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Homework 1

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Answer to the Question No. 1

Given,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1-x}, \forall n \geq 1 \text{ and } x \neq 1$$

we find this in geomatric progression.

Let $P(n) : 1 + x + x^2 + \cdots + x^{(n-1)} = \frac{1-x^n}{1-x}, \forall n \geq 1$

Base case ($n=1$):

$$R.H.S = \frac{1-x^1}{1-x} = \frac{1-x}{1-x} = 1 = L.H.S$$

Assume $n = k$,

$$1 + x + x^2 + \cdots + x^{k-1} = \frac{1-x^k}{1-x}$$

Now,

$$L.H.S = 1 + x + x^2 + \cdots + x^{(k-1)} + x^k$$

$$= \left[1 + x + x^2 + \cdots + x^{k-1}\right] + x^k$$

$$= \frac{1-x^k}{1-x} + x^k \quad \text{[By induction hypothesis]}$$

$$= \frac{1 - x^k + x^k(1-x)}{(1-x)}$$

$$= \frac{1 - x^k + x^k - x^{(k+1)}}{1-x}$$

$$= \frac{1-x^{(k+1)}}{(1-x)}$$

$$= \frac{1-x^{n}}{1-x} \quad [\because n = k+1]$$

$$= R.H.S$$

[Proved]

Answer to the Question No. 2

Geomatric Series Sum,

$$1 + x + \dots \dots + x^{n-1} = \frac{1-x^n}{1-x}$$

Assume, $A = 1 + x + x^2 + \dots \dots + x^{(n-1)}$

$$\Rightarrow \frac{dA}{dx} = 0 + 1 + 2x + 3x^2 + \dots \dots + (n-1) \cdot x^{(n-2)}$$

$$\Rightarrow \frac{dA}{dx} = 1 + 2x + 3x^2 + \dots \dots + (n-1)x^{(n-2)}$$

$$\Rightarrow x \cdot \frac{dA}{dx} = x + 2x^2 + 3x^3 + \dots \dots + (n-1)x^{(n-1)}$$

$$\Rightarrow \frac{dA^2}{dx^2} = 0 + 2 + 6x + 12x^2 + \dots \dots + (n-1)(n-2)x^{(n-3)}$$

$$\Rightarrow x^2 \frac{dA^2}{dx^2} = 2x^2 + 6x^3 + 12x^4 + \cdots\cdots + (n-1)(n-2)\, x^{(n-3)}$$

$$\Rightarrow \cancel{x^2} A'' = \sum_{k=2}^{n} k(k-1)x^{(k-2)} \quad \left[\text{If } k=2 \text{ to } k=0\right]$$

Therefore,

$$2 + 6x + 12x^2 + \cdots\cdots + n(n-1)x^{n-2} = \sum_{k=2}^{n} k(k-1)x^{(k-2)}$$

We also know,

$$\sum_{k=2}^{n} k(k-1)x^{(k-2)} = \frac{d}{dx^2}\left(\frac{1-x^{n+1}}{1-x}\right)$$

$$\text{so,} \quad A'' = \frac{d^2}{dx^2}\left(\frac{1-x^{n+1}}{1-x}\right)$$

$$\Rightarrow x^2 A'' = 2x^2 + 6x^3 + \cdots\cdots + n(n-1)x^n$$

Now,

$$S = A'' = \frac{d^2}{dx^2}\left(\frac{1-x^n}{1-x}\right)$$

$$\Rightarrow 2 + 6x + 12x^2 + \cdots\cdots + n(n-1)x^{(n-2)} = \frac{d^2}{dx^2}\left(\frac{1-x^n}{1-x}\right)$$

$$\Rightarrow 2 + 6x + 12x^2 + \cdots\cdots + n(n-1)x^{(n-2)}$$

$$= \sum_{k=2}^{n} k(k-1)x^{(k-2)}$$

[Expanded form]

Answer to the Question No. 3

$$\frac{e^k}{3^{k-1}} = e \cdot \left(\frac{e}{3}\right)^{k-1}$$

The geomatric Series of the form is,

$$\sum_{k=1}^{\infty} ar^{k-1}$$

formula for the infinite geomatric series for $|r| < 1$:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

