfrac{20 - 6.1}{6.1} = \dfrac{139}{61} \approx 2.2787$. $\quad P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{20}{1 + \frac{139}{61}e^{-t/305}}$, so

$P(10) = \dfrac{20}{1 + \frac{139}{61}e^{-10/305}} \approx 6.24$ billion, which underestimates the actual 2010 population of 6.9 billion.

(c) The years 2100 and 2500 correspond to $t = 100$ and $t = 500$, respectively. $P(100) = \dfrac{20}{1 + \frac{139}{61}e^{-100/305}} \approx 7.57$ billion

and $P(500) = \dfrac{20}{1 + \frac{139}{61}e^{-500/305}} \approx 13.87$ billion.

10. (a) Let $t = 0$ correspond to the year 2000. $A = \dfrac{M - P_0}{P_0} = \dfrac{800 - 282}{282} = \dfrac{259}{141} \approx 1.8369$.

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{800}{1 + \frac{259}{141}e^{-kt}} \text{ with } P \text{ in millions.}$$

(b) $P(10) = 309 \quad \Leftrightarrow \quad \dfrac{800}{1 + \frac{259}{141}e^{-10k}} = 309 \quad \Leftrightarrow \quad \dfrac{800}{309} = 1 + \dfrac{259}{141}e^{-10k} \quad \Leftrightarrow \quad \dfrac{491}{309} = \dfrac{259}{141}e^{-10k} \quad \Leftrightarrow$

$\dfrac{491 \cdot 141}{309 \cdot 259} = e^{-10k} \quad \Leftrightarrow \quad -10k = \ln \dfrac{491 \cdot 47}{103 \cdot 259} \quad \Leftrightarrow \quad k = -\dfrac{1}{10} \ln \dfrac{23{,}077}{26{,}677} \approx 0.0145$.

(c) The years 2100 and 2200 correspond to $t = 100$ and $t = 200$, respectively. $P(100) = \dfrac{800}{1 + \frac{259}{141}e^{-100k}} \approx 559$ million and

$P(200) = \dfrac{800}{1 + \frac{259}{141}e^{-200k}} \approx 727$ million.

(d) $P(t) = 500 \quad \Leftrightarrow \quad \dfrac{800}{1 + \frac{259}{141}e^{-kt}} = 500 \quad \Leftrightarrow \quad \dfrac{800}{500} = 1 + \dfrac{259}{141}e^{-kt} \quad \Leftrightarrow \quad \dfrac{3}{5} = \dfrac{259}{141}e^{-kt} \quad \Leftrightarrow \quad \dfrac{3 \cdot 141}{5 \cdot 259} = e^{-kt} \quad \Leftrightarrow$

$-kt = \ln \dfrac{423}{1295} \quad \Leftrightarrow \quad t = 10\,\dfrac{\ln(423/1295)}{\ln(23{,}077/26{,}677)} \approx 77.18$ years. Our logistic model predicts that the US population will exceed 500 million in 77.18 years; that is, in the year 2077.

11. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1 - y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (4), $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$, we substitute $y = \dfrac{P}{M}$, $P = My$, and $\dfrac{dP}{dt} = M\,\dfrac{dy}{dt}$,

to obtain $M\,\dfrac{dy}{dt} = k(My)(1 - y) \quad \Leftrightarrow \quad \dfrac{dy}{dt} = ky(1 - y)$, our equation in part (a).

Now the solution to (4) is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$.

We use the same substitution to obtain $My = \dfrac{M}{1 + \dfrac{M - My_0}{My_0}e^{-kt}} \quad \Rightarrow \quad y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in the solution of Equation 4.

(c) Let t be the number of hours since 8 AM. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$,

so $y = \dfrac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \dfrac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2 - 2y}{23y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4 - 1} = \frac{1 - y}{y}.$$

It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1 - y)/y]}{\ln \frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1 - y)/y)}{\ln \frac{2}{23}}\right]$.

When $y = 0.9$, $\dfrac{1 - y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln \frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

12. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $M = 10{,}000$. From the solution to the logistic differential equation

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kt}}, \text{ we get } P = \frac{400\,(10{,}000)}{400 + (9600)e^{-kt}} = \frac{10{,}000}{1 + 24e^{-kt}}. \quad P(1) = 1200 \quad \Rightarrow$$

$$1 + 24e^{-k} = \tfrac{100}{12} \quad \Rightarrow \quad e^k = \tfrac{288}{88} \quad \Rightarrow \quad k = \ln \tfrac{36}{11}. \text{ So } P = \frac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \frac{10{,}000}{1 + 24 \cdot (11/36)^t}.$$

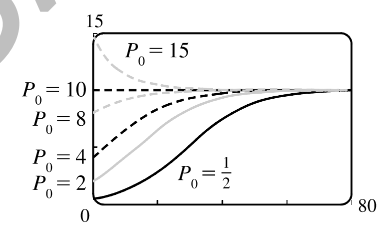
(b) $5000 = \dfrac{10{,}000}{1 + 24(11/36)^t} \quad \Rightarrow \quad 24\left(\tfrac{11}{36}\right)^t = 1 \quad \Rightarrow \quad t \ln \tfrac{11}{36} = \ln \tfrac{1}{24} \quad \Rightarrow \quad t \approx 2.68$ years.

13. (a) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) \quad \Rightarrow \quad \dfrac{d^2P}{dt^2} = k\left[P\left(-\dfrac{1}{M}\dfrac{dP}{dt}\right) + \left(1 - \dfrac{P}{M}\right)\dfrac{dP}{dt}\right] = k\,\dfrac{dP}{dt}\left(-\dfrac{P}{M} + 1 - \dfrac{P}{M}\right)$

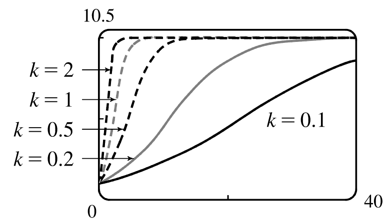
$$= k\left[kP\left(1 - \frac{P}{M}\right)\right]\left(1 - \frac{2P}{M}\right) = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \quad \Leftrightarrow \quad P = 0$, $P = M$, or $P = M/2$. Since $0 < P < M$, we see that $P'' = 0 \quad \Leftrightarrow \quad P = M/2$.

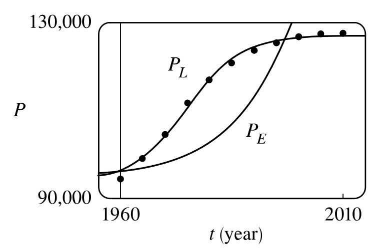
14. First we keep k constant (at 0.1, say) and change P_0 in the function $P = \dfrac{10P_0}{P_0 + (10 - P_0)e^{-0.1t}}$. (Notice that P_0 is the P-intercept.) If $P_0 = 0$, the function is 0 everywhere. For $0 < P_0 < 5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5 < P_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $P_0 = 10$, the function is the constant function $P = 10$, and if $P_0 > 10$, the function decreases. For all $P_0 \neq 0$, $\lim_{t\to\infty} P = 10$.



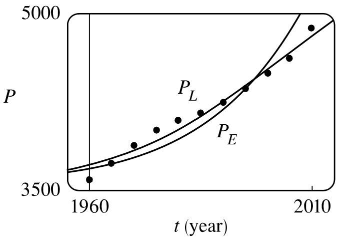
Now we instead keep P_0 constant (at $P_0 = 1$) and change k in the function $P = \dfrac{10}{1 + 9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x-scales, the graphs all look the same.)



15. Following the hint, we choose $t = 0$ to correspond to 1960 and subtract 94,000 from each of the population figures. We then use a calculator to obtain the models and add 94,000 to get the exponential function $P_E(t) = 1909.7761(1.0796)^t + 94{,}000$ and the logistic function $P_L(t) = \dfrac{33{,}086.4394}{1 + 12.3428e^{-0.1657t}} + 94{,}000$. P_L is a reasonably accurate model, while P_E is not, since an exponential model would only be used for the first few data points.



16. Following the hint, we choose $t = 0$ to correspond to 1960 and subtract 3500 from each of the population figures. We then use a calculator to obtain the models and add 3500 to get the exponential function $P_E(t) = 180.9934(1.0445)^t + 3500$ and the logistic function $P_L(t) = \dfrac{1348.9650}{1 + 6.2784e^{-0.0721t}} + 3500$. P_L is a reasonably accurate accurate model, while P_E is not, since an exponential model would only be used for the first few data points.



17. (a) $\dfrac{dP}{dt} = kP - m = k\left(P - \dfrac{m}{k}\right)$. Let $y = P - \dfrac{m}{k}$, so $\dfrac{dy}{dt} = \dfrac{dP}{dt}$ and the differential equation becomes $\dfrac{dy}{dt} = ky$.

The solution is $y = y_0 e^{kt} \quad \Rightarrow \quad P - \dfrac{m}{k} = \left(P_0 - \dfrac{m}{k}\right)e^{kt} \quad \Rightarrow \quad P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$.

(b) Since $k > 0$, there will be an exponential expansion $\Leftrightarrow \quad P_0 - \dfrac{m}{k} > 0 \quad \Leftrightarrow \quad m < kP_0$.

(c) The population will be constant if $P_0 - \dfrac{m}{k} = 0 \quad \Leftrightarrow \quad m = kP_0$. It will decline if $P_0 - \dfrac{m}{k} < 0 \quad \Leftrightarrow \quad m > kP_0$.

(d) $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000 \quad \Rightarrow \quad m > kP_0$ $(= 128{,}000)$, so by part (c), the population was declining.

18. (a) $\dfrac{dy}{dt} = ky^{1+c} \quad \Rightarrow \quad y^{-1-c}\,dy = k\,dt \quad \Rightarrow \quad \dfrac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \dfrac{y_0^{-c}}{-c}$. Thus,

$\dfrac{y^{-c}}{-c} = kt + \dfrac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \dfrac{1}{y_0^{-c} - ckt} = \dfrac{y_0^c}{1 - cy_0^c kt}$ and $y(t) = \dfrac{y_0}{(1 - cy_0^c kt)^{1/c}}$.

(b) $y(t) \to \infty$ as $1 - cy_0^c kt \to 0$, that is, as $t \to \dfrac{1}{cy_0^c k}$. Define $T = \dfrac{1}{cy_0^c k}$. Then $\lim\limits_{t \to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus,

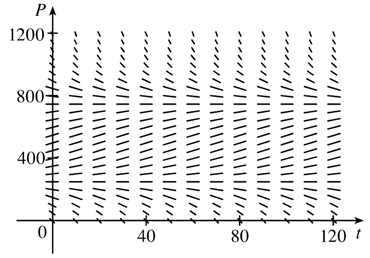
$y_0 = 2$ and $16 = y(3) = \dfrac{y_0}{(1 - cy_0^c k \cdot 3)^{1/c}}$. Since $T = \dfrac{1}{cy_0^c k}$, we will solve for $cy_0^c k$. $\quad 16 = \dfrac{2}{(1 - 3cy_0^c k)^{100}} \quad \Rightarrow$

$1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \quad \Rightarrow \quad cy_0^c k = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when

$t = T = \dfrac{1}{cy_0^c k} = \dfrac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

19. (a) The term -15 represents a harvesting of fish at a constant rate—in this case, 15 fish/week. This is the rate at which fish are caught.

(b)



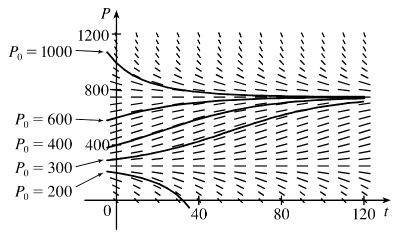
(c) From the graph in part (b), it appears that $P(t) = 250$ and $P(t) = 750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt = 0$ as follows: $0.08P(1 - P/1000) - 15 = 0 \quad \Rightarrow$

$0.08P - 0.00008P^2 - 15 = 0 \quad \Rightarrow$

$-0.00008(P^2 - 1000P + 187{,}500) = 0 \quad \Rightarrow$

$(P - 250)(P - 750) = 0 \quad \Rightarrow \quad P = 250$ or 750.

(d)



For $0 < P_0 < 250$, $P(t)$ decreases to 0. For $P_0 = 250$, $P(t)$ remains constant. For $250 < P_0 < 750$, $P(t)$ increases and approaches 750. For $P_0 = 750$, $P(t)$ remains constant. For $P_0 > 750$, $P(t)$ decreases and approaches 750.

(e) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right) - 15 \quad \Leftrightarrow \quad -\dfrac{100{,}000}{8} \cdot \dfrac{dP}{dt} = (0.08P - 0.00008P^2 - 15) \cdot \left(-\dfrac{100{,}000}{8}\right) \quad \Leftrightarrow$

$-12{,}500\,\dfrac{dP}{dt} = P^2 - 1000P + 187{,}500 \quad \Leftrightarrow \quad \dfrac{dP}{(P-250)(P-750)} = -\dfrac{1}{12{,}500}\,dt \quad \Leftrightarrow$

$\displaystyle\int \left(\frac{-1/500}{P-250} + \frac{1/500}{P-750}\right) dP = -\frac{1}{12{,}500}\,dt \quad \Leftrightarrow \quad \int \left(\frac{1}{P-250} - \frac{1}{P-750}\right) dP = \tfrac{1}{25}\,dt \quad \Leftrightarrow$

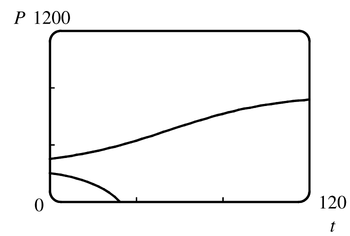
$\ln|P-250| - \ln|P-750| = \frac{1}{25}t + C \quad \Leftrightarrow \quad \ln\left|\dfrac{P-250}{P-750}\right| = \frac{1}{25}t + C \quad \Leftrightarrow \quad \left|\dfrac{P-250}{P-750}\right| = e^{t/25+C} = ke^{t/25} \quad \Leftrightarrow$

$\dfrac{P-250}{P-750} = ke^{t/25} \quad \Leftrightarrow \quad P - 250 = Pke^{t/25} - 750ke^{t/25} \quad \Leftrightarrow \quad P - Pke^{t/25} = 250 - 750ke^{t/25} \quad \Leftrightarrow$

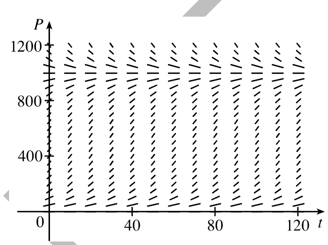
$P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$. If $t = 0$ and $P = 200$, then $200 = \dfrac{250 - 750k}{1-k} \quad \Leftrightarrow \quad 200 - 200k = 250 - 750k \quad \Leftrightarrow$

$550k = 50 \quad \Leftrightarrow \quad k = \frac{1}{11}$. Similarly, if $t = 0$ and $P = 300$, then $k = -\frac{1}{9}$. Simplifying P with these two values of k gives us

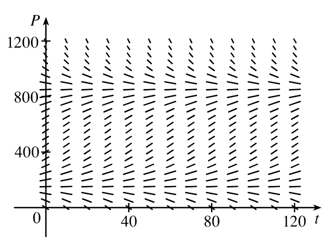
$P(t) = \dfrac{250(3e^{t/25} - 11)}{e^{t/25} - 11}$ and $P(t) = \dfrac{750(e^{t/25} + 3)}{e^{t/25} + 9}$.



