Stochastic Process

In a ~~rand~~ random variable, the outcome of the experiment performed is random. That is, every time the experiment performed the outcome is unpredictable and hence, is probabilistic.

For instance, everytime we roll a die the number appears is random. By definition, we map the outcomes to a real number as

$$\{X(\omega) : \omega \in S \mid X(\omega) \in \mathbb{R}\}$$

The idea of a Stochastic process is an extension of the idea of a random variable

Instead of mapping each outcome $\omega \in S$, to a number $X(\omega)$, for stochastic process we map this to $X(t, \omega)$. Here, t is time, so $X(t, \omega)$ is a function of time.

Definition: A stochastic process (also, synonymously known as the random process) defined on the random experiment $(S, \mathcal{F}, P)$ ~~of random variables~~ is a $\boxed{\text{family of random variables}}$ $\{X(t) ; t \in \mathbb{R}\}$ defined on probability space $(S, \mathcal{F}, P)$ and indexed by t.

Random Variable

A RV $X : S \to \mathbb{R}$ for a particular outcome ω_0, ~~is~~ where $\omega_0 \in S$, is just a real number $X(\omega_0)$. So, $X(\omega_0) \in \mathbb{R}$

Here, at any given time if the outcome ω_0, in an immediate ~~a~~ next instant the outcome ω_0 may also change because of the random behavior of the experiment. In general,

A stochastic process $X(t, \omega_0)$ is a function of two variables: time t and the outcome $\omega \in S$

So, How do we form pair (t, ω) ? We can take Cartesian product of their relevant sets.

$$t \in \mathbb{R}, \quad \omega \in S$$

So, we take $\mathbb{R} \times S$ and the random process

$$X(t, \omega) : \underbrace{\mathbb{R}}_{\text{Real time values}} \times \underbrace{S}_{\text{outcomes of random experiment}} \rightarrow \mathbb{R}$$

Notations for Stochastic Process:

$X(\omega)$: Random variable and can be written in short form as X

hiding the explicit dependence on $\omega \in S$

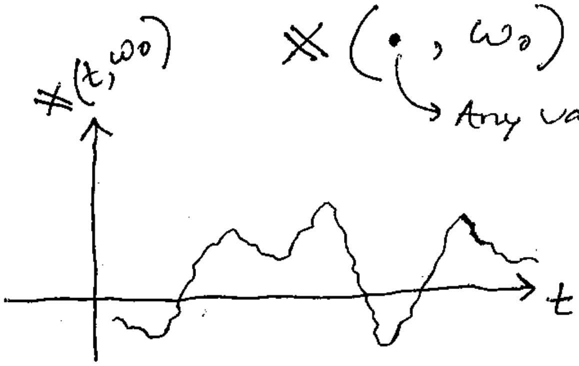
$X(t, \omega)$: Stochastic/random process, written as $X(t)$. Again hiding dependence of $\omega \in S$

⊞ Suppose in $X(t, \omega)$, we fix our outcome as ω_0. Then, we obtain

$$X(\underbrace{\cdot}_{\text{Any value of } t}, \omega_0) : \mathbb{R} \rightarrow \mathbb{R}$$

it was $\mathbb{R} \times S$ when ω was not fixed.

That is, outcome is fixed.



For each $\omega \in S$, $X(\cdot, \omega)$ is a different function of time.

Definition: Let's assume that we are interested about the fixed outcome $\omega_0 \in S$. Then, the time function $X(\cdot) = X(\cdot, \omega_0)$, also denoted as $X(t, \omega_0)$ for any time t, is called the

Sample Path of the random process $X(t)$

corresponding to outcomes ω_0.

We can also define the $\underbrace{\text{set of all sample paths}}_{\mathcal{E}}$ of $\omega \in S$.

$$\mathcal{E} = \{X(\cdot) : X(\cdot, \omega),\ \forall \omega \in S\}$$

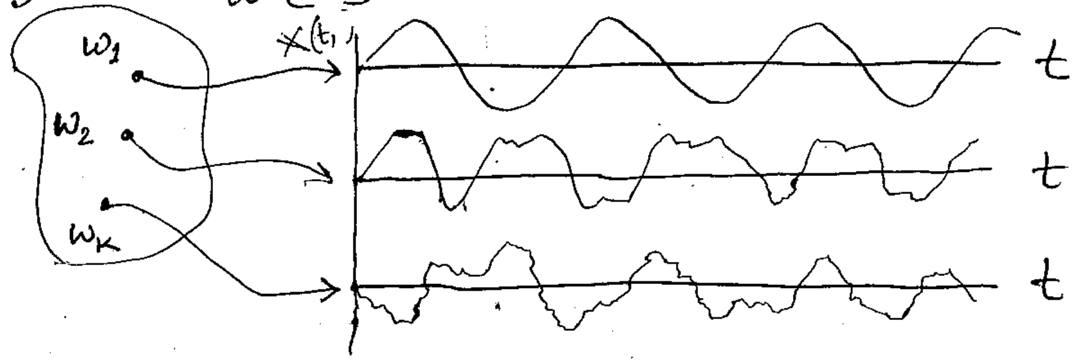
The $\mathcal{E}$ is called as the ensemble of random process $X(t)$.

