APPENDIXES

A Numbers, Inequalities, and Absolute Values

1. $|5-23| = |-18| = 18$

2. $|5| - |-23| = 5 - 23 = -18$

3. $|-\pi| = \pi$ because $\pi > 0$.

4. $|\pi - 2| = \pi - 2$ because $\pi - 2 > 0$.

5. $\left|\sqrt{5} - 5\right| = -\left(\sqrt{5} - 5\right) = 5 - \sqrt{5}$ because $\sqrt{5} - 5 < 0$.

6. $\Big||-2| - |-3|\Big| = |2-3| = |-1| = 1$

7. If $x < 2$, $x - 2 < 0$, so $|x-2| = -(x-2) = 2 - x$.

8. If $x > 2$, $x - 2 > 0$, so $|x-2| = x - 2$.

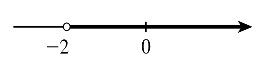
9. $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 < 0 \end{cases} = \begin{cases} x+1 & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1 \end{cases}$

10. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$

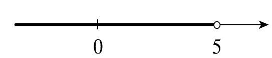
11. $\left|x^2+1\right| = x^2 + 1$ [since $x^2 + 1 \geq 0$ for all x].

12. Determine when $1 - 2x^2 < 0 \iff 1 < 2x^2 \iff x^2 > \frac{1}{2} \iff \sqrt{x^2} > \sqrt{\frac{1}{2}} \iff |x| > \sqrt{\frac{1}{2}} \iff$ $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$. Thus, $\left|1-2x^2\right| = \begin{cases} 1-2x^2 & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ 2x^2 - 1 & \text{if } x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}} \end{cases}$

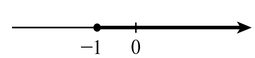
13. $2x + 7 > 3 \iff 2x > -4 \iff x > -2$, so $x \in (-2, \infty)$.



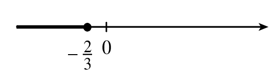
14. $3x - 11 < 4 \iff 3x < 15 \iff x < 5$, so $x \in (-\infty, 5)$.



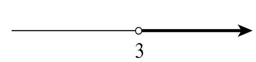
15. $1 - x \leq 2 \iff -x \leq 1 \iff x \geq -1$, so $x \in [-1, \infty)$.



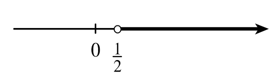
16. $4 - 3x \geq 6 \iff -3x \geq 2 \iff x \leq -\frac{2}{3}$, so $x \in \left(-\infty, -\frac{2}{3}\right]$.



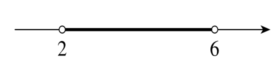
17. $2x + 1 < 5x - 8 \iff 9 < 3x \iff 3 < x$, so $x \in (3, \infty)$.



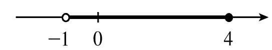
18. $1 + 5x > 5 - 3x \iff 8x > 4 \iff x > \frac{1}{2}$, so $x \in \left(\frac{1}{2}, \infty\right)$.



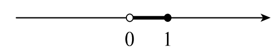
19. $-1 < 2x - 5 < 7 \iff 4 < 2x < 12 \iff 2 < x < 6$, so $x \in (2, 6)$.



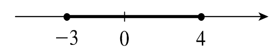
20. $1 < 3x + 4 \le 16 \quad \Leftrightarrow \quad -3 < 3x \le 12 \quad \Leftrightarrow \quad -1 < x \le 4$, so $x \in (-1, 4]$.



21. $0 \le 1 - x < 1 \quad \Leftrightarrow \quad -1 \le -x < 0 \quad \Leftrightarrow \quad 1 \ge x > 0$, so $x \in (0, 1]$.

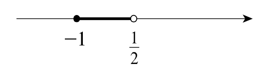


22. $-5 \le 3 - 2x \le 9 \quad \Leftrightarrow \quad -8 \le -2x \le 6 \quad \Leftrightarrow \quad 4 \ge x \ge -3$, so $x \in [-3, 4]$.



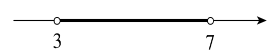
23. $4x < 2x + 1 \le 3x + 2$. So $4x < 2x + 1 \quad \Leftrightarrow \quad 2x < 1 \quad \Leftrightarrow \quad x < \frac{1}{2}$, and

$2x + 1 \le 3x + 2 \quad \Leftrightarrow \quad -1 \le x$. Thus, $x \in \left[-1, \frac{1}{2}\right)$.



24. $2x - 3 < x + 4 < 3x - 2$. So $2x - 3 < x + 4 \quad \Leftrightarrow \quad x < 7$, and

$x + 4 < 3x - 2 \quad \Leftrightarrow \quad 6 < 2x \quad \Leftrightarrow \quad 3 < x$, so $x \in (3, 7)$.



25. $(x - 1)(x - 2) > 0$.

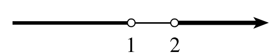
Case 1: (both factors are positive, so their product is positive) $x - 1 > 0 \quad \Leftrightarrow \quad x > 1$,

and $x - 2 > 0 \quad \Leftrightarrow \quad x > 2$, so $x \in (2, \infty)$.

Case 2: (both factors are negative, so their product is positive) $x - 1 < 0 \quad \Leftrightarrow \quad x < 1$,

and $x - 2 < 0 \quad \Leftrightarrow \quad x < 2$, so $x \in (-\infty, 1)$.

Thus, the solution set is $(-\infty, 1) \cup (2, \infty)$.

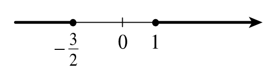


26. $(2x + 3)(x - 1) \ge 0$.

Case 1: $2x + 3 \ge 0 \quad \Leftrightarrow \quad x \ge -\frac{3}{2}$, and $x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge 1$, so $x \in [1, \infty)$.

Case 2: $2x + 3 \le 0 \quad \Leftrightarrow \quad x \le -\frac{3}{2}$, and $x - 1 \le 0 \quad \Leftrightarrow \quad x \le 1$, so $x \in \left(-\infty, -\frac{3}{2}\right]$.

Thus, the solution set is $\left(-\infty, -\frac{3}{2}\right] \cup [1, \infty)$.



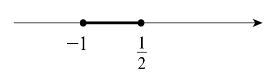
27. $2x^2 + x \le 1 \quad \Leftrightarrow \quad 2x^2 + x - 1 \le 0 \quad \Leftrightarrow \quad (2x - 1)(x + 1) \le 0$.

Case 1: $2x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge \frac{1}{2}$, and $x + 1 \le 0 \quad \Leftrightarrow \quad x \le -1$,

which is an impossible combination.

Case 2: $2x - 1 \le 0 \quad \Leftrightarrow \quad x \le \frac{1}{2}$, and $x + 1 \ge 0 \quad \Leftrightarrow \quad x \ge -1$, so $x \in \left[-1, \frac{1}{2}\right]$.

Thus, the solution set is $\left[-1, \frac{1}{2}\right]$.

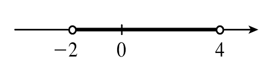


28. $x^2 < 2x + 8 \quad \Leftrightarrow \quad x^2 - 2x - 8 < 0 \quad \Leftrightarrow \quad (x - 4)(x + 2) < 0$.

Case 1: $x > 4$ and $x < -2$, which is impossible.

Case 2: $x < 4$ and $x > -2$.

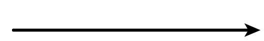
Thus, the solution set is $(-2, 4)$.



29. $x^2 + x + 1 > 0 \quad \Leftrightarrow \quad x^2 + x + \frac{1}{4} + \frac{3}{4} > 0 \quad \Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. But since

$\left(x + \frac{1}{2}\right)^2 \ge 0$ for every real x, the original inequality will be true for all real x as well.

Thus, the solution set is $(-\infty, \infty)$.



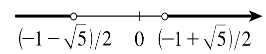
30. $x^2 + x > 1 \quad \Leftrightarrow \quad x^2 + x - 1 > 0$. Using the quadratic formula, we obtain

$x^2 + x - 1 = \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0.$

Case 1: $x - \frac{-1-\sqrt{5}}{2} > 0$ and $x - \frac{-1+\sqrt{5}}{2} > 0$, so that $x > \frac{-1+\sqrt{5}}{2}$.

Case 2: $x - \frac{-1-\sqrt{5}}{2} < 0$ and $x - \frac{-1+\sqrt{5}}{2} < 0$, so that $x < \frac{-1-\sqrt{5}}{2}$.

Thus, the solution set is $\left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \infty\right)$.

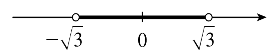


31. $x^2 < 3 \quad \Leftrightarrow \quad x^2 - 3 < 0 \quad \Leftrightarrow \quad (x - \sqrt{3})(x + \sqrt{3}) < 0$.

Case 1: $x > \sqrt{3}$ and $x < -\sqrt{3}$, which is impossible.

Case 2: $x < \sqrt{3}$ and $x > -\sqrt{3}$.

Thus, the solution set is $(-\sqrt{3}, \sqrt{3})$.



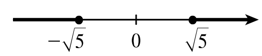
Another method: $x^2 < 3 \quad \Leftrightarrow \quad |x| < \sqrt{3} \quad \Leftrightarrow \quad -\sqrt{3} < x < \sqrt{3}$.

32. $x^2 \geq 5 \quad \Leftrightarrow \quad x^2 - 5 \geq 0 \quad \Leftrightarrow \quad (x - \sqrt{5})(x + \sqrt{5}) \geq 0$.

Case 1: $x \geq \sqrt{5}$ and $x \geq -\sqrt{5}$, so $x \in [\sqrt{5}, \infty)$.

Case 2: $x \leq \sqrt{5}$ and $x \leq -\sqrt{5}$, so $x \in (-\infty, -\sqrt{5}]$.

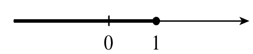
Thus, the solution set is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$.



Another method: $x^2 \geq 5 \quad \Leftrightarrow \quad |x| \geq \sqrt{5} \quad \Leftrightarrow \quad x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.

33. $x^3 - x^2 \leq 0 \quad \Leftrightarrow \quad x^2(x-1) \leq 0$. Since $x^2 \geq 0$ for all x, the inequality is satisfied when $x - 1 \leq 0 \quad \Leftrightarrow \quad x \leq 1$.

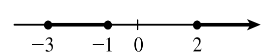
Thus, the solution set is $(-\infty, 1]$.



34. $(x+1)(x-2)(x+3) = 0 \quad \Leftrightarrow \quad x = -1, 2$, or -3. Construct a chart:

Interval	$x+1$	$x-2$	$x+3$	$(x+1)(x-2)(x+3)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < -1$	$-$	$-$	$+$	$+$
$-1 < x < 2$	$+$	$-$	$+$	$-$
$x > 2$	$+$	$+$	$+$	$+$

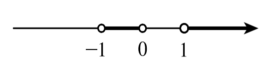
Thus, $(x+1)(x-2)(x+3) \geq 0$ on $[-3, -1]$ and $[2, \infty)$, and the solution set is $[-3, -1] \cup [2, \infty)$.



35. $x^3 > x \quad \Leftrightarrow \quad x^3 - x > 0 \quad \Leftrightarrow \quad x(x^2 - 1) > 0 \quad \Leftrightarrow \quad x(x-1)(x+1) > 0$. Construct a chart:

Interval	x	$x-1$	$x+1$	$x(x-1)(x+1)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

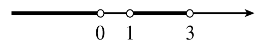
Since $x^3 > x$ when the last column is positive, the solution set is $(-1, 0) \cup (1, \infty)$.



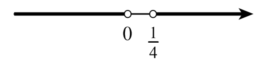
36. $x^3 + 3x < 4x^2 \quad \Leftrightarrow \quad x^3 - 4x^2 + 3x < 0 \quad \Leftrightarrow \quad x\left(x^2 - 4x + 3\right) < 0 \quad \Leftrightarrow \quad x(x-1)(x-3) < 0.$

Interval	x	$x-1$	$x-3$	$x(x-1)(x-3)$
$x < 0$	$-$	$-$	$-$	$-$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

Thus, the solution set is $(-\infty, 0) \cup (1, 3)$.

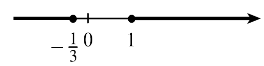


37. $1/x < 4$. This is clearly true for $x < 0$. So suppose $x > 0$. then $1/x < 4 \quad \Leftrightarrow$ $1 < 4x \quad \Leftrightarrow \quad \frac{1}{4} < x$. Thus, the solution set is $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$.



38. $-3 < 1/x \le 1$. We solve the two inequalities separately and take the intersection of the solution sets. First, $-3 < 1/x$ is clearly true for $x > 0$. So suppose $x < 0$. Then $-3 < 1/x \quad \Leftrightarrow \quad -3x > 1 \quad \Leftrightarrow \quad x < -\frac{1}{3}$, so for this inequality, the solution set is $\left(-\infty, -\frac{1}{3}\right) \cup (0, \infty)$. Now $1/x \le 1$ is clearly true if $x < 0$. So suppose $x > 0$. Then $1/x \le 1 \quad \Leftrightarrow \quad 1 \le x$, and the solution set here is $(-\infty, 0) \cup [1, \infty)$.

Taking the intersection of the two solution sets gives the final solution set: $\left(-\infty, -\frac{1}{3}\right) \cup [1, \infty)$.



39. $C = \frac{5}{9}(F - 32) \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. So $50 \le F \le 95 \quad \Rightarrow \quad 50 \le \frac{9}{5}C + 32 \le 95 \quad \Rightarrow \quad 18 \le \frac{9}{5}C \le 63 \quad \Rightarrow$ $10 \le C \le 35$. So the interval is $[10, 35]$.

40. Since $20 \le C \le 30$ and $C = \frac{5}{9}(F - 32)$, we have $20 \le \frac{5}{9}(F - 32) \le 30 \quad \Rightarrow \quad 36 \le F - 32 \le 54 \quad \Rightarrow \quad 68 \le F \le 86$. So the interval is $[68, 86]$.

41. (a) Let T represent the temperature in degrees Celsius and h the height in km. $T = 20$ when $h = 0$ and T decreases by 10°C for every km (1°C for each 100-m rise). Thus, $T = 20 - 10h$ when $0 \le h \le 12$.

(b) From part (a), $T = 20 - 10h \quad \Rightarrow \quad 10h = 20 - T \quad \Rightarrow \quad h = 2 - T/10$. So $0 \le h \le 5 \quad \Rightarrow \quad 0 \le 2 - T/10 \le 5 \quad \Rightarrow$ $-2 \le -T/10 \le 3 \quad \Rightarrow \quad -20 \le -T \le 30 \quad \Rightarrow \quad 20 \ge T \ge -30 \quad \Rightarrow \quad -30 \le T \le 20$. Thus, the range of temperatures (in $^\circ$C) to be expected is $[-30, 20]$.

42. The ball will be at least 32 ft above the ground if $h \ge 32 \quad \Leftrightarrow \quad 128 + 16t - 16t^2 \ge 32 \quad \Leftrightarrow \quad 16t^2 - 16t - 96 \le 0 \quad \Leftrightarrow$ $16(t-3)(t+2) \le 0$. $t = 3$ and $t = -2$ are endpoints of the interval we're looking for, and constructing a table gives $-2 \le t \le 3$. But $t \ge 0$, so the ball will be at least 32 ft above the ground in the time interval $[0, 3]$.

43. $|2x| = 3 \quad \Leftrightarrow \quad$ either $2x = 3$ or $2x = -3 \quad \Leftrightarrow \quad x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

44. $|3x + 5| = 1 \quad \Leftrightarrow \quad$ either $3x + 5 = 1$ or -1. In the first case, $3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$, and in the second case, $3x = -6 \quad \Leftrightarrow \quad x = -2$. So the solutions are -2 and $-\frac{4}{3}$.

45. $|x+3| = |2x+1| \quad \Leftrightarrow \quad$ either $x+3 = 2x+1$ or $x+3 = -(2x+1)$. In the first case, $x = 2$, and in the second case, $x+3 = -2x-1 \quad \Leftrightarrow \quad 3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$. So the solutions are $-\frac{4}{3}$ and 2.

46. $\left|\dfrac{2x-1}{x+1}\right| = 3 \quad \Leftrightarrow \quad$ either $\dfrac{2x-1}{x+1} = 3$ or $\dfrac{2x-1}{x+1} = -3$. In the first case, $2x-1 = 3x+3 \quad \Leftrightarrow \quad x = -4$, and in the second case, $2x-1 = -3x-3 \quad \Leftrightarrow \quad x = -\frac{2}{5}$.

47. By Property 5 of absolute values, $|x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$, so $x \in (-3, 3)$.

48. By Properties 4 and 6 of absolute values, $|x| \geq 3 \quad \Leftrightarrow \quad x \leq -3$ or $x \geq 3$, so $x \in (-\infty, -3] \cup [3, \infty)$.

49. $|x-4| < 1 \quad \Leftrightarrow \quad -1 < x-4 < 1 \quad \Leftrightarrow \quad 3 < x < 5$, so $x \in (3, 5)$.

