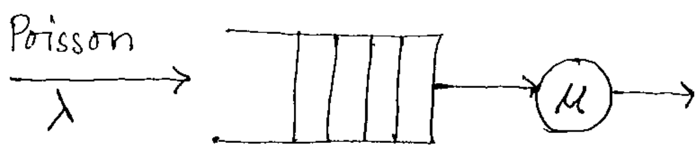
M/M/1

As per the Kendall notation

M : Memoryless, Arrivals, Poisson arrivals
M : Memoryless, Service time, Exponential service-time
1 : Single-server

Here, buffer size infinitely large



- Poisson arrivals at the queue and service-time is exponentially distributed with mean μ.

- Suppose, we consider that n^{th} ~~poo~~ arrival occurs, and it gets serviced for τ_n time. Representing the service time using variable s, we define the quantity $$P\{\tau_n \leq s\}$$

As service time is exponentially distributed then for the $f(t) = \mu e^{-\mu t}$] pdf

So,

$$P\{\tau_n \leq s\} = \int_0^s f(t)\, dt = \int_0^s \mu e^{-\mu t} dt$$

$$= \mu \int_0^s e^{-\mu t} dt = \mu \left[\frac{1}{-\mu} e^{-\mu t}\right]_0^s$$

$$= -\left[e^{-\mu s} - e^0\right] = -\left[e^{-\mu s} - 1\right]$$

$$= 1 - e^{-\mu s}, \text{ where } s \geq 0$$

... (1)

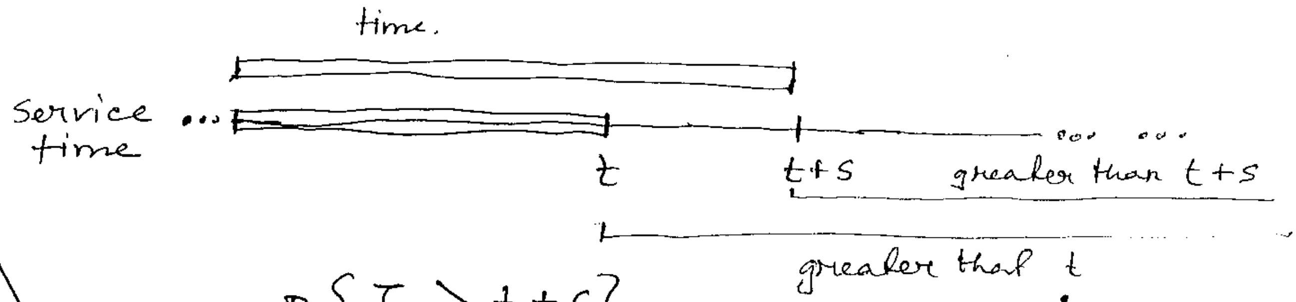
- Given that an arrival (customer/packet/call etc.) is already in service for t seconds.

we might want to know the probability ~~how~~ that ~~long~~ ~~will~~ it waits for additional service time s

That is,

$P\{\tau_n > t+s \mid \tau_n > t\}$ is the term that we are interested in. So,

$$P\{\tau_n > t+s \mid \tau_n > t\} = \frac{P\{\tau_n > t+s \cap \tau_n > t\}}{P\{\tau_n \geq t\}}$$



$$\text{so,} \quad = \frac{P\{\tau_n > t+s\}}{P\{\tau_n > t\}}$$

$$= \frac{1 - P\{\tau_n \leq t+s\}}{1 - P\{\tau_n \leq t\}}$$

$$= \frac{1 - (1 - e^{-\mu(s+t)})}{1 - (1 - e^{-\mu t})} \quad \text{As per Eq. 1}$$

$$= \frac{e^{-\mu(s+t)}}{e^{-\mu t}} = e^{-\mu s}$$

What does it say?

Consider $P\{\tau_n > s\} = 1 - P\{\tau_n \leq s\}$

$= 1 - (1 - e^{-\mu s})$ Eq. 1

$= e^{-\mu s}$

So, $P\{\tau_n > t+s \mid \tau_n > t\} = P\{\tau_n > s\}$ as the derivation suggests.

It means that additional time an arrival in service waits in the service does not depend on the ~~time~~ amount of time it is already in service

Some quantities of a queue that we are interested in

① $\boxed{\text{Queue length distribution.}}$

That is, the number of arrivals being waiting in the queue for service

We calculate P_n, that is probability that there are n-arrivals in the queue ~~[illegible]~~ waiting for their service to start.