⊞ The above information can be used to form an equivalent definition of a random/stochastic process as below –

Definition: Consider a random experiment on the probability space $(S, \mathcal{F}, P)$. A function

$X : S \to \mathcal{E}$, where $\mathcal{E}$ is a set of $\underbrace{\text{functions of time}}_{\text{sample path}}$, is called a random process.

Simply, X is a mapping that assigns a function of time $X(\cdot, \omega)$ to each $\omega \in S$



Example : Consider an experiment of rolling a die

Sample space $S = \{ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \}$

$$\begin{array}{cccccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

We construct a random process $X(t) = X(t, \omega)$ for $\omega \in S$ as :

numerical outcome on die

$$X(t, \omega_K) = \underbrace{\cos(2\pi K t)}_{\text{exact functional expression used}}, \quad K = 1, 2, 3, 4, 5, 6$$

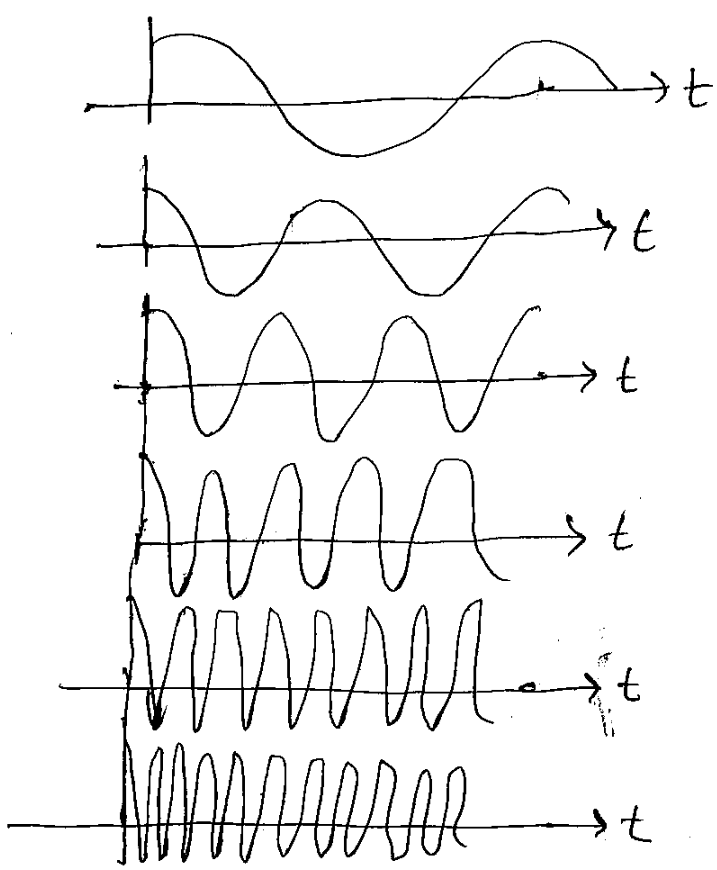
could be something else

So, as defined above

we have a stochastic/random process $X(t)$ that is a cosine function and its frequency is selected at random. A pictorial representation would be as follows

$$X(t, \omega_K) = \cos(2\pi K t)$$

ω	$X(t,\omega)$
ω_1	$\cos 2\pi t = \cos 2\pi t$
ω_2	$\cos 2\pi 2t = \cos 4\pi t$
ω_3	$\cos 2\pi 3t = \cos 6\pi t$
ω_4	$\cos 2\pi 4t = \cos 8\pi t$
ω_5	$\cos 2\pi 5t = \cos 10\pi t$
ω_6	$\cos 2\pi 6t = \cos 12\pi t$



Finally, the Objectives

- We are interested to know the state of $\mathbb{N}$ the queueing system and the associated statistical quantities.

$$\Pr\{\mathbb{N} = n\}$$

- $\mathbb{E}[\mathbb{N}]$, $\mathrm{Var}(\mathbb{N})$ etc.

KENDALL NOTATION

Queues implemented in many systems are represented using a common notation, known as Kendall Notation.

Specifically, KENDALL NOTATION : A/B/C/D

Where,

A : Corresponds to arrival process — relates the inter-arrival time between events, packets, requests.