50. $|x-6| < 0.1 \quad \Leftrightarrow \quad -0.1 < x-6 < 0.1 \quad \Leftrightarrow \quad 5.9 < x < 6.1$, so $x \in (5.9, 6.1)$.

51. $|x+5| \geq 2 \quad \Leftrightarrow \quad x+5 \geq 2$ or $x+5 \leq -2 \quad \Leftrightarrow \quad x \geq -3$ or $x \leq -7$, so $x \in (-\infty, -7] \cup [-3, \infty)$.

52. $|x+1| \geq 3 \quad \Leftrightarrow \quad x+1 \geq 3$ or $x+1 \leq -3 \quad \Leftrightarrow \quad x \geq 2$ or $x \leq -4$, so $x \in (-\infty, -4] \cup [2, \infty)$.

53. $|2x-3| \leq 0.4 \quad \Leftrightarrow \quad -0.4 \leq 2x-3 \leq 0.4 \quad \Leftrightarrow \quad 2.6 \leq 2x \leq 3.4 \quad \Leftrightarrow \quad 1.3 \leq x \leq 1.7$, so $x \in [1.3, 1.7]$.

54. $|5x-2| < 6 \quad \Leftrightarrow \quad -6 < 5x-2 < 6 \quad \Leftrightarrow \quad -4 < 5x < 8 \quad \Leftrightarrow \quad -\frac{4}{5} < x < \frac{8}{5}$, so $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$.

55. $1 \leq |x| \leq 4$. So either $1 \leq x \leq 4$ or $1 \leq -x \leq 4 \quad \Leftrightarrow \quad -1 \geq x \geq -4$. Thus, $x \in [-4, -1] \cup [1, 4]$.

56. $0 < |x-5| < \frac{1}{2}$. Clearly $0 < |x-5|$ for $x \neq 5$. Now $|x-5| < \frac{1}{2} \quad \Leftrightarrow \quad -\frac{1}{2} < x-5 < \frac{1}{2} \quad \Leftrightarrow \quad 4.5 < x < 5.5$. So the solution set is $(4.5, 5) \cup (5, 5.5)$.

57. $a(bx-c) \geq bc \quad \Leftrightarrow \quad bx - c \geq \dfrac{bc}{a} \quad \Leftrightarrow \quad bx \geq \dfrac{bc}{a} + c = \dfrac{bc+ac}{a} \quad \Leftrightarrow \quad x \geq \dfrac{bc+ac}{ab}$

58. $a \leq bx + c < 2a \quad \Leftrightarrow \quad a - c \leq bx < 2a - c \quad \Leftrightarrow \quad \dfrac{a-c}{b} \leq x < \dfrac{2a-c}{b}$ (since $b > 0$)

59. $ax + b < c \quad \Leftrightarrow \quad ax < c - b \quad \Leftrightarrow \quad x > \dfrac{c-b}{a}$ [since $a < 0$]

60. $\dfrac{ax+b}{c} \leq b \quad \Leftrightarrow \quad ax + b \geq bc$ [since $c < 0$] $\quad \Leftrightarrow \quad ax \geq bc - b \quad \Leftrightarrow \quad x \leq \dfrac{b\,(c-1)}{a}$ [since $a < 0$]

61. $|(x+y) - 5| = |(x-2) + (y-3)| \leq |x-2| + |y-3| < 0.01 + 0.04 = 0.05$

62. Use the Triangle Inequality: $|x+3| < \frac{1}{2} \quad \Rightarrow$

$|4x+13| = |4\,(x+3) + 1| \leq |4\,(x+3)| + |1| = 4\,|x+3| + 1 < 4\left(\frac{1}{2}\right) + 1 = 3$

Another method: $|x+3| < \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2} < x+3 < \frac{1}{2} \quad \Rightarrow \quad -2 < 4x+12 < 2 \quad \Rightarrow \quad -1 < 4x+13 < 3 \quad \Rightarrow$ $|4x+13| < 3$

63. If $a < b$ then $a + a < a + b$ and $a + b < b + b$. So $2a < a + b < 2b$. Dividing by 2, we get $a < \frac{1}{2}\,(a+b) < b$.

64. If $0 < a < b$, then $\dfrac{1}{ab} > 0$. So $a < b \;\Rightarrow\; \dfrac{1}{ab} \cdot a < \dfrac{1}{ab} \cdot b \;\Leftrightarrow\; \dfrac{1}{b} < \dfrac{1}{a}$.

65. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2 b^2} = \sqrt{a^2}\,\sqrt{b^2} = |a|\,|b|$

66. $\left|\dfrac{a}{b}\right| |b| = \left|\dfrac{a}{b} \cdot b\right| = |a|$ [using the result of Exercise 65]. Dividing the equation through by $|b|$ gives $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. If $0 < a < b$, then $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$ [using Rule 3 of Inequalities]. So $a^2 < ab < b^2$ and hence $a^2 < b^2$.

68. Following the hint, the Triangle Inequality becomes $|(x - y) + y| \le |x - y| + |y| \;\Leftrightarrow\; |x| \le |x - y| + |y| \;\Leftrightarrow$ $|x - y| \ge |x| - |y|$.

69. Observe that the sum, difference and product of two integers is always an integer. Let the rational numbers be represented by $r = m/n$ and $s = p/q$ (where m, n, p and q are integers with $n \ne 0$, $q \ne 0$). Now $r + s = \dfrac{m}{n} + \dfrac{p}{q} = \dfrac{mq + pn}{nq}$, but $mq + pn$ and nq are both integers, so $\dfrac{mq + pn}{nq} = r + s$ is a rational number by definition. Similarly, $r - s = \dfrac{m}{n} - \dfrac{p}{q} = \dfrac{mq - pn}{nq}$ is a rational number. Finally, $r \cdot s = \dfrac{m}{n} \cdot \dfrac{p}{q} = \dfrac{mp}{nq}$ but mp and nq are both integers, so $\dfrac{mp}{nq} = r \cdot s$ is a rational number by definition.

70. (a) No. Consider the case of $\sqrt{2}$ and $-\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} + \left(-\sqrt{2}\right) = 0$ and 0, being an integer, is not irrational.

(b) No. Consider the case of $\sqrt{2}$ and $\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational.

B Coordinate Geometry and Lines

1. Use the distance formula with $P_1(x_1, y_1) = (1, 1)$ and $P_2(x_2, y_2) = (4, 5)$ to get

$$|P_1 P_2| = \sqrt{(4 - 1)^2 + (5 - 1)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

2. The distance from $(1, -3)$ to $(5, 7)$ is $\sqrt{(5 - 1)^2 + [7 - (-3)]^2} = \sqrt{4^2 + 10^2} = \sqrt{116} = 2\sqrt{29}$.

3. The distance from $(6, -2)$ to $(-1, 3)$ is $\sqrt{-1 - 6)^2 + [3 - (-2)]^2} = \sqrt{(-7)^2 + 5^2} = \sqrt{74}$.

4. The distance from $(1, -6)$ to $(-1, -3)$ is $\sqrt{(-1 - 1)^2 + [-3 - (-6)]^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$.

5. The distance from $(2, 5)$ to $(4, -7)$ is $\sqrt{(4 - 2)^2 + (-7 - 5)^2} = \sqrt{2^2 + (-12)^2} = \sqrt{148} = 2\sqrt{37}$.

6. The distance from (a, b) to (b, a) is $\sqrt{(b - a)^2 + (a - b)^2} = \sqrt{(a - b)^2 + (a - b)^2} = \sqrt{2(a - b)^2} = \sqrt{2}\,|a - b|$.

7. The slope m of the line through $P(1, 5)$ and $Q(4, 11)$ is $m = \dfrac{11 - 5}{4 - 1} = \dfrac{6}{3} = 2$.

8. The slope m of the line through $P(-1, 6)$ and $Q(4, -3)$ is $m = \dfrac{-3 - 6}{4 - (-1)} = -\dfrac{9}{5}$.

9. The slope m of the line through $P(-3, 3)$ and $Q(-1, -6)$ is $m = \dfrac{-6 - 3}{-1 - (-3)} = -\dfrac{9}{2}$.

10. The slope m of the line through $P(-1, -4)$ and $Q(6, 0)$ is $m = \dfrac{0 - (-4)}{6 - (-1)} = \dfrac{4}{7}$.

11. Using $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$, we have $|AC| = \sqrt{(-4 - 0)^2 + (3 - 2)^2} = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$ and $|BC| = \sqrt{[-4 - (-3)]^2 + [3 - (-1)]^2} = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$, so the triangle has two sides of equal length, and is isosceles.

12. (a) Using $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$, we have

$|AB| = \sqrt{(11 - 6)^2 + [-3 - (-7)]^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$,

$|AC| = \sqrt{(2 - 6)^2 + [-2 - (-7)]^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$, and

$|BC| = \sqrt{(2 - 11)^2 + [-2 - (-3)]^2} = \sqrt{(-9)^2 + 1^2} = \sqrt{82}$.

Thus, $|AB|^2 + |AC|^2 = 41 + 41 = 82 = |BC|^2$ and so $\triangle ABC$ is a right triangle.

(b) $m_{AB} = \dfrac{-3 - (-7)}{11 - 6} = \dfrac{4}{5}$ and $m_{AC} = \dfrac{-2 - (-7)}{2 - 6} = -\dfrac{5}{4}$. Thus $m_{AB} \cdot m_{AC} = -1$ and so AB is perpendicular to AC and $\triangle ABC$ must be a right triangle.

(c) Taking lengths from part (a), the base is $\sqrt{41}$ and the height is $\sqrt{41}$. Thus the area is $\frac{1}{2}bh = \frac{1}{2}\sqrt{41}\sqrt{41} = \frac{41}{2}$.

13. Using $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$, we have

$|AB| = \sqrt{[4 - (-2)]^2 + (6 - 9)^2} = \sqrt{6^2 + (-3)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|BC| = \sqrt{(1 - 4)^2 + (0 - 6)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|CD| = \sqrt{(-5 - 1)^2 + (3 - 0)^2} = \sqrt{(-6)^2 + 3^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$, and

$|DA| = \sqrt{[-2 - (-5)]^2 + (9 - 3)^2} = \sqrt{3^2 + 6^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$. So all sides are of equal length and we have a rhombus. Moreover, $m_{AB} = \dfrac{6 - 9}{4 - (-2)} = -\dfrac{1}{2}$, $m_{BC} = \dfrac{0 - 6}{1 - 4} = 2$, $m_{CD} = \dfrac{3 - 0}{-5 - 1} = -\dfrac{1}{2}$, and

$m_{DA} = \dfrac{9 - 3}{-2 - (-5)} = 2$, so the sides are perpendicular. Thus, A, B, C, and D are vertices of a square.

14. (a) Using $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$, we have

$|AB| = \sqrt{[3 - (-1)]^2 + (11 - 3)^2} = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$,

$|BC| = \sqrt{(5 - 3)^2 + (15 - 11)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$, and

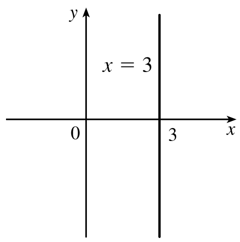
$|AC| = \sqrt{[5 - (-1)]^2 + (15 - 3)^2} = \sqrt{6^2 + 12^2} = \sqrt{180} = 6\sqrt{5}$. Thus, $|AC| = |AB| + |BC|$.

(b) $m_{AB} = \dfrac{11 - 3}{3 - (-1)} = \dfrac{8}{4} = 2$ and $m_{AC} = \dfrac{15 - 3}{5 - (-1)} = \dfrac{12}{6} = 2$. Since the segments AB and AC have the same slope, A, B and C must be collinear.

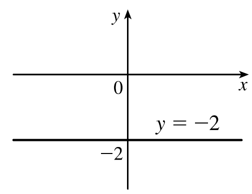
15. For the vertices $A(1,1)$, $B(7,4)$, $C(5,10)$, and $D(-1,7)$, the slope of the line segment AB is $\dfrac{4-1}{7-1}=\dfrac{1}{2}$, the slope of CD is $\dfrac{7-10}{-1-5}=\dfrac{1}{2}$, the slope of BC is $\dfrac{10-4}{5-7}=-3$, and the slope of DA is $\dfrac{1-7}{1-(-1)}=-3$. So AB is parallel to CD and BC is parallel to DA. Hence $ABCD$ is a parallelogram.

16. For the vertices $A(1,1)$, $B(11,3)$, $C(10,8)$, and $D(0,6)$, the slopes of the four sides are $m_{AB}=\dfrac{3-1}{11-1}=\dfrac{1}{5}$, $m_{BC}=\dfrac{8-3}{10-11}=-5$, $m_{CD}=\dfrac{6-8}{0-10}=\dfrac{1}{5}$, and $m_{DA}=\dfrac{1-6}{1-0}=-5$. Hence $AB \parallel CD$, $BC \parallel DA$, $AB \perp BC$, $BC \perp CD$, $CD \perp DA$, and $DA \perp AB$, and so $ABCD$ is a rectangle.

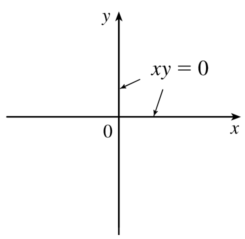
17. The graph of the equation $x=3$ is a vertical line with x-intercept 3. The line does not have a slope.



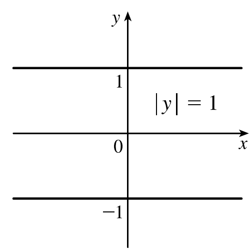
18. The graph of the equation $y=-2$ is a horizontal line with y-intercept -2. The line has slope 0.



19. $xy=0 \quad \Leftrightarrow \quad x=0$ or $y=0$. The graph consists of the coordinate axes.



20. $|y|=1 \quad \Leftrightarrow \quad y=1$ or $y=-1$



21. By the point-slope form of the equation of a line, an equation of the line through $(2,-3)$ with slope 6 is $y-(-3)=6(x-2)$ or $y=6x-15$.

22. $y-4=-3[x-(-1)]$ or $y=-3x+1$

23. $y-7=\frac{2}{3}(x-1)$ or $y=\frac{2}{3}x+\frac{19}{3}$

24. $y-(-5)=-\frac{7}{2}[x-(-3)]$ or $y=-\frac{7}{2}x-\frac{31}{2}$

25. The slope of the line through $(2,1)$ and $(1,6)$ is $m=\dfrac{6-1}{1-2}=-5$, so an equation of the line is $y-1=-5(x-2)$ or $y=-5x+11$.

26. For $(-1,-2)$ and $(4,3)$, $m = \dfrac{3-(-2)}{4-(-1)} = 1$. An equation of the line is $y - 3 = 1(x-4)$ or $y = x - 1$.

27. By the slope-intercept form of the equation of a line, an equation of the line is $y = 3x - 2$.

28. By the slope-intercept form of the equation of a line, an equation of the line is $y = \frac{2}{5}x + 4$.

29. Since the line passes through $(1,0)$ and $(0,-3)$, its slope is $m = \dfrac{-3-0}{0-1} = 3$, so an equation is $y = 3x - 3$.

Another method: From Exercise 61, $\dfrac{x}{1} + \dfrac{y}{-3} = 1 \quad\Rightarrow\quad -3x + y = -3 \quad\Rightarrow\quad y = 3x - 3$.

30. For $(-8,0)$ and $(0,6)$, $m = \dfrac{6-0}{0-(-8)} = \dfrac{3}{4}$. So an equation is $y = \frac{3}{4}x + 6$.

Another method: From Exercise 61, $\dfrac{x}{-8} + \dfrac{y}{6} = 1 \quad\Rightarrow\quad -3x + 4y = 24 \quad\Rightarrow\quad y = \frac{3}{4}x + 6$.

31. The line is parallel to the x-axis, so it is horizontal and must have the form $y = k$. Since it goes through the point $(x,y) = (4,5)$, the equation is $y = 5$.

32. The line is parallel to the y-axis, so it is vertical and must have the form $x = k$. Since it goes through the point $(x,y) = (4,5)$, the equation is $x = 4$.

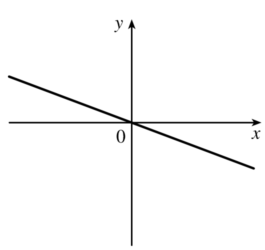
33. Putting the line $x + 2y = 6$ into its slope-intercept form gives us $y = -\frac{1}{2}x + 3$, so we see that this line has slope $-\frac{1}{2}$. Thus, we want the line of slope $-\frac{1}{2}$ that passes through the point $(1,-6)$: $y - (-6) = -\frac{1}{2}(x-1) \quad\Leftrightarrow\quad y = -\frac{1}{2}x - \frac{11}{2}$.

34. $2x + 3y + 4 = 0 \quad\Leftrightarrow\quad y = -\frac{2}{3}x - \frac{4}{3}$, so $m = -\frac{2}{3}$ and the required line is $y = -\frac{2}{3}x + 6$.