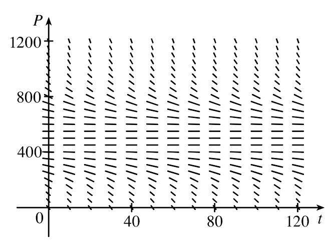
20. (a)



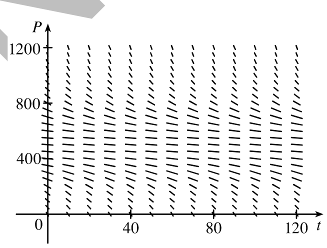
$c = 0$



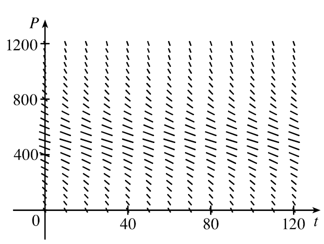
$c = 10$



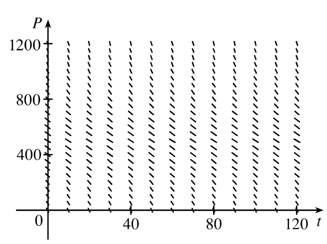
$c = 20$



$c = 21$



$c = 25$



$c = 30$

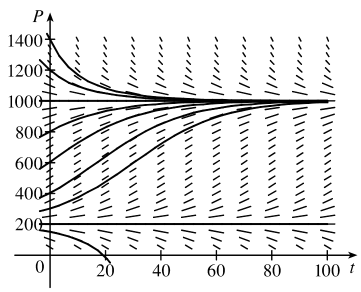
(b) For $0 \le c \le 20$, there is at least one equilibrium solution. For $c > 20$, the population always dies out.

(c) $\dfrac{dP}{dt} = 0.08P - 0.00008P^2 - c$. $\dfrac{dP}{dt} = 0 \quad \Leftrightarrow \quad P = \dfrac{-0.08 \pm \sqrt{(0.08)^2 - 4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow$ $0.0064 - 0.00032c \geq 0$ $\Leftrightarrow$ $c \leq 20$. For $0 \leq c \leq 20$, there is at least one value of P such that $dP/dt = 0$ and hence, at least one equilibrium solution. For $c > 20$, $dP/dt < 0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

21. (a) $\dfrac{dP}{dt} = (kP)\left(1 - \dfrac{P}{M}\right)\left(1 - \dfrac{m}{P}\right)$. If $m < P < M$, then $dP/dt = (+)(+)(+) = +$ $\Rightarrow$ P is increasing. If $0 < P < m$, then $dP/dt = (+)(+)(-) = -$ $\Rightarrow$ P is decreasing.

(b)



$k = 0.08$, $M = 1000$, and $m = 200$ $\Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)\left(1 - \dfrac{m}{P}\right) = kP\left(\dfrac{M-P}{M}\right)\left(\dfrac{P-m}{P}\right) = \dfrac{k}{M}(M-P)(P-m) \quad \Leftrightarrow$

$\displaystyle\int \frac{dP}{(M-P)(P-m)} = \int \frac{k}{M}\,dt$. By partial fractions, $\dfrac{1}{(M-P)(P-m)} = \dfrac{A}{M-P} + \dfrac{B}{P-m}$, so $A(P-m) + B(M-P) = 1$.

If $P = m$, $B = \dfrac{1}{M-m}$; if $P = M$, $A = \dfrac{1}{M-m}$, so $\displaystyle\frac{1}{M-m}\int\left(\frac{1}{M-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{M}\,dt \quad \Rightarrow$

$\dfrac{1}{M-m}\left(-\ln|M-P| + \ln|P-m|\right) = \dfrac{k}{M}t + C \quad \Rightarrow \quad \dfrac{1}{M-m}\ln\left|\dfrac{P-m}{M-P}\right| = \dfrac{k}{M}t + C \quad \Rightarrow$

$\ln\left|\dfrac{P-m}{M-P}\right| = (M-m)\dfrac{k}{M}t + C_1 \quad \Leftrightarrow \quad \dfrac{P-m}{M-P} = De^{(M-m)(k/M)t} \quad [D = \pm e^{C_1}]$.

Let $t = 0$: $\dfrac{P_0 - m}{M - P_0} = D$. So $\dfrac{P-m}{M-P} = \dfrac{P_0 - m}{M - P_0}e^{(M-m)(k/M)t}$.

Solving for P, we get $P(t) = \dfrac{m(M - P_0) + M(P_0 - m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0 - m)e^{(M-m)(k/M)t}}$.

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0) = P_0(M-m) > 0$, and $P_0 - m < 0 \quad \Leftrightarrow \quad \lim\limits_{t\to\infty} M(P_0 - m)e^{(M-m)(k/M)t} = -\infty \quad \Rightarrow \quad \lim\limits_{t\to\infty} N(t) = -\infty$.

Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

22. (a) $\dfrac{dP}{dt} = c\ln\left(\dfrac{M}{P}\right)P \quad \Rightarrow \quad \displaystyle\int \frac{dP}{P\ln(M/P)} = \int c\,dt$. Let $u = \ln\left(\dfrac{M}{P}\right) = \ln M - \ln P \quad \Rightarrow \quad du = -\dfrac{dP}{P} \quad \Rightarrow$

$\displaystyle\int -\frac{du}{u} = ct + D \quad \Rightarrow \quad \ln|u| = -ct - D \quad \Rightarrow \quad |u| = e^{-(ct+D)} \quad \Rightarrow \quad |\ln(M/P)| = e^{-(ct+D)} \quad \Rightarrow$

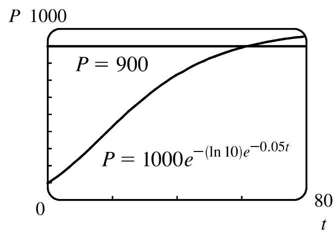
$\ln(M/P) = \pm e^{-(ct+D)}$. Letting $t = 0$, we get $\ln(M/P_0) = \pm e^{-D}$, so

$\ln(M/P) = \pm e^{-ct-D} = \pm e^{-ct}e^{-D} = \ln(M/P_0)e^{-ct} \Rightarrow M/P = e^{\ln(M/P_0)e^{-ct}} \Rightarrow$

$P(t) = Me^{-\ln(M/P_0)e^{-ct}}$, $c \neq 0$.

(b) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} Me^{-\ln(M/P_0)e^{-ct}} = Me^{-\ln(M/P_0)\cdot 0} = Me^0 = M$

(c)



The graphs look very similar. For the Gompertz function, $P(40) \approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P = 900$ at $t \approx 61.7$ and its value at $t = 80$ is about 959, so it doesn't increase quite as fast as the logistic curve.

(d) $\dfrac{dP}{dt} = c\ln\left(\dfrac{M}{P}\right)P = cP(\ln M - \ln P) \Rightarrow$

$$\frac{d^2P}{dt^2} = c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right) + (\ln M - \ln P)\frac{dP}{dt}\right] = c\frac{dP}{dt}\left[-1 + \ln\left(\frac{M}{P}\right)\right]$$
$$= c[c\ln(M/P)P][\ln(M/P) - 1] = c^2P\ln(M/P)\,[\ln(M/P) - 1]$$

Since $0 < P < M$, $P'' = 0 \Leftrightarrow \ln(M/P) = 1 \Leftrightarrow M/P = e \Leftrightarrow P = M/e$. $P'' > 0$ for $0 < P < M/e$ and $P'' < 0$ for $M/e < P < M$, so P' is a maximum (and P grows fastest) when $P = M/e$.

Note: If $P > M$, then $\ln(M/P) < 0$, so $P''(t) > 0$.

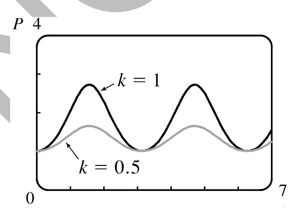
23. (a) $dP/dt = kP\cos(rt - \phi) \Rightarrow (dP)/P = k\cos(rt - \phi)\,dt \Rightarrow \int (dP)/P = k\int \cos(rt - \phi)\,dt \Rightarrow$

$\ln P = (k/r)\sin(rt - \phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \Rightarrow C = \ln P_0 + (k/r)\sin\phi$. Thus, $\ln P = (k/r)\sin(rt - \phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt - \phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

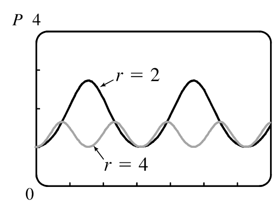
(b) As k increases, the amplitude increases, but the minimum value stays the same.

As r increases, the amplitude and the period decrease.

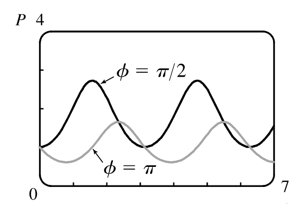
A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0e^{(k/r)(1+\sin\phi)}$ and $P_0e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim\limits_{t\to\infty} P(t)$ does not exist.

24. (a) $dP/dt = kP\cos^2(rt-\phi) \quad\Rightarrow\quad (dP)/P = k\cos^2(rt-\phi)\,dt \quad\Rightarrow\quad \int (dP)/P = k\int \cos^2(rt-\phi)\,dt \quad\Rightarrow$

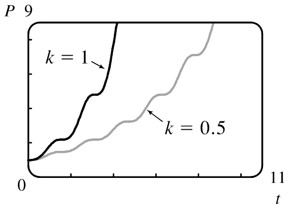
$\ln P = k\displaystyle\int \frac{1+\cos(2(rt-\phi))}{2}\,dt = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt-\phi)) + C$. From $P(0)=P_0$, we get

$\ln P_0 = \dfrac{k}{4r}\sin(-2\phi) + C = C - \dfrac{k}{4r}\sin 2\phi$, so $C = \ln P_0 + \dfrac{k}{4r}\sin 2\phi$ and

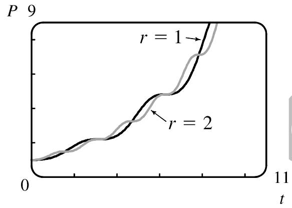
$\ln P = \dfrac{k}{2}t + \dfrac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \dfrac{k}{4r}\sin 2\phi$. Simplifying, we get

$\ln\dfrac{P}{P_0} = \dfrac{k}{2}t + \dfrac{k}{4r}\left[\sin(2(rt-\phi)) + \sin 2\phi\right] = f(t)$, or $P(t) = P_0e^{f(t)}$.

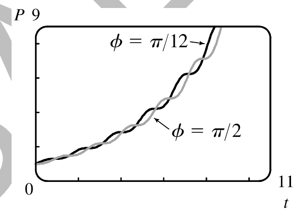
(b) An increase in k stretches the graph of P vertically while maintaining $P(0)=P_0$. An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 19. As in Exercise 23, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \ge 0$. Since $P(t) = P_0e^{f(t)}$, we have $P'(t) = P_0f'(t)e^{f(t)} \ge 0$, with equality only when $\cos(2(rt-\phi)) = -1$; that is, when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0e^{kt/2}e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim\limits_{t\to\infty} P(t) = \infty$.

25. By Equation 7, $P(t) = \dfrac{K}{1+Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t-c)$, then

$$1+\tanh u = 1 + \frac{e^u-e^{-u}}{e^u+e^{-u}} = \frac{e^u+e^{-u}}{e^u+e^{-u}} + \frac{e^u-e^{-u}}{e^u+e^{-u}} = \frac{2e^u}{e^u+e^{-u}}\cdot\frac{e^{-u}}{e^{-u}} = \frac{2}{1+e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc}e^{-kt} = e^{\ln A}e^{-kt} = Ae^{-kt}$, so

$$\tfrac{1}{2}K\left[1+\tanh\left(\tfrac{1}{2}k(t-c)\right)\right] = \frac{K}{2}[1+\tanh u] = \frac{K}{2}\cdot\frac{2}{1+e^{-2u}} = \frac{K}{1+e^{-2u}} = \frac{K}{1+Ae^{-kt}} = P(t).$$

9.5 Linear Equations

1. $y' + x\sqrt{y} = x^2$ is not linear since it cannot be put into the standard form (1), $y' + P(x)\,y = Q(x)$.

2. $y' - x = y\tan x \quad\Leftrightarrow\quad y' + (-\tan x)y = x$ is linear since it can be put into the standard form (1), $y' + P(x)\,y = Q(x)$.

3. $ue^t = t + \sqrt{t}\,\dfrac{du}{dt} \quad\Leftrightarrow\quad \sqrt{t}\,u' - e^tu = -t \quad\Leftrightarrow\quad u' - \dfrac{e^t}{\sqrt{t}}\,u = -\sqrt{t}$ is linear since it can be put into the standard form, $u' + P(t)\,u = Q(t)$.

4. $\dfrac{dR}{dt} + t\cos R = e^{-t} \quad\Leftrightarrow\quad R' + t\cos R = e^{-t}$ is not linear since it cannot be put into the standard form $R' + P(t)\,R = Q(t)$.

5. Comparing the given equation, $y' + y = 1$, with the general form, $y' + P(x)\,y = Q(x)$, we see that $P(x) = 1$ and the integrating factor is $I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation by $I(x)$ gives

$e^x y' + e^x y = e^x \quad\Rightarrow\quad (e^x y)' = e^x \quad\Rightarrow\quad e^x y = \int e^x\,dx \quad\Rightarrow\quad e^x y = e^x + C \quad\Rightarrow\quad \dfrac{e^x y}{e^x} = \dfrac{e^x}{e^x} + \dfrac{C}{e^x} \quad\Rightarrow$

$y = 1 + Ce^{-x}$.

6. $y' - y = e^x \quad\Leftrightarrow\quad y' + (-1)y = e^x \quad\Rightarrow\quad P(x) = -1.\quad I(x) = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$. Multiplying the original differential equation by $I(x)$ gives $e^{-x}y' - e^{-x}y = e^0 \quad\Rightarrow\quad (e^{-x}y)' = 1 \quad\Rightarrow\quad e^{-x}y = \int 1\,dx \quad\Rightarrow$

$e^{-x}y = x + C \quad\Rightarrow\quad y = \dfrac{x + C}{e^{-x}} \quad\Rightarrow\quad y = xe^x + Ce^x$.

7. $y' = x - y \quad\Rightarrow\quad y' + y = x \;(\star).\;\; I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation $(\star)$ by $I(x)$ gives

$e^x y' + e^x y = xe^x \quad\Rightarrow\quad (e^x y)' = xe^x \quad\Rightarrow\quad e^x y = \int xe^x\,dx \quad\Rightarrow\quad e^x y = xe^x - e^x + C \quad$ [by parts] $\quad\Rightarrow$

$y = x - 1 + Ce^{-x} \quad$ [divide by e^x].

