Homework No. 2

Course Title: Modeling & Simulation

Course No: CSE 562

Submitted to: Md. Shahriar Karim (MSK1)

Submitted by: Sumaiya Tarannum Noor

Section: 01

ID: 2425410650

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Name: Sumaiya Tarannum Noor
ID: 2425410650

CSE-562

Homework 2

Answer to the Question No. 1

ⓐ $(A\cap\overline{B})\cup(B\cap\overline{A})$

$=(A\cup B)\cap(\overline{A}\cup\overline{B})$

$=(A\cup B)\cap(A\cap B)$

$=(A\cup B)-(A\cap B)$

ⓑ $(A\cap\overline{B})\cap(A\cap B)$

$=(A\cap A)\cap(\overline{B}\cap B)$

$=A\cap\emptyset$

$=\emptyset$

ⓒ Using De Morgans law to solve:

ⓐ $\overline{A\cap(B\cup C)}=(\overline{A}\cup\overline{B})\cap(\overline{A}\cup\overline{C})$

L.H.S $=\overline{A\cap(B\cup C)}$

$=\overline{A}\cup\overline{(B\cup C)}$

$=\overline{A}\cup(\overline{B}\cap\overline{C})$ $\left[\because \overline{B\cup C}=\overline{B}\cap\overline{C}\right]$

Now, by using Distributive law, we can write,

$\overline{A}\cup(\overline{B}\cap\overline{C})$

$=(\overline{A}\cup\overline{B})\cap(\overline{A}\cup\overline{C})$

$=$ R.H.S

[Showed]

(b) $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

$L.H.S = \overline{A \cap B \cap C}$

$= \overline{(A \cap (B \cap C))}$

$= \bar{A} \cup \overline{(B \cap C)}$

$= \bar{A} \cup (\bar{B} \cup \bar{C})$ [Applying De Morgan's law]

$= \bar{A} \cup \bar{B} \cup \bar{C}$

$= R.H.S$

$\therefore L.H.S = R.H.S$

[Showed]

Answer to the Question No. 2

To show that $\{B_1, B_2, \ldots\ldots, B_n\}$ is a partition of set G, we need to demonstrate two things:

1. The sets $B_1, B_2, \ldots\ldots, B_n$ are disjoint, i.e., $B_i \cap B_j = \phi$ for $i \neq j$.

2. The union of all sets $B_1, B_2, \ldots\ldots, B_n$ equals G, i.e., $B_1 \cup B_2 \cup \ldots\ldots \cup B_n = G$.

By definition, each set B_j is given by:

$$B_j = G \cap A_j \text{ for } j = 1, 2, \ldots\ldots, n$$

Now, consider the intersection of any two sets B_i and B_j for $i \neq j$ we want to show that:

$$B_i \cap B_j = \phi$$

Using the definition of sets B_i and B_j, we have:

$$B_i \cap B_j = (G \cap A_i) \cap (G \cap A_j)$$

By the associativity and commutativity of set intersection, this simplifies to:

$$B_i \cap B_j = G \cap (A_i \cap A_j)$$

Since $\{A_1, A_2, \ldots\ldots, A_n\}$ is a partition of space S, the sets A_i and A_j are disjoint for $i \neq j$, i.e.,

$$A_i \cap A_j = \phi \text{ for } i \neq j$$

Therefore,

$$B_i \cap B_j = G \cap \phi = \phi$$

Thus, B_i and B_j are disjoint for $i \neq j$.

Next, we need to show that the union of all the sets $B_1, B_j, \ldots\ldots, B_n$ equals G, i.e.,

$$B_1 \cup B_2 \cup \ldots\ldots \cup B_n = G$$

B_j is defined as:

$$B_j = G \cap A_j \text{ for } j = 1, 2, \ldots\ldots\ldots, n.$$

Now, considering the union:

$$B_1 \cup B_2 \cup \ldots\ldots \cup B_n = (G \cap A_1) \cup (G \cap A_2) \cup \ldots\ldots \cup (G \cap A_n)$$

Using the distributive law property of set union and intersection, we get:

$$B_1 \cup B_2 \cup \ldots\ldots\ldots \cup B_n = G \cap (A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n)$$

Since, $\{A_1, A_2, \ldots\ldots, A_n\}$ is a partition of S, the union of all sets $A_1 \cup A_2 \cup \ldots\ldots \cup A_n = S$.