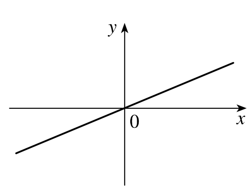
35. $2x + 5y + 8 = 0 \quad\Leftrightarrow\quad y = -\frac{2}{5}x - \frac{8}{5}$. Since this line has slope $-\frac{2}{5}$, a line perpendicular to it would have slope $\frac{5}{2}$, so the required line is $y - (-2) = \frac{5}{2}[x - (-1)] \quad\Leftrightarrow\quad y = \frac{5}{2}x + \frac{1}{2}$.

36. $4x - 8y = 1 \quad\Leftrightarrow\quad y = \frac{1}{2}x - \frac{1}{8}$. Since this line has slope $\frac{1}{2}$, a line perpendicular to it would have slope -2, so the required line is $y - \left(-\frac{2}{3}\right) = -2\left(x - \frac{1}{2}\right) \quad\Leftrightarrow\quad y = -2x + \frac{1}{3}$.

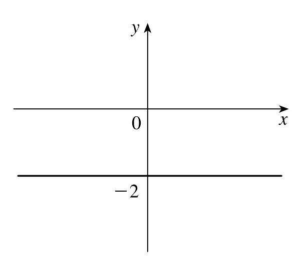
37. $x + 3y = 0 \quad\Leftrightarrow\quad y = -\frac{1}{3}x$, so the slope is $-\frac{1}{3}$ and the y-intercept is 0.



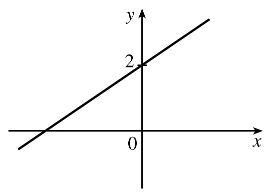
38. $2x - 5y = 0 \quad\Leftrightarrow\quad y = \frac{2}{5}x$, so the slope is $\frac{2}{5}$ and the y-intercept is 0.



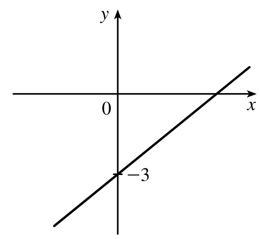
39. $y = -2$ is a horizontal line with slope 0 and y-intercept -2.



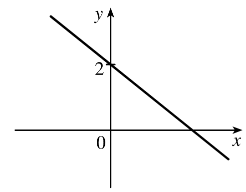
40. $2x - 3y + 6 = 0 \quad \Leftrightarrow$
$y = \frac{2}{3}x + 2$, so the slope is $\frac{2}{3}$ and the y-intercept is 2.



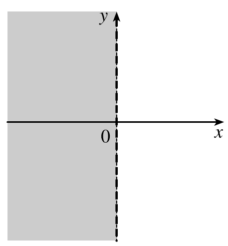
41. $3x - 4y = 12 \quad \Leftrightarrow$
$y = \frac{3}{4}x - 3$, so the slope is $\frac{3}{4}$ and the y-intercept is -3.



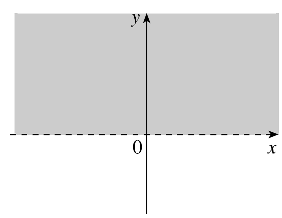
42. $4x + 5y = 10 \quad \Leftrightarrow$
$y = -\frac{4}{5}x + 2$, so the slope is $-\frac{4}{5}$ and the y-intercept is 2.



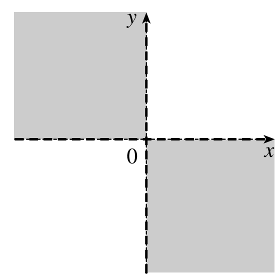
43. $\{(x, y) \mid x < 0\}$



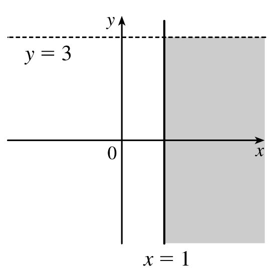
44. $\{(x, y) \mid y > 0\}$



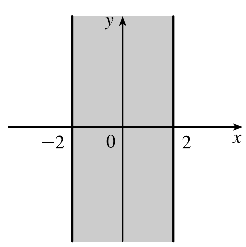
45. $\{(x, y) \mid xy < 0\} =$
$\{(x, y) \mid x < 0 \text{ and } y > 0\}$
$\cup \{(x, y) \mid x > 0 \text{ and } y < 0\}$



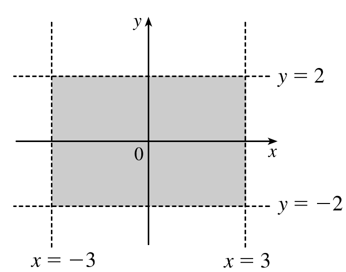
46. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$



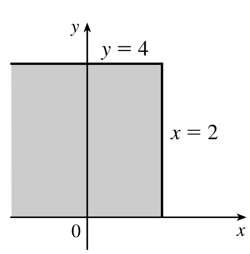
47. $\left\{(x, y) \,\middle|\, |x| \leq 2\right\} =$
$\{(x, y) \mid -2 \leq x \leq 2\}$



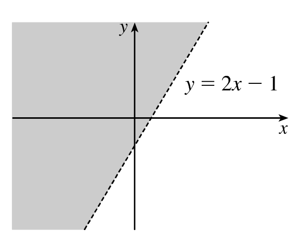
48. $\left\{(x, y) \,\middle|\, |x| < 3 \text{ and } |y| < 2\right\}$



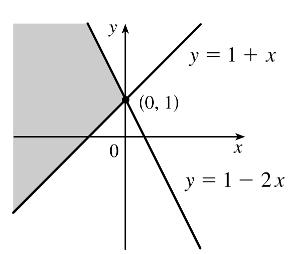
49. $\{(x, y) \mid 0 \leq y \leq 4, x \leq 2\}$



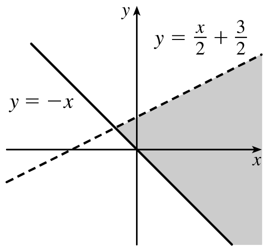
50. $\{(x, y) \mid y > 2x - 1\}$



51. $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$



52. $\left\{(x,y) \mid -x \le y < \frac{1}{2}(x+3)\right\}$



53. Let $P\,(0,y)$ be a point on the y-axis. The distance from P to $(5,-5)$ is $\sqrt{(5-0)^2+(-5-y)^2} = \sqrt{5^2+(y+5)^2}$. The distance from P to $(1,1)$ is $\sqrt{(1-0)^2+(1-y)^2} = \sqrt{1^2+(y-1)^2}$. We want these distances to be equal: $\sqrt{5^2+(y+5)^2} = \sqrt{1^2+(y-1)^2} \;\Leftrightarrow\; 5^2+(y+5)^2 = 1^2+(y-1)^2 \;\Leftrightarrow\; 25+(y^2+10y+25) = 1+(y^2-2y+1) \;\Leftrightarrow\; 12y=-48 \;\Leftrightarrow\; y=-4$. So the desired point is $(0,-4)$.

54. Let M be the point $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$. Then

$$|MP_1|^2 = \left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(y_1 - \frac{y_1+y_2}{2}\right)^2 = \left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2$$
$$|MP_2|^2 = \left(x_2 - \frac{x_1+x_2}{2}\right)^2 + \left(y_2 - \frac{y_1+y_2}{2}\right)^2 = \left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2$$

Hence, $|MP_1| = |MP_2|$; that is, M is equidistant from P_1 and P_2.

55. (a) Using the midpoint formula from Exercise 54 with $(1,3)$ and $(7,15)$, we get $\left(\frac{1+7}{2}, \frac{3+15}{2}\right) = (4,9)$.

(b) Using the midpoint formula from Exercise 54 with $(-1,6)$ and $(8,-12)$, we get $\left(\frac{-1+8}{2}, \frac{6+(-12)}{2}\right) = \left(\frac{7}{2},-3\right)$.

56. With $A(1,0)$, $B(3,6)$, and $C(8,2)$, the midpoint M_1 of AB is $\left(\frac{1+3}{2}, \frac{0+6}{2}\right) = (2,3)$, the midpoint M_2 of BC is $\left(\frac{3+8}{2}, \frac{6+2}{2}\right) = \left(\frac{11}{2},4\right)$, and the midpoint M_3 of CA is $\left(\frac{8+1}{2}, \frac{2+0}{2}\right) = \left(\frac{9}{2},1\right)$. The lengths of the medians are

$$|AM_2| = \sqrt{\left(\tfrac{11}{2}-1\right)^2+(4-0)^2} = \sqrt{\left(\tfrac{9}{2}\right)^2+4^2} = \sqrt{\tfrac{145}{4}} = \tfrac{\sqrt{145}}{2}$$
$$|BM_3| = \sqrt{\left(\tfrac{9}{2}-3\right)^2+(1-6)^2} = \sqrt{\left(\tfrac{3}{2}\right)^2+(-5)^2} = \sqrt{\tfrac{109}{4}} = \tfrac{\sqrt{109}}{2}$$
$$|CM_1| = \sqrt{(2-8)^2+(3-2)^2} = \sqrt{(-6)^2+1^2} = \sqrt{37}$$

57. $2x-y=4 \;\Leftrightarrow\; y=2x-4 \;\Rightarrow\; m_1=2$ and $6x-2y=10 \;\Leftrightarrow\; 2y=6x-10 \;\Leftrightarrow\; y=3x-5 \;\Rightarrow\; m_2=3$. Since $m_1 \ne m_2$, the two lines are not parallel. To find the point of intersection: $2x-4=3x-5 \;\Leftrightarrow\; x=1 \;\Rightarrow\; y=-2$. Thus, the point of intersection is $(1,-2)$.

58. $3x-5y+19=0 \;\Leftrightarrow\; 5y=3x+19 \;\Leftrightarrow\; y=\frac{3}{5}x+\frac{19}{5} \;\Rightarrow\; m_1=\frac{3}{5}$ and $10x+6y-50=0 \;\Leftrightarrow\; 6y=-10x+50 \;\Leftrightarrow\; y=-\frac{5}{3}x+\frac{25}{3} \;\Rightarrow\; m_2=-\frac{5}{3}$. Since $m_1m_2=\frac{3}{5}\left(-\frac{5}{3}\right)=-1$, the two lines are perpendicular. To find the point of intersection: $\frac{3}{5}x+\frac{19}{5}=-\frac{5}{3}x+\frac{25}{3} \;\Leftrightarrow\; 9x+57=-25x+125 \;\Leftrightarrow\; 34x=68 \;\Leftrightarrow\; x=2 \;\Rightarrow\; y=\frac{3}{5}\cdot 2+\frac{19}{5}=\frac{25}{5}=5$. Thus, the point of intersection is $(2,5)$.

59. With $A(1,4)$ and $B(7,-2)$, the slope of segment AB is $\frac{-2-4}{7-1}=-1$, so its perpendicular bisector has slope 1. The midpoint of AB is $\left(\frac{1+7}{2}, \frac{4+(-2)}{2}\right)=(4,1)$, so an equation of the perpendicular bisector is $y-1=1(x-4)$ or $y=x-3$.

60. (a) Side PQ has slope $\frac{4-0}{3-1} = 2$, so its equation is $y - 0 = 2(x-1) \Leftrightarrow y = 2x - 2$. Side QR has slope $\frac{6-4}{-1-3} = -\frac{1}{2}$, so its equation is $y - 4 = -\frac{1}{2}(x-3) \Leftrightarrow y = -\frac{1}{2}x + \frac{11}{2}$. Side RP has slope $\frac{0-6}{1-(-1)} = -3$, so its equation is $y - 0 = -3(x-1) \Leftrightarrow y = -3x + 3$.

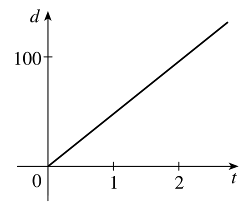
(b) M_1 (the midpoint of PQ) has coordinates $\left(\frac{1+3}{2}, \frac{0+4}{2}\right) = (2, 2)$. M_2 (the midpoint of QR) has coordinates $\left(\frac{3-1}{2}, \frac{4+6}{2}\right) = (1, 5)$. M_3 (the midpoint of RP) has coordinates $\left(\frac{1-1}{2}, \frac{0+6}{2}\right) = (0, 3)$. RM_1 has slope $\frac{2-6}{2-(-1)} = -\frac{4}{3}$ and hence equation $y - 2 = -\frac{4}{3}(x-2) \Leftrightarrow y = -\frac{4}{3}x + \frac{14}{3}$. PM_2 is a vertical line with equation $x = 1$. QM_3 has slope $\frac{3-4}{0-3} = \frac{1}{3}$ and hence equation $y - 3 = \frac{1}{3}(x-0) \Leftrightarrow y = \frac{1}{3}x + 3$. PM_2 and RM_1 intersect where $x = 1$ and $y = -\frac{4}{3}(1) + \frac{14}{3} = \frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$. PM_2 and QM_3 intersect where $x = 1$ and $y = \frac{1}{3}(1) + 3 = \frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$, so this is the point where all three medians intersect.

61. (a) Since the x-intercept is a, the point $(a, 0)$ is on the line, and similarly since the y-intercept is b, $(0, b)$ is on the line. Hence, the slope of the line is $m = \dfrac{b-0}{0-a} = -\dfrac{b}{a}$. Substituting into $y = mx + b$ gives $y = -\dfrac{b}{a}x + b \Leftrightarrow \dfrac{b}{a}x + y = b \Leftrightarrow \dfrac{x}{a} + \dfrac{y}{b} = 1$.

(b) Letting $a = 6$ and $b = -8$ gives $\dfrac{x}{6} + \dfrac{y}{-8} = 1 \Leftrightarrow -8x + 6y = -48$ [multiply by -48] $\Leftrightarrow 6y = 8x - 48 \Leftrightarrow 3y = 4x - 24 \Leftrightarrow y = \frac{4}{3}x - 8$.

62. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus, we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40-0}{5/6-0} = 48$ and $d = 48t$.

(b)



(c) The slope is 48 and represents the car's speed in $\mathrm{mi/h}$.

C Graphs of Second-Degree Equations

1. An equation of the circle with center $(3, -1)$ and radius 5 is $(x-3)^2 + (y+1)^2 = 5^2 = 25$.

2. An equation of the circle with center $(-2, -8)$ and radius 10 is $(x+2)^2 + (y+8)^2 = 10^2 = 100$.

3. The equation has the form $x^2 + y^2 = r^2$. Since $(4, 7)$ lies on the circle, we have $4^2 + 7^2 = r^2 \Rightarrow r^2 = 65$. So the required equation is $x^2 + y^2 = 65$.

4. The equation has the form $(x+1)^2 + (y-5)^2 = r^2$. Since $(-4, -6)$ lies on the circle, we have $r^2 = (-4+1)^2 + (-6-5)^2 = 130$. So an equation is $(x+1)^2 + (y-5)^2 = 130$.

5. $x^2 + y^2 - 4x + 10y + 13 = 0 \Leftrightarrow x^2 - 4x + y^2 + 10y = -13 \Leftrightarrow (x^2 - 4x + 4) + (y^2 + 10y + 25) = -13 + 4 + 25 = 16 \Leftrightarrow (x-2)^2 + (y+5)^2 = 4^2$. Thus, we have a circle with center $(2, -5)$ and radius 4.

6. $x^2 + y^2 + 6y + 2 = 0 \Leftrightarrow x^2 + (y^2 + 6y + 9) = -2 + 9 \Leftrightarrow x^2 + (y+3)^2 = 7$. Thus, we have a circle with center $(0, -3)$ and radius $\sqrt{7}$.

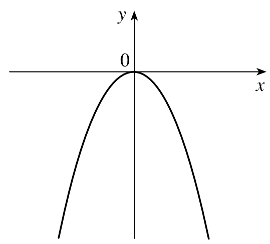
7. $x^2+y^2+x=0 \quad\Leftrightarrow\quad \left(x^2+x+\frac{1}{4}\right)+y^2=\frac{1}{4} \quad\Leftrightarrow\quad \left(x+\frac{1}{2}\right)^2+y^2=\left(\frac{1}{2}\right)^2$. Thus, we have a circle with center $\left(-\frac{1}{2},0\right)$ and radius $\frac{1}{2}$.

8. $16x^2+16y^2+8x+32y+1=0 \quad\Leftrightarrow\quad 16\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+16(y^2+2y+1)=-1+1+16 \quad\Leftrightarrow$ $16\left(x+\frac{1}{4}\right)^2+16(y+1)^2=16 \quad\Leftrightarrow\quad \left(x+\frac{1}{4}\right)^2+(y+1)^2=1$. Thus, we have a circle with center $\left(-\frac{1}{4},-1\right)$ and radius 1.

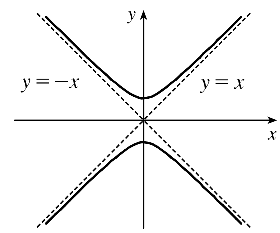
9. $2x^2+2y^2-x+y=1 \quad\Leftrightarrow\quad 2\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)+2\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)=1+\frac{1}{8}+\frac{1}{8} \quad\Leftrightarrow$ $2\left(x-\frac{1}{4}\right)^2+2\left(y+\frac{1}{4}\right)^2=\frac{5}{4} \quad\Leftrightarrow\quad \left(x-\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2=\frac{5}{8}$. Thus, we have a circle with center $\left(\frac{1}{4},-\frac{1}{4}\right)$ and radius $\frac{\sqrt{5}}{2\sqrt{2}}=\frac{\sqrt{10}}{4}$.

