17 □ SECOND-ORDER DIFFERENTIAL EQUATIONS

17.1 Second-Order Linear Equations

1. The auxiliary equation is $r^2 - r - 6 = 0 \quad \Rightarrow \quad (r-3)(r+2) = 0 \quad \Rightarrow \quad r = 3, r = -2$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-2x}$.

2. The auxiliary equation is $r^2 - 6r + 9 = 0 \quad \Rightarrow \quad (r-3)^2 = 0 \quad \Rightarrow \quad r = 3$. Then by (10), the general solution is $y = c_1 e^{3x} + c_2 x e^{3x}$.

3. The auxiliary equation is $r^2 + 2 = 0 \quad \Rightarrow \quad r = \pm\sqrt{2}\, i$. Then by (11) the general solution is $y = e^{0x}\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right) = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$.

4. The auxiliary equation is $r^2 + r - 12 = 0 \quad \Rightarrow \quad (r-3)(r+4) = 0 \quad \Rightarrow \quad r = 3, r = -4$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-4x}$.

5. The auxiliary equation is $4r^2 + 4r + 1 = 0 \quad \Rightarrow \quad (2r+1)^2 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}$. Then by (10), the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.

6. The auxiliary equation is $9r^2 + 4 = 0 \quad \Rightarrow \quad r^2 = -\frac{4}{9} \quad \Rightarrow \quad r = \pm\frac{2}{3}i$, so the general solution is $y = e^{0x}\left[c_1 \cos\left(\frac{2}{3}x\right) + c_2 \sin\left(\frac{2}{3}x\right)\right] = c_1 \cos\left(\frac{2}{3}x\right) + c_2 \sin\left(\frac{2}{3}x\right)$.

7. The auxiliary equation is $3r^2 - 4r = r(3r-4) = 0 \quad \Rightarrow \quad r = 0, r = \frac{4}{3}$, so $y = c_1 e^{0x} + c_2 e^{4x/3} = c_1 + c_2 e^{4x/3}$.

8. The auxiliary equation is $r^2 - 1 = (r-1)(r+1) = 0 \quad \Rightarrow \quad r = 1, r = -1$. Then the general solution is $y = c_1 e^{x} + c_2 e^{-x}$.

9. The auxiliary equation is $r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$, so $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$.

10. The auxiliary equation is $3r^2 + 4r - 3 = 0 \quad \Rightarrow \quad r = \dfrac{-4 \pm \sqrt{52}}{6} = \dfrac{-2 \pm \sqrt{13}}{3}$, so $y = c_1 e^{(-2+\sqrt{13})x/3} + c_2 e^{(-2-\sqrt{13})x/3}$.

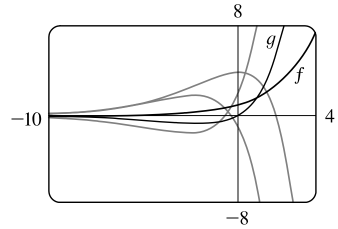
11. The auxiliary equation is $2r^2 + 2r - 1 = 0 \quad \Rightarrow \quad r = \dfrac{-2 \pm \sqrt{12}}{4} = \dfrac{-1 \pm \sqrt{3}}{2}$, so $y = c_1 e^{\left(-1+\sqrt{3}\right)t/2} + c_2 e^{\left(-1-\sqrt{3}\right)t/2}$.

12. The auxiliary equation is $r^2 + 6r + 34 = 0 \quad \Rightarrow \quad r = \dfrac{-6 \pm \sqrt{-100}}{2} = -3 \pm 5i$, so $R = e^{-3t}(c_1 \cos 5t + c_2 \sin 5t)$.

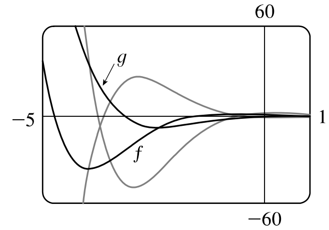
13. The auxiliary equation is $3r^2 + 4r + 3 = 0 \quad \Rightarrow \quad r = \dfrac{-4 \pm \sqrt{-20}}{6} = -\frac{2}{3} \pm \frac{\sqrt{5}}{3}i$, so

$V = e^{-2t/3}\left[c_1 \cos\left(\frac{\sqrt{5}}{3}t\right) + c_2 \sin\left(\frac{\sqrt{5}}{3}t\right)\right]$.

14. The auxiliary equation is $4r^2 - 4r + 1 = (2r-1)^2 = 0 \quad \Rightarrow \quad r = \frac{1}{2}$, so the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. We graph the basic solutions $f(x) = e^{x/2}$, $g(x) = xe^{x/2}$ as well as $y = 2e^{x/2} + 3xe^{x/2}$, $y = -e^{x/2} - 3xe^{x/2}$, and $y = 4e^{x/2} - 2xe^{x/2}$. The graphs are all asymptotic to the x-axis as $x \to -\infty$, and as $x \to \infty$ the solutions approach $\pm\infty$.

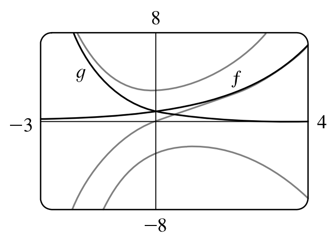


15. The auxiliary equation is $r^2 + 2r + 2 = 0 \quad \Rightarrow$ $r = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$, so the general solution is $y = e^{-x}\left(c_1 \cos x + c_2 \sin x\right)$. We graph the basic solutions $f(x) = e^{-x}\cos x$, $g(x) = e^{-x}\sin x$ as well as



$y = e^{-x}\left(-\cos x - 2\sin x\right)$ and $y = e^{-x}\left(2\cos x + 3\sin x\right)$. All the solutions oscillate with amplitudes that become arbitrarily large as $x \to -\infty$ and the solutions are asymptotic to the x-axis as $x \to \infty$.

16. The auxiliary equation is $2r^2 + r - 1 = (2r-1)(r+1) = 0 \quad \Rightarrow$ $r = \frac{1}{2}$, $r = -1$, so the general solution is $y = c_1 e^{x/2} + c_2 e^{-x}$. We graph the basic solutions $f(x) = e^{x/2}$, $g(x) = e^{-x}$ as well as $y = 2e^{x/2} + e^{-x}$, $y = -e^{x/2} - 2e^{-x}$, and $y = e^{x/2} - e^{-x}$. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \to \pm\infty$.



17. $r^2 + 3 = 0 \quad \Rightarrow \quad r = \pm\sqrt{3}\,i$ and the general solution is

$y = e^{0x}\left[c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)\right] = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$. Then $y(0) = 1 \quad \Rightarrow \quad c_1 = 1$ and, since $y' = -\sqrt{3}\,c_1 \sin\left(\sqrt{3}\,x\right) + \sqrt{3}\,c_2 \cos\left(\sqrt{3}\,x\right)$, $y'(0) = 3 \quad \Rightarrow \quad \sqrt{3}\,c_2 = 3 \quad \Rightarrow \quad c_2 = \frac{3}{\sqrt{3}} = \sqrt{3}$, so the solution to the initial-value problem is $y = \cos\left(\sqrt{3}\,x\right) + \sqrt{3}\,\sin\left(\sqrt{3}\,x\right)$.

18. $r^2 - 2r - 3 = (r-3)(r+1) = 0$, so $r = 3$, $r = -1$ and the general solution is $y = c_1 e^{3x} + c_2 e^{-x}$. Then $y' = 3c_1 e^{3x} - c_2 e^{-x}$, so $y(0) = 2 \quad \Rightarrow \quad c_1 + c_2 = 2$ and $y'(0) = 2 \quad \Rightarrow \quad 3c_1 - c_2 = 2$, giving $c_1 = 1$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = e^{3x} + e^{-x}$.

19. $9r^2 + 12r + 4 = (3r+2)^2 = 0 \quad \Rightarrow \quad r = -\frac{2}{3}$ and the general solution is $y = c_1 e^{-2x/3} + c_2 x e^{-2x/3}$. Then $y(0) = 1 \quad \Rightarrow$ $c_1 = 1$ and, since $y' = -\frac{2}{3}c_1 e^{-2x/3} + c_2\left(1 - \frac{2}{3}x\right)e^{-2x/3}$, $y'(0) = 0 \quad \Rightarrow \quad -\frac{2}{3}c_1 + c_2 = 0$, so $c_2 = \frac{2}{3}$ and the solution to the initial-value problem is $y = e^{-2x/3} + \frac{2}{3}xe^{-2x/3}$.

20. $3r^2 - 2r - 1 = (3r+1)(r-1) = 0 \quad \Rightarrow \quad r = -\frac{1}{3}, r = 1$ and the general solution is $y = c_1 e^{-x/3} + c_2 e^x$. Then $y' = -\frac{1}{3}c_1 e^{-x/3} + c_2 e^x$, so $y(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$ and $y'(0) = -4 \quad \Rightarrow \quad -\frac{1}{3}c_1 + c_2 = -4$, giving $c_1 = 3$ and $c_2 = -3$. Thus the solution to the initial-value problem is $y = 3e^{-x/3} - 3e^x$.

21. $r^2 - 6r + 10 = 0 \quad \Rightarrow \quad r = 3 \pm i$ and the general solution is $y = e^{3x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $3 = y'(0) = c_2 + 3c_1 \quad \Rightarrow \quad c_2 = -3$ and the solution to the initial-value problem is $y = e^{3x}(2\cos x - 3\sin x)$.

22. $4r^2 - 20r + 25 = (2r-5)^2 = 0 \quad \Rightarrow \quad r = \frac{5}{2}$ and the general solution is $y = c_1 e^{5x/2} + c_2 x e^{5x/2}$. Then $2 = y(0) = c_1$ and $-3 = y'(0) = \frac{5}{2}c_1 + c_2 \quad \Rightarrow \quad c_2 = -8$. The solution to the initial-value problem is $y = 2e^{5x/2} - 8xe^{5x/2}$.