Therefore,

$$B_1 \cup B_2 \cup \cdots\cdots \cup B_n = G \cap S = G.$$

Since we have shown that $B_1 \cup B_2 \cup \cdots\cdots \cup B_n = G$ and

$B_i \cup B_j = \phi$ for $i \neq j$, the sets $\{B_1, B_2, \cdots\cdots, B_n\}$

form a partition of set G.

Answer to the Question No. 3

Let $S = \{a_1, a_2, \cdots\cdots, a_n\}$ be a finite set with n elements.

We want to show that the number of distinct subsets of S is 2^n.

Each subset of S is formed by either or including or excluding each element of S.

For each element $a_i \in S$, we have two choices:

- include a_i in the subsets, or
- exclude a_i from the subset.

Since there are n elements and for each we have 2 choices, the total number of different combination we can make is:

$$2 \times 2 \times \cdots \cdots \times 2 \ (n \text{ time}) = 2^n$$

So, there are 2^n possible subsets.

This includes the empty set $\emptyset$ and the full set S, as well as all other subsets in between.

Therefore, the number of distinct subsets of a finite set with n elements is 2^n.

Answer to the Question No. 4

ⓐ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let $x \in (\overline{A \cup B})$

By the definition of complement:

$$x \notin A \cup B$$

By the definition of union:

$$x \notin A \text{ and } x \notin B$$

This implies :

$$x \in \overline{A} \text{ and } x \in \overline{B}$$

So,

$$x \in \overline{A} \cap \overline{B}$$

Thus,

$$x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$$

Now the reverse!

Let $x \in \overline{A} \cap \overline{B}$

Then,

$$x \in \overline{A} \text{ and } x \in \overline{B} \Rightarrow x \notin A \text{ and } x \notin B$$

So,

$$x \notin A \cup B \Rightarrow x \in (\overline{A \cup B})$$

Therefore,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

(ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Let $x \in \overline{A \cap B}$

Then :

$$x \notin A \cap B$$

By definition of intersection:

$$x \notin A \text{ or } x \notin B$$

So,

$$x \in \overline{A} \text{ or } x \in \overline{B} \Rightarrow x \in \overline{A} \cup \overline{B}$$

Now the reverse;

Let $x \in \overline{A} \cup \overline{B}$

Then,

$$x \in \overline{A} \text{ or } x \in \overline{B} \Rightarrow x \notin A \text{ or } x \notin B$$

So,

$$x \notin A \cap B \Rightarrow x \in (\overline{A \cap B})$$

Therefore

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Answer to the Question No. 5

If $A = \{2 \leq x \leq 5\}$ $B = \{3 \leq x \leq 6\}$

$S = \{-\alpha < x < \alpha\}$ then

$A \cup B = \{2 \leq x \leq 6\}$

$A \cap B = \{3 \leq x \leq 5\}$

$(A \cup B) \cap (\overline{A \cap B})$

$A \cup B = \{2 \leq x \leq 6\}$

$A \cap B = \{3 \le x \le 5\}$

$(A \cup B) \cap (\overline{A \cap B})$

$A \cup B = \{2 \le x \le 6\}$

$A \cap B = \{3 \le x \le 5\}$

So, $\overline{A \cap B}$ relative to the universe $\mathbb{R}$ is,

$$(-\infty, 3) \cup (5, \infty)$$

$\overline{A \cap B} = \{x < 3 \text{ or } x > 5\}$

Now intersect it with $A \cup B = \{2 \le x \le 6\}$

$$A \cup B \cap (\overline{A \cap B}) = \{2 \le x \le 6\} \cap [(-\infty < x < 3) \cup (5 < x < \infty)]$$

$$= \{2 \le x < 3\} \cup \{5 < x \le 6\}$$

Answer to the Question No. 6

Given,

$A \cap B = \phi$, meaning event A and B are mutually exclusive.

Since, $A \cap B = \phi$, it means no element of A are in B.

That implies : $A \subseteq \overline{B}$

Every element of A is not in B, which is exactly the definition of $\overline{B}$.

If $A \subseteq \overline{B}$, then by a basic property of probability:

$$P(A) < \cancel{P(\overline{B})}\ P(\overline{B})$$

This holds for any probability measure.