10. $x^2+y^2+ax+by+c=0 \quad\Leftrightarrow\quad \left(x^2+ax+\frac{1}{4}a^2\right)+\left(y^2+by+\frac{1}{4}b^2\right)=-c+\frac{1}{4}a^2+\frac{1}{4}b^2 \quad\Leftrightarrow$ $\left(x+\frac{1}{2}a\right)^2+\left(y+\frac{1}{2}b\right)^2=\frac{1}{4}(a^2+b^2-4c)$. For this to represent a nondegenerate circle, $\frac{1}{4}(a^2+b^2-4c)>0$ or $a^2+b^2>4c$. If this condition is satisfied, the circle has center $\left(-\frac{1}{2}a,-\frac{1}{2}b\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2-4c}$.

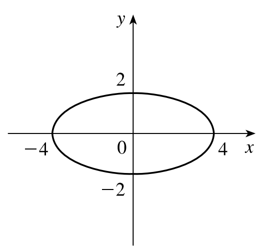
11. $y=-x^2$. Parabola



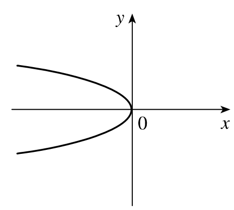
12. $y^2-x^2=1$. Hyperbola



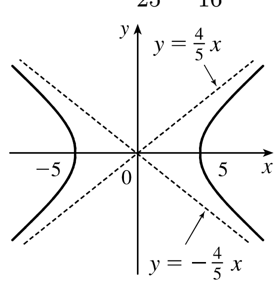
13. $x^2+4y^2=16 \quad\Leftrightarrow\quad \dfrac{x^2}{16}+\dfrac{y^2}{4}=1$. Ellipse



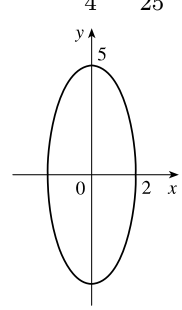
14. $x=-2y^2$. Parabola



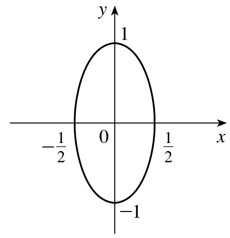
15. $16x^2-25y^2=400 \quad\Leftrightarrow\quad \dfrac{x^2}{25}-\dfrac{y^2}{16}=1$. Hyperbola



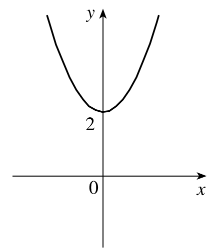
16. $25x^2+4y^2=100 \quad\Leftrightarrow\quad \dfrac{x^2}{4}+\dfrac{y^2}{25}=1$. Ellipse



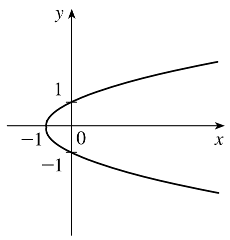
17. $4x^2 + y^2 = 1 \quad \Leftrightarrow \quad \dfrac{x^2}{1/4} + y^2 = 1$. Ellipse



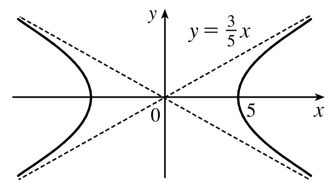
18. $y = x^2 + 2$. Parabola with vertex at $(0, 2)$



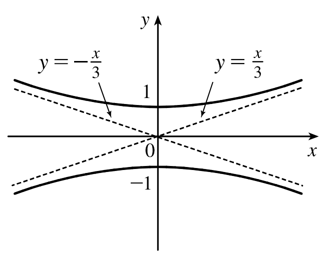
19. $x = y^2 - 1$. Parabola with vertex at $(-1, 0)$



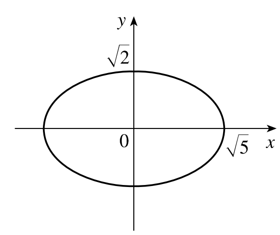
20. $9x^2 - 25y^2 = 225 \quad \Leftrightarrow \quad \dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$. Hyperbola



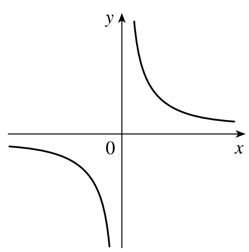
21. $9y^2 - x^2 = 9 \quad \Leftrightarrow \quad y^2 - \dfrac{x^2}{9} = 1$. Hyperbola



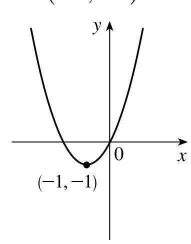
22. $2x^2 + 5y^2 = 10 \quad \Leftrightarrow \quad \dfrac{x^2}{5} + \dfrac{y^2}{2} = 1$. Ellipse



23. $xy = 4$. Hyperbola

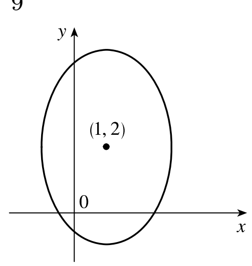


24. $y = x^2 + 2x = \left(x^2 + 2x + 1\right) - 1 = (x + 1)^2 - 1$.
Parabola with vertex at $(-1, -1)$



25. $9(x - 1)^2 + 4(y - 2)^2 = 36 \quad \Leftrightarrow$

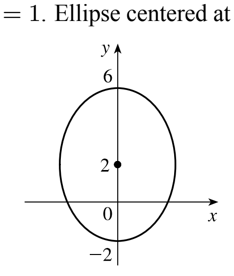
$\dfrac{(x - 1)^2}{4} + \dfrac{(y - 2)^2}{9} = 1$. Ellipse centered at $(1, 2)$



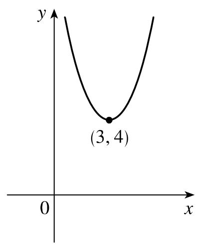
26. $16x^2 + 9y^2 - 36y = 108 \quad \Leftrightarrow$

$16x^2 + 9(y^2 - 4y + 4) = 108 + 36 = 144 \quad \Leftrightarrow$

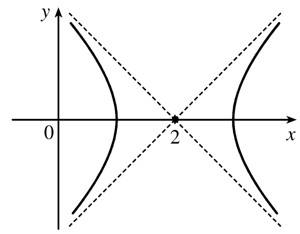
$\dfrac{x^2}{9} + \dfrac{(y - 2)^2}{16} = 1$. Ellipse centered at $(0, 2)$



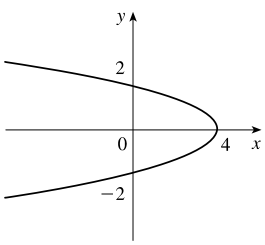
27. $y = x^2 - 6x + 13 = \left(x^2 - 6x + 9\right) + 4 = (x - 3)^2 + 4.$
Parabola with vertex at $(3, 4)$



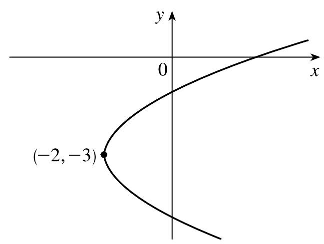
28. $x^2 - y^2 - 4x + 3 = 0 \quad \Leftrightarrow$
$\left(x^2 - 4x + 4\right) - y^2 = -3 + 4 = 1 \quad \Leftrightarrow$
$(x - 2)^2 - y^2 = 1$. Hyperbola centered at $(2, 0)$



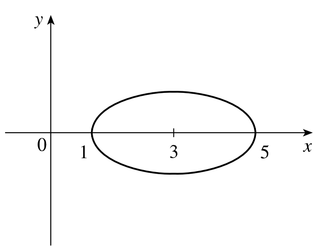
29. $x = 4 - y^2 = -y^2 + 4$. Parabola with vertex at $(4, 0)$



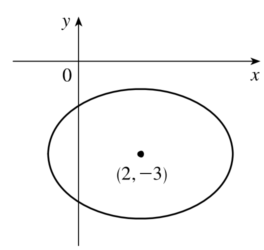
30. $y^2 - 2x + 6y + 5 = 0 \quad \Leftrightarrow \quad y^2 + 6y + 9 = 2x + 4 \quad \Leftrightarrow$
$(y + 3)^2 = 2\,(x + 2)$. Parabola with vertex $(-2, -3)$



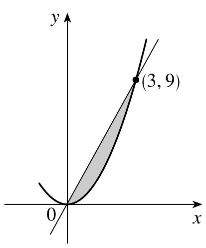
31. $x^2 + 4y^2 - 6x + 5 = 0 \quad \Leftrightarrow$
$(x^2 - 6x + 9) + 4y^2 = -5 + 9 = 4 \quad \Leftrightarrow$
$\dfrac{(x - 3)^2}{4} + y^2 = 1$. Ellipse centered at $(3, 0)$



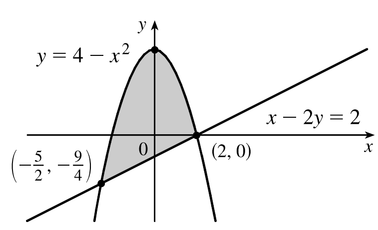
32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0 \quad \Leftrightarrow$
$4(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -61 + 16 + 81 = 36$
$\Leftrightarrow \quad \dfrac{(x - 2)^2}{9} + \dfrac{(y + 3)^2}{4} = 1$. Ellipse centered at $(2, -3)$



33. $y = 3x$ and $y = x^2$ intersect where $3x = x^2 \quad \Leftrightarrow$
$0 = x^2 - 3x = x(x - 3)$, that is, at $(0, 0)$ and $(3, 9)$.



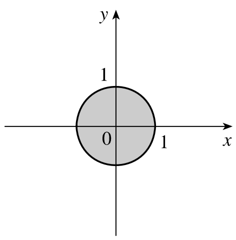
34. $y = 4 - x^2$, $x - 2y = 2$. Substitute y from the first equation into the second: $x - 2(4 - x^2) = 2 \quad \Leftrightarrow$
$2x^2 + x - 10 = 0 \quad \Leftrightarrow \quad (2x + 5)(x - 2) = 0 \quad \Leftrightarrow$
$x = -\frac{5}{2}$ or 2. So the points of intersection are $\left(-\frac{5}{2}, -\frac{9}{4}\right)$ and $(2, 0)$.



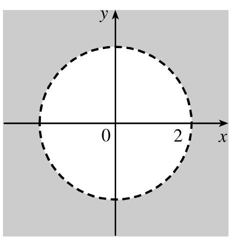
35. The parabola must have an equation of the form $y = a(x-1)^2 - 1$. Substituting $x = 3$ and $y = 3$ into the equation gives $3 = a(3-1)^2 - 1$, so $a = 1$, and the equation is $y = (x-1)^2 - 1 = x^2 - 2x$. Note that using the other point $(-1, 3)$ would have given the same value for a, and hence the same equation.

36. The ellipse has an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Substituting $x = 1$ and $y = -\dfrac{10\sqrt{2}}{3}$ gives $\dfrac{1^2}{a^2} + \dfrac{\left(-10\sqrt{2}/3\right)^2}{b^2} = \dfrac{1}{a^2} + \dfrac{200}{9b^2} = 1$. Substituting $x = -2$ and $y = \dfrac{5\sqrt{5}}{3}$ gives $\dfrac{(-2)^2}{a^2} + \dfrac{\left(5\sqrt{5}/3\right)^2}{b^2} = \dfrac{4}{a^2} + \dfrac{125}{9b^2} = 1$. From the first equation, $\dfrac{1}{a^2} = 1 - \dfrac{200}{9b^2}$. Putting this into the second equation gives $4\left(1 - \dfrac{200}{9b^2}\right) + \dfrac{125}{9b^2} = 1 \quad\Leftrightarrow$ $3 = \dfrac{675}{9b^2} \quad\Leftrightarrow\quad b^2 = \dfrac{675}{27} = 25$, so $b = 5$. Hence $\dfrac{1}{a^2} = 1 - \dfrac{200}{9(5)^2} = \dfrac{1}{9}$ and so $a = 3$. The equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

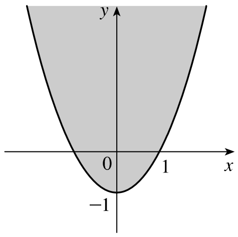
37. $\{(x, y) \mid x^2 + y^2 \le 1\}$



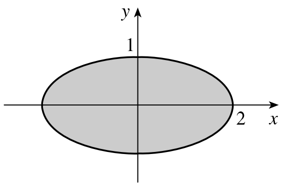
38. $\{(x, y) \mid x^2 + y^2 > 4\}$



39. $\{(x, y) \mid y \ge x^2 - 1\}$



40. $\{(x, y) \mid x^2 + 4y^2 \le 4\}$



D Trigonometry

1. $210^\circ = 210^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{7\pi}{6}$ rad

2. $300^\circ = 300^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{3}$ rad

3. $9^\circ = 9^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{20}$ rad

4. $-315^\circ = -315^\circ\left(\frac{\pi}{180^\circ}\right) = -\frac{7\pi}{4}$ rad

5. $900^\circ = 900^\circ\left(\frac{\pi}{180^\circ}\right) = 5\pi$ rad

6. $36^\circ = 36^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{5}$ rad

7. $4\pi \text{ rad} = 4\pi\left(\frac{180^\circ}{\pi}\right) = 720^\circ$

8. $-\frac{7\pi}{2} \text{ rad} = -\frac{7\pi}{2}\left(\frac{180^\circ}{\pi}\right) = -630^\circ$

9. $\frac{5\pi}{12} \text{ rad} = \frac{5\pi}{12}\left(\frac{180^\circ}{\pi}\right) = 75^\circ$

10. $\frac{8\pi}{3} \text{ rad} = \frac{8\pi}{3}\left(\frac{180^\circ}{\pi}\right) = 480^\circ$

11. $-\frac{3\pi}{8}$ rad $= -\frac{3\pi}{8}\left(\frac{180^\circ}{\pi}\right) = -67.5^\circ$

12. 5 rad $= 5\left(\frac{180^\circ}{\pi}\right) = \left(\frac{900}{\pi}\right)^\circ$

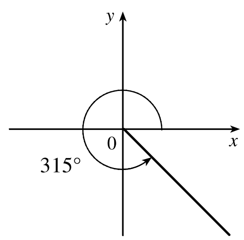
13. Using Formula 3, $a = r\theta = 36 \cdot \frac{\pi}{12} = 3\pi$ cm.

14. Using Formula 3, $a = r\theta = 10 \cdot 72^\circ\left(\frac{\pi}{180^\circ}\right) = 4\pi$ cm.

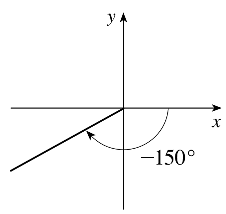
15. Using Formula 3, $\theta = a/r = \frac{1}{1.5} = \frac{2}{3}$ rad $= \frac{2}{3}\left(\frac{180^\circ}{\pi}\right) = \left(\frac{120}{\pi}\right)^\circ \approx 38.2^\circ$.

16. $a = r\theta \quad \Rightarrow \quad r = \frac{a}{\theta} = \frac{6}{3\pi/4} = \frac{8}{\pi}$ cm

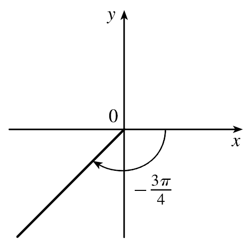
17.



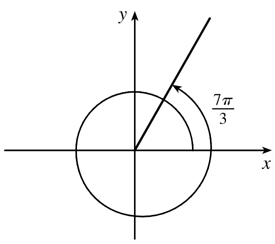
18.



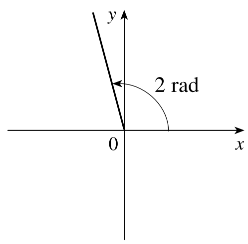
19.



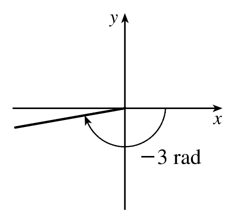
20.



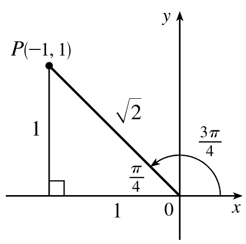
21.



22.

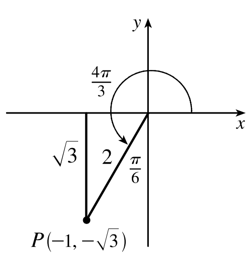


23.