B : Corresponds to the time needed for the service at the server

C : Corresponds to number of servers being used/available in the queueing system.

D : D corresponds to the buffer size.

↳ No. of packets in the queue/waiting line + the number of servers

So, A/B/1/D

↳ buffer size D

↳ single server

A/B/1 : No D, means infinite buffer

↳ single server

Features of a Queueing System:

Arrival Process: New arrivals occur following stochastic process.

↳ Poisson process

↳ Bernoulli trial

Service Time: We define Random Variable (RV) denoting the service time for an arrival.

↳ could be a packet

If the arrival is a packet, the $\boxed{\text{length of}}$ → random variable the packet is related to the service time

Service Discipline: How is the service offered to the arrivals (for instance, packet) — a few options

- First in First out (FIFO)
- Last in Firrstout (LIFO)
- First come First Served (FIFO)

Number of Servers: Could be single server or multi-server system.

State of the Queueing System: Number of packets in the buffer, and packet numbers in the buffer depend on issues such as Arrival process, service time, number of servers.

↳ arrivals

So, If we sample the random process $X(t)$ at n given points in time, we have n- RVs

$$\underbrace{X(t_1),\ X(t_2)\ \dots\ X(t_n)}$$

These n-RVs are jointly distributed

Definition: Let's consider $X(t)$ is a stochastic process. As already stated $X(t) \equiv X(t, \omega)$. so $X(t)$ represents sample path for specific outcomes. Consider that we fix n time instant

$$t_1, t_2, \dots t_n.$$

Then, the n^{th}-order CDF of $X(t)$ at times $t_1, t_2, \dots, t_n$ is the joint CDF of the n jointly distributed RVs $X(t_1), X(t_2) \dots X(t_n)$

So,

$$F_{X(t_1), X(t_2) \dots X(t_n)}(x_1, x_2, \dots x_n) = P\left(\{X(t_1) \le x_1, X(t_2) \le x_2 \dots X(t_n) \le x_n\right)$$

$$= F_{X(t)}(x_1, x_2, \dots x_n)$$

and the corresponding n^{th} order pdf of $X(t)$ is

$$f_{X(t_1), X(t_2) \dots X(t_n)}(x_1, x_2, \dots x_n) = \frac{\partial^n F_{X(t_1), X(t_2) \dots X(t_n)}(x_1, x_2 \dots x_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$$

$$= f_{X(t)}(x_1, x_2, \dots x_n)$$

Theorem: Let's consider $X(t)$ is a random process
The probabilistic behavior of a random process $X(t)$ is completely characterized by the collection of all n-th

Stochastic Process: Example

Consider that $X_0, X_1, X_2, \ldots$ is a sequence of random variables that takes values from a countable set.

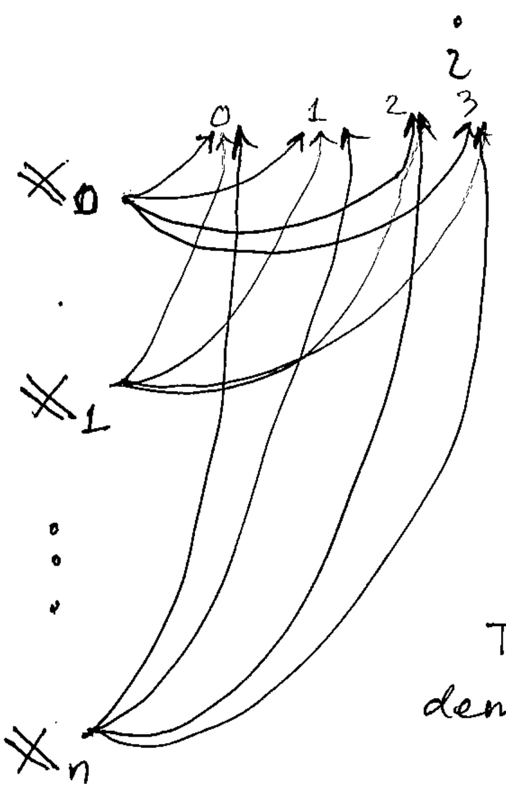
Let's say that the values of the RVs will be from the set of non-negative integers $\{0, 1, 2 \ldots \ldots\} : S$

So, RV sequence $\{X_n, n = 0, 1, 2, \ldots\}$, where $n = 0, 1, 2 \ldots$ may be analogous to time $t_0, t_1, t_2 \ldots$ given that the continuum is time. So, if we express as

$X_n = i$, it means that the

process is $\underbrace{\text{at state}}_{\text{value}}$ at i at time n.

is the value from non-negative set S as stated above



At any timepoint n X_n can hold any value from S

The above arrangements denote a stochastic process.

