Homework No. 3

Course Title: Modeling & Simulation

Course No: CSE 562

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Answer to the Question No. 1

Given,

Gaussian PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

we need to show, it satisfies

$$I = \int_{-\alpha}^{\alpha} f_X(x)\, dx = 1$$

Let,

$$z = \frac{x-\mu}{\sigma}$$

$$\Rightarrow x = \sigma z + \mu$$

$$\therefore dx = \sigma dz$$

Then,

$$f_X(x)\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

Therefore,

$$I = \int_{-\alpha}^{\alpha} \frac{1}{\cancel{2}\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

This is standard normal distribution, and it's known fact that:

$$\int_{-\alpha}^{\alpha} \frac{1}{\cancel{2}\sqrt{2\pi}} e^{-z^2/2}\, dz = 1$$

Now,

$$I = \int_{-\alpha}^{\alpha} e^{-x} dx$$

$$I^2 = \left(\int_{-\alpha}^{\alpha} e^{-x^2} dx\right)\left(\int_{-\alpha}^{\alpha} e^{-y^2} dx\right)$$

$$= \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} e^{-(x^2+y^2)} dx\, dy$$

Change to polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$

$x^2+y^2 = r^2$

$$I^2 = \int_0^{2\pi}\int_0^{\alpha} e^{-r^2} dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\alpha} r e^{-r^2} dr$$

Now,

let $u = r^2$

$\Rightarrow du = 2r\, dr$

$\Rightarrow r\, dr = \frac{1}{2} du$

So, $$\int_0^{\alpha} r e^{-r^2} dr = \frac{1}{2}\int_0^{\alpha} e^{-u} du$$

$$= \frac{1}{2}$$

Therefore, $I^2 = \int_0^{2\pi} d\theta \cdot \frac{1}{2}$

$= 2\pi \cdot \frac{1}{2}$

$\therefore I = \sqrt{\pi}$

Thus, $\int_{-\alpha}^{\alpha} e^{-x^2} dx$

$= \sqrt{\pi}$

So, the standard normal pdf integrates to:

$$\int_{-\alpha}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

$$\Rightarrow I = \int_{-\alpha}^{\alpha} f_X(x)\, dx = 1$$

[Showed]

Answer to the Question No. 2

Given,

$$f_X(x) = \frac{1}{\mu} e^{-x/\mu} \, 1_{[0,\alpha)}(x)$$

We need to find, conditional density $f_X(x \mid \mu < X \leq 2\mu)$.

The conditional density of X given $\mu < X \leq 2\mu$ is:

$$f_X(x \mid \mu < X \leq 2\mu) = \begin{cases} \frac{f_X(x)}{P(\mu < X \leq 2\mu)} & \text{if } \mu < x \leq 2\mu \\ 0 & \text{otherwise.} \end{cases}$$

Computing the denominator $P(\mu < X \leq 2\mu)$

$$P(\mu < X \leq 2\mu) = \int_{\mu}^{2\mu} \frac{1}{\mu} e^{-x/\mu} dx$$

Let,

$$u = \frac{x}{\mu}$$

$$\Rightarrow du = \frac{dx}{\mu}$$

Therefore, $\int_1^2 e^{-u} du$

$$= \left[-e^{-u}\right]_1^2$$

$$= e^{-1} - e^{-2}$$

So, $P(\mu < X \leq 2\mu) = e^{-1} - e^{-2}$

Now,

Plugging into the formula we get:

$$f_X(x \mid \mu < X \leq 2\mu) = \begin{cases} \dfrac{\frac{1}{\mu} e^{-x/\mu}}{} & \mu < x \leq 2\mu \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$f_X(x \mid \mu < X \leq 2\mu) = \begin{cases} \dfrac{1}{\mu(e^{-1} - e^{-2})} e^{-x/\mu} & \text{for } \mu < x < 2\mu \\ 0 & \text{otherwise} \end{cases}$$

Answer to the Question No.3

We are asked to prove the identity:

$$F_X(x|A) = \frac{P(A|\{X \le x\}\, F_X(x)}{P(A)}$$

Let,

$F_X(x) = P(X \le x)$: CDF of X

$F_X(x|A) = \frac{P(X \le x | A)}{p(X \le x)}$: conditional Probability of A given $X \le x$,

Now, we can write,

$$F_X(x|A) = P(X \le x | A) = \frac{P(X \le x \cap A)}{P(A)} \quad \text{--- (I)}$$

according to the definition of conditional probability.