Since $A \cap B = \phi \Rightarrow A \subseteq \overline{B}$

and since $A \subseteq \overline{B} \Rightarrow P(A) \leq P(\overline{B})$

[Showed]

Answer to the Question No. 7

(a) We are given,

$$P(A) = P(B) = P(A \cap B)$$

starting with total probability expressions:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$
$$P(B) = P(A \cap B) + P(B \cap \overline{A})$$

Since we are given,

$$P(A) = P(B) = P(A \cap B)$$

That means :

$$P(A\cap B)+P(A\cap \bar{B}) = P(A\cap B)$$

$\Rightarrow P(A\cap \bar{B}) = 0$ [substracting $P(A\cap B)$ from both side]

And,

$$P(A\cap B)+P(B\cap \bar{A}) = P(A\cap B)$$

$\Rightarrow P(B\cap \bar{A}) = 0$ [substracting $P(A\cap B)$ from both side]

Now consider,

$$P((A\cap \bar{B})\cup(B\cap \bar{A})) = P(A\cap \bar{B})+P(B\cap \bar{A})$$

[since the sets are disjoint]

So,

$$P((A\cap \bar{B})\cup(B\cap \bar{A})) = 0+0 = 0$$

[Showed]

(b) We are given:

$$P(A) = P(B) = 1$$

we also know:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the given values:

$$P(A \cup B) = 1 + 1 - P(A \cap B)$$
$$= 2 - P(A \cap B)$$

But since $(A \cup B) \subseteq S$ where S is and by the axioms of probability:

$$P(S) = 1$$
$$\Rightarrow P(A \cup B) \leq 1$$

Therefore,

$$2 - P(A \cap B) \leq 1$$
$$\Rightarrow P(A \cap B) \geq 1$$

But, probabilities can never be greater than 1,

so, $P(A \cap B) \leq 1$

Combining both inequalities:

$P(A \cap B) \geq 1$ and $P(A \cap B) \leq 1 \Rightarrow P(A \cap B) = 1$

[Showed] 12

Answer to Question No. 8

To prove and generalize the identity

$$P(A\cup B\cup C)= P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

and extend it to the union of n events, we proceed in two parts:

① Proving the identity for 3 events:

We use the principal of inclusion-exclusion. The identity for the union of three events:

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

We apply for two events

$$P(A\cup B)=P(A)+P(B) \cancel{+} -P(A\cap B)$$

Now for three events,

$$P(A\cup B\cup C)= P(A)+P(B\cup C)-P(A\cap(B\cup C))$$

We compute:

$$-\ P(B\cup C)=P(B)+P(C)-P(B\cap C)$$

$$-\ P(A\cap(B\cup C))=P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)$$

Putting it together:

$$P(A \cup B \cup C) = P(A) + [P(B) + P(C) - P(B \cap C)]$$
$$- P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$= P(A) + P(B) + P(C)$$
$$- P(B \cap C) - P(A \cap B) - P(A \cap C)$$
$$+ P(A \cap B \cap C)$$

(11) Generalize to n events:

Let $A_1, A_2, \ldots\ldots, A_n$ be n events.

Then,
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum P(A_i) - \sum P(A_i \cap A_j \cap A_k)$$
$$\ldots\ldots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n)$$

This is known as inclusion - exclusion principal.

Let $S_n = P\left(\bigcup_{i=1}^{n} A_i\right)$

We show the inclusion-exclusion formula holds for all n by mathematical induction.

Base case : $n = 1$

$$P(A_1) = P(A_1) \quad \text{True}$$

Inductive step:

Assume the formula holds for $n = k$

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{m=1}^{k} (-1)^{m+1} \sum_{1 \le i_1 < \cdots < i_m < k} P(A_{i_1} \cap \cdots\cdots \cap A_{i_m})$$

Now consider $n = k+1$, then,

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$

Applying inclusion-exclusion to the last term:

$$P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right) = \sum_{m=1}^{k} (-1)^{m+1} \sum_{1 \le i_1 < \cdots < i_m < k} P(A_{i_1} \cap \cdots\cdots \cap A_{i_m} \cap A_{k+1})$$

Now plugging all the values:

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = (\text{terms upto } A_k) + P(A_{k+1})$$

$$+ \text{ all intersection terms with } A_{k+1}$$

which reconstructs the inclusion-exclusion formula for $k+1$.