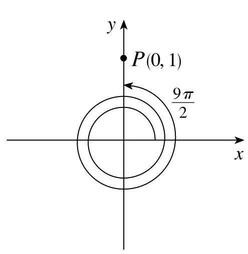
From the diagram we see that a point on the terminal side is $P(-1, 1)$. Therefore, taking $x = -1$, $y = 1$, $r = \sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan\frac{3\pi}{4} = -1$, $\csc\frac{3\pi}{4} = \sqrt{2}$, $\sec\frac{3\pi}{4} = -\sqrt{2}$, and $\cot\frac{3\pi}{4} = -1$.

24.



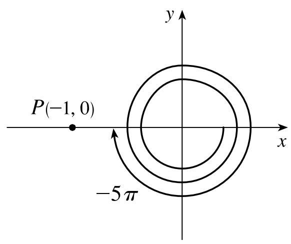
From the diagram and Figure 8, we see that a point on the terminal side is $P\left(-1, -\sqrt{3}\right)$. Therefore, taking $x = -1$, $y = -\sqrt{3}$, $r = 2$ in the definitions of the trigonometric ratios, we have $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = -\frac{1}{2}$, $\tan\frac{4\pi}{3} = \sqrt{3}$, $\csc\frac{4\pi}{3} = -\frac{2}{\sqrt{3}}$, $\sec\frac{4\pi}{3} = -2$, and $\cot\frac{4\pi}{3} = \frac{1}{\sqrt{3}}$.

25.



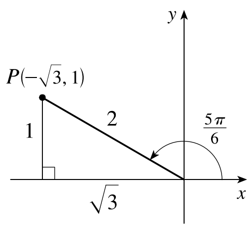
From the diagram we see that a point on the terminal side is $P(0, 1)$. Therefore taking $x = 0$, $y = 1$, $r = 1$ in the definitions of the trigonometric ratios, we have $\sin\frac{9\pi}{2} = 1$, $\cos\frac{9\pi}{2} = 0$, $\tan\frac{9\pi}{2} = y/x$ is undefined since $x = 0$, $\csc\frac{9\pi}{2} = 1$, $\sec\frac{9\pi}{2} = r/x$ is undefined since $x = 0$, and $\cot\frac{9\pi}{2} = 0$.

26.



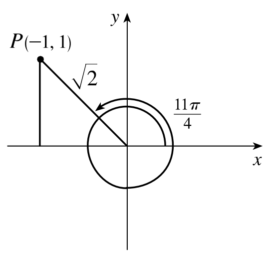
From the diagram, we see that a point on the terminal side is $P(-1, 0)$. Therefore taking $x = -1$, $y = 0$, $r = 1$ in the definitions of the trigonometric ratios we have $\sin(-5\pi) = 0$, $\cos(-5\pi) = -1$, $\tan(-5\pi) = 0$, $\csc(-5\pi)$ is undefined, $\sec(-5\pi) = -1$, and $\cot(-5\pi)$ is undefined.

27.



Using Figure 8 we see that a point on the terminal side is $P\left(-\sqrt{3}, 1\right)$. Therefore taking $x = -\sqrt{3}$, $y = 1$, $r = 2$ in the definitions of the trigonometric ratios, we have $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$, $\csc\frac{5\pi}{6} = 2$, $\sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$, and $\cot\frac{5\pi}{6} = -\sqrt{3}$.

28.



From the diagram, we see that a point on the terminal side is $P(-1, 1)$. Therefore taking $x = -1$, $y = 1$, $r = \sqrt{2}$ in the definitions of the trigonometric ratios we have $\sin\frac{11\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{11\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan\frac{11\pi}{4} = -1$, $\csc\frac{11\pi}{4} = \sqrt{2}$, $\sec\frac{11\pi}{4} = -\sqrt{2}$, and $\cot\frac{11\pi}{4} = -1$.

29. $\sin\theta = y/r = \frac{3}{5} \quad\Rightarrow\quad y = 3$, $r = 5$, and $x = \sqrt{r^2 - y^2} = 4$ (since $0 < \theta < \frac{\pi}{2}$). Therefore taking $x = 4$, $y = 3$, $r = 5$ in the definitions of the trigonometric ratios, we have $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, and $\cot\theta = \frac{4}{3}$.

30. Since $0 < \alpha < \frac{\pi}{2}$, α is in the first quadrant where x and y are both positive. Therefore, $\tan\alpha = y/x = \frac{2}{1} \quad\Rightarrow\quad y = 2$, $x = 1$, and $r = \sqrt{x^2 + y^2} = \sqrt{5}$. Taking $x = 1$, $y = 2$, $r = \sqrt{5}$ in the definitions of the trigonometric ratios, we have $\sin\alpha = \frac{2}{\sqrt{5}}$, $\cos\alpha = \frac{1}{\sqrt{5}}$, $\csc\alpha = \frac{\sqrt{5}}{2}$, $\sec\alpha = \sqrt{5}$, and $\cot\alpha = \frac{1}{2}$.

31. $\frac{\pi}{2} < \phi < \pi \quad\Rightarrow\quad \phi$ is in the second quadrant, where x is negative and y is positive. Therefore $\sec\phi = r/x = -1.5 = -\frac{3}{2} \quad\Rightarrow\quad r = 3$, $x = -2$, and $y = \sqrt{r^2 - x^2} = \sqrt{5}$. Taking $x = -2$, $y = \sqrt{5}$, and $r = 3$ in the definitions of the trigonometric ratios, we have $\sin\phi = \frac{\sqrt{5}}{3}$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\frac{\sqrt{5}}{2}$, $\csc\phi = \frac{3}{\sqrt{5}}$, and $\cot\theta = -\frac{2}{\sqrt{5}}$.

32. Since $\pi < x < \frac{3\pi}{2}$, x is in the third quadrant where x and y are both negative. Therefore $\cos x = x/r = -\frac{1}{3} \quad\Rightarrow\quad x = -1$, $r = 3$, and $y = -\sqrt{r^2 - x^2} = -\sqrt{8} = -2\sqrt{2}$. Taking $x = -1$, $r = 3$, $y = -2\sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = 2\sqrt{2}$, $\csc x = -\frac{3}{2\sqrt{2}}$, $\sec x = -3$, and $\cot x = \frac{1}{2\sqrt{2}}$.

33. $\pi < \beta < 2\pi$ means that β is in the third or fourth quadrant where y is negative. Also since $\cot\beta = x/y = 3$ which is positive, x must also be negative. Therefore $\cot\beta = x/y = \frac{3}{1} \quad\Rightarrow\quad x = -3$, $y = -1$, and $r = \sqrt{x^2 + y^2} = \sqrt{10}$. Taking $x = -3$, $y = -1$ and $r = \sqrt{10}$ in the definitions of the trigonometric ratios, we have $\sin\beta = -\frac{1}{\sqrt{10}}$, $\cos\beta = -\frac{3}{\sqrt{10}}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, and $\sec\beta = -\frac{\sqrt{10}}{3}$.

34. Since $\frac{3\pi}{2} < \theta < 2\pi$, θ is in the fourth quadrant where x is positive and y is negative. Therefore $\csc\theta = r/y = -\frac{4}{3} \Rightarrow$ $r = 4$, $y = -3$, and $x = \sqrt{r^2 - y^2} = \sqrt{7}$. Taking $x = \sqrt{7}$, $y = -3$, and $r = 4$ in the definitions of the trigonometric ratios, we have $\sin\theta = -\frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$, $\sec\theta = \frac{4}{\sqrt{7}}$, and $\cot\theta = -\frac{\sqrt{7}}{3}$.

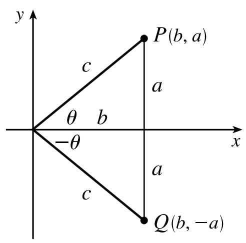
35. $\sin 35^\circ = \dfrac{x}{10} \quad\Rightarrow\quad x = 10\sin 35^\circ \approx 5.73576$ cm

36. $\cos 40^\circ = \dfrac{x}{25} \quad\Rightarrow\quad x = 25\cos 40^\circ \approx 19.15111$ cm

37. $\tan\frac{2\pi}{5} = \dfrac{x}{8} \quad\Rightarrow\quad x = 8\tan\frac{2\pi}{5} \approx 24.62147$ cm

38. $\cos\dfrac{3\pi}{8} = \dfrac{22}{x} \quad\Rightarrow\quad x = \dfrac{22}{\cos\frac{3\pi}{8}} \approx 57.48877$ cm

39.



(a) From the diagram we see that $\sin\theta = \dfrac{y}{r} = \dfrac{a}{c}$, and $\sin(-\theta) = \dfrac{-a}{c} = -\dfrac{a}{c} = -\sin\theta$.

(b) Again from the diagram we see that $\cos\theta = \dfrac{x}{r} = \dfrac{b}{c} = \cos(-\theta)$.

40. (a) Using (12a) and (12b), we have

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\,\cos y + \cos x\,\sin y}{\cos x\,\cos y - \sin x\,\sin y} = \frac{\dfrac{\sin x\,\cos y}{\cos x\,\cos y} + \dfrac{\cos x\,\sin y}{\cos x\,\cos y}}{\dfrac{\cos x\,\cos y}{\cos x\,\cos y} - \dfrac{\sin x\,\sin y}{\cos x\,\cos y}} = \frac{\tan x + \tan y}{1 - \tan x\,\tan y}$$

(b) From (10a) and (10b), we have $\tan(-\theta) = -\tan\theta$, so (14a) implies that

$$\tan(x-y) = \tan(x + (-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x\,\tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x\,\tan y}$$

41. (a) Using (12a) and (13a), we have

$\frac{1}{2}[\sin(x+y) + \sin(x-y)] = \frac{1}{2}[\sin x\,\cos y + \cos x\,\sin y + \sin x\,\cos y - \cos x\,\sin y] = \frac{1}{2}(2\sin x\,\cos y) = \sin x\,\cos y.$

(b) This time, using (12b) and (13b), we have

$\frac{1}{2}[\cos(x+y) + \cos(x-y)] = \frac{1}{2}[\cos x\,\cos y - \sin x\,\sin y + \cos x\,\cos y + \sin x\,\sin y] = \frac{1}{2}(2\cos x\,\cos y) = \cos x\,\cos y.$

(c) Again using (12b) and (13b), we have

$$\begin{aligned}\tfrac{1}{2}[\cos(x-y) - \cos(x+y)] &= \tfrac{1}{2}[\cos x\,\cos y + \sin x\,\sin y - \cos x\,\cos y + \sin x\,\sin y] \\ &= \tfrac{1}{2}(2\sin x\,\sin y) = \sin x\,\sin y\end{aligned}$$

42. Using (13b), $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = 0\cdot\cos x + 1\cdot\sin x = \sin x$.

43. Using (12a), we have $\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x = 1\cdot\cos x + 0\cdot\sin x = \cos x$.

44. Using (13a), we have $\sin(\pi - x) = \sin\pi\cos x - \cos\pi\,\sin x = 0\cdot\cos x - (-1)\sin x = \sin x$.

45. Using (6), we have $\sin\theta\cot\theta = \sin\theta\cdot\dfrac{\cos\theta}{\sin\theta} = \cos\theta$.

46. $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\,\cos x + \cos^2 x = (\sin^2 x + \cos^2 x) + \sin 2x$ [by (15a)] $= 1 + \sin 2x$ [by (7)]

47. $\sec y - \cos y = \dfrac{1}{\cos y} - \cos y$ [by (6)] $= \dfrac{1 - \cos^2 y}{\cos y} = \dfrac{\sin^2 y}{\cos y}$ [by (7)] $= \dfrac{\sin y}{\cos y} \sin y = \tan y \sin y$ [by (6)]

48. $\tan^2 \alpha - \sin^2 \alpha = \dfrac{\sin^2 \alpha}{\cos^2 \alpha} - \sin^2 \alpha = \dfrac{\sin^2 \alpha - \sin^2 \alpha \cos^2 \alpha}{\cos^2 \alpha} = \dfrac{\sin^2 \alpha\,(1 - \cos^2 \alpha)}{\cos^2 \alpha} = \tan^2 \alpha \sin^2 \alpha$ [by (6), (7)]

49. $\cot^2 \theta + \sec^2 \theta = \dfrac{\cos^2 \theta}{\sin^2 \theta} + \dfrac{1}{\cos^2 \theta}$ [by (6)] $= \dfrac{\cos^2 \theta \cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$

$= \dfrac{(1 - \sin^2 \theta)(1 - \sin^2 \theta) + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$ [by (7)] $= \dfrac{1 - \sin^2 \theta + \sin^4 \theta}{\sin^2 \theta \cos^2 \theta}$

$= \dfrac{\cos^2 \theta + \sin^4 \theta}{\sin^2 \theta \cos^2 \theta}$ [*by* (7)] $= \dfrac{1}{\sin^2 \theta} + \dfrac{\sin^2 \theta}{\cos^2 \theta} = \csc^2 \theta + \tan^2 \theta$ [by (6)]

50. $2 \csc 2t = \dfrac{2}{\sin 2t} = \dfrac{2}{2 \sin t \cos t}$ [by (15a)] $= \dfrac{1}{\sin t \cos t} = \sec t \csc t$

51. Using (14a), we have $\tan 2\theta = \tan(\theta + \theta) = \dfrac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$.

52. $\dfrac{1}{1 - \sin \theta} + \dfrac{1}{1 + \sin \theta} = \dfrac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} = \dfrac{2}{1 - \sin^2 \theta} = \dfrac{2}{\cos^2 \theta}$ [by (7)] $= 2 \sec^2 \theta$

53. Using (15a) and (16a),

$$\begin{aligned}
\sin x \sin 2x + \cos x \cos 2x &= \sin x\,(2 \sin x \cos x) + \cos x\,(2 \cos^2 x - 1) = 2 \sin^2 x \cos x + 2 \cos^3 x - \cos x \\
&= 2(1 - \cos^2 x) \cos x + 2 \cos^3 x - \cos x \quad \text{[by (7)]} \\
&= 2 \cos x - 2 \cos^3 x + 2 \cos^3 x - \cos x = \cos x
\end{aligned}$$

Or: $\sin x \sin 2x + \cos x \cos 2x = \cos(2x - x)$ [by 13(b)] $= \cos x$

54. We start with the right side using equations (12a) and (13a):

$$\begin{aligned}
\sin(x + y) \sin(x - y) &= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) \\
&= \sin^2 x \cos^2 y - \sin x \cos y \cos x \sin y + \cos x \sin y \sin x \cos y - \cos^2 x \sin^2 y \\
&= \sin^2 x\,(1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y \quad \text{[by (7)]} \\
&= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y = \sin^2 x - \sin^2 y
\end{aligned}$$

55. $\dfrac{\sin \phi}{1 - \cos \phi} = \dfrac{\sin \phi}{1 - \cos \phi} \cdot \dfrac{1 + \cos \phi}{1 + \cos \phi} = \dfrac{\sin \phi\,(1 + \cos \phi)}{1 - \cos^2 \phi} = \dfrac{\sin \phi\,(1 + \cos \phi)}{\sin^2 \phi}$ [by (7)]

$= \dfrac{1 + \cos \phi}{\sin \phi} = \dfrac{1}{\sin \phi} + \dfrac{\cos \phi}{\sin \phi} = \csc \phi + \cot \phi$ [by (6)]

56. $\tan x + \tan y = \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y} = \dfrac{\sin(x + y)}{\cos x \cos y}$ [by (12a)]

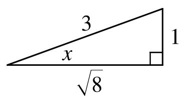
57. Using (12a),

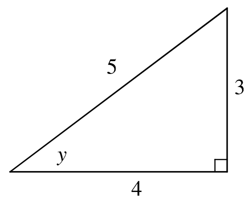
$$\begin{aligned}
\sin 3\theta + \sin \theta &= \sin(2\theta + \theta) + \sin \theta = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta + \sin \theta \\
&= \sin 2\theta \cos \theta + (2 \cos^2 \theta - 1) \sin \theta + \sin \theta \quad \text{[by (16a)]} \\
&= \sin 2\theta \cos \theta + 2 \cos^2 \theta \sin \theta - \sin \theta + \sin \theta = \sin 2\theta \cos \theta + \sin 2\theta \cos \theta \quad \text{[by (15a)]} \\
&= 2 \sin 2\theta \cos \theta
\end{aligned}$$

58. We use (12b) with $x = 2\theta$, $y = \theta$ to get

$$\begin{aligned} \cos 3\theta = \cos(2\theta + \theta) &= \cos 2\theta \, \cos\theta - \sin 2\theta \, \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \, \cos\theta \quad \text{[by (16a) and (15a)]} \\ &= (2\cos^2\theta - 1)\cos\theta - 2(1 - \cos^2\theta) \, \cos\theta \quad \text{[by (7)]} \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta \end{aligned}$$

59. Since $\sin x = \frac{1}{3}$ we can label the opposite side as having length 1, the hypotenuse as having length 3, and use the Pythagorean Theorem to get that the adjacent side has length $\sqrt{8}$. Then, from the diagram, $\cos x = \frac{\sqrt{8}}{3}$. Similarly we have that $\sin y = \frac{3}{5}$. Now use (12a):





$\sin(x + y) = \sin x \, \cos y + \cos x \, \sin y = \frac{1}{3} \cdot \frac{4}{5} + \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4}{15} + \frac{3\sqrt{8}}{15} = \frac{4 + 6\sqrt{2}}{15}$.