Now, by plugging in the values:

$$\sum_{k=1}^{\infty} \frac{e^k}{3^{k-1}} = \frac{e}{1-\frac{e}{3}}$$

$$= \frac{e}{\frac{3-e}{3}}$$

$$= \frac{e \cdot 3}{3-e}$$

$$= \frac{3e}{3-e}$$

This series converges only if $|r| < 1$, i.e., $|e/3| < 1$ which is true since $e \approx 2.718 < 3$

Therefore, $\sum_{k=1}^{\infty} \frac{e^k}{3^{k-1}} = \frac{3e}{3-e}$

Answer to the Question No. 4

If $f(x,y)$ is continuous on the rectangle

$R = [a, b] \times [c \times d]$ then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Now,

$$I = \iint_R (1-(x-1)^2 + 4y^2).\,dA$$

where, $R = [0,4] \times [0,4]$

By Fubini's Theorem,

$$I = \int_{x=0}^{4} \int_{y=0}^{4} [1 - \cancel{(-1)}(x-1)^2 + 4y^2]\,dy\,dx$$

Integrate with respect to y,

$$\int_{y=0}^{4} [1-(x-1)^2 + 4y^2]$$

$$= \int_{y=0}^{4} 1, dy - \int_{y=0}^{4} (x-1)^2, dy + \int_{y=0}^{4} 4y^2, dy$$

$$= y\Big|_0^4 - (x-1)^2 \cdot y\Big|_0^4 + 4\left[\frac{y^3}{3}\right]_0^4$$

$$= 4 - 4(x-1)^2 + 4 \cdot \frac{64}{3}$$

$$= 4 - 4(x-1)^2 + \frac{256}{3}$$

Integrate with respect to x,

$$\int_{x=0}^{4} \left[1 - (x-1)^2 + \frac{25.6}{3}\right] dx$$

$$= \int_{x=0}^{4} 4, dx - \int_{x=0}^{4} 4(x-1)^2, dx + \int_{x=0}^{4} \frac{256}{3}$$

$$= 4x\Big|_0^4 - \int_{x=0}^{4} (x^2 - 2x + 1) dx + \frac{256}{3}(4-0)$$

$$= 4 \cdot 4 - \left[\frac{x^3}{3} - x^2 + x\right]_0^4 + \frac{1024}{3}$$

$$= 16 - \left[\frac{4^3}{3} - 4^2 + 4\right] + \frac{1024}{3}$$

$$= 16 - \frac{64 - 48 + 12}{3} + \frac{1024}{3}$$

$$= 16 - \frac{112}{3} + \frac{1024}{3}$$

$$= 16 + \frac{1024 - 112}{3}$$

$$= 16 + \frac{912}{3}$$

$$= 16 + 304$$

$$= 320$$

Therefore, $\iint_R \left[1 - (x-1)^2 + 4y^2\right] dA = 320$ Ans.

Answer to the Question No. 5

(a) The region is a square as both (x, y) go from 0 to 2.

Region = R where $R = (x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2$

Thus the region R can be split as:

$R_1 = (x, y) \mid 0 \le x \le 2,\ 0 \le y \le x$

$R_2 = (x, y) \mid 0 \le x \le 2,\ x \le y \le 2$

So, $I = \iint_{R_1} x \, dA + \iint_{R_2} y \, dA$

$$= \int_{x=0}^{2} \int_{y=0}^{x} x \, dy \, dx + \int_{x=0}^{2} \int_{y=x}^{2} y \, dy \, dx$$

Calculating the first integral;

$$\int_0^2 \left[\int_0^x x \, dy \right] dx$$

$$= \int_0^2 \left[xy \right]_0^x dx$$

$$= \int_0^2 \left[x \cdot x \right] dx$$

$$= \int_0^2 x^2 \, dx$$

$$= \left[\frac{x^3}{3} \right]_0^2$$

$$= 8/3$$

Calculating the second integral:

$$\int_0^2 \left[\int_x^2 y \, dy\right] dx$$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_x^2 dx$$

$$= \int_0^2 \left[4/2 - x^2/2\right] dx$$

$$= \int_0^2 2 \, dx - \int_0^2 x^2/2 \, dx$$

$$= \left[2x\right]_0^2 - 1/2 \left[x^3/3\right]_0^2$$

$$= 4 - 1/2 \times 8/3$$

$$= 4 - 4/3$$

$$= 8/3$$

So, $I = 8/3 + 8/3 = 16/3$

Double Integral over $0 \leq x \leq 2$, $0 \leq y \leq 2$ of $\max(x, y) \, dA = \frac{16}{3}$ Ans.