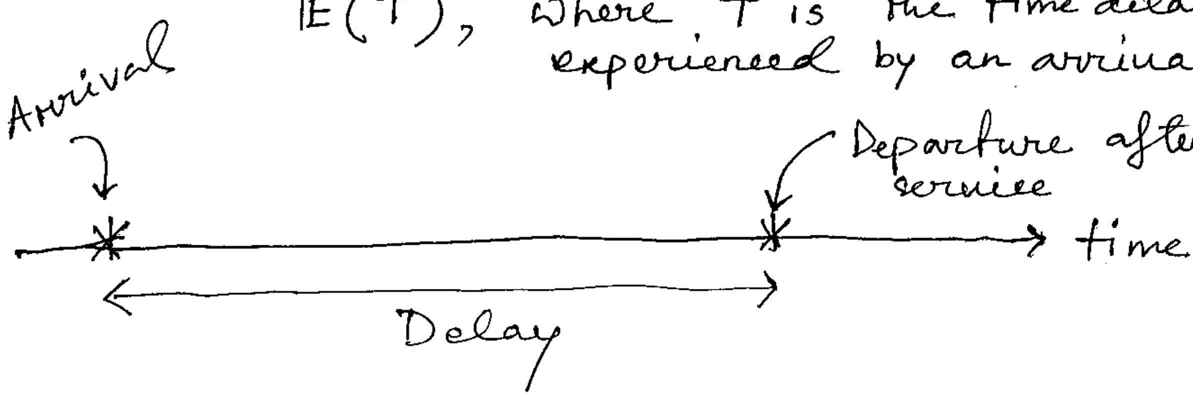
② Average Queue Length

It means the average number of customers/arrivals waiting in the system to get the required service from the ~~[illegible]~~ server.

Simply, Average number of customers in the system.

③ Average Delay Time, also known as Time Delay

$\mathbb{E}(T)$, where T is the time delay experienced by an arrival



④ Throughput : mean number of arrivals/customers processed per unit time.

⊞ Let's consider that P_n stands for
the steady state probability of having n packets/arrivals/calls in the system.

If $P_n(t)$ stands for "n" arrivals in the system at time t, then the steady state probability would be

$$P_n = \lim_{t \to \infty} P_n(t)$$

Question: Suppose at time $t+\Delta t$, the queue is in state n, then we can ask —

Given that the system is at state n at time $t+\Delta t$, What is the ~~probability~~ possible states of the system at time t ?

queue

⊞ Consider that at time $t+\Delta t$ $n \geq 1$, that is there are at least one arrival in the queue. So, the queue state could be any state greater than 1

A few scenario could happen —

- At time t, the queue could be in state $\geq n+2$ or $\leq n-2$ suggesting at least two new arrival or two departure.

However, it happens with probability $o(\Delta t)$

- The queue is in state n at time t, and the following ~~hap:~~ can happen within the next Δt time

    i) No arrival or departure in $(t, t+\Delta t)$

    ii) 1 arrival and 1 departure in $(t, t+\Delta t)$

    iii) Any other scenario other than i and ii is $o(\Delta t)$

- Queue is in state $n-1$ in time t

    i) One arrival and no-departure

    ii) Any other scenario $o(\Delta t)$

- Queue is in state $n+1$ at time t

    i) One departure and no arrival

    ii) Any other scenario $o(\Delta t)$

So,

$$P_n(t+\Delta t) = o(\Delta t) + P_n(t)\left[\mu\Delta t\,\lambda\Delta t + (1-\lambda\Delta t)(1-\mu\Delta t) + o(\Delta t)\right]$$
$$+ P_{n-1}(t)\left[\lambda\Delta t\,(1-\mu\Delta t) + \cancel{\Delta t}\; o(\Delta t)\right]$$
$$+ P_{n+1}(t)\left[\mu\Delta t(1-\lambda\Delta t) + o(\Delta t)\right]$$

$$= o(\Delta t) + P_n(t)\left[\mu\lambda(\Delta t)^2 + \left(1-\mu\Delta t-\lambda\Delta t+\mu\lambda(\Delta t)^2\right) + o(\Delta t)\right]$$
$$+ P_{n-1}\left[\lambda\Delta t - \mu\lambda(\Delta t)^2 + o(\Delta t)\right] + P_{n+1}\left[\mu\Delta t - \mu\lambda(\Delta t)^2 + o(\Delta t)\right]$$