23. $r^2 - r - 12 = (r-4)(r+3) = 0 \quad \Rightarrow \quad r = 4, r = -3$ and the general solution is $y = c_1 e^{4x} + c_2 e^{-3x}$. Then $0 = y(1) = c_1 e^4 + c_2 e^{-3}$ and $1 = y'(1) = 4c_1 e^4 - 3c_2 e^{-3}$ so $c_1 = \frac{1}{7}e^{-4}$, $c_2 = -\frac{1}{7}e^3$ and the solution to the initial-value problem is $y = \frac{1}{7}e^{-4}e^{4x} - \frac{1}{7}e^3 e^{-3x} = \frac{1}{7}e^{4x-4} - \frac{1}{7}e^{3-3x}$.

24. $4r^2 + 4r + 3 = 0 \quad \Rightarrow \quad r = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}i$ and the general solution is $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x\right)$. Then $0 = y(0) = c_1$ and $1 = y'(0) = \frac{\sqrt{2}}{2}c_2 - \frac{1}{2}c_1 \quad \Rightarrow \quad c_2 = \sqrt{2}$ and the solution to the initial-value problem is $y = e^{-x/2}\left(0 + \sqrt{2}\sin \frac{\sqrt{2}}{2}x\right) = \sqrt{2}\,e^{-x/2}\sin \frac{\sqrt{2}}{2}x$.

25. $r^2 + 16 = 0 \quad \Rightarrow \quad r = \pm 4i$ and the general solution is $y = c_1 \cos 4x + c_2 \sin 4x$. Then $-3 = y(0) = c_1$ and $2 = y(\pi/8) = c_2$, so the solution of the boundary-value problem is $y = -3\cos 4x + 2\sin 4x$.

26. $r^2 + 6r = r(r+6) = 0 \quad \Rightarrow \quad r = 0, r = -6$ and the general solution is $y = c_1 + c_2 e^{-6x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(1) = c_1 + c_2 e^{-6}$ so $c_1 = \dfrac{1}{1-e^6}$, $c_2 = -\dfrac{e^6}{1-e^6}$. The solution of the boundary-value problem is

$$y = \frac{1}{1-e^6} - \frac{e^6}{1-e^6} \cdot e^{-6x} = \frac{1}{1-e^6} - \frac{e^{6-6x}}{1-e^6}.$$

27. $r^2 + 4r + 4 = (r+2)^2 = 0 \quad \Rightarrow \quad r = -2$ and the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$. Then $2 = y(0) = c_1$ and $0 = y(1) = c_1 e^{-2} + c_2 e^{-2}$ so $c_2 = -2$, and the solution of the boundary-value problem is $y = 2e^{-2x} - 2xe^{-2x}$.

28. $r^2 - 8r + 17 = 0 \quad \Rightarrow \quad r = 4 \pm i$ and the general solution is $y = e^{4x}(c_1 \cos x + c_2 \sin x)$. But $3 = y(0) = c_1$ and $2 = y(\pi) = -c_1 e^{4\pi} \quad \Rightarrow \quad c_1 = -2/e^{4\pi}$, so there is no solution.

29. $r^2 - r = r(r-1) = 0 \quad \Rightarrow \quad r = 0, r = 1$ and the general solution is $y = c_1 + c_2 e^x$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e$ so $c_1 = \dfrac{e-2}{e-1}$, $c_2 = \dfrac{1}{e-1}$. The solution of the boundary-value problem is $y = \dfrac{e-2}{e-1} + \dfrac{e^x}{e-1}$.

30. $4r^2 - 4r + 1 = (2r-1)^2 = 0 \quad \Rightarrow \quad r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $4 = y(0) = c_1$ and $0 = y(2) = c_1 e + 2c_2 e \quad \Rightarrow \quad c_2 = -2$. The solution of the boundary-value problem is $y = 4e^{x/2} - 2xe^{x/2}$.

31. $r^2 + 4r + 20 = 0 \quad \Rightarrow \quad r = -2 \pm 4i$ and the general solution is $y = e^{-2x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1 e^{-2\pi} \quad \Rightarrow \quad c_1 = 2e^{2\pi}$, so there is no solution.

32. $r^2 + 4r + 20 = 0 \quad \Rightarrow \quad r = -2 \pm 4i$ and the general solution is $y = e^{-2x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $e^{-2\pi} = y(\pi) = c_1 e^{-2\pi} \quad \Rightarrow \quad c_1 = 1$, so c_2 can vary and the solution of the boundary-value problem is $y = e^{-2x}(\cos 4x + c \sin 4x)$, where c is any constant.

33. (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\quad \Rightarrow$ $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ $(*)$ and $0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$. Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\,L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,L = n\pi$ where n is an integer $\quad \Rightarrow \quad \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $\lim_{x\to\infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $r = -b/(2a) < 0$ since a, b are positive. Hence $\lim_{x\to\infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ where $\alpha = -b/(2a) < 0$ since a and b are positive. Thus $\lim_{x\to\infty} y(x) = 0$.

35. (a) $r^2 - 2r + 2 = 0 \quad \Rightarrow \quad r = 1 \pm i$ and the general solution is $y = e^x\,(c_1 \cos x + c_2 \sin x)$. If $y(a) = c$ and $y(b) = d$ then $e^a\,(c_1 \cos a + c_2 \sin a) = c \quad \Rightarrow \quad c_1 \cos a + c_2 \sin a = ce^{-a}$ and $e^b\,(c_1 \cos b + c_2 \sin b) = d \quad \Rightarrow$ $c_1 \cos b + c_2 \sin b = de^{-b}$. This gives a linear system in c_1 and c_2 which has a unique solution if the lines are not parallel. If the lines are not vertical or horizontal, we have parallel lines if $\cos a = k \cos b$ and $\sin a = k \sin b$ for some nonzero constant k or $\dfrac{\cos a}{\cos b} = k = \dfrac{\sin a}{\sin b} \quad \Rightarrow \quad \dfrac{\sin a}{\cos a} = \dfrac{\sin b}{\cos b} \quad \Rightarrow \quad \tan a = \tan b \quad \Rightarrow \quad b - a = n\pi$, n any integer. (Note that none of $\cos a$, $\cos b$, $\sin a$, $\sin b$ are zero.) If the lines are both horizontal then $\cos a = \cos b = 0 \quad \Rightarrow \quad b - a = n\pi$, and similarly vertical lines means $\sin a = \sin b = 0 \quad \Rightarrow \quad b - a = n\pi$. Thus the system has a unique solution if $b - a \neq n\pi$.

(b) The linear system has no solution if the lines are parallel but not identical. From part (a) the lines are parallel if $b-a=n\pi$. If the lines are not horizontal, they are identical if $ce^{-a}=kde^{-b} \Rightarrow \dfrac{ce^{-a}}{de^{-b}}=k=\dfrac{\cos a}{\cos b} \Rightarrow \dfrac{c}{d}=e^{a-b}\dfrac{\cos a}{\cos b}$. (If $d=0$ then $c=0$ also.) If they are horizontal then $\cos b=0$, but $k=\dfrac{\sin a}{\sin b}$ also (and $\sin b\neq 0$) so we require $\dfrac{c}{d}=e^{a-b}\dfrac{\sin a}{\sin b}$. Thus the system has no solution if $b-a=n\pi$ and $\dfrac{c}{d}\neq e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b=0$, in which case $\dfrac{c}{d}\neq e^{a-b}\dfrac{\sin a}{\sin b}$.

(c) The linear system has infinitely many solution if the lines are identical (and necessarily parallel). From part (b) this occurs when $b-a=n\pi$ and $\dfrac{c}{d}=e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b=0$, in which case $\dfrac{c}{d}=e^{a-b}\dfrac{\sin a}{\sin b}$.

17.2 Nonhomogeneous Linear Equations

1. The auxiliary equation is $r^2+2r-8=(r-2)(r+4)=0 \Rightarrow r=2, r=-4$, so the complementary solution is $y_c(x)=c_1e^{2x}+c_2e^{-4x}$. We try the particular solution $y_p(x)=Ax^2+Bx+C$, so $y_p'=2Ax+B$ and $y_p''=2A$. Substituting into the differential equation, we have $(2A)+2(2Ax+B)-8(Ax^2+Bx+C)=1-2x^2$ or $-8Ax^2+(4A-8B)x+(2A+2B-8C)=-2x^2+1$. Comparing coefficients gives $-8A=-2 \Rightarrow A=\frac{1}{4}$, $4A-8B=0 \Rightarrow B=\frac{1}{8}$, and $2A+2B-8C=1 \Rightarrow C=-\frac{1}{32}$, so the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^{-4x}+\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{32}$.

2. The auxiliary equation is $r^2-3r=r(r-3)=0 \Rightarrow r=0, r=3$, so the complementary solution is $y_c(x)=c_1+c_2e^{3x}$. We try the particular solution $y_p(x)=A\cos 2x+B\sin 2x$, so $y_p'=-2A\sin 2x+2B\cos 2x$ and $y_p''=-4A\cos 2x-4B\sin 2x$. Substitution into the differential equation gives $(-4A\cos 2x-4B\sin 2x)-3(-2A\sin 2x+2B\cos 2x)=\sin 2x \Rightarrow (-4A-6B)\cos 2x+(6A-4B)\sin 2x=\sin 2x$. Then $-4A-6B=0$ and $6A-4B=1 \Rightarrow A=\frac{3}{26}$ and $B=-\frac{1}{13}$. Thus the general solution is $y(x)=y_c(x)+y_p(x)=c_1+c_2e^{3x}+\frac{3}{26}\cos 2x-\frac{1}{13}\sin 2x$.

3. The auxiliary equation is $9r^2+1=0$ with roots $r=\pm\frac{1}{3}i$, so the complementary solution is $y_c(x)=c_1\cos(x/3)+c_2\sin(x/3)$. Try the particular solution $y_p(x)=Ae^{2x}$, so $y_p'=2Ae^{2x}$ and $y_p''=4Ae^{2x}$. Substitution into the differential equation gives $9\left(4Ae^{2x}\right)+\left(Ae^{2x}\right)=e^{2x}$ or $37Ae^{2x}=e^{2x}$. Thus $37A=1 \Rightarrow A=\frac{1}{37}$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1\cos(x/3)+c_2\sin(x/3)+\frac{1}{37}e^{2x}$.