Hence, by induction, the formula is proven for all n.

Final General Identity,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{m=1}^{n} (-1)^{m+1} \sum_{1 \le i_1 < i_2 < \cdots < i_m < n} P(A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_m})$$

Answer to the Question No. 9

Let $A \subseteq S$ be any subset

Since S is countable, A is also countable (either finite or countably infinite).

So we can write,

$$A = \{ \varepsilon_{i_1}, \varepsilon_{i_2}, \varepsilon_{i_3}, \ldots\ldots \}$$

But then,

$$A = \{\varepsilon_{i_1}\} \cup \{\varepsilon_{i_2}\} \cup \{\varepsilon_{i_3}\} \cup \ldots\ldots$$

Since each singleton $\{\varepsilon_{i_k}\} \in \mathcal{F}$, and since $\mathcal{F}$ is closed under countable unions, it follows that $A \in \mathcal{F}$.

Thus, every subset of S is in $\mathcal{F}$

So, $\mathcal{F} = P(S)$, the power set of S.

Every subset of a countable sample space S is an event if all singletons are events. because all subsets can be expressed as countable unions of singletons, and the σ-algebra is closed under such union.

Answer to the Question No. 10

We are given:

$$A = \{1\}$$

$$B = \{2, 3\}$$

Sample Space : $S = \{1, 2, 3, 4\}$

We need to construct the smallest σ-field that contains both A and B.

The complements:

$$\overline{\{1\}} = \{2, 3, 4\}$$

$$\overline{\{2, 3\}} = \{1, 4\}$$

Unions :

$$\{1\} \cup \{2, 3\} = \{1, 2, 3\}$$

$$\{1\} \cup \{1, 4\} = \{1, 4\}$$

$$\{1\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\} = S$$

$$\{2, 3\} \cup \{1, 4\} = \{1, 2, 3, 4\} = S \text{ etc.}$$

Intersections:

$\{1\} \cap \{2,3\} = \phi$

$\{1\} \cap \{1,2,3,4\} = \{1\}$

$\{2,3\} \cap \{1,4\} = \phi$ etc.

From above combinations, the smallest σ-field must contain the following distinct sets:

ϕ

$\{1\}$

$\{2,3\}$

$\{4\}$ - from complement of $\{1,2,3\}$

$\{1,4\}$

$\{2,3,4\}$

$\{1,2,3\}$

$S = \{1,2,3,4\}$

The smallest σ-field containing $\{1\}$ and $\{2,3\}$ is:

$$\mathcal{F} = \{\phi, \{1\}, \{2,3\}, \{4\}, \{1,4\}, \{2,3,4\}, \{1,2,3\}, \{1,2,3,4\}\}$$

Answer to the Question NO. 11

If $A \subset B$, $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$ then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Ans.

Answer to the Question NO. 12

Proving: $P(A \cap B|C) = P(A|B \cap C) \cdot P(B|C)$

Starting from chain rule and definition of conditional probability:

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)}$$

Now applying the chain rule inside the numerator:

$$P(A \cap B \cap C) = P(A|B \cap C) \cdot P(B \cap C)$$

So, $$P(A \cap B|C) = \frac{P(A|B \cap C) \cdot P(B \cap C)}{P(C)}$$

$$= P(A|B \cap C) \cdot \frac{P(B \cap C)}{P(C)}$$

$$= P(A|B \cap C) \cdot P(B|C)$$

$$\therefore P(A \cap B|C) = P(A|B \cap C) \cdot P(B|C)$$

$$P(A\cap B\cap C) = P(A|B\cap C)\cdot P(B|C)\cdot P(C)$$

This is direct from the chain rule of probability.

$$P(A\cap B\cap C) = P(A|B\cap C)\cdot P(B\cap C)$$

And,

$$P(B\cap C) = P(B|C)\cdot P(C)$$

Now after combining,

$$P(A\cap B\cap C) = P(A|B\cap C)\cdot P(B|C)\cdot P(C)$$

[Showed]

Answer to the Question No. 13

Let's prove the identity:

$$P(A)\cdot P(B) - P(A\cap B) = P(\bar{A}\cap B) - P(\bar{A})\cdot P(B)$$
$$= P(A\cap \bar{B}) - P(A)\cdot P(\bar{B})$$

For any two events A and B in a probability space (S, F, P).

First Identity:

$$P(A)P(B) - P(A\cap B) = P(\bar{A}\cap B) - P(\bar{A})P(B)$$

$$L.H.S = P(A)\cdot P(B) - P(A\cap B)$$

We simplifying R.H.S:

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$P(\bar{A}) = 1 - P(A)$$

So,

$$R.H.S = [P(B) - P(A \cap B)] - [1 - P(A)] \cdot P(B)$$

$$= P(B) - P(A \cap B) - P(B) + P(A)P(B)$$

$$= P(A)P(B) - P(A \cap B)$$

Now ~~tat~~, $L.H.S = P(A)P(B) - P(A \cap B) = R.H.S$

Second Identity:

$$P(A)P(B) - P(A \cap B) = P(A \cap \bar{B}) - P(A)P(\bar{B})$$

Let,

$$R.H.S = P(A \cap \bar{B}) - P(A) \cdot P(\bar{B})$$

we are simplifying R.H.S,

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$P(\bar{B}) = 1 - P(B)$$

So, $R.H.S = [P(A) - P(A \cap B)] - PA(1 - P(B))$

$$= P(A) - P(A \cap B) - P(A) + P(A) \cdot P(B)$$

$$= P(A)P(B) - P(A \cap B)$$

$\therefore L.H.S = R.H.S$

Answer to the Question No: 14

ⓐ At least one of A, B or C occurs

This means, A or B or C happens

$\therefore \quad A \cup B \cup C$

ⓑ At most one of A, B or C occurs

This means, zero or one of the events occurs,

So either:

— None occur $\rightarrow \bar{A} \cap \bar{B} \cap \bar{C}$

— Only one occurs:

— $A \cap \bar{B} \cap \bar{C}$

— $\bar{A} \cap B \cap \bar{C}$

— $\bar{A} \cap \bar{B} \cap C$

So, the full expression:

$$(\bar{A} \cap \bar{B} \cap \bar{C}) \cup (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$$

ⓒ None of the events A, B or C occurs

This means: $\bar{A} \cap \bar{B} \cap \bar{C}$

ⓓ All three event occurs

This means: $A \cap B \cap C$

(e) Exactly one of A, B or C occurs

This means, only one occurs, the other two do not:

$(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)$

(f) A and B occurs but not C

$A\cap B\cap\bar{C}$

(g) A occurs; if not, then B does not occur either:

This is equivalent to, "If A does not occur, then B must also not occur."

$\bar{A}\Rightarrow\bar{B}$

which is logically equivalent to:

$A\cup\bar{B}$

Answer to the Question No. 15

Sample Space S

Tossing a coin 3 times gives:

$S=\{HHH, HHT, HTH, \cancel{THH}, HTT, THH, THT, TTH, TTT\}$

(a) List the element of A, B and C

Event A: Outcomes with exactly 2 H's

$A=\{HHT, HTH, THH\}$

Event B: At least two heads.

so, outcomes with 2 or 3 H's

$B=\{HHT, HTH, THH, HHH\}$

Event C: Head appears when tail has appeared at least once.
This means, Tail appears somewhere, also Head appears somewhere.

So removing all H and all T outcomes we get,

$C = \{HHT, HTH, HTT, THH, THT, TTH\}$

(b) Describe the following events:

(i) $\bar{A} \cap B$

$\bar{A}$ = everything not in A

$\bar{A} = \{HHH, HTT, THT, TTH, TTT\}$

$B = \{HHT, HTH, THH, HHH\}$

$\therefore \bar{A} \cap B = \{HHH\}$ Ans.

(ii) $\bar{A} \cap \bar{B}$

from (i) we get $\bar{A} = \{HHH, HTT, THT, TTH, TTT\}$

$\bar{B}$ = Outcomes with fewer than 2 heads

$\bar{B} = \{HTT, THT, TTH, TTT\}$

$\therefore \bar{A} \cap \bar{B} = \{HTT, THT, TTH, TTT\}$ Ans.

(iii) $A \cap C$

$A = \{HHT, HTH, THH\}$

$C = \{HHT, HTH, HTT, THH, THT, TTH\}$

$\therefore A \cap C = \{HHT, HTH, THH\}$ Ans.