Considering that $(\Delta t)^2$ is very small, we obtain $P_n(t+\Delta t)$ ~~as~~

$$P_n(t+\Delta t) = o(\Delta t) + P_n(t)\left[(1-\mu\Delta t-\lambda\Delta t)\right] + P_{n-1}\left[\lambda\Delta t\right] + P_{n+1}\left[\mu\Delta t\right]$$

$$\text{(1)} \quad \ldots = P_n(t)\left[1-(\mu+\lambda)\Delta t\right] + P_{n-1}(t)\left[\lambda\Delta t\right] + P_{n+1}(t)\left[\mu\Delta t\right] + o(\Delta t)$$

Applying Taylor's expansion to $P_n(t+\Delta t)$ and ignoring Δt^2 terms and beyond

$$P_n(t+\Delta t) = P_n(t) + \frac{d}{dt}P_n(t).\Delta t + o(\Delta t) \quad \ldots \text{(2)}$$

Equating ① and ②

$$P_n(t) + \frac{d}{dt}P_n(t)\Delta t + o(\Delta t) = P_n(t)\left[1-(\mu+\lambda)\Delta t\right] + P_{n-1}(t)[\lambda\Delta t] + P_{n+1}(t)[\mu\Delta t] + o(\Delta t)$$

$$\Rightarrow \frac{d}{dt}P_n(t)\Delta t + o(\Delta t) = -P_n(t)(\mu+\lambda)\Delta t + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)[\mu\Delta t] + o(\Delta t)$$

By dividing Δt, we obtain

$$\frac{\Delta t}{\Delta t}\frac{d}{dt}P_n(t) + \frac{o(\Delta t)}{\Delta t} = -\frac{P_n(t)(\mu+\lambda)\Delta t}{\Delta t} + \frac{P_{n-1}(t)\lambda\Delta t}{\Delta t} + \frac{P_{n+1}(t)\mu\Delta t}{\Delta t} + \frac{o(\Delta t)}{\Delta t}$$

$$\Rightarrow \frac{d}{dt}P_n(t) + \frac{o(\Delta t)}{\Delta t} = -P_n(t)(\mu+\lambda) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu + \frac{o(\Delta t)}{\Delta t}$$

$$\Rightarrow \frac{d}{dt}P_n(t) = -P_n(t)(\mu+\lambda) + P_{n-1}(t)\cdot\lambda + P_{n+1}(t)\cdot\mu$$

Rate of change of Probability that the system state is at n in time t

For stationary probability, $\frac{d}{dt}P_n(t) = 0$

$$0 = -P_n(t)(\mu+\lambda) + P_{n-1}(t)\cdot\lambda + P_{n+1}(t)\cdot\mu$$

$$\Rightarrow P_n(t)(\mu+\lambda) = P_{n-1}(t)\cdot\lambda + P_{n+1}(t)\cdot\mu.$$

As it is stationary, we can get rid of t notation

$$P_n(\mu+\lambda) = P_{n-1}\cdot\lambda + P_{n+1}\cdot\mu, \quad n \geq 1 \quad \ldots ③$$

For, $n=0$, a similar approach yield

$$P_0\lambda = P_1\lambda \quad \ldots \ldots ④$$

By rearranging Eq.3, we obtain

$$\lambda P_n = P_{n-1}\cdot\lambda + P_{n+1}\cdot\mu - P_n\cdot\mu$$

$$\Rightarrow \lambda P_n = P_{n+1}\cdot\mu + [\lambda\cdot P_{n-1} - \mu\cdot P_n] \quad \ldots ⑤$$

M/M/1 : Interpretations of the Balance Equations

The ~~Balance~~ stationary P_n equation as obtained for $n \geq 1$.

$$\underbrace{(\lambda+\mu)P_n}_{\text{LHS}} = \lambda P_{n-1} + \mu P_{n+1}, \quad n \geq 1$$