4. The auxiliary equation is $r^2-2r+2=0$ with roots $r=1\pm i$, so the complementary solution is $y_c(x)=e^x(c_1\cos x+c_2\sin x)$. Try the particular solution $y_p(x)=Ax+B+Ce^x$, so $y_p'=A+Ce^x$ and $y_p''=Ce^x$. Substitution into the differential equation gives $(Ce^x)-2(A+Ce^x)+2(Ax+B+Ce^x)=x+e^x \Rightarrow$

$2Ax + (-2A + 2B) + Ce^x = x + e^x$. Comparing coefficients, we have $2A = 1 \Rightarrow A = \frac{1}{2}$, $-2A + 2B = 0 \Rightarrow$ $B = \frac{1}{2}$, and $C = 1$, so the general solution is $y(x) = y_c(x) + y_p(x) = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}x + \frac{1}{2} + e^x$.

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$.

6. The auxiliary equation is $r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow r = 2$, so the complementary solution is $y_c(x) = c_1 e^{2x} + c_2 x e^{2x}$. For $y'' - 4y' + 4y = x$ try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}' = A$ and $y_{p_1}'' = 0$, and substitution into the differential equation gives $0 - 4A + 4(Ax + B) = x$ or $4Ax + (4B - 4A) = x$, so $4A = 1 \Rightarrow A = \frac{1}{4}$ and $4B - 4A = 0 \Rightarrow B = \frac{1}{4}$. Thus $y_{p_1}(x) = \frac{1}{4}x + \frac{1}{4}$. For $y'' - 4y' + 4y = -\sin x$ try $y_{p_2}(x) = A\cos x + B \sin x$. Then $y_{p_2}' = -A \sin x + B \cos x$ and $y_{p_2}'' = -A \cos x - B \sin x$. Substituting, we have
$(-A\cos x - B \sin x) - 4(-A \sin x + B \cos x) + 4(A \cos x + B \sin x) = -\sin x \Rightarrow$
$(3A - 4B)\cos x + (4A + 3B)\sin x = -\sin x$. Thus $3A - 4B = 0$ and $4A + 3B = -1$,
giving $A = -\frac{4}{25}$ and $B = -\frac{3}{25}$, so $y_{p_2}(x) = -\frac{4}{25}\cos x - \frac{3}{25}\sin x$. The general solution is
$y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{4}x + \frac{1}{4} - \frac{4}{25}\cos x - \frac{3}{25}\sin x$.

7. The auxiliary equation is $r^2 - 2r + 5 = 0$ with roots $r = 1 \pm 2i$, so the complementary solution is $y_c(x) = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Try the particular solution $y_p(x) = A\cos x + B \sin x$, so $y_p' = -A\sin x + B \cos x$ and $y_p'' = -A\cos x - B \sin x$. Substituting, we have
$(-A\cos x - B\sin x) - 2(-A \sin x + B \cos x) + 5(A\cos x + B \sin x) = \sin x \Rightarrow$
$(4A - 2B)\cos x + (2A + 4B)\sin x = \sin x$. Then $4A - 2B = 0$, $2A + 4B = 1 \Rightarrow A = \frac{1}{10}$, $B = \frac{1}{5}$ and the general solution is $y(x) = y_c(x) + y_p(x) = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{10}\cos x + \frac{1}{5}\sin x$. But $1 = y(0) = c_1 + \frac{1}{10} \Rightarrow c_1 = \frac{9}{10}$ and $1 = y'(0) = 2c_2 + c_1 + \frac{1}{5} \Rightarrow c_2 = -\frac{1}{20}$. Thus the solution to the initial-value problem is
$y(x) = e^x\left(\frac{9}{10}\cos 2x - \frac{1}{20}\sin 2x\right) + \frac{1}{10}\cos x + \frac{1}{5}\sin x$.

8. The auxiliary equation is $r^2 - 1 = 0$ with roots $r = \pm 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 e^{-x}$. Try the particular solution $y_p(x) = (Ax + B)e^{2x}$, so $y_p' = (2Ax + A + 2B)e^{2x}$ and $y_p'' = (4Ax + 4A + 4B)e^{2x}$. Substituting, we have $(4Ax + 4A + 4B)e^{2x} - (Ax + B)e^{2x} = xe^{2x} \Rightarrow (3Ax + 4A + 3B)e^{2x} = xe^{2x}$. Then $3A = 1 \Rightarrow A = \frac{1}{3}$ and $4A + 3B = 0 \Rightarrow B = -\frac{4}{9}$, and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 e^{-x} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^{2x}$. But $0 = y(0) = c_1 + c_2 - \frac{4}{9}$ and $1 = y'(0) = c_1 - c_2 - \frac{5}{9} \Rightarrow c_1 = 1$, $c_2 = -\frac{5}{9}$. Thus the solution to the initial-value problem is $y(x) = e^x - \frac{5}{9}e^{-x} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^{2x}$.

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0, r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2e^x$.

Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then

$y_p' = (Ax^2 + (2A + B)x + B)e^x$ and $y_p'' = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \quad \Rightarrow \quad (2Ax + (2A + B))e^x = xe^x \quad \Rightarrow$ $A = \frac{1}{2}, B = -1$. Thus $y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x$ and the general solution is $y(x) = c_1 + c_2e^x + \left(\frac{1}{2}x^2 - x\right)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right)$.

10. $y_c(x) = c_1e^x + c_2e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}' = A$, $y_{p_1}'' = 0$, and substitution gives $0 + A - 2(Ax + B) = x \quad \Rightarrow \quad A = -\frac{1}{2}, B = -\frac{1}{4}$, so $y_{p_1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try $y_{p_2}(x) = A\cos 2x + B\sin 2x$. Then $y_{p_2}' = -2A\sin 2x + 2B\cos 2x$, $y_{p_2}'' = -4A\cos 2x - 4B\sin 2x$, and substitution gives $(-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = \sin 2x \quad \Rightarrow \quad A = -\frac{1}{20}$, $B = -\frac{3}{20}$. Thus $y_{p_2}(x) = -\frac{1}{20}\cos 2x + -\frac{3}{20}\sin 2x$ and the general solution is

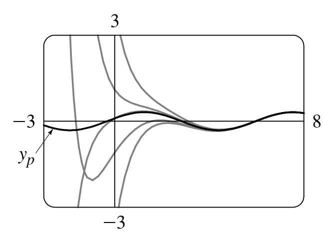
$y(x) = c_1e^x + c_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and $0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \quad \Rightarrow \quad c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is

$y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$.

11. The auxiliary equation is $r^2 + 3r + 2 = (r + 1)(r + 2) = 0$, so $r = -1, r = -2$ and $y_c(x) = c_1e^{-x} + c_2e^{-2x}$.

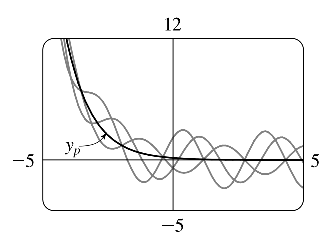
Try $y_p = A\cos x + B\sin x \quad \Rightarrow \quad y_p' = -A\sin x + B\cos x$, $y_p'' = -A\cos x - B\sin x$. Substituting into the differential equation gives $(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \cos x$ or

$(A + 3B)\cos x + (-3A + B)\sin x = \cos x$. Then solving the equations $A + 3B = 1, -3A + B = 0$ gives $A = \frac{1}{10}$, $B = \frac{3}{10}$ and the general solution is $y(x) = c_1e^{-x} + c_2e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$. The graph shows y_p and several other solutions. Notice that all solutions are asymptotic to y_p as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.



12. The auxiliary equation is $r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$, so $y_c(x) = c_1\cos 2x + c_2\sin 2x$. Try $y_p = Ae^{-x} \quad \Rightarrow$ $y_p' = -Ae^{-x}$, $y_p'' = Ae^{-x}$. Substituting into the differential equation gives $Ae^{-x} + 4Ae^{-x} = e^{-x} \quad \Rightarrow$

$5A = 1 \quad \Rightarrow \quad A = \frac{1}{5}$, so $y_p = \frac{1}{5}e^{-x}$ and the general solution is $y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{5}e^{-x}$. We graph y_p along with several other solutions. All of the solutions except y_p oscillate around $y_p = \frac{1}{5}e^{-x}$, and all solutions approach ∞ as $x \to -\infty$.



13. Here $y_c(x) = c_1 e^{2x} + c_2 e^{-x}$, and a trial solution is $y_p(x) = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$.

14. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = \cos 4x$ try $y_{p_1}(x) = A\cos 4x + B\sin 4x$ and for $y'' + 4y = \cos 2x$ try $y_{p_2}(x) = x\,(C\cos 2x + D\sin 2x)$ (so that no term of y_{p_2} is a solution of the complementary equation). Thus a trial solution is $y_p(x) = y_{p_1}(x) + y_{p_2}(x) = A\cos 4x + B\sin 4x + Cx\cos 2x + Dx\sin 2x$.

15. Here $y_c(x) = c_1 e^{2x} + c_2 e^x$. For $y'' - 3y' + 2y = e^x$ try $y_{p_1}(x) = Axe^x$ (since $y = Ae^x$ is a solution of the complementary equation) and for $y'' - 3y' + 2y = \sin x$ try $y_{p_2}(x) = B\cos x + C\sin x$. Thus a trial solution is $y_p(x) = y_{p_1}(x) + y_{p_2}(x) = Axe^x + B\cos x + C\sin x$.

16. Since $y_c(x) = c_1 e^x + c_2 e^{-4x}$ try $y_p(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.

17. Since $y_c(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$ we try $y_p(x) = x(Ax^2 + Bx + C)e^{-x}\cos 3x + x(Dx^2 + Ex + F)e^{-x}\sin 3x$ (so that no term of y_p is a solution of the complementary equation).

18. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = e^{3x}$ try $y_{p_1}(x) = Ae^{3x}$ and for $y'' + 4y = x\sin 2x$ try $y_{p_2}(x) = x(Bx + C)\cos 2x + x(Dx + E)\sin 2x$ (so that no term of y_{p_2} is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1' = -\frac{Gy_2}{a\,(y_1y_2' - y_2y_1')} \qquad \text{and} \qquad u_2' = \frac{Gy_1}{a\,(y_1y_2' - y_2y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) Here $4r^2 + 1 = 0 \;\Rightarrow\; r = \pm\frac{1}{2}i$ and $y_c(x) = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right)$. We try a particular solution of the form $y_p(x) = A\cos x + B\sin x \;\Rightarrow\; y_p' = -A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$. Then the equation $4y'' + y = \cos x$ becomes $4(-A\cos x - B\sin x) + (A\cos x + B\sin x) = \cos x$ or $-3A\cos x - 3B\sin x = \cos x \;\Rightarrow\; A = -\frac{1}{3}, B = 0$. Thus, $y_p(x) = -\frac{1}{3}\cos x$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right) - \frac{1}{3}\cos x$.

(b) From (a) we know that $y_c(x) = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}$. Setting $y_1 = \cos\frac{x}{2}$, $y_2 = \sin\frac{x}{2}$, we have $y_1y_2' - y_2y_1' = \frac{1}{2}\cos^2\frac{x}{2} + \frac{1}{2}\sin^2\frac{x}{2} = \frac{1}{2}$. Thus $u_1' = -\dfrac{\cos x\,\sin\frac{x}{2}}{4\cdot\frac{1}{2}} = -\frac{1}{2}\cos\left(2\cdot\frac{x}{2}\right)\sin\frac{x}{2} = -\frac{1}{2}\left(2\cos^2\frac{x}{2} - 1\right)\sin\frac{x}{2}$ and $u_2' = \dfrac{\cos x\,\cos\frac{x}{2}}{4\cdot\frac{1}{2}} = \frac{1}{2}\cos\left(2\cdot\frac{x}{2}\right)\cos\frac{x}{2} = \frac{1}{2}\left(1 - 2\sin^2\frac{x}{2}\right)\cos\frac{x}{2}$. Then

$u_1(x) = \int\left(\frac{1}{2}\sin\frac{x}{2} - \cos^2\frac{x}{2}\,\sin\frac{x}{2}\right)dx = -\cos\frac{x}{2} + \frac{2}{3}\cos^3\frac{x}{2}$ and

$u_2(x) = \int\left(\frac{1}{2}\cos\frac{x}{2} - \sin^2\frac{x}{2}\,\cos\frac{x}{2}\right)dx = \sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2}$. Thus

$$\begin{aligned} y_p(x) &= \left(-\cos\tfrac{x}{2} + \tfrac{2}{3}\cos^3\tfrac{x}{2}\right)\cos\tfrac{x}{2} + \left(\sin\tfrac{x}{2} - \tfrac{2}{3}\sin^3\tfrac{x}{2}\right)\sin\tfrac{x}{2} = -\left(\cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2}\right) + \tfrac{2}{3}\left(\cos^4\tfrac{x}{2} - \sin^4\tfrac{x}{2}\right) \\ &= -\cos\left(2\cdot\tfrac{x}{2}\right) + \tfrac{2}{3}\left(\cos^2\tfrac{x}{2} + \sin^2\tfrac{x}{2}\right)\left(\cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2}\right) = -\cos x + \tfrac{2}{3}\cos x = -\tfrac{1}{3}\cos x \end{aligned}$$

and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2} - \frac{1}{3}\cos x$.

20. (a) Here $r^2 - 2r - 3 = (r-3)(r+1) = 0 \quad \Rightarrow \quad r = 3, r = -1$ and the complementary solution is

$y_c(x) = c_1 e^{3x} + c_2 e^{-x}$. A particular solution is of the form $y_p(x) = Ax + B \quad \Rightarrow \quad y_p' = A, y_p'' = 0$, and substituting into the differential equation gives $0 - 2A - 3(Ax + B) = x + 2$ or $-3Ax + (-2A - 3B) = x + 2$, so $A = -\frac{1}{3}$ and $-2A - 3B = 2 \quad \Rightarrow \quad B = -\frac{4}{9}$. Thus $y_p(x) = -\frac{1}{3}x - \frac{4}{9}$ and the general solution is

$y(x) = y_c(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3}x - \frac{4}{9}$.

(b) In (a), $y_c(x) = c_1 e^{3x} + c_2 e^{-x}$, so set $y_1 = e^{3x}$, $y_2 = e^{-x}$. Then $y_1 y_2' - y_2 y_1' = -e^{3x}e^{-x} - 3e^{3x}e^{-x} = -4e^{2x}$ so

$u_1' = -\dfrac{(x+2)e^{-x}}{-4e^{2x}} = \frac{1}{4}(x+2)e^{-3x} \quad \Rightarrow \quad u_1(x) = \frac{1}{4}\int (x+2)e^{-3x}\,dx = \frac{1}{4}\left[-\frac{1}{3}(x+2)e^{-3x} - \frac{1}{9}e^{-3x}\right]$ [by parts]

and $u_2' = \dfrac{(x+2)e^{3x}}{-4e^{2x}} = -\frac{1}{4}(x+2)e^{x} \quad \Rightarrow \quad u_2(x) = -\frac{1}{4}\int (x+2)e^{x}\,dx = -\frac{1}{4}[(x+2)e^{x} - e^{x}]$ [by parts].

Hence $y_p(x) = \frac{1}{4}\left[\left(-\frac{1}{3}x - \frac{7}{9}\right)e^{-3x}\right]e^{3x} - \frac{1}{4}[(x+1)e^{x}]\,e^{-x} = -\frac{1}{3}x - \frac{4}{9}$ and

$y(x) = y_c(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3}x - \frac{4}{9}$.

21. (a) $r^2 - 2r + 1 = (r-1)^2 = 0 \quad \Rightarrow \quad r = 1$, so the complementary solution is $y_c(x) = c_1 e^{x} + c_2 x e^{x}$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \quad \Rightarrow \quad Ae^{2x} = e^{2x} \quad \Rightarrow \quad A = 1 \quad \Rightarrow \quad y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{x} + c_2 x e^{x} + e^{2x}$.

(b) From (a), $y_c(x) = c_1 e^{x} + c_2 x e^{x}$, so set $y_1 = e^{x}$, $y_2 = xe^{x}$. Then, $y_1 y_2' - y_2 y_1' = e^{2x}(1+x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^{x} \quad \Rightarrow \quad u_1(x) = -\int xe^{x}\,dx = -(x-1)e^{x}$ [by parts] and $u_2' = e^{x} \quad \Rightarrow \quad u_2(x) = \int e^{x}\,dx = e^{x}$. Hence $y_p(x) = (1-x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{x} + c_2 x e^{x} + e^{2x}$.

22. (a) Here $r^2 - r = r(r-1) = 0 \quad \Rightarrow \quad r = 0, 1$ and $y_c(x) = c_1 + c_2 e^{x}$ and so we try a particular solution of the form $y_p(x) = Axe^{x}$. Thus, after calculating the necessary derivatives, we get $y'' - y' = e^{x} \quad \Rightarrow$

$Ae^{x}(2+x) - Ae^{x}(1+x) = e^{x} \quad \Rightarrow \quad A = 1$. Thus $y_p(x) = xe^{x}$ and the general solution is $y(x) = c_1 + c_2 e^{x} + xe^{x}$.

(b) From (a) we know that $y_c(x) = c_1 + c_2 e^{x}$, so setting $y_1 = 1$, $y_2 = e^{x}$, then $y_1 y_2' - y_2 y_1' = e^{x} - 0 = e^{x}$. Thus $u_1' = -e^{2x}/e^{x} = -e^{x}$ and $u_2' = e^{x}/e^{x} = 1$. Then $u_1(x) = -\int e^{x}\,dx = -e^{x}$ and $u_2(x) = x$. Thus $y_p(x) = -e^{x} + xe^{x}$ and the general solution is $y(x) = c_1 + c_2 e^{x} - e^{x} + xe^{x} = c_1 + c_3 e^{x} + xe^{x}$.

23. As in Example 5, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$,

so $u_1' = -\dfrac{\sec^2 x\,\cos x}{-1} = \sec x \quad \Rightarrow \quad u_1(x) = \int \sec x\,dx = \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$,

and $u_2' = \dfrac{\sec^2 x\,\sin x}{-1} = -\sec x\,\tan x \quad \Rightarrow \quad u_2(x) = -\sec x$. Hence

$y_p(x) = \ln(\sec x + \tan x) \cdot \sin x - \sec x \cdot \cos x = \sin x \ln(\sec x + \tan x) - 1$ and the general solution is

$y(x) = c_1 \sin x + c_2 \cos x + \sin x\,\ln(\sec x + \tan x) - 1$.

24. As in Exercise 23, $y_c(x) = c_1 \sin x + c_2 \cos x$, $y_1 = \sin x$, $y_2 = \cos x$, and $y_1 y_2' - y_2 y_1' = -1$. Then

$u_1' = -\dfrac{\sec^3 x \, \cos x}{-1} = \sec^2 x \quad \Rightarrow \quad u_1(x) = \tan x$ and $u_2' = \dfrac{\sec^3 x \, \sin x}{-1} = -\sec^2 x \tan x \quad \Rightarrow$

$u_2(x) = -\int \tan x \, \sec^2 x \, dx = -\frac{1}{2} \tan^2 x$. Hence

$y_p(x) = \tan x \, \sin x - \frac{1}{2} \tan^2 x \, \cos x = \tan x \, \sin x - \frac{1}{2} \tan x \, \sin x = \frac{1}{2} \tan x \, \sin x$ and the general solution

is $y(x) = c_1 \sin x + c_2 \cos x + \frac{1}{2} \tan x \, \sin x$.