8. $4x^3 y + x^4 y' = \sin^3 x \quad\Rightarrow\quad (x^4 y)' = \sin^3 x \quad\Rightarrow\quad x^4 y = \int \sin^3 x\,dx \quad\Rightarrow$

$$x^4 y = \int \sin x\,(1 - \cos^2 x)\,dx = \int (1 - u^2)(-du) \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$$

$$= \int (u^2 - 1)\,du = \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}u(u^2 - 3) + C = \tfrac{1}{3}\cos x\,(\cos^2 x - 3) + C \quad\Rightarrow$$

$$y = \frac{1}{3x^4}\cos x\,(\cos^2 x - 3) + \frac{C}{x^4}$$

9. Since $P(x)$ is the derivative of the coefficient of y' $\;[P(x) = 1$ and the coefficient is $x]$, we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \quad\Rightarrow\quad xy = \frac{2}{3}x^{3/2} + C \quad\Rightarrow\quad y = \frac{2}{3}\sqrt{x} + C/x$.

10. $2xy' + y = 2\sqrt{x} \quad\Rightarrow\quad y' + \dfrac{1}{2x}\,y = \dfrac{1}{\sqrt{x}} \quad [x > 0] \quad\Rightarrow\quad P(x) = \dfrac{1}{2x}$.

$I(x) = e^{\int P(x)\,dx} = e^{\int 1/(2x)\,dx} = e^{(1/2)\ln|x|} = (e^{\ln x})^{1/2} = \sqrt{x}$. Multiplying the differential equation by $I(x)$ gives

$\sqrt{x}\,y' + \dfrac{1}{2\sqrt{x}}\,y = 1 \quad\Rightarrow\quad (\sqrt{x}\,y)' = 1 \quad\Rightarrow\quad \sqrt{x}\,y = \int 1\,dx \quad\Rightarrow\quad \sqrt{x}\,y = x + C \quad\Rightarrow\quad y = \dfrac{x + C}{\sqrt{x}}$.

11. $xy' - 2y = x^2 \quad\Rightarrow\quad y' - \dfrac{2}{x}\,y = x \quad\Rightarrow\quad P(x) = -\dfrac{2}{x}$.

$I(x) = e^{\int P(x)\,dx} = e^{\int -2/x\,dx} = e^{-2\ln x} \quad [x > 0] \quad = x^{-2} = \dfrac{1}{x^2}$. Multiplying the differential equation by $I(x)$ gives

$\dfrac{1}{x^2}\,y' - \dfrac{2}{x^3}\,y = \dfrac{1}{x} \quad\Rightarrow\quad \left(\dfrac{1}{x^2}\,y\right)' = \dfrac{1}{x} \quad\Rightarrow\quad \dfrac{1}{x^2}\,y = \displaystyle\int \dfrac{1}{x}\,dx \quad\Rightarrow\quad \dfrac{1}{x^2}\,y = \ln x + C \quad\Rightarrow\quad y = x^2(\ln x + C)$.

12. $y' + 2xy = 1 \quad\Rightarrow\quad P(x) = 2x.\quad I(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$. Multiplying the differential equation by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = e^{x^2} \quad\Rightarrow\quad \left(e^{x^2}y\right)' = e^{x^2} \quad\Rightarrow\quad e^{x^2}y = \int_0^x e^{t^2}\,dt + C \quad$ [see page 507] $\quad\Rightarrow$

$y = e^{-x^2}\int_0^x e^{t^2}\,dt + Ce^{-x^2}$.

13. $t^2\,\dfrac{dy}{dt} + 3ty = \sqrt{1+t^2} \quad\Rightarrow\quad y' + \dfrac{3}{t}\,y = \dfrac{\sqrt{1+t^2}}{t^2} \quad\Rightarrow\quad P(t) = \dfrac{3}{t}.$

$I(t) = e^{\int P(t)\,dt} = e^{\int 3/t\,dt} = e^{3\ln t} \quad [t>0] \quad = t^3$. Multiplying by t^3 gives $t^3y' + 3t^2y = t\sqrt{1+t^2} \quad\Rightarrow$

$(t^3y)' = t\sqrt{1+t^2} \quad\Rightarrow\quad t^3y = \int t\sqrt{1+t^2}\,dt \quad\Rightarrow\quad t^3y = \frac{1}{3}(1+t^2)^{3/2} + C \quad\Rightarrow\quad y = \frac{1}{3}t^{-3}(1+t^2)^{3/2} + Ct^{-3}.$

14. $t\ln t\,\dfrac{dr}{dt} + r = te^t \quad\Rightarrow\quad \dfrac{dr}{dt} + \dfrac{1}{t\ln t}\,r = \dfrac{e^t}{\ln t}.\quad I(t) = e^{\int dt/(t\ln t)} = e^{\ln(\ln t)} = \ln t$. Multiplying by $\ln t$ gives

$\ln t\,\dfrac{dr}{dt} + \dfrac{1}{t}\,r = e^t \quad\Rightarrow\quad [(\ln t)r]' = e^t \quad\Rightarrow\quad (\ln t)r = e^t + C \quad\Rightarrow\quad r = \dfrac{e^t + C}{\ln t}.$

15. $x^2y' + 2xy = \ln x \quad\Rightarrow\quad (x^2y)' = \ln x \quad\Rightarrow\quad x^2y = \int \ln x\,dx \quad\Rightarrow\quad x^2y = x\ln x - x + C$ [by parts]. Since $y(1) = 2$,

$1^2(2) = 1\ln 1 - 1 + C \quad\Rightarrow\quad 2 = -1 + C \quad\Rightarrow\quad C = 3$, so $x^2y = x\ln x - x + 3$, or $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$.

16. $t^3\,\dfrac{dy}{dt} + 3t^2y = \cos t \quad\Rightarrow\quad (t^3y)' = \cos t \quad\Rightarrow\quad t^3y = \int \cos t\,dt \quad\Rightarrow\quad t^3y = \sin t + C$. Since $y(\pi) = 0$,

$\pi^3(0) = \sin\pi + C \quad\Rightarrow\quad C = 0$, so $t^3y = \sin t$, or $y = \dfrac{\sin t}{t^3}$.

17. $t\,\dfrac{du}{dt} = t^2 + 3u \quad\Rightarrow\quad u' - \dfrac{3}{t}u = t \ (\star).\quad I(t) = e^{\int -3/t\,dt} = e^{-3\ln|t|} = (e^{\ln|t|})^{-3} = t^{-3} \quad [t>0] \ = \dfrac{1}{t^3}$. Multiplying $(\star)$

by $I(t)$ gives $\dfrac{1}{t^3}\,u' - \dfrac{3}{t^4}u = \dfrac{1}{t^2} \quad\Rightarrow\quad \left(\dfrac{1}{t^3}u\right)' = \dfrac{1}{t^2} \quad\Rightarrow\quad \dfrac{1}{t^3}\,u = \displaystyle\int \dfrac{1}{t^2}\,dt \quad\Rightarrow\quad \dfrac{1}{t^3}\,u = -\dfrac{1}{t} + C$. Since $u(2) = 4$,

$\dfrac{1}{2^3}(4) = -\dfrac{1}{2} + C \quad\Rightarrow\quad C = 1$, so $\dfrac{1}{t^3}\,u = -\dfrac{1}{t} + 1$, or $u = -t^2 + t^3$.

18. $xy' + y = x\ln x \quad\Rightarrow\quad (x\,y)' = x\ln x \quad\Rightarrow\quad xy = \int x\ln x\,dx \quad\Rightarrow\quad xy = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \quad \left[\begin{array}{c}\text{by parts}\\ \text{with } u = \ln x\end{array}\right] \quad\Rightarrow$

$y = \dfrac{1}{2}x\ln x - \dfrac{1}{4}x + \dfrac{C}{x}.\quad y(1) = 0 \quad\Rightarrow\quad 0 = 0 - \dfrac{1}{4} + C \quad\Rightarrow\quad C = \dfrac{1}{4}$, so $y = \dfrac{1}{2}x\ln x - \dfrac{1}{4}x + \dfrac{1}{4x}$.

19. $xy' = y + x^2\sin x \quad\Rightarrow\quad y' - \dfrac{1}{x}\,y = x\sin x.\quad I(x) = e^{\int(-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}.$

Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x}\,y' - \dfrac{1}{x^2}\,y = \sin x \quad\Rightarrow\quad \left(\dfrac{1}{x}\,y\right)' = \sin x \quad\Rightarrow\quad \dfrac{1}{x}\,y = -\cos x + C \quad\Rightarrow\quad y = -x\cos x + Cx.$

$y(\pi) = 0 \quad\Rightarrow\quad -\pi\cdot(-1) + C\pi = 0 \quad\Rightarrow\quad C = -1$, so $y = -x\cos x - x$.

20. $(x^2+1)\,\dfrac{dy}{dx} + 3x(y-1) = 0 \quad\Rightarrow\quad (x^2+1)\,y' + 3xy = 3x \quad\Rightarrow\quad y' + \dfrac{3x}{x^2+1}\,y = \dfrac{3x}{x^2+1}.$

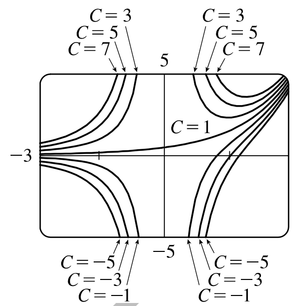
$I(x) = e^{\int 3x/(x^2+1)\,dx} = e^{(3/2)\ln|x^2+1|} = \left(e^{\ln(x^2+1)}\right)^{3/2} = (x^2+1)^{3/2}$. Multiplying by $(x^2+1)^{3/2}$ gives

$(x^2+1)^{3/2}\,y' + 3x(x^2+1)^{1/2}\,y = 3x(x^2+1)^{1/2} \quad\Rightarrow\quad \left[(x^2+1)^{3/2}y\right]' = 3x(x^2+1)^{1/2} \quad\Rightarrow$

$(x^2+1)^{3/2}\,y = \int 3x(x^2+1)^{1/2}\,dx = (x^2+1)^{3/2} + C \quad\Rightarrow\quad y = 1 + C(x^2+1)^{-3/2}$. Since $y(0) = 2$, we have

$2 = 1 + C(1) \quad\Rightarrow\quad C = 1$ and hence, $y = 1 + (x^2+1)^{-3/2}$.

21. $xy' + 2y = e^x \quad \Rightarrow \quad y' + \dfrac{2}{x}\,y = \dfrac{e^x}{x}$.



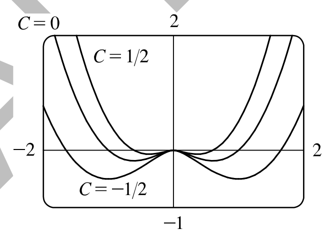
$I(x) = e^{\int (2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$.

Multiplying by $I(x)$ gives $x^2\,y' + 2xy = xe^x \quad \Rightarrow \quad (x^2 y)' = xe^x \quad \Rightarrow$

$x^2 y = \int xe^x\,dx = (x-1)e^x + C \quad$ [by parts] $\quad \Rightarrow$

$y = [(x-1)e^x + C]/x^2$. The graphs for $C = -5, -3, -1, 1, 3, 5$, and 7 are shown. $C = 1$ is a transitional value. For $C < 1$, there is an inflection point and for $C > 1$, there is a local minimum. As $|C|$ gets larger, the "branches" get further from the origin.

22. $xy' = x^2 + 2y \quad \Leftrightarrow \quad xy' - 2y = x^2 \quad \Leftrightarrow \quad y' - \dfrac{2}{x}\,y = x$.



$I(x) = e^{\int -2/x\,dx} = e^{-2\ln|x|} = (e^{\ln|x|})^{-2} = |x|^{-2} = \dfrac{1}{x^2}$. Multiplying by

$I(x)$ gives $\dfrac{1}{x^2}\,y' - \dfrac{2}{x^3}\,y = \dfrac{1}{x} \quad \Rightarrow \quad \left(\dfrac{1}{x^2}\,y\right)' = \dfrac{1}{x} \;\Rightarrow\; \dfrac{1}{x^2}\,y = \displaystyle\int \frac{1}{x}\,dx \quad \Rightarrow$

$\dfrac{1}{x^2}\,y = \ln|x| + C \quad \Rightarrow \quad y = (\ln|x| + C)x^2$. For all values of C, as $|x| \to 0$, $y \to 0$, and as $|x| \to \infty$, $y \to \infty$. As $|x|$ increases from 0, the function decreases and attains an absolute minimum. The inflection points, absolute minimums, and x-intercepts all move farther from the origin as C decreases.

23. Setting $u = y^{1-n}$, $\dfrac{du}{dx} = (1-n)\,y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{y^n}{1-n}\dfrac{du}{dx} = \dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

24. Here $xy' + y = -xy^2 \quad \Rightarrow \quad y' + \dfrac{y}{x} = -y^2$, so $n = 2$, $P(x) = \dfrac{1}{x}$ and $Q(x) = -1$. Setting $u = y^{-1}$, u satisfies $u' - \dfrac{1}{x}u = 1$. Then $I(x) = e^{\int(-1/x)dx} = \dfrac{1}{x}$ (for $x > 0$) and $u = x\left(\displaystyle\int \frac{1}{x}\,dx + C\right) = x(\ln|x| + C)$. Thus,

$y = \dfrac{1}{x(C + \ln|x|)}$.

25. Here $y' + \dfrac{2}{x}\,y = \dfrac{y^3}{x^2}$, so $n = 3$, $P(x) = \dfrac{2}{x}$ and $Q(x) = \dfrac{1}{x^2}$. Setting $u = y^{-2}$, u satisfies $u' - \dfrac{4u}{x} = -\dfrac{2}{x^2}$.

Then $I(x) = e^{\int(-4/x)\,dx} = x^{-4}$ and $u = x^4\left(\displaystyle\int -\frac{2}{x^6}\,dx + C\right) = x^4\left(\dfrac{2}{5x^5} + C\right) = Cx^4 + \dfrac{2}{5x}$.

Thus, $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$.

26. $xy'' + 2y' = 12x^2$ and $u = y' \quad \Rightarrow \quad xu' + 2u = 12x^2 \quad \Rightarrow \quad u' + \dfrac{2}{x}\,u = 12x$.

$I(x) = e^{\int(2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$. Multiplying the last differential equation by x^2 gives

$x^2u' + 2xu = 12x^3 \quad \Rightarrow \quad (x^2u)' = 12x^3 \quad \Rightarrow \quad x^2u = \int 12x^3\,dx = 3x^4 + C \quad \Rightarrow \quad u = 3x^2 + C/x^2 \quad \Rightarrow$

$y' = 3x^2 + C/x^2 \quad \Rightarrow \quad y = x^3 - C/x + D.$

27. (a) $2\,\dfrac{dI}{dt} + 10I = 40$ or $\dfrac{dI}{dt} + 5I = 20$. Then the integrating factor is $e^{\int 5\,dt} = e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\,\dfrac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \quad \Rightarrow \quad (e^{5t}I)' = 20e^{5t} \quad \Rightarrow$

$I(t) = e^{-5t}\left[\int 20e^{5t}\,dt + C\right] = 4 + Ce^{-5t}$. But $0 = I(0) = 4 + C$, so $I(t) = 4 - 4e^{-5t}$.

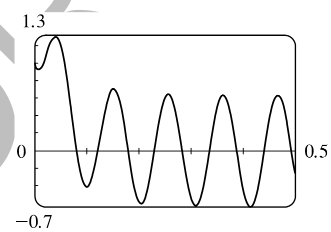
(b) $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$ A

28. (a) $\dfrac{dI}{dt} + 20I = 40\sin 60t$, so the integrating factor is e^{20t}. Multiplying the differential equation by the integrating factor gives $e^{20t}\,\dfrac{dI}{dt} + 20Ie^{20t} = 40e^{20t}\sin 60t \quad \Rightarrow \quad (e^{20t}I)' = 40e^{20t}\sin 60t \quad \Rightarrow$

$$\begin{aligned} I(t) &= e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt + C\right] = e^{-20t}\left[40e^{20t}\left(\tfrac{1}{4000}\right)(20\sin 60t - 60\cos 60t)\right] + Ce^{-20t} \\ &= \frac{\sin 60t - 3\cos 60t}{5} + Ce^{-20t} \end{aligned}$$

But $1 = I(0) = -\frac{3}{5} + C$, so $I(t) = \dfrac{\sin 60t - 3\cos 60t\ + 8e^{-20t}}{5}$.

(b) $I(0.1) = \dfrac{\sin 6 - 3\cos 6 + 8e^{-2}}{5} \approx -0.42$ A

(c) 

29. $5\,\dfrac{dQ}{dt} + 20Q = 60$ with $Q(0) = 0$ C. Then the integrating factor is $e^{\int 4\,dt} = e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\,\dfrac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \quad \Rightarrow \quad (e^{4t}Q)' = 12e^{4t} \quad \Rightarrow$

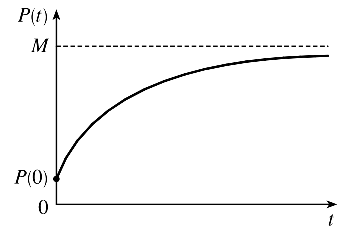
$Q(t) = e^{-4t}\left[\int 12e^{4t}\,dt + C\right] = 3 + Ce^{-4t}$. But $0 = Q(0) = 3 + C$ so $Q(t) = 3(1 - e^{-4t})$ is the charge at time t and $I = dQ/dt = 12e^{-4t}$ is the current at time t.

30. $2\,\dfrac{dQ}{dt} + 100Q = 10\sin 60t$ or $\dfrac{dQ}{dt} + 50Q = 5\sin 60t$. Then the integrating factor is $e^{\int 50\,dt} = e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\,\dfrac{dQ}{dt} + 50e^{50t}Q = 5e^{50t}\sin 60t \quad \Rightarrow \quad (e^{50t}Q)' = 5e^{50t}\sin 60t \quad \Rightarrow$

$$\begin{aligned} Q(t) &= e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt + C\right] = e^{-50t}\left[5e^{50t}\left(\tfrac{1}{6100}\right)(50\sin 60t - 60\cos 60t)\right] + Ce^{-50t} \\ &= \tfrac{1}{122}(5\sin 60t - 6\cos 60t) + Ce^{-50t} \end{aligned}$$

But $0 = Q(0) = -\frac{6}{122} + C$ so $C = \frac{3}{61}$ and $Q(t) = \dfrac{5\sin 60t - 6\cos 60t}{122} + \dfrac{3e^{-50t}}{61}$ is the charge at time t, while the current is $I(t) = \dfrac{dQ}{dt} = \dfrac{150\cos 60t + 180\sin 60t - 150e^{-50t}}{61}$.

31. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \;\Rightarrow\; (e^{kt}P)' = kMe^{kt} \;\Rightarrow$ $P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.



32. Since $P(0) = 0$, we have $P(t) = M(1 - e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1) = 25$ and $P_1(2) = 45$. Hence, $25 = M_1(1 - e^{-k})$ and $45 = M_1(1 - e^{-2k})$, so $1 - 25/M_1 = e^{-k}$ or $k = -\ln\left(1 - \dfrac{25}{M_1}\right) = \ln\left(\dfrac{M_1}{M_1 - 25}\right)$. But $45 = M_1(1 - e^{-2k})$ so $45 = M_1\left[1 - \left(\dfrac{M_1 - 25}{M_1}\right)^2\right]$ or $45 = \dfrac{50M_1 - 625}{M_1}$. Thus, $M_1 = 125$ is the maximum number of units per hour Jim is capable of processing. Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1) = 35$ and $P_2(2) = 50$. So $k = \ln\left(\dfrac{M_2}{M_2 - 35}\right)$ and $50 = M_2\left[1 - \left(\dfrac{M_2 - 35}{M_2}\right)^2\right]$ or $M_2 = 61.25$. Hence the maximum number of units per hour for Mark is approximately 61. Another approach would be to use the midpoints of the intervals so that $P_1(0.5) = 25$ and $P_1(1.5) = 45$. Doing so gives us $M_1 \approx 52.6$ and $M_2 \approx 51.8$.

33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\,\dfrac{\text{kg}}{\text{L}}\right)\left(5\,\dfrac{\text{L}}{\text{min}}\right) = 2\,\dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}\right)\left(3\,\dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100 + 2t}\,\dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\dfrac{dy}{dt} = 2 - \dfrac{3y}{100 + 2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100 + 2t}\right)y = 2$, we see that it is linear.

$$I(t) = \exp\left(\int \frac{3\,dt}{100 + 2t}\right) = \exp\left(\tfrac{3}{2}\ln(100 + 2t)\right) = (100 + 2t)^{3/2}$$

Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2}\dfrac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \;\Rightarrow$ $[(100 + 2t)^{3/2}y]' = 2(100 + 2t)^{3/2} \;\Rightarrow\; (100 + 2t)^{3/2}y = \frac{2}{5}(100 + 2t)^{5/2} + C \;\Rightarrow$ $y = \frac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$. Now $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \;\Rightarrow\; C = -40{,}000$, so $y = \left[\frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \dfrac{y(t)}{100 + 2t} = \left[\dfrac{-40{,}000}{(100 + 2t)^{5/2}} + \dfrac{2}{5}\right]\dfrac{\text{kg}}{\text{L}}$. In particular, $C(20) = \dfrac{-40{,}000}{140^{5/2}} + \dfrac{2}{5} \approx 0.2275\,\dfrac{\text{kg}}{\text{L}}$ and $y(20) = \frac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05 \text{ g/L})\,(400 \text{ L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\dfrac{y(t)}{400-6t}\,\dfrac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\dfrac{y(t)}{400-6t}\,\dfrac{\text{g}}{\text{L}}\right]\left[10\,\dfrac{\text{L}}{\text{s}}\right] = \dfrac{10\,y(t)}{400-6t}\,\dfrac{\text{g}}{\text{s}} = \dfrac{5\,y(t)}{200-3t}\,\dfrac{\text{g}}{\text{s}}$. Therefore, $\dfrac{dy}{dt} = -\dfrac{5y}{200-3t} \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{-5\,dt}{200-3t} \;\Rightarrow$ $\ln y = \frac{5}{3}\ln(200-3t) + C \;\Rightarrow\; y = \exp\!\left(\frac{5}{3}\ln(200-3t)+C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \;\Rightarrow$ $e^C = \dfrac{20}{200^{5/3}}$, so $y(t) = 20\dfrac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

35. (a) $\dfrac{dv}{dt} + \dfrac{c}{m}v = g$ and $I(t) = e^{\int (c/m)\,dt} = e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives $e^{(c/m)t}\dfrac{dv}{dt} + \dfrac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \;\Rightarrow\; \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence, $v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$ and $K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)[1 - e^{-(c/m)t}]$.

(b) $\lim\limits_{t\to\infty} v(t) = mg/c$

(c) $s(t) = \int v(t)\,dt = (mg/c)[t + (m/c)e^{-(c/m)t}] + c_1$ where $c_1 = s(0) - m^2g/c^2$. $s(0)$ is the initial position, so $s(0) = 0$ and $s(t) = (mg/c)[t + (m/c)e^{-(c/m)t}] - m^2g/c^2$.

36. $v = (mg/c)(1 - e^{-ct/m}) \;\Rightarrow$

$$\frac{dv}{dm} = \frac{mg}{c}\left(0 - e^{-ct/m}\cdot\frac{ct}{m^2}\right) + \frac{g}{c}\left(1 - e^{-ct/m}\right)\cdot 1 = -\frac{gt}{m}\,e^{-ct/m} + \frac{g}{c} - \frac{g}{c}\,e^{-ct/m}$$
$$= \frac{g}{c}\left(1 - e^{-ct/m} - \frac{ct}{m}\,e^{-ct/m}\right) \;\Rightarrow$$

$\dfrac{c}{g}\dfrac{dv}{dm} = 1 - \left(1 + \dfrac{ct}{m}\right)e^{-ct/m} = 1 - \dfrac{1+ct/m}{e^{ct/m}} = 1 - \dfrac{1+Q}{e^Q}$, where $Q = \dfrac{ct}{m} \ge 0$. Since $e^Q > 1 + Q$ for all $Q > 0$, it follows that $dv/dm > 0$ for $t > 0$. In other words, for all $t > 0$, v increases as m increases.

37. (a) $z = \dfrac{1}{P} \;\Rightarrow\; P = \dfrac{1}{z} \;\Rightarrow\; P' = -\dfrac{z'}{z^2}$. Substituting into $P' = kP(1 - P/M)$ gives us $-\dfrac{z'}{z^2} = k\,\dfrac{1}{z}\left(1 - \dfrac{1}{zM}\right) \;\Rightarrow$ $z' = -kz\left(1 - \dfrac{1}{zM}\right) \;\Rightarrow\; z' = -kz + \dfrac{k}{M} \;\Rightarrow\; z' + kz = \dfrac{k}{M}\;(\star)$.

(b) The integrating factor is $e^{\int k\,dt} = e^{kt}$. Multiplying $(\star)$ by e^{kt} gives $e^{kt}z' + ke^{kt}z = \dfrac{ke^{kt}}{M} \;\Rightarrow\; (e^{kt}z)' = \dfrac{k}{M}\,e^{kt} \;\Rightarrow$ $e^{kt}z = \displaystyle\int \frac{k}{M}\,e^{kt}\,dt \;\Rightarrow\; e^{kt}z = \frac{1}{M}\,e^{kt} + C \;\Rightarrow\; z = \frac{1}{M} + Ce^{-kt}$. Since $P = \dfrac{1}{z}$, we have $P = \dfrac{1}{\dfrac{1}{M} + Ce^{-kt}} \;\Rightarrow\; P = \dfrac{M}{1 + MCe^{-kt}}$, which agrees with Equation 9.4.7, $P = \dfrac{M}{1 + Ae^{-kt}}$, when $MC = A$.

38. (a) $z = \dfrac{1}{P} \;\Rightarrow\; P = \dfrac{1}{z} \;\Rightarrow\; P' = \dfrac{-z'}{z^2}$. Substituting into $\dfrac{dP}{dt} = k(t)P\left(1 - \dfrac{P}{M(t)}\right)$ gives us

$$-\frac{z'}{z^2} = \frac{k(t)}{z}\left(1 - \frac{1}{M(t)z}\right) \;\Rightarrow\; z' = -k(t)z\left(1 - \frac{1}{M(t)z}\right) \;\Rightarrow\; z' = -k(t)z + \frac{k(t)}{M(t)} \;\Rightarrow$$

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)} \quad (\star).$$

(b) The integrating factor is $e^{K(t)}$, where $K(t) = \int_0^t k(s)\,ds$, so that $K'(t) = k(t)$. Multiplying $(\star)$ by

$e^{K(t)}$ gives $e^{K(t)}\dfrac{dz}{dt} + e^{K(t)}k(t)z = \dfrac{e^{K(t)}k(t)}{M(t)} \;\Rightarrow\; (e^{K(t)}z)' = \dfrac{K'(t)e^{K(t)}}{M(t)} \;\Rightarrow$

$e^{K(t)}z = \displaystyle\int_0^t \frac{K'(s)e^{K(s)}}{M(s)}\,ds + C$, so $P = \dfrac{1}{z} = \dfrac{e^{K(t)}}{\displaystyle\int_0^t \frac{K'(s)e^{K(s)}}{M(s)}\,ds + C}$. Now suppose that M is a constant. Then

$P(t) = \dfrac{Me^{K(t)}}{\int_0^t K'(s)e^{K(s)}\,ds + CM} = \dfrac{Me^{K(t)}}{e^{K(t)} + CM} = \dfrac{M}{1 + CMe^{-K(t)}}$. If $\int_0^\infty k(t)\,dt = \infty$, then $\lim\limits_{t\to\infty} K(t) = \infty$, so

$$\lim_{t\to\infty} P(t) = \frac{M}{1 + CM\lim\limits_{t\to\infty} e^{-K(t)}} = \frac{M}{1 + CM\cdot 0} = M.$$

(c) If k is constant, but M varies, then $K(t) = kt$ and we get $e^{kt}z = \displaystyle\int_0^t \frac{ke^{ks}}{M(s)}\,ds + C \;\Rightarrow$

$z(t) = \dfrac{\displaystyle\int_0^t \frac{ke^{ks}}{M(s)}\,ds + C}{e^{kt}} \;\Rightarrow\; z(t) = e^{-kt}\displaystyle\int_0^t \frac{ke^{ks}}{M(s)}\,ds + Ce^{-kt}$. Suppose $M(t)$ has a limit as $t \to \infty$,

say $\lim\limits_{t\to\infty} M(t) = L$. Then

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\frac{1}{z(t)} = \lim_{t\to\infty}\frac{e^{kt}}{\displaystyle\int_0^t \frac{ke^{ks}}{M(s)}\,ds + C} = \lim_{t\to\infty}\frac{ke^{kt}}{\dfrac{ke^{kt}}{M(t)} + 0} \quad \left[\begin{array}{c}\text{l'Hospital's}\\ \text{and FTC 1}\end{array}\right] = \lim_{t\to\infty} M(t) = L.$$

9.6 Predator-Prey Systems

1. (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y, x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x, y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

3. (a) $dx/dt = 0.5x - 0.004x^2 - 0.001xy = 0.5x(1 - x/125) - 0.001xy$.
$dy/dt = 0.4y - 0.001y^2 - 0.002xy = 0.4y(1 - y/400) - 0.002xy$.

The system shows that x and y have carrying capacities of 125 and 400. An increase in x reduces the growth rate of y due to the negative term $-0.002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.001xy$. Hence the system describes a competition model.

(b) $dx/dt = 0 \quad\Rightarrow\quad x(0.5 - 0.004x - 0.001y) = 0 \quad\Rightarrow\quad x(500 - 4x - y) = 0$ **(1)** and $dy/dt = 0 \quad\Rightarrow$
$y(0.4 - 0.001y - 0.002x) = 0 \quad\Rightarrow\quad y(400 - y - 2x) = 0$ **(2).**

From **(1)** and **(2)**, we get four equilibrium solutions.