Let's consider that there's a fixed probability P_{ij} exists that describes ~~that~~ the probability of transition to state j from state i, in the next transition (time).

So, we can write as

$$P_{ij} = P\{\underbrace{X_{n+1} = j}_{\text{Next state}} \,/\, \underbrace{X_n = i}_{\text{Current state}}, \underbrace{X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2} \ldots ; X_0 = i_0}_{\text{Other past states : Memory}}\}$$

for all possible states $i_0, i_1, i_2, \ldots i_{n-1}, i, j$ and considering $n \geq 0$.

As we see, the transition probability P_{ij} is independent of the past states $X_{n-1}, X_{n-2}, \ldots X_0$. Instead, it is solely dependent on the current state. Such a sequence of RVs, defined as Random Process, is known as $\boxed{\text{Markov Chain}}$

So, a concise definition of the Markov Chain goes as follows —

Definition :

A sequence of RVs (also known as stochastic process) on the countable set S (gives the states for each of RVs) is a Markov Chain for $i, j, n \geq 0$ and $i, j \in S$ if the below conditions hold.

① $P\{X_{n+1} = j / X_0, X_1, \ldots X_n\} = P\{X_{n+1} = j / X_n\}$.

and ② $P\{X_{n+1} = j / X_n = i\} = P_{ij}$

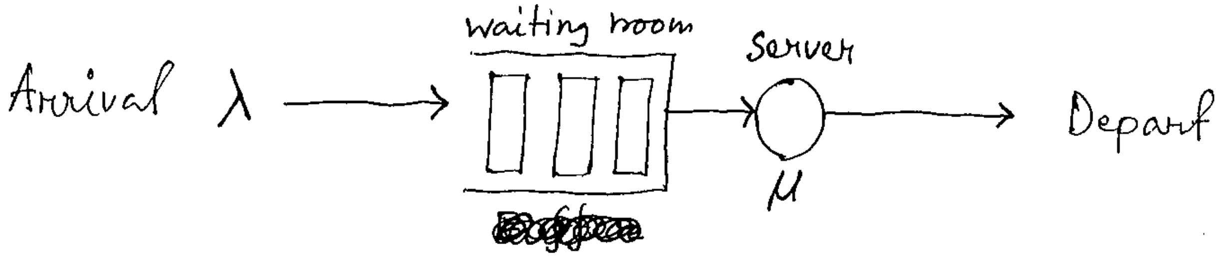
We interpret it as follows :

If we want to predict the future ~~for~~ in the sequence, it depends only on the current state — The memoryless notion of the process.

Queueing Theory

Queueing arises naturally in many systems that we encounter in our daily life. For instance, queue can form in

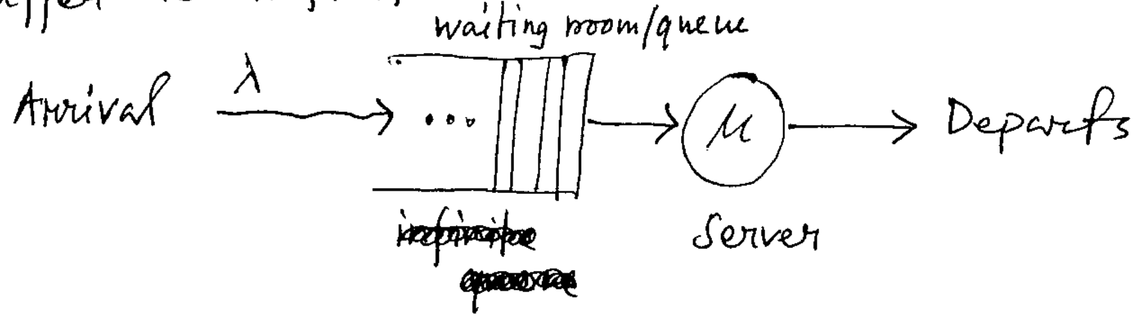
- Packet switched network
- Teller's window in a bank or coffee shops.
- Transmission network — as in router switch etc.



Generally, buffer is the waiting space where an arrival waits to get the service. Theoretically, if $\mu > \lambda$, queue does not form.

However, in a real-world, $\lambda > \mu$, so, queue forms and hence, we must characterize the queue behavior to ensure the quality of service (QOS). For instance, because of buffer overflow, packet loss can occur in packet communication, call drop can occur in mobile communication, inefficient resource distribution may happen in any distributed system.