60. Use (12b) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59 to get

$$\cos(x + y) = \cos x \, \cos y - \sin x \, \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} - \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2} - 3}{15}$$

61. Using (13b) and the values for $\cos x$ and $\sin y$ obtained in Exercise 59, we have

$$\cos(x - y) = \cos x \, \cos y + \sin x \, \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2} + 3}{15}$$

62. Using (13a) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59, we get

$$\sin(x - y) = \sin x \, \cos y - \cos x \, \sin y = \frac{1}{3} \cdot \frac{4}{5} - \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4 - 6\sqrt{2}}{15}$$

63. Using (15a) and the values for $\sin y$ and $\cos y$ obtained in Exercise 59, we have $\sin 2y = 2\sin y \, \cos y = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$.

64. Using (16a) with $\cos y = \frac{4}{5}$, we have $\cos 2y = 2\cos^2 y - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$.

65. $2\cos x - 1 = 0 \quad \Leftrightarrow \quad \cos x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$ for $x \in [0, 2\pi]$.

66. $3\cot^2 x = 1 \quad \Leftrightarrow \quad 3 = 1/\cot^2 x \quad \Leftrightarrow \quad \tan^2 x = 3 \quad \Leftrightarrow \quad \tan x = \pm\sqrt{3} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

67. $2\sin^2 x = 1 \quad \Leftrightarrow \quad \sin^2 x = \frac{1}{2} \quad \Leftrightarrow \quad \sin x = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

68. $|\tan x| = 1 \quad \Leftrightarrow \quad \tan x = -1$ or $\tan x = 1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

69. Using (15a), we have $\sin 2x = \cos x \quad \Leftrightarrow \quad 2\sin x \, \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x(2\sin x - 1) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $2\sin x - 1 = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Therefore, the solutions are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

70. By (15a), $2\cos x + \sin 2x = 0 \quad \Leftrightarrow \quad 2\cos x + 2\sin x\cos x = 0 \quad \Leftrightarrow \quad 2\cos x\,(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $1 + \sin x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = -1 \quad \Rightarrow \quad x = \frac{3}{2}\pi$. So the solutions are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

71. $\sin x = \tan x \quad \Leftrightarrow \quad \sin x - \tan x = 0 \quad \Leftrightarrow \quad \sin x - \dfrac{\sin x}{\cos x} = 0 \quad \Leftrightarrow \quad \sin x\left(1 - \dfrac{1}{\cos x}\right) = 0 \quad \Leftrightarrow \quad \sin x = 0$ or $1 - \dfrac{1}{\cos x} = 0 \quad \Rightarrow \quad x = 0, \pi, 2\pi$ or $1 = \dfrac{1}{\cos x} \quad \Rightarrow \quad \cos x = 1 \quad \Rightarrow \quad x = 0, 2\pi$. Therefore the solutions are $x = 0, \pi, 2\pi$.

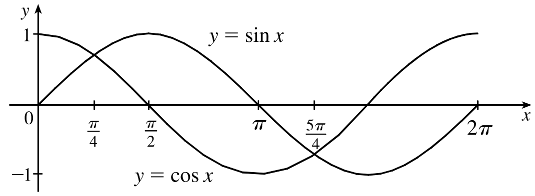
72. By (16a), $2 + \cos 2x = 3\cos x \quad \Leftrightarrow \quad 2 + 2\cos^2 x - 1 = 3\cos x \quad \Leftrightarrow \quad 2\cos^2 x - 3\cos x + 1 = 0 \quad \Leftrightarrow$ $(2\cos x - 1)(\cos x - 1) = 0 \quad \Leftrightarrow \quad \cos x = 1$ or $\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 0, 2\pi$ or $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

73. We know that $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, and from Figure 13(a), we see that $\sin x \le \frac{1}{2} \quad \Rightarrow \quad 0 \le x \le \frac{\pi}{6}$ or $\frac{5\pi}{6} \le x \le 2\pi$ for $x \in [0, 2\pi]$.

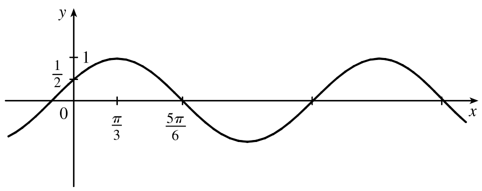
74. $2\cos x + 1 > 0 \quad \Rightarrow \quad 2\cos x > -1 \quad \Rightarrow \quad \cos x > -\frac{1}{2}$. $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ and from Figure 13(b), we see that $\cos x > -\frac{1}{2}$ when $0 \le x < \frac{2\pi}{3}$, $\frac{4\pi}{3} < x \le 2\pi$.

75. $\tan x = -1$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, and $\tan x = 1$ when $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. From Figure 14(a) we see that $-1 < \tan x < 1 \quad \Rightarrow$ $0 \le x < \frac{\pi}{4}$, $\frac{3\pi}{4} < x < \frac{5\pi}{4}$, and $\frac{7\pi}{4} < x \le 2\pi$.

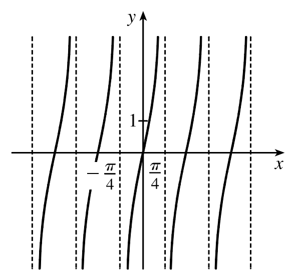
76. We know that $\sin x = \cos x$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$, and from the diagram we see that $\sin x > \cos x$ when $\frac{\pi}{4} < x < \frac{5\pi}{4}$.



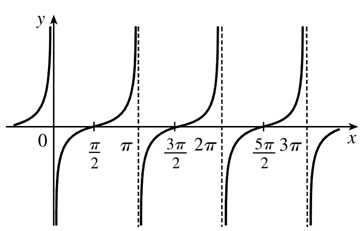
77. $y = \cos\left(x - \frac{\pi}{3}\right)$. We start with the graph of $y = \cos x$ and shift it $\frac{\pi}{3}$ units to the right.



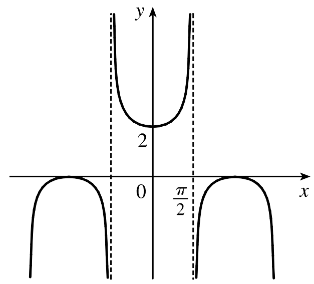
78. $y = \tan 2x$. Start with the graph of $y = \tan x$ with period π and compress it to a period of $\frac{\pi}{2}$.



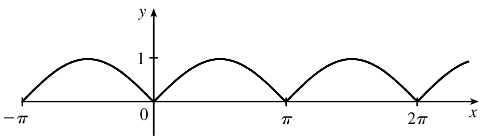
79. $y = \frac{1}{3}\tan\left(x - \frac{\pi}{2}\right)$. We start with the graph of $y = \tan x$, shift it $\frac{\pi}{2}$ units to the right and compress it to $\frac{1}{3}$ of its original vertical size.



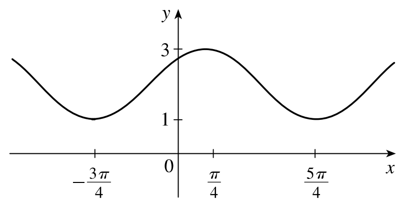
80. $y = 1 + \sec x$. Start with the graph of $y = \sec x$ and raise it by one unit.



81. $y = |\sin x|$. We start with the graph of $y = \sin x$ and reflect the parts below the x-axis about the x-axis.



82. $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$. Start with the graph of $y = \sin x$, and shift it $\frac{\pi}{4}$ units to the left and 2 units up.



83. From the figure in the text, we see that $x = b\cos\theta$, $y = b\sin\theta$, and from the distance formula we have that the distance c from (x, y) to $(a, 0)$ is $c = \sqrt{(x-a)^2 + (y-0)^2} \quad \Rightarrow$

$$\begin{aligned} c^2 &= (b\cos\theta - a)^2 + (b\sin\theta)^2 = b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta \\ &= a^2 + b^2(\cos^2\theta + \sin^2\theta) - 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta \quad \text{[by (7)]} \end{aligned}$$

84. $|AB|^2 = |AC|^2 + |BC|^2 - 2\,|AC|\,|BC|\cos\angle C = (820)^2 + (910)^2 - 2(820)(910)\cos 103^\circ \approx 1{,}836{,}217 \quad \Rightarrow$ $|AB| \approx 1355$ m

85. Using the Law of Cosines, we have $c^2 = 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta) = 2\,[1 - \cos(\alpha - \beta)]$. Now, using the distance formula, $c^2 = |AB|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$. Equating these two expressions for c^2, we get

$2[1 - \cos(\alpha - \beta)] = \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\,\cos\beta - 2\sin\alpha\,\sin\beta \quad \Rightarrow$

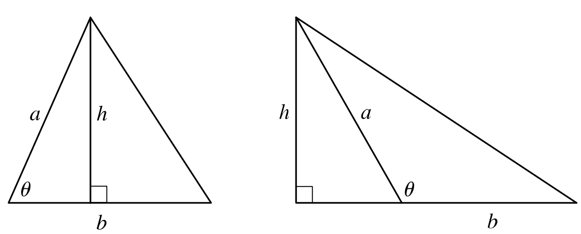
$1 - \cos(\alpha - \beta) = 1 - \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \quad \Rightarrow \quad \cos(\alpha - \beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta$.

86.
$$\begin{aligned} \cos(x + y) &= \cos(x - (-y)) = \cos x\,\cos(-y) + \sin x\,\sin(-y) \\ &= \cos x\,\cos y - \sin x\,\sin y \quad \text{[using Equations (10a) and (10b)]} \end{aligned}$$

87. In Exercise 86 we used the subtraction formula for cosine to prove the addition formula for cosine. Using that formula with $x = \frac{\pi}{2} - \alpha$, $y = \beta$, we get $\cos\left[\left(\frac{\pi}{2} - \alpha\right) + \beta\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \quad \Rightarrow$

$\cos\left[\frac{\pi}{2} - (\alpha - \beta)\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$. Now we use the identities given in the problem, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, to get $\sin(\alpha - \beta) = \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta$.

88. If $0 < \theta < \frac{\pi}{2}$, we have the case depicted in the first diagram. In this case, we see that the height of the triangle is $h = a\sin\theta$. If $\frac{\pi}{2} \leq \theta < \pi$, we have the case depicted in the second diagram. In this case, the height of the triangle is $h = a\sin(\pi - \theta) = a\sin\theta$ (by the identity proved in Exercise 44). So in either case, the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}ab\sin\theta$.



89. Using the formula from Exercise 88, the area of the triangle is $\frac{1}{2}(10)(3)\sin 107^\circ \approx 14.34457\ \text{cm}^2$.

E Sigma Notation

1. $\sum_{i=1}^{5} \sqrt{i} = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$

2. $\sum_{i=1}^{6} \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

3. $\sum_{i=4}^{6} 3^i = 3^4 + 3^5 + 3^6$

4. $\sum_{i=4}^{6} i^3 = 4^3 + 5^3 + 6^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1} = -1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$

6. $\sum_{k=5}^{8} x^k = x^5 + x^6 + x^7 + x^8$

7. $\sum_{i=1}^{n} i^{10} = 1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

8. $\sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$

9. $\sum_{j=0}^{n-1} (-1)^j = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + f(x_3)\,\Delta x_3 + \cdots + f(x_n)\,\Delta x_n$

11. $1 + 2 + 3 + 4 + \cdots + 10 = \sum_{i=1}^{10} i$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} = \sum_{i=3}^{7} \sqrt{i}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20} = \sum_{i=1}^{19} \frac{i}{i+1}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4}$

15. $2 + 4 + 6 + 8 + \cdots + 2n = \sum_{i=1}^{n} 2i$

16. $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^{n} (2i-1)$

17. $1 + 2 + 4 + 8 + 16 + 32 = \sum_{i=0}^{5} 2^i$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} = \sum_{i=1}^{6} \frac{1}{i^2}$

19. $x + x^2 + x^3 + \cdots + x^n = \sum_{i=1}^{n} x^i$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{i=0}^{n} (-1)^i x^i$

21. $\sum_{i=4}^{8} (3i-2) = [3(4)-2] + [3(5)-2] + [3(6)-2] + [3(7)-2] + [3(8)-2] = 10 + 13 + 16 + 19 + 22 = 80$

22. $\sum_{i=3}^{6} i(i+2) = 3 \cdot 5 + 4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 = 15 + 24 + 35 + 48 = 122$

23. $\sum_{j=1}^{6} 3^{j+1} = 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 = 9 + 27 + 81 + 243 + 729 + 2187 = 3276$

(For a more general method, see Exercise 47.)

24. $$\begin{aligned}\sum_{k=0}^{8} \cos k\pi &= \cos 0 + \cos \pi + \cos 2\pi + \cos 3\pi + \cos 4\pi + \cos 5\pi + \cos 6\pi + \cos 7\pi + \cos 8\pi \\ &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1\end{aligned}$$

25. $\sum_{n=1}^{20} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 0$

26. $\sum_{i=1}^{100} 4 = \underbrace{4 + 4 + 4 + \cdots + 4}_{\text{(100 summands)}} = 100 \cdot 4 = 400$

27. $\sum_{i=0}^{4}(2^i + i^2) = (1+0) + (2+1) + (4+4) + (8+9) + (16+16) = 61$

28. $\sum_{i=-2}^{4} 2^{3-i} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} = 63.5$

29. $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = 2 \cdot \dfrac{n(n+1)}{2}$ [by Theorem 3(c)] $= n(n+1)$

30. $\sum_{i=1}^{n}(2-5i) = \sum_{i=1}^{n} 2 - \sum_{i=1}^{n} 5i = 2n - 5\sum_{i=1}^{n} i = 2n - \dfrac{5n(n+1)}{2} = \dfrac{4n}{2} - \dfrac{5n^2+5n}{2} = -\dfrac{n(5n+1)}{2}$

31. $\sum_{i=1}^{n}(i^2+3i+4) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4 = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{3n(n+1)}{2} + 4n$

$$= \tfrac{1}{6}[(2n^3+3n^2+n) + (9n^2+9n) + 24n] = \tfrac{1}{6}(2n^3+12n^2+34n) = \tfrac{1}{3}n(n^2+6n+17)$$

32. $\sum_{i=1}^{n}(3+2i)^2 = \sum_{i=1}^{n}(9+12i+4i^2) = \sum_{i=1}^{n} 9 + 12\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2 = 9n + 6n(n+1) + \dfrac{2n(n+1)(2n+1)}{3}$

$$= \frac{27n+18n^2+18n+4n^3+6n^2+2n}{3} = \tfrac{1}{3}(4n^3+24n^2+47n) = \tfrac{1}{3}n(4n^2+24n+47)$$

33. $\sum_{i=1}^{n}(i+1)(i+2) = \sum_{i=1}^{n}(i^2+3i+2) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{3n(n+1)}{2} + 2n$

$$= \frac{n(n+1)}{6}[(2n+1)+9] + 2n = \frac{n(n+1)}{3}(n+5) + 2n$$

$$= \frac{n}{3}[(n+1)(n+5)+6] = \frac{n}{3}(n^2+6n+11)$$

34. $\sum_{i=1}^{n} i(i+1)(i+2) = \sum_{i=1}^{n}(i^3+3i^2+2i) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right] = \frac{n(n+1)}{4}(n^2+n+4n+2+4)$$

$$= \frac{n(n+1)}{4}(n^2+5n+6) = \frac{n(n+1)(n+2)(n+3)}{4}$$

35. $\sum_{i=1}^{n}(i^3-i-2) = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 = \left[\dfrac{n(n+1)}{2}\right]^2 - \dfrac{n(n+1)}{2} - 2n$

$$= \tfrac{1}{4}n(n+1)[n(n+1)-2] - 2n = \tfrac{1}{4}n(n+1)(n+2)(n-1) - 2n$$

$$= \tfrac{1}{4}n[(n+1)(n-1)(n+2)-8] = \tfrac{1}{4}n[(n^2-1)(n+2)-8] = \tfrac{1}{4}n(n^3+2n^2-n-10)$$

36. By Theorem 3(c) we have that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2} = 78 \quad\Leftrightarrow\quad n(n+1) = 156 \quad\Leftrightarrow\quad n^2+n-156 = 0 \quad\Leftrightarrow$

$(n+13)(n-12) = 0 \quad\Leftrightarrow\quad n = 12$ or -13. But $n = -13$ produces a negative answer for the sum, so $n = 12$.