(i) $x = 0$ and $y = 0$: If the populations are zero, there is no change.

(ii) $x = 0$ and $400 - y - 2x = 0 \quad\Rightarrow\quad x = 0$ and $y = 400$: In the absence of an x-population, the y-population stabilizes at 400.

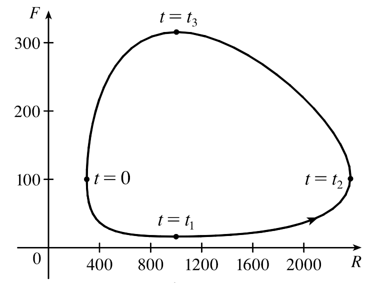
(iii) $500 - 4x - y = 0$ and $y = 0 \quad\Rightarrow\quad x = 125$ and $y = 0$: In the absence of y-population, the x-population stabilizes at 125.

(iv) $500 - 4x - y = 0$ and $400 - y - 2x = 0 \quad\Rightarrow\quad y = 500 - 4x$ and $y = 400 - 2x \quad\Rightarrow\quad 500 - 4x = 400 - 2x \quad\Rightarrow$ $100 = 2x \quad\Rightarrow\quad x = 50$ and $y = 300$: A y-population of 300 is just enough to support a constant x-population of 50.

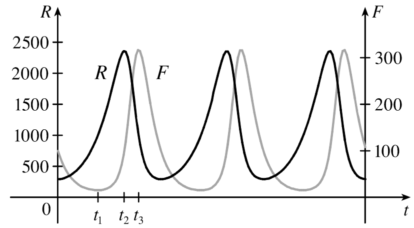
4. Let $L(t)$, $H(t)$, and $W(t)$ represent the populations of lynx, hares, and willows at time t. Let the k_i's and the c_i's denote positive constants, so that a plus sign means an increase and a minus sign means a decrease in the corresponding growth rate. "In the absence of hares, the willow population will grow exponentially and the lynx population will decay exponentially" gives us $dW/dt = +k_1W$ and $dL/dt = -k_2L$. "In the absence of lynx and willow, the hare population will decay exponentially" gives us $dH/dt = -k_3H$. "Lynx eat snowshoe hares and snowshoe hares eat woody plants like willows" gives us encounters that lynx win, hares lose and win, and willows lose. In terms of growth rates, this means that $dL/dt = +c_1LH$, $dH/dt = -c_2LH + c_3HW$, and $dW/dt = -c_4HW$. Putting this information together gives us the following system of differential equations.

$$\begin{aligned} dL/dt &= -k_2L \quad + c_1LH \\ dH/dt &= -k_3H \quad - c_2LH + c_3HW \\ dW/dt &= +k_1W \qquad\qquad - c_4HW \end{aligned}$$

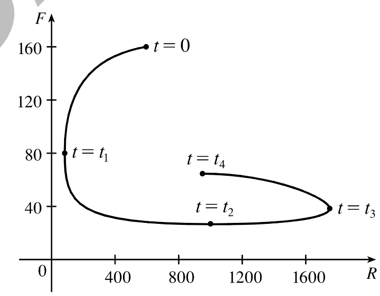
5. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



(b)



6. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.



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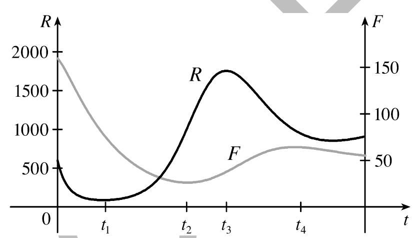

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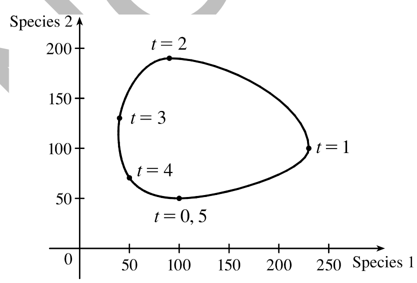

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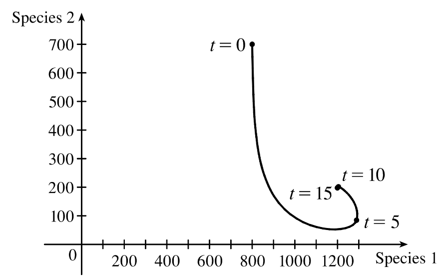


9. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \quad \Leftrightarrow \quad (0.08 - 0.001W)R\,dW = (-0.02 + 0.00002R)W\,dR \quad \Leftrightarrow$

$\dfrac{0.08 - 0.001W}{W}\,dW = \dfrac{-0.02 + 0.00002R}{R}\,dR \quad \Leftrightarrow \quad \displaystyle\int\left(\frac{0.08}{W} - 0.001\right)dW = \int\left(-\frac{0.02}{R} + 0.00002\right)dR \quad \Leftrightarrow$

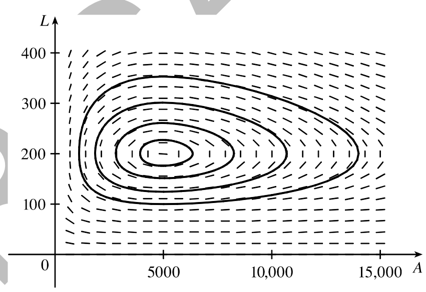
$0.08\ \ln|W| - 0.001W = -0.02\ \ln|R| + 0.00002R + K \quad\Leftrightarrow\quad 0.08\ \ln W + 0.02\ \ln R = 0.001W + 0.00002R + K \quad\Leftrightarrow$
$\ln\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \quad\Leftrightarrow\quad W^{0.08}R^{0.02} = e^{0.00002R+0.001W+K} \quad\Leftrightarrow$
$R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \quad\Leftrightarrow\quad \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$. In general, if $\dfrac{dy}{dx} = \dfrac{-ry+bxy}{kx-axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

10. (a) A and L are constant $\Rightarrow$ $A' = 0$ and $L' = 0$ $\Rightarrow$ $\left\{\begin{array}{l} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{array}\right.$

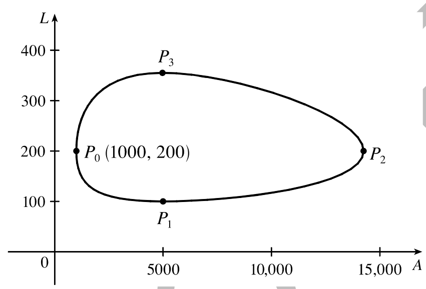
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point $(5000, 200)$ inside them.



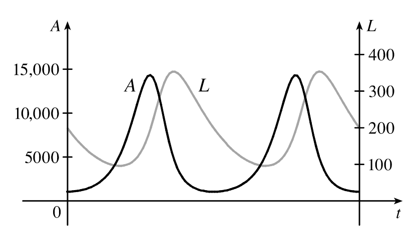
(d)



At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14, 250, 200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



11. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \quad \Leftrightarrow \quad R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\Rightarrow \quad R' = 0$ and $W' = 0 \quad \Rightarrow$

$$\left.\begin{cases} 0 = 0.08R(1 - 0.0002R) - 0.001RW \\ 0 = -0.02W + 0.00002RW \end{cases}\right\} \Rightarrow \begin{cases} 0 = R[0.08(1 - 0.0002R) - 0.001W] \\ 0 = W(-0.02 + 0.00002R) \end{cases}$$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \quad \Leftrightarrow$ $0 = 80(1 - 0.2) - W \quad \Leftrightarrow \quad W = 64$.

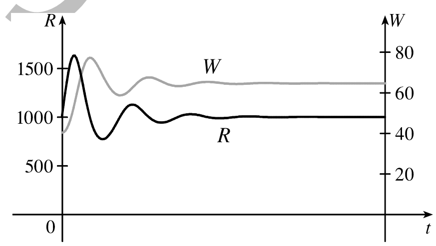
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



12. (a) If $L = 0$, $dA/dt = 2A(1 - 0.0001A)$, so $dA/dt = 0 \quad \Leftrightarrow \quad A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. Since $dA/dt > 0$ for $0 < A < 10{,}000$, we expect the aphid population to *increase* to 10,000 for these values of A. Since $dA/dt < 0$ for $A > 10{,}000$, we expect the aphid population to *decrease* to 10,000 for these values of A. Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10,000.

(b) A and L are constant $\Rightarrow \quad A' = 0$ and $L' = 0 \quad \Rightarrow$

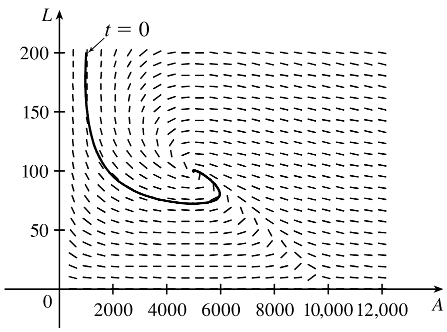
$$\left.\begin{cases} 0 = 2A(1 - 0.0001A) - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{cases}\right\} \Rightarrow \begin{cases} 0 = A[2(1 - 0.0001A) - 0.01L] \\ 0 = L(-0.5 + 0.0001A) \end{cases}$$

The second equation is true if $L = 0$ or $A = \frac{0.5}{0.0001} = 5000$. If $L = 0$ in the first equation, then either $A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. If $A = 5000$, then $0 = 5000[2(1 - 0.0001 \cdot 5000) - 0.01L] \quad \Leftrightarrow$ $0 = 10{,}000(1 - 0.5) - 50L \quad \Leftrightarrow \quad 50L = 5000 \quad \Leftrightarrow \quad L = 100$.

The equilibrium solutions are: (i) $L = 0, A = 0$ (ii) $L = 0, A = 10{,}000$ (iii) $A = 5000, L = 100$

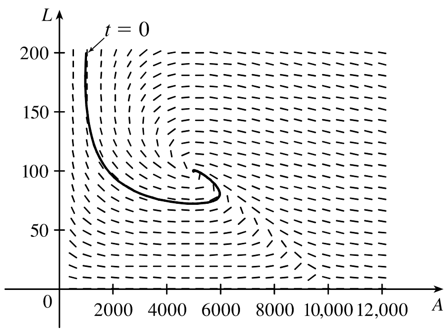
(c) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A(1 - 0.0001A) - 0.01AL}$

(d)



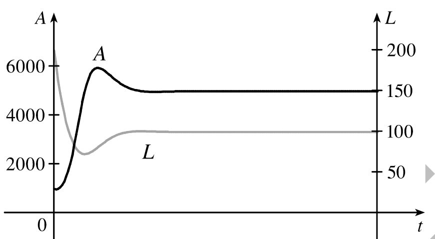
All of the phase trajectories spiral tightly around the equilibrium solution $(5000, 100)$.

(e)



At $t = 0$, the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about $(5000, 75)$. The ladybug population starts to increase and quickly stabilizes at 100, while the aphid population stabilizes at 5000.

(f)



The graph of A peaks just after the graph of L has a minimum.

9 Review

TRUE-FALSE QUIZ

1. True. Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions.

2. True. $f(x) = y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{1 - \ln x}{x^2}$.

LHS $= x^2 y' + xy = x^2 \cdot \dfrac{1 - \ln x}{x^2} + x \cdot \dfrac{\ln x}{x} = (1 - \ln x) + \ln x = 1 =$ RHS, so $y = \dfrac{\ln x}{x}$ is a solution of $x^2 y' + xy = 1$.

3. False. $x + y$ cannot be written in the form $g(x)f(y)$.

4. True. $y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x)f(y)$, and hence, is separable.

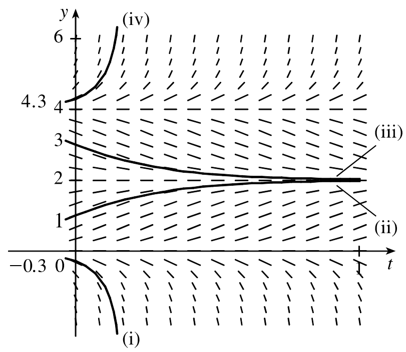
5. True. $e^x y' = y \Rightarrow y' = e^{-x}y \Rightarrow y' + (-e^{-x})y = 0$, which is of the form $y' + P(x)\,y = Q(x)$, so the equation is linear.

6. False. $y' + xy = e^y$ cannot be put in the form $y' + P(x)\,y = Q(x)$, so it is not linear.

7. True. By comparing $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{5}\right)$ with the logistic differential equation (9.4.4), we see that the carrying capacity is 5; that is, $\lim\limits_{t\to\infty} y = 5$.

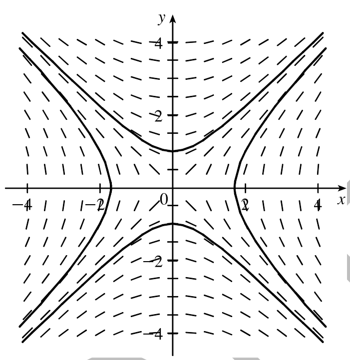
EXERCISES

1. (a)



(b) $\lim\limits_{t\to\infty} y(t)$ appears to be finite for $0 \le c \le 4$. In fact $\lim\limits_{t\to\infty} y(t) = 4$ for $c = 4$, $\lim\limits_{t\to\infty} y(t) = 2$ for $0 < c < 4$, and $\lim\limits_{t\to\infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

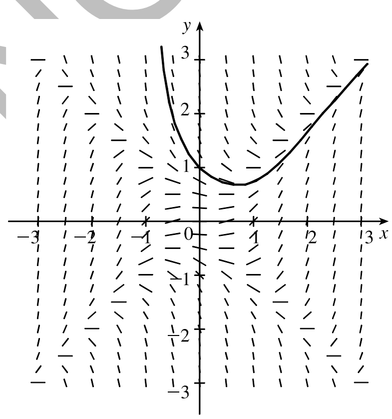
2. (a)



We sketch the direction field and four solution curves, as shown. Note that the slope $y' = x/y$ is not defined on the line $y = 0$.

(b) $y' = x/y \Leftrightarrow y\,dy = x\,dx \Leftrightarrow y^2 = x^2 + C$. For $C = 0$, this is the pair of lines $y = \pm x$. For $C \neq 0$, it is the hyperbola $x^2 - y^2 = -C$.

3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - y_{n-1}^2\right)$. Thus,

$y_1 = 1 + 0.1\left(0^2 - 1^2\right) = 0.9$, $y_2 = 0.9 + 0.1\left(0.1^2 - 0.9^2\right) = 0.82$, $y_3 = 0.82 + 0.1\left(0.2^2 - 0.82^2\right) = 0.75676$.

This is close to our graphical estimate of $y(0.3) \approx 0.8$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

4. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = 2xy^2$. We need y_2.

$y_1 = 1 + 0.2(2 \cdot 0 \cdot 1^2) = 1$, $y_2 = 1 + 0.2(2 \cdot 0.2 \cdot 1^2) = 1.08 \approx y(0.4)$.

