Homework No. 4

Course Title: Modeling & Simulation

Course No: CSE 562

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Homework-4

Answer to the Question No: 1

ⓐ Gaussian: $X \sim N(\mu, \sigma^2), \sigma > 0$

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), x \in \mathbb{R}$$

MGF:

$$M_X(t) = \int_{\mathbb{R}} e^{tx} f(x)\, dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + tx\right) dx$$

Complete the square, we write:

$$-\frac{(x-\mu)^2}{2\sigma^2} + tx = -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu + \sigma^2 t)x + \mu^2\right]$$

$$= \frac{(\mu+\sigma^2 t)^2 - \mu^2}{2\sigma^2} - \frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}$$

Hence

$$M_X(t) = \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right) \cdot \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx$$

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Make the change $y = \frac{x - (\mu + \sigma^2 t)}{\sigma}$.

Then $dx = \sigma\, dy$ and

$$\int_{\mathbb{R}} \exp\left(-\frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx$$

$$= \sigma \int_{\mathbb{R}} e^{-y^2/2}\, dy$$

$$= \sigma \sqrt{2\pi}$$

$$\therefore\ M_X(t) = \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)\quad t \in \mathbb{R}$$

CF:

Replace $t \to it$, so we get,

$$\varphi_X(t) = \exp\left(i\mu t - \tfrac{1}{2}\sigma^2 t^2\right)\quad t \in \mathbb{R}$$

(ii) Poisson: $X \sim \mathrm{Pois}(\lambda),\ \lambda > 0$

PMF:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!},\quad k \in \mathbb{N}_0.$$

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MGF:

Definition and absolute convergence (ratio test) justify exchanging sum and expectation:

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} P(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} \exp(\lambda e^t)$$

$$= \exp(\lambda(e^t - 1)) \quad t \in \mathbb{R}$$

CF:

Same computation with e^{it}:

$$\varphi_X(t) = \exp(\lambda(e^{it} - 1)) \quad t \in \mathbb{R}$$

© Exponential: $X \sim Exp(\lambda)$, $\lambda > 0$

PDF:

$$f(x) = \lambda e^{-\lambda x} 1_{[0,\infty)}(x)$$

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MGF:

For $t < \lambda$ (to ensure integrability of $e^{(t-\lambda)x}$ on $[0, \infty)$):

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \lambda \cdot \frac{1}{\lambda - t}$$

$$= \frac{\lambda}{\lambda - t} ; t < \lambda$$

(The integral equals $1/(\lambda - t)$ since $\int_0^\infty e^{-ax} dx = 1/a$ for $a > 0$)

CF:

For all $t \in \mathbb{R}$ (since $R(\lambda - it) = \lambda > 0$):

$$\varphi_X(t) = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - it)x} dx$$

$$= \lambda \cdot \frac{1}{\lambda - it}$$

$$= \frac{\lambda}{\lambda - it}.$$

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(d) Binomial : $X \sim Bin(n, p)$, $n \in \mathbb{N}$, $0 < p < 1$

PMF:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

MGF:

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$

$$= ((1-p) + pe^t)^n \quad \text{(binomial theorem)}$$

$$= (1 - p + pe^t)^n \quad t \in \mathbb{R}$$

CF:

Replace $t \to it$:

$$\varphi_X(t) = (1 - p + pe^{it})^n; \; t \in \mathbb{R}$$

Answer to the Question No 2

ⓐ Gaussian $X \sim N(\mu, \sigma^2)$

$\mathbb{E}[X] = \mu, \quad var(X) = \sigma^2$

(One can compute $\mathbb{E}[X] = \int x f(x)\, dx$ by symmetry/shift and $var(X) = \int (x-\mu)^2 f(x)\, dx = \sigma^2$)

From MGF: $M_X(t) = exp(\mu t + \frac{1}{2}\sigma^2 t^2)$

compute derivatives:

$$M'_X(t) = (\mu + \sigma^2 t)\, M_X(t)$$

so,

$$M'_X(0) = \mu \cdot M_X(0) = \mu \cdot 1 = \mu$$

Thus $E[X] = \mu$

Next,

$$M''_X(t) = \frac{d}{dt}\left((\mu + \sigma^2 t) M_X(t)\right) = \sigma^2 M_X(t) + (\mu + \sigma^2 t)^2 M_X(t).$$

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So,

$$M''_X(0) = \sigma^2 + \mu^2$$

Thus

$$\text{Var}(X) = M''_X(0) - (M'_X(0))^2$$

$$= (\sigma^2 + \mu^2) - \mu^2$$

$$= \sigma^2$$

which matches the analytic result.

(b) Poisson $X \sim \text{Pois}(\lambda)$

Mean:

$$E[X] = \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

Second moment:

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} + e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!}$$

Evaluate sums:

$$e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = \lambda \text{ (we already found)}$$

and

$$e^{-\lambda} \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda^2$$

so,

$$E[X^2] = \lambda^2 + \lambda$$

Hence

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= (\lambda^2 + \lambda) - \lambda^2$$

$$= \lambda$$

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From MGF:

$$M_X'(t) = \lambda e^t M_X(t) \quad , \quad M_X''(t) = (\lambda e^t + \lambda^2 e^{2t}) M_X(t)$$

Evaluate at $t=0$

$$M_X'(0) = \lambda, \quad M_X''(0) = \lambda + \lambda^2$$

So, $E[X] = \lambda$, $E[X^2] = \lambda + \lambda^2$ and

$$\begin{aligned} Var(X) &= M_X''(0) - (M_X'(0))^2 \\ &= (\lambda + \lambda^2) - \lambda^2 \\ &= \lambda \end{aligned}$$

matching the analytic calculation.