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1+e^{-x}}$ and

$u_1(x) = \displaystyle\int -\frac{e^{-x}}{1+e^{-x}} \, dx = \ln(1+e^{-x})$. $\quad u_2' = \dfrac{e^x}{(1+e^{-x})e^{3x}} = \dfrac{e^x}{e^{3x}+e^{2x}}$ so

$u_2(x) = \displaystyle\int \frac{e^x}{e^{3x}+e^{2x}} \, dx = \ln\!\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}$. Hence

$y_p(x) = e^x \ln(1+e^{-x}) + e^{2x}[\ln(1+e^{-x}) - e^{-x}]$ and the general solution is

$y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}$.

26. $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1 y_2' - y_2 y_1' = -e^{-3x}$. So $u_1' = -\dfrac{(\sin e^x)e^{-2x}}{-e^{-3x}} = e^x \sin e^x$

and $u_2' = \dfrac{(\sin e^x)e^{-x}}{-e^{-3x}} = -e^{2x} \sin e^x$. Hence $u_1(x) = \int e^x \sin e^x dx = -\cos e^x$ and

$u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x$. Then $y_p(x) = -e^{-x} \cos e^x - e^{-2x}[\sin e^x - e^x \cos e^x]$

and the general solution is $y(x) = (c_1 - \cos e^x)e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]e^{-2x}$.

27. $r^2 - 2r + 1 = (r-1)^2 = 0 \quad \Rightarrow \quad r = 1$ so $y_c(x) = c_1 e^x + c_2 x e^x$. Thus $y_1 = e^x$, $y_2 = xe^x$ and

$y_1 y_2' - y_2 y_1' = e^x(x+1)e^x - xe^x e^x = e^{2x}$. So $u_1' = -\dfrac{xe^x \cdot e^x/(1+x^2)}{e^{2x}} = -\dfrac{x}{1+x^2} \quad \Rightarrow$

$u_1 = -\displaystyle\int \frac{x}{1+x^2} \, dx = -\frac{1}{2} \ln\left(1+x^2\right)$, $u_2' = \dfrac{e^x \cdot e^x/(1+x^2)}{e^{2x}} = \dfrac{1}{1+x^2} \quad \Rightarrow \quad u_2 = \displaystyle\int \frac{1}{1+x^2} \, dx = \tan^{-1} x$ and

$y_p(x) = -\frac{1}{2} e^x \ln(1+x^2) + xe^x \tan^{-1} x$. Hence the general solution is $y(x) = e^x \left[c_1 + c_2 x - \frac{1}{2} \ln(1+x^2) + x \tan^{-1} x\right]$.

28. $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$ and $y_1 y_2' - y_2 y_1' = e^{-4x}$. Then $u_1' = \dfrac{-e^{-2x} x e^{-2x}}{x^3 e^{-4x}} = -\dfrac{1}{x^2}$ so $u_1(x) = x^{-1}$ and

$u_2' = \dfrac{e^{-2x} e^{-2x}}{x^3 e^{-4x}} = \dfrac{1}{x^3}$ so $u_2(x) = -\dfrac{1}{2x^2}$. Thus $y_p(x) = \dfrac{e^{-2x}}{x} - \dfrac{xe^{-2x}}{2x^2} = \dfrac{e^{-2x}}{2x}$ and the general solution is

$y(x) = e^{-2x}[c_1 + c_2 x + 1/(2x)]$.

17.3 Applications of Second-Order Differential Equations

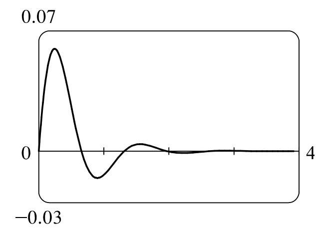
1. By Hooke's Law $k(0.25) = 25$ so $k = 100$ is the spring constant and the differential equation is $5x'' + 100x = 0$. The auxiliary equation is $5r^2 + 100 = 0$ with roots $r = \pm 2\sqrt{5}\,i$, so the general solution to the differential equation is $x(t) = c_1 \cos\left(2\sqrt{5}\,t\right) + c_2 \sin\left(2\sqrt{5}\,t\right)$. We are given that $x(0) = 0.35 \quad \Rightarrow \quad c_1 = 0.35$ and $x'(0) = 0 \quad \Rightarrow$ $2\sqrt{5}\,c_2 = 0 \quad \Rightarrow \quad c_2 = 0$, so the position of the mass after t seconds is $x(t) = 0.35\cos\left(2\sqrt{5}\,t\right)$.

2. By Hooke's Law $k(0.4) = 32$ so $k = \frac{32}{0.4} = 80$ is the spring constant and the differential equation is $8x'' + 80x = 0$. The general solution is $x(t) = c_1 \cos\left(\sqrt{10}\,t\right) + c_2 \sin\left(\sqrt{10}\,t\right)$. But $0 = x(0) = c_1$ and $1 = x'(0) = \sqrt{10}\,c_2 \quad \Rightarrow$ $c_2 = \frac{1}{\sqrt{10}}$, so the position of the mass after t seconds is $x(t) = \frac{1}{\sqrt{10}} \sin\left(\sqrt{10}\,t\right)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

4. (a) $k(0.25) = 13 \quad \Rightarrow \quad k = 52$, so the differential equation is $2x'' + 8x' + 52x = 0$ with general solution $x(t) = e^{-2t}\left[c_1 \cos\left(\sqrt{22}\,t\right) + c_2 \sin\left(\sqrt{22}\,t\right)\right]$. Then $0 = x(0) = c_1$ and $0.5 = x'(0) = \sqrt{22}\,c_2 \quad \Rightarrow \quad c_2 = \frac{1}{2\sqrt{22}}$, so the position is given by $x(t) = \frac{1}{2\sqrt{22}} e^{-2t} \sin\left(\sqrt{22}\,t\right)$.

(b)



5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

6. For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{2 \cdot 52} = 4\sqrt{26}$.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\dfrac{d^2x}{dt^2} + c\,\dfrac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$.

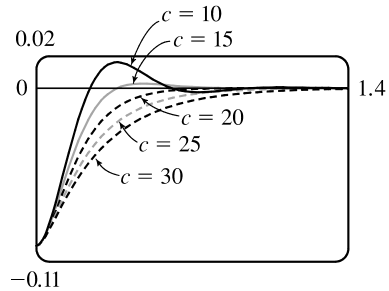
If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}\,i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos\left(5\sqrt{3}\,t\right) + c_2 \sin\left(5\sqrt{3}\,t\right)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}\,c_2 - 5c_1 \quad \Rightarrow \quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos\left(5\sqrt{3}\,t\right) - \frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}\,t\right)\right]$.

If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1 \cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x\,(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}\,c_1 \quad \Rightarrow \quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$.

For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2 t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \Rightarrow c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$.

If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1 e^{-5t} + c_2 e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \Rightarrow c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$, so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$.

If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1 e^{(-15+5\sqrt{5})t} + c_2 e^{(-15-5\sqrt{5})t}$.
Then $-0.1 = x(0) = c_1 + c_2$ and
$0 = x'(0) = (-15 + 5\sqrt{5})\, c_1 + (-15 - 5\sqrt{5})\, c_2 \Rightarrow$
$c_1 = \frac{-5-3\sqrt{5}}{100}$ and $c_2 = \frac{-5+3\sqrt{5}}{100}$, so
$x = \left(\frac{-5-3\sqrt{5}}{100}\right) e^{(-15+5\sqrt{5})t} + \left(\frac{-5+3\sqrt{5}}{100}\right) e^{(-15-5\sqrt{5})t}$.



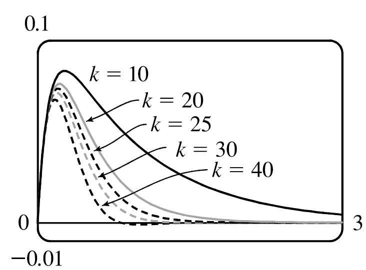
8. We are given $m = 1$, $c = 10$, $x(0) = 0$ and $x'(0) = 1$. The differential equation is $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + kx = 0$ with auxiliary equation $r^2 + 10r + k = 0$. $k = 10$: the auxiliary equation has roots $r = -5 \pm \sqrt{15}$ so we have overdamping and the solution is $x = c_1 e^{(-5+\sqrt{15})t} + c_2 e^{(-5-\sqrt{15})t}$. Entering the initial conditions gives $c_1 = \frac{1}{2\sqrt{15}}$ and $c_2 = -\frac{1}{2\sqrt{15}}$, so $x = \frac{1}{2\sqrt{15}} e^{(-5+\sqrt{15})t} - \frac{1}{2\sqrt{15}} e^{(-5-\sqrt{15})t}$.

$k = 20$: $r = -5 \pm \sqrt{5}$ and the solution is $x = c_1 e^{(-5+\sqrt{5})t} + c_2 e^{(-5-\sqrt{5})t}$ so again the motion is overdamped. The initial conditions give $c_1 = \frac{1}{2\sqrt{5}}$ and $c_2 = -\frac{1}{2\sqrt{5}}$, so $x = \frac{1}{2\sqrt{5}} e^{(-5+\sqrt{5})t} - \frac{1}{2\sqrt{5}} e^{(-5-\sqrt{5})t}$.

$k = 25$: we have equal roots $r_1 = r_2 = -5$, so the motion is critically damped and the solution is $x = (c_1 + c_2 t)e^{-5t}$. The initial conditions give $c_1 = 0$ and $c_2 = 1$, so $x = te^{-5t}$.

$k = 30$: $r = -5 \pm \sqrt{5}\, i$ so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos(\sqrt{5}\, t) + c_2 \sin(\sqrt{5}\, t)\right]$. The initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{5}}$, so $x = \frac{1}{\sqrt{5}} e^{-5t} \sin(\sqrt{5}\, t)$.

$k = 40$: $r = -5 \pm \sqrt{15}\, i$ so we again have underdamping.
The solution is $x = e^{-5t}\left[c_1 \cos(\sqrt{15}\, t) + c_2 \sin(\sqrt{15}\, t)\right]$,
and the initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{15}}$.
Thus $x = \frac{1}{\sqrt{15}} e^{-5t} \sin(\sqrt{15}\, t)$.