(b) $h = 0.1$ now, so $y_1 = 1 + 0.1(2 \cdot 0 \cdot 1^2) = 1$, $y_2 = 1 + 0.1(2 \cdot 0.1 \cdot 1^2) = 1.02$,

$y_3 = 1.02 + 0.1(2 \cdot 0.2 \cdot 1.02^2) \approx 1.06162$, $y_4 = 1.06162 + 0.1(2 \cdot 0.3 \cdot 1.06162^2) \approx 1.1292 \approx y(0.4)$.

(c) The equation is separable, so we write $\dfrac{dy}{y^2} = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int 2x\,dx \;\Leftrightarrow\; -\frac{1}{y} = x^2 + C$, but $y(0) = 1$, so $C = -1$ and $y(x) = \dfrac{1}{1 - x^2} \;\Leftrightarrow\; y(0.4) = \dfrac{1}{1 - 0.16} \approx 1.1905$. From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.

5. $y' = xe^{-\sin x} - y\cos x \;\Rightarrow\; y' + (\cos x)\,y = xe^{-\sin x}$ $(\star)$. This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying $(\star)$ by $e^{\sin x}$ gives $e^{\sin x}\,y' + e^{\sin x}(\cos x)\,y = x \;\Rightarrow\; (e^{\sin x}\,y)' = x \;\Rightarrow$ $e^{\sin x}\,y = \frac{1}{2}x^2 + C \;\Rightarrow\; y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$.

6. $\dfrac{dx}{dt} = 1 - t + x - tx = 1(1 - t) + x(1 - t) = (1 + x)(1 - t) \;\Rightarrow\; \dfrac{dx}{1 + x} = (1 - t)\,dt \;\Rightarrow$

$\displaystyle\int \frac{dx}{1 + x} = \int (1 - t)\,dt \;\Rightarrow\; \ln|1 + x| = t - \tfrac{1}{2}t^2 + C \;\Rightarrow\; |1 + x| = e^{t - t^2/2 + C} \;\Rightarrow$

$1 + x = \pm e^{t - t^2/2} \cdot e^C \;\Rightarrow\; x = -1 + Ke^{t - t^2/2}$, where K is any nonzero constant.

7. $2ye^{y^2}y' = 2x + 3\sqrt{x} \;\Rightarrow\; 2ye^{y^2}\dfrac{dy}{dx} = 2x + 3\sqrt{x} \;\Rightarrow\; 2ye^{y^2}\,dy = \left(2x + 3\sqrt{x}\right)dx \;\Rightarrow$

$\int 2ye^{y^2}\,dy = \int \left(2x + 3\sqrt{x}\right)dx \;\Rightarrow\; e^{y^2} = x^2 + 2x^{3/2} + C \;\Rightarrow\; y^2 = \ln(x^2 + 2x^{3/2} + C) \;\Rightarrow$

$y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

8. $x^2y' - y = 2x^3e^{-1/x} \;\Rightarrow\; y' - \dfrac{1}{x^2}\,y = 2xe^{-1/x}$ $(\star)$. This is a linear equation and the integrating factor is $I(x) = e^{\int(-1/x^2)\,dx} = e^{1/x}$. Multiplying $(\star)$ by $e^{1/x}$ gives $e^{1/x}\,y' - e^{1/x} \cdot \dfrac{1}{x^2}\,y = 2x \;\Rightarrow\; (e^{1/x}\,y)' = 2x \;\Rightarrow$ $e^{1/x}\,y = x^2 + C \;\Rightarrow\; y = e^{-1/x}(x^2 + C)$.

9. $\dfrac{dr}{dt} + 2tr = r \;\Rightarrow\; \dfrac{dr}{dt} = r - 2tr = r(1 - 2t) \;\Rightarrow\; \displaystyle\int \frac{dr}{r} = \int (1 - 2t)\,dt \;\Rightarrow\; \ln|r| = t - t^2 + C \;\Rightarrow$

$|r| = e^{t - t^2 + C} = ke^{t - t^2}$. Since $r(0) = 5$, $5 = ke^0 = k$. Thus, $r(t) = 5e^{t - t^2}$.

10. $(1+\cos x)y' = (1+e^{-y})\sin x \quad\Rightarrow\quad \dfrac{dy}{1+e^{-y}} = \dfrac{\sin x\,dx}{1+\cos x} \quad\Rightarrow\quad \displaystyle\int \frac{dy}{1+1/e^y} = \int \frac{\sin x\,dx}{1+\cos x} \quad\Rightarrow$

$\displaystyle\int \frac{e^y\,dy}{1+e^y} = \int \frac{\sin x\,dx}{1+\cos x} \quad\Rightarrow\quad \ln|1+e^y| = -\ln|1+\cos x| + C \quad\Rightarrow\quad \ln(1+e^y) = -\ln(1+\cos x) + C \quad\Rightarrow$

$1+e^y = e^{-\ln(1+\cos x)}\cdot e^C \quad\Rightarrow\quad e^y = ke^{-\ln(1+\cos x)} - 1 \quad\Rightarrow\quad y = \ln[ke^{-\ln(1+\cos x)} - 1]$. Since $y(0)=0$,

$0 = \ln[ke^{-\ln 2} - 1] \quad\Rightarrow\quad e^0 = k\left(\frac{1}{2}\right) - 1 \quad\Rightarrow\quad k=4$. Thus, $y(x) = \ln[4e^{-\ln(1+\cos x)} - 1]$. An equivalent form

is $y(x) = \ln\dfrac{3-\cos x}{1+\cos x}$.

11. $xy' - y = x\ln x \quad\Rightarrow\quad y' - \dfrac{1}{x}y = \ln x.\quad I(x) = e^{\int(-1/x)\,dx} = e^{-\ln|x|} = \left(e^{\ln|x|}\right)^{-1} = |x|^{-1} = 1/x$ since the condition

$y(1)=2$ implies that we want a solution with $x>0$. Multiplying the last differential equation by $I(x)$ gives

$\dfrac{1}{x}y' - \dfrac{1}{x^2}y = \dfrac{1}{x}\ln x \quad\Rightarrow\quad \left(\dfrac{1}{x}y\right)' = \dfrac{1}{x}\ln x \quad\Rightarrow\quad \dfrac{1}{x}y = \displaystyle\int\frac{\ln x}{x}\,dx \quad\Rightarrow\quad \dfrac{1}{x}y = \frac{1}{2}(\ln x)^2 + C \quad\Rightarrow$

$y = \frac{1}{2}x(\ln x)^2 + Cx$. Now $y(1)=2 \quad\Rightarrow\quad 2 = 0 + C \quad\Rightarrow\quad C=2$, so $y = \frac{1}{2}x(\ln x)^2 + 2x$.

12. $y' = 3x^2e^y \quad\Rightarrow\quad \dfrac{dy}{dx} = 3x^2e^y \quad\Rightarrow\quad e^{-y}\,dy = 3x^2\,dx \quad\Rightarrow$

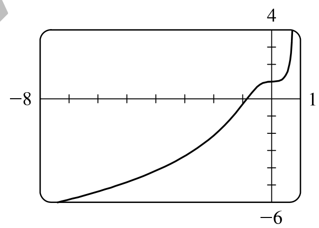
$\int e^{-y}\,dy = \int 3x^2\,dx \quad\Rightarrow\quad -e^{-y} = x^3 + C$. Now $y(0)=1 \quad\Rightarrow$

$-e^{-1} = C$, so $-e^{-y} = x^3 - e^{-1} \quad\Rightarrow\quad e^{-y} = -x^3 + e^{-1} \quad\Rightarrow$

$-y = \ln(-x^3+e^{-1}) \quad\Rightarrow\quad y = -\ln(-x^3+e^{-1})$. To find the domain,

solve $-x^3 + e^{-1} > 0 \quad\Rightarrow\quad x^3 < e^{-1} \quad\Rightarrow\quad x < e^{-1/3}$, so the domain is

$(-\infty, e^{-1/3})$ and $x = e^{-1/3}$ [≈ 0.72] is a vertical asymptote.



13. $\dfrac{d}{dx}(y) = \dfrac{d}{dx}(ke^x) \quad\Rightarrow\quad y' = ke^x = y$, so the orthogonal trajectories must have $y' = -\dfrac{1}{y} \quad\Rightarrow\quad \dfrac{dy}{dx} = -\dfrac{1}{y} \quad\Rightarrow$

$y\,dy = -dx \quad\Rightarrow\quad \int y\,dy = -\int dx \quad\Rightarrow\quad \frac{1}{2}y^2 = -x + C \quad\Rightarrow\quad x = C - \frac{1}{2}y^2$, which are parabolas with a horizontal axis.

14. $\dfrac{d}{dx}(y) = \dfrac{d}{dx}(e^{kx}) \quad\Rightarrow\quad y' = ke^{kx} = ky = \dfrac{\ln y}{x}\cdot y$, so the orthogonal trajectories must have $y' = -\dfrac{x}{y\ln y} \quad\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{x}{y\ln y} \quad\Rightarrow\quad y\ln y\,dy = -x\,dx \quad\Rightarrow\quad \int y\ln y\,dy = -\int x\,dx \quad\Rightarrow\quad \frac{1}{2}y^2\ln y - \frac{1}{4}y^2$ [parts with $u=\ln y$,

$dv = y\,dy$] $= -\frac{1}{2}x^2 + C_1 \quad\Rightarrow\quad 2y^2\ln y - y^2 = C - 2x^2$.

15. (a) Using (4) and (7) in Section 9.4, we see that for $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{2000}\right)$ with $P(0)=100$, we have $k=0.1$,

$M = 2000$, $P_0 = 100$, and $A = \dfrac{2000-100}{100} = 19$. Thus, the solution of the initial-value problem is

$P(t) = \dfrac{2000}{1+19e^{-0.1t}}$ and $P(20) = \dfrac{2000}{1+19e^{-2}} \approx 560$.

(b) $P = 1200 \quad\Leftrightarrow\quad 1200 = \dfrac{2000}{1+19e^{-0.1t}} \quad\Leftrightarrow\quad 1+19e^{-0.1t} = \dfrac{2000}{1200} \quad\Leftrightarrow\quad 19e^{-0.1t} = \dfrac{5}{3} - 1 \quad\Leftrightarrow$

$e^{-0.1t} = \left(\frac{2}{3}\right)/19 \quad\Leftrightarrow\quad -0.1t = \ln\frac{2}{57} \quad\Leftrightarrow\quad t = -10\ln\frac{2}{57} \approx 33.5$.

16. (a) Let $t = 0$ correspond to the year 2000. An exponential model is $P(t) = ae^{kt}$. $P(0) = 6.1$, so $P(t) = 6.1e^{kt}$.

$P(10) = 6.1e^{10k}$ and $P(10) = 6.9$, so $6.1e^{10k} = 6.9 \quad \Leftrightarrow \quad \dfrac{6.9}{6.1} = e^{10k} \quad \Rightarrow \quad 10k = \ln \dfrac{69}{61} \quad \Rightarrow$

$k = \dfrac{1}{10} \ln \dfrac{69}{61} \approx 0.0123$. Thus, $P(t) = 6.1e^{kt}$ and $P(20) = 6.1e^{20k} \approx 7.8$. Our model predicts that the world population in the year 2020 will be 7.8 billion.

(b) $P(t) = 10 \quad \Leftrightarrow \quad 6.1e^{kt} = 10 \quad \Leftrightarrow \quad e^{kt} = \dfrac{10}{6.1} \quad \Leftrightarrow \quad kt = \ln \dfrac{10}{6.1} \quad \Leftrightarrow \quad t = 10\, \dfrac{\ln(10/6.1)}{\ln(69/61)} \approx 40.11$ years. Our exponential model predicts that the world population will exceed 10 billion in 40.11 years; that is, in the year 2040.

(c) $A = \dfrac{M - P_0}{P_0} = \dfrac{20 - 6.1}{6.1} = \dfrac{139}{61}$ and from part (a) , $k = \dfrac{1}{10} \ln \dfrac{69}{61}$, so $P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{20}{1 + \frac{139}{61} e^{-kt}}$. Thus,

$P(20) = \dfrac{20}{1 + \frac{139}{61} e^{-20k}} \approx 7.2$ billion, which is less than our prediction of 7.8 billion from the exponential model in part (a).

(d) $P(t) = 10 \quad \Leftrightarrow \quad \dfrac{20}{1 + \frac{139}{61} e^{-kt}} = 10 \quad \Leftrightarrow \quad \dfrac{20}{10} = 1 + \dfrac{139}{61} e^{-kt} \quad \Leftrightarrow \quad 1 = \dfrac{139}{61} e^{-kt} \quad \Leftrightarrow \quad \dfrac{61}{139} = e^{-kt} \quad \Leftrightarrow$

$\ln \dfrac{61}{139} = -kt \quad \Leftrightarrow \quad t = -10\, \dfrac{\ln(61/139)}{\ln(69/61)} \approx 66.83$ years. Our logistic model predicts that the world population will exceed 10 billion in 66.83 years; that is, in the year 2066, which is considerably later than our prediction of 2040 from the exponential model in part (b).

17. (a) $\dfrac{dL}{dt} \propto L_\infty - L \quad \Rightarrow \quad \dfrac{dL}{dt} = k(L_\infty - L) \quad \Rightarrow \quad \displaystyle\int \frac{dL}{L_\infty - L} = \int k\, dt \quad \Rightarrow \quad -\ln|L_\infty - L| = kt + C \quad \Rightarrow$

$\ln|L_\infty - L| = -kt - C \quad \Rightarrow \quad |L_\infty - L| = e^{-kt-C} \quad \Rightarrow \quad L_\infty - L = Ae^{-kt} \quad \Rightarrow \quad L = L_\infty - Ae^{-kt}$.

At $t = 0$, $L = L(0) = L_\infty - A \quad \Rightarrow \quad A = L_\infty - L(0) \quad \Rightarrow \quad L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$.

(b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \quad \Rightarrow \quad L(t) = 53 - (53 - 10)e^{-0.2t} = 53 - 43e^{-0.2t}$.

18. Denote the amount of salt in the tank (in kg) by y. $y(0) = 0$ since initially there is only water in the tank.

The rate at which y increases is equal to the rate at which salt flows into the tank minus the rate at which it flows out.

That rate is $\dfrac{dy}{dt} = 0.1 \dfrac{\text{kg}}{\text{L}} \times 10 \dfrac{\text{L}}{\text{min}} - \dfrac{y}{100} \dfrac{\text{kg}}{\text{L}} \times 10 \dfrac{\text{L}}{\text{min}} = 1 - \dfrac{y}{10} \dfrac{\text{kg}}{\text{min}} \quad \Rightarrow \quad \displaystyle\int \frac{dy}{10 - y} = \int \frac{1}{10}\, dt \quad \Rightarrow$

$-\ln|10 - y| = \frac{1}{10}t + C \quad \Rightarrow \quad 10 - y = Ae^{-t/10}$. $y(0) = 0 \quad \Rightarrow \quad 10 = A \quad \Rightarrow \quad y = 10(1 - e^{-t/10})$.

At $t = 6$ minutes, $y = 10(1 - e^{-6/10}) \approx 4.512$ kg.