Some observations: Consider that we have a single server queue and the waiting space, that is the buffer is infinite

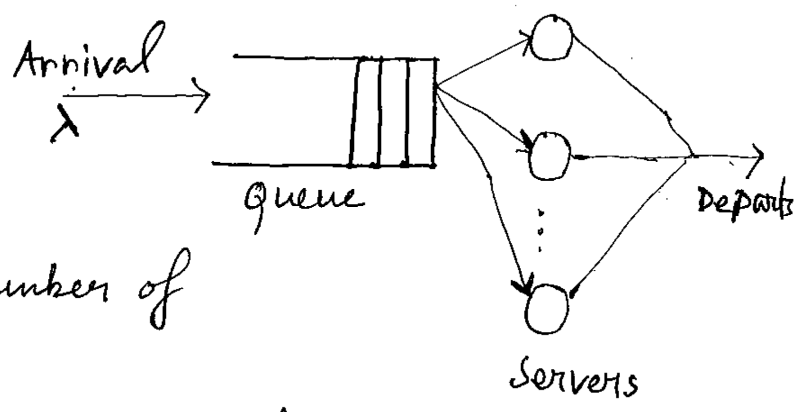


- If $\lambda > \mu$, the queue forms and is of enormous size. Precisely, the queue goes to infinity and the system becomes <u>unstable</u>

- Suppose, the $\boxed{\text{buffer}}$ size is finite, and $\lambda > \mu$ that implies that the rate of arrivals is higher than the rate of service (μ),
  then, probability of packet loss does not reach to 1.

For single server, we can have 1 packet in the server.

Multi-server Queue:



Here,
- Minimum buffer space is the number of servers.
- New arrivals first go to empty servers.
  If more than one server is empty, server can be chosen randomly, or can be done otherwise.

Important observation:

- From the previous example, it is evident that once we know the particular outcome $w_K \in S$, we don't have any other source of randomness in the random/stochastic process.

- So, the randomness ~~exists~~ in a random process enters only through the random selection of $\omega \in S$ from the underlying random experiment.

Comments: A random process is completely analogous to a Random Variable.

Random variable: $X : S \to \mathbb{R}$ (Real number line)

Random Process: $X(t) : S \to \mathcal{E}$, where $\mathcal{E}$ is the set of functions of time

Example: Suppose, we flip a coin three times and construct a random process by making a waveform taking a value once per second in that second corresponding to the H and T outcomes of the coin toss. The H and T are mapped to values as