9. The differential equation is $mx'' + kx = F_0 \cos\omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\, i = \pm\omega i$ so $x_c(t) = c_1 \cos\omega t + c_2 \sin\omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Then we need $(m)(-\omega_0^2)(A\cos\omega_0 t + B\sin\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0 \cos\omega_0 t$ or $A(k - m\omega_0^2) = F_0$ and

$B(k - m\omega_0^2) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$.

10. As in Exercise 9, $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t(A \cos \omega t + B \sin \omega t)$. Then we need

$m(2\omega B - \omega^2 At) \cos \omega t - m(2\omega A + \omega^2 Bt) \sin \omega t + kAt \cos \omega t + kBt \sin \omega t = F_0 \cos \omega t$ or $2m\omega B = F_0$ and $-2m\omega A = 0$ [noting $-m\omega^2 A + kA = 0$ and $-m\omega^2 B + kB = 0$ since $\omega^2 = k/m$]. Hence the general solution is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + [F_0 t/(2m\omega)] \sin \omega t$.

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $g(t) = \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$. Then f is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \;\Rightarrow\; a = \frac{b\omega}{\omega_0}$ where a and b are non-zero integers. Then

$x\left(t + a \cdot \frac{2\pi}{\omega}\right) = f\left(t + a \cdot \frac{2\pi}{\omega}\right) + g\left(t + a \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + \frac{b\omega}{\omega_0} \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + b \cdot \frac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t)$

so $x(t)$ is periodic.

12. (a) The graph of $x = c_1 e^{rt} + c_2 t e^{rt}$ has a t-intercept when $c_1 e^{rt} + c_2 t e^{rt} = 0 \;\Leftrightarrow\; e^{rt}(c_1 + c_2 t) = 0 \;\Leftrightarrow\; c_1 = -c_2 t$. Since $t > 0$, x has a t-intercept if and only if c_1 and c_2 have opposite signs.

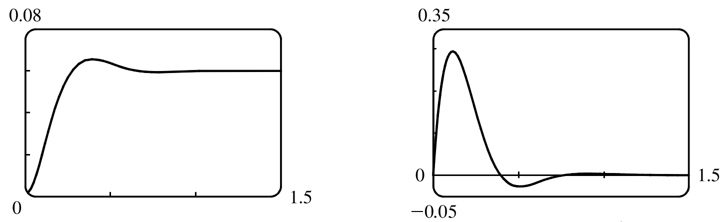
(b) For $t > 0$, the graph of x crosses the t-axis when $c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0 \;\Leftrightarrow\; c_2 e^{r_2 t} = -c_1 e^{r_1 t} \;\Leftrightarrow\; c_2 = -c_1 \dfrac{e^{r_1 t}}{e^{r_2 t}} = -c_1 e^{(r_1 - r_2)t}$. But $r_1 > r_2 \;\Rightarrow\; r_1 - r_2 > 0$ and since $t > 0$, $e^{(r_1 - r_2)t} > 1$. Thus $|c_2| = |c_1|\, e^{(r_1 - r_2)t} > |c_1|$, and the graph of x can cross the t-axis only if $|c_2| > |c_1|$.

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ and try $Q_p(t) = A \;\Rightarrow\; 500A = 12$ or $A = \frac{3}{125}$.

The general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250} e^{-10t}(6 \cos 20t + 3 \sin 20t) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right) \sin 20t$.

14. (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$. Then $Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$ and try $Q_p(t) = A \;\Rightarrow\; A = \frac{3}{50}$ and the general solution is $Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + \frac{3}{50}$. But $0.001 = Q(0) = c + \frac{3}{50}$ so $c_1 = -0.059$. Also $Q'(t) = I(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t]$ and $0 = Q'(0) = -6c_1 + 8c_2$ so $c_2 = -0.04425$. Hence the charge is $Q(t) = -e^{-6t}(0.059 \cos 8t + 0.04425 \sin 8t) + \frac{3}{50}$ and the current is $I(t) = e^{-6t}(0.7375) \sin 8t$.

(b)

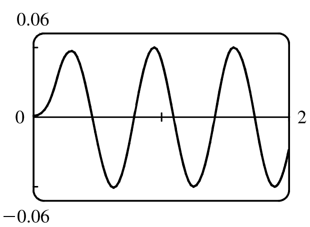


charge, $Q(t)$ current, $I(t) = Q'(t)$

15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives $(-100A + 200B + 500A) \cos 10t + (-100B - 200A + 500B) \sin 10t = 12 \sin 10t \quad \Rightarrow$ $400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$. Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ and $0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by $Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$.

16. (a) As in Exercise 14, $Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$ but try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives $(-200A + 240B + 200A) \cos 10t + (-200B - 240A + 200B) \sin 10t = 12 \sin 10t$, so $B = 0$ and $A = -\frac{1}{20}$. Hence, the general solution is $Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{20} \cos 10t$. But $0.001 = Q(0) = c_1 - \frac{1}{20}$, $Q'(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] - \frac{1}{2} \sin 10t$ and $0 = Q'(0) = -6c_1 + 8c_2$, so $c_1 = 0.051$ and $c_2 = 0.03825$. Thus the charge is given by $Q(t) = e^{-6t}(0.051 \cos 8t + 0.03825 \sin 8t) - \frac{1}{20} \cos 10t$.

(b)



17. $x(t) = A \cos(\omega t + \delta) \quad \Leftrightarrow \quad x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \quad \Leftrightarrow \quad x(t) = A\left(\dfrac{c_1}{A} \cos \omega t + \dfrac{c_2}{A} \sin \omega t\right)$ where $\cos \delta = c_1/A$ and $\sin \delta = -c_2/A \quad \Leftrightarrow \quad x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. [Note that $\cos^2 \delta + \sin^2 \delta = 1 \quad \Rightarrow \quad c_1^2 + c_2^2 = A^2$.]

18. (a) We approximate $\sin \theta$ by θ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\dfrac{d^2\theta}{dt^2} + 9.8\theta = 0$. The auxiliary equation is $r^2 + 9.8 = 0 \quad \Rightarrow \quad r = \pm\sqrt{9.8}\, i$, so the general solution is $\theta(t) = c_1 \cos\left(\sqrt{9.8}\, t\right) + c_2 \sin\left(\sqrt{9.8}\, t\right)$.

Then $0.2 = \theta(0) = c_1$ and $1 = \theta'(0) = \sqrt{9.8}\,c_2 \quad \Rightarrow \quad c_2 = \frac{1}{\sqrt{9.8}}$, so the equation is

$\theta(t) = 0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)$.

(b) $\theta'(t) = -0.2\sqrt{9.8}\sin\left(\sqrt{9.8}\,t\right) + \cos\left(\sqrt{9.8}\,t\right) = 0$ or $\tan\left(\sqrt{9.8}\,t\right) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are

$t = \frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}}\pi$ (n any integer). The maximum angle from the vertical is

$\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right) \approx 0.377$ radians (or about 21.7°).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}} \approx 2.007$ seconds.

(d) $\theta(t) = 0 \quad \Rightarrow \quad 0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right) = 0 \quad \Rightarrow \quad \tan\left(\sqrt{9.8}\,t\right) = -0.2\sqrt{9.8} \quad \Rightarrow$

$t = \frac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right) + \pi\right] \approx 0.825$ seconds.

(e) $\theta'(0.825) \approx -1.180$ rad/s.

17.4 Series Solutions

1. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum\limits_{n=1}^{\infty} nc_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$\sum\limits_{n=1}^{\infty} nc_n x^{n-1} - \sum\limits_{n=0}^{\infty} c_n x^n = 0$. Replacing n by $n+1$ in the first sum gives $\sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum\limits_{n=0}^{\infty} c_n x^n = 0$, so

$\sum\limits_{n=0}^{\infty} [(n+1)c_{n+1} - c_n]x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is

$c_{n+1} = \dfrac{c_n}{n+1}$, $n = 0, 1, 2, \ldots$. Then $c_1 = c_0$, $c_2 = \dfrac{1}{2}c_1 = \dfrac{c_0}{2}$, $c_3 = \dfrac{1}{3}c_2 = \dfrac{1}{3}\cdot\dfrac{1}{2}c_0 = \dfrac{c_0}{3!}$, $c_4 = \dfrac{1}{4}c_3 = \dfrac{c_0}{4!}$, and

in general, $c_n = \dfrac{c_0}{n!}$. Thus, the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} \dfrac{c_0}{n!}x^n = c_0 \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = c_0 e^x$.

2. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y' = xy \quad \Rightarrow \quad y' - xy = 0 \quad \Rightarrow \quad \sum\limits_{n=1}^{\infty} nc_n x^{n-1} - x\sum\limits_{n=0}^{\infty} c_n x^n = 0$ or

$\sum\limits_{n=1}^{\infty} nc_n x^{n-1} - \sum\limits_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing n with $n+1$ in the first sum and n with $n-1$ in the second

gives $\sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum\limits_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum\limits_{n=1}^{\infty} (n+1)\,c_{n+1}x^n - \sum\limits_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus,

$c_1 + \sum\limits_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}]\,x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)\,c_{n+1} - c_{n-1} = 0$. Thus, the

recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also,

$c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4 \cdot 2} = \dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6 \cdot 4 \cdot 2} = \dfrac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n \cdot n!}$. Thus, the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{2n} x^{2n} = \sum\limits_{n=0}^{\infty} \dfrac{c_0}{2^n \cdot n!}\, x^{2n} = c_0 \sum\limits_{n=0}^{\infty} \dfrac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$.