The LHS denotes that the rate at which the system moves out of state n

- Can happen because of 1 arrival and no departure — a)
- Can happen because of no arrival and one departure — b

for a) Probability that the system moves up (from n to n+1) is

$$\underbrace{\lambda \Delta T}_{\text{1 arrival}} \, (1-\underbrace{\mu \Delta T)}_{\text{no departure}} + o(\Delta T)$$

$$= \lambda \Delta T - \mu\lambda(\Delta T)^2 + o(\Delta T)$$

$$= \lambda \Delta T + o(\Delta T)$$

$(\Delta T)^2$ falls under $o(\Delta T)$

b) Probability that the system moves down (from n to n-1) by one state during ΔT.

$$\underbrace{(1-\lambda\Delta T)}_{\text{No arrival}} \, \underbrace{\mu \Delta T}_{\text{1 departure}}$$

$$= \mu \Delta T - \lambda \Delta T \, \mu \Delta T + o(\Delta T)$$

$$= \mu \Delta T - \mu\lambda(\Delta T)^2 + o(\Delta T)$$

$$= \mu \Delta T + o(\Delta T)$$

So, by considering Eq. 4 and Eq. 5 together, and considering different values of n, we can write

$$\lambda P_0 = \mu P_1 \quad \cdots \quad (6)$$

$$\Rightarrow P_1 = \lambda/\mu \cdot P_0$$

$$\lambda P_1 = \mu P_2 + [\lambda P_0 - \mu P_1] \quad \text{Considering } \lambda P_0 = \mu P_1$$

$$\Rightarrow \lambda P_1 = \mu P_2 \quad \cdots \quad (7)$$

$$\Rightarrow P_2 = \frac{\lambda}{\mu} P_1$$

$$\Rightarrow P_2 = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} P_0$$

$$\Rightarrow P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Again, for n = 2

$$\lambda P_2 = \mu P_3 + [\lambda P_1 - \mu P_2]$$

$$\Rightarrow \lambda P_2 = \mu P_3$$

$$\Rightarrow P_3 = \frac{\lambda}{\mu} P_2$$

$$\Rightarrow P_3 = \frac{\lambda}{\mu} \cdot \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$\Rightarrow P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

... following the recursive derivation, we finally obtain

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Consider $\frac{\lambda}{\mu} = \rho$ and apply the normalization of the probabilities

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1 \quad \cdots \quad (8)$$

Simplifying further, we obtain the below from equation (8)

$$\sum_{n=0}^{\infty} (\rho)^n P_0 = 1$$

$$\Rightarrow P_0 \sum_{n=0}^{\infty} (\rho)^n = 1$$

$$\Rightarrow P_0 \cdot \frac{1}{1-\rho} = 1$$

$$\Rightarrow P_0 = 1 - \rho$$

So, stationary probability that the system is at state n is

$$\boxed{P_n = \rho^n (1-\rho)}$$

For a stable Queue, $\rho < 1$. That is, arrival rate must be less than the service rate

So, Probability of leaving state n in ΔT time

$= P\{\text{leaving state } n \text{ in } \Delta T\}$

$= P\{$ ~~Being~~ staying in state n and moving up or down in $\Delta T\}$

$= P_n(\lambda \Delta T + \mu \Delta T) + o(\Delta T)$

So, Rate of leaving state n is

$$= \lim_{\Delta T \to 0} \frac{P\{\text{leaving state } n \text{ in } \Delta T\}}{\Delta T}$$

$$= \lim_{\Delta T \to 0} \frac{P_n(\lambda \Delta T + \mu \Delta T) + o(\Delta T)}{\Delta T}$$

$$= \lim_{\Delta T \to 0} \left[\frac{P_n(\lambda + \mu)\Delta T}{\Delta T} + \frac{o(\Delta T)}{\Delta T} \right]$$

$$= \lim_{\Delta T \to 0} \left[P_n(\lambda + \mu) + \frac{o(\Delta T)}{\Delta T} \right]$$

$$= P_n(\lambda + \mu) + \lim_{\Delta T \to 0} \frac{o(\Delta T)}{\Delta T}$$

$$= P_n(\lambda + \mu)$$

Rate of leaving state n

Similarly, $\lambda P_{n-1} + \mu P_{n+1}$ stands for the transition rate into P_n.