19. Let P represent the population and I the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $\dfrac{dI}{dt} = kI(P - I) \quad \Rightarrow \quad I(t) = \dfrac{I_0 P}{I_0 + (P - I_0)e^{-kPt}}$ [from the discussion of logistic growth in Section 9.4].

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k:

$$1200 = \frac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000 \cdot 7 \cdot k}} \quad \Leftrightarrow \quad 3 = \frac{2000}{160 + 4840e^{-35{,}000k}} \quad \Leftrightarrow \quad 480 + 14{,}520e^{-35{,}000k} = 2000 \quad \Leftrightarrow$$

$$e^{-35{,}000k} = \frac{2000 - 480}{14{,}520} \quad \Leftrightarrow \quad -35{,}000k = \ln\frac{38}{363} \quad \Leftrightarrow \quad k = \frac{-1}{35{,}000}\ln\frac{38}{363} \approx 0.00006448.$$ Next, let

$I = 5000 \times 80\% = 4000$, and solve for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-k \cdot 5000 \cdot t}} \quad \Leftrightarrow \quad 1 = \dfrac{200}{160 + 4840e^{-5000kt}} \quad \Leftrightarrow$

$$160 + 4840e^{-5000kt} = 200 \quad \Leftrightarrow \quad e^{-5000kt} = \frac{200 - 160}{4840} \quad \Leftrightarrow \quad -5000kt = \ln\frac{1}{121} \quad \Leftrightarrow$$

$t = \dfrac{-1}{5000k}\ln\dfrac{1}{121} = \dfrac{1}{\frac{1}{7}\ln\frac{38}{363}} \cdot \ln\dfrac{1}{121} = 7 \cdot \dfrac{\ln 121}{\ln\frac{363}{38}} \approx 14.875$. So it takes about 15 days for 80% of the population to be infected.

20. $\dfrac{1}{R}\dfrac{dR}{dt} = \dfrac{k}{S}\dfrac{dS}{dt} \quad \Rightarrow \quad \dfrac{d}{dt}(\ln R) = \dfrac{d}{dt}(k\ln S) \quad \Rightarrow \quad \ln R = k\ln S + C \quad \Rightarrow$

$R = e^{k\ln S + C} = e^C\left(e^{\ln S}\right)^k \quad \Rightarrow \quad R = AS^k$, where $A = e^C$ is a positive constant.

21. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k+h}\right) \quad \Rightarrow \quad \displaystyle\int \frac{k+h}{h}\,dh = \int\left(-\frac{R}{V}\right)dt \quad \Rightarrow \quad \int\left(1 + \frac{k}{h}\right)dh = -\frac{R}{V}\int 1\,dt \quad \Rightarrow$

$h + k\ln h = -\dfrac{R}{V}t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

22. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

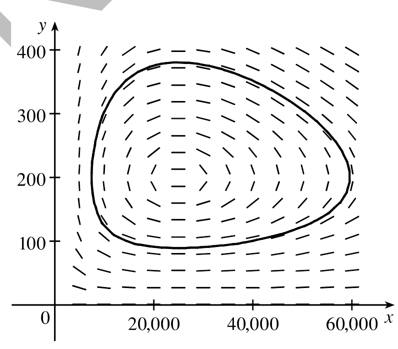
(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left\{\begin{matrix} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{matrix}\right\} \quad \Rightarrow \quad \begin{cases} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{cases} \quad \Rightarrow \quad y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

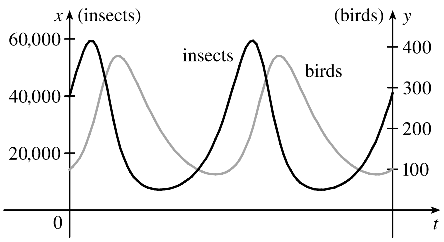
(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

(d)



At $(x, y) = (40{,}000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59{,}646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e)



Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

23. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0 \quad \Leftrightarrow \quad x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left\{\begin{array}{l} 0 = 0.4x(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{array}\right.$$

The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4(25{,}000)[(1 - 0.000005 \cdot 25{,}000) - 0.005y] \quad \Rightarrow$ $0 = 10{,}000[(1 - 0.125) - 0.005y] \quad \Rightarrow \quad 0 = 8750 - 50y \quad \Rightarrow \quad y = 175$.

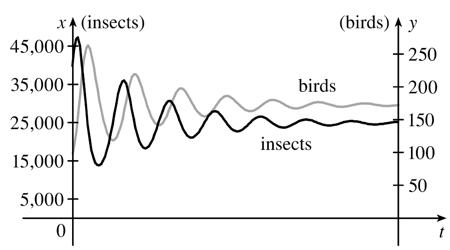
Case (i): $y = 0$, $x = 0$: Zero populations

Case (ii): $y = 0$, $x = 200{,}000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: The predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d)



24. First note that, in this question, "weighs" is used in the informal sense, so what we really require is Barbara's mass m in kg as a function of t. Barbara's net intake of calories per day at time t (measured in days) is $c(t) = 1600 - 850 - 15m(t) = 750 - 15m(t)$, where $m(t)$ is her mass at time t. We are given that $m(0) = 60$ kg and $\dfrac{dm}{dt} = \dfrac{c(t)}{10{,}000}$, so $\dfrac{dm}{dt} = \dfrac{750 - 15m}{10{,}000} = \dfrac{150 - 3m}{2000} = \dfrac{-3(m - 50)}{2000}$ with $m(0) = 60$. From $\displaystyle\int \frac{dm}{m - 50} = \int \frac{-3\,dt}{2000}$, we get $\ln|m - 50| = -\frac{3}{2000}t + C$. Since $m(0) = 60$, $C = \ln 10$. Now $\ln\dfrac{|m - 50|}{10} = -\dfrac{3t}{2000}$, so $|m - 50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous, initially positive, and the right-hand side is never zero. Thus, $m - 50$ is positive for all t, and $m(t) = 50 + 10e^{-3t/2000}$ kg. As $t \to \infty$, $m(t) \to 50$ kg. Thus, Barbara's mass gradually settles down to 50 kg.

☐ PROBLEMS PLUS

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \quad \Rightarrow \quad 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \quad \Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \quad \Rightarrow \quad [f(x) - f'(x)]^2 = 0 \quad \Leftrightarrow \quad f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 9.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

2. $(fg)' = f'g'$, where $f(x) = e^{x^2} \quad \Rightarrow \quad \left(e^{x^2}g\right)' = 2xe^{x^2}g'$. Since the student's mistake did not affect the answer,

$\left(e^{x^2}g\right)' = e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'$. So $(2x-1)g' = 2xg$, or $\dfrac{g'}{g} = \dfrac{2x}{2x-1} = 1 + \dfrac{1}{2x-1} \quad \Rightarrow$

$\ln|g(x)| = x + \frac{1}{2}\ln(2x-1) + C \quad \Rightarrow \quad g(x) = Ae^x\sqrt{2x-1}$.

3. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)\,[f(h) - 1]}{h} \quad$ [since $f(x+h) = f(x)f(h)$]

$= f(x) \lim\limits_{h\to 0} \dfrac{f(h) - 1}{h} = f(x) \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 9.4.2 we get $f(x) = Ae^x$.

Now $f(0) = 1 \quad \Rightarrow \quad A = 1 \quad \Rightarrow \quad f(x) = e^x$.

4. $\left(\displaystyle\int f(x)\,dx\right)\left(\displaystyle\int \frac{dx}{f(x)}\right) = -1 \quad \Rightarrow \quad \displaystyle\int \frac{dx}{f(x)} = \frac{-1}{\int f(x)\,dx} \quad \Rightarrow \quad \frac{1}{f(x)} = \frac{f(x)}{\left[\int f(x)\,dx\right]^2}$ [after differentiating] $\Rightarrow$

$\int f(x)\,dx = \pm f(x)$ [after taking square roots] $\Rightarrow \quad f(x) = \pm f'(x)$ [after differentiating again] $\Rightarrow \quad y = Ae^x$ or $y = Ae^{-x}$ by Theorem 9.4.2. Therefore, $f(x) = Ae^x$ or $f(x) = Ae^{-x}$, for all nonzero constants A, are the functions satisfying the original equation.

5. "The area under the graph of f from 0 to x is proportional to the $(n+1)$st power of $f(x)$" translates to

$\int_0^x f(t)\,dt = k[f(x)]^{n+1}$ for some constant k. By FTC1, $\dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt = \frac{d}{dx}\left\{k[f(x)]^{n+1}\right\} \quad \Rightarrow$

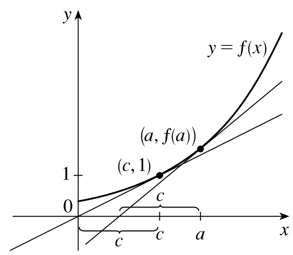
$f(x) = k(n+1)[f(x)]^n f'(x) \quad \Rightarrow \quad 1 = k(n+1)[f(x)]^{n-1}f'(x) \quad \Rightarrow \quad 1 = k(n+1)y^{n-1}\dfrac{dy}{dx} \quad \Rightarrow$

$k(n+1)y^{n-1}\,dy = dx \quad \Rightarrow \quad \int k(n+1)y^{n-1}\,dy = \int dx \quad \Rightarrow \quad k(n+1)\dfrac{1}{n}y^n = x + C$.

Now $f(0) = 0 \quad \Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0$ and then $f(1) = 1 \quad \Rightarrow \quad k(n+1)\dfrac{1}{n} = 1 \quad \Rightarrow \quad k = \dfrac{n}{n+1}$,

so $y^n = x$ and $y = f(x) = x^{1/n}$.

6. Let $y = f(x)$ be a curve that passes through the point $(c, 1)$ and whose subtangents all have length c. The tangent line at $x = a$ has equation $y - f(a) = f'(a)(x - a)$. Assuming $f(a) \neq 0$ and $f'(a) \neq 0$, it has x-intercept $a - \dfrac{f(a)}{f'(a)}$ [let $y = 0$ and solve for x]. Thus, the length of the subtangent is c, so $\left| a - \left(a - \dfrac{f(a)}{f'(a)} \right) \right| = \left| \dfrac{f(a)}{f'(a)} \right| = c \;\Rightarrow\; \dfrac{f'(a)}{f(a)} = \pm\dfrac{1}{c}$.



Now $\dfrac{f'(x)}{f(x)} = \pm\dfrac{1}{c} \;\Rightarrow\; f'(x) = \pm\dfrac{1}{c}\, f(x) \;\Rightarrow\; \dfrac{dy}{dx} = \pm\dfrac{1}{c}\, y \;\Rightarrow\; \dfrac{dy}{y} = \pm\dfrac{1}{c}\, dx \;\Rightarrow\; \displaystyle\int \frac{1}{y}\, dy = \pm\frac{1}{c}\int dx \;\Rightarrow$

$\ln|y| = \pm\dfrac{1}{c}\, x + K$. Since $f(c) = 1$, $\ln 1 = \pm 1 + K \;\Rightarrow\; K = \mp 1$. Thus, $y = e^{\pm x/c \mp 1}$, or $y = e^{\pm(x/c-1)}$. One curve is an increasing exponential (as shown in the figure) and the other curve is its reflection about the line $x = c$.

7. Let $y(t)$ denote the temperature of the peach pie t minutes after 5:00 PM and R the temperature of the room. Newton's Law of Cooling gives us $dy/dt = k(y - R)$. Solving for y we get $\dfrac{dy}{y - R} = k\, dt \;\Rightarrow\; \ln|y - R| = kt + C \;\Rightarrow$

$|y - R| = e^{kt+C} \;\Rightarrow\; y - R = \pm e^{kt} \cdot e^{C} \;\Rightarrow\; y = Me^{kt} + R$, where M is a nonzero constant. We are given temperatures at three times.

$$\begin{aligned} y(0) &= 100 \;\Rightarrow\; 100 = M + R \;\Rightarrow\; R = 100 - M \\ y(10) &= 80 \;\Rightarrow\; 80 = Me^{10k} + R \qquad (\mathbf{1}) \\ y(20) &= 65 \;\Rightarrow\; 65 = Me^{20k} + R \qquad (\mathbf{2}) \end{aligned}$$

Substituting $100 - M$ for R in **(1)** and **(2)** gives us

$$-20 = Me^{10k} - M \;\; \mathbf{(3)} \quad \text{and} \quad -35 = Me^{20k} - M \;\; \mathbf{(4)}$$

Dividing **(3)** by **(4)** gives us $\dfrac{-20}{-35} = \dfrac{M(e^{10k} - 1)}{M(e^{20k} - 1)} \;\Rightarrow\; \dfrac{4}{7} = \dfrac{e^{10k} - 1}{e^{20k} - 1} \;\Rightarrow\; 4e^{20k} - 4 = 7e^{10k} - 7 \;\Rightarrow$

$4e^{20k} - 7e^{10k} + 3 = 0$. This is a quadratic equation in e^{10k}. $(4e^{10k} - 3)(e^{10k} - 1) = 0 \;\Rightarrow\; e^{10k} = \frac{3}{4}$ or $1 \;\Rightarrow$

$10k = \ln\frac{3}{4}$ or $\ln 1 \;\Rightarrow\; k = \frac{1}{10}\ln\frac{3}{4}$ since k is a nonzero constant of proportionality. Substituting $\frac{3}{4}$ for e^{10k} in **(3)** gives us

$-20 = M \cdot \frac{3}{4} - M \;\Rightarrow\; -20 = -\frac{1}{4}M \;\Rightarrow\; M = 80$. Now $R = 100 - M$ so $R = 20^\circ\text{C}$.

8. Let b be the number of hours before noon that it began to snow, t the time measured in hours after noon, and $x = x(t) =$ distance traveled by the plow at time t. Then $dx/dt =$ speed of plow. Since the snow falls steadily, the height at time t is $h(t) = k(t + b)$, where k is a constant. We are given that the rate of removal is constant, say R (in m^3/h).

If the width of the path is w, then $R = \text{height} \times \text{width} \times \text{speed} = h(t) \times w \times \dfrac{dx}{dt} = k(t + b)w\,\dfrac{dx}{dt}$. Thus, $\dfrac{dx}{dt} = \dfrac{C}{t + b}$,

where $C = \dfrac{R}{kw}$ is a constant. This is a separable equation. $\displaystyle\int dx = C\int \frac{dt}{t + b} \;\Rightarrow\; x(t) = C\ln(t + b) + K$.

Put $t = 0$: $0 = C\ln b + K \;\Rightarrow\; K = -C\ln b$, so $x(t) = C\ln(t + b) - C\ln b = C\ln(1 + t/b)$.

Put $t = 1$: $6000 = C\ln(1 + 1/b)$ $\quad[x = 6 \text{ km}]$.

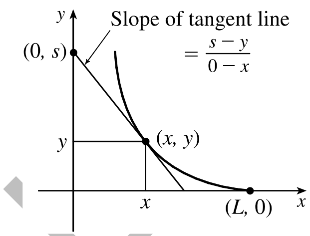
Put $t = 2$: $9000 = C\ln(1 + 2/b)$ $\quad[x = (6 + 3)\ \text{km}]$.