(6) $\int_0^1 \left[\int_0^x \sqrt{1-x^2}\, dy \right] dx$

$= \int_0^1 \left[\sqrt{1-x^2} \times \int_0^x 1\, dy \right] dx$

$= \int_0^1 x\sqrt{1-x^2}\, dx$

Let, $u = 1-x^2$

$\Rightarrow du = -2x\, dx$

$\Rightarrow -du/2 = x\, dx$

when $x = 0$, $u = 1$

when $x = 1$, $u = 0$

So, $\int_0^1 x \cdot \sqrt{1-x^2}\, dx$

$= \int_0^1 \sqrt{u}\, -\frac{du}{2}$

$= \frac{1}{2} \left[2/3\, u^{3/2} \right]_0^1$

$= 1/2 \times 2/3 \times (1-0)$

$= 1/3$ Ans.

Answer to the Question No. 6

Surface, $z = \sqrt{4-r^2}$

Region: Under this surface, above the part of the circle $x^2+y^2=9$ in the first quadrant.

The circle $x^2+y^2=9$ has radius 3 and center (0,0) in first quadrant means $x \geq 0$, $y \geq 0$.

In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$

first quadrant: $0 \leq \theta \leq \pi/2$

for the circle: $0 \leq r \leq 3$

Volume $V = \iint (\text{over } R)\, z\, dA$

where in cylindrical coordinates:

$dA = r\,dr\,d\theta$

$z = \sqrt{(4-r^2)}$

R = Described Region.

Limits:

$\theta : 0$ to $\pi/2$

$r : 0$ to 3

So, $v = \int_0^{\pi/2} \left[\int_0^3 \sqrt{4-r^2} \times r \, dr \right] d\theta$

Let, $I = \int_0^3 \sqrt{4-r^2} \times r \, dr$

Let, $u = 4 - r^2$, so $du = -2r\,dr$

$\Rightarrow \frac{du}{2} = r\,dr$

when $r=0$, $u=4$

when $r=3$, $u = 4-9 = -5$

So, $I = \int_4^{-5} \sqrt{u} \left(-\frac{du}{2}\right)$

$= \frac{1}{2} \int_{-5}^{4} \sqrt{u}\, du$

But $\sqrt{u}$ is not real for $u<0$, so the surface only exist where $4-r^2 \geq 0 \Rightarrow r \leq 2$

Now,

$r: 0$ to 2

when, $r = \cancel{0}2$, $u=0$ (so surface meets the plane $z=0$

when, $r=0$, $u=4$

so, $I = \int_0^2 \sqrt{4-r^2}\, r\, dr$

Let $u = 4 - r^2$, $r\,dr = -du/2$

when $r = 0$, $u = 4$

when $r = 2$, $u = 0$

$$I = \int_4^0 \sqrt{u}\; -\left(du/2\right)$$

$$= 1/2 \int_0^4 \sqrt{u}\, du$$

$$= 1/2 \int_0^4 u^{1/2}\, du$$

$$= \frac{1}{2}\left[2/3\, u^{3/2}\right]_0^4$$

$$= 1/2\left(2/3 \times 4^{3/2}\right) - 4$$

$$= 1/3 \times 8$$

$$= 8/3$$

Now, computing the θ integral,

$$V = \int_0^{\pi/2} 8/3\, d\theta$$

$$= 8/3\left[\theta\right]_0^{\pi/2}$$

$$= 8/3\left(\pi/2 - 0\right)$$

$$= 4\pi/3 \text{ Ans.}$$

Answer to the Question No. 7

ⓐ $\int \tan^{-1} x \, dx$

Let $u = \tan^{-1} x$, $du = dx$

then, $du = \left(\frac{1}{1+x^2}\right) dx$, $v = x$

By integration by parts:

$$\int u \, dv = uv - \int v \, du$$

So, $\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \left[\frac{1}{1-x^2}\right] dx$

Now,

$$\int \frac{x}{1-x^2} dx$$

Let, $w = 1 + x^2$

$\Rightarrow dw = 2x \, dx$

$\Rightarrow \frac{1}{2} dw = x \, dx$

So, $\int \frac{x}{1-x^2} dx$

$= \frac{1}{2} \int \frac{1}{w} dw$

$= \frac{1}{2} \ln|w| + c$

$= \frac{1}{2} \ln(1+x^2) + c$ $\quad \therefore \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$

Ans.