37. By Theorem 2(a) and Example 3, $\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = cn$.

38. Let S_n be the statement that $\sum\limits_{i=1}^{n} i^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

1. S_1 is true because $1^3 = \left(\dfrac{1 \cdot 2}{2}\right)^2$.

2. Assume S_k is true. Then $\sum\limits_{i=1}^{k} i^3 = \left[\dfrac{k(k+1)}{2}\right]^2$, so

$$\sum_{i=1}^{k+1} i^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 = \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}(k+2)^2 = \left(\frac{(k+1)[(k+1)+1]}{2}\right)^2$$

showing that S_{k+1} is true.

Therefore, S_n is true for all n by mathematical induction.

39.
$$\begin{aligned}\sum_{i=1}^{n}[(i+1)^4 - i^4] &= (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \cdots + \left[(n+1)^4 - n^4\right] \\ &= (n+1)^4 - 1^4 = n^4 + 4n^3 + 6n^2 + 4n\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{n}[(i+1)^4 - i^4] &= \sum_{i=1}^{n}(4i^3 + 6i^2 + 4i + 1) = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= 4S + n(n+1)(2n+1) + 2n(n+1) + n \qquad \left[\text{where } S = \sum_{i=1}^{n} i^3\right] \\ &= 4S + 2n^3 + 3n^2 + n + 2n^2 + 2n + n = 4S + 2n^3 + 5n^2 + 4n\end{aligned}$$

Thus, $n^4 + 4n^3 + 6n^2 + 4n = 4S + 2n^3 + 5n^2 + 4n$, from which it follows that

$4S = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$ and $S = \left[\dfrac{n(n+1)}{2}\right]^2$.

40. The area of G_i is

$$\begin{aligned}\left(\sum_{k=1}^{i} k\right)^2 - \left(\sum_{k=1}^{i-1} k\right)^2 &= \left[\frac{i(i+1)}{2}\right]^2 - \left[\frac{(i-1)i}{2}\right]^2 = \frac{i^2}{4}\left[(i+1)^2 - (i-1)^2\right] \\ &= \frac{i^2}{4}\left[(i^2 + 2i + 1) - (i^2 - 2i + 1)\right] = \frac{i^2}{4}(4i) = i^3\end{aligned}$$

Thus, the area of $ABCD$ is $\sum\limits_{i=1}^{n} i^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

41. (a) $\sum\limits_{i=1}^{n}\left[i^4 - (i-1)^4\right] = (1^4 - 0^4) + (2^4 - 1^4) + (3^4 - 2^4) + \cdots + \left[n^4 - (n-1)^4\right] = n^4 - 0 = n^4$

(b) $\sum\limits_{i=1}^{100}(5^i - 5^{i-1}) = (5^1 - 5^0) + (5^2 - 5^1) + (5^3 - 5^2) + \cdots + (5^{100} - 5^{99}) = 5^{100} - 5^0 = 5^{100} - 1$

(c) $\sum\limits_{i=3}^{99}\left(\dfrac{1}{i} - \dfrac{1}{i+1}\right) = \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \left(\dfrac{1}{4} - \dfrac{1}{5}\right) + \left(\dfrac{1}{5} - \dfrac{1}{6}\right) + \cdots + \left(\dfrac{1}{99} - \dfrac{1}{100}\right) = \dfrac{1}{3} - \dfrac{1}{100} = \dfrac{97}{300}$

(d) $\sum\limits_{i=1}^{n}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) = a_n - a_0$

42. Summing the inequalities $-|a_i| \le a_i \le |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum\limits_{i=1}^{n}|a_i| \le \sum\limits_{i=1}^{n} a_i \le \sum\limits_{i=1}^{n}|a_i|$. Since $|x| \le c \iff -c \le x \le c$, we have $\left|\sum\limits_{i=1}^{n} a_i\right| \le \sum\limits_{i=1}^{n}|a_i|$. *Another method:* Use mathematical induction.

43. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2 = \lim\limits_{n\to\infty}\frac{1}{n^3}\sum\limits_{i=1}^{n} i^2 = \lim\limits_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \lim\limits_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$

44. $$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n}\left[\left(\frac{i}{n}\right)^3+1\right] = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left[\frac{i^3}{n^4}+\frac{1}{n}\right] = \lim\limits_{n\to\infty}\left[\frac{1}{n^4}\sum\limits_{i=1}^{n} i^3 + \frac{1}{n}\sum\limits_{i=1}^{n} 1\right] = \lim\limits_{n\to\infty}\left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{n}(n)\right]$$

$$= \lim\limits_{n\to\infty}\frac{1}{4}\left(1+\frac{1}{n}\right)^2 + 1 = \frac{1}{4}+1 = \frac{5}{4}$$

45. $$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{2}{n}\left[\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right] = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left[\frac{16}{n^4}i^3+\frac{20}{n^2}i\right] = \lim\limits_{n\to\infty}\left[\frac{16}{n^4}\sum\limits_{i=1}^{n} i^3 + \frac{20}{n^2}\sum\limits_{i=1}^{n} i\right]$$

$$= \lim\limits_{n\to\infty}\left[\frac{16}{n^4}\frac{n^2(n+1)^2}{4} + \frac{20}{n^2}\frac{n(n+1)}{2}\right] = \lim\limits_{n\to\infty}\left[\frac{4(n+1)^2}{n^2} + \frac{10n(n+1)}{n^2}\right]$$

$$= \lim\limits_{n\to\infty}\left[4\left(1+\frac{1}{n}\right)^2 + 10\left(1+\frac{1}{n}\right)\right] = 4\cdot 1 + 10\cdot 1 = 14$$

46. $$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3 - 2\left(1+\frac{3i}{n}\right)\right] = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{3}{n}\left[1+\frac{9i}{n}+\frac{27i^2}{n^2}+\frac{27i^3}{n^3}-2-\frac{6i}{n}\right]$$

$$= \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left[\frac{81}{n^4}i^3+\frac{81}{n^3}i^2+\frac{9}{n^2}i-\frac{3}{n}\right]$$

$$= \lim\limits_{n\to\infty}\left[\frac{81}{n^4}\frac{n^2(n+1)^2}{4}+\frac{81}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{9}{n^2}\frac{n(n+1)}{2}-\frac{3}{n}n\right]$$

$$= \lim\limits_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2+\frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{9}{2}\left(1+\frac{1}{n}\right)-3\right]$$

$$= \frac{81}{4}+\frac{54}{2}+\frac{9}{2}-3 = \frac{195}{4}$$

47. Let $S = \sum\limits_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplying both sides by r gives us $rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$. Subtracting the first equation from the second, we find $(r-1)S = ar^n - a = a(r^n-1)$, so $S = \dfrac{a(r^n-1)}{r-1}$ [since $r \neq 1$].

48. $\sum\limits_{i=1}^{n}\frac{3}{2^{i-1}} = 3\sum\limits_{i=1}^{n}\left(\frac{1}{2}\right)^{i-1} = \dfrac{3\left[\left(\frac{1}{2}\right)^n-1\right]}{\frac{1}{2}-1}$ [using Exercise 47 with $a=3$ and $r=\frac{1}{2}$] $= 6\left[1-\left(\frac{1}{2}\right)^n\right]$

49. $\sum\limits_{i=1}^{n}(2i+2^i) = 2\sum\limits_{i=1}^{n} i + \sum\limits_{i=1}^{n} 2\cdot 2^{i-1} = 2\dfrac{n(n+1)}{2} + \dfrac{2(2^n-1)}{2-1} = 2^{n+1}+n^2+n-2.$

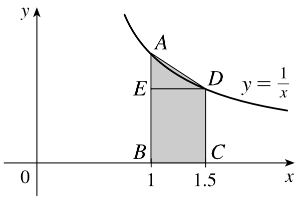
For the first sum we have used Theorems 2(a) and 3(c), and for the second, Exercise 47 with $a=r=2$.

50. $\sum\limits_{i=1}^{m}\left[\sum\limits_{j=1}^{n}(i+j)\right] = \sum\limits_{i=1}^{m}\left[\sum\limits_{j=1}^{n} i + \sum\limits_{j=1}^{n} j\right]$ [Theorem 2(b)] $= \sum\limits_{i=1}^{m}\left[ni + \dfrac{n(n+1)}{2}\right]$ [Theorem 3(b) and 3(c)]

$$= \sum\limits_{i=1}^{m} ni + \sum\limits_{i=1}^{m}\frac{n(n+1)}{2} = \frac{nm(m+1)}{2} + \frac{nm(n+1)}{2} = \frac{nm}{2}(m+n+2)$$

G The Logarithm Defined as an Integral

1. (a)



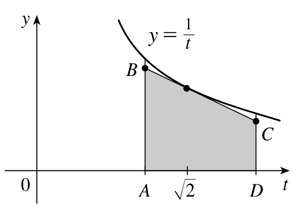
We interpret $\ln 1.5$ as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

(b) $\ln x = \int_1^x (1/t)\,dt$, so $\ln 1.5 = \int_1^{1.5} (1/t)\,dt$. With $f(t) = 1/t$, $n = 10$, and $\Delta t = \frac{1.5 - 1}{10} = 0.05$, we have

$$\begin{aligned}\ln 1.5 = \int_1^{1.5} (1/t)\,dt &\approx (0.05)[f(1.025) + f(1.075) + \cdots + f(1.475)] = (0.05)\left[\tfrac{1}{1.025} + \tfrac{1}{1.075} + \cdots + \tfrac{1}{1.475}\right] \\ &\approx 0.4054\end{aligned}$$

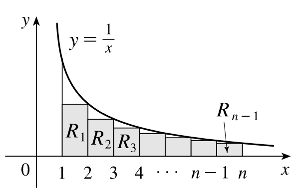
2. (a) $y = \dfrac{1}{t}$, $y' = -\dfrac{1}{t^2}$. The slope of the line through $A(1, 1)$ and $D\left(2, \frac{1}{2}\right)$ is $\dfrac{1/2 - 1}{2 - 1} = -\dfrac{1}{2}$. Let c be the t-coordinate of the point on $y = \dfrac{1}{t}$ with slope $-\dfrac{1}{2}$. Then $-\dfrac{1}{c^2} = -\dfrac{1}{2} \quad\Rightarrow\quad c^2 = 2 \quad\Rightarrow\quad c = \sqrt{2}$ since $c > 0$. Therefore, the tangent line is given by $y - \frac{1}{\sqrt{2}} = -\frac{1}{2}\left(t - \sqrt{2}\right)$, or $y = -\frac{1}{2}t + \sqrt{2}$.

(b)

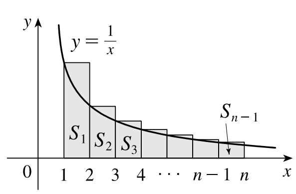


Since the graph of $y = 1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC. Now $|AB| = -\frac{1}{2} + \sqrt{2}$ and $|CD| = -1 + \sqrt{2}$. The area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2} + \sqrt{2}\right) + \left(-1 + \sqrt{2}\right)\right]1 = -\frac{3}{4} + \sqrt{2} \approx 0.6642$. So $\ln 2 >$ area of trapezoid $ABCD > 0.66$.

3.



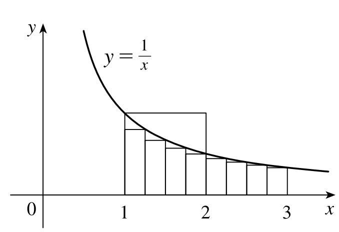
The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} > \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.

Thus, $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \ln n < 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1}$.

4. (a) From the diagram, we see that the area under the graph of $y = 1/x$ between $x = 1$ and $x = 2$ is less than the area of the square, which is 1. So $\ln 2 = \int_1^2 (1/x)\,dx < 1$. To show the other side of the inequality, we must find an area larger than 1 which lies under the graph of $y = 1/x$ between $x = 1$ and $x = 3$. One way to do this is to partition the interval $[1, 3]$ into



8 intervals of equal length and calculate the resulting Riemann sum, using the right endpoints:

$$\frac{1}{4}\left(\frac{1}{5/4}+\frac{1}{3/2}+\frac{1}{7/4}+\frac{1}{2}+\frac{1}{9/4}+\frac{1}{5/2}+\frac{1}{11/4}+\frac{1}{3}\right)=\frac{28{,}271}{27{,}720}>1$$

and therefore $1 < \int_1^3 (1/x)\,dx = \ln 3$.

A slightly easier method uses the fact that since $y = 1/x$ is concave upward, it lies above all its tangent lines. Drawing two such tangent lines at the points $\left(\frac{3}{2}, \frac{2}{3}\right)$ and $\left(\frac{5}{2}, \frac{2}{5}\right)$, we see that the area under the curve from $x = 1$ to $x = 3$ is more than the sum of the areas of the two trapezoids, that is, $\frac{2}{3} + \frac{2}{5} = \frac{16}{15}$. Thus, $1 < \frac{16}{15} < \int_1^3 (1/x)\,dx = \ln 3$.

(b) By part (a), $\ln 2 < 1 < \ln 3$. But e is defined such that $\ln e = 1$, and because the natural logarithm function is increasing, we have $\ln 2 < \ln e < \ln 3 \quad \Leftrightarrow \quad 2 < e < 3$.

5. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)\left(rx^{r-1}\right) = r/x$. But if $g(x) = r\ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r\ln x + C$. Put $x = 1$: $\ln(1^r) = r\ln 1 + C \quad \Rightarrow \quad C = 0$, so $\ln(x^r) = r\ln x$.

6. Using the second law of logarithms and Equation 10, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since $\ln$ is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

7. Using the third law of logarithms and Equation 10, we have $\ln e^{rx} = rx = r\ln e^x = \ln(e^x)^r$. Since $\ln$ is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

8. Using Definition 13 and the second law of exponents for e^x, we have $a^{x-y} = e^{(x-y)\ln a} = e^{x\ln a - y\ln a} = \dfrac{e^{x\ln a}}{e^{y\ln a}} = \dfrac{a^x}{a^y}$.

9. Using Definition 13, the first law of logarithms, and the first law of exponents for e^x, we have $(ab)^x = e^{x\ln(ab)} = e^{x(\ln a + \ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x$.

10. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \quad \Rightarrow \quad \log_a(xy) = r + s = \log_a x + \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^r}{a^s} = a^{r-s} \quad \Rightarrow \quad \log_a \dfrac{x}{y} = r - s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \quad \Rightarrow \quad \log_a(x^y) = ry = y\log_a x$

H Complex Numbers

1. $(5 - 6i) + (3 + 2i) = (5 + 3) + (-6 + 2)i = 8 + (-4)i = 8 - 4i$

2. $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right) = (4 - 9) + \left(-\frac{1}{2} - \frac{5}{2}\right)i = -5 + (-3)i = -5 - 3i$

3. $(2 + 5i)(4 - i) = 2(4) + 2(-i) + (5i)(4) + (5i)(-i) = 8 - 2i + 20i - 5i^2 = 8 + 18i - 5(-1)$
$= 8 + 18i + 5 = 13 + 18i$

4. $(1 - 2i)(8 - 3i) = 8 - 3i - 16i + 6(-1) = 2 - 19i$

5. $\overline{12 + 7i} = 12 - 7i$

6. $2i\left(\frac{1}{2}-i\right)=i-2(-1)=2+i \quad\Rightarrow\quad \overline{2i\left(\frac{1}{2}-i\right)}=\overline{2+i}=2-i$

7. $\dfrac{1+4i}{3+2i}=\dfrac{1+4i}{3+2i}\cdot\dfrac{3-2i}{3-2i}=\dfrac{3-2i+12i-8(-1)}{3^2+2^2}=\dfrac{11+10i}{13}=\dfrac{11}{13}+\dfrac{10}{13}i$

8. $\dfrac{3+2i}{1-4i}=\dfrac{3+2i}{1-4i}\cdot\dfrac{1+4i}{1+4i}=\dfrac{3+12i+2i+8(-1)}{1^2+4^2}=\dfrac{-5+14i}{17}=-\dfrac{5}{17}+\dfrac{14}{17}i$

9. $\dfrac{1}{1+i}=\dfrac{1}{1+i}\cdot\dfrac{1-i}{1-i}=\dfrac{1-i}{1-(-1)}=\dfrac{1-i}{2}=\dfrac{1}{2}-\dfrac{1}{2}i$

10. $\dfrac{3}{4-3i}=\dfrac{3}{4-3i}\cdot\dfrac{4+3i}{4+3i}=\dfrac{12+9i}{16-9(-1)}=\dfrac{12}{25}+\dfrac{9}{25}i$

11. $i^3=i^2\cdot i=(-1)i=-i$

12. $i^{100}=(i^2)^{50}=(-1)^{50}=1$

13. $\sqrt{-25}=\sqrt{25}\,i=5i$

14. $\sqrt{-3}\,\sqrt{-12}=\sqrt{3}\,i\,\sqrt{12}\,i=\sqrt{3\cdot 12}\,i^2=\sqrt{36}\,(-1)=-6$

15. $\overline{12-5i}=12+15i$ and $|12-15i|=\sqrt{12^2+(-5)^2}=\sqrt{144+25}=\sqrt{169}=13$

16. $\overline{-1+2\sqrt{2}\,i}=-1-2\sqrt{2}\,i$ and $\left|-1+2\sqrt{2}\,i\right|=\sqrt{(-1)^2+\left(2\sqrt{2}\right)^2}=\sqrt{1+8}=\sqrt{9}=3$

17. $\overline{-4i}=\overline{0-4i}=0+4i=4i$ and $|-4i|=\sqrt{0^2+(-4)^2}=\sqrt{16}=4$

18. Let $z=a+bi$ and $w=c+di$.

(a) $\overline{z+w}=\overline{(a+bi)+(c+di)}=\overline{(a+c)+(b+d)i}=(a+c)-(b+d)i=(a-bi)+(c-di)=\overline{z}+\overline{w}$

(b) $\overline{zw}=\overline{(a+bi)(c+di)}=\overline{(ac-bd)+(ad+bc)i}=(ac-bd)-(ad+bc)i.$

On the other hand, $\overline{z}\,\overline{w}=(a-bi)(c-di)=(ac-bd)-(ad+bc)i=\overline{zw}.$

(c) Use mathematical induction and part (b): Let S_n be the statement that $\overline{z^n}=\overline{z}^{\,n}$. S_1 is true because $\overline{z^1}=\overline{z}=\overline{z}^{\,1}$.