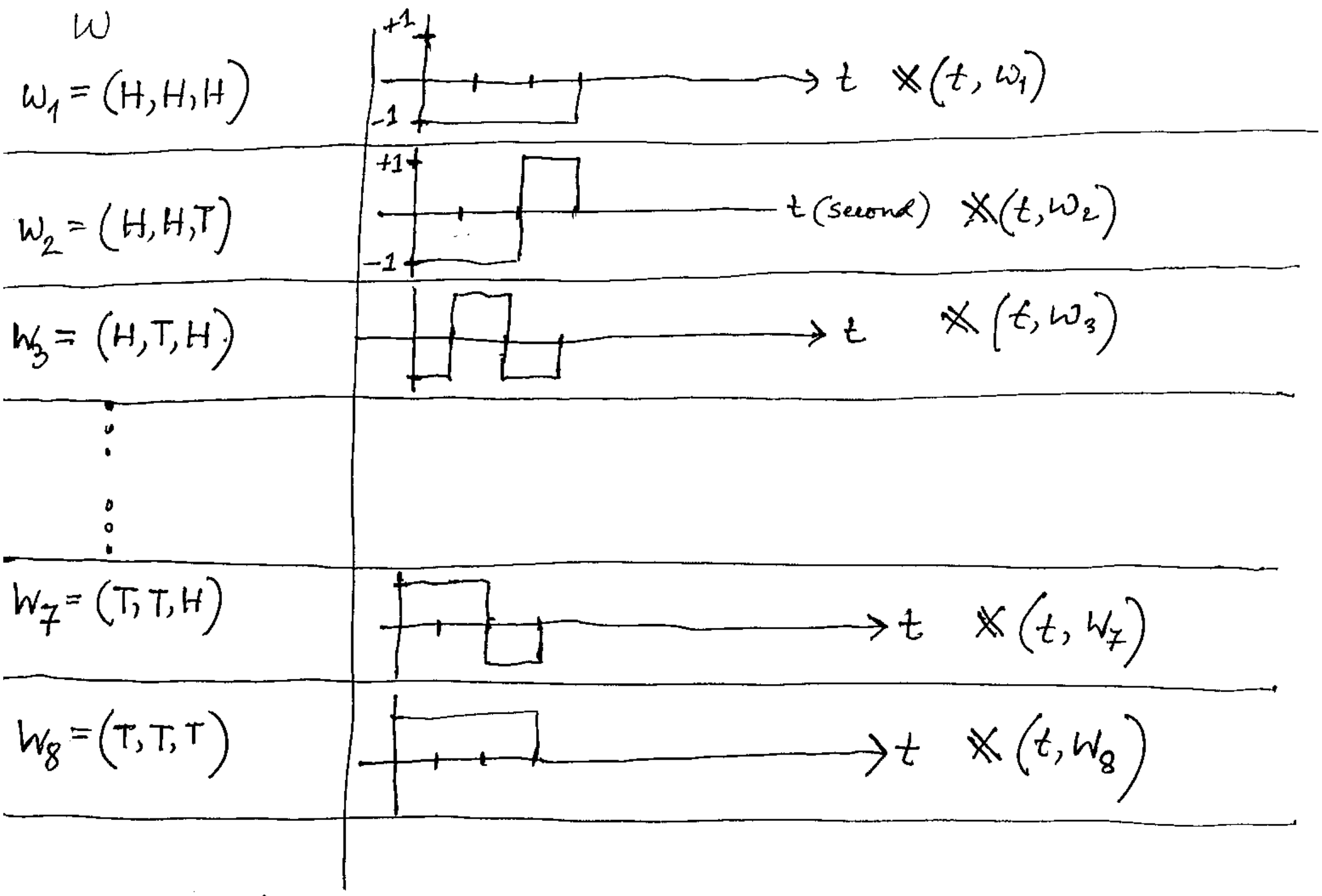
$$H \to -1$$

$$T \to 1$$

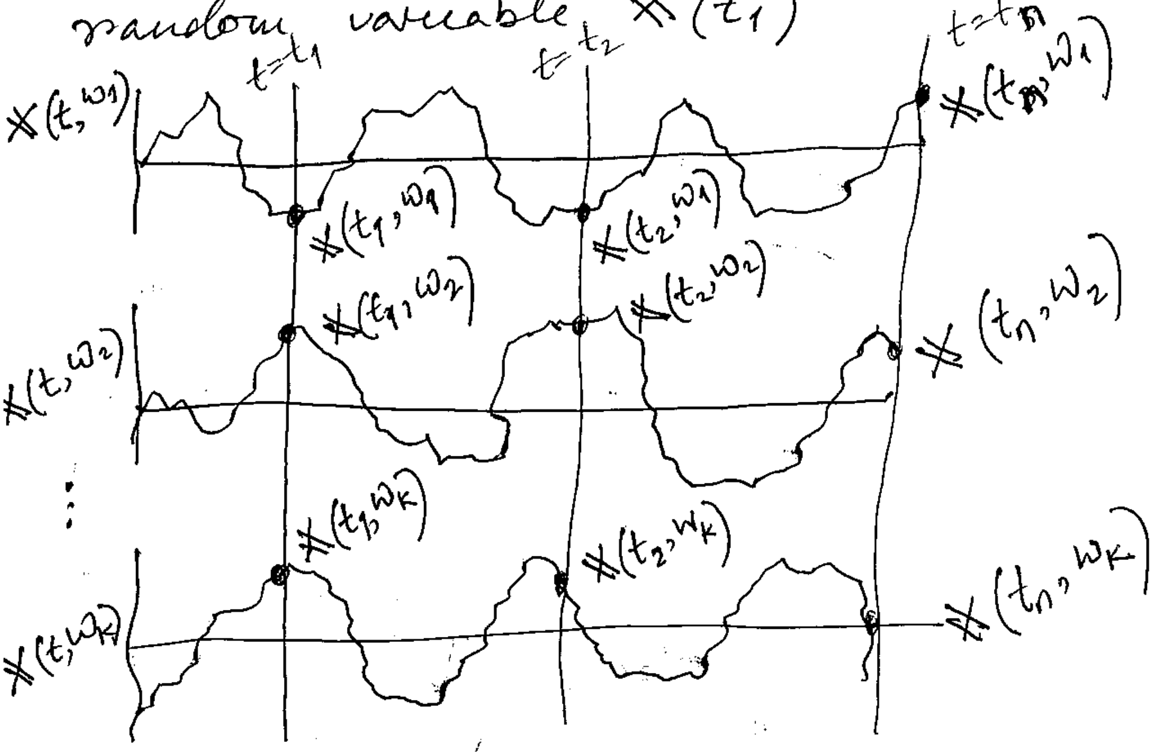
So, for that 3 second, we can construct $2^3 = 8$ possible outcomes considering that we treat it as combined experiment.

So, 3 consecutive coin-tosses create a sample space S as : $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$$\begin{matrix} HHH & HHT & HTH & HTT & THH & THT & TTH & TTT \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{matrix}$$



☐ Comments: If we evaluate a random process $X(t,w)$ at any given time t_1, we obtain a random variable $X(t_1)$



The two properties we stated are known as Markov properties.

Property (1) states that X_{n+1} is conditionally ~~independent~~ ~~on~~ independent of the states $X_0, X_1, \ldots \ldots X_{n-1}$ for the present state X_n

Memoryless Property [So, next state is dependent only on the current state

Another aspect is the time-parameter "n". If the transition probability from i to j, denoted as P_{ij}, does not depend on the time-parameter n, the Markov Process is known as time-homogeneous process.

However, if the transition probability depends on time-parameter "n", the process is known as ~~time~~ non-time-homogeneous Markov chain.

Overall, P_{ij} : Probability of transition to state j given that the current state is i

$$P_{ij} \geq 0, \qquad \sum_{j=0}^{\infty} P_{ij} = 1, \quad \text{where } i = 0, 1, \ldots$$

The transition Matrix.

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \end{bmatrix}$$

Example:

Suppose a communication system transmits 0 or 1.

comprises of different stages

In each stage, the input digit might get altered.

Assume, p is the probability that the digit enters remain unchanged. So

p: Probability that digit is unchanged

1-p: " " " is changed

Let's consider X_n represents the digit entering n^{th} stage then $\{X_n, n = 0, 1, 2, \ldots\}$ a two-state Markov process

Changed Unchanged

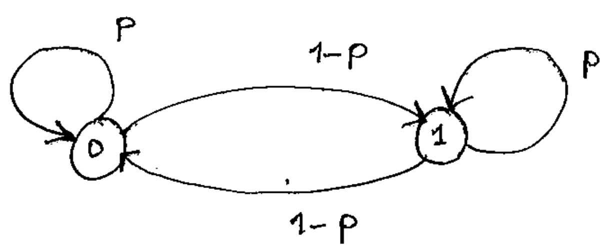
$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & p & 1-p \\ 1 & 1-p & p \end{array}$$

= transition probability

Where, $0 \leq p \leq 1$

				Prob.
State 0	Sent 0	o/p 0 :		p
	Sent 0	o/p 1 :		1-p
State 1	Sent 1	o/p 0 :		1-p
	Sent 1	o/p 1 :		p

If we draw the Markov chain that provides a visual representation of the state ~~of~~ transition



Example: Suppose a state-space of a Markov process is given by integers $i = 0, \pm 1, \pm 2, \pm 3 \ldots$

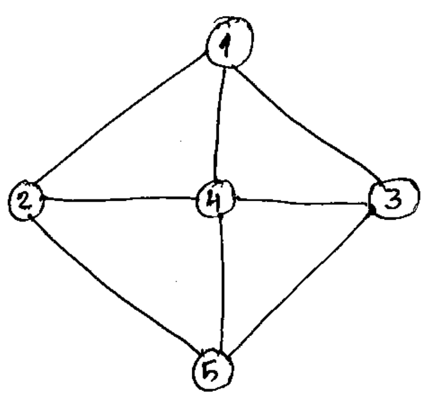
The process would be a random walk if for some number $0 < P < 1$

probability

$$P_{i,\,i+1} = p = 1 - P_{i,\,i-1}$$

An individual is walking on a straight line and moving to either left or right with probability P, or $1-P$, respectively.

Suppose, we traverse graph of 5 nodes randomly with probality calculated from frequentist's view.



Let P_{ij} denotes the probability of transition from state i to state j in the next time parameter.

Assume the transitions to any nodes from a given node is uniform.

Also, assume that transition must occur.

$P \equiv$ state transition matrix

$$\equiv \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

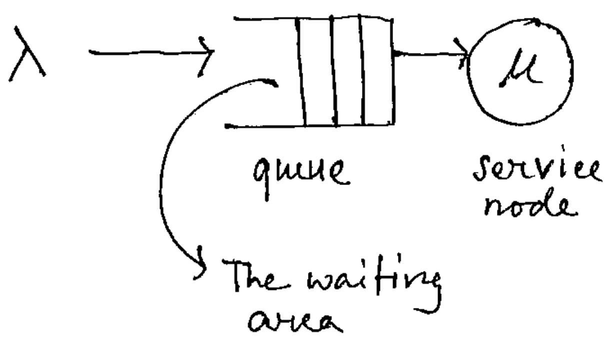
Given that it is a random walk on the graph or a Markov process, we can ask ~~a~~ questions as —

How quickly we can cover the whole ~~node~~ graph ?

How quickly can we reach a particular node in the graph

More examples :

. Consider a single server queue M/M/1



μ : Mean service rate

λ : Mean arrival rate.