Solve for b: $\dfrac{\ln(1+1/b)}{6000} = \dfrac{\ln(1+2/b)}{9000} \quad\Rightarrow\quad 3\ln\left(1+\dfrac{1}{b}\right) = 2\ln\left(1+\dfrac{2}{b}\right) \quad\Rightarrow\quad \left(1+\dfrac{1}{b}\right)^3 = \left(1+\dfrac{2}{b}\right)^2 \quad\Rightarrow$

$1+\dfrac{3}{b}+\dfrac{3}{b^2}+\dfrac{1}{b^3} = 1+\dfrac{4}{b}+\dfrac{4}{b^2} \quad\Rightarrow\quad \dfrac{1}{b}+\dfrac{1}{b^2}-\dfrac{1}{b^3}=0 \quad\Rightarrow\quad b^2+b-1=0 \quad\Rightarrow\quad b=\frac{-1\pm\sqrt{5}}{2}$.

But $b>0$, so $b=\frac{-1+\sqrt{5}}{2}\approx 0.618$ h ≈ 37 min. The snow began to fall $\frac{\sqrt{5}-1}{2}$ hours before noon; that is, at about 11:23 AM.

9. (a) While running from $(L,0)$ to (x,y), the dog travels a distance



$s=\int_x^L \sqrt{1+(dy/dx)^2}\,dx = -\int_L^x \sqrt{1+(dy/dx)^2}\,dx$, so

$\dfrac{ds}{dx} = -\sqrt{1+(dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0,s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx}=\dfrac{s-y}{0-x}$ (see the figure).

Thus, $s = y - x\,\dfrac{dy}{dx} \quad\Rightarrow\quad \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\,\dfrac{d^2y}{dx^2}+1\,\dfrac{dy}{dx}\right) = -x\,\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$ gives us $x\,\dfrac{d^2y}{dx^2} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z=\dfrac{dy}{dx}$, we obtain the differential equation $x\,\dfrac{dz}{dx}=\sqrt{1+z^2}$, or $\dfrac{dz}{\sqrt{1+z^2}}=\dfrac{dx}{x}$. Integrating:

$\ln x = \displaystyle\int \frac{dz}{\sqrt{1+z^2}} \overset{25}{=} \ln\left(z+\sqrt{1+z^2}\right)+C$. When $x=L$, $z=dy/dx=0$, so $\ln L=\ln 1+C$. Therefore,

$C=\ln L$, so $\ln x = \ln\left(\sqrt{1+z^2}+z\right)+\ln L = \ln\left[L\left(\sqrt{1+z^2}+z\right)\right] \quad\Rightarrow\quad x = L\left(\sqrt{1+z^2}+z\right) \quad\Rightarrow$

$\sqrt{1+z^2}=\dfrac{x}{L}-z \quad\Rightarrow\quad 1+z^2=\left(\dfrac{x}{L}\right)^2-\dfrac{2xz}{L}+z^2 \quad\Rightarrow\quad \left(\dfrac{x}{L}\right)^2-2z\left(\dfrac{x}{L}\right)-1=0 \quad\Rightarrow$

$z=\dfrac{(x/L)^2-1}{2(x/L)}=\dfrac{x^2-L^2}{2Lx}=\dfrac{x}{2L}-\dfrac{L}{2}\dfrac{1}{x}$ [for $x>0$]. Since $z=\dfrac{dy}{dx}$, $y=\dfrac{x^2}{4L}-\dfrac{L}{2}\ln x+C_1$.

Since $y=0$ when $x=L$, $0=\dfrac{L}{4}-\dfrac{L}{2}\ln L+C_1 \quad\Rightarrow\quad C_1=\dfrac{L}{2}\ln L-\dfrac{L}{4}$. Thus,

$y=\dfrac{x^2}{4L}-\dfrac{L}{2}\ln x+\dfrac{L}{2}\ln L-\dfrac{L}{4}=\dfrac{x^2-L^2}{4L}-\dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$.

(c) As $x\to 0^+$, $y\to\infty$, so the dog never catches the rabbit.

10. (a) If the dog runs twice as fast as the rabbit, then the rabbit's position when the dog has traveled a distance s is $(0,s/2)$.

Since the dog runs straight toward the rabbit, the tangent line to the dog's path has slope $\dfrac{dy}{dx}=\dfrac{s/2-y}{0-x}$.

Thus, $s=2y-2x\,\dfrac{dy}{dx} \quad\Rightarrow\quad \dfrac{ds}{dx}=2\,\dfrac{dy}{dx}-\left(2x\,\dfrac{d^2y}{dx^2}+2\,\dfrac{dy}{dx}\right)=-2x\,\dfrac{d^2y}{dx^2}$.

From Problem 9(a), $\dfrac{ds}{dx}=-\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$, so $2x\,\dfrac{d^2y}{dx^2}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$.

Letting $z=\dfrac{dy}{dx}$, we obtain the differential equation $2x\,\dfrac{dz}{dx}=\sqrt{1+z^2}$, or $\dfrac{2\,dz}{\sqrt{1+z^2}}=\dfrac{dx}{x}$. Integrating, we get

$\ln x = \displaystyle\int \frac{2\,dz}{\sqrt{1+z^2}} = 2\ln\Big(\sqrt{1+z^2}+z\Big) + C$. [See Problem 9(b).] When $x = L$, $z = dy/dx = 0$, so $\ln L = 2\ln 1 + C = C$. Thus,

$$\ln x = 2\ln\big(\sqrt{1+z^2}+z\big) + \ln L = \ln\Big(L\big(\sqrt{1+z^2}+z\big)^2\Big) \quad\Rightarrow\quad x = L\big(\sqrt{1+z^2}+z\big)^2 \quad\Rightarrow$$

$$\sqrt{1+z^2} = \sqrt{\frac{x}{L}} - z \quad\Rightarrow\quad 1+z^2 = \frac{x}{L} - 2\sqrt{\frac{x}{L}}\,z + z^2 \quad\Rightarrow\quad 2\sqrt{\frac{x}{L}}\,z = \frac{x}{L} - 1 \quad\Rightarrow$$

$$\frac{dy}{dx} = z = \frac{1}{2}\sqrt{\frac{x}{L}} - \frac{1}{2\sqrt{x/L}} = \frac{1}{2\sqrt{L}}\,x^{1/2} - \frac{\sqrt{L}}{2}\,x^{-1/2} \quad\Rightarrow\quad y = \frac{1}{3\sqrt{L}}\,x^{3/2} - \sqrt{L}\,x^{1/2} + C_1.$$

When $x = L$, $y = 0$, so $0 = \dfrac{1}{3\sqrt{L}}\,L^{3/2} - \sqrt{L}\,L^{1/2} + C_1 = \dfrac{L}{3} - L + C_1 = C_1 - \frac{2}{3}L$. Therefore, $C_1 = \frac{2}{3}L$ and $y = \dfrac{x^{3/2}}{3\sqrt{L}} - \sqrt{L}\,x^{1/2} + \frac{2}{3}L$. As $x \to 0$, $y \to \frac{2}{3}L$, so the dog catches the rabbit when the rabbit is at $\big(0, \frac{2}{3}L\big)$. (At that point, the dog has traveled a distance of $\frac{4}{3}L$, twice as far as the rabbit has run.)

(b) As in the solutions to part (a) and Problem 9, we get $z = \dfrac{dy}{dx} = \dfrac{x^2}{2L^2} - \dfrac{L^2}{2x^2}$ and hence $y = \dfrac{x^3}{6L^2} + \dfrac{L^2}{2x} - \frac{2}{3}L$.

We want to minimize the distance D from the dog at (x, y) to the rabbit at $(0, 2s)$. Now $s = \frac{1}{2}y - \frac{1}{2}x\dfrac{dy}{dx} \quad\Rightarrow$

$2s = y - xz \quad\Rightarrow\quad y - 2s = xz = x\left(\dfrac{x^2}{2L^2} - \dfrac{L^2}{2x^2}\right) = \dfrac{x^3}{2L^2} - \dfrac{L^2}{2x}$, so

$$D = \sqrt{(x-0)^2 + (y-2s)^2} = \sqrt{x^2 + \left(\frac{x^3}{2L^2} - \frac{L^2}{2x}\right)^2} = \sqrt{\frac{x^6}{4L^4} + \frac{x^2}{2} + \frac{L^4}{4x^2}} = \sqrt{\left(\frac{x^3}{2L^2} + \frac{L^2}{2x}\right)^2}$$
$$= \frac{x^3}{2L^2} + \frac{L^2}{2x}$$

$$D' = 0 \quad\Leftrightarrow\quad \frac{3x^2}{2L^2} - \frac{L^2}{2x^2} = 0 \quad\Leftrightarrow\quad \frac{3x^2}{2L^2} = \frac{L^2}{2x^2} \quad\Leftrightarrow\quad x^4 = \frac{L^4}{3} \quad\Leftrightarrow\quad x = \frac{L}{\sqrt[4]{3}},\ x > 0,\ L > 0.$$

Since $D''(x) = \dfrac{3x}{L^2} + \dfrac{L^2}{x^3} > 0$ for all $x > 0$, we know that $D\left(\dfrac{L}{\sqrt[4]{3}}\right) = \dfrac{(L\cdot 3^{-1/4})^3}{2L^2} + \dfrac{L^2}{2L\cdot 3^{-1/4})} = \dfrac{2L}{3^{3/4}}$ is the minimum value of D, that is, the closest the dog gets to the rabbit. The positions at this distance are

$$\text{Dog: } (x, y) = \left(\frac{L}{\sqrt[4]{3}}, \left(\frac{5}{3^{7/4}} - \frac{2}{3}\right)L\right) = \left(\frac{L}{\sqrt[4]{3}}, \frac{5\sqrt[4]{3} - 6}{9}L\right)$$

$$\text{Rabbit: } (0, 2s) = \left(0, \frac{8\sqrt[4]{3}L}{9} - \frac{2L}{3}\right) = \left(0, \frac{8\sqrt[4]{3} - 6}{9}L\right)$$

11. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi$ ft^3/h, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\big(\frac{3}{2}h\big)^2 h = \frac{3}{4}\pi h^3 \quad\Rightarrow$

$\dfrac{dV}{dt} = \frac{3}{4}\pi\cdot 3h^2\,\dfrac{dh}{dt} = \frac{9}{4}\pi h^2\,\dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\star) \quad\Rightarrow$

$\int 3h^2\,dh = \int 80{,}000\,dt \quad\Rightarrow\quad h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so

$h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000 \Rightarrow$ $80{,}000t = 784{,}000 \Rightarrow t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi$ ft^2, and the area of the base of the pile is $A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is $F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217$ ft^2. Now $A = \frac{9\pi}{4}h^2 \Rightarrow dA/dt = \frac{9\pi}{4} \cdot 2h\,(dh/dt)$, and from ($\star$) in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore, $dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283$ ft^2/h.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi$ ft^3/h. From ($\star$) in part (a),

$\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{4(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2} \Rightarrow \int 9h^2\,dh = \int 160{,}000\,dt \Rightarrow 3h^3 = 160{,}000t + C$. When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top, $h = 100 \Rightarrow$ $3(100)^3 = 160{,}000t + 2{,}187{,}000 \Rightarrow t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about 5.1 h.

12. Let $P(a, b)$ be any first-quadrant point on the curve $y = f(x)$. The tangent line at P has equation $y - b = f'(a)(x - a)$, or equivalently, $y = mx + b - ma$, where $m = f'(a)$. If $Q(0, c)$ is the y-intercept, then $c = b - am$. If $R(k, 0)$ is the x-intercept, then $k = \dfrac{am - b}{m} = a - \dfrac{b}{m}$. Since the tangent line is bisected at P, we know that $|PQ| = |PR|$; that is, $\sqrt{(a-0)^2 + [b - (b - am)]^2} = \sqrt{[a - (a - b/m)]^2 + (b - 0)^2}$. Squaring and simplifying gives us $a^2 + a^2m^2 = b^2/m^2 + b^2 \Rightarrow a^2m^2 + a^2m^4 = b^2 + b^2m^2 \Rightarrow a^2m^4 + \left(a^2 - b^2\right)m^2 - b^2 = 0 \Rightarrow$ $\left(a^2m^2 - b^2\right)\left(m^2 + 1\right) = 0 \Rightarrow m^2 = b^2/a^2$. Since m is the slope of the line from a positive y-intercept to a positive x-intercept, m must be negative. Since a and b are positive, we have $m = -b/a$, so we will solve the equivalent differential equation $\dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x} \Rightarrow \displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + C\ [x, y > 0] \Rightarrow$ $y = e^{-\ln x + C} = e^{\ln x^{-1}} \cdot e^C = x^{-1} \cdot A \Rightarrow y = A/x$. Since the point $(3, 2)$ is on the curve, $3 = A/2 \Rightarrow A = 6$ and the curve is $y = 6/x$ with $x > 0$.

13. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P, then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\dfrac{1}{m}(x - a)$, or equivalently, $y = -\dfrac{1}{m}x + b + \dfrac{a}{m}$. The y-intercept is always 6, so $b + \dfrac{a}{m} = 6 \Rightarrow \dfrac{a}{m} = 6 - b \Rightarrow m = \dfrac{a}{6 - b}$. We will solve the equivalent differential equation $\dfrac{dy}{dx} = \dfrac{x}{6 - y} \Rightarrow$ $(6 - y)\,dy = x\,dx \Rightarrow \displaystyle\int (6 - y)\,dy = \int x\,dx \Rightarrow 6y - \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \Rightarrow 12y - y^2 = x^2 + K$.

Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \Rightarrow K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \Rightarrow$ $x^2 + y^2 - 12y + 36 = -11 + 36 \Rightarrow x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.

14. Let $P(x_0, y_0)$ be a point on the curve. Since the midpoint of the line segment determined by the normal line from (x_0, y_0) to its intersection with the x-axis has x-coordinate 0, the x-coordinate of the point of intersection with the x-axis must be $-x_0$. Hence, the normal line has slope $\dfrac{y_0 - 0}{x_0 - (-x_0)} = \dfrac{y_0}{2x_0}$. So the tangent line has slope $-\dfrac{2x_0}{y_0}$. This gives the differential equation $y' = -\dfrac{2x}{y} \Rightarrow y\,dy = -2x\,dx \Rightarrow \int y\,dy = \int(-2x)\,dx \Rightarrow \frac{1}{2}y^2 = -x^2 + C \Rightarrow x^2 + \frac{1}{2}y^2 = C$ $[C > 0]$. This is a family of ellipses.

15. From the figure, slope $OA = \dfrac{y}{x}$. If triangle OAB is isosceles, then slope AB must be $-\dfrac{y}{x}$, the negative of slope OA. This slope is also equal to $f'(x)$, so we have $\dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow \displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow$

$\ln|y| = -\ln|x| + C \Rightarrow |y| = e^{-\ln|x| + C} \Rightarrow$

$|y| = (e^{\ln|x|})^{-1} e^C \Rightarrow |y| = \dfrac{1}{|x|} e^C \Rightarrow y = \dfrac{K}{x}, K \neq 0.$