Assume S_k is true, that is $\overline{z^k}=\overline{z}^{\,k}$. Then $\overline{z^{k+1}}=\overline{z^{1+k}}=\overline{zz^k}=\overline{z}\,\overline{z^k}$ [part (b) with $w=z^k$] $=\overline{z}^1\overline{z}^k=\overline{z}^{1+k}=\overline{z}^{k+1}$,

which shows that S_{k+1} is true. Therefore, by mathematical induction, $\overline{z^n}=\overline{z}^n$ for every positive integer n.

Another proof: Use part (b) with $w=z$, and mathematical induction.

19. $4x^2+9=0 \quad\Leftrightarrow\quad 4x^2=-9 \quad\Leftrightarrow\quad x^2=-\frac{9}{4} \quad\Leftrightarrow\quad x=\pm\sqrt{-\frac{9}{4}}=\pm\sqrt{\frac{9}{4}}\,i=\pm\frac{3}{2}i.$

20. $x^4=1 \quad\Leftrightarrow\quad x^4-1=0 \quad\Leftrightarrow\quad (x^2-1)(x^2+1)=0 \quad\Leftrightarrow\quad x^2-1=0$ or $x^2+1=0 \quad\Leftrightarrow\quad x=\pm 1$ or $x=\pm i.$

21. By the quadratic formula, $x^2+2x+5=0 \quad\Leftrightarrow\quad x=\dfrac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}=\dfrac{-2\pm\sqrt{-16}}{2}=\dfrac{-2\pm 4i}{2}=-1\pm 2i.$

22. $2x^2-2x+1=0 \quad\Leftrightarrow\quad x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(2)(1)}}{2(2)}=\dfrac{2\pm\sqrt{-4}}{4}=\dfrac{2\pm 2i}{4}=\dfrac{1}{2}\pm\dfrac{1}{2}i$

23. By the quadratic formula, $z^2 + z + 2 = 0 \quad \Leftrightarrow \quad z = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \dfrac{-1 \pm \sqrt{-7}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{7}}{2}i.$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0 \quad \Leftrightarrow \quad 4z^2 + 2z + 1 = 0 \quad \Leftrightarrow$

$$z = \frac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2\sqrt{3}\,i}{8} = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

25. For $z = -3 + 3i$, $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$ and $\tan\theta = \frac{3}{-3} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}$ (since z lies in the second quadrant). Therefore, $-3 + 3i = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

26. For $z = 1 - \sqrt{3}\,i$, $r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$ and $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \quad \Rightarrow \quad \theta = \frac{5\pi}{3}$ (since z lies in the fourth quadrant). Therefore, $1 - \sqrt{3}\,i = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

27. For $z = 3 + 4i$, $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\theta = \frac{4}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$ (since z lies in the first quadrant). Therefore, $3 + 4i = 5\left[\cos\left(\tan^{-1}\frac{4}{3}\right) + i\sin\left(\tan^{-1}\frac{4}{3}\right)\right]$.

28. For $z = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan\theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2}$ (since z lies on the positive imaginary axis). Therefore, $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

29. For $z = \sqrt{3} + i$, $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ and $\tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \frac{\pi}{6} \quad \Rightarrow \quad z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

For $w = 1 + \sqrt{3}\,i$, $r = 2$ and $\tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \quad \Rightarrow \quad w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

Therefore, $zw = 2 \cdot 2\left[\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right] = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$,

$z/w = \frac{2}{2}\left[\cos\left(\frac{\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\right] = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$, and $1 = 1 + 0i = 1(\cos 0 + i\sin 0) \quad \Rightarrow$

$1/z = \frac{1}{2}\left[\cos\left(0 - \frac{\pi}{6}\right) + i\sin\left(0 - \frac{\pi}{6}\right)\right] = \frac{1}{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$. For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.

30. For $z = 4\sqrt{3} - 4i$, $r = \sqrt{\left(4\sqrt{3}\right)^2 + (-4)^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{-4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \frac{11\pi}{6} \quad \Rightarrow$

$z = 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$. For $w = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan\theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2} \quad \Rightarrow$

$w = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Therefore, $zw = 8 \cdot 8\left[\cos\left(\frac{11\pi}{6} + \frac{\pi}{2}\right) + i\sin\left(\frac{11\pi}{6} + \frac{\pi}{2}\right)\right] = 64\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

$z/w = \frac{8}{8}\left[\cos\left(\frac{11\pi}{6} - \frac{\pi}{2}\right) + i\sin\left(\frac{11\pi}{6} - \frac{\pi}{2}\right)\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, and

$1 = 1 + 0i = 1(\cos 0 + i\sin 0) \quad \Rightarrow \quad 1/z = \frac{1}{8}\left[\cos\left(0 - \frac{11\pi}{6}\right) + i\sin\left(0 - \frac{11\pi}{6}\right)\right] = \frac{1}{8}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.

For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{8}\left(\cos\frac{11\pi}{6} - i\sin\frac{11\pi}{6}\right)$.

31. For $z = 2\sqrt{3} - 2i$, $r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$ and $\tan\theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = -\frac{\pi}{6} \quad \Rightarrow$

$z = 4\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$. For $w = -1 + i$, $r = \sqrt{2}$, $\tan\theta = \frac{1}{-1} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4} \quad \Rightarrow$

$w = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. Therefore, $zw = 4\sqrt{2}\left[\cos\left(-\frac{\pi}{6} + \frac{3\pi}{4}\right) + i\sin\left(-\frac{\pi}{6} + \frac{3\pi}{4}\right)\right] = 4\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$,

$z/w = \frac{4}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right) + i\sin\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right)\right] = \frac{4}{\sqrt{2}}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right] = 2\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$, and

$1/z = \frac{1}{4}\left[\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right] = \frac{1}{4}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

32. For $z = 4(\sqrt{3} + i) = 4\sqrt{3} + 4i$, $r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$ $z = 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. For $w = -3 - 3i$, $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ and $\tan\theta = \frac{-3}{-3} = 1 \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow$ $w = 3\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$. Therefore, $zw = 8 \cdot 3\sqrt{2}\left[\cos\left(\frac{\pi}{6} + \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{5\pi}{4}\right)\right] = 24\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$, $z/w = \frac{8}{3\sqrt{2}}\left[\cos\left(\frac{\pi}{6} - \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{6} - \frac{5\pi}{4}\right)\right] = \frac{4\sqrt{2}}{3}\left[\cos\left(-\frac{13\pi}{12}\right) + i\sin\left(-\frac{13\pi}{12}\right)\right]$, and $1/z = \frac{1}{8}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.

33. For $z = 1 + i$, $r = \sqrt{2}$ and $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. So by De Moivre's Theorem,

$$\begin{aligned}(1+i)^{20} &= \left[\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\right]^{20} = (2^{1/2})^{20}\left(\cos\tfrac{20\cdot\pi}{4} + i\sin\tfrac{20\cdot\pi}{4}\right) = 2^{10}(\cos 5\pi + i\sin 5\pi) \\ &= 2^{10}[-1 + i(0)] = -2^{10} = -1024\end{aligned}$$

34. For $z = 1 - \sqrt{3}\,i$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ and $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

So by De Moivre's Theorem,

$$\begin{aligned}\left(1 - \sqrt{3}\,i\right)^5 &= \left[2\left(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}\right)\right]^5 = 2^5\left(\cos\tfrac{5\cdot 5\pi}{3} + i\sin\tfrac{5\cdot 5\pi}{3}\right) = 2^5\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) \\ &= 32\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,i\right) = 16 + 16\sqrt{3}\,i\end{aligned}$$

35. For $z = 2\sqrt{3} + 2i$, $r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$ and $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

So by De Moivre's Theorem,

$\left(2\sqrt{3} + 2i\right)^5 = \left[4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5 = 4^5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 1024\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = -512\sqrt{3} + 512i.$

36. For $z = 1 - i$, $r = \sqrt{2}$ and $\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = \frac{7\pi}{4} \Rightarrow z = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \Rightarrow$

$(1-i)^8 = \left[\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\right]^8 = 2^4\left(\cos\frac{8\cdot 7\pi}{4} + i\sin\frac{8\cdot 7\pi}{4}\right) = 16(\cos 14\pi + i\sin 14\pi) = 16(1 + 0i) = 16.$

37. $1 = 1 + 0i = 1(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 1$, $n = 8$, and $\theta = 0$, we have

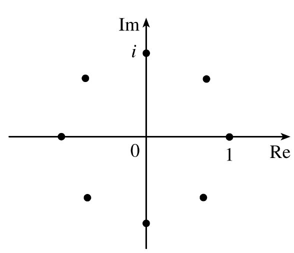
$w_k = 1^{1/8}\left[\cos\left(\dfrac{0 + 2k\pi}{8}\right) + i\sin\left(\dfrac{0 + 2k\pi}{8}\right)\right] = \cos\dfrac{k\pi}{4} + i\sin\dfrac{k\pi}{4}$, where $k = 0, 1, 2, \ldots, 7$.

$w_0 = 1(\cos 0 + i\sin 0) = 1$, $w_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_2 = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i$, $w_3 = 1\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_4 = 1(\cos\pi + i\sin\pi) = -1$, $w_5 = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$,

$w_6 = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -i$, $w_7 = 1\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



38. $32 = 32 + 0i = 32(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 32$, $n = 5$, and $\theta = 0$, we have

$w_k = 32^{1/5}\left[\cos\left(\dfrac{0 + 2k\pi}{5}\right) + i\sin\left(\dfrac{0 + 2k\pi}{5}\right)\right] = 2\left(\cos\frac{2}{5}\pi k + i\sin\frac{2}{5}\pi k\right)$, where $k = 0, 1, 2, 3, 4$.

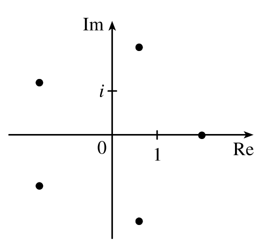
$w_0 = 2(\cos 0 + i\sin 0) = 2$

$w_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$w_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$w_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$w_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$



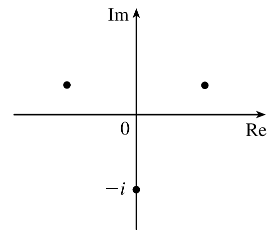
39. $i = 0 + i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Using Equation 3 with $r = 1$, $n = 3$, and $\theta = \frac{\pi}{2}$, we have

$$w_k = 1^{1/3}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$$w_0 = \left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$w_1 = \left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right) = -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$w_2 = \left(\cos\tfrac{9\pi}{6} + i\sin\tfrac{9\pi}{6}\right) = -i$$



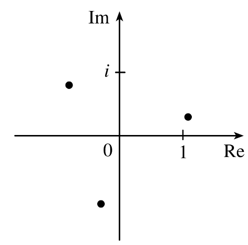
40. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Using Equation 3 with $r = \sqrt{2}$, $n = 3$, and $\theta = \frac{\pi}{4}$, we have

$$w_k = \left(\sqrt{2}\right)^{1/3}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$$w_0 = 2^{1/6}\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right)$$

$$w_1 = 2^{1/6}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right) = 2^{1/6}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) = -2^{-1/3} + 2^{-1/3}i$$

$$w_2 = 2^{1/6}\left(\cos\tfrac{17\pi}{12} + i\sin\tfrac{17\pi}{12}\right)$$



41. Using Euler's formula (6) with $y = \frac{\pi}{2}$, we have $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + 1i = i$.

42. Using Euler's formula (6) with $y = 2\pi$, we have $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$.

43. Using Euler's formula (6) with $y = \dfrac{\pi}{3}$, we have $e^{i\pi/3} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

44. Using Euler's formula (6) with $y = -\pi$, we have $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1$.

45. Using Equation 7 with $x = 2$ and $y = \pi$, we have $e^{2+i\pi} = e^2 e^{i\pi} = e^2(\cos\pi + i\sin\pi) = e^2(-1 + 0) = -e^2$.

46. Using Equation 7 with $x = \pi$ and $y = 1$, we have $e^{\pi+i} = e^{\pi}\cdot e^{1i} = e^{\pi}(\cos 1 + i\sin 1) = e^{\pi}\cos 1 + (e^{\pi}\sin 1)i$.

47. Take $r = 1$ and $n = 3$ in De Moivre's Theorem to get

$$\begin{aligned}
[1(\cos\theta + i\sin\theta)]^3 &= 1^3(\cos 3\theta + i\sin 3\theta)\\
(\cos\theta + i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta\\
\cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta\\
\cos^3\theta + (3\cos^2\theta\,\sin\theta)i - 3\cos\theta\,\sin^2\theta - (\sin^3\theta)i &= \cos 3\theta + i\sin 3\theta\\
(\cos^3\theta - 3\sin^2\theta\,\cos\theta) + (3\sin\theta\,\cos^2\theta - \sin^3\theta)i &= \cos 3\theta + i\sin 3\theta
\end{aligned}$$

Equating real and imaginary parts gives $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\,\cos\theta$ and $\sin 3\theta = 3\sin\theta\,\cos^2\theta - \sin^3\theta$.

48. Using Formula 6,

$$e^{ix} + e^{-ix} = (\cos x + i\sin x) + [\cos(-x) + i\sin(-x)] = \cos x + i\sin x + \cos x - i\sin x = 2\cos x$$

Thus, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$. Similarly,

$$e^{ix} - e^{-ix} = (\cos x + i\sin x) - [\cos(-x) + i\sin(-x)] = \cos x + i\sin x - \cos x - (-i\sin x) = 2i\sin x$$

Therefore, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

49. $F(x) = e^{rx} = e^{(a+bi)x} = e^{ax+bxi} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + i(e^{ax}\sin bx) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= (e^{ax}\cos bx)' + i(e^{ax}\sin bx)' \\ &= (ae^{ax}\cos bx - be^{ax}\sin bx) + i(ae^{ax}\sin bx + be^{ax}\cos bx) \\ &= a[e^{ax}(\cos bx + i\sin bx)] + b[e^{ax}(-\sin bx + i\cos bx)] \\ &= ae^{rx} + b[e^{ax}(i^2\sin bx + i\cos bx)] \\ &= ae^{rx} + bi[e^{ax}(\cos bx + i\sin bx)] = ae^{rx} + bie^{rx} = (a+bi)e^{rx} = re^{rx} \end{aligned}$$

50. (a) From Exercise 49, $F(x) = e^{(1+i)x} \quad \Rightarrow \quad F'(x) = (1+i)e^{(1+i)x}$. So

$$\int e^{(1+i)x}\,dx = \frac{1}{1+i}\int F'(x)\,dx = \frac{1}{1+i}\,F(x) + C = \frac{1-i}{2}\,F(x) + C = \frac{1-i}{2}\,e^{(1+i)x} + C$$

(b) $\int e^{(1+i)x}\,dx = \int e^x e^{ix}\,dx = \int e^x(\cos x + i\sin x)\,dx = \int e^x\cos x\,dx + i\int e^x\sin x \quad$ (**1**).

Also,

$$\begin{aligned} \frac{1-i}{2}\,e^{(1+i)x} &= \tfrac{1}{2}e^{(1+i)x} - \tfrac{1}{2}ie^{(1+i)x} = \tfrac{1}{2}e^{x+ix} - \tfrac{1}{2}ie^{x+ix} \\ &= \tfrac{1}{2}e^x(\cos x + i\sin x) - \tfrac{1}{2}ie^x(\cos x + i\sin x) \\ &= \tfrac{1}{2}e^x\cos x + \tfrac{1}{2}e^x\sin x + \tfrac{1}{2}ie^x\sin x - \tfrac{1}{2}ie^x\cos x \\ &= \tfrac{1}{2}e^x(\cos x + \sin x) + i\left[\tfrac{1}{2}e^x(\sin x - \cos x)\right] \quad \mathbf{(2)} \end{aligned}$$

Equating the real and imaginary parts in (**1**) and (**2**), we see that $\int e^x\cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x) + C$ and $\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.