Now,

$$P(A) X \le x) = \frac{P(A \cap \{X \le x\})}{P(X \le x)}$$

$$\Rightarrow P(A \cap \{X \le x\}) = P(A|X \le x) \cdot P(X \le x) \quad \text{--- (II)}$$

Plugging (II) into (I),

$$F_X(x|A) = \frac{P(A \cap \{X \le x\})}{P(A)}$$

Now, as $F_X(x) = P(X \le x)$

So,

$$F_X(x|A) = \frac{P(A|X \le x)\, F_X(x)}{P(A)}$$

[Showed]

5

Answer to the Question No. 4

We wish to find the conditional probability $P(0.3 \le P \le 0.7) | A)$ where $A = \{6$ heads in 10 tosses$\}$. clearly $P(A | P = P) = p^6 (1-P)^4$

$$f(P|A) = \frac{p^6 (1-P)^4}{\int_0^1 p^6 (1-P)^4 dp} = \frac{p^6 (1-P)^4}{4329 \times 10^{-7}}$$

This yeilds,

$$P(0.3 \le P \le 0.7 | A) = \int_{0.3}^{0.7} f(P|A) dp$$

$$= \frac{10^7}{4329} \int_{0.3}^{0.7} p^6 (1-P)^4 dp$$

$$= 0.768 \text{ Ans.}$$

Answer to the Question No. 5

Given, $Y = -4X + 3$, we solve for X:

$$X = \frac{3 - Y}{4}$$

Since, $X \in [0, \alpha)$, plug into the following

$$Y = -4X + 3$$

We need to find the CDF and PDF of Y, using the transformation $Y = -4X + 3\}$.

Now,

$$Y = -4X + 3$$

$$\therefore X = \frac{3-Y}{4}$$

Since, $X \geq 0$, we get,

$$\frac{3-Y}{4} \geq 0$$

$$\Rightarrow 3 - Y \geq 0$$

$$\Rightarrow Y \leq 3$$

As $X \to 0$, $Y \to 3$

As $X \to \infty$, $Y \to -\infty$

So, $Y \in (-\infty, 3]$

We define,

$$F_Y(y) = P(Y \leq y) = P(-4X + 3 \leq y)$$

Solving inside the probability,

$$-4X + 3 \leq y$$

$$\Rightarrow X \geq \frac{3-y}{4}$$

So, $F_Y(y) = P\left(X \geq \frac{3-y}{4}\right)$

Using the exponential tail probability:

$$P(X \geq a) = \int_a^{\infty} 2e^{-2x}\,dx = e^{-2a}$$

$$\text{So, } F_Y(y) = \begin{cases} e^{-2\cdot\frac{3-y}{4}} = e^{-\frac{3-y}{2}}, & y \le 3 \\ 1 & y > 3 \end{cases}$$

Differentiate the CDF for $y < 3$,

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

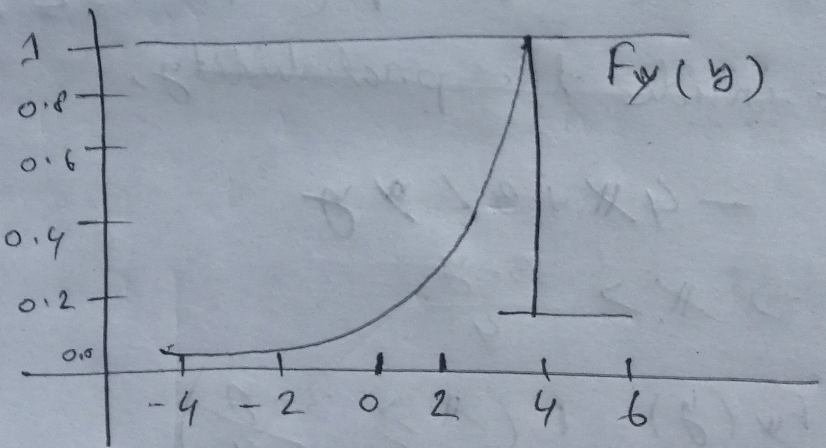
$$= \frac{d}{dy} e^{-\frac{3-y}{2}}$$

$$= \frac{1}{2} e^{-\frac{3-y}{2}}, \quad y < 3$$

So, the full PDF is,

$$f_Y(y) = \begin{cases} \frac{1}{2} e^{-\frac{3-y}{2}}, & y < 3 \\ 0 & y > 3 \end{cases}$$

CDF Plot:



Here's the plot of the CDF $F_Y(y)$ for the transformed variable $Y = -4X+3$, where $X \sim$ Exponential (2).

Answer to the Question No. 6

Since X ranges from $-2c$ to $2c$, $Y = X^2$ will range from 0 to $(2c)^2 = 4c^2$. Therefore, the support for Y is $y \in [0, 4c^2]$.

The CDF $F_Y(y)$ is the probability that $Y \le y$.
To find this, we first express it in terms of X:

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4c}\,dx$$

$$= \frac{1}{4c}\, 2\sqrt{y}$$

$$= \frac{\sqrt{y}}{2c} \quad \text{for } 0 \le y \le 4c^2$$

The PDF $f_Y(y)$ is the derivative of CDF $F_Y(y)$ with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y}}{2c}\right)$$

This gives:

$$f_Y(y) = \frac{1}{4c\sqrt{y}}, \quad \text{for } 0 < y < 4c^2$$

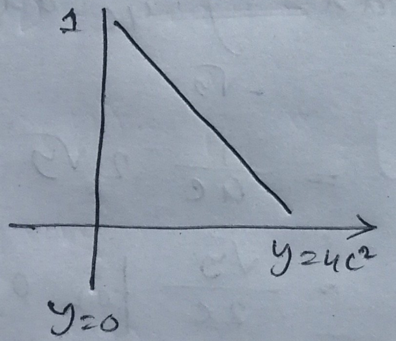
For $y = 0$, $f_Y(y) = 0$, since $Y = X^2$ takes values strictly greater than 0 with positive probability.