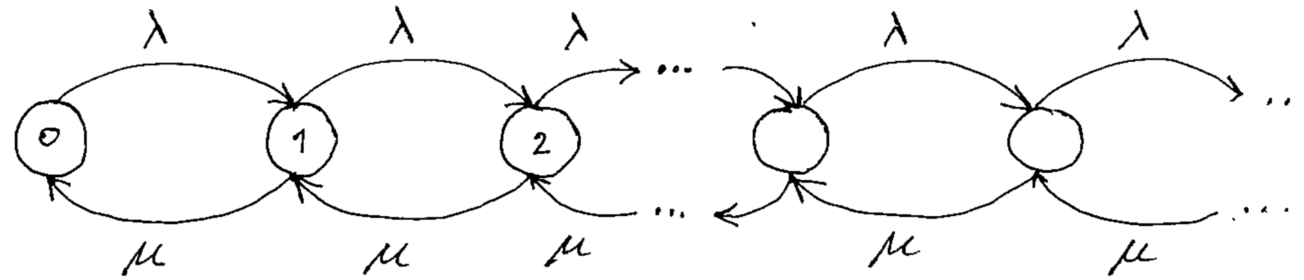
For stationary probability

Rate of leaving = Rate of transition

$$\Rightarrow (\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$$

conservation of rates

State Transition Diagram (STD)



In the STD, state i represents the number of arrivals/packets/calls/customers in the queue.

An arc from i to j represents the rate at which the transition occurs

An arc from j to i represents the rate at which the departure occurs.

Given the above, we can evaluate P_n, the stationary probability that the state is n in two ways:

(1) Global Balance Equations

(2) Local Balance Equations

Global Balance Approach:

We obtained $\lambda P_0 = \mu P_1$

For state $n=2$: From eq^n. $(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$

For $n=1$, $(\lambda+\mu)P_1 = \lambda P_0 + \mu P_2$

$\Rightarrow \lambda P_1 + \mu P_1 = \lambda P_0 + \mu P_2$

$\Rightarrow \lambda P_1 = \mu P_2 + (\lambda P_0 - \mu P_1)$

$\Rightarrow \lambda P_1 = \mu P_2 \quad \cdots (1)$

$\Rightarrow (\lambda + \mu) P_2 = \lambda P_1 + \mu P_3$

$\Rightarrow \lambda P_2 + \mu P_2 = \lambda P_1 + \mu P_3$

$\Rightarrow \lambda P_2 = \lambda P_1 + \mu P_3 - \mu P_2$

$\Rightarrow \lambda P_2 = \mu P_3 + \underbrace{(\lambda P_1 - \mu P_2)}_{\text{From } (1)}$

$\Rightarrow \lambda P_2 = \mu P_3$

So, we obtain following the steps in a recursive way

$$\lambda P_0 = \mu P_1 \quad \Rightarrow \quad P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$$
$$\lambda P_1 = \mu P_2 \quad \Rightarrow \quad P_2 = \frac{\lambda}{\mu} P_1 = \frac{\lambda}{\mu}\cdot\frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$
$$\lambda P_2 = \mu P_3 \quad \Rightarrow \quad P_3 = \frac{\lambda}{\mu} P_2 = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^2 P_0 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$
$$\vdots \qquad\qquad \vdots$$
$$\lambda P_n = \mu P_{n+1} \qquad P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Considering $\left(\frac{\lambda}{\mu}\right) = \rho$ = traffic intensity, we can write $P_n = \rho^n P_0$. By applying normalization of probability $\sum_{n=0}^{\infty} P_n = 1$, we obtain

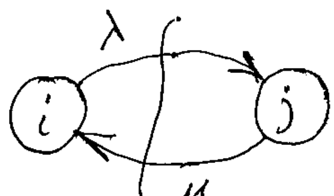
$$P_0 + P_1 + P_2 + \dots + P_n \dots = \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0 \sum_{n=0}^{\infty} (\rho)^n = 1 \quad \Rightarrow P_0 (1-\rho) = 1$$
$$\Rightarrow P_0 = \frac{1}{1/(1-\rho)} = 1-\rho$$

Finally,
$$P_n = \rho^n (1-\rho)$$

Local Balance Equation:

We separate the transition from state $i_{=0}$ to $j_{=1}$ as



$P_0 \lambda = P_1 \mu$

rate from left to right

rate from right to left.

So, $P_0 \lambda = P_1 \mu$
$$\Rightarrow P_1 = \frac{\lambda}{\mu} P_0$$
$$\Rightarrow P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$
$$\vdots$$
Following previous explanations & $\sum_{n=0}^{\infty} P_n = 1$

We obtain $P_n = (1-\rho)\rho^n$