Where : M^{1st} : Arrival is Markovian
↳ denotes memory-less

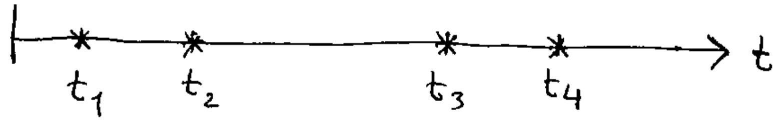
M^{2nd} : Service time distribution is also Markovian / memoryless

C : 1 : Number of server

K : size of the queue

The Poisson Point Process

Definition: A point process is a set of random points $\{\mathbf{t}_i\}$ on the time axis.



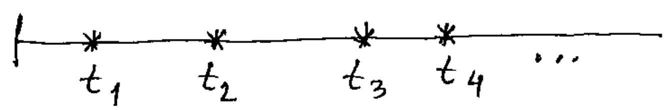
The points are RVs that represent times at which random events occur.

So, a point process is a set of random points along the time axis: $\{\mathbf{t}_I\}$

For instance, events would be customer arrival at tellers window of a bank.

Time at which the light bulbs in a house burn out.

However, counting process can be formed, or occur, in any continuum. For instance, it can be over a space as well.



In short, the points are ordered in time.

$$t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n$$

and $\{\mathbf{t}_i\}$ are a collection of RVs defined on some probability space $(S, \mathcal{F}, P)$

Definition

$\hookrightarrow$ Counting Process: To each point process $\{\mathbf{t}_i\}$ we can assign a random process $\mathbf{X}(t)$ called the counting process defined

$$\mathbf{X}(t) \triangleq \text{number of points in the interval}$$

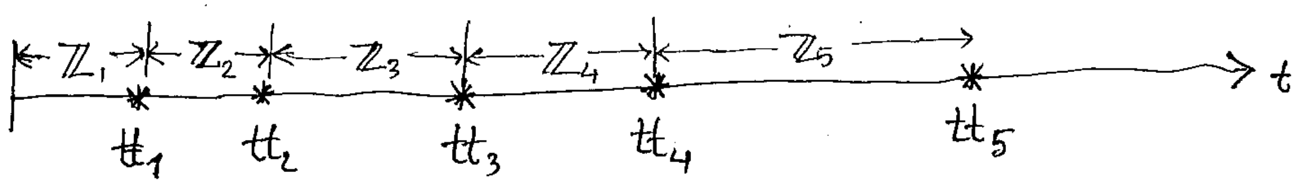
Definition: Renewal Process

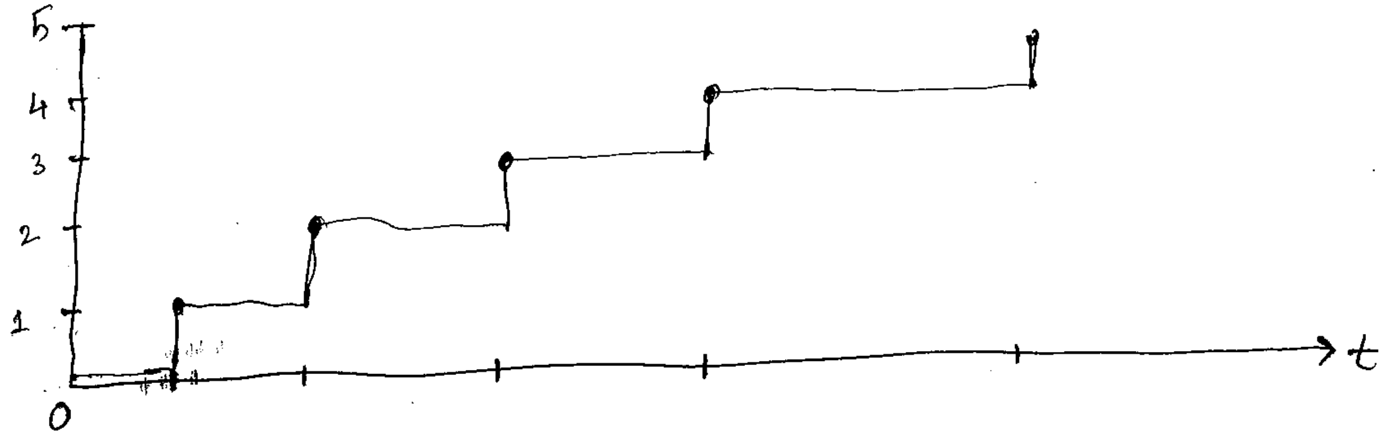
To any point process $\{tt_i\}$ we can associate a renewal process,

↳ A sequence of RVs defined by

$$Z_n = \begin{cases} tt_1, & n = 1 \\ tt_n - tt_{n-1}, & n = 2, 3, \ldots \end{cases}$$

Here, the renewal process represents ~~represents~~ time between the events in the point process $\{tt_i\}$. The sequence $\{Z_n\}$ are RVs defined on $(S, \mathcal{F}, P)$





So,