The PDF $f_Y(y)$ is a decreasing function of y, and it has the form $\frac{1}{4c\sqrt{y}}$ for $y \in (0, 4c^2]$

The CDF $F_Y(y)$ is a square root function, increasing from 0 at $y=0$ to 1 at $y=4c^2$

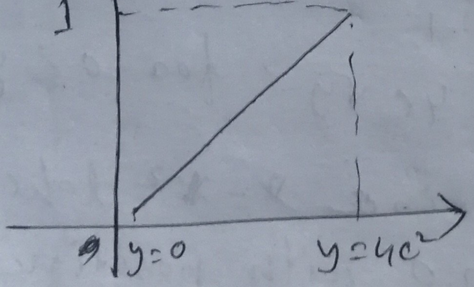
PDF : $f_Y(y) = \frac{1}{4c\sqrt{y}}$

The function starts at $y=0$ and decreases towards zero as $y \to 4c^2$



CDF : $F_Y(y)$

The CDF starts at 0 and increases, eventually reaching 1 at $y = 4c^2$



Answer to the Question No. 7

Let, $X_1, X_2, \ldots\ldots, X_{200}$ be independent random variables uniformly distributed on $(0, 100)$

Let, $z = \min(X_1, X_2, \ldots\ldots, X_{100})$

This means z is a minimum of 200 independent and identically distributed. Uniform $(0, 100)$ random variables.

If $X_1, \ldots\ldots, X_n$ are independent identically independent with CDF $F_X(x)$, then the CDF of $z = \min(X_1, \ldots\ldots, X_n)$ is

$$F_Z(z) = 1 - [1 - F_X(z)]^n$$

Each $X_i \sim$ Uniform $(0, 100)$, so:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/100 & 0 \le x \le 100 \\ 1 & x > 100 \end{cases}$$

Now, applying the formula for $n = 200$:

For ~~f_Z~~ $z \in [0, 100]$:

$$F_Z(z) = 1 - \left(1 - \frac{z}{100}\right)^{200}$$

for $z < 0$: $F_Z(z) = 0$

for $z > 100$: $F_Z(z) = 1$

Therefore,

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 - (1 - z/100) & \text{if } 0 \le z \le 100 \\ 1 & \text{if } z > 100 \end{cases}$$ 11

Answer to the Question No. 8

(a) $Y = |X|$

The function $g(x) = |x|$ is not monotonic. It is, decreasing on $(-\infty, 0)$ and increasing on $[0, \infty)$

So, for $y > 0$, both $x = y$ and $x = -y$ satisfy $g(x) = y$.

For $y = 0$ only $x = 0$ works.

Using CDF ~~format~~ method:

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y)$$

This means:

$$F_Y(y) = P(-y \leq X \leq y) = \int_{-y}^{y} f_X(x)\, dx \text{ for } y \geq 0$$

Differentiate with respect to y to get the PDF:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(y) + f_X(-y), \text{ for } y > 0$$

At $y = 0$, since both $f_X(y)$ and $f_X(-y)$ collapse to $f_X(0)$:

$$f_X(0) = f_X(0)$$

$$\therefore f_Y(y) = \begin{cases} f_X(y), f_X(-y) & , y > 0 \\ f_X(0) & , y = 0 \\ 0 & , y < 0 \end{cases}$$

(b) Define the CDF of Y:

$$F_Y(y) = P(Y \leq y)$$

Now, since $Y = e^{-X}$ only for $X \geq 0$, and $Y \in (0, 1]$, we focus on $y \in (0, 1]$

$$F_Y(y) = P(e^{-x} \leq y) = P(X - \ln y)$$

Because $e^{-X} \leq y$ implies $X \geq -\ln y$ (as the exponential function is decreasing).

As $y = e^{-x}$
$\Rightarrow \ln y = \ln e^{-x}$
$\therefore \ln y = -X$

Thus,

$$F_Y(y) = P(X \geq -\ln y)$$
$$= \int_{-\ln y}^{\infty} f_X(x)\,dx$$

For PDF:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[\int_{-\ln y}^{\infty} f_X(x)\,dx\right]$$

We treat is a composition:

$$y \rightarrow -\ln y$$

then plug into $f_X(x)$.

So, by chain rule:

$$f_Y(y) = -\frac{d}{dy}(-\ln y) \cdot f_X(-\ln y) = \frac{1}{y} \cdot f_X(-\ln y)$$

Therefore,

$$f_Y(y) = \begin{cases} 1/y \cdot f_X(-\ln y) & \text{for } 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$