3. Assuming $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1} = \sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n$ and $-x^2 y = -\sum\limits_{n=0}^{\infty} c_n x^{n+2} = -\sum\limits_{n=2}^{\infty} c_{n-2}x^n$. Hence, the equation $y' = x^2 y$ becomes $\sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum\limits_{n=2}^{\infty} c_{n-2}x^n = 0$ or $c_1 + 2c_2 x + \sum\limits_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}]\, x^n = 0$. Equating coefficients gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{3^2 \cdot 2!}$, $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 6 \cdot 3} = \dfrac{c_0}{3^3 \cdot 3!}, \ldots$, and $c_{3n} = \dfrac{c_0}{3^n \cdot n!}$. Thus, the solution is $y\,(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{3n} x^{3n} = \sum\limits_{n=0}^{\infty} \dfrac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum\limits_{n=0}^{\infty} \dfrac{x^{3n}}{3^n n!} = c_0 \sum\limits_{n=0}^{\infty} \dfrac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}$.

4. Let $y\,(x) = \sum\limits_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad y'\,(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1} = \sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n$. Then the differential equation becomes

$$(x-3)\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad\Rightarrow\quad \sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1} - 3\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad\Rightarrow$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \quad\Rightarrow\quad \sum_{n=0}^{\infty}[(n+2)c_n - 3(n+1)c_{n+1}]x^n = 0$$

$\left[\text{since } \sum\limits_{n=1}^{\infty} n c_n x^n = \sum\limits_{n=0}^{\infty} n c_n x^n\right]$. Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = c_0 \sum\limits_{n=0}^{\infty} \dfrac{n+1}{3^n}\, x^n$.

$\left[\text{Note that } c_0 \sum\limits_{n=0}^{\infty} \dfrac{n+1}{3^n}\, x^n = \dfrac{9c_0}{(3-x)^2} \text{ for } |x| < 3.\right]$

5. Let $y\,(x) = \sum\limits_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad y'\,(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ and $y''\,(x) = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x\sum\limits_{n=1}^{\infty} n c_n x^{n-1} + \sum\limits_{n=0}^{\infty} c_n x^n = 0$ or $\sum\limits_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + n c_n + c_n]x^n = 0$ $\left[\text{since } \sum\limits_{n=1}^{\infty} n c_n x^n = \sum\limits_{n=0}^{\infty} n c_n x^n\right]$. Equating coefficients gives $(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$, thus the recursion relation is $c_{n+2} = \dfrac{-(n+1)c_n}{(n+2)(n+1)} = -\dfrac{c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Then the even coefficients are given by $c_2 = -\dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2 \cdot 4}$, $c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2 \cdot 4 \cdot 6}$, and in general,

$c_{2n} = (-1)^n \dfrac{c_0}{2 \cdot 4 \cdot \cdots \cdot 2n} = \dfrac{(-1)^n c_0}{2^n\, n!}$. The odd coefficients are $c_3 = -\dfrac{c_1}{3}$, $c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3 \cdot 5}$, $c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3 \cdot 5 \cdot 7}$,

and in general, $c_{2n+1} = (-1)^n \dfrac{c_1}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} = \dfrac{(-2)^n\, n!\, c_1}{(2n+1)!}$. The solution is

$y\,(x) = c_0 \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{2^n\, n!}\, x^{2n} + c_1 \sum\limits_{n=0}^{\infty} \dfrac{(-2)^n\, n!}{(2n+1)!}\, x^{2n+1}$.

6. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$. Hence, the equation $y'' = y$ becomes $\sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum\limits_{n=0}^{\infty} c_n x^n = 0$ or $\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - c_n]x^n = 0$. So the recursion relation is $c_{n+2} = \dfrac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2 \cdot 1}$, $c_4 = \dfrac{c_2}{4 \cdot 3} = \dfrac{c_0}{4!}$, $c_6 = \dfrac{c_4}{6 \cdot 5} = \dfrac{c_0}{6!}, \ldots,$ $c_{2n} = \dfrac{c_0}{(2n)!}$ and $c_3 = \dfrac{c_1}{3 \cdot 2}$, $c_5 = \dfrac{c_3}{5 \cdot 4} = \dfrac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \dfrac{c_1}{5!}$, $c_7 = \dfrac{c_5}{7 \cdot 6} = \dfrac{c_1}{7!}, \ldots, c_{2n+1} = \dfrac{c_1}{(2n+1)!}$. Thus, the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{2n}x^{2n} + \sum\limits_{n=0}^{\infty} c_{2n+1}x^{2n+1} = c_0 \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!} + c_1 \sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$. The solution can be written as $y(x) = c_0 \cosh x + c_1 \sinh x$ $\left[\text{or } y(x) = c_0\, \dfrac{e^x + e^{-x}}{2} + c_1\, \dfrac{e^x - e^{-x}}{2} = \dfrac{c_0 + c_1}{2}\, e^x + \dfrac{c_0 - c_1}{2}\, e^{-x}\right]$.

7. Let $y\,(x) = \sum\limits_{n=0}^{\infty} c_n x^n \;\Rightarrow\; y'\,(x) = \sum\limits_{n=1}^{\infty} nc_n x^{n-1} = \sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n$ and $y''\,(x) = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$. Then $(x-1)y''(x) = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^{n+1} - \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n = \sum\limits_{n=1}^{\infty} n(n+1)c_{n+1}x^n - \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$. Since $\sum\limits_{n=1}^{\infty} n(n+1)c_{n+1}x^n = \sum\limits_{n=0}^{\infty} n(n+1)c_{n+1}x^n$, the differential equation becomes

$$\sum_{n=0}^{\infty} n(n+1)c_{n+1}x^n - \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n = 0 \quad\Rightarrow$$

$$\sum_{n=0}^{\infty} [n(n+1)c_{n+1} - (n+2)(n+1)c_{n+2} + (n+1)c_{n+1}]x^n = 0 \;\text{ or }\; \sum_{n=0}^{\infty} [(n+1)^2 c_{n+1} - (n+2)(n+1)c_{n+2}]x^n = 0.$$

Equating coefficients gives $(n+1)^2 c_{n+1} - (n+2)(n+1)c_{n+2} = 0$ for $n = 0, 1, 2, \ldots$. Then the recursion relation is $c_{n+2} = \dfrac{(n+1)^2}{(n+2)(n+1)} c_{n+1} = \dfrac{n+1}{n+2} c_{n+1}$, so given c_0 and c_1, we have $c_2 = \frac{1}{2}c_1$, $c_3 = \frac{2}{3}c_2 = \frac{1}{3}c_1$, $c_4 = \frac{3}{4}c_3 = \frac{1}{4}c_1$, and in general $c_n = \dfrac{c_1}{n}$, $n = 1, 2, 3, \ldots$. Thus the solution is $y(x) = c_0 + c_1 \sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$. Note that the solution can be expressed as $c_0 - c_1 \ln(1-x)$ for $|x| < 1$.

8. Assuming $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and $-xy(x) = -\sum\limits_{n=0}^{\infty} c_n x^{n+1} = -\sum\limits_{n=1}^{\infty} c_{n-1}x^n$. The equation $y'' = xy$ becomes

$\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum\limits_{n=1}^{\infty}c_{n-1}x^n = 0$ or $2c_2 + \sum\limits_{n=1}^{\infty}[(n+2)(n+1)c_{n+2} - c_{n-1}]x^n = 0$. Equating coefficients gives $c_2 = 0$ and $c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \ldots$. Since $c_2 = 0$, $c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Given c_0, $c_3 = \dfrac{c_0}{3 \cdot 2}$, $c_6 = \dfrac{c_3}{6 \cdot 5} = \dfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \ldots, c_{3n} = \dfrac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdot \cdots \cdot 6 \cdot 5 \cdot 3 \cdot 2}$. Given c_1, $c_4 = \dfrac{c_1}{4 \cdot 3}$, $c_7 = \dfrac{c_4}{7 \cdot 6} = \dfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \ldots, c_{3n+1} = \dfrac{c_1}{(3n+1)3n(3n-2)(3n-3) \ldots 7 \cdot 6 \cdot 4 \cdot 3}$. The solution can be written as $y(x) = c_0 \sum\limits_{n=0}^{\infty} \dfrac{(3n-2)(3n-5) \cdot \cdots \cdot 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum\limits_{n=0}^{\infty} \dfrac{(3n-1)(3n-4) \cdot \cdots \cdot 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}$.

9. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum\limits_{n=1}^{\infty} nc_n x^{n-1} = -\sum\limits_{n=1}^{\infty} nc_n x^n = -\sum\limits_{n=0}^{\infty} nc_n x^n$,

$y''(x) = \sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$, and the equation $y'' - xy' - y = 0$ becomes

$\sum\limits_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{nc_n + c_n}{(n+2)(n+1)} = \dfrac{c_n(n+1)}{(n+2)(n+1)} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. One of the given conditions is $y(0) = 1$. But $y(0) = \sum\limits_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0$, so $c_0 = 1$. Hence, $c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2 \cdot 4}$, $c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}, \ldots$, $c_{2n} = \dfrac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But $y'(0) = \sum\limits_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1$, so $c_1 = 0$. By the recursion relation, $c_3 = \dfrac{c_1}{3} = 0$, $c_5 = 0, \ldots, c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus, the solution to the initial-value problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{2n} x^{2n} = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{2^n n!} = \sum\limits_{n=0}^{\infty} \dfrac{(x^2/2)^n}{n!} = e^{x^2/2}$.

10. Assuming that $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum\limits_{n=0}^{\infty} c_n x^{n+2}$ and

$y''(x) = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=-2}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2} = 2c_2 + 6c_3 x + \sum\limits_{n=0}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}$.

Thus, the equation $y'' + x^2 y = 0$ becomes $2c_2 + 6c_3 x + \sum\limits_{n=0}^{\infty}[(n+4)(n+3)c_{n+4} + c_n]\,x^{n+2} = 0$. So $c_2 = c_3 = 0$ and the recursion relation is $c_{n+4} = -\dfrac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation, $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \ldots$. Also, $c_0 = y(0) = 1$, so $c_4 = -\dfrac{c_0}{4 \cdot 3} = -\dfrac{1}{4 \cdot 3}$, $c_8 = -\dfrac{c_4}{8 \cdot 7} = \dfrac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3}, \ldots, c_{4n} = \dfrac{(-1)^n}{4n(4n-1)(4n-4)(4n-5) \cdot \cdots \cdot 4 \cdot 3}$. Thus, the solution to the initial-value problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = c_0 + \sum\limits_{n=0}^{\infty} c_{4n} x^{4n} = 1 + \sum\limits_{n=1}^{\infty}(-1)^n \dfrac{x^{4n}}{4n(4n-1)(4n-4)(4n-5) \cdot \cdots \cdot 4 \cdot 3}$.

11. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$, $x^2 y' = x^2 \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} nc_n x^{n+1}$,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1} \qquad \text{[replace } n \text{ with } n+3\text{]}$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1},$$

and the equation $y'' + x^2y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2)c_{n+3} + nc_n + c_n\right] x^{n+1} = 0$. So $c_2 = 0$ and the recursion relation is $c_{n+3} = \dfrac{-nc_n - c_n}{(n+3)(n+2)} = -\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Also, $c_1 = y'(0) = 1$, so $c_4 = -\dfrac{2c_1}{4 \cdot 3} = -\dfrac{2}{4 \cdot 3}$,

$c_7 = -\dfrac{5c_4}{7 \cdot 6} = (-1)^2 \dfrac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \dfrac{2^2 5^2}{7!}, \ldots, c_{3n+1} = (-1)^n \dfrac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}$. Thus, the solution is

$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \dfrac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2 x^{3n+1}}{(3n+1)!}\right]$.

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^{n+2}$,

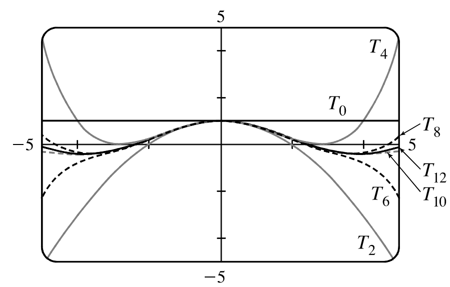
$xy'(x) = \sum_{n=1}^{\infty} nc_n x^n = \sum_{n=-1}^{\infty} (n+2)c_{n+2}x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2)c_{n+2}x^{n+2}$, and the equation

$x^2y'' + xy' + x^2y = 0$ becomes $c_1 x + \sum_{n=0}^{\infty} \{[(n+2)(n+1) + (n+2)]c_{n+2} + c_n\}x^{n+2} = 0$. So $c_1 = 0$ and the recursion relation is $c_{n+2} = -\dfrac{c_n}{(n+2)^2}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$.

Also, $c_0 = y(0) = 1$, so $c_2 = -\dfrac{1}{2^2}$, $c_4 = -\dfrac{c_2}{4^2} = (-1)^2 \dfrac{1}{4^2 2^2} = (-1)^2 \dfrac{1}{2^4\,(2!)^2}$, $c_6 = -\dfrac{c_4}{6^2} = (-1)^3 \dfrac{1}{2^6\,(3!)^2}, \ldots,$

$c_{2n} = (-1)^n \dfrac{1}{2^{2n}\,(n!)^2}$. The solution is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{2^{2n}\,(n!)^2}$.

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph. Because T_{10} and T_{12} are close together throughout the interval $[-5, 5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.



17 Review

TRUE-FALSE QUIZ

1. True. See Theorem 17.1.3.
2. False. The differential equation is not homogeneous.
3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.
4. False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

EXERCISES

1. The auxiliary equation is $4r^2 - 1 = 0 \quad\Rightarrow\quad (2r+1)(2r-1) = 0 \quad\Rightarrow\quad r = \pm\frac{1}{2}$. Then the general solution is $y = c_1 e^{x/2} + c_2 e^{-x/2}$.
2. The auxiliary equation is $r^2 - 2r + 10 = 0 \quad\Rightarrow\quad r = 1 \pm 3i$, so $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$.
3. The auxiliary equation is $r^2 + 3 = 0 \quad\Rightarrow\quad r = \pm\sqrt{3}\,i$. Then the general solution is $y = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$.
4. The auxiliary equation is $r^2 + 8r + 16 = 0 \quad\Rightarrow\quad (r+4)^2 = 0 \quad\Rightarrow\quad r = -4$, so the general solution is $y = c_1 e^{-4x} + c_2 x e^{-4x}$.
5. $r^2 - 4r + 5 = 0 \quad\Rightarrow\quad r = 2 \pm i$, so $y_c\,(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p\,(x) = Ae^{2x} \quad\Rightarrow\quad y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \quad\Rightarrow\quad A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.
6. $r^2 + r - 2 = 0 \quad\Rightarrow\quad r = 1, r = -2$ and $y_c(x) = c_1 e^x + c_2 e^{-2x}$. Try $y_p(x) = Ax^2 + Bx + C \quad\Rightarrow\quad y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitution gives $2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 \quad\Rightarrow\quad A = B = -\frac{1}{2}, C = -\frac{3}{4}$ so the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.
7. $r^2 - 2r + 1 = 0 \quad\Rightarrow\quad r = 1$ and $y_c(x) = c_1 e^x + c_2 x e^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \quad\Rightarrow$ $y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \quad\Rightarrow\quad A = 0, B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2}\cos x - \frac{1}{2}(x+1)\sin x$.
8. $r^2 + 4 = 0 \quad\Rightarrow\quad r = \pm 2i$ and $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p(x) = Ax\cos 2x + Bx\sin 2x$ so that no term of y_p is a solution of the complementary equation. Then $y_p' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$ and $y_p'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$. Substitution gives $4B\cos 2x - 4A\sin 2x = \sin 2x \quad\Rightarrow$ $A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x\cos 2x$.

9. $r^2 - r - 6 = 0 \quad \Rightarrow \quad r = -2, r = 3$ and $y_c(x) = c_1 e^{-2x} + c_2 e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p_1}(x) = A$. Then $y'_{p_1}(x) = y''_{p_1}(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p_2}(x) = Bxe^{-2x}$ [since $y = Be^{-2x}$ satisfies the complementary equation]. Then $y'_{p_2} = (B - 2Bx)e^{-2x}$ and $y''_{p_2} = (4Bx - 4B)e^{-2x}$, and substitution gives $-5Be^{-2x} = e^{-2x} \quad \Rightarrow \quad B = -\frac{1}{5}$. The general solution then is $y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_{p_1}(x) + y_{p_2}(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5} x e^{-2x}$.

10. Using variation of parameters, $y_c(x) = c_1 \cos x + c_2 \sin x$, $u'_1(x) = -\csc x \sin x = -1 \quad \Rightarrow \quad u_1(x) = -x$, and $u'_2(x) = \dfrac{\csc x \cos x}{x} = \cot x \quad \Rightarrow \quad u_2(x) = \ln|\sin x| \quad \Rightarrow \quad y_p = -x \cos x + \sin x \ln|\sin x|$. The solution is $y(x) = (c_1 - x)\cos x + (c_2 + \ln|\sin x|)\sin x$.

11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right)$.

13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.

14. $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p_2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p_2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9\cos(x/3) - 27\sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.

15. $r^2 + 4r + 29 = 0 \quad \Rightarrow \quad r = -2 \pm 5i$ and the general solution is $y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$. But $1 = y(0) = c_1$ and $-1 = y(\pi) = -c_1 e^{-2\pi} \quad \Rightarrow \quad c_1 = e^{2\pi}$, so there is no solution.

16. $r^2 + 4r + 29 = 0 \quad \Rightarrow \quad r = -2 \pm 5i$ and the general solution is $y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$. But $1 = y(0) = c_1$ and $-e^{-2\pi} = y(\pi) = -c_1 e^{-2\pi} \quad \Rightarrow \quad c_1 = 1$, so c_2 can vary and the solution of the boundary-value problem is $y = e^{-2x}(\cos 5x + c \sin 5x)$, where c is any constant.

17. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3 \cdot 5}$,

$c_7 = \dfrac{(-1)^3}{3 \cdot 5 \cdot 7} = \dfrac{(-1)^3 2^3 3!}{7!}, \ldots, c_{2n+1} = \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}\, x^{2n+1}$.

18. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=0}^{\infty} n(n-1)\, c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$ and the differential equation becomes $\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, we have $c_2 = \dfrac{c_0}{1}$, $c_4 = \dfrac{c_2}{3} = \dfrac{c_0}{1 \cdot 3}$, $c_6 = \dfrac{c_4}{5} = \dfrac{c_0}{1 \cdot 3 \cdot 5}, \ldots,$

$c_{2n} = \dfrac{c_0}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = c_0 \dfrac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \dfrac{c_1}{2}$, $c_5 = \dfrac{c_3}{4} = \dfrac{c_1}{2 \cdot 4}$,

$c_7 = \dfrac{c_5}{6} = \dfrac{c_1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n+1} = \dfrac{c_1}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} = \dfrac{c_1}{2^n\, n!}$. Thus the general solution is

$y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum\limits_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\, x^{2n}}{(2n-1)!} + c \sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n\, n!}$. But $\sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{2^n\, n!} = x \sum\limits_{n=0}^{\infty} \dfrac{\left(\frac{1}{2}x^2\right)^n}{n!} = xe^{x^2/2}$,

so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \sum\limits_{n=1}^{\infty} \dfrac{2^{n-1}(n-1)!\, x^{2n}}{(2n-1)!}$.

19. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}$. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

20. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$, $x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \quad \Rightarrow \quad c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

21. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3} r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion, $m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3}\, y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so $g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 17.1, not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 17.1, the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 17.3) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960 \text{ mi} = 3960 \cdot 5280 \text{ ft}$ and $g = 32 \text{ ft/s}^2$, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3} \text{ s}^{-1}$ and $T = 2\pi/k \approx 5079 \text{ s} \approx 85 \text{ min}$.

(d) $y(t) = 0 \quad \Leftrightarrow \quad \cos kt = 0 \quad \Leftrightarrow \quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad \Rightarrow \quad y'(t) = -kR \sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899 \text{ mi/s} \approx 17{,}600 \text{ mi/h}$.