(c) Exponential $X \sim Exp(\lambda)\ (\lambda > 0)$

Mean:

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

Using integration by parts,

let, $u = x$, $dv = \lambda e^{-\lambda x} dx$

Then $du = dx$, $v = -e^{-\lambda x}$

$$\begin{aligned} E[X] &= \left[-x e^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 + \frac{1}{\lambda} \\ &= \frac{1}{\lambda} \end{aligned}$$

Second moment:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

Integration by parts twice: $E[X^2] = \frac{2}{\lambda^2}$

(Equivalently, recognize $X \sim$ Gamma $(k = 1, \theta = 1/\lambda)$

so $E[X^2] = Var + (E^2)$

$$= 1/\lambda^2 + \frac{1}{\lambda^2}$$

$$= 2/\lambda^2$$

Hence

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

From MGF:

$M_X(t) = \frac{\lambda}{\lambda - t}$ (valid for $t < \lambda$)

compute derivatives:

$$M'_X(t) = \frac{\lambda}{(\lambda-t)^2}, \quad M''_X(t) = \frac{2\lambda}{(\lambda-t)^3}$$

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Evaluate at $t=0$:

$$M_X'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}, \quad M_X''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

So, $E[X] = \frac{1}{\lambda}$, $E[X^2] = \frac{2}{\lambda^2}$ and

$$\cancel{Var}\ Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$
$$= \frac{1}{\lambda^2}$$

matching the analytic result.

(d) Binomial $X \sim Bin(n, p)$

Mean:

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad \left[\text{using } k\binom{n}{k} = n\binom{n-1}{k-1}\right]$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= np \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}}_{1}$$

$$= np$$

Second moment:

$$E[X(X-1)] = \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k}$$

$$= n(n-1)p^2 \text{ (a standard identity/ combinatorial argument)}$$

Then

$$E[X^2] = E[X(X-1)] + E[X].$$

$$= n(n-1)p^2 + np$$

Hence

$$Var(X) = E[X^2] - (E[X])^2$$

$$= (n(n-)p^2 + np) - (np)^2$$

$$= np(1-p)$$

From MGF:

$$M_X(t) = (1-p+pe^t)^n$$

Differentiate:

$$M'_X(t) = npe^t(1-p+pe^t)^{n-1}$$

So,

$$M'_X(0) = np$$

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Differentiate again (product + chainrule)

$$M''_X(t) = npe^t(1-p+pe^t)^{n-1} + npe^t(n-1)pe^t(1-p+pe^t)^{n-2}$$

$$= npe^t(1-p+pe^t)^{n-2}\left((1-p+pe^t)+(n-1)pe^t\right)$$

Evaluate at $t=0$ (note: $e^0=1$ and $1-p+p=1$)

$$M''_X(0) = np(1+(n-1)p)$$
$$= np + n(n-1)p^2.$$

Thus

$$\mathrm{var}(X) = M''_X(0) - (M'_X(0))^2$$
$$= (np+n(n-1)p^2) - (np)^2$$
$$= np(1-p)$$

matching with the anlytic computation.

Answer to the Question No. 3

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as stats

np.random.seed(42)

# Define parameter sets for each distribution
params = {
    "Gaussian": [(0, 1), (2, 1), (0, 2)],  # (mu, sigma)
    "Poisson": [2, 5, 10],                # lambda
    "Exponential": [0.5, 1, 2],           # lambda
    "Binomial": [(10, 0.3), (10, 0.5), (20, 0.7)]  # (n, p)
}

def print_stats(dist_name, distribution, *params):
    mean, var = distribution.stats(*params, moments="mv")
    print(f"{dist_name} params={params} => Mean={mean:.3f}, Var={var:.3f}")

fig, axes = plt.subplots(4, 1, figsize=(4, 8))

# Gaussian
x = np.linspace(-10, 10, 500)
for mu, sigma in params["Gaussian"]:
    pdf = stats.norm.pdf(x, mu, sigma)
    axes[0].plot(x, pdf, label=f"μ={mu}, σ={sigma}")
    print_stats("Gaussian", stats.norm, mu, sigma)
```

```
axes[0].set_title("Gaussian Distribution")
axes[0].legend()
axes[0].grid()

# Poisson
x = np.arange(0, 25)
for lam in params["Poisson"]:
    pmf = stats.poisson.pmf(x, lam)
    axes[1].stem(x, pmf, basefmt=" ", label=f"λ={lam}")
    print_stats("Poisson", stats.poisson, lam)
axes[1].set_title("Poisson Distribution")
axes[1].legend()
axes[1].grid()

# Exponential
x = np.linspace(0, 10, 500)
for lam in params["Exponential"]:
    pdf = stats.expon.pdf(x, scale=1/lam)
    axes[2].plot(x, pdf, label=f"λ={lam}")
    print_stats("Exponential", stats.expon, 0, 1/lam)  # loc=0, scale=1/λ
axes[2].set_title("Exponential Distribution")
axes[2].legend()
axes[2].grid()

# Binomial
x = np.arange(0, 21)
for n, p in params["Binomial"]:
    pmf = stats.binom.pmf(x, n, p)
```

```
    axes[3].stem(x, pmf, basefmt=" ", label=f"n={n}, p={p}")
    print_stats("Binomial", stats.binom, n, p)
axes[3].set_title("Binomial Distribution")
axes[3].legend()
axes[3].grid()

plt.tight_layout()
plt.show()
```

Plotted Figure: