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Subject _____

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Date: 16/08/2025

Homework - 4

Answer to the Question No. 1

Q Gaussian: $X \sim N(\mu, \sigma^2), \sigma > 0$

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), x \in \mathbb{R}$$

MGF:

$$M_X(t) = \int_{\mathbb{R}} e^{tx} f(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + tx\right) dx$$

Complete the square, we write:

$$\begin{aligned} -\frac{(x-\mu)^2}{2\sigma^2} + tx &= -\frac{1}{2\sigma^2} [x^2 - 2(\mu + \sigma^2 t)x + \mu^2] \\ &= \frac{(\mu + \sigma^2 t)^2 - \mu^2}{2\sigma^2} - \frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2} \end{aligned}$$

Hence

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \cdot \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(-\frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx$$

①

Make the change $y = \frac{x - (\mu + \sigma^2 t)}{\sigma}$,

Then $dx = \sigma dy$ and

$$\int_{\mathbb{R}} \exp\left(-\frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx$$

$$= \sigma \int_{\mathbb{R}} e^{-y^2/2} dy$$

$$= \sigma \sqrt{2\pi}$$

$$\therefore M_x(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2) \quad t \in \mathbb{R}$$

cF:

Replace $t \rightarrow it$, so we get,

$$\varphi_x(t) = \exp(it\mu - \frac{1}{2}\sigma^2 t^2) \quad t \in \mathbb{R}$$

(ii) Poisson: $X \sim \text{Pois}(\lambda)$, $\lambda > 0$

PMF:

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}_0$$

(1)

(2)

MGF:

Definition and absolute convergence (ratio test)
 justify exchanging sum and expectation:

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} P(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$x(t) = e^{-\lambda} \exp(\lambda e^t)$$

$$= \exp(\lambda(e^t - 1)) \quad t \in \mathbb{R}$$

CF:

Same computation with e^{it} :

$$\varphi_X(t) = \exp(\lambda(e^{it} - 1)) \quad t \in \mathbb{R}$$

④ Exponential: $X \sim \text{Exp}(\lambda), \lambda > 0$

PDF:

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0, \infty)}(x)$$

(3)

MGF:

For $t < \lambda$, (to ensure integrability of $e^{(t-\lambda)x}$ on $[0, \alpha]$):

$$\begin{aligned} M_X(t) &= \int_0^\alpha e^{tx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\alpha e^{-(\lambda-t)x} dx \end{aligned}$$

$$= \lambda \cdot \frac{1}{\lambda - t}$$

$$= \frac{\lambda}{\lambda - t}; t < \lambda$$

(The integral equals $\lambda/(\lambda - t)$ since $\int_0^\infty e^{-ax} dx = 1/a$ for $a > 0$)

CF:

For all $t \in \mathbb{R}$ (since $R(\lambda - it) = \lambda > 0$):

$$\gamma_X(t) = \int_0^\alpha e^{itx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\alpha e^{-(\lambda-it)x} dx$$

$$= \lambda \cdot \frac{1}{\lambda - it}$$

$$= \frac{\lambda}{\lambda - it}$$

(4)

Subject _____

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(d) Binomial : $X \sim \text{Bin}(n, p)$, $n \in \mathbb{N}$, $0 < p < 1$

PMF :

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, k=0, 1, \dots, n$$

MGF :

$$\begin{aligned} M_X(t) &= \sum_{k=0}^n e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\ &\stackrel{\text{(binomial theorem)}}{=} ((1-p) + pe^t)^n \\ &= (1 - p + pe^t)^n, t \in \mathbb{R} \end{aligned}$$

CF :

Replace $t \rightarrow it$:

$$\varphi_X(t) = (1 - p + pe^{it})^n; t \in \mathbb{R}$$

(5)

Answer to the Question no.2

① Gaussian $X \sim N(\mu, \sigma^2)$

$$\mathbb{E}[X] = \mu, \quad \text{var}(X) = \sigma^2$$

(One can compute $\mathbb{E}[X] = \int x f(x) dx$ by symmetry/shift and $\text{var}(X) = \int (x - \mu)^2 f(x) dx = \sigma^2$)

$$\text{From MGF: } M_X(t) = \exp(\mu t + \frac{1}{2} \sigma^2 t^2)$$

compute derivatives:

$$M'_X(t) = (\mu + \sigma^2 t) M_X(t)$$

so,

$$\begin{aligned} M'_X(0) &= \mu \cdot M_X(0) \\ &= \mu \cdot 1 \\ &= \mu \end{aligned}$$

$$\text{Thus } \mathbb{E}[X] = \mu$$

Next,

$$\begin{aligned} M''_X(t) &= \frac{d}{dt} ((\mu + \sigma^2 t) M_X(t)) \\ &= \sigma^2 M_X(t) + (\mu + \sigma^2 t)^2 M_X(t). \end{aligned}$$

⑥

So,

$$M''_{\bar{X}}(0) = \sigma^2 + \mu^2$$

Thus $\text{var}(\bar{X}) = M''_{\bar{X}}(0) - (M'_{\bar{X}}(0))^2$

$$= (\sigma^2 + \mu^2) - \mu^2$$

$$= \sigma^2$$

which matches the analytic result.

⑩ Poisson $X \sim \text{Pois}(\lambda)$

Mean:

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \\ &= e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \end{aligned}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda (e^{\lambda} - 1) = \lambda = (X)_{\text{avg}}$$

$$\lambda = (R + S) \cdot$$

⑩

Second moment:

$$\begin{aligned} E[X^2] &= \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} + e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} \end{aligned}$$

Evaluate sums:

$$e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} = \lambda \quad (\text{we already found})$$

and

$$\begin{aligned} e^{-\lambda} \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} &= e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} \\ &= e^{-\lambda} \lambda^2 \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \\ &= \lambda^2 \end{aligned}$$

so,

$$E[X^2] = \lambda^2 + \lambda$$

Hence

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= (\lambda^2 + \lambda) - \lambda^2 \\ &= \lambda \end{aligned} \tag{8}$$

From MGF:

$$M_X(t) = \lambda e^t M_X^{(t)}, \quad M''_X(t) = (\lambda e^t + \lambda^2 e^{2t}) M_X^{(t)}$$

Evaluate at $t=0$

$$M'_X(0) = \lambda, \quad M''_X(0) = \lambda + \lambda^2$$

$$\text{So, } E[X] = \lambda, \quad E[X^2] = \lambda + \lambda^2 \text{ and}$$

$$\begin{aligned} \text{Var}(X) &= M''_X(0) - (M'_X(0))^2 \\ &= (\lambda + \lambda^2) - \lambda^2 \\ &= \lambda \end{aligned}$$

matching the analytic calculation.

① Exponential $X \sim \text{Exp}(\lambda) (\lambda > 0)$

Mean :

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx$$

Using integration by parts,

$$\text{let, } u = x \quad dv = \lambda e^{-\lambda x} dx$$

$$\text{Then } du = dx, \quad v = -e^{-\lambda x}$$

$$E[X] = \left[-x e^{-\lambda x} \right]_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}$$

①

Second moment:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

Integration by parts twice: $E[X^2] = \frac{2}{\lambda^2}$

(Equivalently, recognize $X \sim \text{Gamma}(\kappa=1, \theta=\lambda)$)

$$\begin{aligned} \text{so } E[X^2] &= \text{Var} + (E[X])^2 \\ &= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Hence

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

From MGF:

$$M_X(t) = \frac{1}{1-t} \quad (\text{valid for } t < 1)$$

Compute derivatives:

$$M'_X(t) = \frac{1}{(1-t)^2}, \quad M''_X(t) = \frac{2}{(1-t)^3}$$

Evaluate at $t=0$:

$$M'_X(0) = \frac{1}{\lambda^2} = \frac{1}{1}, \quad M''_X(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

so, $E[X] = \frac{1}{\lambda}$, $E[X^2] = \frac{2}{\lambda^2}$ and

$$\text{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ = \frac{1}{\lambda^2}$$

matching the analytic result.

② Binomial $X \sim \text{Bin}(n, p)$

Mean:

$$\begin{aligned} E[X] &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n n \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad \left[\text{using } k \binom{n}{k} = n \binom{n-1}{k-1} \right] \\ &= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\ &= np \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}}, \end{aligned}$$

$$= np$$

⑪

Second moment:

$$\begin{aligned} E[X(X-1)] &= \sum_{k=0}^n k(k-1) \binom{n}{k} p^k (1-p)^{n-k} \\ &= n(n-1)p^2 \quad (\text{a standard identity/combinatorial argument}) \end{aligned}$$

Then

$$\begin{aligned} E[X^2] &= E[X(X-1)] + E[X] \\ &= n(n-1)p^2 + np \end{aligned}$$

Hence

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= (n(n-1)p^2 + np) - (np)^2 \\ &= np(1-p) \end{aligned}$$

From MGF:

$$M_X(t) = (1-p + pe^t)^n$$

Differentiate:

$$M'_X(t) = npe^t(1-p+pe^t)^{n-1}$$

So,

$$M'_X(0) = np$$

(12)

Differentiate again (product + chainrule)

$$M''_X(t) = np e^t (1-p+pe^t)^{n-1} + np e^t (n-1)p e^t \\ (1-p+pe^t)^{n-2}$$

$$= np e^t (1-p+pe^t)^{n-2} ((1-p+pe^t) + (n-1)p e^t)$$

Evaluate at $t=0$ (note: $e^0=1$ and $1-p+p=1$)

$$M''_X(0) = np (1 + (n-1)p) \\ = np + n(n-1)p^2.$$

Thus

$$\text{var}(X) = M''_X(0) - (M'_X(0))^2 \\ = (np + n(n-1)p^2) - (np)^2 \\ = np(1-p)$$

matching with the analytic computation.