(ii) $\int x^2 e^x dx$

Let $u = x^2$, $dv = e^x dx$

Then $du = 2x\,dx$, $v = e^x$

By parts:

$\int x^2 e^x dx$

$= \int x^2 e^x - \int 2x e^x dx$

Now, Solving the second part again:

let $u = x$, $dv = e^x dx$

Then $du = dx$, $v = e^x$

So, $\int x e^x dx$

$= 2(xe^x - e^x)$

$= 2xe^x - 2e^x$

Going back: $\int x^2 e^x dx$

$= x^2 e^x - [2xe^x - 2e^x]$

$= x^2 e^x - 2xe^x + 2e^x + c$

$= (x^2 - 2x + 2)e^x + c$ Ans.

(c) $\int_0^{\infty} \lambda x \, e^{(-\lambda x)} dx$

Let $u = x$, $dv = \lambda e^{(-\lambda x)} dx$

Then, $du = dx$, $v = -e^{(-\lambda x)}$

By parts: $\int x \lambda e^{(-\lambda x)} dx$

$$= -x e^{-\lambda x} - \int -e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} + \int e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} - 1/\lambda \, e^{-\lambda x}$$

Evaluate from $x = 0$ to $x = \infty$

As $x \to \infty$, $e^{-\lambda x} \to 0$, so both

$x e^{-\lambda x}$ and $e^{-\lambda x} \to 0$

At $x = 0$:

$-x e^{-\lambda x} = 0$

$-\frac{1}{\lambda} e^{-\lambda x} = -\left(1/\lambda\right) \times 1 = -\frac{1}{\lambda}$

So, value of $\infty = 0$

value of $0 = 0 - \left(-1/\lambda\right)$

$= \frac{1}{\lambda}$

$\therefore \int_0^{\infty} \lambda x \, e^{(-\lambda x)} dx = 1$. Ans.

(d) $\int_{-\infty}^{\infty} \left(\frac{\lambda x}{2}\right) e^{-\lambda|x|} dx$

for $x < 0$, $|x| = -x$

for $x > 0$, $|x| = x$

so, $\int_{-\infty}^{\infty} \left(\frac{\lambda x}{2}\right) e^{-\lambda|x|} dx$

$= \int_{-\infty}^{0} \frac{\lambda x}{2} e^{\lambda x} dx$

Now, $\int_{0}^{\infty} \frac{\lambda x}{2} e^{\lambda x} dx$

Computing first part:

Let $I_1 = \int_{-\infty}^{0} \lambda x/2 \, e^{\lambda x} dx$

let $u = x$, $dv = e^{\lambda x} dx$

$du = dx$, $v = \frac{1}{\lambda} e^{\lambda x}$

Integration by parts

$I_1 = \lambda/2 \int x e^{\lambda x} dx$

$= \lambda/2 \left[x \cdot 1/\lambda \, e^{\lambda x} - \int 1/\lambda \, e^{\lambda x} dx\right]$

$$= \lambda/2 \left[x/_\lambda \, e^{\lambda x} - \frac{1}{\lambda^2} e^{\lambda x} \right]$$

$$= 1/2 \left[x \cdot e^{\lambda x} - \left(\frac{1}{\lambda}\right) e^{\lambda x} \right]$$

[Evaluated from $x = -\alpha$ to $x = 0$]

At $x = 0$:

$$x e^{\lambda x} = 0$$

$$1/_\lambda \, e^{\lambda x} = \frac{1}{\lambda}$$

At $x = \alpha$:

$x \cdot e^{\lambda x} \longrightarrow 0$ (since $x \to -\alpha$ and $e^{(\lambda x)} \to 0$ very fast)

$$1/_\lambda \, e^{\lambda x} \longrightarrow 0$$

So, $I_1 = 1/2 [0 - 1/_\lambda] - 0$

$$= -1/_{2\lambda}$$

Now, the second part:

$$I_2 = \int_0^\alpha \frac{\lambda x}{2} e^{-\lambda x} dx$$

Again let $u = x$, $dv = e^{-\lambda x} dx$

$du = dx$, $v = -(1/_\lambda) e^{-\lambda x}$

$$I_2 = \lambda/2 \int x e^{-\lambda x}\, dx$$

$$= \lambda/2 \left[-\frac{x}{\lambda} e^{-\lambda x} + \left(\frac{1}{\lambda^2}\right) e^{-\lambda x} \right]$$

$$= 1/2 \left[-x e^{-\lambda x} - \left(1/\lambda\right) e^{-\lambda x} \right]$$

[Evaluate from $x=0$ to $x=\alpha$]

At $x = \alpha$:

$$-x e^{-\lambda x} \longrightarrow 0$$

$$(1/\lambda)\, e^{-\lambda x} \longrightarrow 0$$

At $x = 0$:

$$-x e^{-\lambda x} = 0$$

$$(1/\lambda)\, e^{-\lambda x} = \frac{1}{\lambda}$$

So, $I_2 = 1/2 \left[0 - 0\right] - \left[0 - 1/\lambda\right]$

$$= 1/2 \times 1/\lambda$$

$$= 1/2\lambda$$

Adding part ① and ②:

$$I_1 + I_2$$

$$= -1/2\lambda + 1/2\lambda$$

$$= 0$$

$\therefore \int_{-\alpha}^{\alpha} \lambda x/2 \, e^{-\lambda|x|}\, dx = 0$ Ans.

Counting

Answer to the Question No 1

Ways to split 10 students into three teams (Team 1, Team 2, Team 3):

Choose 2 students for Team 1

Number of ways $= c(10,2)$

Choose 4 students for Team 2 from remaining 8 students

Number of ways $= c(8,4)$

The ~~remaing~~ remaining 4 students go to Team 3

Number of ways $= 1$

So, total number of ways $= c(10\times2)\times c(8,4)\times 1$

But, if teams are unlabled, we can divide by 2!

since Team 2 and Team 3 are of equal size and swapping them doesn't split into a new one.

So, total number of ways $= \dfrac{[c(10,2)\times c(8,4)]}{2}$

where,

$$c(n,k) = \frac{n!}{k!\,(n-k)!}$$

By calculating we get,

$C(10, 2) = 45$

$C(8, 4) = 70$

So, total $= \frac{45 \times 70}{2}$

$= \frac{3150}{2}$

$= 1575$ Ans.

Therefore, 1575 ways to split 10 students into teams of 2, 4 and 4 Ans.

Answer to the Question NO 2

I want to blend the five other flavours with the plain cake, and I can use any subset (including none) of the five flavours.

Each flavour can either be included or not included in the blend.

Number of possible combinations $= 2^5 = 32$

So, I can make 32 different types of cake including the plain cake with no flavour at all. A.

Answer to the Question No. 3

ⓐ Let number of red dice chosen = r

Let number of white dice chosen = w

Total dice chosen : $r + w = 4$

At least three red dice : $r = 3$ or 4

Case 1 :

$r = 3$, $w = 1$

Number of ways = $C(12, 3) \times C(6 \times 1)$

$= 220 \times 6$

$= 1320$

Case 2 :

$r = 4$, $w = 0$

Number of ways = $C(12, 4) \times C(6, 0)$

$= 495$

∴ Total number of ways = $1320 + 495$

$= 1815$

Therefore, 1815 ways to choose 4 dice with at least 3 red. Ans.

(6) Total number of ways to choose 4 dice from 18 (12 red + 6 white):

$$\text{Total} = C(18, 4) = 3060$$

Number of ways with no red dice (all white):

$$C(6, 4) = 15$$

Number of ways with at least one red dice

$$= \text{Total} - \text{all white}$$
$$= 3060 - 15$$
$$= 3045$$

Therefore, 3045 ways to choose 4 dice with at least one red in each choice.

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